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Completeness of Propositional Provability Logic

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Abstract. Propositional provability logic stems from the desire to investigate what mathematical theories can say about themselves. In this thesis, we will discuss multiple papers on the completeness of propositional provability logic. It turns out that propositional provability logic is complete with respect to the class of finite irreflexive trees and arithmetically complete with respect to Peano Arithmetic. The last result gave rise to exploring the boundaries of propositional provability logic with respect to weaker arithmetics. We will discuss a class of theories discovered by D. de Jongh who has proven its arithmetical completeness based on the proof that R.M. Solovay provided for Peano Arithmetic.

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1 Introduction

In this report, we will discuss papers on propositional provability logic and its completeness with respect to finite irreflexive trees, Peano Arithmetic and weaker arithmetics. Propositional provability logic originates from trying to answer the following question:
What can mathematical theories say about themselves by encoding interesting properties?

Before we will explain how this question was (partly) answered, we will first provide some background information. Propositional provability logic is a modal logic which is also called GL after Gödel and Löb who are responsible for its existence. For the rest of this paper we will use the name GL instead of propositional provability logic. The logic GL is based on modal logic K (which will be recalled in Section 2.1) with one additional axiom. This axiom is called Löb's axiom, after the man who has proposed it. This axiom is a result of the above question. It says that if the following implication can be proven: a sentence is provable implies that this sentence is indeed true in that language; then the sentence can indeed be proven.

We will discuss the papers on the completeness of GL in a chronological fashion in order to understand the results and implications of the papers. In this way, we will create an overview on how undecidable theories can be explored using decidable logic.

In section 2, we introduce some definitions and notation. The notation we have chosen is important to state, because this varies a lot for different papers and books on the subject. In section 3, we recall the proof on finite trees by C. Smoryński. In section 4, we discuss Solovay's arithmetical completeness theorem of GL with respect to Peano Arithmetic. And in section 5, we discuss a paper by de Jongh and others that investigates the limits of Solovay's result.

2 Preliminaries

In this section, we discuss some definitions and concepts that will help us to fully understand the proofs by Smoryński, Solovay and de Jongh. First, we give the definition of the modal logic GL. We introduce the concept of frames and trees and their notation. We will briefly discuss the definitions of soundness and completeness. Finally, we discuss the formal system of Peano Arithmetic and the concept of Σ_1-sentences, which will be used in Solovay's theorem and proof.

2.1 Propositional provability logic

The propositional provability logic, which we will call GL, is an extension of the modal logic K. Additionally to the operators in propositional logic, the modal logic K has the necessity operator and the possibility operator, which are denoted by $\Box$ and $\Diamond$, respectively.

The modal logic K is formed from propositional logic by adding the following two principles [6]:

1. The Necessitation Rule: If A is a theorem of K, then so is $\Box A$.
2. The Distribution Axiom: $\Box(A \to B) \to (\Box A \to \Box B)$.

We extend the logic K with the following axiom in order to form logic GL:

$$\text{Löb's Axiom: } \Box(\Box A \to A) \to \Box A \tag{1}$$

The $\Box$-operator in GL is translated to "is provable in T" in arithmetic, where T is a sufficiently strong formal theory. The $\Diamond$-operator in $\Diamond A$ will just be a shorthand notation for $\neg\Box\neg A$.

Propositional logic can be axiomatized using the following (nonmodal) axioms:

1. $A \to (B \to A)$
2. $(C \to (A \to B)) \to ((C \to A) \to (C \to B))$
3. $((A \to \bot) \to \bot) \to A$

and the inference rule Modus Ponens, in short MP, which allows us to derive B from $A \to B$ and A. Since GL is an extension of propositional logic, inference rule MP and the above, so called nonmodal axioms, are valid in GL [14]. Furthermore, K contains all instantiations of theorems of propositional logic and therefore so does GL.

2.2 Frames and trees

For frames, models and trees there are many different notations in use. In this section we not only give their definitions, but also the notation, which we will use throughout this paper. The notation will partly be from Smoryński [10], and partly from Segerberg [9].

Definition 2.1. *A frame F is a triple (W, R, α_0), where W is a non-empty set of nodes (including α_0), R is a transitive binary relation on W (i.e. for $\alpha, \beta, \gamma \in W$, $\alpha R \beta$ and $\beta R \gamma$ imply $\alpha R \gamma$), and α_0 is a minimum element of W with respect to R (i.e. for any $\beta \in W$ other than α_0, $\alpha_0 R \beta$).*

Often, a frame is only denoted by (W, R) where R need not be transitive. However, in this paper R will always be transitive and the notation with α_0 proves to be useful.

In order to evaluate formulas on our nodes, we will add the notion of valuation to the frame, which we will then call a Kripke model.

Definition 2.2. *A model, short for Kripke model, is a quadruple $\mathfrak{M} = (W, R, \alpha_0, V)$, where (W, R, α_0) is a frame and V is a function which assigns to a propositional atom the set of all nodes in which the formula is true. We will extend the concept of truth to all formulas using the operator $\models$:*

i. *For each $n \in \mathbb{N}$, $\mathfrak{M}, \alpha \models p_n$ iff $\alpha \in V(p_n)$, where p_n is a propositional atom.*

ii. $\mathfrak{M}, \alpha \not\models \bot$.

iii. $\mathfrak{M}, \alpha \models A \to B$ *iff (if* $\mathfrak{M}, \alpha \models A$ *then* $\mathfrak{M}, \alpha \models B$*).*

iv. $\mathfrak{M}, \alpha \models \Box A$ *iff for all* $\beta \in W$ *(if* $\alpha R \beta$ *then* $\mathfrak{M}, \beta \models A$)

where α is some node in W.

Definition 2.3. *A formula A is true in a model $\mathfrak{M}$, we write $\mathfrak{M} \models A$, if it is true at every point of the model.*

We will put some restrictions on the notion of a frame in order to get a smaller class, which we call trees.

Definition 2.4. *A tree $\mathcal{T}$ is a frame $(W, <, \alpha_0)$ in which*

i. *$<$ is a strict partial ordering which means that it is transitive and asymmetric.*

ii. *the set of predecessors of any element is finite and linearly ordered by $<$, i.e. for $\alpha \in W$, $\{\beta \in W | \beta < \alpha\}$ consists of $\beta_1 < \beta_2 < ... < \beta_k$ for some k.*

Since $\mathcal{T}$ is a frame, the node α_0 is the minimum element, which means that for any $\beta \in W$ other than α_0, $\alpha_0 < \beta$. We will call α_0 the root of the tree.

2.3 Soundness and completeness

We would like to be able to compare a logic to a class of frames. To do this we will define the concepts validity, soundness and completeness [9].

Definition 2.5. *A formula is valid in a frame if it is true in every model on the frame.*

Definition 2.6. *A logic $\mathcal{L}$ is sound with respect to a class of frames $\mathcal{C}$ if every frame in $\mathcal{C}$ is a frame for $\mathcal{L}$.*

A frame F is a frame for a set of formulas Σ if every formula in Σ is valid in F. This means that a logic is sound with respect to a class of frames if every formula in the logic is valid in every frame in the class.

Definition 2.7. *A logic $\mathcal{L}$ is complete with respect to a class of frames $\mathcal{C}$ if every formula valid in $\mathcal{C}$ is a theorem of $\mathcal{L}$.*

2.4 Peano Arithmetic

"Number theory or arithmetic may be described as the branch of mathematics that deals with the natural numbers and other (categorically defined) enumerable systems of objects, such as the integers or the rational numbers" [8].
Peano arithmetic, in short PA, is a consensus about "obviously true" claims about the natural numbers. PA is formulated in a first-order language that has the constant symbol 0, variables: $x, y, z, ...$, the identity predicate (=), function symbols: + (plus), $\times$ (times) and the unary successor function s. All logical symbols from before are used, as well as the existential and universal quantifiers, $\exists$ and $\forall$. The first seven axioms are:

1. $\forall x(s(x) \neq 0)$
2. $\forall x \forall y(s(x) = s(y) \leftrightarrow x = y)$
3. $\forall x(x + 0 = x)$
4. $\forall x \forall y[x + s(y) = s(x + y)]$
5. $\forall x(x \times 0 = 0)$
6. $\forall x \forall y[x \times s(y) = (x \times y) + x]$
7. $\forall x \forall y \forall z(x = y \to (x = z \to y = z))$

The axioms 3 to 6 define the binary function symbols + and $\times$.
PA has an axiom scheme capturing the principle of mathematical induction on the natural numbers (the seventh axiom). This can be stated as follows:

8. $[Q(0) \wedge \forall x(Q(x) \to Q(s(x)))] \to \forall x Q(x)$

Note that all tautologies in propositional logic are axioms.

9. all propositional tautologies

Finally, the inference rule on PA is Modus Ponens [1] [8]. We will show in Section 4.1 that the Necessitation Rule also preserves validity on PA.

It would be inconvenient to write multiple applications of s to the constant 0 as it is unclear and can take up a lot of space. Therefore, we shall take $1, 2, ..., \mathbf{k}$ to be formal expressions of $s(0), s(s(0)), ..., s(...s(0))$. We will call $1, 2, ..., \mathbf{k}$ numerals.

We would like to investigate provability for PA, therefore we define the notion 'provable' as follows [8]:

Definition 2.8. *A sentence D in PA is provable if*

1. *D is an axiom of PA, or*
2. *D is an immediate consequence (by a rule of inference) of a sentence E in PA and E is provable, or*
3. *D is an immediate consequence (by a rule of inference) of sentences E and F in PA and both E and F are provable.*

A formula is provable only as required by the above three statements.

2.4.1 Gödel numbering

Gödel came up with a numbering that allows us to code every proof uniquely. Note that there need not be a unique number to prove a sentence in Peano Arithmetic since there might exist different proofs for the same formula. We will use the Gödel numbering to define a proof predicate. Namely, we will denote the Gödel number of an arithmetical formula A by $\ulcorner A \urcorner$. We define the predicate $\mathbf{Prov}(x)$ to be $\exists p \mathbf{Proof}(p, x)$, where $\mathbf{Proof}(p, x)$ stands for "Gödel number p codes a correct proof from the axioms of Peano Arithmetic of the formula with Gödel number x" [14].

