The Theory of (2+1)-Dimensional Quantum Electrodynamics and Corresponding β-Function

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Abstract

This paper quantizes the theory of (2+1)-dimensional quantum electro dynamics (QED$_3$), derives the theories β-function and discusses the physical ramifications of this function. The dimensional dependence of classical electrodynamics is first discussed. Subsequently the path integral formalism is derived and used to define the renormalization group and the β-function. QED$_3$ is then quantized, the counter terms are calculated to 1-loop accuracy by using the ϵ-expansion from 4 dimensions. The β-function is computed and compared to previous results. Chiral symmetry breaking for QED$_3$ and its relation to the number of massless fermion flavors to which the theory is coupled is discussed. QED$_3$ as a model for high T_c cuprate superconductors is briefly touched upon.

Contents

1 Introduction

Einsteins theory of general relativity postulates that the universe that we inhabit is a 4-dimensional manifold, despite this it is often useful to examine how physical processes vary if the amount of dimensions is reduced. Especially when moving into the realm of quantum field theory the dimensional dependence of many effective theories ensures that the properties of even rather simple models may change drastically as the number of spatial dimensions is changed. One of these theories, and the focus of this thesis, is that of quantum electrodynamics (QED), which describes the theory of the electromagnetic field coupled to a set of Dirac fermion fields.

In 4 dimensions the theory of QED has researched extensively over the last few decades and has offered predictions that have been experimentally verified with high degrees of accuracy. In the case that the fermions with which the electromagnetic field couples are massless and the charge e is zero the theory is invariant under conformal transformations, ie. conformally invariant (see [1]), however interactions often break this invariance. The perturbation theory that is used to describe the theory results in the coupling constant, the electric charge e, to be dependent on the energy scale of the theory. The β-function, which is a measure of how the coupling constant varies with changing energy, is strictly non-negative for QED and increases as the energy scale increases. This implying that the theory becomes free at larger distances (infrared regime; IR) and becomes strongly coupled at short distances (ultraviolet regime; UV).

In a dimension $d < 4$ the physics that describes QED begins to change drastically. Wilson and Fisher [2] were able to show that by using a technique known as the ϵ-expansion, in which the theory is calculated with a dimension of $d = 4 - 2\epsilon$ and $\epsilon \ll 1$, that certain theories such as the XY model have additional renormalization group (RG) fixed points in dimensions even slightly lower than 4. These RG fixed points correspond to the zeros of the β-function. This was shown to be true in QED [3] as well, in addition to the fact that the ϵ may be taken to larger values such as $\frac{1}{2}$ to explore three dimensional QED (QED_3) with reasonable accuracy. Similar results had also been calculated using the $1/N_f$-expansion of QED [4] (see also [5]), in which N_f is the number of Dirac fermions. Each resulting in the fact that the β-function for QED in dimension lower than 4 has a zero at a nonzero coupling, therefore the theory is quantized, conformal, and interacting at this fixed point.

The implications of this zero, and the corresponding fixed point in the renormalization group, result in QED_3 having interesting properties not found in its 4-dimensional counterpart. QED_3 has a global $SU(2N_f)$ symmetry group, in which N_f is the number of massless fermion flavours to which the electromagnetic field is coupled. Under certain values of N_f the existence of this fixed point, referred to as the Wilson-Fixed point, results in a spontaneous symmetry breaking in the from.

$$SU(2N_f) \to SU(N_f) \times SU(N_f) \times U(1) \tag{1.1}$$

This breaking of symmetry is known as the chiral symmetry breaking, and can result in the spontaneous generation of mass. The condition placed upon N_f for this to occur was initially though to be that $N_f > 0$ [6] but a paper by Appelquist et al. (1988) [7] showed that chiral symmetry breaking occurs only when $N_f < N_f^c$, where N_f^c is a critical value. There are various estimates for the value of N_f^c ranging from 2 to 10 [3, 7–10]

It has also been shown that QED_3 is equivalent to the theory of quasiparticles in high critical temperature (T_c) cuprate superconductors [11–13]. The chiral symmetry breaking may actually play a role in the spontaneous generation of gaps for fermionic interactions at $T = 0$.

In this thesis we derive the β-function for QED_3 and discuss some of the physical ramifications of this theory. The following section will offer a brief discussion of classical electrodynamics and how it depends on the number of spatial and temporal dimensions classically. In addition, it also discusses some of the properties of the classical electrodynamic lagrangian which will be the starting point for when we quantize the theory.

Section 3 introduces the path integral formulation of quantum field theory. This alternative formulation to the operator based view of the Heisenberg or Schrödinger models is particularly useful in the computation of correlation functions. In section 4 we describe the principles of the renormalization group and how this results in the definition of the β-function and the anomalous dimension. Furthermore, we discuss the existence of renormalization group flow from and towards to fixed points and how these correspond to the zeros of the β-functions.

In section 5 we quantize the theory of electrodynamics and obtain the general partition function. We also discuss how the partition function may be interpreted in terms of connected Feynman diagrams. In section 6 we compute the β-function for QED. We begin by calculating the various 1-loop corrections to QED_3 to determine the value of the counterterms. These counterterms are subsequently used to determine the β-function. The β-function computed here is compared to β-functions calculated through other methods, such as different renormalization schemes. In pertubation theory the β-function of QED takes a polynomial form as an expansion in terms of the coupling constant e, but the exact location of the zero differs. Finally in section 7 we discuss the physical relevance of the QED_3 theory and β-function, such as the spontaneous chiral symmetry breaking and the relation to high T_c super conductors.

2 Classical Electrodynamics in Arbitrary Dimension

Classical electrodynamics is the study of how charged particles interact with each other and with external electric and magnetic fields. Before we quantize this theory to examine the interactions between various fields on a quantum scale, and how this quantization is dependent upon dimensions, it is important to first understand the dimensional dependence of classical electrodynamics.
In the classical formulation of $(3+1)$-dimensional electrodynamics the equations that govern the behaviour of the electric and magnetic field are the Maxwell equations (in Gaussian units):

$$\nabla \cdot \mathbf{E} = 4\pi\rho \quad \nabla \cdot \mathbf{B} = 0 \tag{2.1}$$
$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{B} = \frac{1}{c}\left(4\pi\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}\right)$$

Where $\mathbf{E},\ \mathbf{B}$ are the electric and magnetic fields, ρ is the charge density and $\mathbf{J}$ is the current density. These four equations in conjunction with the conservation of charge and equation for the Lorentz force density

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \qquad \mathbf{f} = \rho\mathbf{E} + \frac{1}{c}\mathbf{J} \times \mathbf{B} \tag{2.2}$$

are generally sufficient to describe all classical electrodynamics phenomena. The physics changes substantially if the number of dimensions is lowered. In $(2+1)$-dimensions the electric field naturally becomes a 2-dimensional vector however the magnetic field becomes a scalar field. In $(1+1)$ dimensions the magnetic field ceases to exist (unless magnetic monopoles exist), and the electric field becomes a scalar. It may be shown that in $(1+1)$ dimensions that for any region of space that does not contain a charge the electric field is constant, as such electromagnetic waves cannot occur.

In $(3+1)$ dimensions the introduction of the scalar potential ϕ and the vector potential $\mathbf{A}$ defined by the relations:

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{2.3}$$

Similar definition may be formulated for $(2+1)$ and $(1+1)$ dimensional cases. Through these definitions is possible to reformulate the Maxwell equations in a significantly more compact relativistic description. Defining the vector $A_\mu = (\phi, \mathbf{A})$ we may define the electromagnetic field tensor as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.4}$$

From this definition it is clear that the tensor F is anti-symmetric. If we further define the vector $J_\mu = (\rho, \mathbf{J})$ as the collection of the charge and current, a relativistic formulation, in Minkowski spacetime, of the Maxwell equations (2.1) is given by the following equations:

$$\partial^\mu F_{\mu\nu} = \frac{4\pi}{c} J_\mu \tag{2.5}$$

$$\partial^\lambda F_{\mu\nu} + \partial^\mu F_{\nu\lambda} + \partial^\nu F_{\lambda\mu} = 0 \tag{2.6}$$

where (2.6) is the Bianchi identity. The conservation of current may be simply written as $\partial_\mu J_\mu = 0$. The conservation of current directly implies the conservation of charge due to the relation

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int \rho(\mathbf{x}, t)\ \mathrm{d}\mathbf{x} = 0 \tag{2.7}$$

Finally the Lorentz force may be written as:

$$f_\mu = F_{\mu\nu} \frac{J^\nu}{c} \tag{2.8}$$

where the Einstein summation convention is in effect.
In attempting to extend this to arbitrary $(n+1)$-dimensions we might naively assume that the equations will hold simply by allowing μ, ν to run from $1 \ldots n+1$. This is almost sufficient, however as pointed out by [14] the factor 4π that appears in (2.5) is associated with the surface area of a 3-dimensional unit sphere. As such in arbitrary dimension this factor must be replaced by the equivalent surface area of an n-dimensional unit sphere. This is given by:

$$C_n = \frac{n\pi^{n/2}}{\Gamma(n/2+1)} \tag{2.9}$$

As such classical electrodynamics in $(n+1)$ is given by equations (2.4), (2.6), (2.8) and:

$$\partial_\mu F_{\mu\nu} = \frac{C_n}{c} J_\mu \tag{2.10}$$

where $\mu, \nu = 1 \ldots n+1$. For the classical langagrian density for electrodynamics we choose:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{C_n}{c} A_\mu J^\mu \tag{2.11}$$

the Euler-Lagrange equations of this lagrangian give equation (2.10) justifying this choice. For the case of (2+1) dimensions, C_n takes the value 2π. As such this constant may simply be absorbed into the source term J^μ or may be removed by changing the unit from Gaussian to Heaveside-Lorentz units. If we further adopt the convention that $c = 1$ the general Lagrangian density for a non interacting electromagnetic field with a source J^μ is given by:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A_\mu J^\mu \tag{2.12}$$

This implies that the lagrangian to be quantized is itself independent of dimension except for the scaling of some units.
Now we note that if A_μ is shifted by the derivative of an arbitrary function $\partial_\mu \Gamma$ the field strength is invariant:

$$F'_{\mu\nu} = \partial_\mu(A_\nu - \partial_\nu \Gamma) - \partial_\nu(A_\mu - \partial_\mu \Gamma) = F_{\mu\nu} - (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\Gamma = F_{\mu\nu}$$

A transformation of the potential that does not change the field strength tensor, $F_{\mu\nu}$, is defined to be a gauge transformation, and F is therefore said to be gauge invariant. It may further be seen that if the source term J_μ is conserved the Lagrangian is gauge invariant.
Due to the fact that the field strength $F^{\mu\nu}$ is anti-symmetric, the Lagrangian (2.12) does not contain any time derivatives of A^0. As such it is impossible to construct a canonical conjugate momentum and in turn makes the theory difficult to quantise. The gauge invariance of the Lagrangian has resulted in the system having too many degrees of freedom. This is solved by choosing an explicit gauge condition such that $F^{\mu\nu}$ may only be constructed by a single choice of A^μ. The most common choice is the Lorenz gauge given by the condition $\partial^\mu A_\mu = 0$. Throughout this paper we will use the Lorentz gauge and the closely associated Landau gauge in section 5.
The imposition of the gauge and the fact that A^0 is not a dynamic field fixes 2 degrees of freedom. The remaining degrees of freedom will determine the number of polarization states that photons may exhibit. In $(3+1)$ dimensions this obviously results in two polarization states, while in $(1+1)$ dimensions no degrees of freedom remain corresponding to the fact that electromagnetic waves do not exist in $(1+1)$ dimensions.

