# On the Geometry of String Theory

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**Abstract**

The aim of this Bachelor Thesis is to provide a mathematical description for the space-time in which strings described by string theory life. Firstly a detailed description of string theory will be given. A consequence of string theory known as T-duality will be discussed. Secondly a mathematical framework for the geometry of stringy space-time will be provided which tries to encapture T-duality as a natural symmetry. Generalized Geometry will be discussed and this will be related to Double Field Theory and Metastring theory descriptions of the string theory by means of para-Hermitian manifolds.

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# 1 Introduction

String theory is the number one contender for the "Theory of Everything" in the current theoretical physics paradigm. Its earliest history can be found around the fifties in order to describe the laws of the strong interaction. A detailed description of the history of string theory can be found in [17].

String theory can be seen as a minor extension of the way one thinks of fundamental particles. These particles are not thought of as points in space but are changed to describe strings in space. This minor extension gives stunning theoretical results. First of all the equations that describe the string is a combination of the usual motion one expects for a point particle, with additional oscillator terms often called 'ghost' oscillators. When one quantizes the string analogously to the quantization of the point particle, the different oscillations of the string determine the type of fundamental particle one has at hand.

Besides this representation of the fundamental particles, Einsteins theory of relativity follows naturally out of the quantization of the string. This will be discussed in this thesis in chapter 2. Chapter 2 will give a detailed description of string theory starting from the physics of a non-relativistic classical string following closely the book by Zwiebach [24]. This string will then be discussed in a relativistic setting and subsequently quantized in a way analogous to the point particle. Finally a consequence of string theory known as T-duality is discussed. This T-duality will be the key ingredient to a mathematical formulation of string theory.

In 2002, Nigel Hitchin introduced new field of Geometry, called Generalized geometry [5], [7]. This new type of geometry was largely motivated by a duality property of string theory called T-Duality. Soon after the introduction of Generalized Geometry, physicists have tried to incorporate this theory in turn to describe the geometry of space-time in which strings live. The space-time geometry of strings will be referred to as Target-Space geometry. This generalized geometry and its prerequisites will be discussed in chapters 4 and 5.

Motivated by Generalized geometry, string physicist such as Olaf Hohm and Barton Zwiebach, introduced a specific framework for the Target-Space geometry called Double Field Theory [4], [16], [15]. While this double field theory is inspired by Generalized Geometry, its formulation does not specifically make use of it. Another framework similar to Double Field theory is therefore introduced by Friedel et al. This theory is called metastring theory [12] . Double Field Theory and Metastring theory are discussed in chapter 5.

The framework of Metastring theory is further carried out by the likes of Svoboda and Friedel to combine this with Generalized Geometry by means of para-Hermitian manifolds [13], [11], [20]. This will be discussed in chapter 6 and concludes this thesis.

# 2 Quantization of the classical relativistic string

## 2.1 Nonrelativistic String

We start by considering a classical string which can move in two directions, transversally (y-direction) and longitudinal (x-direction), for simplicity we only consider the transverse movement. Furthermore the string moves in time. We assume that the string has constant mass density $\mu_0$ and string tension $T_0$. One can write the kinetic energy as the sum over all infinitesimal kinetic terms

$$T = \int_0^a \frac{1}{2}\mu_0 dx \left(\frac{\partial y}{\partial t}\right)^2 .$$

The potential term is given by :

$$V = \int_0^a \frac{1}{2}T_0 \left(\frac{\partial y}{\partial x}\right)^2 dx.$$

The Langrangian density for the string therefore becomes

$$\mathcal{L} = \frac{1}{2}\mu_0 dx \left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T_0 \left(\frac{\partial y}{\partial x}\right)^2 .$$

The corresponding equation of motion is the well known wave equation with $v_0 = \sqrt{T_0/\mu_0}$:

$$\frac{\partial^2 y}{\partial x^2} - \frac{\mu_0}{T_0}\frac{\partial^2 y}{\partial t^2} = 0 \tag{1}$$

This equation is a second order partial differential equation. To specify a solution one must impose initial and boundary conditions. We classify two different types of boundary conditions.

$$\frac{\partial y}{\partial t}(t,0) = \frac{\partial y}{\partial t}(t,a) = 0 \qquad \text{Dirichlet boundary conditions}$$

This boundary condition dictates that the string is fixed at both end point $x = 0$, $x = a$.

$$\frac{\partial y}{\partial x}(t,0) = \frac{\partial y}{\partial x}(t,a) = 0 \qquad \text{Neumann boundary conditions}$$

This boundary dictates that the string endpoints move freely with constant velocity along the string endpoints.

The general solution to the wave equation 1 can be written as a superpostion of two waves, one moving to the left and one moving to the right.

$$y(t,x) = h_R(x - v_0 t) + h_L(x + v_0 t)$$

With this equation one can naturally impose initial conditions as follows

$$y(0,x) = h_R(x) + h_L(x)$$
$$\frac{\partial y}{\partial t}(0,x) = -v_0 h_R'(x) + v_0 h_L'(x)$$

Where the primes stand for the derivatives with respect to the full argument $x \pm v_0 t$. Now, we could like to conclude this discussion by introducing the conjugate momentum densities and writing the equation of motion differently

$$\mathcal{P}^t \equiv \frac{\partial \mathcal{L}}{\partial \dot{y}}, \qquad \mathcal{P}^x \equiv \frac{\partial \mathcal{L}}{\partial y'}$$

with $y' = \frac{\partial y}{\partial x}$. For the nonrelativistic lagrangian one can write them explicitly as

$$\mathcal{P}^t = \mu_0 \frac{\partial y}{\partial t}, \qquad \mathcal{P}^x = -T_0 \frac{\partial y}{\partial x}.$$

One can now apply the principle of least action, taking $S = \int \mathcal{L} dt$, subsequently substituting the conjugate momentum densities one arrives at the following equation of motion

$$\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x} = 0. \qquad (2)$$

## 2.2 Relativistic String

### 2.2.1 Relativistic point particle

Consider a point particle living in Minkowski spacetime (d,1) i.e. one time dimension and d-spatial dimensions. This particle will trace out a 1-dimensional line called the world line. The worldline is parametrized by parameter $\tau$ which we can think of as the time parameter. The metric for this case will be a scalar

$$g = \frac{\partial x^\mu}{\partial \tau} \eta_{\mu\nu} \frac{\partial x^\nu}{\partial \tau}.$$

The corresponding arc length is then given by

$$ds = \sqrt{g} d\tau$$

this arc length is called the proper time in the language of special relativity. This proper time is related with the action by considering the right units. Action has units of (energy $\times$ time) whereas the units of proper time is time itself. We therefore need to multiply the proper time by a Lorentz invariant scalar which has units of energy. The rest mass $m$ has units of energy using relativistic units such that $c = 1$, and is indeed Lorentz invariant, we therefore arrive at the following expression for the action

$$S = -m \int ds = -m \int_{\tau_i}^{\tau_f} \sqrt{\frac{\partial x^\mu}{\partial \tau} \eta_{\mu\nu} \frac{\partial x^\nu}{\partial \tau}} d\tau.$$

Using the $(+, -, \cdots, -)$ convention for $\eta$, and furthermore identifying $\tau$ with time $x^0$ such that $x^0(\tau) = \tau$, we can write the corresponding Lagrangian as follows

$$L = -m\sqrt{g} = -m\sqrt{1 - \vec{v}^2}.$$

This is indeed the familiar Lagrangian for the free relativistic particle. If one takes the non-relativistic limit such that $v \ll 1$, one can expand the square root and take the first linear term to see

$$L \approx -m\left(1 - \frac{1}{2}\vec{v}^2\right).$$

Constant terms in the Lagrangian will not affect the equations of motion therefore for the free particle we can leave out the mass $m$ and conclude that indeed the Lagrangian is the total kinetic energy as expected.

### 2.2.2 Relativistic string

Just as the lifetime of a point particle will trace out a wordline in Minkowski spacetime, the lifetime of string will trace out a two-dimensional surface called the world-sheet of the string. The surface will be parametrized by $\tau$ and $\sigma$. We will denote the coordinate maps simply by $X^\mu(\tau, \sigma)$. For an arbitrary surface embedded in some higher dimensional space one could determine the area as

$$dA = \sqrt{g} d\tau d\sigma.$$

Where $g$ represents the determinant of the induced metric $g_{ij}$ from the ambient space. Since our ambient space is Minkowski (d,1), the metric can be written as follows

$$g_{ij} = \frac{\partial X^\mu}{\partial \xi^i}\eta_{\mu\nu}\frac{\partial X^\mu}{\partial \xi^j}$$

where, $\xi^1 = \tau$, $\xi_2 = \sigma$ and $\eta_{\mu\nu}$ is the standard metric for Minkowski spacetime. Since we are dealing with two parameters, the metric $g_{ij}$ will be a 2x2 matrix and its corresponding determinant will be

$$g = g_{11}g_{22} - g_{12}g_{21}.$$

However, this will yield a negative value. Changing $g \to -g$ will have no effect on the Lorentz invariance and therefore the proper area is denoted by

$$A = \int d\tau d\sigma \sqrt{\left(\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma}\right)^2 - \left(\frac{\partial X}{\partial \tau}\right)^2 \left(\frac{\partial X}{\partial \sigma}\right)^2}.$$

Where the dot product and squares are defined as usual with respect to the Minkowski innerproduct. The parameter $\tau$ can be seen as a time parameter and $\sigma$ can be seen as a space parameter. We therefore write the following short hand notation.

$$\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau}, \quad X'^\mu \equiv \frac{\partial X^\mu}{\partial \sigma}$$

Similar to the relativistic point particle we need to multiply with a factor of the right units to get the action. The area has units of length squared and to get the right units we need to multiply with a Lorentz invariant constant that has relativistic units of (energy/time). The string tension $T_0$ is a force and therefore naturally has the right units. The action, also called Nambu-Goto action is then

$$S = -\frac{T_0}{c} \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2} \tag{3}$$

The corresponding equation of motion to this action is very similar to (2).

$$\frac{\partial \mathcal{P}_\mu^\tau}{\partial \tau} + \frac{\partial \mathcal{P}_\mu^\sigma}{\partial \sigma} = 0 \tag{4}$$

For every $\mu \neq 0$ one can impose Dirichlet and Neumann boundary conditions.

The conjugate momentum densities are

$$\mathcal{P}_\mu^\tau = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')\partial_\tau(X_\mu) - (\dot{X})^2 \partial_\sigma(X_\mu)}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}} \tag{5}$$

$$\mathcal{P}_\mu^\sigma = \frac{\partial \mathcal{L}}{\partial X'^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')\partial_\sigma(X_\mu) - (\dot{X})^2 \partial_\tau(X_\mu)}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}}. \tag{6}$$

The Nambu-Goto action is reparametrization invariant [24], so we can parametrize the world-sheet in a useful way. We will do this by means of light-cone coordinates. We will parametrize in such a way that firstly, the euquations of motion will be turned into a wave equation, and secondly using specifically the light-cone coordinates, the quantization of the string will be handled.

The string tension is for historical reason turned into $\alpha'$ via

$$T_0 = \frac{1}{2\pi\alpha'\hbar c}.$$

Subsequently the parameter $\alpha'$ is thought of as the length of the string squared.