2.4.2 Interpretation

In order to compare the logic GL to Peano Arithmetic, we define an interpretation that maps every propositional atom p in modal logic to a sentence p^* of arithmetic. We define the following restriction on the interpretation:

$$\begin{aligned} &1)\ (\bot)^* = \text{"}0 = 1\text{"} \\ &2)\ (A \to B)^* = \text{"}A^* \to B^*\text{"} \\ &3)\ (\Box A)^* = \text{"}\mathbf{Prov}(\ulcorner A^* \urcorner)\text{"} \end{aligned}$$

where A and B are modal formulas and $\ulcorner A^* \urcorner$ is the Gödel number of a A^* [12].

Based on an interpretation we can determine whether a logic is arithmetically sound.

Definition 2.9. *A logic $\mathcal{L}$ is called arithmetically sound with respect to Peano Arithmetic if every formula that can be proven in $\mathcal{L}$ is logically valid with respect to every interpretation of the logic in Peano Arithmetic.*

Thus $\mathcal{L}$ is arithmetically sound if for an arbitrary formula C, if $\vdash_{\mathcal{L}} C$ then $\vdash_{PA} C^*$ for all interpretations $*$.

Definition 2.10. *A deductive theory is said to be consistent if no two asserted statements of this theory contradict each other, or, in other words, if of any two contradictory sentences at least one cannot be proven [13].*

Thus Peano Arithmetic is consistent if for any formula A we do not have A and $\neg A$ both provable in the system.

2.5 Arithmetical hierarchy

We will not only discuss Peano Arithmetic but also other arithmetical theories. A theory T, which is a set of sentences, can be classified based on its complexity. A classification we will look into is the one based on the existential and universal quantifier. Specifically, we will define the classes Σ_0 and Σ_1.

Before we do so, we will first add a predicate to the language of Peano Arithmetic. We define $\leq$ to be a predicate such that $x \leq y$ is equivalent to the formula $\exists z(x + z = y)$ [3].

"For any variable v_i and any variable or numeral c, we write $(\forall v_i \leq c)(...)$ as an abbreviation of $\forall v_i(v_i \leq c \rightarrow (...))$ and $(\exists v_i \leq c)(...)$ as an abbreviation of $\exists v_i(v_i \leq c \wedge (...))$. We refer to $(\forall v_i \leq c)$ and $(\exists v_i \leq c)$ as bounded quantifiers. By an atomic Σ_0-formula, we mean a formula of any of the four forms $c_1 + c_2 = c_3$, $c_1 \times c_2 = c_3$, $c_1 = c_2$ or $c_1 \leq c_2$. The class of Σ_0-formulas is then defined recursively:

1. Every atomic Σ_0-formula is a Σ_0-formula.
2. If F_1 and G_1 are Σ_0-formulas, then so are $F_1 \wedge G_1$, $F_1 \vee G_1$, $F_1 \rightarrow G_1$ and $\neg F_1$.
3. If F is a Σ_0-formula, v_i is a variable and c is a numeral or a variable distinct from v_i, then $(\forall v_i \leq c)F$ and $(\exists v_i \leq c)F$ are Σ_0-formulas.

Thus all quantifiers of Σ_0-formulas are bounded." [11]

A Σ_1-formula is a Σ_0-formula or a formula of the form $\exists v_i F$, where F is a Σ_0-formula [11].

Definition 2.11. *A theory T is Σ_1-sound if every Σ_1-sentence provable in T is true in the standard model of arithmetic $\mathbb{N}$ (i.e., the structure of the usual natural numbers with addition and multiplication).*

Definition 2.12. *A theory T is Σ_1-complete if every Σ_1-sentence that is true in the standard model $\mathbb{N}$ is provable in T [3].*

3 Smoryński on finite trees

Now that we have established the necessary knowledge and notation, let us take a look at the soundness and completeness of logic GL with respect to the class of all finite trees. Originally it was K. Segerberg who was able to prove this completeness [9]. However, a more constructive proof is given in the book "Self-Reference and Modal Logic" by C. Smoryński [10]. Smoryński gives three equivalent statements in his theorem. We will leave out one of the equivalent statements since it is not of importance to us. However, this means that the proof is slightly modified. In this new proof we will extend on Smoryński's proof by providing more detail about the soundness of GL with respect to finite trees and about the construction of a tree in the completeness part.

For the soundness proof, we will use Theorem 3.3 of K. Segerberg [9]. We will first give the definition of a classical system, since it is part of Segerberg's terminology.

Definition 3.1. *A classical system is a modal system with an additional inference rule: from $A \leftrightarrow B$ infer $\Box A \leftrightarrow \Box B$. This inference rule is called RE, which is short for replacements of material equivalents.*

Note that a classical system is not equivalent to the modal logic K, since it does not include the Distribution Axiom and the Necessitation Rule. However, the modal logic K is a classical system since validity of the inference rule RE can be proven in K.

Definition 3.2. *We define logic E to be the smallest classical system.*

It follows that modal logic K is an extension of the logic E.

Theorem 3.3. *Suppose that $\mathcal{L}$ is a classical system. Let $\mathcal{C}$ be any class of frames. If every modal axiom of $\mathcal{L}$ is valid in $\mathcal{C}$, then $\mathcal{L}$ is sound with respect to $\mathcal{C}$.*

The logic GL is an extension of modal logic K and thereby also an extension of logic E. We can conclude that GL is an extension of a classical system and we can use Theorem 3.3. We still need to prove that all additional inference rules preserve validity in the class $\mathcal{C}$. Thus we will have to prove that the modal axioms of GL are valid in the class of finite trees and that the Necessitation rule preserves validity in the class of finite trees. First we will show that Axiom 4 of modal logic follows from Löb's axiom such that the validity of Axiom 4 follows from the validity of Löb's axiom. The proof works out the steps given in [14]. Often the logic GL is defined to include Axiom 4, however, after we have proven this implication, we do not need Axiom 4 in our definition of GL. Axiom 4 is the axiom

$$\Box A \to \Box\Box A$$

We let RR denote the inference rule: from $A \to B$, infer $\Box A \to \Box B$. This rule can be proven using the Distribution axiom and the Necessitation Rule.

Theorem 3.4. *Axiom 4 is provable in GL.*

Proof. Recall that Löb's axiom states the validity of formula

$$\Box(\Box A \to A) \to \Box A$$

By substituting $A \wedge \Box A$ for A in Löb's axiom we get

$$\vdash \Box(\Box(A \wedge \Box A) \to (A \wedge \Box A)) \to \Box(A \wedge \Box A) \tag{2}$$

We will use the Distribution axiom, the Necessitation rule and some propositional logic to prove the following two statements:
(i) $\vdash \Box A \to \Box(\Box(A \wedge \Box A) \to (A \wedge \Box A))$
(ii) $\Box(A \wedge \Box A) \to \Box\Box A$

Let us start with (i). We will use the nonmodal axiom $A \to (B \to A)$ and substitute $C \wedge D$ for A. (Note that all nonmodal axioms are true in GL)

$$(C \wedge D) \to (B \to (C \wedge D)) \tag{3}$$

Using the Exportation rule in propositional logic, which states that $A \to (B \to C) \leftrightarrow (A \wedge B) \to C)$ [7], it is easily shown that Formula (3) is equivalent to

$$C \to ((D \wedge B) \to (C \wedge D))$$

If we let $C := A$, $D := \Box A$ and $B := \Box\Box A$ we get

$$\vdash A \to ((\Box A \wedge \Box\Box A) \to (A \wedge \Box A))$$

By modal axiom RR we have that

$$\vdash \Box A \to \Box((\Box A \wedge \Box\Box A) \to (A \wedge \Box A))$$

For part (ii), we will again use the nonmodal axiom $A \to (B \to A)$, inference rule RR and the Exportation rule.
Let $A := \Box A$ and $B := A$.

$$\begin{aligned} &\Box A \to (A \to \Box A) \Leftrightarrow \{\textit{by the Exportation rule}\} \\ &(A \wedge \Box A) \to \Box A \end{aligned}$$

By inference rule RR we have that $\vdash \Box(A \wedge \Box A) \to \Box\Box A$

Because all instantiations of theorems of propositional logic are axioms in GL, we have that from $\vdash A \to B$ and $\vdash B \to C$ we can infer $\vdash A \to C$. We can use this three times in a row to conclude that by Equation (2), statements (i) and (ii) we have that

$$\vdash \Box A \to \Box\Box A$$

which is exactly Axiom 4. □

3.1 Modal soundness

Now we will go on proving that the modal axioms of GL that are left are indeed valid in the class of finite trees and that the Necessitation rule preserves validity. The validity of the modal axioms is proven by contraposition. The proof for Löb's axiom is based on the proof of the characterisation theorem of Smoryński [10].

Theorem 3.5. *The logic GL is sound with respect to the class of all finite trees.*

Proof. By Theorem 3.3, we have that inference rules Modus Ponens and RE preserve validity in any class of frames and therefore certainly for the class of finite trees.

Distribution Axiom: suppose that the Distribution Axiom does not hold at some point α_i in some model $\mathfrak{M}_\mathcal{T}$ for some tree $\mathcal{T} = (W, <, \alpha_0)$ then $\mathfrak{M}_\mathcal{T}, \alpha_i \not\models \Box(A \to B) \to (\Box A \to \Box B)$. By definition of $\models$, we have that $\mathfrak{M}_\mathcal{T}, \alpha_i \models \Box(A \to B)$ and $\mathfrak{M}_\mathcal{T}, \alpha_i \not\models \Box A \to \Box B$. Thus $\mathfrak{M}_\mathcal{T}, \alpha_i \models \Box A$ and $\mathfrak{M}_\mathcal{T}, \alpha_i \not\models \Box B$. $\mathfrak{M}_\mathcal{T}, \alpha_i \not\models \Box B$ implies that there exists a point $\alpha_j \in W$ such that $\alpha_i < \alpha_j$ and $\mathfrak{M}_\mathcal{T}, \alpha_j \not\models B$. Since we have the relation $\alpha_i < \alpha_j$, we have that $\mathfrak{M}_\mathcal{T}, \alpha_j \models A \to B$ and $\mathfrak{M}_\mathcal{T}, \alpha_j \models A$. By Modus Ponens we have that $\mathfrak{M}_\mathcal{T}, \alpha_j \models B$, which leads to a contradiction. Therefore there is no point in any model for any finite tree where the Distribution Axiom does not hold. So the Distribution Axiom is valid in every finite tree.