3 Path Integral Formalism

As mentioned in the previous section, it is impossible to construct a canonical conjugate momentum for the electrodynamic lagrangian. As such it is difficult, but by no means impossible, to quantize the theory through an operator based model, such as through the construction of creation and destruction operators. The quantization of electrodynamics may instead be completed through use of the path integral formalism , which has the distinct advantage of working with integrals and functions in contrast to operators. In addition, this formalism allows for the derivation and physical intuition behind the β-function, which will be explored in the subsequent section. To derive the path integral formalism we first consider the Hamiltonian for one dimensional (non-relativistic) quantum mechanics:

$$H(P,Q) = \frac{1}{2m}P^2 + V(Q) \tag{3.1}$$

such that P and Q are momentum and position operators satisfying the commutator relation $[Q,P] = i$, where in our notation $\hbar = 1$. Working in the

Heisenberg picture we let $|q,t\rangle = e^{+iHt}|q\rangle$ represent the state that a particle exists at position q at time t. We further transform the operators into their time dependent form, $Q(t) = e^{iHt}Qe^{-iHt}$, such that

$$Q(t)|q,t\rangle = e^{iHt}Q|q\rangle = qe^{iHt}|q\rangle = q|q,t\rangle$$

Then we specifically wish to calculate the transition amplitude between a state initially given by $|q,t\rangle$ and finally given by $|q',t'\rangle$. This amplitude is given by $\langle q',t'|q,t\rangle$ in the chosen Heisenberg picture. Using the identity function $1 = \int \mathrm{d}q\,|q\rangle\langle q|$ we may split the total time interval $T = t' - t$ into $N+1$ equal intervals of time $\Delta t = \frac{T}{N+1}$ to rewrite our transition amplitude as:

$$\left\langle q',t'\middle|q,t\right\rangle = \int \prod_{j=1}^{N} \mathrm{d}q_j \,\langle q'|\, e^{-iH\Delta t}\,|q_N\rangle\,\langle q_N|\, e^{iH\Delta t}\,|q_{N-1}\rangle \ldots \langle q_1|\, e^{-iH\Delta t}\,|q\rangle \tag{3.2}$$

One may use the Campbell-Baker-Hausdorf formula to show an equivalence between [15] :

$$\langle q_N|\, e^{iH\Delta t}\,|q_{N-1}\rangle = \int \frac{\mathrm{d}p_{N-1}}{2\pi} e^{-iH(p_{N-1},q_{N-1})\Delta t} e^{ip_{N-1}(q_N - q_{N-1})} \tag{3.3}$$

in the limit that Δt approaches 0. This relation, in conjunction with Weyl ordering which gives a relation between the quantum operator based hamiltonian and the classical hamiltonian to be:

$$H(P,Q) = \int \frac{\mathrm{d}x}{2\pi}\frac{\mathrm{d}k}{2\pi} e^{ixP+ikQ} \int \mathrm{d}q\mathrm{d}p e^{-ixp-ikq} H(p,q) \tag{3.4}$$

allows us to rewrite the transition amplitude as:

$$\left\langle q',t'\middle|q,t\right\rangle = \int \prod_{j=0}^{N} \mathrm{d}q_j \prod_{k=0}^{N} \frac{\mathrm{d}p}{2\pi} e^{ip_k(q_{j+1}-q_j)} e^{-iH(p_k,q_j)\Delta t} \tag{3.5}$$

in which we take $q = q_0$ and $q' = q_{N+1}$. Defining $\dot{q}_j = \frac{q_{j+1}-q_j}{\Delta t}$ and taking the formal limit of the amplitude as $\Delta t \to 0$ we obtain:

$$\left\langle q',t'\middle|q,t\right\rangle = \int \mathcal{D}q\mathcal{D}p \,\exp\left[i\int_t^{t'} \mathrm{d}t\,(p(t)\dot{q}(t) - H(p(t),q(t)))\right] \tag{3.6}$$

in which $\mathcal{D}q$ and $\mathcal{D}p$ are (Lebesgue) integral measures defined as the formal limit of the infinite products of dq_js and dp_ks respectively.
In the special cases in which the Hamiltonian (3.1) is at most quadratic in momentum and if that quadratic term is independent of the position q then it is possible to integrate out p. In this case the integrals over p are gaussian [15] and all constant factors maybe absorbed into $\mathcal{D}q$ to obtain:

$$\left\langle q',t'\middle|q,t\right\rangle = \int_{q(t)=q}^{q(t')=q'} \mathcal{D}q \,\exp\left[i\int_t^{t'} \mathrm{d}t L(\dot{q}(t),q(t))\right] = \int_q^{q'} \mathcal{D}q e^{iS} \tag{3.7}$$

In which S is the action. We may consider this path integral as a weighted average over all possible paths between the initial and final states.
we may examine some of the properties of this transition amplitude. Examining the case $\langle q', t' | Q(t_1) | q, t\rangle$ where $t < t_1 < t'$ we see that this is equivalent to

$$\langle q', t' | Q(t_1) | q, t\rangle = \langle q', t' | e^{-iH(t'-t_1)} Q e^{iH(t_1-t)} | q, t\rangle = \int_q^{q'} \mathcal{D}q \; q(t_1) e^{iS} \quad (3.8)$$

As such the operator is transformed into a simple function under the path integral. Generalizing this to arbitrarily many position operators:

$$\langle q', t' | T \; Q(t_1)Q(t_2)\ldots Q(t_n) | q, t\rangle = \int_q^{q'} \mathcal{D}q \; q(t_1)q(t_2)\ldots q(t_n) e^{iS} \quad (3.9)$$

Where T is the time ordering symbol, that orders the operators such that the later times are placed to the left of the earlier times.
Examining further if we replace our Lagrangian with $L \to L + f(t)q(t)$, which is equivalent to a Lagrangian with an external source $f(t)$, we see that:

$$\frac{1}{i}\frac{\delta}{\delta f(t_1)} \langle q', t' | q, t\rangle_f = \int_q^{q'} \mathcal{D}q \; q(t_1) e^{i\int \mathrm{d}t L + fq} \quad (3.10)$$

This allows us to further generalize the time ordered product of operators to

$$\langle q', t' | T \; Q(t_1)\ldots Q(t_n) | q, t\rangle = \frac{1}{i}\frac{\delta}{\delta f(t_1)} \ldots \frac{1}{i}\frac{\delta}{\delta f(t_n)} \langle q', t' | q, t\rangle\Big|_{f=0} \quad (3.11)$$

In particular we wish to calculate the transition amplitude from the ground state to the ground state, and further to take the times to the limits $t \to -\infty$ and $t' \to \infty$. As the path integral is interpreted as an integral over all possible paths from the initial to the final state there is the possibility that the boundary conditions of these paths are not well behaved in taking the time limits to infinity. However [15] shows that any reasonable boundary condition will results in the ground state being the initial and final state provided that we replace our Hamiltonian in (3.6) with $H \to (1 - i\epsilon)H$, where ϵ is an infinitesimal constant. This allows us to write:

$$\langle 0|0\rangle_f = \int \mathcal{D}q \; e^{i\int_{-\infty}^{\infty} dt \; (p\dot{q} - (1-i\epsilon)H(p,q) + fq)} \quad (3.12)$$

Assuming that the hamiltonion is, as previously described, at most quadratic in momentum and if this quadratic term exits it is independent of q, and further assuming the corresponding lagrangian may be written in the form $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$, such that $\mathcal{L}_1$ is only dependent on q and may be treated as a perturbation of

$\mathcal{L}_0$, then we obtain the following relation

$$\langle 0|0\rangle_f = \int \mathcal{D}q \, \exp\left[i\int_{-\infty}^{\infty} \mathrm{d}t \left(\mathcal{L}_0(q,\dot{q}) + \mathcal{L}_1(q) + fq\right)\right]$$
$$= \exp\left[i\int_{-\infty}^{\infty} \mathrm{d}t\mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta f(t)}\right)\right] \times \int \mathcal{D}q \, \exp\left[i\int_{-\infty}^{\infty} \mathrm{d}t \left(\mathcal{L}_0(q,\dot{q}) + fq\right)\right] \quad (3.13)$$

This equation may be derived by noting that if the exponential pre-factor is taken inside of the integral, then in accordance with equation (3.10), the correct factor of q will be pulled out.

In the case of relativistic field theory the Lagrangian is replaced with the appropriate Lagrangian density. Similarly the position functions are replaced with fields $q(t) \to \phi(\mathbf{x}, t)$, where ϕ is an arbitrary field. The operators are replaced with corresponding operator fields $Q(t) \to \phi(\mathbf{x}, t)$ and the functions f are replaced by sources $f(t) \to J(\mathbf{x}, t)$, in d-dimension space this allows us to write in general (suppressing the $i\epsilon$)

$$Z_0(J) \equiv \langle 0|0\rangle_J = \int \mathcal{D}\phi e^{i\int \mathrm{d}^d x \ [\mathcal{L}+J\phi]} \quad (3.14)$$

Where $Z_0(J)$ is generally called the partition function or generating functional (of the correlation functions). The usefulness of the partition function over the more traditional method of quantization using the creation and annihilation operators is in its ability to transform operators into functions. As previously mentioned the position operators simply transform into position functions under the partition functions integral. Furthermore similar relationships exist for other operators, such as those for momentum or the number of particles. As will be shown in the subsequent section, this allows us to calculate physical objects such as propagators with relative ease and computational clarity.