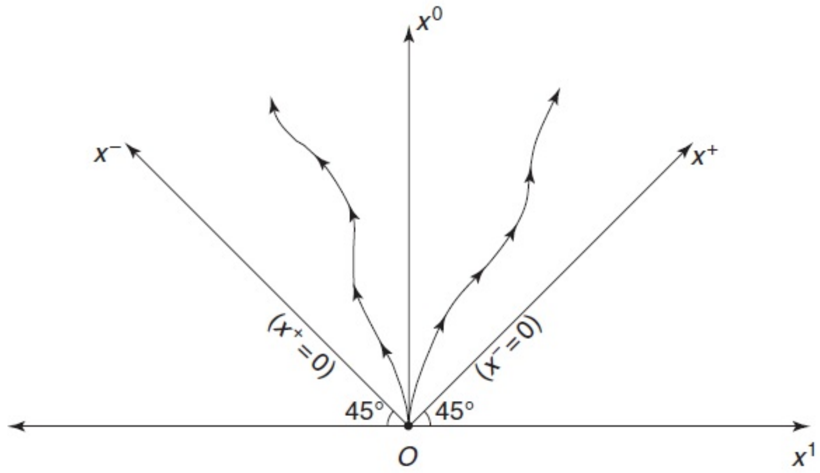
## 2.3 Light-cone relativistic strings

### 2.3.1 Light-cone coordinates

In this subsection we will make a change of coordinates that is useful for the quantization of the string. We introduce so-called light-cone coordinates as follows

$$x^+ = \frac{x^0 + x^1}{\sqrt{2}}, \quad x^- = \frac{x^0 - x^1}{\sqrt{2}}. \tag{7}$$

These new coordinates are called light cone coordinates because they correspond to rays of light going in the $+x^1$ or $-x^1$ direction respectively. They can be represented in the following picture




The standard Minkowski metric $\eta$ also needs to be changed to recover the same physics. For a 4-dimensional space the metric becomes

$$\hat{\eta} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

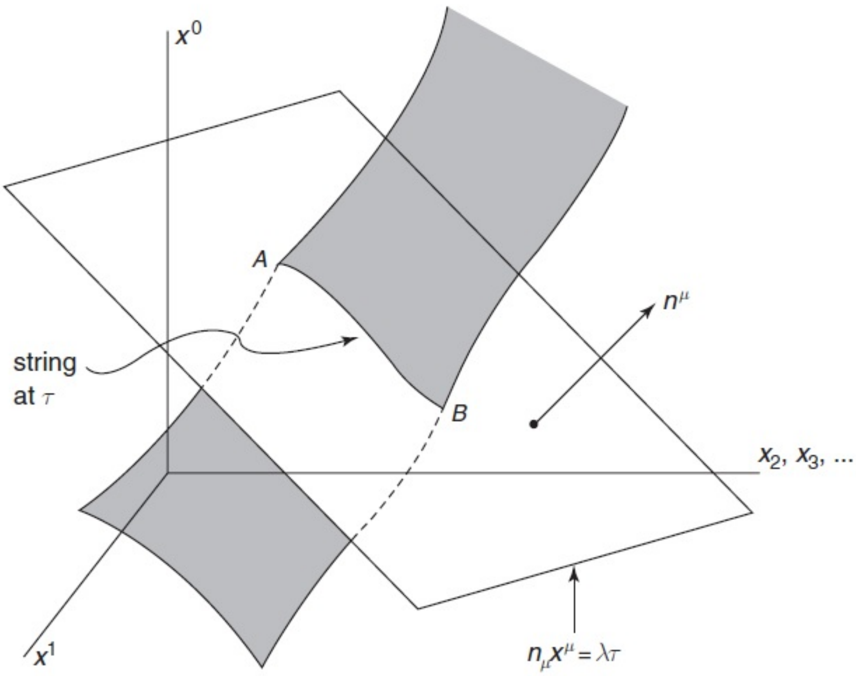
Consequently any Lorentz vector can be expresses in light-cone components by changing its coordinates analogous to (7).

### 2.3.2 Reparametrizations and Gauge conditions

The Nambu-Goto action is parametrization invariant, therefore we can setup the parametrization in a 'clever' way. We will do this by means of gauge conditions. These are equations such that they determine a world sheet parametrization for our closed string. We will look at a particular set of gauges that will turn the equation of motion into a wave equation. Firstly consider the equation

$$n_\mu X^\mu(\tau, \sigma) = \lambda \tau \tag{8}$$

where, $n$ is some constant vector. Later on we will take a particular vector $n$ which will induce the so-called light-cone gauge.




If we consider two solutions $x_1$, $x_2$ to equation (8) for a fixed $\tau_0$, then $n_\mu(x_1^\mu - x_2^\mu) = 0$. The difference between to solutions is a vector which is orthogonal to $n$, we can therefore conclude that all the solutions lie in the hyperplane with $n$ as its normal vector. We can therefore see the string as a curve on $\gamma(\sigma)$ on the hyperplane when we freeze time. Similar to the relativistic particle, the momentum $p$ of the closed bosonic string is a conserved charge. Therefore $n \cdot p$ is a constant. We substitute this into the gauge condition to get

$$n \cdot X(\tau, \sigma) = \tilde{\lambda}(n \cdot p)\tau$$

where, the dot product is the standard Minkowski product. In order for both sides of the equation to have the same dimension we take $\tilde{\lambda} = \alpha'$, where $\alpha'$ is equal to the length of the string squared.. Therefore the full gauge condition for $\tau$ is

$$n \cdot X(\tau, \sigma) = \alpha'(n \cdot p)\tau. \tag{9}$$

For every $\tau$ we found an hyperplane such that the physical string can be seen as a curve on this hyperplane. We have therefore found a parametrization for $\tau$. Now we wish to find the right curve out of all these possible curves. This will again be done by a gauge condition which in turn then gives the full parametrization.

At every point $\tau$ we formulated a hyperplane depending on our gauge vector $n$. The parametrization for $\sigma$ therefore can be seen as a curve $\gamma(\sigma)$ on this hyperplane. An example of a certain type of gauge is the static gauge. In this gauge the $\sigma$ parametrization is the one where the energy density $\mathcal{P}^{\tau 0}$ is constant. This can equivalently seen as

$$n \cdot \mathcal{P}^\tau = const \tag{10}$$

where, $n = (1, 0, \cdots, 0)$. We now generalize this for any gauge related to the hyperplane determined by $n$. The gauge condition therefore becomes (10) for any $n$. This gauge condition is set for a specific time $\tau$. Therefore one can write (10) as

$$n \cdot \mathcal{P}^\tau(\tau, \sigma) = a(\tau) \tag{11}$$

for some function $a(\tau)$. Furthermore we want $\sigma \in [0, 2\pi]$ for the closed string. This can be satisfied by adjusting $\sigma$ with parameter $b$ such that $\sigma \mapsto b\sigma$. If we now integrate (11) over the full closed string $\sigma \in [0, 2\pi]$ we find

$$\int_0^{2\pi} n \cdot \mathcal{P}^\tau(\tau, \sigma) d\sigma = n \cdot p = 2\pi a(\tau).$$

We can consequently write $a(\tau) = (n \cdot p)/2\pi$ and arrive at the final gauge condition

$$n \cdot \mathcal{P}^\tau = \frac{n \cdot p}{2\pi}. \tag{12}$$

Now writing the equations of motion (4) and taking the inner product with $n$

$$\frac{\partial}{\partial \tau}(n \cdot \mathcal{P}^\tau) + \frac{\partial}{\partial \sigma}(n \cdot \mathcal{P}^\sigma) = 0$$

Using the gauge conditions $n \cdot \mathcal{P}^\tau = const$, we see that $n \cdot \mathcal{P}^\sigma = const$. In fact we will prove that on the closed string one can choose the point $\sigma = 0$ in such a way that $n \cdot \mathcal{P}^\sigma(\tau, 0) = 0$, and therefore $n \cdot \mathcal{P}^\sigma = 0$ for all $\sigma \in [0, 2\pi]$.

Now one would like to see what these constraint will do to $\dot{X}$ and $X'$. Dotting (5) with $n$,

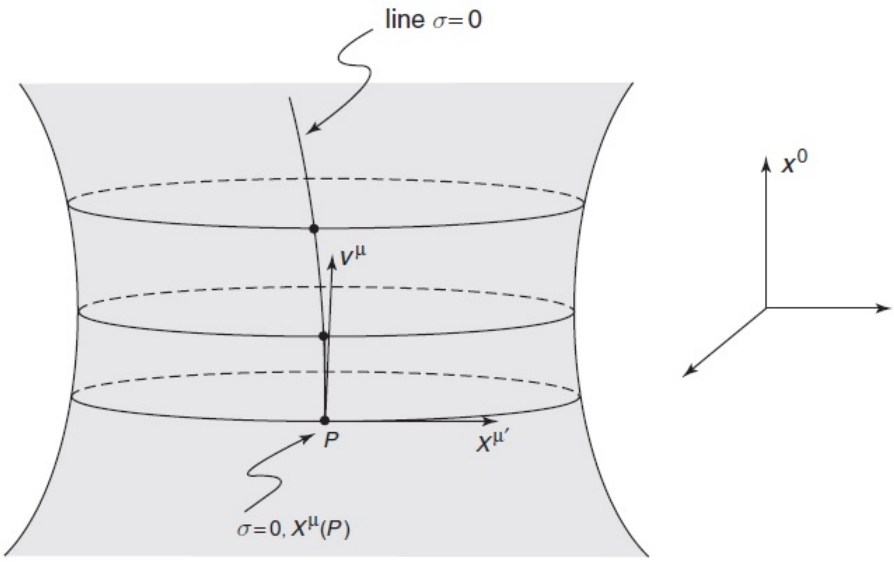
$$n \cdot \mathcal{P}^\sigma = \frac{\partial \mathcal{L}}{\partial X'} = -\frac{1}{2\pi\alpha'} \frac{(\dot{X} \cdot X')\partial_\tau(n \cdot X) - (\dot{X})^2 \partial_\sigma(n \cdot X)}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}}$$

From (9), we see that $\partial_\sigma n \cdot X = 0$ and $\partial_\tau X = const$, therefore we must require that $\dot{X} \cdot X' = 0$ for $\sigma = 0$. We can indeed identify an $X(\tau, 0)$ line such that $\dot{X} \cdot X' = 0$. Consider a point $P$ on the world-sheet such that for some specific time $\tau_0$, $X(\tau_0, 0) = P$. At this point there exist a time like tangent vector $t^\mu$ which cannot be parallel to the space-like tangent vector $X'^\mu$. Therefore they span the tangent-space of the world-sheet at point P. These vectors need not be orthogonal but this can be done by means of the Gramm-Schmidt procedure. One can take a vector $v^\mu$ such that

$$v^\mu = t^\mu - \frac{t \cdot X'}{X' \cdot X'} X'^\mu.$$

This vector $v^\mu$ is naturally orthogonal to $X'^\mu$. The line $X(\tau, 0)$ in a neighborhood of $P$ can therefore be linearly written as

$$X^\mu(\tau_0 + \varepsilon, 0) = X(\tau_0, 0) + \varepsilon v^\mu.$$




For small enough $\varepsilon$. The tangent vector along this line is then given by $v^\mu$. We denote all other points on the line $X(\tau,0)$ in such a way that at each time $\tau$, the tangent vector must be orthogonal to $X'^\mu$. The tangent vector to the line $X(\tau,0)$ at time $\tau_0$ is given by $\dot{X}(\tau_0,0)$ and we therefore conclude that with this construction $\dot{X}\cdot X'=0$ and consequently $n\cdot\mathcal{P}^\sigma=0$.