Löb's Axiom: suppose that the axiom does not hold at some point α_i in some model $\mathfrak{M}_\mathcal{T}$ of some finite tree $\mathcal{T} = (W, <, \alpha_0)$. Then $\mathfrak{M}_\mathcal{T}, \alpha_i \not\models \Box(\Box A \to A) \to \Box A$. This implies that $\mathfrak{M}_\mathcal{T}, \alpha_i \models \Box(\Box A \to A)$ and $\mathfrak{M}_\mathcal{T}, \alpha_i \not\models \Box A$. The second statement guarantees the existence of a point $\alpha_j \in W$ such that $\alpha_i < \alpha_j$ and $\mathfrak{M}_\mathcal{T}, \alpha_j \not\models A$. Using the first statement and the relation $\alpha_i < \alpha_j$ we have that $\mathfrak{M}_\mathcal{T}, \alpha_j \models \Box A \to A$. Hence, $\mathfrak{M}_\mathcal{T}, \alpha_j \not\models \Box A$. This means that there is a point $\alpha_k \in W$ such that $\alpha_j < \alpha_k$ and $\mathfrak{M}_\mathcal{T}, \alpha_k \not\models A$. Since the relation $<$ is transitive we also have that $\alpha_i < \alpha_k$. Therefore by $\mathfrak{M}_\mathcal{T}, \alpha_i \models \Box(\Box A \to A)$ we have again that $\mathfrak{M}_\mathcal{T}, \alpha_k \models \Box A \to A$ which leads to an infinite sequence $\alpha_i < \alpha_j < \alpha_k < ...$ This is a contradiction with tree $\mathcal{T}$ being finite. We can conclude that Löb's axiom is valid in every finite tree.

Necessitation rule: suppose that A is valid in the class of finite trees. This means that A is true at every point in any finite tree for logic GL. To this end, let $\mathcal{T} = (W, <, \alpha_0)$ be any tree in the class of finite trees and $\mathfrak{M}_\mathcal{T}$ be any model for $\mathcal{T}$. Then $\mathfrak{M}_\mathcal{T}, \alpha \models A$ for all $\alpha \in W$. Suppose that $\mathfrak{M}_\mathcal{T}, \alpha_i \not\models \Box A$ for some point $\alpha_i \in W$. Then there must exist a point $\alpha_j \in W$ such that $\alpha_i < \alpha_j$ and $\mathfrak{M}_\mathcal{T}, \alpha_j \not\models A$. This is in contradiction with the validity of A. Therefore, $\mathfrak{M}_\mathcal{T}, \alpha_i \models \Box A$ for all points $\alpha_i \in W$. Thus the Necessitation Rule preserves validity in the class of finite trees.

□

3.2 Modal completeness

Now that we have proven that logic GL is sound with respect to the class of finite trees, we have left to show that logic GL is also complete with respect to the class of finite trees. We will prove completeness by contraposition. To this end we will choose a formula which is not derivable in logic GL and construct a finite tree in which the formula is not valid. The construction of the tree differs a bit from Smoryński's construction because we have chosen to prove validity directly instead of using the equivalence that every formula which is true in the root of all models on finite trees is also valid on all finite trees (true at all points of the model). We will prove validity directly because it saves us the trouble of proving the above equivalence. However, by changing the construction of the tree, the proof of completeness that Smoryński provides does not change [10].

Theorem 3.6. *Logic GL is complete with respect to the class of finite trees.*

Proof. We will assume a formula A is not derivable in GL and we will construct a tree in which the formula A is not true at the root (and thus not valid in the tree).

Let $GL \nvdash A$ and let $\mathfrak{M} = (W, R, \beta_0, V)$ be a countermodel to A. Recall that GL is an extension of logic E, the smallest classical logic by Definition 3.2. Since E is proven to be complete with respect to the class of all frames [9], we have that GL is complete with respect to the class of all frames and thus such a countermodel exists. Therefore, there is a point $\alpha \in W$ such that A is not true at α. Then by the Fundamental theorem for normal logics [9], we have that $\mathfrak{M}, \alpha \not\models A$.

We will construct a submodel $\mathfrak{M}_\alpha = (W_\alpha, R_\alpha, \alpha, V_\alpha)$ from countermodel $\mathfrak{M}$ the following way:
$W_\alpha = \{\alpha\} \cup \{\beta \in W \mid \alpha R \beta\}$
$R_\alpha = R{\restriction}W_\alpha \times W_\alpha$ which is the restriction of R to W_α.
V_α means that for a propositional atom p, $V(p) = V_\alpha(p)$ thus for $\beta \in W_\alpha$, $\mathfrak{M}_\alpha, \beta \models p$ iff $\mathfrak{M}, \beta \models p$.

We denote α by α_0 and construct a finite tree model $\mathfrak{M}_\mathcal{T} = (W_T, <_T, (\alpha_0), V_T)$ from model $\mathfrak{M}_\alpha$ in a similar fashion as Smoryński.
Let $S = \{B | B \text{ is a subformula of } A\}$.
Stage 0: Put sequence (α_0) into W_T.
Stage $n+1$:
For each sequence $(\alpha_0, ..., \alpha_n) \in W_T$, look at the set $\{\Box B \in S | \mathfrak{M}_\alpha, \alpha_n \not\models \Box B\}$. If this set is empty, do not extend the sequence $(\alpha_0, ..., \alpha_n)$. Otherwise, for each such $\Box B$, we have that $\mathfrak{M}_\alpha, \alpha_n \models \Box(\Box B \to B) \to \Box B$ (Löb's axiom) thus that $\mathfrak{M}_\alpha, \alpha_n \not\models \Box(\Box B \to B)$. So there is a node $\beta \in K_\alpha$ such that $\alpha_n R \beta$ and $\mathfrak{M}_\alpha, \beta \not\models \Box B \to B$. Thus we have that $\alpha_n R \beta$ and $\mathfrak{M}_\alpha, \beta \models \Box B$ and $\mathfrak{M}_\alpha, \beta \not\models B$.
In order to construct the tree, we will add the sequence $(\alpha_0, ..., \alpha_n, \beta)$ to W_T.
The relation $<_T$ will be the strict partial ordering by extension of finite sequences.
The sequence (α_0) will be the root of the tree.
The valuation function V_T is defined: for every propositional atom p, $V_T(p) = (\alpha_0, ..., \alpha_n)$ iff $V(p) = \alpha_n$. Thus we have that $\mathfrak{M}_\mathcal{T}, (\alpha_0, ..., \alpha_n) \models p$ iff $\mathfrak{M}_\alpha, \alpha_n \models p$.

We need to show that the constructed tree is indeed a tree and that it is finite. Also, we need to show that formula A is not valid in the tree to prove completeness.

First we will show that $\mathcal{T} = (W_T, <_T, (\alpha_0))$ is indeed a tree with origin (α_0). This is pretty straightforward. Then we will show that the tree is finite.
For W_T to be a tree it should comply with statements (i) and (ii) of the definition of a tree. Since $<_T$ is the usual strict partial ordering by extension of finite sequences we have that (i) holds. Since any element in W_T is a sequence $(\alpha_0, ..., \alpha_n, \beta)$ or (α_0), we have that for the first case the sequence $(\alpha_0, ..., \alpha_i)$ with $i \leq n$ is a predecessor and also an element of W_T. Since $(\alpha_0) <_T ... <_T (\alpha_0, ..., \alpha_n) <_T (\alpha_0, ..., \alpha_n, \beta)$, we have a linear order of predecessors of any element. The set of predecessors is finite since they are in the same branch and a branch correlates with the elements of the finite set S.
We will show that the tree is finite using the fact that the tree is finitely branching and König's Tree lemma [5]. König's Tree lemma states that any finitely branching, infinite tree contains an infinite branch.
We know that the tree is finitely branching since the set S is finite. There are no infinite branches because the succession from sequence $(\alpha_0, ..., \alpha_n)$ to $(\alpha, ..., \alpha_n, \alpha_{n+1})$ results in at least one additional sentence $\Box B \in S$ being forced by α_{n+1}; thus after one has gone through all such sentences, the process stops. So we can conclude that the tree is not infinite and therefore finite.

We will now show that the formula A is not valid in the constructed tree by proving the following statement:
For all $B \in S$ and all $(\alpha_0, ..., \alpha_n) \in W_T$, $\mathfrak{M}_\mathcal{T}(\alpha_0, ..., \alpha_n) \models B$ iff $\mathfrak{M}_\alpha, \alpha_n \models B$.
We prove this statement by induction on the length of formula B. Thus it is only of interest to look at the case that $B = \Box C$. This means that our induction hypothesis will be: for some arbitrary formula C and some arbitrary sequence $(\alpha_0, ..., \alpha_n)$, $\mathfrak{M}_\mathcal{T}, (\alpha_0, ..., \alpha_n) \models C$ iff $\mathfrak{M}_\alpha, \alpha_n \models C$.
We will first prove the statement from right to left. Suppose that $\mathfrak{M}_\alpha, \alpha_n \models \Box C$. For any point $\beta \in W_\alpha$, if $\alpha_n R_\alpha \beta$ then $\mathfrak{M}_\alpha, \beta \models C$. Since the sequence $(\alpha_0, ..., \alpha_n, \beta)$ is only an element of W_T if $\alpha_n R_\alpha \beta$, for any $\beta \in W_\alpha$ if $(\alpha_0, ..., \alpha_n, \beta) \in W_T$ then $\mathfrak{M}_\alpha, \beta \models C$. By induction hypothesis we have that $\mathfrak{M}_\alpha, \beta \models C$ implies $\mathfrak{M}_\mathcal{T}, \beta \models C$. Thus for any $\beta \in W_\alpha$ if $(\alpha_0, ..., \alpha_n, \beta) \in W_T$ then $\mathfrak{M}_\mathcal{T}, \beta \models C$. Therefore, $\mathfrak{M}_\mathcal{T}, (\alpha_0, ..., \alpha_n) \models \Box B$.
Now we will prove the statement from left to right by contraposition. Suppose that $\mathfrak{M}_\alpha, \alpha_n \not\models \Box C$. Then there exists a point $\beta \in W_\alpha$ such that $\alpha_n R_\alpha \beta$ and $\mathfrak{M}_\alpha, \beta \not\models C$. Then by the construction of the model $\mathfrak{M}_\mathcal{T}$ we have that there is a sequence $(\alpha_0, ..., \alpha_n, \beta)$ in W_T such that $\mathfrak{M}_\mathcal{T}, \beta \models \Box C$ and $\mathfrak{M}_\mathcal{T}, \beta \not\models C$. Thus there exists a point $\beta \in W_\alpha$ such that the sequence $(\alpha_0, ..., \alpha_n, \beta) \in W_T$ and $\mathfrak{M}_\mathcal{T}, \beta \not\models C$. Therefore, $\mathfrak{M}_\mathcal{T}, (\alpha_0, ..., \alpha_n) \not\models \Box C$.