4 The Renormalization Group

The partition function, as calculated in the previous section, describes the theory based on the lagrangian that defines it. However parts of this lagrangian, such as the coupling constants for the interaction terms, may depend on the energy scale. As such it useful to determine how consequently the partition function is dependent on the energy scale. In addition, many theories are solved or simplified through the usage of perturbation theory, which implies that the theory breaks down as sufficiently high energy scales. How the partition function evolves as the energy is changed is described by the β-function(s), which measures how the coupling constant(s) vary with respect to the energy scale. For example, in the case of (3+1)-dimensional QED the beta function is non-negative and increasing. This implies that at sufficiently high energy the coupling constant, in this case the electric charge e, becomes infinite at which point the perturbation theory

breaks down.
To obtain the β-function we begin by operating under the assumption that the action takes a generic form of

$$S_{\Lambda_0}[\psi] = \int \mathrm{d}^d x \left[\frac{1}{2}\partial^\mu \psi \partial_\mu \psi + \sum_i \Lambda_0^{d-d_i} g_{i0} \mathcal{O}_i(x) \right] \tag{4.1}$$

In this equation we have an arbitrary set of operators $\mathcal{O}_i$ each with canonical dimension $d_i > 0$. The operators are all of the form $\mathcal{O}_i \sim (\partial_\mu \psi)^{p_i} (\partial^\mu \psi)^{p_i} \psi^{q_i}$ where p_i and q_i are both integers and sum to $n_i = p_i + q_i$. g_{i0} is the dimensionless coupling constant of the operator. The value Λ_0 is an energy scale used to ensure that the coupling constant is dimensionless. We may use this action to define a regularized partition function

$$Z_{\Lambda_0}(g_{i0}) = \int_{C^\infty(M) \leq \Lambda_0} \mathcal{D}\psi e^{iS_{\Lambda_0}[\psi]} \tag{4.2}$$

As the partition function may be interpreted as the integral over all possible curves on some d-dimensional manifold M averaged by the factor e^{iS} the regularized partition function only integrates over all smooth function with total energy less than or equal to Λ_0.
The space $C^\infty(M) \leq \Lambda_0$ may be equipped with pointwise addition and constant multiplication and as such may be interpreted as a vector space. This allows to consider the path integral in two steps, first by integrating over all smooth functions with energy less that some $\Lambda < \Lambda_0$ and then over the region between $(\Lambda, \Lambda_0]$. The field may be split accordingly by means of a Fourier transform

$$\psi(x) = \int_{|p| \leq \Lambda} \frac{\mathrm{d}^d p}{(2\pi)^d} e^{ipx} \phi(p) + \int_{\Lambda < |p| \leq \Lambda_0} \frac{\mathrm{d}^d p}{(2\pi)^d} e^{ipx} \phi(p) = \phi(x) + \chi(x) \tag{4.3}$$

such that $\phi(x) \in C^\infty(M)_{\leq \Lambda}$ and $\chi(x) \in C^\infty(M)_{(\Lambda, \Lambda_0]}$ are the low and high energy regions of the field. The measure may likewise be factorized to $\mathcal{D}\psi = \mathcal{D}\phi\mathcal{D}\chi$. Using this fact we may integrate over only the high energy modes χ; from this we obtain the effective action at an energy scale Λ, one version of the so called renormalization group equation

$$S_\Lambda^{\mathrm{eff}}[\phi] = -i \log \left[\int_{C^\infty(M)_{(\Lambda, \Lambda_0]}} \mathcal{D}\chi \exp(iS_{\Lambda_0}[\phi + \chi]) \right] \tag{4.4}$$

This process is referred to as changing the scale of the theory, in reference to the fact that the energy scale of the partition function has been decreased. This process may be iteratively performed to probe the theory at lower and lower energy modes, this low energy region is generally called the IR or infrared region. The new partion function

$$Z_\Lambda(g_i(\Lambda)) = \int_{C^\infty(M)_{\leq \Lambda}} \mathcal{D}\phi e^{iS_\Lambda^{\mathrm{eff}}[\phi]} \tag{4.5}$$

is now dependent upon the lower energy scale Λ. However this equation is equivalent to the original partition function $Z_\Lambda(g_i(\Lambda)) = Z_{\Lambda_0}(g_{i0};\Lambda_0)$. This may be seen by considering that despite the fact that the new partition functions path integral is only perfomed over the low energy modes $C^\infty(M)_{\leq\Lambda}$ all information concerning the higher energy modes is now contained in the effective action S_Λ^{eff}. As such no information is lost. The effective action takes a general form similar to the action initially defined in (4.1)

$$S_\Lambda^{\text{eff}}[\phi] = \int \mathrm{d}^d x \left[\frac{C_\Lambda}{2}\partial^\mu\phi\partial_\mu\phi + \sum_i \Lambda_0^{d-d_i} C_\Lambda^{n_i/2} g_{i0}\mathcal{O}_i(x)\right] \tag{4.6}$$

Where C_Λ is called the wavefunction renormalization factor which accounts for the possibility that the integration of the higher modes resulted in quantum corrections to the various terms in the action. This allows us to define a renormalized field $\xi = C_\Lambda^{1/2}\phi$
The renormalization of the wavefunction becomes relevant in the computation of the correlation functions of operators. Suppose we wish to compute the n-point correlator, the correlation function of n field operators

$$\langle 0|\,\phi(x_1)\cdots\phi(x_n)\,|0\rangle = \int_{C^\infty(M)_{\leq\Lambda}} \mathcal{D}\phi\; e^{iS_\Lambda^{\text{eff}}[C_\Lambda^{1/2}\phi;g_i(\Lambda)]}\phi(x_1)\cdots\phi(x_n) \tag{4.7}$$

In terms of the previously defined renormalized field we obtain that

$$\langle 0|\,\phi(x_1)\cdots\phi(x_n)\,|0\rangle = C_\Lambda^{-n/2}\,\langle 0|\,\xi(x_1)\cdots\xi(x_n)\,|0\rangle$$

We may subsequently integrate out all of the modes in the region $(s\Lambda,\Lambda]$ for some value of $s<1$. Noting that in evaluating of the correlation function it will result in some function $\Gamma_\Lambda^{(n)}(x_1,\ldots,x_n;g_i(\Lambda))$, that is dependent on the scale Λ, the fixed points x_i, and the set of coupling constants $g_i(\Lambda)$. This results in the relation

$$C_{s\Lambda}^{-n/2}\Gamma_\Lambda^{(n)}(x_1,\ldots,x_n;g_i(s\Lambda)) = C_\Lambda^{-n/2}\Gamma_\Lambda^{(n)}(x_1,\ldots,x_n;g_i(\Lambda)) \tag{4.8}$$

When $s\to 0$ this relation leads to a differential equation, which is a single example of a Callan-Symanzik equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\log\Lambda}&\Gamma_\Lambda^{(n)}(x_1,\ldots,x_n;g_i(\Lambda))\\ &= \left(\frac{\partial}{\partial\log\Lambda} + \beta_i\frac{\partial}{\partial g_i} + n\gamma\right)\Gamma_\Lambda^{(n)}(x_1,\ldots,x_n;g_i(\Lambda))\end{aligned} \tag{4.9}$$

In which $\beta_i = \frac{\mathrm{d}g_i}{\mathrm{d}\log\lambda}$ and $\gamma = \frac{\mathrm{d}\log C_\Lambda}{\mathrm{d}\log\Lambda}$ are the beta function of the running coupling $g_i(\Lambda)$ and the anomalous dimension of the field respectively. The anomalous dimension γ gives the difference between the scaling dimension of the field and the classical dimension. The anomolous dimensions are particularly interesting once a fixed point has been found as they contain much of the relevant physics. However the β-function is more interesting for the purpose of this paper.

In general the lagrangian that describes the system will have define a set of coupling constants $\{g_i\}$, each of these coupling constant will have a corresponding β-function $\beta_i(g_1, g_2, \ldots, g_n)$. At specific energy values the coupling constants take particular values $\{g_{1*}, \ldots, g_{n*}\}$ such that the beta functions at those points vanish $\beta_j(g_{1*}, \ldots, g_{n*}) = 0$. These values are are called critical or fixed points of the renormalization groups flow. As such the process of determining fixed points consists of determining the zeros of a set of couple ordinary differential equations. A fixed point is always reached when the energy scale goes to infinity or zero. When the scale goes to zero the region is called infrared (IR). If an IR fixed point exists it is reached in the limit of far IR. Equivalently the energy scale approaching infinity is referred to as far into the ultraviolet (UV) region, if an UV fixed point exists it is reached in the limit of this process. Theories at fixed points are independent of scale. Moreover all Lorentz invariant, unitary theories, such as quantum electrodynamics, are at critical points invariant under the larger group of conformal transformations. This implies that theories at critical points are, in all the mentioned cases, conformal field theories (CFT).
When the theory is very close to a critical point the beta functions take the form

$$\beta_j(g_{i*} + \delta g_i) = B_{ij}\delta g_i + \mathcal{O}(\delta g_i^2) \tag{4.10}$$

where $\delta g_i = g_i - g_i*$ is an infinitesimal transformation of g_i. The matrix B_{ij} is constant with eigenvectors σ_i and corresponding eigenvalues $\Delta_i - d$. Classically we would assume that $\Delta_i = d_i$ however the effect of integrating out the high energy modes in the quantum theory results in the dimension of the operators to change at the critical points. The quantity Δ_i is called the scaling dimension of the operator and the value $\gamma_i = \Delta_i - d_i$ is known as the anomalous dimension of the operator.
Due to the fact that σ_i is an eigenvector of B_{ij} this implies that

$$\frac{\partial \sigma_i}{\partial \log \Lambda} = (\Delta_i - d)\sigma_i + \mathcal{O}(\sigma^2) \tag{4.11}$$

which implies that the dependence of σ_i on the energy scale, in effect how it flows under the renormalization group is given by

$$\sigma_i(\Lambda) = \left(\frac{\Lambda}{\Lambda_0}\right)^{\Delta_i - d} \sigma_i(\Lambda_0) \tag{4.12}$$

in the region defined by perturbing the theory around the critical point (which defines a basis of attraction). This gives three possibilities for operators. Firstly we consider operators with $\Delta_i > d$. Accordingly the value of the associated coupling constant decreases as the energy scale is lowered, implying that deep into the IR these operators do not play any role. These operators are called irrelevant. Secondly we consider operators with $\Delta_i < d$. These operators coupling constants increase as the energy scale decrease. These operators are known as relevant. Finally, operators with $\Delta_i = d$ are called marginal operators. Near critical points quantum corrections can result in a scale dependence in these operators resulting in them either becoming marginally relevant or marginally irrelevant.