The equation for $\mathcal{P}^\tau$ (5) then simplifies to

$$\mathcal{P}^{\tau\mu}=\frac{1}{2\pi\alpha'}\frac{X'^2\dot{X}^\mu}{\sqrt{-\dot{X}^2X'^2}}.$$

Using gauge condition (12) we can write

$$n\cdot p=\frac{1}{\alpha'}\frac{X'^2(n\cdot\dot{X})}{\sqrt{-\dot{X}^2X'^2}}.$$

Now, using (9) and differentiating with respect to $\tau$ such that $n\cdot\dot{X}=n\cdot p$ we see that

$$1=\frac{X'^2}{\sqrt{-\dot{X}^2X'^2}}\longrightarrow\dot{X}^2+X'^2=0.$$

Hence we transformed the constraint into

$$\dot{X}\cdot X'=0,\quad \dot{X}^2+X'^2=0$$

which can be written as

$$(\dot{X}\pm X')^2=0. \tag{13}$$

Since $X'^2=-\dot{X}^2$, $\sqrt{-\dot{X}^2X'^2}=X'^2$ , our momentum densities (5),(6) simplify to

$$\mathcal{P}^{\tau\mu}=\frac{1}{2\pi\alpha'}\dot{X}^\mu \tag{14}$$

$$\mathcal{P}^{\sigma\tau}=-\frac{1}{2\pi\alpha}X'^\mu \tag{15}$$

The field equation (4) then becomes

$$\ddot{X}^\mu - X''^\mu = 0 \tag{16}$$

This is the well known wave-equation. We see that introducing the specific gauge conditions using a hyperplane determined by normal vector $n$ allows us to rewrite the equations of motion into a wave equation.

### 2.3.3 Solutions of the wave equation

The equation of motion (16) is a wave equation and the solution can be decomposed in a so-called 'left-mover' and 'right-mover' as follows

$$X^\mu(\tau, \sigma) = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma).$$

Furthermore since we are dealing with a closed string, the parameter space for $\sigma$ is a circle $S^1$ which identifies point which differ by $2\pi$ as the same, in other words $\sigma \sim \sigma + 2\pi$. Therefore our coordinates must be periodic in $\sigma$ with period $2\pi$

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi).$$

This periodicity of $X$ is only valid in non-compactified target spaces. In the chapter about T-Duality we will deal with compactified spaces and see that the winding of a closed string brings an additional term that makes $X$ quasi-periodic. Now , set $u = \tau + \sigma$ and $v = \tau - \sigma$ and using that $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$,

$$X_L^\mu(u) + X_R^\mu(v) = X_L^\mu(u + 2\pi) + X_R^\mu(v - 2\pi)$$

this immplies,

$$X_L^\mu(u + 2\pi) - X_L^\mu(u) = X_R^\mu(v) - X_R^\mu(v - 2\pi). \tag{17}$$

Because $u$ and $v$ are independant variables this equation is equal to a constant. Hence if we take the $u$-derivative on the left side, this should vanish and therefore one can conclude that $X_L'^\mu(u)$ is periodic with period $2\pi$, a similar argument can be made for $X_R'^\mu(v)$. One can therefore write the Fourier expansion as follows,

$$X_L'^\mu(u) = \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \bar{\alpha}_n^\mu e^{-inu} \tag{18}$$

$$X_R'^\mu(v) = \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu e^{-inv}. \tag{19}$$

Integrating this yields

$$X_L^\mu(u) = \frac{1}{2} x_0^{L\mu} + \sqrt{\frac{\alpha'}{2}} \bar{\alpha}_0^\mu u + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\bar{\alpha}_n^\mu}{n} e^{-inu} \tag{20}$$

$$X_R^\mu(v) = \frac{1}{2} x_0^{R\mu} + \sqrt{\frac{\alpha'}{2}} \alpha_0^\mu v + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-inv}. \tag{21}$$

Using the periodicity condition (17), $\bar{\alpha}_0^\mu = \alpha_0^\mu$ this is the so-called level-matching condition. Substituting $\tau$ and $\sigma$ again we arrive at the final solution

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}(\alpha_n^\mu e^{in\sigma} + \bar{\alpha}_n^\mu e^{-in\sigma}) \tag{22}$$

Lastly, the zero mode $\alpha_0$ can be identified as the momentum. The momentum density can be written as

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu = \frac{1}{2\pi\alpha'}(\sqrt{2\alpha'}\alpha_0^\mu + \cdots)$$

where, the dots represent terms that will vanish when we integrate the momentum density over $\sigma \in [0, 2\pi]$. Therefore

$$p^\mu = \int_0^{2\pi}\mathcal{P}^{\tau\mu}(\tau,\sigma)d\sigma = \int_0^{2\pi}\frac{\sqrt{2}}{2\pi\sqrt{\alpha'}}\alpha_0^\mu d\sigma = \sqrt{\frac{2}{\alpha'}}\alpha_0^\mu. \tag{23}$$

Remember that all these solution and equations where determined for which we set $\tau$ equal to a linear combination of string coordinates. Now we want to take our favourite linear combination: the light-cone coordinates. For this we take

$$n_\mu = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0)$$

therefore,

$$n\cdot X = \frac{X^0 + X^1}{\sqrt{2}} = X^+, \quad n\cdot p = \frac{p^0 + p^1}{\sqrt{2}}.$$

Rewriting the parametrization conditions (9) and (12) for the light-cone gauge,

$$X^+(\tau,\sigma) = \alpha' p^+\tau, \quad p^+ = 2\pi\mathcal{P}^{\tau +}.$$

It turns out that for this particular light cone gauge, there will be no dynamics in the $X^-$ coordinate up to a constant $x_0^-$. All other dynamics will be determined by the transverse coordinates $X^I$ [24].

## 2.4 Canonical Quantization and Classical analogy

Classical analogy was first introduced by Paul Dirac in his famous book "The principles of Quantum Mechanics" [3]. Dirac noted that classical mechanics should be limiting case of quantum mechanics, more specifically the limit is when $h \to 0$. In Quantum mechanics, the dynamical variables one observes are non-commutative operators acting on a state space called Hilbert space. We will see that in fact this non-commutativity results in the canonical quantization relations.

Classical mechanics can be formulated in terms of Poisson geometry as follows. In order to know a trajectory of a particle or rigid body in space one needs the know there position $q$ and momentum $p$ at a given point in time. Assuming one has an $n$-dimensional real physical space, one can take the position and momentum coordinates together to make a $2n$-dimensional space called phase space. One defines a natural Poisson bracket on functions of $p$ and $q$ as follows

$$\{f, g\} = \sum_i \left[ \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right].$$

This Poisson bracket together with the space of smooth functions defines a Lie-algebra called the Poisson algebra.

The coordinates $(q^i, p_i)$ are taken to be independent variables in this phase space description. Therefore $\frac{\partial q^i}{\partial p_j} = 0$ and $\frac{\partial q^i}{\partial q^j} = \frac{\partial p^i}{\partial p^j} = \delta^i_j$. A quick calculation then gives us a set of Poisson bracket relations

$$\begin{aligned} \{q_i, q_j\} = \{p^i, p^j\} &= 0 \\ \{q_i, p^j\} &= \delta^i_j \end{aligned}$$

This set of relations determines all bracket relations since the smooth functions are only dependent on the phase-space coordinates and therefore one can find the bracket relations by doing some algebraic manipulations on the brackets.

Given a Hamiltonian function $H$ and a function $f$ in the Poisson algebra dependent on phase-space coordinates and time. The total time derivative of $f$ is then given by

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}.$$

In the classical analogy quantization procedure, the phase-space coordinates $q$ and $p$ are being promoted to operators acting on a Hilbert space. These operators are in general non-commutative. The Poisson bracket $\{\cdot, \cdot\}$ transformed into the commutator $[\cdot, \cdot]$ and the defining bracket relations now become the canonical commutation relations

$$\begin{aligned} [q_i, q_j] = [p^i, p^j] &= 0 \\ [q_i, p^j] &= i\hbar\delta^i_j \end{aligned}$$

We can write the time derivative of an Heisenberg operator in Hilbert space as follows

$$\frac{dO}{dt} = i[H, O].$$

In the Heisenberg picture, operators are time dependent whereas in the Schrödinger picture, states are time dependent. There is a natural correspondence between Heisenberg operators $O_h$ and Schrödinger operators $O_S$

$$O_h = e^{iHt} O_s e^{-iHt}$$

where, $H$ denotes the Hamiltonian of the system.

### 2.4.1 Harmonic Oscillator

*"The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction."* Sidney Coleman. [21]

In this section we will treat the standard example of a harmonic oscillator and how this effects canonical quantization. As the quote already hints, we would take the result of this harmonic oscillator generalize the properties derived by it. Consider the following Hamiltonian for only two coordinates $p$, $q$

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2.$$

We now define new variables $a$ and $a^\dagger$ as follows

$$a = \sqrt{\frac{\omega}{2}} + \frac{i}{\sqrt{2\omega}}p, \quad a^\dagger = \sqrt{\frac{\omega}{2}} - \frac{i}{\sqrt{2\omega}}p.$$

One can rewrite the commuation relations for $p$ and $q$ in terms of $a$ and $a^\dagger$

$$[a.a^\dagger] = 1$$

One can write the Hamiltonian as follows

$$H = \omega(a^\dagger a + \frac{1}{2}).$$

Consequently,

$$[H, a^\dagger] = \omega a^\dagger, \quad [H, a] = -\omega a.$$

This shows that $a$ can be thought of as an annihilation operator and $a^\dagger$ as a creation operator. Indeed let $|E\rangle$ be an eigenstate of the hamiltonian $H$ with eigenvalue $E$.

$$Ha^\dagger |E\rangle = ([H, a^\dagger] + a^\dagger H) |E\rangle = (\omega a^\dagger + a^\dagger E) |E\rangle = (E + \omega)a^\dagger |E\rangle .$$

A similar calculation for $a$ shows that $Ha |E\rangle = (E - \omega)a |E\rangle$. These operators $a$ and $a^\dagger$ turn the state $|E\rangle$ into a state with more or less energy. In this description, the $a$-operators are in the Schrödinger picture. We can change it to the Heisenberg picture by solving the following differential equation:

$$\frac{da}{dt} = i[H(t), a(t)] = i\omega[a(t)^\dagger a(t), a(t)] = -i\omega a(t).$$

Similar analysis can be carried out for $a^\dagger$. We set $a(0) = a$ and $a^\dagger(0) = a^\dagger$. We therefore find that

$$a(t) = e^{-i\omega t}a, \quad a^\dagger(t) = e^{i\omega t}a^\dagger.$$

We can put this back into our operator $q$ and see that

$$q(t) = \frac{1}{\sqrt{2\omega}}(a(t) + a^\dagger(t)) = \frac{1}{\sqrt{2\omega}}(ae^{-i\omega t} + a^\dagger e^{i\omega t}).$$

This expression is the motivating example for the quantization of the string. We started from an string action which is equivalent to starting with an Hamiltonian. For the string we found that firstly coordinates depend on $\tau$ and $\sigma$, subsequently we can write the string coordinates $X^\mu$ in a similar fashion as the Harmonic oscillator but now with infinitely many creation and annihilation operators, now denoted by $\alpha$.