$\Box$

4 Solovay's completeness theorem

We have just shown that the logic GL is complete and sound with respect to finite trees. Solovay was inspired by this result to create an even more interesting theorem. Solovay was able to prove that the logic GL is arithmetically sound and complete with respect to Peano Arithmetic. The significance of Solovay's theorem is that it allows us to study an undecidable formal theory like Peano Arithmetic by studying a decidable modal logic, namely GL [14].

We will use a predicate $\mathbf{Con}(x)$ in our proofs of the arithmetical soundness lemma and completeness theorem. We shall define the predicate $\mathbf{Con}(x)$ to state that x is consistent. For example, $PA \vdash \mathbf{Con}(PA)$ would state that Peano Arithmetic proves its own consistency. Note that this statement is false by the second Gödel incompleteness theorem [8]:

Theorem 4.1. (Gödel's second incompleteness theorem) *If the arithmetic formal system is consistent, then there is no consistency proof for it by methods formalizable in the system.*

A formal system consists of a list of formal symbols, a set of formation rules that defines the syntax of the grammar and a set of transformation rules, which gives the formal system the structure of a deductive theory. The set of transformation rules consists of axioms and inference rules [8].

4.1 Arithmetical soundness

At the time of Solovay's article, it was already known that the theorems of GL are all PA-valid [12]. Only the last part of the arithmetical soundness proof was used in this paper, namely to prove the validity of Löb's Axiom. However, the proof by Solovay was incomplete, therefore the last part of this proof is based on a proof by G. Boolos [3].

Lemma 4.2. (arithmetical soundness) *Every theorem of GL is PA-valid.*

Proof. We need to show that all axioms and inference rules of GL are valid in PA. Let us start with inference rule Modus Ponens, MP. MP is an inference rule of the language PA and therefore it preserves validity in PA.

We will now show that the other inference rule, the Necessitation Rule, preserves validity in PA. Suppose that $PA \vdash A$ for an arbitrary sentence A. If we can derive A in PA it is by definition provable, see Section 2.4. So there exists a (formal) proof of a finite sequence of one or more formulas such that each formula is either an axiom or an immediate consequence of preceding formulas of the sequence [8]. We can code this proof with some Gödel number n, $PA \vdash \mathbf{Proof}(\mathbf{n}, A)$. Thus $\exists n \mathbf{Proof}(n, A)$ in PA. Therefore, $PA \vdash \mathbf{Prov}(\ulcorner A \urcorner)$. We can conclude that $PA \vdash A$ implies $PA \vdash \mathbf{Prov}(\ulcorner A \urcorner)$ so the Necessitation Rule preserves validity in PA.

We will move on to the axioms. For the proof of the Distribution Axiom, we will reason in PA such that the following argument can be formalized in PA. Suppose that $\mathbf{Prov}(\ulcorner A \to B \urcorner)$ is true in PA for arbitrary sentences A and B then there exists a Gödel number n such that $\mathbf{Proof}(n, A \to B)$. Then suppose that $\mathbf{Prov}(\ulcorner A \urcorner)$ is true. This means that there is a Gödel number m such that $\mathbf{Proof}(m, A)$. Thus concatenating the proofs of $A \to B$ and A together, we can add the step of using Modus Ponens to conclude B. Then we can code this proof with some Gödel number k. Thus $\exists k \mathbf{Proof}(k, B)$, which means that $\mathbf{Prov}(\ulcorner B \urcorner)$. Formalizing the above proof, we have that $PA \vdash \mathbf{Prov}(\ulcorner A \to B \urcorner) \to (\mathbf{Prov}(\ulcorner A \urcorner) \to \mathbf{Prov}(\ulcorner B \urcorner))$.

Finally, we need to show validity of Löb's Axiom. For this proof we will use the newly introduced predicate **Con**. We will first prove that Löb's rule, from $\Box A \to A$ infer A, is sound with respect to PA and then we will show that the logic K4LR has exactly the same theorems as GL.

The logic K4LR is the modal logic K with Axiom 4, recall that $\Box A \to \Box\Box A$, and with Löb's rule.

Suppose that for an arbitrary sentence A we have that $PA \vdash \mathbf{Prov}(\ulcorner A \urcorner) \to A$. We must show that $PA \vdash A$. By some propositional logic, from $PA \vdash \mathbf{Prov}(\ulcorner A \urcorner) \to A$ we can conclude $PA \vdash \neg A \to \neg \mathbf{Prov}(\ulcorner A \urcorner)$. If there does not exist a proof for A then we can safely add $\neg A$ to PA without this leading to a contradiction. In other words, by adding $\neg A$, the system is still consistent. Therefore we have that $PA \vdash \neg A \to \mathbf{Con}(PA + \ulcorner \neg A \urcorner)$. This statement means that $PA + \ulcorner \neg A \urcorner$ proves its own consistency. However, by Gödel's second incompleteness theorem, Theorem 4.1, an arithmetic formal system that proves its own consistency must be inconsistent. Thus $PA + \ulcorner \neg A \urcorner$ is inconsistent. This means that $PA \vdash A$. This proves the validity of Löb's rule in PA [12].

If we could do the above reasoning in PA, we could formalize it and prove Löb's axiom. However, if we would reason in PA, we should first prove Gödel's second incompleteness theorem in PA. We will now show that logic K4LR has exactly the same theorems as GL. First note that by formalizing the proof of the Necessitation Rule, we have that Axiom 4 preserves validity in PA. Therefore, we have shown that K4LR is sound with respect to PA.

We have already shown that Axiom 4 is derivable in GL. We will now show that GL is closed under Löb's rule: Suppose $GL \vdash \Box A \to A$. By necessitation, $GL \vdash \Box(\Box A \to A)$. Löb's theorem is an axiom of GL thus $GL \vdash \Box(\Box A \to A) \to \Box A$. By Modus Ponens, $GL \vdash \Box A$. Again by Modus Ponens, $GL \vdash A$. Thus if $K4LR \vdash A$, then $GL \vdash A$.
To show the converse, let $B := \Box(\Box A \to A)$, $C := \Box A$, $D := B \to C$. We must show $K4LR \vdash D$. By the Distribution Axiom: $K \vdash \Box D \to (\Box B \to \Box C)$. Also by Distribution Axiom: $K \vdash \Box(\Box A \to A) \to (\Box\Box A \to \Box A)$, which is equivalent to $K \vdash B \to (\Box C \to C)$. By Axiom 4, $K4 \vdash B \to \Box B$, since B begins with $\Box$.

Thus by some propositional logic, $K4 \vdash \Box D \to (B \to C)$. By definition of D, $K4 \vdash \Box D \to D$. Finally, by Löb's rule, $K4LR \vdash D$ [3].

We have shown that the logic K4LR is arithmetically sound with respect to PA and that it has exactly the same theorems as logic GL, therefore, GL is arithmetically sound with respect to PA. (A more direct proof can be given using the diagonal lemma, which will be introduced shortly. The proof can be found in [14].) □

4.2 Arithmetical completeness

The main result of Solovay's article was his arithmetical completeness proof for GL with respect to Peano Arithmetic. In his proof he makes use of the completeness of GL with respect to the class of all finite trees. Solovay proves the arithmetical completeness by contrapostion, similar to the modal completeness proof for the class of finite trees. For a formula which is not derivable in GL, he sets out to prove that for a certain interpretation in Peano Arithmetic, the interpretation of the formula is not derivable in PA. Therefore, he constructs a finite tree countermodel using this modal completeness of GL. He comes up with a certain function for which he can prove statements about its limit. These statements help him prove the desired contradiction for a certain interpretation in Peano Arithmetic. The finite tree countermodel and the limit of the function are used for the construction of this interpretation. Solovay has proven the following lemma about the limit l.

Lemma 4.3. *Let $\prec$ be a strict partial ordering on $\{1, ..., n\}$. If $1 < j \leq n$, then $1 \prec j$. There is a term l of PA such that:*

1. *$PA \vdash 0 \leq l \leq n$.*

2. *In the standard model of PA, $l = 0$.*

3. *If $0 \leq i \leq n$, "$PA +$ 'l=i' " is consistent.*

   For $1 \leq i \leq n$, let $S_i = \{j : i \prec j\}$. Let $S_0 = \{1, ..., n\}$

4. *Let $0 \leq i \leq n$. Let $j \in S_i$.*
   Then $PA \vdash$ (If $l = i$, then "$PA +$ 'l=j' " is consistent).

5. *Let $0 < i \leq n$. Let $j \notin S_i$. Then $PA \vdash$* (If $l = i$, then $PA \vdash$ "$l \neq j$").

By the standard model of PA we mean the natural numbers with constant '0' being zero and + and × are addition and multiplication, respectively.

We will not provide the whole proof of the lemma here, as the proof that Solovay gives in his article is straightforward [12]. We will discuss the first part of his proof as he constructs a function h which gives us an intuitive meaning of the term l. Let h be a primitive recursive function which maps the natural numbers to the set $\{0, ..., n\}$. We define $h(0) = 0$. Moreover, if $h(m) = i$ then $h(m+1) = i$ unless for some $j \in S_i$, $m+1$ is the Gödel number of a proof in Peano Arithmetic of $l \neq j$, then $h(m+1) = j$. We let l denote $\lim_{m\to\infty} h(m)$ if the limit exists, otherwise, $l = n+1$. We now intuitively understand that l is updated to some value i only if it is guaranteed that at a later point in the algorithm the term l will be updated to a value other than i. Therefore, we have in the standard model that $l = 0$ as stated in (2). The proof of statement (5) uses Σ_1-completeness, which is important information in Section 5.