5 Quantization of Electrodynamics

We begin the quantization of the theory of electrodynamics by first examining the path integral of a photon in (2+1)-dimensions

$$Z_0(J) = \int \mathcal{D}A e^{iS} \tag{5.1}$$

where S is the action given by

$$S = \int d^d x - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu \tag{5.2}$$

Such that the time coordinate is given by x^0 and that the Einstein summation convention is in effect. To simplify this expression we first take the Fourier transform of the fields, using that

$$\phi(k) = \int \mathrm{d}^3 x e^{-ikx} \phi(x), \qquad \phi(x) = \int \frac{\mathrm{d}^3 x}{(2\pi)^3} e^{ikx} \phi(k) \tag{5.3}$$

Using equation (2.4) the Fourier transform of the action is found to be

$$\begin{aligned} S = \frac{1}{2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} (&-A_\mu(k)(k^2 g^{\mu\nu} - k^\mu k^\nu) A_\nu(-k) \\ &+ J^\mu(k) A_\mu(-k) + J^\mu(-k) A_\mu(k)) \end{aligned} \tag{5.4}$$

where $g_{\mu\nu}$ is the space-time metric. To further simplify this expression we examine the matrix $P^{\mu\nu}(k)$ given by the relation

$$k^2 P^{\mu\nu}(k) = k^2 g^{\mu\nu} - k^\mu k^\nu \tag{5.5}$$

We may see the P matrix satisfies $P^{\mu\nu}(k) P^\rho_\nu(k) = P^{\mu\rho}$ implying that it is a projection matrix, hence its eigenvalues may only take the values of 1 and 0. Noting further that $P^{\mu\nu}(k) k_\nu = 0$ and that $g_{\mu\nu} P^{\mu\nu} = 2$ we see that one of the eigenvalues is equal to 0 and the other two equal to 1. In equation (5.4) we may split the field $A(k)$ into its components with reference to a linearly independent basis specified by three vectors, one of which may be given explicitly by k_μ. In

this basis the term quadratic in $A_\mu(k)$ in the action (5.4) will be independent of the component in the direction of k_μ because of the fact that $P^{\mu\nu}k_\nu = 0$. Similarly the terms that are linear in A will not contribute either due to the fact that $\partial^\mu J_\mu(x) = 0$, which implies that $k^\mu J_\mu(k) = 0$. Hence the k_μ component does not contribute to observables. When computing the path integral (5.1) it is now irrelevant to complete the calculation over the k_μ components, hence we redefine the path integral to only integrate over the components spanned by two other basis vectors. These components now all satisfy $k^\mu A_\mu(k) = 0$, which when Fourier transformed into the position space is equivalent to the imposition of the Lorentz gauge $\partial^\mu A_\mu(x) = 0$. As shown in section 2 the imposition of a gauge is necessary to ensure the correct number of physical degrees of freedom in the action, and as such this redefinition is legitimate.
Due to the fact that $P^{\mu\nu}(k)$ is a projection matrix and that $P^{\mu\nu}(k)k_\nu = 0$ it will project vectors into a subspace that is orthogonal to the vector k_μ. While the fact that $P^{\mu\nu}(k)$ has a 0 eigenvalue would make it impossible to invert, the redefinition of the path integral ensures that in the subspace that we are working $P^{\mu\nu}(k)$ is equivalent to the identity matrix. As such the inverse of $k^2 P^{\mu\nu}(k)$ is given by $\frac{1}{k^2}P^{\mu\nu}(k)$. To account for vacuum boundary condition in which k^2 may take a 0 value we use the same technique as specified in the previous section and replace k^2 with $k^2 + i\epsilon$. Now we make the following field redefinition

$$B^\mu(k) = A^\mu(k) - (k^2 P^{\mu\nu})^{-1} J_\nu(k) = A^\mu(k) - \frac{P^{\mu\nu} J_\nu}{k^2 + i\epsilon} \tag{5.6}$$

This results in equation (5.4) taking the form

$$S = \frac{1}{2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} J_\mu(k) \frac{P^{\mu\nu}}{k^2 + i\epsilon} J_\nu(-k) + B^\mu \text{ terms} \tag{5.7}$$

Furthermore as A^μ has only been shifted by a constant the measure $\mathcal{D}B$ is equivalent to $\mathcal{D}A$. When computing the path integral (5.1) the integral over B^μ is simply given by $Z_0(0) = \langle 0|0\rangle_{J=0} = 1$. Hence the path integral is given by

$$\begin{aligned} Z_0(J) = \exp &\left[\frac{i}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^3} J_\mu(k) \frac{P^{\mu\nu}}{k^2 + i\epsilon} J_\nu(-k) \right] \\ &= \exp \left[\frac{i}{2} \int \mathrm{d}^\mathrm{d} x \mathrm{d}^\mathrm{d} y J_\mu(x) \Delta^{\mu\nu}(x-y) J_\nu(x) \right] \end{aligned} \tag{5.8}$$

in which

$$\Delta^{\mu\nu}(x-y) = \int \frac{\mathrm{d}^d k}{(2\pi)^3} e^{ik(x-y)} \frac{P^{\mu\nu}}{k^2 + i\epsilon} \tag{5.9}$$

and is called the photon propagator for the Lorentz gauge. We may also compute a simple calculation to show that the propagator is equivalent to $\langle 0| \mathrm{T} A^\mu(x) A^\nu(y) |0\rangle = \frac{1}{i}\Delta^{\mu\nu}(x-y)$, in effect the probability that a field in state $A^\nu(y)$ will become a state $A^\mu(x)$.

We may subsequently examine the theory of 1/2 spin charged Dirac fields (eg. positrons and electrons) and their interactions with photons. The general lagrangian for Dirac particles takes the form of

$$\mathcal{L}_D = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \tag{5.10}$$

in which γ^μ are $d \times d$ dimensional matrices that satisfy the d-dimensional Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I$, such that η is the Minkovski metric and I is the identity matrix. Similarly to the photon case the partition function of the dirac field takes the form

$$Z_{0,D}(\overline{\eta}, \eta) = \exp\left[i \int \mathrm{d}^d x \mathrm{d}^d y \overline{\eta}(x) S(x-y) \eta(y)\right] \tag{5.11}$$

Where η and $\overline{\eta}$ are two source terms for particles and antiparticles respectively, and S the propogator for dirac particles given by

$$S(x-y) = \int \frac{\mathrm{d}^d p}{(2\pi)^3} \frac{(-\not{p} + m)}{p^2 + m^2 - i\epsilon} e^{ip(x-y)} \tag{5.12}$$

We now attempt to quantize the theory where the electromagnetic field interacts with the fermion field. We assume that the source terms J^μ for the photon theory is proportional to the conserved Noether current resulting from the $U(1)$ symmetry of the Dirac field such that

$$J^\mu = e\bar{\Psi}\gamma^\mu \Psi \tag{5.13}$$

where e is a coupling constant that is assumed to take the value of the elementary charge. However in the construction of the total lagrangian we assume that the creation and destruction operators for electromagnetic and Dirac fields remain the same when the fields are interacting. The LSZ theorem [15] implies that this assumption is valid provided that the total lagranian is renormalized such that it takes the form

$$\mathcal{L} = -\frac{1}{4} Z_3 F^{\mu\nu} F_{\mu\nu} + iZ_2 \overline{\Psi} \not{\partial} \Psi - m Z_m \overline{\Psi}\Psi + Z_1 e \overline{\Psi} \not{A} \Psi \tag{5.14}$$

with the factors Z_1, Z_2, Z_3, Z_m chosen such that the the following conditions are satisfied

$$\langle 0| A^i(x) |0\rangle = 0, \qquad \langle k, \lambda| A^i(x) |0\rangle = \epsilon^i_\lambda(\mathbf{k}) e^{ikx} \tag{5.15}$$

Where $|k, \lambda\rangle$ represents the state of a single photon of momentum k and helicity λ, and where $\epsilon^i_\lambda(\mathbf{k})$ is the ith polarization vector as a function of momentum k. The states are normalized by:

$$\langle k_1, \lambda_1 | k_2, \lambda_2\rangle = (2\pi)^2 2k^0 \delta^2(\mathbf{k}_2 - \mathbf{k}_1)\delta_{\lambda_1\lambda_2}$$

However as may be shown all factors of Z_i take the form $Z_i = 1 + \mathcal{O}(e^2)$ [15]. Due to the fact that tree level processes, or processes that do not contain loops in their representative Feynman diagrams, do not have factors of e^2 in their partition function. As such when examining purely tree level processes we may treat the factors of Z_i as the identity.
We treat the interaction term $\mathcal{L}_1 = e\overline{\Psi}\not{A}\Psi$ as perturbation of the lagrangian in which no interaction takes place. Then using equation (3.13) we find that the partition function takes the form

$$\begin{aligned} Z(\overline{\eta}, \eta, J) &\propto \exp\left[i\int \mathrm{d}^3x\mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}(x)}, i\frac{\delta}{\delta\eta}, \frac{1}{i}\frac{\delta}{\delta J}\right)\right] \times Z_0(\overline{\eta}, \eta, J) \\ &\propto \exp\left[ie\int \mathrm{d}^3x\left(\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right)\left(i\frac{\delta}{\delta\eta_\alpha(x)}\right)(\gamma^\mu)_{\alpha\beta}\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\beta(x)}\right)\right] \times Z_0(\overline{\eta}, \eta, J) \end{aligned} \tag{5.16}$$

The proportionality sign is necessary due to the fact that normalization is not guarenteed by the partition function and must instead be set manunally by the requirement that $Z(0,0,0) = 1$. In addition $Z_0(\overline{\eta}, \eta, J)$ is the partition function for a non interacing dirac field and A field given by

$$\begin{aligned} Z_0(\overline{\eta}, \eta, J) = \exp&\left[i\int \mathrm{d}^3x\mathrm{d}^3y\overline{\eta}(x)S(x-y)\eta(y)\right] \\ &\times \exp\left[\frac{i}{2}\int \mathrm{d}^3x\mathrm{d}^3yJ_\mu(x)\Delta^{\mu\nu}(x-y)J_\nu(x)\right] \end{aligned} \tag{5.17}$$

Through a series of clever calculations it may be shown that (5.16) is equivalent to

$$Z(\overline{\eta}, \eta, J) = \exp[iW(\overline{\eta}, \eta, J)] \tag{5.18}$$

Were $iW(\overline{\eta}, \eta, J)$ is the sum of all connected Feynman diagrams with photon, positron, and electron sources. As we have assumed that $\mathcal{L}_1$ functions as a perturbation it implies that e is generically small. Equation (5.18) that we may expand the partition function to greater degrees of accuracy by calculating more complex diagrams. The $\mathcal{O}(1)$ order corresponds to all tree level diagrams, diagrams containing no loops. For increased accuracy we may calculate the $\mathcal{O}(e^2)$ order, which corresponds to all diagrams that, in accordance to the Feynman rules, have one loop. We may continue this process to the desired degree of accuracy.

6 Calculation of the β-function

6.1 1-loop Corrections to spinor QED3

The β-function is a function that describes how a coupling constant, in the present case e, varies with a changing energy scale. At the tree level of quantization that we have calculated previously the coupling constant remains constant in contrast to the physically observed data. As such, to begin to approximate the beta function with $\mathcal{O}(e^2)$ precision we renormalize the electrodynamic Lagrangian with respect to the LSZ theorem, and calculate the renormalization terms Z_i to the $\mathcal{O}(e^2)$ order.

To complete this calculation we will use the method of ϵ-expansion, a technique derived from statistical mechanics. The epsilon expansion assumes that the dimension of the system is given by $d = 4 - \epsilon$ in which $\epsilon << 1$ [2], and then assumes that the properties of the system continue to hold as ϵ is increased to larger values such as 1 to examine systems of dimension $d = 3$.

The general electrodynamic lagrangian that we will use for this renormalization is given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi + \mathcal{L}_1 \tag{6.1}$$

In which $\mathcal{L}_1$ is the interaction term given by:

$$\mathcal{L}_1 = Z_1 e\overline{\Psi}A\!\!\!/\Psi + \mathcal{L}_{ct} \tag{6.2}$$

and $\mathcal{L}_{ct}$ consists of the counter terms,

$$\mathcal{L}_{ct} = -\frac{1}{4}(Z_3 - 1)F^{\mu\nu}F_{\mu\nu} + i(Z_2 - 1)\overline{\Psi}\partial\!\!\!/\Psi - (Z_m - 1)m\overline{\Psi}\Psi \tag{6.3}$$

We write the lagrangian in this manner such that the $\mathcal{L}_1$ term, consisting of the interacting term in addition to the counterterms may be treated as a perturbation of the bare lagrangian (6.1) such that equation (5.16) holds with the new value of $\mathcal{L}_1$.