## 2.5 Quantization

We have seen that the solution to the equation of motions is determined as an expansion with modes $\alpha$ or $\bar{\alpha}$. In the quantization of the relativistic closed string, these modes will become operators. Now we would like to proceed the quantization process in the same way as that is done for a point particle. Our phase-space coordinates now however depend not only on time but also on $\sigma$. Furthermore we introduced a light cone gauge such that we have coordinates $X^+$ and $X^-$. As seen previously, the independent variables can be seen as the following operators $X^I, x_0^-, \mathcal{P}^{\tau I}, p^+$. We set up the canonical commutator relations. for all transverse string-coordinates we have the following relation

$$\left[X^I(\tau,\sigma), \mathcal{P}^{\tau I}(\tau,\sigma')\right] = i\eta^{IJ}\delta(\sigma-\sigma').$$

Furthermore we have the relation

$$\left[x_0^-, p^+\right] = -i.$$

All other other relations between operators vanish.

From these relations the commutator relations for the transverse mode-operators follow

$$\begin{aligned}
\left[\bar{\alpha}_m^I, \bar{\alpha}_n^J\right] &= m\delta_{m+n,0}\eta^{IJ} \\
\left[\alpha_m^I, \alpha_n^J\right] &= m\delta_{m+n,0}\eta^{IJ} \\
\left[\alpha_m^I, \bar{\alpha}_n^J\right] &= 0
\end{aligned}$$

These $\alpha$ modes will be turned into operators $a$ as follows

$$\begin{aligned}
\bar{\alpha}_n^I &= \bar{a}_n^I\sqrt{n} \\
\bar{\alpha}_{-n}^I &= \bar{a}_n^{\dagger I}\sqrt{n} \\
\alpha_n^I &= a_n^I\sqrt{n} \\
\alpha_{-n}^I &= a^{\dagger I}\sqrt{n}
\end{aligned}$$

he Hamiltonian is given by [24]

$$H = \alpha' p^+ p^-.$$

This quantization procedure can be seen as the first quantization of the string. First quantization refers to the translation from dynamical ( such as momentum, position) to operators. There also exist a quantization procedure which in some sense is follows after first quantization. This quantization is called second quantization and is the quantization of fields. In Quantum field theory, fields are being translated into field operators in an analogous manner. This second quantization lets us identify quantum states as excitations of the vacuum state $|\Omega\rangle$

$$\left|p^{+},\vec{p}_T\right\rangle \longleftrightarrow a^{\dagger}_{p^{+},\vec{p}_T}\left|\Omega\right\rangle . \tag{24}$$

This creation operator $a^{\dagger}$ can be interpreted as coming from a scalar field as follows. A general state $|\Psi,\tau\rangle$ is given by

$$|\Psi,\tau\rangle = \int dp^{+}d\vec{p}_T\psi(\tau,p^{+},\vec{p}_T)\left|p^{+},\vec{p}_T\right\rangle .$$

This state will satisfy the Schrödinger equation

$$i\frac{\partial}{\partial\tau}|\Psi,\tau\rangle = H|\Psi,\tau\rangle .$$

Consequently the wave function $\psi$ will satisfy an correlated equation. This equation turns out to be the scalar-field equation for scalar field $\phi$. We therefore found the identification

$$\psi(\tau,p^{+},\vec{p}_T) \longleftrightarrow \phi(\tau,p^{+},\vec{p}_T).$$

This scalar field in turn can be quantized resulting in creation and annihilation operators $a^{\dagger}$, $a$.

For the string however we will see that instead of scalar fields coming from one-string states we will have tensor fields. This will be discussed in the next chapter.

### 2.5.1 Constructing state space

Similarly to quantization of a point particle we act on the ground state with creation operators $a_n^{\dagger I}$ and $\bar{a}_n^{\dagger I}$. The state space is then written as

$$\left|\lambda,\bar{\lambda}\right\rangle = \left[\prod_{n=1}^{\infty}\prod_{I=2}^{25}\left(a_n^{\dagger I}\right)^{\lambda_{n,I}}\right]\times\left[\prod_{m=1}^{\infty}\prod_{J=2}^{25}\left(\bar{a}_m^{\dagger I}\right)^{\bar{\lambda}_{m,J}}\right]\left|p^{+},\vec{p}_T\right\rangle$$

The eigenvalues of the corresponding number operators are then given by

$$N^{\perp} = \sum_{n=1}^{\infty}\sum_{I=2}^{25} n\lambda_{n,I}, \quad \bar{N}^{\perp} = \sum_{m=1}^{\infty}\sum_{J=2}^{25} m\bar{\lambda}_{n,I}$$

In theory the state space can have as many excitations but this will result in very high enerrgies. The physics that we know today is encaptured in the low-energy spectrum of the string state space, in particular the mass-less sector is of interest since this would result in bosonic fields. The mass squared is given as follows [24]

$$M^2 = \frac{2}{\alpha'}(N^{\perp} + \bar{N}^{\perp} - 2)$$

If one sets $M = 0$, and uses the constraint $\bar{N}^{\perp} = N^{\perp}$ we arrive at only one possible state

$$a_1^{I\dagger}\bar{a}_1^{J\dagger}\left|p^+, \vec{p}_T\right\rangle.$$

Very similar to (24) we can make the identification here as follows

$$a_1^{I\dagger}\bar{a}_1^{J\dagger}\left|p^+, \vec{p}_T\right\rangle \longleftrightarrow a^{\dagger IJ}_{p^+,\vec{p}_T}\left|\Omega\right\rangle \tag{25}$$

resulting in a tensor field identification $R_{IJ}$

$$R_{IJ}a_1^{I\dagger}\bar{a}_1^{J\dagger}\left|p^+, \vec{p}_T\right\rangle$$

where the summation convention is used. The symbol $R_{IJ}$ can be seen as a square matrix of size $(D-2)$. For every matrix one could write it as a sum of its symmetric and anti-symmetric part

$$R_{IJ} = \frac{1}{2}(R_{IJ} + R_{JI}) + \frac{1}{2}(R_{IJ} - R_{JI}) \equiv S_{IJ} + A_{IJ}$$

where, $R_{JI}$ is the transpose of $R_{IJ}$ and , $S_{IJ}$ and $A_{IJ}$ are defined to be the symmetric and anti-symmetric parts. Furthermore one can remove the trace of $S_{IJ}$ by subtracting $\frac{1}{(D-2)}tr(S_{IJ})$ from every diagonal entry

$$S_{IJ} = (S_{IJ} - \frac{1}{(D-2)}\delta_{IJ}tr(S_{IJ})) + \frac{1}{(D-2)}\delta_{IJ}tr(S_{IJ}).$$

If we define the traceless part to be $\hat{S}_{IJ}$ and furthermore define $S' = tr(S_{IJ})/(D-2)$, the original matrix $R_{IJ}$ splits into three parts

$$R_{IJ} = \hat{S}_{IJ} + A_{IJ} + \delta_{IJ}S'.$$

Consequently, the state space is split into three parts which will be the first hints of Generalized Geometry as we will see later on

$$\hat{S}_{IJ}a_1^{I\dagger}\bar{a}_1^{J\dagger}\left|p^+, \vec{p}_T\right\rangle \tag{26}$$

$$A_{IJ}a_1^{I\dagger}\bar{a}_1^{J\dagger}\left|p^+, \vec{p}_T\right\rangle \tag{27}$$

$$S'\delta_{IJ}a_1^{I\dagger}\bar{a}_1^{J\dagger}\left|p^+, \vec{p}_T\right\rangle \tag{28}$$

$$\tag{29}$$

These states can be identified with bosonic one-particle states corresponding to the gravitation field, the Kalb-Ramond field and the dilaton respectively. As will be discussed later the Kalb-Ramond and gravity fields will arrise naturally when we consider Generalized Geometry.

## 2.6 T-Duality

A consequence of string theory is that the world we live in would be 26-dimensional if one considers bosonic string theory only and 10-dimensional if one allows fermionic strings aswell. However, the world we observe seems to have dimension 4. These other dimensions then would be compactified. The compacification can come in many ways, the easiest example being a generalization of a torus. If one does this toroidal compactification one observes the striking result that it does not matter whether the compactification is done using a large radius $R$ or a small radius $r = 1/R$, this result is known as T-duality (T stands for toriodal). We will first look at the easiest example to consider, two spatial dimension of which one is compactified. The result would be a infinite long cylinder. Let $x$ be the compactified space coordinate such that

$$x \sim x + 2\pi R.$$

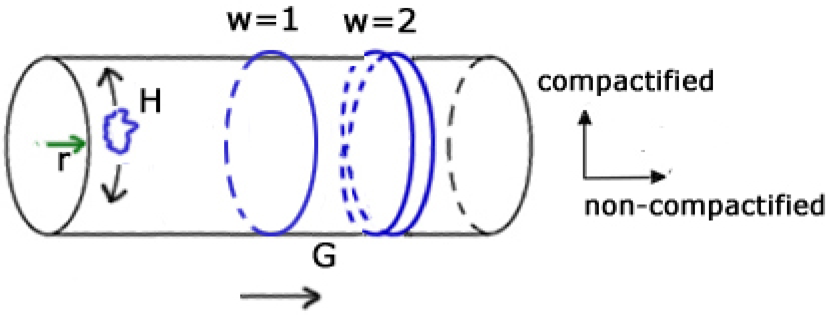



Figure 1: scource: redshift academy

Now, closed oriented strings can wrap around this cylinder. Hence strings can be wrapped around $n$ times positively, negatively or not at all. The corresponding string coordinate $X$ would have the following identification

$$\begin{aligned} \text{not wrapped:} \quad & X(\tau, \sigma = 2\pi) - X(\tau, \sigma = 0) = 0 \\ \text{positively wrapped:} \quad & X(\tau, \sigma = 2\pi) - X(\tau, \sigma = 0) = m(2\pi R) \\ \text{negatively wrapped:} \quad & X(\tau, \sigma = 2\pi) - X(\tau, \sigma = 0) = -m(2\pi R). \end{aligned}$$

Hence for the compact dimension one can define an integer $m$ which denotes the winding behaviour of the string. We define the associated winding $w$ to be

$$w \equiv \frac{mR}{\alpha'}.$$

Our string-coordinate identification then becomes

$$X(\tau, \sigma + 2\pi) = X(\tau, \sigma) + 2\pi\alpha' w.$$

Now we will bring this compactification to our original setting of 25 spatial dimensions. Let $x = x^{25}$ be the compactified coordinate and $X(\tau, \sigma) = X^{25}(\tau, \sigma)$ the corresponding string coordinate. We denote the transervse coordinates by $X^i$ and the light-cone coordinate are as usual $X^+, X^-$. For the solution to the wave equation, compactification has given us a replacement for the level-matching condition

$$X_L(u+2\pi) + X_R(v-2\pi) = X_L(u) + X_R(v) + 2\pi\alpha' w$$

hence,

$$X_L(u+2\pi) - X_L(u) = X_R(v) - X_R(v-2\pi) + 2\pi\alpha' w.$$

The level-matching condition therefore becomes

$$\bar{\alpha}_0 - \alpha_0 = \sqrt{2\alpha'} w.$$

Recall that the momentum can be written as

$$p = \frac{1}{2\pi\alpha'} \int_0^{2\pi} \dot{X} d\sigma = \frac{1}{2\pi\alpha'} \int_0^{2\pi} \dot{X}_L + \dot{X}_R d\sigma = \frac{1}{\sqrt{2\alpha'}} (\bar{\alpha}_0 + \alpha_0).$$

Hence we see that one can treat momentum $p \sim \bar{\alpha}_0 + \alpha_0$ and winding $w \sim \bar{\alpha}_0 - \alpha_0$ on the same footing. A natural question might arise at this stage, we defined the winding $w$ to have a discrete spectrum yet this is never mentioned for the momentum. It turns out that the string, analogous to a point particle, will be quantized in the compactified dimension.