One might think the definition of function h to be circular as h is defined in terms of proofs of $l \neq j$, while $l = j$ asserts that the limit of h is equal to j. Solovay uses the recursion theorem to handle this apparent circularity. However, it is easier to use the diagonal lemma.

Lemma 4.4. (The generalized diagonal lemma [3]) *Suppose that $y_0, .., y_n, z_1, ..., z_m$ are distinct variables and that $P_0(y_0, ..., y_n, \boldsymbol{z}), ..., P_n(y_0, ..., y_n, \boldsymbol{z})$ are formulas of the language PA in which all free variables are among $y_0, .., y_n, \boldsymbol{z}$. ($\boldsymbol{z}$ abbreviates $z_1, ..., z_m$.) Then there exist formulas $S_0(\boldsymbol{z}), ..., S_n(\boldsymbol{z})$ of the language of PA in which all free variables are among $\boldsymbol{z}$, such that*

$$PA \vdash S_0(\boldsymbol{z}) \leftrightarrow P_0(\ulcorner S_0(\boldsymbol{z})\urcorner, ..., \ulcorner S_n(\boldsymbol{z})\urcorner, \boldsymbol{z}), ..., \text{ and}$$
$$PA \vdash S_n(\boldsymbol{z}) \leftrightarrow P_n(\ulcorner S_0(\boldsymbol{z})\urcorner, ..., \ulcorner S_n(\boldsymbol{z})\urcorner, \boldsymbol{z})$$

In [3], G. Boolos comes up with a formula $H(a, b)$ of PA which is defined using the diagonal lemma and defines the definition of the function h as described above.

We will prove a result from Lemma 4.3 that we will need in our completeness proof. The proof uses Σ_1-completeness.

Lemma 4.5. *Let $0 < i \leq n$, then $PA \vdash$ (If $l = i$, then $PA \vdash l \in S_i$)*

Proof. By construction of term l in Lemma 4.3, $l = i$ iff (1) for some m, $h(m) = i$ and it is not the case that (2) for some m, $h(m) \in S_i$. Since (1) and (2) only have unbounded existential quantifier, they are Σ_1 sentences and therefore $l = i$ is a Boolean combination of Σ_1 sentences [12].
Because (1) is a Σ_1 sentence, by Σ_1-completeness we have $PA \vdash \exists m(h(m) = i) \rightarrow \mathbf{Prov}(\ulcorner \exists m(h(m) = i)\urcorner)$. We also know that in PA $\exists m(h(m) = i)$ implies that $l = i$ or that $l = j$ for some $j \in S_i$, in other words $l \in S_i$. Formalizing this argument gives, $PA \vdash \mathbf{Prov}(\ulcorner \exists m(h(m) = i)\urcorner) \rightarrow \mathbf{Prov}(\ulcorner l = i \vee l \in S_i\urcorner)$. By definition of l, we have that $PA \vdash l = i \rightarrow \exists m(h(m) = i)$. Combining these results gives $PA \vdash l = i \rightarrow \mathbf{Prov}(\ulcorner l = i \vee l \in S_i\urcorner)$ [3].
By propositional logic we have that for $j \in S_i$, $PA \vdash ((l \neq i) \wedge (l = i \vee l \in S_i)) \rightarrow (l \in S_i)$. Then by the Distribution Axiom and Necessitation Rule and some propositional logic, we have $PA \vdash (\mathbf{Prov}(\ulcorner l \neq i\urcorner) \wedge \mathbf{Prov}(\ulcorner l = i \vee l \in S_i\urcorner)) \rightarrow \mathbf{Prov}(\ulcorner l \in S_i\urcorner)$.
Take $j = i$ then $j \notin S_i$ thus by statements (1) and (5) of Lemma 4.3, we have that $PA \vdash l = i \rightarrow \mathbf{Prov}(\ulcorner l \neq i\urcorner)$.

Therefore, we have our desired result, $PA \vdash l = i \to \mathbf{Prov}(\ulcorner l \in S_i \urcorner)$.

$\square$

We are now able to prove the arithmetical completeness of GL with respect to Peano Arithmetic. Again, our proof will be by contraposition. Therefore we assume that there is a modal formula that is not a theorem of GL. Then we come up with a certain interpretation such that the interpretation of this formula in PA is not a theorem of PA. We construct a finite tree model in which the formula is false at the root. Then we will define an interpretation on PA in which the initial modal formula will not be PA-valid. This interpretation is based on the finite tree model and the term l from Lemma 4.3.

Theorem 4.6. (arithmetical completeness) *A modal formula A is PA-valid iff A is a theorem of GL.*

Proof. By Lemma 4.2 every theorem of GL is PA-valid. We shall prove the completeness by contraposition. Now let A be a modal formula which is not a theorem of GL. We shall find an interpretation that maps propositional modal logic to P such that A^* is not a sentence of PA.
By Theorem 3.6 we have that there exists a finite tree model
$\mathfrak{M}_{\mathcal{T}} = (W, <_T, \alpha_0, V_T)$ in which A is false at the root, $\mathfrak{M}_{\mathcal{T}}, \alpha_0 \not\models A$. We will rename the points in W such that $1 := \alpha_0$, $W = \{1, ..., n\}$ and for $1 < j \leq n$ we have $1 <_T j$.

We let l denote the term defined in Lemma 4.3. Now we start with the construction of the interpretation in PA which is based on the valuation of the finite tree model. Let the assignment of a sentence in PA of a propositional variable p be: p^* is a disjunction of "sentences $l = i$",

$$p^* := \vee\{l = i : 0 \leq i \leq n \text{ and } \mathfrak{M}_{\mathcal{T}}, i \models p\}$$

However, if this is an empty disjunction, set $p^* =$ "$0 = 1$". We want to prove that this assignment gives rise to the following lemma:

Lemma 4.7. *Let B be a modal formula. Let $1 \leq i \leq n$. Then*
1) If $\mathfrak{M}_{\mathcal{T}}, i \models B$, then $PA \vdash (l = i) \to B^*$.
2) If $\mathfrak{M}_{\mathcal{T}}, i \not\models B$, then $PA \vdash (l = i) \to \neg B^*$.

We will prove the lemma shortly. First, we will show that this lemma finalizes the proof of the theorem. We had that $\mathfrak{M}_{\mathcal{T}}, 1 \not\models A$ thus by the lemma, $PA \vdash (l = 1) \to \neg A^*$. By Lemma 4.3 (3), $PA +$ "$l = 1$" is consistent. Therefore, we have that $PA + \neg A^*$ is consistent so A^* is not a theorem of PA. Thus A is not PA-valid, which proves our theorem.
We will now show that Lemma 4.7 holds. The proof is by induction on the length of sentence B.

Case 1: For arbitrary propositional variable p, the first statement is clear from the definition of p^*. For the second part, suppose that $\mathfrak{M}_{\mathcal{T}}, i \not\models p$ then $l = i$ is not part of the disjunction of p^*. Because of the definition of l, it equals a unique number and therefore we have that if $0 \leq i, j \leq n$ and $i \neq j$ then $PA \vdash (l = i \rightarrow l \neq j)$. By this result and some propositional logic we have the desired result, $PA \vdash (l = i) \rightarrow \neg p^*$.

Case 2: For $B := \bot$, we have that $\mathfrak{M}_{\mathcal{T}}, i \not\models B$ for all $1 \leq i \leq n$. Also, $\neg B^* =$ "$0 \neq 1$" which is always true so $PA \vdash (l = i) \rightarrow \neg B^*$.

Case 3: For arbitrary modal formulas C and D, let $B := C \rightarrow D$. The sentence $B^* = (C \rightarrow D)^*$ is equal to $B^* =$ "$C^* \rightarrow D^*$" because of the restrictions on all interpretations.
Induction hypothesis: suppose that the lemma holds for modal formula C and D.
Suppose that $\mathfrak{M}_{\mathcal{T}}, i \models B$, then we have $\mathfrak{M}_{\mathcal{T}}, i \not\models C$ or $\mathfrak{M}_{\mathcal{T}}, i \models D$. It is now straightforward to see that $PA \vdash (l = i) \rightarrow B^*$ holds. Suppose that $\mathfrak{M}_{\mathcal{T}}, i \not\models B$, then we have $\mathfrak{M}_{\mathcal{T}}, i \models C$ and $\mathfrak{M}_{\mathcal{T}}, i \not\models D$. It is now straightforward to see that $PA \vdash (l = i) \rightarrow \neg B^*$ holds.

Case 4: Now suppose that $B := \Box C$ for an arbitrary modal formula C.
Induction hypothesis: the lemma holds for modal formula C.
Suppose that $\mathfrak{M}_{\mathcal{T}}, i \models \Box C$. By Lemma 4.5 if $i > 0$ then
(i) $PA \vdash$ (If $l = i$, then $PA \vdash l \in S_i$)
Since $\mathfrak{M}_{\mathcal{T}}, i \models \Box C$, $\mathfrak{M}_{\mathcal{T}}, j \models C$ for all $j \in S_i$.
Hence, by the induction hypothesis,
(ii) $PA \vdash (l = j) \rightarrow C^*$ where $j \in S_i$
Thus,
(iii) $PA \vdash (l \in S_i) \rightarrow C^*$
By (i) and (iii), we have that
(iv) $PA \vdash$ (If $l = i$, then $PA \vdash C^*$)
which is equivalent to saying $P \vdash (l = i) \rightarrow$ "$PA \vdash C^*$". Then by definition of $\Box$-operator,
(v) $PA \vdash (l = i) \rightarrow (\Box C)^*$
This proves the first statement of the lemma.
Now suppose that $\mathfrak{M}_{\mathcal{T}}, i \not\models \Box C$. Then for some $j \in S_i$, we have that $\mathfrak{M}_{\mathcal{T}}, j \not\models C$.
But then by the induction hypothesis,
(i) $PA \vdash (l = j) \rightarrow \neg C^*$
By Lemma 4.3 (4),
(ii) $PA \vdash$ (If $l = i$, then "$PA +' l = j'$" is consistent)
which we will denote by $PA \vdash (l = i) \rightarrow \mathbf{Con}(PA +$ "$l = j$").
Then by (i) and (ii),
(iii) $PA \vdash (l = i) \rightarrow \mathbf{Con}(PA + \neg C^*)$
since $l = i$ implies that "$PA + \neg C^*$" is consistent, we have that $PA \nvdash C^*$. Thus $PA \vdash (l = i) \rightarrow \neg(\Box C)^*$. This proves the second statement of the lemma. $\Box$

By the arithmetical completeness that we have just proven for GL with respect to Peano Arithmetic, we can compare the meaning of the $\Box$-operator in frames and the translation in Peano Arithmetic. For instance, when $\mathfrak{M}, w \models \Box A$ for some finite tree model $\mathfrak{M}$ at some node w, it can be shown that $PA \nvdash$ $\mathbf{Prov}(\ulcorner \neg A^* \urcorner)$ for some interpretation *.