In order to evaluate the values of the renormalization factors Z_i we will calculate 1-loop corrections to propagators and vertex terms that have definite conditions placed upon them by the renormalization group. We begin by examining the photon propagator. We may show that the following relation holds:

$$\frac{1}{i}\mathbf{\Delta}(x_1 - x_2) = \langle 0|\,\mathrm{T}\ A^{\mu}(x_1)A^{\nu}(x_2)\,|0\rangle = \frac{1}{i}\frac{\delta}{\delta J^{\mu}(x_1)}\frac{1}{i}\frac{\delta}{\delta J^{\mu}(x_2)}iW(J)\bigg|_{J=0} \tag{6.4}$$

As previously discussed $iW(J)$ is the sum of all connected feynman diagrams. The effect of the derivatives is to remove sources from the diagram and to label the propagator corresponding to the removed source as an endpoint x_i. To the $\mathcal{O}(e^2)$ order the diagrams that this condition corresponds to are given by figure 6.1. By examining these figures with respect to the feynman rules, and working in the fourier transformed momentum space we see that the exact propagator takes the form:

$$\frac{1}{i}\mathbf{\Delta}_{\mu\nu}(k) = \frac{1}{i}\Delta_{\mu\nu}(k) + \frac{1}{i}\Delta_{\mu\rho}(k)i\tilde{\Pi}^{\rho\sigma}(k)\frac{1}{i}\Delta_{\sigma\nu}(k) + \mathcal{O}(e^4) \tag{6.5}$$

in which $\Delta_{\mu\nu}(k^2)$ is the Fourier transform of the propagator calculated in the previous section; equation (5.9), and $\tilde{\Pi}^{\mu\nu}(k^2)$ is the self energy of the loop. A more useful definition is to define $\Pi^{\mu\nu}$ as the sum of all one-particle irreducible (1PI) diagrams. These are diagrams that remain connected if any one internal line of the diagram is cut. This allows the exact photon propagator to be written as a geometric series

$$\frac{1}{i}\boldsymbol{\Delta}_{\mu\nu}(k) = \frac{1}{i}\Delta_{\mu\nu}(k) + \frac{1}{i}\Delta_{\mu\rho}(k)i\Pi^{\rho\sigma}(k)\frac{1}{i}\Delta_{\sigma\nu}(k) + \frac{1}{i}\Delta_{\mu\alpha}(k)i\Pi^{\alpha\beta}(k)\frac{1}{i}\Delta_{\beta\gamma}(k)i\Pi^{\gamma\sigma}(k)\frac{1}{i}\Delta_{\sigma\nu}(k) + \dots \quad (6.6)$$

Due to the Ward identity for the electromagnetic current it may be shown that $k_\mu\Pi^{\mu\nu}(k) = k_\nu P^{\mu\nu}(k) = 0$ implying that we may write $\Pi^{\mu\nu}(k)$ as

$$\Pi^{\mu\nu} = \Pi(k^2)(k^2 g^{\mu\nu} - k^\mu k^\nu) = k^2\Pi(k^2)P^{\mu\nu}(k) \quad (6.7)$$

Where $P^{\mu\nu}$ is the projection matrix discussed in the previous section. Using this redefinition of $\Pi^{\mu\nu}(k)$ allows us to solve for the sum of the geometric series given by

$$\boldsymbol{\Delta}_{\mu\nu}(k) = \frac{P^{\mu\nu}}{k^2(1-\Pi(k^2)) - i\epsilon} \quad (6.8)$$

This equation has a pole at $k^2 = 0$ with a corresponding residue of $\frac{P^{\mu\nu}}{1-\Pi(0)}$. As we will complete this calculation using the on-shell renormalization scheme which requires that that the exact photon propagator must have a pole at $k^2 = 0$ with residue of $P^{\mu\nu}$, which fixes the condition that

$$\Pi(0) = 0 \quad (6.9)$$

This condition may be used to determine the value of the constant Z_3.

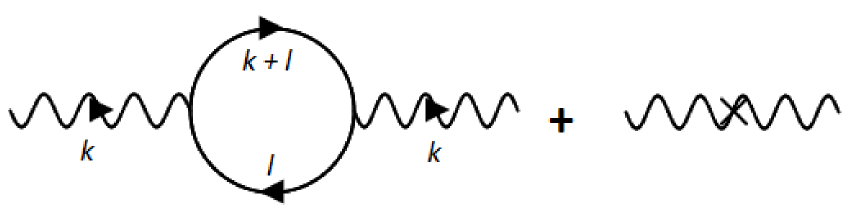

Figure 6.1: 1-loop 1PI contributions to the self energy of the photon propagator

Examining figure 6.1 we may note that $i\Pi^{\mu\nu}$ may be written as

$$i\Pi^{\mu\nu}(k) = -(iZ_1 e)^2(\frac{1}{i})^2 \int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\,[S(\not{l} + \not{k})\gamma^\mu S(\not{l})\gamma^\nu] - i(Z_3 - 1)(k^2 g^{\mu\nu} - k^\mu k^\nu) + \mathcal{O}(e^4) \quad (6.10)$$

As previously mentioned S is the fermion propagator. In addition the first term is negative due to the fact that the closed loop is fermionic. Noting that $Z_1 = 1 + \mathcal{O}(e^2)$ as previously discussed implies that we may equivalently set $Z_1 = 1$ and absorb the rest of Z_1 into the $\mathcal{O}(e^4)$ term.
The trace may equivalently written as:

$$\mathrm{Tr}\left[S(\not{l} + \not{k})\gamma^\mu S(\not{l})\gamma^\nu\right] = \int_0^1 \mathrm{d}x \frac{4N^{\mu\nu}}{(q^2 + D)^2} \tag{6.11}$$

in which $q = l + xk$ and $D = x(1-x)k^2 + m^2 - i\varepsilon$, and

$$N^{\mu\nu} = \mathrm{Tr}\left[(-\not{l} - \not{k} + m)\gamma^\mu(-\not{l} + m)\gamma^\nu\right] = 2x(1-x)(k^2 g^{\mu\nu} - k^\mu k^\nu) \tag{6.12}$$

Where the result of the second equals sign is derived in [15]. Using the relation that $\Pi^{\mu\nu}(k) = \Pi(k^2)(k^2 g^{\mu\nu} - k^\mu k^\nu)$ we may rewrite equation (6.10) as:

$$i\Pi(k^2) = -e^2\tilde{\mu}^\epsilon \int_0^1 \mathrm{d}x 8x(1-x) \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(q^2 + D)^2} - i(Z_3 - 1) + \mathcal{O}(e^4) \tag{6.13}$$

This integral diverges for dimensions $d \geq 4$, as such to obtain a solution it has been analytically continued into $d = 4 - \epsilon$ dimensions. Taking the limit of $\epsilon \to 0$ will result in the correct integral for the 4-dimensional case. In the present case we will examine the result of $\epsilon \to 1$ to examine the three dimensional result. In order to ensure that the coupling constant e will remain dimensionless it has been replaced by $e\tilde{\mu}^{\epsilon/2}$. Using the general relation that:

$$\int \frac{\mathrm{d}^d x}{(2\pi)^d} \frac{(x^2)^a}{(x^2 + D)^b} = \frac{i\Gamma(b - a - \frac{1}{2}d)\Gamma(a + \frac{1}{2}d)}{(4\pi)^{d/2}\Gamma(b)\Gamma(\frac{1}{2}d)} D^{-(b-a-\frac{1}{2}d)} \tag{6.14}$$

this integral may be evaluated over q as:

$$\tilde{\mu}^\epsilon \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(q^2 + D)^2} = i\frac{\Gamma(\frac{\epsilon}{2})}{16\pi^2} \left(\frac{4\pi\tilde{\mu}}{D}\right)^{\epsilon/2} \tag{6.15}$$

Operating under the assumption that ϵ is small we may make use of the following property of the Γ function:

$$\Gamma(-n + x) = \frac{(-1)^n}{n!}\left[\frac{1}{x} - \gamma + \sum_{k=1}^{n} k^{-1} + \mathcal{O}(x)\right] \tag{6.16}$$

In which n is a non-negative integer and x is a small value and γ is the Euler-Mascheroni constant which has the value $\gamma = 0.5772$. Further using the approximation that, provided that ϵ is small:

$$A^{\epsilon/2} = 1 + \frac{\epsilon}{2}\ln A + \mathcal{O}(\epsilon^2) \tag{6.17}$$

we may rewrite equation (6.15) as:

$$\frac{i}{16\pi^2}\left(\frac{2}{\epsilon}+\ln\!\left(\frac{4\pi\tilde{\mu}}{D}\right)-\gamma\right)=\frac{i}{8\pi^2}\left(\frac{1}{\epsilon}-\frac{1}{2}\ln\!\left(\mu^2/D\right)\right) \tag{6.18}$$

In which $\mu^2 = 4\pi e^{-\gamma}\tilde{\mu}^2$. In addition we have dropped any terms of $\mathcal{O}(\epsilon)$ and higher. As such we obtain that for equation (6.13)

$$\Pi(k^2) = -\frac{e^2}{\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\left(\frac{1}{\epsilon}-\frac{1}{2}\ln\!\left(\mu^2/D\right)\right)-(Z_3-1)+\mathcal{O}(e^4) \tag{6.19}$$

Using the on-shell renormalization scheme in which $\Pi(0)=0$ we obtain that:

$$Z_3 = 1-\frac{e^2}{6\pi^2}\left(\frac{1}{\epsilon}-\ln(m/\mu)\right)+\mathcal{O}(e^4) \tag{6.20}$$

In this expression the m in the logarithmic term is obtained by evaluating D at $k^2=0$.

Next we turn to the exact fermion propagator. Using the Lehmann-Källen form of the exact fermion propagator we may obtain the realtion

$$\mathbf{S}(\not{p})^{-1} = \not{p}+m-i\epsilon-\Sigma(\not{p}) \tag{6.21}$$

Similarly to the photon propagator $i\Sigma(\not{p})$ is the sum of all 1PI diagrams, consisting of two external fermion lines such that the external propagators have been removed. The use of the on-shell renormalization scheme in addition to the fact that the exact fermion propagator $\mathbf{S}(\not{p})$ has a pole with residue 1 at $\not{p}=-m$ implies the following two conditions

$$\Sigma(-m)=0 \qquad \Sigma'(-m)=0 \tag{6.22}$$

where the apostrophe implies the derivative of $\Sigma(\not{p})$ with respect to $\not{p}$. The two conditions allow us to further specify two renormalization constant Z_2 and Z_m.