The zero mode $x_0$ is the conjugate coordinate operator of the momentum operator and now lives on a circle. We can see this as the linear approximation of the string. Considering this one can treat the string as a quantum mechanical point particle with phase-space coordinates $(x_0, p)$ and the string property is encaptured by the winding mode $w$.

Now considering a point particle with position coordinates $x_0$, we can identify a position ket vector $|x_0\rangle$ which has $x_0$ as position eigenvalue. On this ket vector one can define the translation operator $T$ such that

$$T(a) |x_0\rangle = |x_0 + a\rangle .$$

This operator $T$ can subsequently be written in terms of the momentum operator $p$ as follows [14]

$$T(a) = e^{-\frac{i}{\hbar} ap}.$$

In our standard units we take $\hbar = 1$. Furthermore since the dimension in which $x_0$ lives is compactified we have the identification of ket vectors

$$|x_0\rangle = |x_0 + 2\pi R\rangle = T(2\pi R) |x_0\rangle .$$

Therefore the translation operator should equal the identity operator is one translates by $2\pi R$ i.e.

$$T(2\pi R) = e^{-i2\pi Rp} = 1.$$

Hence, $2\pi Rp = 2\pi n$ for some integer $n$ and therefore

$$p = \frac{n}{R} \quad n \in \mathbb{Z}. \tag{30}$$

Hence we have seen that it is very natural to treat $p$ and $w$ on the same footing.

Indeed one can write

$$\bar{\alpha}_0 = \sqrt{\frac{2}{\alpha'}}(p+w) \tag{31}$$

$$\alpha_0 = \sqrt{\frac{2}{\alpha'}}(p-w). \tag{32}$$

This allows us to write the left and right mover solutions to the wave equation in terms of $p$ and $w$. Introduce coordinates $x_0$ and $q_0$ such that $x_0^L = x_0 + q_0$ and $x_0^R = x_0 - q_0$. For the compactified string coordinate, (20) and (21) can be written as

$$X_L(\tau,\sigma) = \frac{1}{2}(x_0+q_0) + \frac{\alpha'}{2}(p+w)(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\bar{\alpha}_n}{n}e^{-in(\tau+\sigma)} \tag{33}$$

$$X_R(\tau,\sigma) = \frac{1}{2}(x_0-q_0) + \frac{\alpha'}{2}(p-w)(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n}{n}e^{-in(\tau-\sigma)}. \tag{34}$$

We can now define our usual string coordinate $X = X_L + X_R$, and in addition we can define a new string coordinate $\tilde{X} = X_L - X_R$. This will give a striking symmetry between the parameters.

$$X(\tau,\sigma) = x_0 + \alpha' p\tau + \alpha' w\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}(\bar{\alpha}_n e^{-in\sigma} + \alpha_n e^{in\sigma}) \tag{35}$$

$$\tilde{X}(\tau,\sigma) = q_0 + \alpha' w\tau + \alpha' p\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}(\bar{\alpha}_n e^{-in\sigma} - \alpha_n e^{in\sigma}) \tag{36}$$

We define the conjugate momentum density for string coordinate $\tilde{X}$ as

$$\tilde{\mathcal{P}}^\tau = \frac{1}{2\pi\alpha'}\partial_\tau \tilde{X}.$$

In the integral over $\sigma \in [0, 2\pi]$, the oscillator terms will vanish and therefore

$$\tilde{p} \equiv \int_0^{2\pi} \tilde{\mathcal{P}}^\tau d\sigma = \int_0^{2\pi} \frac{1}{2\pi\alpha'}\partial_\tau \tilde{X} = w.$$

Hence we see that the winding coordinate $w$ for the usual string coordinate $X$ is the momentum coordinate for our new string coordinate $\tilde{X}$. In addition to this we see that

$$\tilde{X}(\tau,\sigma+2\pi) - \tilde{X}(\tau.\sigma) = 2\pi\alpha' p$$

Therefore, for the $\tilde{X}$ string coordinate, the momentum of our usual sting coordinate can be thought of as the winding of the newly defined coordinate. From now on we will call the $\tilde{X}$-coordinate dual coordinate because of the duality between the momentum and winding. The string coordinates and conjugate momentum densities depend on the following parameters

$$(X, \mathcal{P}) \quad : \quad \{x_0, p, w, \bar{\alpha}_n, \alpha_n\} \tag{37}$$

$$(\tilde{X}, \tilde{\mathcal{P}}) \quad : \quad \{q_0, w, p.\bar{\alpha}_n, -\alpha_n\}. \tag{38}$$

The mass-term squared can be consequently written as

$$M^2 = p^2 + w^2 + \frac{2}{\alpha'}(N^{\perp} + \bar{N}^{\perp} - 2)$$

which, using discrete composition of $p$ and $w$ can be written as

$$M^2(R, n, m) = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N^{\perp} + \bar{N}^{\perp} - 2).$$

If one changes the radius $R$ to $\tilde{R} \equiv \alpha'/R$, the mass-spectrum changes only by switching $n$ and $m$ i.e.

$$M^2(R, n, m) = M^2(R, m, n).$$

Hence we see that the mass-spectrum is invariant under the change $R \to \alpha'/R$. Furthermore, $n$ was the quantum number corresponding to momentum $p$ and $m$ was the quantum number corresponding to winding $w$. The change of radius from $R \to \alpha'/R$ is therefore equivalent to the change $p \to w$. We have already encountered what happens if we treat momentum as winding, we simple change from our original description to the dual description

$$(X, \mathcal{P}) \to (\tilde{X}, \tilde{\mathcal{P}}).$$

The dual description $(\tilde{X}, \tilde{\mathcal{P}})$ is a perfectly equivalent description. We can therefore see that the change $R \to \alpha'/R$ on the toroidally compactified dimension results in a symmetry of the whole theory. This symmetry is known as T-duality.

It should be noted that T-duality can be generalized, firstly to toroidal compactification in more than one-dimension leading to momentum $p^{\mu}$ and winding $w^{\mu}$ parameters and furthermore the compactification can be more 'exotic' than a generalization of a torus. This last relaxation of compactification conditions will however not be treated in this thesis.

The next chapters will deal with the geometry of strings in such a way that T-duality is a natural geometrical symmetry. We will introduce a doubled manifold where the $(X, \mathcal{P})$ and $(\tilde{X}, \tilde{\mathcal{P}})$ can be seen as a specific choice of submanifolds.

# 3 Non-generalized differential Geometry

Before looking at the generalized case where one tries to incorporate T-duality and the duality between winding and momentum we want to have a look at what it is one tries to generalize. The geometry of space-time where one considers special relativity is well understood.

Spacetime geometry is based on a set of four element $(M, g, [\cdot,\cdot], \nabla)$. One starts with a basic set of points $M$, ideally this set would be a topological manifold where in addition one would provide a differential structure generated which allows us to move between coordinate charts in a smooth way. Anomalies as black holes might tell us that spacetime as we know it might not be a topological manifolds because for black holes the spacetime might not be Hausdorff. However for our purpose we assume it to be. The differentiable structure is generated by a Lie-bracket on the tangent space of the manifold. This Lie-bracket corresponds to an Lie algebra and consequently to Lie derivatives. Lie derivativives are operators acting on tensors which tell how an infinitesimall change along the flow of a vector field will change the tensor. In particular, if one can define a Lie derivative, one can generate coordinate transformations by choosing the appropriate vector fields. In addition one defines the well-known metric $g$ which gives a measure of distances between point on the manifold. In special relativity this $g$ is denoted by the coordinate independent (3,1) Minkowski metric $\eta$. In the realm of general relativity, $g$ is an coordinate dependent metric which is related to the surrounding mass in a point. Finally one would be interested what path on the manifold would result when one applies no external forces. This idea is encodes by a compatible connection $\nabla$. This connection gives us the notion of parrallel transport.

## 3.1 Vector fields and Lie algebras

Vector fields are of particular interest in this thesis because they are smooth sections of the tangent bundle. On these vector fields one can define the so-called Lie bracket and together they make a Lie algebra. In Generalized Geometry, the tangent bundle will be generalized and therefore these notions will be generalized aswell. We first start with the formal definition of a vector field [8].

**Definition 1. *Vector Field***
*Let $M$ be a manifold. A **vector field** is a smooth section $X : M \to TM$, $X : x \mapsto (x, \vec{\xi})$. We denote the space of all smooth sections of the tangent bundle $TM$ by $\Gamma(TM)$.*

One can think of a vector field as assigning to every point $x \in M$, a vector $\vec{\xi}_x$. In local coordinates one can see a vector field as follows, let $(U, \vec{x})$ be a chart on M. Then $\frac{\partial}{\partial x^i} : U \to TM|_U, p \mapsto \frac{\partial}{\partial x^i}|_p$ are local vector fields defined on U. If one can identify coordinates $(x^1, \cdots, x^n)$ globally, the partial derivatives $\frac{\partial}{\partial x^i}$ can be seen as a basis for the space of vector fields. This can equivalently be seen as follows.

**Lemma 1.** *The space of vector fields $\Gamma(TM)$ can be equivalently seen as the space of all derivations of the algebra of smooth functions on $M$ i.e. $C^\infty(M, \mathbb{R})$. These derivations are $\mathbb{R}$-linear operators $D : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$, with $D(fg) = D(f)g + fD(g)$.*

This natural identification is given by

$$X(f)(x) := X(x)(f) = df(X(x)).$$

This specific derivation will turned out to be the Lie derivative.

**Definition 2.** *(**Lie algebra**) Let $V$ be a linear space equipped with a skew-symmetric bilinear operation $[\cdot,\cdot]$ satisfying the Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

*for all $X, Y, Z \in V$. The set $(V, [\cdot,\cdot])$ is a **Lie algebra**.*

In order to define the Lie derivative we first have to define a flow line of a vector field.