5 Weaker Arithmetics

We have discussed Solovay's proof of the arithmetical completeness of logic GL with respect to Peano Arithmetic. Now we want to explore if his results also hold for weaker arithmetics. In the paper "On the Proof of Solovay's Theorem" by de Jongh and others, a class of theories is discussed for which arithmetical completeness can be proven. However, we are not sure if GL is arithmetically complete with respect to any other theories. The class of theories that de Jongh investigated is those that are $I\Delta_0+$ EXP and the r.e.-extensions (recursively enumerable) that are Σ_1-sound. From now on, T will denote such a theory. Notions about provability, interpretation, soundness and the proof predicate defined in the Preliminaries for Peano Arithmetic are similar for other arithmetical theories. In this section we will show an updated version of one of the proofs from the paper by de Jongh, Jumelet and Montagna. Therefore, all notions, definitions, lemma's and theorems in this section are based on his paper [4].

$I\Delta_0$ is a subtheory of Peano Arithmetic. It has the same axioms as PA, except for the induction axiom, Axiom 7 in Section 2.4, which only holds for formulas A that are Δ_0. Δ_0 is the same classification of sentences as the class Σ_0. The formula EXP expresses that $\forall x\ 2^x$ exists. The extension of $I\Delta_0$ with EXP proves Σ_1-completeness. This is an important characteristic, which we will need in our proof.

In the section on Solovay's proof it is stated that the formula H defined by Boolos mimicking the function h, is found by the diagonalization lemma. However, we will find an alternative way to define the function such that it is a Δ_0 sentence instead of Σ_1. If we want to define the output for $h(n+1)$, we only need to look at the numbers that came before, so $\leq n$. Then $h(n+1) = m$ iff:

1. $n+1$ proves the negation of the limit assertion with respect to m,
2. no such proof concerning a number m' with mRm' (or $m = m'$) is coded by a number $\leq n$ (otherwise, h should have "passed" m already),
3. if any such proof is coded by a number $n' \leq n$ for an m' incomparable to m with respect to R, then there has to be an even smaller number $n'' \leq n'$ that codes such a proof for a number $m''Rm$ (or $m'' = m$) incomparable to m' (otherwise h should have taken a direction from which it could no longer reach m; in other words any proof that could possibly "side-track"

h from its way to m, has to have been preceded by a proof that makes it harmless, by side-tracking it).

We will now introduce some notation in order to give the formula H which satisfies the above requirements.

$\Box_{\leq x}\ulcorner p \urcorner := \exists v \leq x \ \mathbf{Proof}(v, p)$
$\Box_{\leq x}\ulcorner p \urcorner \prec \Box_{\leq x}\ulcorner q \urcorner := \exists v \leq x[\mathbf{Proof}(v, p) \wedge \forall u \leq v \neg \mathbf{Proof}(u, q)]$
$\Diamond_{\leq x}\ulcorner p \urcorner := \neg \Box_{\leq x}\ulcorner \neg p \urcorner$
$i\underline{R}j$ denotes $i = j \vee iRj$
$i \circ j$ denotes $\neg i\underline{R}j \wedge \neg j\underline{R}i$

Also, from now on we will use the shorthand notations

$\Box\ulcorner p \urcorner := \exists v \mathbf{Proof}(v, p)$
$\vdash A := T \vdash A$ for any formula A in the language of T.

Then we can define $H_i(x)$, which we want to denote $h(x) = i$, by self-reference:

$$H_i(x) \leftrightarrow \Box_{\leq x}\neg L_i \wedge \bigwedge_{iRj} \Diamond_{\leq x} L_j \wedge \bigwedge_{i \circ j} \bigvee_{\substack{k\underline{R}i \\ k \circ j}} (\Box_{\leq x}\neg L_k \prec \Box_{\leq x}\neg L_j) \tag{4}$$

where $L_i := \exists y \forall x \geq y H_i(x)$.

We would like to prove a lemma similar to Lemma 4.3 (see page 17), which was proven in Solovay's paper. This lemma is important since it lays the groundwork for the completeness proof. Before we can prove this lemma, we need to introduce the definition of a stable formula and a lemma which will help in our proof.

Definition 5.1. *A formula A is stable iff*

$$\vdash \exists x(\forall y \geq x A y \vee \forall y \geq x \neg A y).$$

Lemma 5.2. *1. Each Boolean combination of stable formulas in the same free variable x is a stable formula.*

2. $\Box_{\leq x}\ulcorner p \urcorner, \Diamond_{\leq x}\ulcorner p \urcorner, \Box_{\leq x}\ulcorner p \urcorner \prec \Box_{\leq x}\ulcorner q \urcorner$ are stable.

3. if $L(A_1(x), ..., A_n(x))$ is a lattice combination of stable formulas $A_1(x), ..., A_n(x)$, and if $L_i \equiv \exists y \forall x \geq y A_i(x)$, then

$$\vdash \exists y \forall x \geq y L(A_1(x), ..., A_n(x)) \leftrightarrow L(L_1, ..., L_n).$$

Proof. (1) and (2) are trivial. We will prove (3) by induction on the complexity of L. A lattice has operators $\wedge$ and $\vee$. The base case and induction on $\wedge$ are trivial. The induction on $\vee$ will be done using the induction hypothesis and

statement (1) of the lemma.
Case $\vee$: Let $L(A_1(x), ..., A_{n+1}(x)) := L(A_1(x), ..., A_n(x)) \vee A_{n+1}(x)$.
By propositional logic, $\vdash L(L_1, ..., L_{n+1}) \leftrightarrow L(L_1, ..., L_{n+1})$.
By definition of L, we have that $\vdash (L(L_1, ..., L_n) \vee L_{n+1}) \leftrightarrow L(L_1, ..., L_{n+1})$.
By definition of L_i and the induction hypothesis,
$\vdash (\exists y \forall x \geq y L(A_1(x), ..., A_n(x)) \vee \exists v \forall w \geq v A_{n+1}(w)) \leftrightarrow L(L_1, ..., L_{n+1})$.
Because L is a boolean combination of stable formulas in the same free variable x, L is a stable formula. Since both L and $A_{n+1}(x)$ are stable formulas, we know that for both there is a certain point such that from that point onwards the formula is true or from that point onwards the formula is false. Take the largest of these two certain points and call it y. Then we know that from that point on ($\forall x \geq y$) L is true or A_{n+1} is true or they are both true or both false. Thus $\exists y \forall x \geq y(L(A_1(x), ..., A_n(x)) \vee A_{n+1}(x))$.
Therefore, $\exists y \forall x \geq y L(A_1(x), ..., A_n(x)) \vee \exists v \forall w \geq v A_{n+1}(w)$ is equivalent to $\exists y \forall x \geq y(L(A_1(x), ..., A_n(x)) \vee A_{n+1}(x))$.
We can conclude $\vdash (\exists y \forall x \geq y(L(A_1(x), ..., A_n(x)) \vee A_{n+1}(x))) \leftrightarrow L(L_1, ..., L_{n+1})$.
So by our definition of lattice combination L,
$\vdash (\exists y \forall x \geq y L(A_1(x), ..., A_{n+1}(x))) \leftrightarrow L(L_1, ..., L_{n+1})$ $\square$

Now that we have proven the lemma on stable formulas we will prove the following lemma, which is in line with Lemma 4.3. In this proof, we need the fact that logic GL is arithmetically sound with respect to theory T. We indeed have this soundness, since de Jongh and colleagues prove that an extension of GL is arithmetically sound with respect to T. The proof of the lemma given by de Jongh et al. [4] is based on a logic called R_F^- which is an extension of GL. Therefore, de Jongh and colleagues do not prove arithmetical completeness for the logic GL but for a logic R. We have proven the lemma without depending on this logic R_F^- and thus the result of this lemma can be used for an arithmetical completeness proof for GL instead of logic R.

Lemma 5.3. *1.* $\vdash \bigvee_{0 \leq i \leq n} L_i$

2. $\mathbb{N} \models L_0$

3. For all i such that $0 \leq i \leq n$, $T + L_i$ *is consistent*

4. $\vdash L_i \to \bigwedge_{iRj} \neg\Box\neg L_j$ *for all* $i \geq 0$

5. $\vdash L_i \to \bigwedge_{\neg iRj} \Box\neg L_j$ *for all* $i > 0$

Proof. Let $H_0(x) := \bigwedge_{i \neq 0} \neg H_i(x)$ where $H_i(x)$ are defined as in Equation (4). By statements (1) and (2) of Lemma 5.2, $H_i(x)$ is stable for $i = 0, ..., n$. Recall that $L_i = \exists y \forall x \geq y H_i(x)$. By the third clause of Lemma 5.2, define the lattice combination

$$L(A_1(x), ..., A_n(x)) := \bigwedge_{1 \leq i \leq n} A_i$$

and the stable formulas

$$A_i := H_i(x),$$

then we have that

$$\vdash L_0 \leftrightarrow \bigwedge_{i \neq 0} \neg L_i \tag{5}$$

If we use another lattice combination, namely $L(A_1(x), ..., A_n(x)) := A_i(x)$, we get $\vdash L_i \leftrightarrow \exists y \forall x \geq y H_i(x)$. Using the facts that $\vdash \Box\ulcorner p \urcorner \leftrightarrow \exists x \Box_{\leq x}\ulcorner p \urcorner$, $\vdash \Diamond\ulcorner p \urcorner \leftrightarrow \forall x \Diamond_{\leq x}\ulcorner p \urcorner$ and $\vdash \Box\ulcorner p \urcorner \prec \Box\ulcorner q \urcorner \leftrightarrow \exists x(\Box_{\leq x}\ulcorner p \urcorner \prec \Box_{\leq x}\ulcorner q \urcorner)$, we get for $i = 1, .., n$:

$$\vdash L_i \leftrightarrow \Box\neg L_i \wedge \bigwedge_{iRj} \neg\Box\neg L_j \wedge \bigwedge_{i \circ j} \bigvee_{\substack{k\underline{R}i \\ k \circ j}} (\Box\neg L_k \prec \Box\neg L_j) \tag{6}$$

Since $L_0 \vee \neg L_0$ is a propositional tautology, we have that $\vdash L_0 \vee \neg L_0$. By Equation 5, we have that $\vdash \neg L_0 \rightarrow \bigvee_{i \neq 0} L_i$, thus $\vdash \bigvee_{0 \leq i \leq n} L_i$. We have proven statement (1) of the lemma.