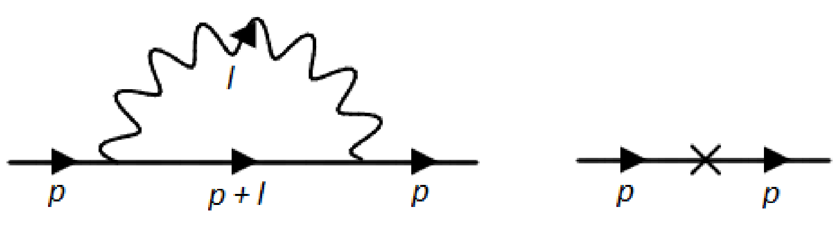

Figure 6.2: 1-loop 1PI contributions to the self energy of the fermion propagator

By examining figure 6.1, which gives the one loop feynman diagrams correcting the fermionic propagator we may obtain the fact that

$$i\Sigma(\not{p}) = (iZ_1 e)^2 \left(\frac{1}{i}\right)^2 \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \left[\gamma^\nu S(\not{p} + \not{l})\gamma^\mu\right] \tilde{\Delta}_{\mu\nu}(l) \\ - i(Z_2 - 1)\not{p} - i(Z_m - 1)m + \mathcal{O}(e^4) \quad (6.23)$$

Similarly to the case of $\Pi^{\mu\nu}(k)$ above we may similarly rewrite this equation to

$$i\Sigma(\not{p}) = e^2 \tilde{\mu}^\epsilon \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{N}{(q^2 + D)^2} - i(Z_2 - 1)\not{p} - i(Z_m - 1)m + \mathcal{O}(e^4) \quad (6.24)$$

in which $q = l + xp$, $D = x(1-x)p^2 + xm^2 + (1-x)m_\gamma^2$. Here m_γ is the mass of the photon, which for the sake of the calculation must be assumed to be positive. Once the integral has been completed $m_\gamma \to 0$ to obtain the correct relation. Finally $N = -(d-2)[\not{q} - (1-x)\not{p}] - dm$. The term linear in N that is linear in q will integrate to 0 [15]. As such we get

$$\Sigma(\not{p}) = -\frac{e^2}{8\pi^2} \int_0^1 \mathrm{d}x \ \left[(2-\epsilon)(1-x)\not{p} + (4-\epsilon)m\right] \left[\frac{1}{\epsilon} - \frac{1}{2}\ln\big(D/\mu^2\big)\right] \\ - (Z_2 - 1)\not{p} - (Z_m - 1)m + \mathcal{O}(e^4) \quad (6.25)$$

This on-shell normalization scheme requires that this expression for $\Sigma(\not{p})$ be finite. As ϵ is taken to be arbitrary to ensure this finitude it requires that the constants Z_2 and Z_m must take the form

$$Z_2 = 1 - \frac{e^2}{8\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \mathcal{O}(e^4) \quad (6.26)$$

and

$$Z_m = 1 - \frac{e^2}{2\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \mathcal{O}(e^4) \quad (6.27)$$

The exact value of these finite terms is unimportant as they do not contribute the to calculation of the β-function. These terms are specified by the conditions of equation (6.22).

Finally to determine the final constant Z_1 we must examine the vertex terms in electrodynamic Feynman diagrams. A vertex in this context is defined by an intersection of three lines necessarily consisting of one incoming and one outgoing fermion with momenta p and p' respectively and one photon with momentum k whose momentum is specified by the conservation of momentum to be equivalent to $k = p' - p$. As in the case of the photon or fermionic propagator we define $iV^\mu(p, p')$ to be the sum of all 1PI diagrams containing a vertex as previously specified. The diagram for the 1-loop correction may be seen in figure

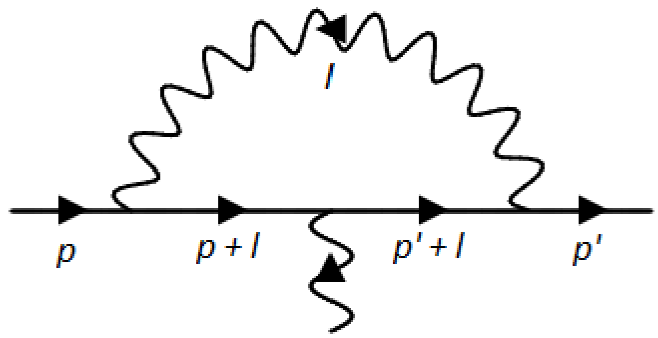

Figure 6.3: 1-loop 1PI contributions to the vertex term

6.1, resulting in a equation

$$iV^{\mu}(p',p) = iZ_1 e\gamma^{\mu} + (ie^3)\left(\frac{1}{i}\right)^3 \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \left[\gamma^{\rho} S(\not{p}' + \not{l})\gamma^{\mu} S(\not{p} + \not{l})\gamma^{\nu}\right] \tilde{\Delta}_{\nu\rho}(l) \tag{6.28}$$

Noting that the first term of this equation is the tree level vertex, while the second term is explicitly given by the diagram in figure 6.1. Rewriting as before we obtain the following equation:

$$iV^{\mu}(p',p) = iZ_1 e\gamma^{\mu} + e^3 \int \mathrm{d}F_3 \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{N^{\mu}}{(q^2+D)^3} \tag{6.29}$$

in which

$$\int \mathrm{d}F_3 = 2\int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \delta(x_1+x_2+x_3-1) \tag{6.30}$$

and $q = l + x_1 p + x_2 p'$,

$$D = x_1(1-x_1)p^2 + x_2(1-x_2)p'^2 - 2x_1x_2 p\cdot p' + (x_1+x_2)m^2 + x_3 m_{\gamma}^2 \tag{6.31}$$

$$N^{\mu} = \frac{(d-2)^2}{d} q^2 \gamma^{\mu} + \tilde{N}^{\mu} \tag{6.32}$$

$$\tilde{N}^{\mu} = \gamma_{\nu}[x_1\not{p} - (1-x_2)\not{p}' + m]\gamma^{\mu}[-(1-x_1)\not{p} + x_2\not{p}' + m]\gamma^{\nu} \tag{6.33}$$

As such we may rewrite the integral as:

$$\frac{i}{e}V^{\mu}(p',p) = iZ_1\gamma^{\mu} + e^2 \int \mathrm{d}F_3 \frac{(d-2)^2}{d}\gamma^{\mu} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q^2}{(q^2+D)^3} + \tilde{N}^{\mu} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(q^2+D)^3} \tag{6.34}$$

We note that

$$\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q^2}{(q^2+D)^3} = i\frac{\Gamma(\epsilon/2)\Gamma(3-\frac{\epsilon}{2})}{2(4\pi)^{2-\epsilon/2}\Gamma(2-\frac{\epsilon}{2})} D^{-\epsilon/2} = \frac{i(2-\epsilon/2)\Gamma(\epsilon/2)}{32\pi^2}\left(\frac{4\pi}{D}\right)^{\epsilon/2} \tag{6.35}$$

where in the second equality we have used the relation $\Gamma(z) = (z-1)\Gamma(z-1)$. Secondly

$$\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(q^2+D)^3} = \frac{i\Gamma(1-\epsilon/2)}{32\pi^2}\left(\frac{4\pi}{D}\right)^{\epsilon/2} \tag{6.36}$$

Using the same manipulations as in the previous sections these relations may be simplified to an expression of $V^\mu(p,p')$

$$\frac{1}{e}V^\mu(p,p') = Z_1\gamma^\mu + \frac{e^2}{8\pi^2}\left[\left(\frac{1}{\epsilon} - 1 - \frac{1}{2}\int \mathrm{d}F_3\ \ln\big(D/\mu^2\big)\right)\gamma^\mu + \frac{1}{4}\int \mathrm{d}F_3\ \frac{\tilde{N}^\mu}{D}\right] + \mathcal{O}(e^4) \tag{6.37}$$

Once again we require that this vertex term is finite for all values of ϵ. This requires that the constant Z_1 must take the form

$$Z_1 = 1 - \frac{e^2}{8\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \mathcal{O}(e^4) \tag{6.38}$$

6.2 The β-function

Now that we have calculated the required renormalization constants we may determine the relationship between the original or "bare" coupling constant e_0 and the renormalized constant e. The coupling constant defines the amount of interaction between the electromagnetic field and the Dirac field through the term $Z_1 e\overline{\Psi}\not{A}\Psi$, such that $e = 0$ would correspond to no interaction. As such the relation between the bare coupling constant and the renormalized constant is given by

$$e_0 = Z_3^{-1/2} Z_2^{-1} Z_1 \tilde{\mu}^{\epsilon/2} e \tag{6.39}$$

in which the Z_3 corresponds to the renormalization of the A field, the Z_2 corresponds to the renormalization of the Ψ fields and the Z_1 corresponds to the renormalization of the interaction term. We may examine the Z_i terms using the minimal subtraction renormalization scheme in which the finite terms have been removed such that

$$Z_1 = 1 - \frac{e^2}{8\pi^2}\frac{1}{\epsilon} + \mathcal{O}(e^4) \tag{6.40}$$

$$Z_2 = 1 - \frac{e^2}{8\pi^2}\frac{1}{\epsilon} + \mathcal{O}(e^4) \tag{6.41}$$

$$Z_3 = 1 - \frac{e^2}{6\pi^2}\frac{1}{\epsilon} + \mathcal{O}(e^4) \tag{6.42}$$

We may take the logarithms of both sides of the equation (6.39)

$$\ln e_0 = \ln Z_3^{-1/2} Z_2^{-1} Z_1 + \ln e + \frac{\epsilon}{2}\ln\tilde{\mu} \tag{6.43}$$

Noting that the bare coupling constant must be independent of the choice of scale $\tilde{\mu}$ we now take the derivative with respect to $\ln\tilde{\mu}$ to obtain

$$0 = \frac{\mathrm{d}}{\mathrm{d}\ln\tilde{\mu}}\ln e_0 = \frac{\partial \ln Z_3^{-1/2} Z_2^{-1} Z_1}{\partial e}\frac{\mathrm{d}e}{\mathrm{d}\ln\tilde{\mu}} + \frac{1}{e}\frac{\mathrm{d}e}{\mathrm{d}\ln\tilde{\mu}} + \frac{\epsilon}{2} \tag{6.44}$$

Because of the fact that Z_1 and Z_2 have the same value up to the $\mathcal{O}(e^2; \frac{1}{\epsilon})$ order and as such cancel. Furthermore, the Ward identity of QED implies that $Z_1 = Z_2$ for all order. We may rewrite the equation and reorganize it to obtain

$$0 = \left(2 - e\frac{\partial \ln Z_3}{\partial e} + \mathcal{O}(\frac{1}{\epsilon^2})\right)\frac{\mathrm{d}e}{\mathrm{d}\ln\tilde{u}} + e\epsilon \tag{6.45}$$

now we may note that to the current order

$$\frac{\partial \ln Z_3}{\partial e} = -\frac{\frac{e}{3\pi^2\epsilon}}{1 - \frac{e^2}{6\pi^2\epsilon}} \tag{6.46}$$

as such we obtain the relation

$$0 = \left(2 + \frac{\frac{e^2}{3\pi^2\epsilon}}{1 - \frac{e^2}{6\pi^2\epsilon}} + \mathcal{O}(\frac{1}{\epsilon^2})\right)\frac{\mathrm{d}e}{\mathrm{d}\ln\tilde{u}} + e\epsilon \tag{6.47}$$