**Definition 3.** *A smooth curve $c : J \to M$ for some interval $J$ is called a **integral curve** or **flow line** of a vector field $X \in \Gamma(TM)$ if $\dot{c}(t) = X(c(t))$.*

**Lemma 2.** *(ref : Natural operation in Differential Geometry) Let $X$ be a vector field on $M$. Then for any $x \in M$ there is an open interval $J_x$ containing $0$ and an integral curve $c_x : J_x \to M$ for $X$ with $c_x(0) = x$. If $J_x$ is maximal, then $c_x$ is unique.*

Hence we see that for every vector field $X$ and every point $x \in M$ we can find a unique integral curve $c_x$. The flow of a vector field is consequently defined as

$$Fl_t^X(x) = Fl^X(t,x) = c_x(t).$$

The Lie derivative on functions is defined as

$$\mathcal{L}_X f := \frac{d}{dt}|_0 (Fl_t^X)^* f = \frac{d}{dt}|_0 (f \circ Fl_t^X)$$

where, $(Fl_t^X)^*$ denotes the pullback. In particular we have that $\mathcal{L}_X f = X(f) = df(X)$. Analogous to the above, the Lie derivative of vector fields can be written as

$$\mathcal{L}_X Y = \frac{d}{dt}|_0 (Fl_t^X)^* Y.$$

For vector fields one can write $\mathcal{L}_X Y = [X,Y]$. Intuitively, the Lie-derivative along $X$ is the infinitesimal change of a tensor fields along the flow of $X$.

Additionally to this geometry one can define extra structure. It is clear that for a physical model one needs extra structure since above no interaction between patricles is determined. When we talk about interactions, we want to introduce some algebraic notion.

## 3.2 Symplectic structure and Poisson geometry

Our theory of strings depends heavily on the canonical quantization procedure. For canonical quantization one starts with a Poisson manifolds such that one can define Poisson brackets on the manifold. Subsequently we turn our dynamical variables into operators on Hilbert spaces, and our Poisson brackets are turned into commutators [2], [18].

**Definition 4.** ***Poisson structure***
*Let $M$ be a manifold and denote the space of continuous functions on this manifold by $C^\infty(M)$. One defines a Poisson structure on this space of continuous functions as follows, define a bracket $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ such that it satisfies the following properties*

1. ***Skew-symmetry*** $\{f, g\} = -\{g, f\}$

2. ***bi-linearity*** $\{f, ag + bh\} = a\{f, g\} + b\{f, h\}$ *for all* $a, b \in \mathbb{R}$

3. ***Jacobi identity*** $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

4. ***Leibniz identity*** $\{f, g \cdot h\} = g \cdot \{f, h\} + \{f, g\} \cdot h$

The pair $(M, \{\cdot,\cdot\})$ is called a Poisson manifold. This bracket is the Lie-algebra over smooth functions on the manifold.

Instead of looking at Poisson structures one can look for a symplectic structure this symplectic structure then defines a poisson structure.

**Definition 5.** ***Symplectic 2-form***
*Let $M$ be a manifolds and $\omega$ an closed 2-form so that for each $p \in M$, $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is bilinear and skew-symmetric, and $d\omega = 0$. Furthermore let $\tilde{\omega} : T_pM \to T_p^*M$ be the map defined by $\tilde{\omega}_p(v)(u) = \omega_p(u, v)$ if for each $p$ this map is bijective, we say that $\omega$ is nondegenerate. A nondegenerate closed 2-form is called a symplectic form. The pair $(M, \omega)$ is called a symplectic manifold.*

One can define a so-called Hamiltonian vector fied $X_f$ on the symplectic manifold as follows

$$i_{X_f}\omega = -df$$

this will result in a Poisson structure as follows

$$\{f, g\} = \omega(X_f, X_g).$$

In rest of this report we are only interested in the symplectic case.

### 3.2.1 Phase space

Phase space is the set of all possible states of a certain system. In classical mechanics the state of a system is characterized by its position and momentum. Lets assume a manifold smooth $n$-dimensional manifold $M$ in which one can introduce coordinates $x^\mu$. The conjugate momenta $p_\mu$ to these position coordinates can be seen as one-forms living in the cotangent plane at a specific point. The total phase space in this setting can be regarded as the cotangent bundle $T^*M$ of the manifold $M$. The cotangent bundle is defined as

$$T^*M = \{(x,p) \quad | \quad x \in M, \quad p \in T_x^*M\}.$$

Locally one can introduce coordinate functions $(x^1, \cdots, x^n, p_1, \cdots, p_n)$ such that the coordinates $x^\mu$ are the corresponding coordinates on the manifold $M$. At a specific point $x = x(x^1, \cdots, x^n)$, the set $(dx^1, \cdots, dx^n)$ forms a basis for the tangent plane $T_xM$ and coordinates $p^\mu$ the coordinates with respect to this basis such that for $p \in T_xM$, $p = p_\mu dx^\mu$ where summation convention is implied. This phase space has natural doubled dimension. On the cotangent bundle one can naturally define a symplectic form as follows

$$\omega = \sum_{i=1}^{n} dx^i \wedge dp_i.$$

This 2-form can also be seen as $\omega = -d\alpha$ where $\alpha$ is defined as

$$\alpha = \sum_{i=1}^{n} p_i dx^i.$$

This $\alpha$ is called the tautological form.

## 3.3 Complex structure

**Definition 6.** ***(Complex structure)*** *Let $M$ be a real differentiable manifold. One can define a tensor field $J$ such that at every point $x \in M$, $J$ is an endomorphism of the tangent space $T_xM$ and furhtermore $J^2 = -Id$. In this sense $J$ is called an almost complex structure. The set $(M, J)$ is called an almost complex manifold.*

An almost complex structure is defined in a coordinate dpendent way locally. If one can define a complex structure in global coordinates on the tangent bundle $TM$ we say that $J$ is a complex structure for the manifold $M$ and the manifold is called complex [22].

With this definition of an almost complex structure one can transform for every $x \in M$, the real vector space $T_xM$ into a complex vector space. Let $X \in T_xM$ be a tangent vector. The scalar multiplication can be defined as follows

$$(a+ib)X = aX + bJX, \quad a,b \in \mathbb{R}.$$

A natural consequence is that an manifold with almost complex structure has to be of even real dimension since the dimension of the complex tangent space can be seen as an even real dimensional space.

## 3.4 Product structure and para-complex manifolds

One type of structure particularly interesting in this report is a product structure, more specifically a para-complex structure [23].

**Definition 7.** ***(Almost product structure)***
*Let $M$ be a manifold and $K$ an tensor field on $M$ with the same properties as the complex structure $J$ defined previously but with the condition $K^2 = Id$. We call $K$ a product structure and $(M, K)$ a product manifold.*

**Definition 8.** ***(Almost para-complex Manifold)*** *Let $(M, K)$ be a product manifold. If the two eigenbundles $T^+M, T^-M$, defined by the +1 and -1 eigenspaces of $K_x$ respectively have the same rank. This means that in every fiber $(T_x^{\pm}M)$, the dimensions of the $\pm 1$ eigenspaces are the same.*

One can reverse the definition above and take the fact that one can split the tangent bundle in two sub bundles $T^{\pm}M$ of same rank as the defining property of an almost para-complex structure.

## 3.5 G - structures

**Definition 9.** ***(Tangent frame bundle)***
*Let $TM$ be the tangent bundle of a manifold $M$ with dimension $m$. The* ***tangent frame bundle*** *$FM$ is defined as a bundle over $M$ such that each fiber in a point $p \in M$ is the set of ordered bases for the tangent space $T_pM$.*

Locally in a chart $U_\alpha$ of the manifold, the frame bundle $FM$ is given by $(p, \{e_a\})$ such that $a = 1, \cdots, dim(M)$. The basis vectors $e_a$ can be written as $e_a = e^i_a \frac{\partial}{\partial x^i}|_p$. This description is called a *local trivialization* [10].

Now consider two different charts $U_\alpha$ and $U_\beta$ with local trivialization $(p, e_a)$ and $(p', e'_a)$. On the intersection $U = U_\alpha \cap U_\beta$, every vector of the tangent bundle is related by a change of coordinates. Specifically, the basis vectors are related as

$$e'^i_a = \frac{\partial x'^i}{\partial x^j} e^j_a.$$

Furthermore, for every pair of vectors, in particular basis vectors $(e_a, e_b)$, these vectors are related by an operator $(A_{\beta\alpha})^b_a \in GL(m, \mathbb{R})$ such that

$$e^i_a = e^i_b (A_{\beta\alpha})^b_a.$$

Therefore,

$$e'^i_a = e^i_b (S A_{\beta\alpha} S^{-1})^b_a$$

where, $S$ denotes the change of coordinates and $A_{\beta\alpha}$ denotes the change of basis vector. We can take this matrices together to get the transition function $t_{\beta\alpha}(p)$ such that

$$e'^i_a = e^i_b (t_{\beta\alpha})^b_a.$$

These transition functions $t_{\beta\alpha}$ form a group called the **structure group**. The structure group encodes the change of the frame bundle of a specific manifold.

**Definition 10.** *A manifold $M$ with dimension $m$, has a G-structure with $G \subset GL(m, \mathbb{R})$ if it is possible to reduce the tangent frame bundle such that is has structure group $G$.*

This reducing of the structure group is due to the extra structures that are defined on the manifold and tangent bundle. Examples of extra structures are an almost complex structure, an almost product structure etc. But also a metric defined on the manifold reduces the structure group, because the transition functions on the manifold with metric should leave the metric invariant. Therefore with defining a metric, the structure group can be reduced to $O(m, \mathbb{R})$.

# 4 Generalized geometry

The existence of winding modes in string theory and the T-duality that connects winding to momentum leads to suggest that in the fundamental geometry of space time should be doubled and T-duality should interplay in this doubling. Two seemingly different approaches to this are proposed. On the one hand we have the generalized geometry introduced by Hitchin and Gualtieri [7] [5], which is based on the doubling of the tangent bundle in a formal sum of the tangent and the co-tangent bundle $TM \oplus T^*M$. On the other hand we have double field theory,introduced by Zwiebach et al [4] [16] [15]. Double field theory is, while inspired by generalized geometry, different in the sense that is doubles the underlying manifold where one uses constraints on the manifold to make physical sense. Very recent work by the likes of Friedel, Rudolph and Svoboda is about combining these two different approaches and makes use of the notion of para-Hermitian manifolds [13] [11] [20].

## 4.1 Bilinear product on the generalized tangent bundle

As mentioned before in generalized geometry one is interested in the formal sum of tangent and cotangent bundles. Sections of this generalized bundle $\mathbb{T}M$ are sums of vector fields and one-forms $X + \xi$ with $X \in \Gamma(TM)$, $\xi \in \Gamma(T^*M)$. With these pairs one can define a very natural symmetric bilinear form

$$(X + \xi, Y + \eta) = \frac{1}{2}(\xi(Y) + \eta(X)).$$

This symmetric inner product has signature $(m, m)$. This means that the metric $\eta_{ij}$ induced by this inner product has $m$ positive and $m$ negative eigenvalues. By defining this inner product, the transition functions of the structure group need to leave the metric invariant. Therefore, the structure group reduces to $O(m, m)$.