By Σ_1-soundness, all theorems of T hold in the standard model. Therefore, by statement (1) of the lemma we must have $\mathbb{N} \models L_i$ for some i such that $0 \leq i \leq n$. By Equation 6, we have that $\vdash L_i \rightarrow \Box\neg L_i$ for $i > 0$. Thus, we must have that $\mathbb{N} \models L_0$. We have proven statement (2) of the lemma.

Equation 6 proves statement (4) for $i > 0$.
For i=0, we need to show that $\vdash L_0 \rightarrow \bigwedge_{i \neq 0} \neg\Box\neg L_i$.
By propositional logic,

$$\vdash \bigvee_{i \neq 0} \Box\neg L_i \rightarrow (\Box\neg L_{i_1} \wedge ... \wedge \Box\neg L_{i_k} \wedge \bigwedge_{g \notin \{i_1,...,i_k\}} \neg\Box\neg L_g). \tag{7}$$

We can order the terms $\Box\neg L_{i_1}, ..., \Box\neg L_{i_k}$ such that

$$\Box\neg L_{i_1} \prec \Box\neg L_{i_2} \wedge ... \wedge \Box\neg L_{i_{k-1}} \prec \Box\neg L_{i_k}.$$

We can now construct a subset $\{m_1, ..., m_l\}$ of the set $\{1, ..., k\}$ as follows:
$m_1 := 1$
$m_{h+1} := m$ if m is the smallest index number in $\{1, ..., k\}$ such that $i_{m_h} R i_m$ and $\vdash \Box\neg L_{i_{m_h}} \prec \Box\neg L_{i_m}$. If no such m exists, set $l = h$ and $m_{h+1} = m_h$. By induction we can prove the following for all p such that $1 \leq p \leq l$:

$$\vdash (\Box\neg L_{i_1} \wedge ... \wedge \Box\neg L_{i_k} \wedge \bigwedge_{g \notin \{i_1,...,i_k\}} \neg\Box\neg L_g) \rightarrow \bigwedge_{j \circ i_{m_p}} \bigvee_{\substack{k\underline{R}i_{m_p} \\ k \circ j}} (\Box\neg L_k \prec \Box\neg L_j)$$

Base case: $p = 1$. Take any j such that $j \circ i_{m_1}$. By definition, $i_{m_1} = i_1$. There are 2 cases, either $j \in \{i_1, ..., i_k\}$ or $j \notin \{i_1, ..., i_k\}$. If $j \in \{i_1, ..., i_k\}$ then by our ordering, $\Box\neg L_{i_1} \prec \Box\neg L_j$. If $j \notin \{i_1, ..., i_k\}$ then

$$\vdash (\Box\neg L_{i_1} \wedge ... \wedge \Box\neg L_{i_k} \wedge \bigwedge_{g \notin \{i_1,...,i_k\}} \neg\Box\neg L_g) \to \neg\Box\neg L_j.$$

Thus we have

$$\vdash (\Box\neg L_{i_1} \wedge ... \wedge \Box\neg L_{i_k} \wedge \bigwedge_{g \notin \{i_1,...,i_k\}} \neg\Box\neg L_g) \to (\Box\neg L_{i_1} \prec \Box\neg L_j).$$

Thus we can conclude that

$$\vdash (\Box\neg L_{i_1} \wedge ... \wedge \Box\neg L_{i_k} \wedge \bigwedge_{g \notin \{i_1,...,i_k\}} \neg\Box\neg L_g) \to \bigwedge_{j \circ i_{m_1}} (\Box\neg L_{i_1} \prec \Box\neg L_j).$$

Since $k\underline{R}i_1$ and $k \circ j$ certainly hold for $k = i_1$, we have

$$\vdash (\Box\neg L_{i_1} \wedge ... \wedge \Box\neg L_{i_k} \wedge \bigwedge_{g \notin \{i_1,...,i_k\}} \neg\Box\neg L_g) \to \bigwedge_{j \circ i_{m_1}} \bigvee_{\substack{k\underline{R}i_{m_1} \\ k \circ j}} (\Box\neg L_k \prec \Box\neg L_j).$$

Induction step: Take any node j such that $j \circ i_{m_{p+1}}$. Either $j \circ i_{m_p}$ or not, so we will split the proof into these two cases.
Case 1: $j \circ i_{m_p}$. By induction hypothesis,

$$\vdash (\Box\neg L_{i_1} \wedge ... \wedge \Box\neg L_{i_k} \wedge \bigwedge_{g \notin \{i_1,...,i_k\}} \neg\Box\neg L_g) \to \bigvee_{\substack{k\underline{R}i_{m_p} \\ k \circ j}} (\Box\neg L_k \prec \Box\neg L_j).$$

By definition of $i_{m_{p+1}}$ we have $i_{m_p} R i_{m_{p+1}}$. Thus $k\underline{R}i_{m_p}$ implies $k\underline{R}i_{m_{p+1}}$, therefore

$$\vdash (\Box\neg L_{i_1} \wedge ... \wedge \Box\neg L_{i_k} \wedge \bigwedge_{g \notin \{i_1,...,i_k\}} \neg\Box\neg L_g) \to \bigvee_{\substack{k\underline{R}i_{m_{p+1}} \\ k \circ j}} (\Box\neg L_k \prec \Box\neg L_j).$$

Case 2: not $j \circ i_{m_p}$, thus $j = i_{m_p}$ or jRi_{m_p} or $i_{m_p}Rj$. However, $j = i_{m_p}$ and jRi_{m_p} both imply that $jRi_{m_{p+1}}$ (since $i_{m_p}Ri_{m_{p+1}}$ by definition of $i_{m_{p+1}}$) which contradicts $j \circ i_{m_{p+1}}$. Thus we have that $i_{m_p}Rj$.
Since $j \circ i_{m_{p+1}}$ we have that that $j \neq i_{m_{p+1}}$. If $j \in \{i_1, ..., i_k\}$, say $j = i_s$ then clearly $s \neq m_{p+1}$. Thus s must violate one of the two conditions for m_{p+1}. For each of these two cases, we will show that $\vdash \Box\neg L_{i_{m_{p+1}}} \prec \Box\neg L_{i_s}$.
By definition of $i_{m_{p+1}}$, m_{p+1} is the smallest index number in $\{1, ..., k\}$ such that $i_{m_p}Ri_{m_{p+1}}$ and $\vdash \Box\neg L_{i_{m_p}} \prec \Box\neg L_{i_{m_{p+1}}}$.
So s must be greater than m_{p+1} or it must violate the condition $\vdash \Box\neg L_{i_{m_p}} \prec \Box\neg L_{i_s}$.

$s > m_{p+1}$: by our ordering we have that $\vdash \Box\neg L_{i_{m_{p+1}}} \prec \Box\neg L_{i_s}$.

$\nvdash \Box\neg L_{i_{m_p}} \prec \Box\neg L_{i_s}$: by our ordering, we have that $s < m_p$. Therefore, we have that $p \neq 1$. If $s = 1$, then $i_s R i_{m_{p+1}}$ by definition of m_{p+1} thus $s \neq 1$. Therefore there must be an $m_t < s$. By our ordering, we have $\vdash \Box\neg L_{i_{m_t}} \prec \Box\neg L_{i_s}$. By definition of set $\{m_1, ..., m_l\}$ and transitivity of R, we have that $i_{m_t} R i_{m_p}$ and thus by transitivity we have that $i_{m_t} R i_s$. By these derivations, $s \in \{m_1, ..., m_l\}$. However, this would imply that $j \circ i_{m_{p+1}}$ is not the case, which contradicts our definition of j such that $j \circ i_{m_{p+1}}$. Therefore, we must have $\vdash \Box\neg L_{i_{m_p}} \prec \Box\neg L_{i_s}$. If $s < m_{p+1}$ then the successor of m_p should be s by our recursive definition of set $\{m_1, ..., m_l\}$ instead of our current m_{p+1}, thus $s > m_{p+1}$. Therefore, we can conclude that $\vdash \Box\neg L_{i_{m_{p+1}}} \prec \Box\neg L_{i_s}$.
We have shown that $\vdash \Box\neg L_{i_{m_{p+1}}} \prec \Box\neg L_j$. By propositional logic, we have that

$$\vdash (\Box\neg L_{i_1} \wedge ... \wedge \Box\neg L_{i_k} \wedge \bigwedge_{g \notin \{i_1,...,i_k\}} \neg\Box\neg L_g) \rightarrow \bigvee_{\substack{k \underline{R} i_{m_{p+1}} \\ k \circ j}} (\Box\neg L_{i_{m_{p+1}}} \prec \Box\neg L_j).$$