We may make the following redefinition in accordance with the standard method of calculating the β-function as given by [15].

$$\frac{\mathrm{d}e}{\mathrm{d}\ln\tilde{\mu}} = -e\epsilon + \beta(e) \tag{6.48}$$

This allows us to simplify the equation to obtain

$$\beta(e)\left(2 + \frac{\frac{e^2}{3\pi^2\epsilon}}{1 - \frac{e^2}{6\pi^2\epsilon}}\right) = \frac{-e\epsilon + \frac{e^3}{6\pi^2}}{1 - \frac{e^2}{6\pi^2\epsilon}} \tag{6.49}$$

This results in the β-function being defined as

$$\beta(e) = -\frac{1}{2}\epsilon e + \frac{e^3}{12\pi^2} + \mathcal{O}(e^5) \tag{6.50}$$

Finally we may take the limit of $\epsilon \to 1$ to obtain that for QED$_3$ the β-function is given by

$$\beta(e) = -\frac{1}{2}e + \frac{e^3}{12\pi^2} + \mathcal{O}(e^5) \tag{6.51}$$

This β-function clearly has a zero at $e_* = \sqrt{6\pi^2}$ implying the existence of a Wilson-fisher fixed point. The β-functions for QED_3 and QED_4 are graphed in figures 6.4 and 6.5 respectively, in which the β-function for QED_4 is competed from equation (6.49) and taking $\epsilon \to 0$. The graph of the QED_4 β-function show that a IR fixed point exists at $e = 0$, implying that the theory becomes free, or non-interacting, in the limit of the IR. In the UV domain the β-function obtains a Landau pole, the point at which the coupling constant becomes infinite and the perturbation theory breaks down. The QED_3 β-function in contrast has an IR fixed point at Wilson-Fisher fixed point previously calculated. As the coupling constant obtains a positive value in the limit of the IR the theory continues to interact even at low energy scales. In addition the β-function has another zero obtained if the coupling constant becomes 0. As the β-function obtain a zero only in the limit of the IR or the UV, this fixed point must correspond to a UV fixed point. This implies that the theory is UV free.

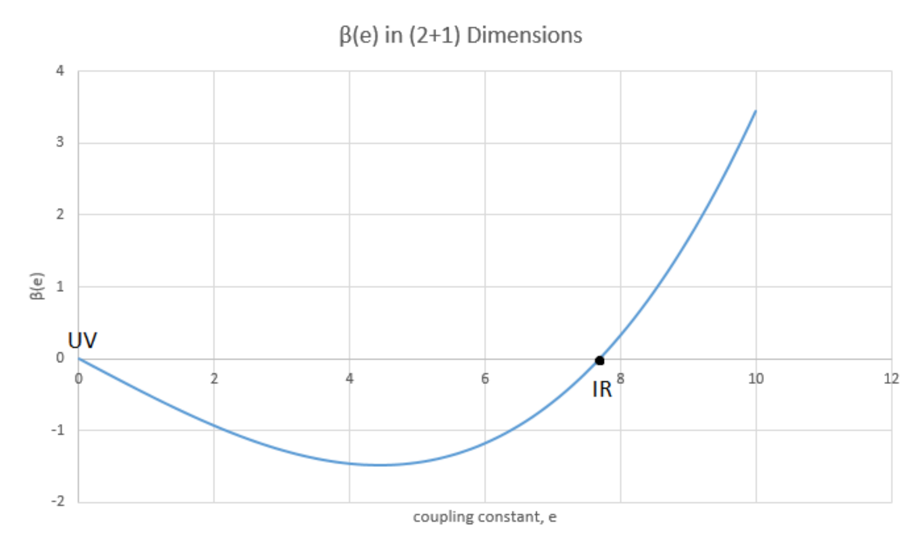

Figure 6.4: β-function for QED_3

The ϵ-expansion used in this paper may be used to further elucidate the properties of three dimensional QED and its corresponding β-function. Di Pietro et al. (2015) [3] noted that for an abstract system in $d = 4$ the correlation function for an operator $\mathcal{O}$ may be expanded in powers of of a coupling constant g that is marginal given by

$$\langle 0| \mathcal{O}(p)\mathcal{O}(-p) |0\rangle = p^{2\Delta-4} \sum_{0\leq m\leq n, n=0}^{\infty} c_{nm} g^n \left(\log \frac{\Lambda^2}{p^2}\right)^m \tag{6.52}$$

Such that Δ is the scaling dimension of the operator under the condition that

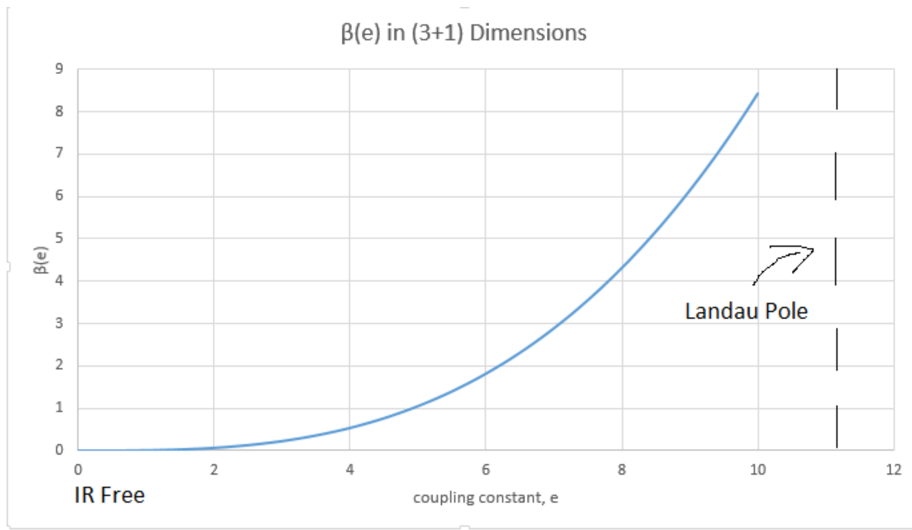

Figure 6.5: β-function for QED_4

$d = 4$ and the coupling constant is equal to 0. In a scale invariant theory (see [1]) under a dilation, or a change in scale, $x \to \lambda x$ each operator picks up a prefactor of $\lambda^{-\Delta}$ such that Δ is the scaling dimension of the operator. Λ is the ultraviolet (UV), or high energy, cutoff; the energy at which the perturbation theory begins to break down. To ensure that the theory is independent of the ultraviolet cutoff the operator may be renormalized to $Z\mathcal{O}$ allowing the renormalization Z and the coupling constant g to evolve as a function of the energy scale such that the Callan-Symanzik equation is satisfiel

$$\left(\frac{\partial}{\partial \log \Lambda} + \beta(g)\frac{\partial}{\partial g} + 2\gamma(g)\right) \langle 0| \mathcal{O}(p)\mathcal{O}(-p) |0\rangle = 0 \tag{6.53}$$

In which $\beta(g) = \frac{\mathrm{d}g}{\mathrm{d}\log\Lambda}$ is the β-function and $\gamma(g) = \frac{\mathrm{d}\log Z}{\mathrm{d}\log\Lambda}$ is called the anomalous dimension. By expanding the beta function and anomalous dimension to the leading order of g

$$\beta(g) = \beta_1 g^3 + \mathcal{O}(g^5), \qquad \gamma(g) = \gamma_1 g^2 + \mathcal{O}(g^4) \tag{6.54}$$

Di Pietro et al. were able to show that for a dimension of $d = 4 - 2\epsilon$ the operator expansion takes the form

$$\langle 0| \mathcal{O}(p)\mathcal{O}(-p) |0\rangle \simeq p^{2\Delta - d}\left(1 + \frac{\beta_1}{c\epsilon}\frac{g}{p^{c\epsilon}}\right)^{-\frac{2\gamma_1}{\beta_1}} \underset{p\to 0}{\approx} p^{2\Delta-d}p^{2\gamma_1\frac{c\epsilon}{\beta_1}} \tag{6.55}$$

In which c is a positive constant given by the positive mass dimension $c\epsilon$ that is obtained by the coupling constant g in the new dimension $d = 4 - 2\epsilon$. As the coupling constant is marginal in 4 dimensions, implying that a dimensional analysis of g is independent of mass, in other dimensions is will obtain a mass dimension. This result of equation (6.55) implies that in the infrared (IR), or low energy, limit (equivalently given by $p \to 0$) the scaling law for the operators changes to become

$$\Delta_{IR} = \Delta + \gamma_1 \frac{c\epsilon}{\beta_1} \tag{6.56}$$

And also that the scaling to this IR scale only occurs when

$$p \ll \left(\frac{\beta_1}{c\epsilon}\right)^{\frac{1}{c\epsilon}} g^{\frac{1}{c\epsilon}} \tag{6.57}$$

The presence of the large factor $(\beta_1/c\epsilon)^{\frac{1}{c\epsilon}}$ implies that the IR fixed point is parametrically close to the UV fixed point. Di Pietro et al. show that the IR fixed point corresponds to one of the zeros of the β-function of $\hat{g} = g\Lambda^{-c\epsilon}$

$$\beta(\hat{g}) = \frac{\mathrm{d}\hat{g}}{\mathrm{d}\log\Lambda} = -c\epsilon\hat{g} + \beta_1\hat{g}^3 + \mathcal{O}(\hat{g}^5) \tag{6.58}$$

$\hat{g}$ was chosen because, unlike g, it is dimensionless.

The lagrangian for 4-dimensional QED chosen by Di Pietro et al. is given by

$$\mathcal{L} = -\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + i\sum_{a=1}^{N_f}\overline{\Psi}_a\gamma^\mu D_\mu\Psi^a \tag{6.59}$$

In which $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative. This lagrangian shows the interaction between the electromagnetic field and N_f flavors of massless fermions. It is an extention of the β-function previously calculated which was coupled to a single Dirac field. This lagrangian in contrast does not have mass term as the fermions are assumed to be massless. This was chosen because the lagrangian exhibits a larger symmetry group without the presence of the mass term. The mass term may be consider as a coupling constant for the bilinear operator $\overline{\Psi}\Psi$ and its renormalization is independent of the re renormalization of e. As such the removal of this term does not substantially change the β-function corresponding to e which, for this extended Lagrangian, in $d = 4 - 2\epsilon$ dimensions was found to be

$$\beta(\hat{e}) = -\epsilon\hat{e} + \frac{N_f}{12\pi^2}\hat{e}^3 + \mathcal{O}(\hat{e}^5) \tag{6.60}$$

such that once again $\hat{e} = e\Lambda^{-\epsilon}$ is chosen to be dimensionless. As such for the present case of (2+1)-dimensions we may choose $\epsilon = \frac{1}{2}$ and obtain the β-function to be

$$\beta(\hat{e}) = -\frac{1}{2}\hat{e} + \frac{N_f}{12\pi^2}\hat{e}^3 + \mathcal{O}(\hat{e}^5) \tag{6.61}$$

Resulting in the existence of a Wilson-Fisher fixed point at $\hat{e}_* = \frac{6\pi^2}{N_f}$.