On the generalized tangent bundle one can introduce a natural volume form i.e. a differential form of degree $2m$. The highest exterior power of the generalized tangent bundle can be split as follows, for $x \in M$,

$$\Lambda^{2m}(T_xM \oplus T_x^*M) = \Lambda^m(T_xM) \otimes \Lambda^m(T_x^*M).$$

Hence it will be split as a product of multi-vectors and multi-dual-vectors. Let $u = u_1 \wedge \cdots \wedge u_m \in \Lambda^m(T_xM)$ and $v^* = v_1^* \wedge \cdots \wedge v_m^* \in \Lambda^m(T_x^*M)$, one then defines the volume form $(u, v^*)$ as

$$(u, v^*) = det(v_i^*(u_j)).$$

If this determinant is positive we have a positive orientation and if it is negative it defines a negative orientation.

The structure group reduces to $SO(m, m)$. The Lie-algebra of the special orthogonal group is determined in the usual way

$$so(T \oplus T^*) = \{ \quad F \in End(T \oplus T^*) \quad | < Fx, y > + < x, Fy >= 0 \quad \forall x, y \in T \oplus T^* \}$$

where, $T$ and $T^*$ denote tangent spaces on an arbitrary point $x \in M$. The corresponding transformation in this Lie-algebra is determined as follows (proof ?)

$$F = \begin{bmatrix} A & \beta \\ B & -A^t \end{bmatrix}.$$

Where, $A \in End(T)$. The transformations $B : T \to T^*$ and $\beta : T^* \to T$ are skew-symmetric and therefore can be regarded as a 2-form and a bi-vector respectively. One can therefore conclude that

$$so(T \oplus T^*) = End(T) \oplus \Lambda^2(T^*) \oplus \Lambda^2(T).$$

The so-called $B$-field transformation is of particular interest. First of all it resembles the Kalb-Ramond field found in string theory whereas, the endomorphisms of the tangent bundle resemble gravity fields. Secondly, as we will see in the next section, the $B$-field transformation is a symmetry of the Courant bracket.

**Example 1.** *(B-**field**) We can see the B-field action as a total action on the generalized tangent bundle as*

$$\tilde{B} = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}.$$

*The corresponding $SO(m, m)$ action therefore is the exponential*

$$e^{\tilde{B}} = \begin{pmatrix} Id & 0 \\ B & Id \end{pmatrix}.$$

*On an arbitrary generalized vector $X + \xi$, this action can be seen as*

$$e^{\tilde{B}} : X + \xi \mapsto X + \xi + i_X B$$

*where, $i_X B = B(X, \cdot)$ is the interior derivative.*

## 4.2 The Courant bracket

The bilinear form on the generalized tangent space generalized gives the notion of a metric on the underlying manifold. In this section we will look at how to generalize Lie-brackets. The natural generalization of the Lie-bracket is the Courant-bracket defined as follows

**Definition 11.** ***(Courant bracket)***
*Let $X + \xi, Y + \eta \in \Gamma(TM \oplus T^*M)$. The Courant bracket is a skew-symmetric bracket defined as follows*

$$[X + \xi, Y + \eta]_C = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi)$$

*where, $[X, Y]$ is the Lie-bracket on the tangent bundle $TM$ and $\mathcal{L}$ defines the corresponding Lie-derivative.*

When restricted to vector fields, the Courant bracket reduced to the usual Lie bracket. In other words, if $\pi : TM \oplus T * M \to T$ is the natural projections, then

$$\pi([x, y]_C) = [\pi(x), \pi(y)].$$

Furthermore one can see that the Courant bracket over 1-forms vanishes.

The Courant brackets fails to satisfy the Jacobi identity and therefore the generalized tangent bundle together with the bracket is not an Lie algebra.

### 4.2.1 Symmetries of the Courant bracket

The Lie bracket is invariant under diffeomorphisms and in fact this is the only invariance a Lie-bracket has. When one looks at the Courant bracket a striking result shows up, it is also invariant under a so called B-field transformation. This B-field transformation corresponds to the Lie algebra action of $\Lambda^2 T^*$. This B-field transformation corresponds to an 2-from and furthermore it needs to be closed.

**Proposition 1.** *Let $B$ be an B-field transformation. The Courant bracket is invariant under $e^B$ as defined previously if and only if $B$ is closed.*

**Proposition 2.** *Let $M$ be a smooth manifolds. Let $(f, F)$ such that $f : M \to M$ and $F : TM \oplus T^*M$ be automorphisms preserving the inner product. Suppose furthermore that $F$ preserves the Courant brackiet i.e. $F([x,y]_C) = [F(x), F(y)]_C$ for all $x, y \in \Gamma(TM \oplus T^*M)$. Then $F$ must be a composition of a diffeomorphism and a closed B-field transformation.*

Proofs for both proposition can be found in [5].

## 4.3 Generalized structures

We now want to generalize structures that have been defined in chapter 3. In particular one can define a generalized complex structure that generalizes the notion of a complex and a symplectic structure.

**Definition 12.** *A generalized complex structure on a linear vector space $T$ is and endomorphism $\mathcal{J}$ of $T \oplus T^*$ such that it is complex $\mathcal{J}^2 = -Id$ and it is symplectic $\mathcal{J}^* = -\mathcal{J}$.*

**Proposition 3.** *An equivalent description for a generalized complex structure is that it is a complex structure on $T \oplus T^*$ which is orthogonal with respect to the inner product defined on $T \oplus T^*$.*

PROOF. If $\mathcal{J}^2 = -Id$ and $\mathcal{J}^* = -\mathcal{J}$ this implies that $\mathcal{J}^*\mathcal{J} = Id$ hence $\mathcal{J}$ is orthogonal. Conversely, if $\mathcal{J}^2 = -Id$ and $\mathcal{J}^*\mathcal{J} = Id$, then $-\mathcal{J}^* = \mathcal{J}^*\mathcal{J}^2 = \mathcal{J}$. Hence it is symplectic. □

One can find generalized complex structures that are induced from the complex or symplectic structure for the vector space $T$. Indeed, consider $\omega$ to be a symplectic structure and $J$ to be a complex structure on $T$. One can then consider

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}.$$

This defines a generalized complex structure. Furthermore one can define

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

This also defines a generalized complex structure.

Subsequently one can also define a generalized product structure $K$.

**Definition 13.** *A generalized product structure $K$ is an endomorphism on $T \oplus T^*$ such that*

$$\langle K(X+\xi), K(Y+\eta)\rangle = -\langle X+\xi, Y+\eta\rangle.$$

# 5 Double Field Theory and Metastring theory

One of the most important properties of closed strings is that the massless sector consists of gravitational fields, Kalb-ramond fields and a dilaton scalar field. In the string theory section about T-duality, we saw that when one does compactification on toroidal backgrounds, a new type of variable arise from the winding of the string. Furthermore we saw that because of T-duality, one needs to treat these winding modes on equal footing with momentum. Double Field theory tries to incorporate all these properties into one field theory. The main idea is to introduce extra coordinates conjugate to the winding modes.

In metastring theory the target space is also doubled but the dual coordinates are related to energy and momentum coordinates. This effectively changes our basic space-time to a phase-space formulation [12].

## 5.1 Double Field Theory

### 5.1.1 Doubled coordinates

Lets assume a $D$-dimensional manifold $M$ with $n$ local non-compact coordinates $X^\mu$ and $d$ compact coordinates $y^m$ on a toroidal background $T^d$. Hence effecitvely one could write is as $\mathcal{R}^{n-1,1} \times T^d$ where, $R^{n-1,1}$ denotes the Minkowski spacetime for n spatial and 1 time dimension. In double field theory one doubles the space of coordinates by introducing coordinates $\tilde{x}_\mu$ and $\tilde{y}_m$. The total space of local coordinates would then be denoted by $X^m = (\tilde{x}_\mu, \tilde{y}_m, x^\mu, y^m)$. The coordinates that are important for T-duality and winding modes are the compactified doubled coordinates $Y^A = (\tilde{y}_m, y^m)$. The coordinates $\tilde{x}_\mu$ would physically make no sense because only when one compactifies dimension, dual winding modes show up.

### 5.1.2 Action and the generalized metric

We are looking for a description of our space-time where the gravity-field $g_{ij}$, Kalb-Ramond field $b_{ij}$ and dilaton field $\phi$ are manifest. An action that incorporates these terms is well known in the literature written as [16]

$$S = \int d^D x \sqrt{-g} e^{-\phi} \left[ R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right]$$

where, $R$ is the scalar curvature with respect to the gravitational field and $H$ a three-form determining the field strength of the Kalb-Ramond field.

In the quantum field theory of the bosonic string upon toroidal compactification, we defined $p = \frac{n}{R}$ and $w = \frac{mR}{\alpha'}$. T-duality will interchange these variables if we change $R \to \frac{1}{R}$. We can see T-duality as an $O(D, D, \mathbb{Z})$ symmetry on the coordinates $(p^\mu, w_\nu)$. The fact that we have an integral symmetry is really a consequence of quantization. We are interested in the classical picture and therefore our T-duality symmetry becomes the $O(D, D, \mathbb{R})$ group. This group has neutral metric (neutral meaning signature (m,m)) $\eta$ defined as

$$\eta^{NM} = \begin{bmatrix} 0 & Id \\ Id & 0 \end{bmatrix}.$$

Additionally to this neutral metric $\eta$, one defines a generalized metric $\mathcal{H}$ that generalized the space-time dynamics of the Kalb-Ramond and gravity fields.

$$\mathcal{H}_{MN} = \begin{bmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{bmatrix}$$

where $B$ and $G$ are the tensors encoding the gravitational field and Kalb-Ramond field respectively [16]. All field dynamics is encoded by this matrix.

Effectively our fields only depend on the real space-time coordinates so in the setting it would depend only on half of the coordinates. One can impose two related constraints. The first constraint, the weak version demands that for all fields $\phi$,

$$\partial^M \partial_M \phi = 0$$

The strong constraint adds that this weak constraint must hold for all products of fields i.e.

$$\partial^M \partial_M \phi_1 \phi_2 = 0.$$

## 5.2 Metastring theory

Metastring theory can be seen as an extension of Double Field theory. The target-space for a worldsheet formulation of the string is phase-space instead of physical space-time. In this sense the manifold we started with also gets doubled but these double coordinates can be thought of as conjugate coordinates. We start by determining a one-form $P_\mu$ such that when integrated on a closed loop $C$, we get the corresponding momentum.

$$\int_C P_\mu = 2\pi p_\mu.$$

This one-form $P_\mu$ is assumed to be exact and therefore can be written as $P_\mu = dY_\mu$ for some coordinate functions $Y_\mu$. The integral over the loop $C$ then just caused the quasi-period of $Y_\mu$

$$Y_\mu(\tau, \sigma + 2\pi n) = Y_\mu(\tau, \sigma) + 2\pi p_\mu n.$$

This construction really resembles the quasi-period of our usual string coordinates $X_\mu$ with winding replaced by momentum. The $Y_\mu$'s can now be regarded as individual momentum coordinates.