Now that we have proven

$$\vdash (\Box\neg L_{i_1} \wedge ... \wedge \Box\neg L_{i_k} \wedge \bigwedge_{g \notin \{i_1,...,i_k\}} \neg\Box\neg L_g) \rightarrow \bigwedge_{j \circ i_{m_p}} \bigvee_{\substack{k \underline{R} i_{m_p} \\ k \circ j}} (\Box\neg L_k \prec \Box\neg L_j)$$

for $p = l$ we have that

$$\vdash (\Box\neg L_{i_1} \wedge ... \wedge \Box\neg L_{i_k} \wedge \bigwedge_{g \notin \{i_1,...,i_k\}} \neg\Box\neg L_g) \rightarrow \bigwedge_{j \circ i_{m_l}} \bigvee_{\substack{k \underline{R} i_{m_l} \\ k \circ j}} (\Box\neg L_k \prec \Box\neg L_j). \quad (8)$$

We know by definition of i_{m_l} that it has no R-successors in $\{i_1, ..., i_k\}$. Thus

$$\vdash (\Box\neg L_{i_1} \wedge ... \wedge \Box\neg L_{i_k} \wedge \bigwedge_{g \notin \{i_1,...,i_k\}} \neg\Box\neg L_g) \rightarrow \bigwedge_{i_{m_l} R j} \neg\Box\neg L_j. \quad (9)$$

Another claim that we can make, by propositional logic, is that

$$\vdash (\Box\neg L_{i_1} \wedge ... \wedge \Box\neg L_{i_k} \wedge \bigwedge_{g \notin \{i_1,...,i_k\}} \neg\Box\neg L_g) \rightarrow \Box\neg L_{i_{m_l}}. \quad (10)$$

Combining statements 8, 9 and 10, we have that

$$\vdash (\Box\neg L_{i_1} \wedge ... \wedge \Box\neg L_{i_k} \wedge \bigwedge_{g \notin \{i_1,...,i_k\}} \neg\Box\neg L_g) \rightarrow (\Box\neg L_{i_{m_l}} \wedge \bigwedge_{i_{m_l} R j} \neg\Box\neg L_j \wedge \bigwedge_{j \circ i_{m_l}} \bigvee_{\substack{k \underline{R} i_{m_l} \\ k \circ j}} (\Box\neg L_k \prec \Box\neg L_j)).$$

By Equation 6, we have

$$\vdash (\Box\neg L_{i_1} \wedge ... \wedge \Box\neg L_{i_k} \wedge \bigwedge_{g \notin \{i_1,...,i_k\}} \neg\Box\neg L_g) \rightarrow L_{i_{m_l}}.$$

Then by Equation 7,

$$\vdash \bigvee_{i \neq 0} \Box\neg L_i \to L_{i_{m_l}}.$$

By Equation 5, we have that

$$\vdash \bigvee_{i \neq 0} \Box\neg L_i \to \neg L_0.$$

By propositional logic,

$$\vdash L_0 \to \neg \bigvee_{i \neq 0} \Box\neg L_i.$$

Again by propositional logic, we can conclude that

$$\vdash L_0 \to \bigwedge_{i \neq 0} \neg\Box\neg L_i.$$

We have proven statement (4) of the lemma.

Since we have proven statements (2) and (4) of the lemma, we obtain $\mathbb{N} \models \bigwedge_{0<i\leq n} \neg\Box\neg L_i$. Thus $\nvdash \Box\neg L_i$ and therefore $T + L_i$ is consistent for all i such that $0 < i \leq n$. Since $\mathbb{N} \models L_0$ we have by Σ_1-completeness that L_0 is provable in T. Thus L_0 is a theorem of T and therefore $T + L_0$ is consistent. We have proven statement (3) of the lemma.

Finally, we need to show statement (5) of the lemma holds. For some node i such that $0 < i \leq n$ we would like to investigate all nodes j such that $\neg iRj$. Thus nodes j such that $i = j$ or $i \circ j$ or jRi. Therefore, we split our prove of statement (5) into three cases.
Case 1: $i = j$. By Equation 6, $\vdash L_i \to \Box\neg L_i$.
Case 2: jRi. By Equation 6, $\vdash L_j \to \bigwedge_{jRk} \neg\Box\neg L_k$. Thus $\vdash L_j \to \neg\Box\neg L_i$.
By propositional logic, $\vdash \Box\neg L_i \to \neg L_j$. By the Necessitation Rule and Distribution Axiom, we have that $\vdash \Box\Box\neg L_i \to \Box\neg L_j$. By soundness of Axiom 4, $\vdash \Box\neg L_i \to \Box\Box\neg L_i$. Thus our statement can be reduced to $\vdash \Box\neg L_i \to \Box\neg L_j$. By Equation 6, $\vdash L_i \to \Box\neg L_i$, thus our statement can be reduced to $\vdash L_i \to \Box\neg L_j$.
Case 3: $i \circ j$. Fix node j. By Equation 6, $\vdash L_i \to \bigwedge_{i \circ j'} \bigvee_{\substack{k\underline{R}i \\ k \circ j'}} (\Box\neg L_k \prec \Box\neg L_{j'})$.

By propositional logic, we can leave out some conjuncts in the consequent, so we obtain

$$\vdash L_i \to \bigwedge_{\substack{i \circ j' \\ j'\underline{R}j}} \bigvee_{\substack{k\underline{R}i \\ k \circ j'}} (\Box\neg L_k \prec \Box\neg L_{j'}). \tag{11}$$

As we did before in the proof of statement (4) of our lemma, we can order formulas $\Box\neg L_k$'s linearly by $\prec$. Therefore, there must be a $\Box\neg L_k$ which is smallest and thus smaller than all $\Box\neg L_{j'}$. Thus we have

$$\vdash \bigwedge_{\substack{i\circ j'\\ j'\underline{R}j}} \bigvee_{\substack{k\underline{R}i\\ k\circ j'}} (\Box\neg L_k \prec \Box\neg L_{j'}) \to \bigvee_{\substack{k\circ j\\ k\underline{R}i}} \bigwedge_{\substack{j'\underline{R}j\\ j'\circ i}} (\Box\neg L_k \prec \Box\neg L_{j'}).$$

By Equation 11 this simplifies to

$$\vdash L_i \to \bigvee_{\substack{k\circ j\\ k\underline{R}i}} \bigwedge_{\substack{j'\underline{R}j\\ j'\circ i}} (\Box\neg L_k \prec \Box\neg L_{j'}) \tag{12}$$

which means that L_i implies that there is a node k such that $k\underline{R}i$ and (this is more important) $k \circ j$ and $\exists p(\mathbf{Proof}(p, L_k) \wedge \neg\exists q \leq p\ \mathbf{Proof}(q, L_{j'})$ for all nodes j' such that $j'\underline{R}j$.
By Equation 6, we have

$$\vdash L_j \to \bigwedge_{j\circ r} \bigvee_{\substack{m\underline{R}j\\ m\circ r}} (\Box\neg L_m \prec \Box\neg L_r). \tag{13}$$

Therefore we have that L_j implies that for each node r such that $r \circ j$ there must be a node m such that $m\underline{R}j$ and $\exists p(\mathbf{Proof}(p, L_m) \wedge \neg\exists q \leq p\ \mathbf{Proof}(q, L_r)$ Since there must be a node k as described before if we have $\bigvee_{\substack{k\circ j\\ k\underline{R}i}} \bigwedge_{\substack{j'\underline{R}j\\ j'\circ i}} (\Box\neg L_k \prec \Box\neg L_{j'})$, it means that there is no node m as described before. However, there must be a node m for $\bigwedge_{j\circ r} \bigvee_{\substack{m\underline{R}j\\ m\circ r}} (\Box\neg L_m \prec \Box\neg L_r)$ to be a theorem of T. Therefore,

$$\vdash \bigvee_{\substack{k\circ j\\ k\underline{R}i}} \bigwedge_{\substack{j'\underline{R}j\\ j'\circ i}} (\Box\neg L_k \prec \Box\neg L_{j'}) \to \neg(\bigwedge_{j\circ r} \bigvee_{\substack{m\underline{R}j\\ m\circ r}} (\Box\neg L_m \prec \Box\neg L_r)).$$

By propositional logic,

$$\vdash \neg(\bigwedge_{j\circ r} \bigvee_{\substack{m\underline{R}j\\ m\circ r}} (\Box\neg L_m \prec \Box\neg L_r)) \to \neg L_j.$$

By concatenating the implications, we have that

$$\vdash L_i \to \neg L_j$$

Since $H_i(x)$ is a Δ_0 sentence, L_j is a Σ_1 sentence so we have that $\vdash \neg L_j \to \Box\neg L_j$. Thus we can conclude that $\vdash L_i \to \Box\neg L_j$. □

Thus we have shown in this section that a lemma similar to the lemma on limit l by Solovay holds for theories T that are $I\Delta_0$+ EXP or r.e.-extensions that are Σ_1-sound. This result was obtained by generalizing a proof by de Jongh and colleagues.

6 Conclusion

In this thesis, we have discussed three papers from which we can conclude the completeness of the logic GL with respect to the class of finite trees and the arithmetical completeness with respect to Peano Arithmetic. We have also laid the groundwork for the proof of arithmetical completeness with respect to a group of weaker arithmetics. For the completeness proof on finite trees we needed to construct a finite tree model from a more general model on an arbitrary frame. This construction was carried out in an intuitive way by Smoryński.

Solovay used this completeness result in his arithmetical completeness result. The arithmetical completeness proof by Solovay is regarded as important because it answers partly the question about what mathematical theories can say about themselves. Also, it gave rise to another arithmetical completeness proof on weaker arithmetics, which is therefore also part of the answer to the question as stated before. Therefore, Segerberg, who was the first to prove the completeness of GL with respect to the class of all finite trees, made a big contribution to answering this question without realizing at the time what he had started.

Not all the limits on the arithmetical completeness result by Solovay are described in this paper or even researched. The paper 'On the provability logic of bounded arithmetics' by A. Berarducci and L.C. Verbrugge [2] is about another group of weaker arithmetics, $I\Delta_0 + \Omega_1$, for which we are not yet able to prove or disprove its arithmetical completeness for GL. The problem has to do with some open problems in computational complexity theory. For further research, it would be interesting to look into the arithmetical completeness proof by de Jongh and his colleagues to verify if this proof can be carried out for logic GL instead of logic R. If this can be done, one can conclude that we have arithmetical completeness not only for logic R but also for logic GL since we have proven Lemma 5.3 without making use of logic R_F^-.

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