Similarly one may compute the β-function using the $\overline{MS}$ renormalization scheme, in constrast to the on-shell scheme used in this paper. This scheme requires the values of the counterterms to be chosen such that the purely divergent part of the loop integrals are removed, for example be subtracting factors of $\frac{1}{\epsilon}$ such that in the limit of $\epsilon \to 0$ no factors in the integral go to infinity. The $\overline{MS}$ β-function was has been determine up to the 4-loop accuracy in $d = 4 - 2\epsilon$ dimensions [4, 16]

$$\beta_{\overline{MS}}(\hat{e}) = -2\epsilon\hat{e} + \frac{2N_f}{3}\hat{e}^2 + \frac{N_f}{3}\hat{e}^3 - \frac{N_f(22N_f+9)}{144}\hat{e}^4 \\ - \frac{N_f}{64}\left[\frac{616N_f^2}{243} + \left(\frac{416\zeta(3)}{9} - \frac{380}{27}\right)N_f + 23\right]\hat{e}^5 + \mathcal{O}(\hat{e}^6) \quad (6.62)$$

In which $\hat{e}$ is once again dimensionless and the $\beta_{\overline{MS}}$-function for the present 3-dimensional case may be found by replacing the ϵ with $\frac{1}{2}$. Once again implying the existence of a Wilson-Fisher fixed point at

$$\hat{e}_* = \frac{3\epsilon}{N_f} - \frac{27\epsilon^2}{4N_f^2} + \frac{99\epsilon^3}{16N_f^2} + \frac{77\epsilon^4}{16N_f^2} + \mathcal{O}\left(\epsilon^5; \frac{1}{N_f^3}\right) \quad (6.63)$$

Which at $\epsilon = \frac{1}{2}$ results in

$$\hat{e}_* = \frac{384N_f - 157}{256N_f^2} \quad (6.64)$$

7 Physical Relevance of the QED$_3$ β-function

As previously discussed in section 4 the presence of fixed point affects the flow of the renormalization group at different energy scales. More specifically as one approaches a fixed point the coupling constants of operators may increase or decrease resulting in certain operators becoming relevant or irrelevant. Primarily we will examine the relevance of certain operators near the Wilson-Fisher fixed point and the implications that this has for the theory of QED$_3$.

The lagrangian of QED$_3$ has $SU(2N_f)$ symmetry group. Requiring that the operators that we examine are invariant under this symmetry group and further requiring them to be of even parity results in two scalar quadrilinear operators given by [3]

$$\mathcal{O}_1 = (\sum_i \overline{\psi}_i\sigma^\mu\psi^i)^2, \qquad \mathcal{O}_2 = (\sum_i \overline{\psi}_i\psi^i)^2 \quad (7.1)$$

in which σ^μ are the Pauli matrices and ψ are given by the decomposition of the diract field according to

$$\Psi^j = \begin{pmatrix} \psi^j \\ i\sigma_2\psi^{j+N_f} \end{pmatrix}, \qquad j = 1 \ldots N_f \quad (7.2)$$

A basis may be constructed using these two operators as the basis vectors. Upon renormalization each of the operators is renormalized by a factor Z_i, based on these factors a anomalous dimension matrix may be computed according to the definition

$$\gamma_{ij} = Z_{ik}\Lambda\frac{\mathrm{d}}{\mathrm{d}\Lambda}(Z_{kj})^{-1} \tag{7.3}$$

When applied to the operators defined by equation (7.1) the result as calculated by [3]

$$\gamma_{\mathcal{O}}(\hat{e}) = \frac{\hat{e}^2}{16\pi^2}\begin{pmatrix} \frac{8}{3}(2N_f+1) & 12 \\ \frac{44}{3} & 0 \end{pmatrix} + \mathcal{O}(\hat{e}^4) \tag{7.4}$$

Whose eigenvalues may be found to be

$$\frac{\hat{e}^2}{12\pi^2}(2N_f + 1 \pm 2\sqrt{N_f^2 + N_f + 25}) \tag{7.5}$$

We may evaluate this at the value of the coupling constant at the Wilson-Fisher fixed point previously found to be $\frac{12\pi^2\epsilon}{N_f}$. This results in the fact that the operator corresponding to the negative eigenvalue only becomes relevant, that its anomalous dimension is negative, when the number of fermion flavours $N_f \leq \frac{9\epsilon}{2} + \mathcal{O}(\epsilon^2)$. In three dimensions, i.e. $\epsilon = \frac{1}{2}$, this implies that the operator becomes relevant if $N_f = 1, 2$. This implies that if the operator becomes relevant it may create a new renormalization group flow to a new IR phase.
A potential result of this new IR phase is the potential for the operator

$$\sum_{a=1}^{N_f}(\overline{\psi}_a\psi^a - \overline{\psi}_{a+N_f}\psi^{a+N_f}) \tag{7.6}$$

to obtain a vacuum expectation value. The operator (7.6) is also known as the number operator and counts the number of particles present in the system. As such the RG flow may result in spontaneous mass generation. This subsequently breaks the global $SU(2N_f)$ symmetry of the lagrangian replacing it with a $SU(N_f) \times SU(N_f) \times U(1)$ symmetry group. This process is known as chiral symmetry breaking and has been shown to only occur below a certain critical value of N_f labelled N_f^c [7]. Various estimates of the value of N_f^c have been estimated ranging from 2-10 [7–10].

The theory of QED_3 has also been used to describe the behaviour of some high T_c cuprate superconductors. Cuprates referring to materials containing anionic copper complexes. At the critical temperature T_c the superconductor undergoes a phase transition and stops superconducting. A paper by Franz et al. [11] showed that for low energy scales, near a temperature of 0, the theory that describes the behaviour of quasi particles inside of the pseudogap of the superconductor was found to be that of (2+1) dimensional QED. The previously discusses chiral symmetry breaking occurs within the cuprates. In these systems it is argued that the number of fermion flavors is given by $N_f = 2n_{\mathrm{CuO}}$, the number of CuO_2 layers that are present per unit cell. In the case that $N_f < N_f^c$ the interactions with a massless Berry gauge field result in the spontaneous creation of a gap for the ferimionic interactions provided that the temperature is $T = 0$ [11]. This is further discussed in other paper on the subject [12, 13].

8 Conclusion

In this paper the β-function for three dimensional quantum electrodynamics (QED_3) was derived and its physical ramifications were discussed.
The classical theory of electrodynamics was first discussed to determine its dependence on the number of dimensions. Examining the relativistic formulation of the Maxwel equation in three and four dimensions resulted in a lagrangian to be quantized of the form

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J_\mu A^\mu \tag{8.1}$$

in which J_μ is a term for external sources. Subsequently using the operator based hamiltoniona for non-relativistic quantum mechanics the partition function, Z, was derived. The partition function gives the probability that the system starts and ends in the ground state as a path integral over all possible paths between the initial and final state weighted by the complex exponent of the action S

$$Z = \langle 0|0\rangle = \int \mathcal{D}\phi \; e^{iS} \tag{8.2}$$

The functionality of the partition function is that operators that would be applied to states become functions under the path integral. This allows for the computation of correlation functions, or other sets of operators, by computing a path integral.
Many of the theories that may be expressed in this path integral formalism are pertubative theories implying that partition function is dependent on the energy scale of the system. The coupling constants define the degree to which interactions between various fields occur and are specified in the lagrangian. They are defined such that a coupling constant of 0 implies no interaction, with greater constant implying greater interactions. By integrating out higher energy modes of partition function the coupling constants become dependent upon the energy scale. This allows for the definition of the β-function to measure the degree of this dependence

$$\beta(g) = \frac{\mathrm{d}g}{\mathrm{d}\log\Lambda} \tag{8.3}$$

such that g is the coupling constant and Λ is the energy scale. Much information concerning the energy scale dependence of the theory may be determined simply from the β-function. The zeros of the β-function correspond to fixed points of

the theory, points at which the theory is conformally invariant. Near these fixed points the beta function may be used to determine which operators become relevant and irrelevant in describing the physical world.
Using the path integral formalism described, QED or the theory of interaction between the electromagnetic field and the Dirac fermion field, was quantized. Be considering the source term J_μ to be proportional to the conserved Noether current corresponding to the $U(1)$ symmetery group of the Dirac field the total lagrangian was found to be

$$\mathcal{L} = -\frac{1}{4}Z_3 F^{\mu\nu}F_{\mu\nu} + iZ_2\overline{\Psi}\partial\!\!\!/\Psi - mZ_m\overline{\Psi}\Psi + eZ_1\overline{\Psi}A\!\!\!/\Psi \tag{8.4}$$

Where Z_i are constants to be determined such that the theory abides by the LSZ theorem. The quatized theory of this lagrangian has been shown to be equivalent to the sum of all connected Feynman diagrams with photon, positron, and electron sources. Thsi allows for the computatiton of the partition function with a $\mathcal{O}(e^{2n})$ degree of accuracy by computing the sum of all Feynman diagrams with n-loops.
By computing the exact photon propagator, exact fermion propagator, and the exact vertex terms to the 1-loop order the Z_i were computed to the $\mathcal{O}(e^2)$ order. This was completed using dimensional expansion by assuming that the dimension of the system is $d = 4 - \epsilon$ and subsequently taking $\epsilon \to 1$ to examine the three dimensional case. The Z_i were used to determine the relationship between the original coupling constant e_0 and the renormalized constant e which was used to determine the energy dependence of the coupling constant, in effect the β-function as

$$\beta(e) = -\frac{1}{2}e + \frac{e^3}{12\pi^2} + \mathcal{O}(e^5) \tag{8.5}$$

By extending the problem to N_f flavors of massless fermions the β-function was found to be

$$\beta(e) = -\frac{1}{2}e + \frac{N_f e^3}{12\pi^2} + \mathcal{O}(e^5) \tag{8.6}$$

This resulted in the Wilson-Fisher fixed point, or the zero of the β-function to take the value of $e_* = \frac{6\pi^2}{N_f}$.
By examining a couple of quadrilinear operators at an energy scale near the Wilson-Fisher fixed point it may be shown that if the number of massless fermion flavors is less than $N_f < \frac{9}{4}$ that one of these operators becomes relevant. This in turn results in spontaneous mass generation resulting in the breaking of the $SU(2N_f)$ symmetry group of QED_3.
Furthermore, several papers, such as [11], have shown that the theory describing high T_c cuprate superconductors may be isometric to that of QED_3.
This paper has hardly been able to discuss the theory of QED_3 in its entirety. We have barely commented on the many aspects of the renormalization group, choosing instead to focus solely on the β-function. More research has been completed in many of these fields, for example, anomalous dimension of the

theory which has been a subject of discussion some papers, see for instance [16]. Nor have we discussed how QED_3 may be used as a prototype or model for more complex system such as quenched QED_4 or three dimensional chromodynamics [17]. In addition there is the distinct possibility that QED_3 may be used to accurately describe other superconductors in addition to the cuprates that have been discussed in papers such as [11–13].

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