With the introduction of the coordinates $Y$, we can now setup the phase space manifold as follows, first we introduce energy and length scale parameters $\varepsilon$ and $\lambda$ respectively such that

$$\hbar = \lambda\varepsilon, \quad \alpha' = \frac{\lambda}{\varepsilon}.$$

On the phase-space manifold, the coordinates can locally be written as

$$\mathbb{X}^A = \begin{pmatrix} X^\mu/\lambda \\ Y_\mu/\epsilon \end{pmatrix}.$$

On this space we define an extension of our Lorentzian metric $H^0$, a neutral metric which encodes the splitting of coordinates $\eta^0$ and a symplectic structure $\omega^0$ as follows

$$\eta^0 = \begin{pmatrix} 0 & Id \\ Id & 0 \end{pmatrix}, \quad H^0 = \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}, \quad \omega^0 = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}.$$

An action can be formulated in these terms [12]

$$S = \frac{1}{4\pi} \int \partial_\tau \mathbb{X}^A (\eta^0_{AB} + \omega^0_{AB}) \partial_\sigma \mathbb{X}^B - \partial_\sigma \mathbb{X}^A H^0_{AB} \partial_\sigma \mathbb{X}^B.$$

These structures introduced can be seen as flat since they do not depend on specific point on the manifold. It should be noted that for a full geometrical picture of space-time these structure might need to become dynamical. This is obvious for the metric $H^0$, if we consider general relativity this must become dynamical. One can transform the metric $H^0$ into the generalized metric $H$ of Double Field theory by considering gravity and B-field action. Indeed let $G = e^T h e$ where $e$ is called the frame field of $G$. A quick calculation the shows that $H = O^T H^0 O$ where,

$$O^T = \begin{pmatrix} Id & B \\ 0 & Id \end{pmatrix} \begin{pmatrix} e^T & 0 \\ 0 & e^{-1} \end{pmatrix}.$$

From now on we will assume that every structure might be dynamical and we denote it by $(\eta, \omega, H)$.

### 5.2.1 Bi-Lagrangian structure

The neutral metric $\eta$ is used to split the phase-space manifold into space-time and into momentum space. We can think of space time as a Lagrangian submanifold $L$.

**Definition 14.** ***Lagrangian submanifold***
*Let $(M, \omega)$ be a symplectic manifold. A Lagrangian submanifold is defined to be a submanifold such that $\omega|L = 0$ and furthermore the dimension of $L$ is half the dimension of $M$.*

Conversely the momentum space of the phase space should also be a Lagrangian submanifold $\tilde{L}$ and such that is transverses the space-time manifold. We thus end up with a bi-Lagrangian structure which has the following properties

$$T\mathcal{P} = TL \oplus T\tilde{L}, \quad TL \cap T\tilde{L} = \{0\}.$$

We will see later on that this structure defines a para-Kähler manifold and allows us to translate double field theory and metastring theory into the language of Generalized Geometry.

#### 5.2.2 T-duality

In this section we will look at a formulation of T-duality specific for the metastring formulation. This T-duality can arrise from a symmetry of a kind of structure $J$. This structure effectively changes coordinates to their dual coordinates. It should be noted that under T-duality the physics remains the same in particular it should leave the metrics invariant i.e. $J^T \eta J = \eta$ and $J^T H J = H$. One can indeed define such a chiral structure as follows take

$$J \equiv \eta^{-1} H.$$

Lets look at the flat case. In this case $\eta^0 = (\eta^0)^-1$ and therefore

$$J_0 \equiv \eta^0 H^0 = \begin{pmatrix} 0 & h^-1 \\ h & 0 \end{pmatrix}.$$

One can see the Lorentzian metric as a map

$$\begin{aligned} h &: T \to T^* \\ x &\mapsto h(x, \cdot) \end{aligned}$$

for some vector space $T$. This map is non-degenerate because $h$ is a non-degenerate metric. Therefore we can conclude that indeed $J_0$ maps coordinates to their duals. Furthermore we can see that $J_0^T \omega_0 J_0 = -\omega_0$. This chiral structure $J$ also needs to be compatible with our bi-Lagrangian structure that can be encoded by a real structure $K$. This real structure determines the splitting of the phase space manifold into Lagrangian submanifolds that can be identified as space-time and energy-momentum space. The compatibility property can be explained as follows. The chiral structure should map one Lagrangian submanifold $L$ to the other one $\tilde{L}$. The Lagrangian submanifolds are determined by the eigenvalues of the real structure $K$. The Lagrangian submanifold $L$ corresponds to eigenvalue $+1$ and

$\tilde{L}$ corresponds to -1. This results in the fact that $K$ and $J$ should anticommute. Indeed let $X \in L$ then $KX = X$ and $JX \in \tilde{L}$ therefore

$$KJX = -JX = -JKX.$$

Similarly let $Y \in \tilde{L}$ then $KY = -Y$ and $JY \in L$

$$KJY = JY = -JKY.$$

Therefore we can conclude that $KJ + JK = 0$.

# 6 Bridge between Generalized Geometry and Double Field Theory

Double Field theory looks very promising and is becoming more and more a well understood topic. The resemblance with Generalized geometry is easily seen but there is still no rigorous formulation of DFT. Recent work has shown that the use of para-Hermitian and para-Kähler manifolds might be the solution. This approach has been worked by Freidel, Rudolph and Svoboda [13] [20]. We will start this section by first introducing para-Hermitian and para-Kähler manifolds and see how it bridges the gap between Generalized Geometry and DFT.

## 6.1 Para-Hermitian and para-Kähler manifolds

**Definition 15.** *Let $\mathcal{P}$ be a manifold equipped with a neutral metric $\eta$ and real structure $K \in End(T\mathcal{P}$ such that*

$$K^2 = +Id, \quad K^T \eta K = -\eta$$

*the triple $(\mathcal{P}, \eta, K)$ is called an almost para-Hermitian manifold.*

**Example 2.** *Let $M$ be any manifold. The generalized tangent bundle $TM \oplus T^*M$ is equipped with a natural para-Hermitian structure given by*

$$K = \begin{bmatrix} Id_T & 0 \\ 0 & -Id_{T^*} \end{bmatrix}, \quad \eta(X_1 + \xi_1, X_2 + \xi_2) = \xi_1(X_2) + \xi_2(X_1).$$

*Ineed one verifies very quickly that $K^2 = Id_{T \oplus T^*}$ and*

$$\eta(K(X_1+\xi_1), K(X_2+\xi_2)) = \eta(X_1-\xi_1, X_2-\xi_2) = -(\xi_1(X_2)+\xi_2(X_1)) = -\eta(X_1+\xi_1, X_2+\xi_2).$$

Since $K$ is a real structure, it has eigenbundles corresponding to eigenvalues $\pm 1$. Consequently the tangent bundle splits as $T\mathcal{P} = L \oplus \tilde{L}$. The second condition of the almost para-Hermitian structure implies that $L$ is maximal isotropic with respect to $\eta$. Indeed, let $x, y \in L$, we see that

$$\eta(Kx, Ky) = \eta(x, y) = -\eta(x, y)$$

and therefore $\eta(x, y) = 0$. The same argument can be made for $\tilde{L}$. Since $\eta$ has signature $(n, n)$, both $L$ and $\tilde{L}$ have the same rank. Furthermore one can define an almost symplectic structure $\omega = \eta K$.This indeed defines a non-degenerate 2-form since both $K$ and $\eta$ are non-degenerate and furthermore

$$\omega(x, x) = \eta(Kx, x) = \pm\eta(x, x) = 0$$

.

The almost para-Hermitian manifold can equivalently be characterized by the triplet $(\mathcal{P}, \eta, \omega)$ such that $\omega = \eta K$. Therefore it can be seen as an almost symplectic manifold with a compatible neutral metric such that there exists an real structure. Considering both $\eta$ with signature $(n, n)$ and an almost symplectic form $\omega$ will reduce the structure group of the tangent bundle from $GL(2n, \mathbb{R})$ to $O(n, n) \cap Sp(2n, \mathbb{R})$. Furthermore, if this symplectic 2-form $\omega$ is closed, we way that the manifold is almost para-Kähler.

One can define natural projections onto $L$ and $\tilde{L}$ as follows

$$P := \frac{1}{2}(Id + K), \quad \tilde{P} := \frac{1}{2}(Id - K).$$

To go from an almost structure to the final structure dropping the 'almost', one need integrability. For para-Hermitian structures, integrability of the $L$ and $\tilde{L}$ are not related to each other. Integrability is related to the Lie-bracket defined on the vector fields. In particular, a real structure $K$ is integrable if the real analogue of the Nijenhuis tensor vanishes $N_K(X, Y) = 0$ for all $X, Y \in T\mathcal{P}$ where,

$$4N_K(X, Y) := [KX, KY] + [X, Y] - K([KX, Y] + [X, KY]).$$

We can rewrite this tensor as $N_K(X, Y) = N_P(X, Y) + N_{\tilde{P}}(X, Y)$ where,

$$N_P(X, Y) := \tilde{P}[PX, PY], \quad N_{\tilde{P}}(X, Y) := P[\tilde{P}X, \tilde{P}Y].$$

Note that $N_P \in \tilde{L}$ and the vanishing of $N_P$ determines the integrability of $L$. The same argument goes for $N_{\tilde{P}}$.

**Definition 16.** *Let $(\mathcal{P}, \eta, \omega)$ be an almost para-Hermitian manifold. It is said to be L-para-Hermitian, dropping the almost, if L is integrable. Furthermore, we say that the almost para-Hermitian manifold is almost L-para-Kähler if*

$$d\omega(PX, PY, PZ) = 0, \quad d\omega(PX, \tilde{P}Y, \tilde{P}Z) = 0.$$

*If the manifold is both L-para-Hermitian and almost L-para-Kähler, it is said to be L-para-Kähler.*

In the definition above one can switch $L$ with $\tilde{L}$ and consequently switch $P$ and $\tilde{P}$ to get the definition for an $\tilde{L}$-para-Hermitian manifold.

It is instructive now to the connection with the Double Field theory setting. Indeed one can think of $L$ as being tangent to the normal coordinates $x^\mu$ whereas $\tilde{L}$ would be seen as tangent to the dual coordinates $\tilde{x}_\alpha$. We will make this clear by looking at foliations of our manifold [9].

**Definition 17.** *Let $M$ be a smooth manifold of dimension $n$. A foliation atlas of codimension $0 \leq q \leq n$ of $M$ is an atlas*

$$(\vec{x}_i : U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q)_{i \in I}$$

*where, the transition diffeomorphisms from one chart to another are locally of the form*

$$\phi_{ij} : \mathbb{R}^{n-q} \times \mathbb{R}^q \to \mathbb{R}^{n-q} \times \mathbb{R}^q \phi_{ij}(x, y) = (g_{ij}(x, y), h_i j(x)).$$

*A foliation of codimension $q$ of $M$ is a maximal foliation atlas of $M$.*

Globally this foliation would split the manifold $M$ into leaves which are smooth manifolds of dimension $n - q$.

Another equivalent definition of a foliation would be by means of an integrable subbundle of the tangent bundle $TM$. Now assume we have a $(L, \tilde{L})$-para-Hermitian structure. This means that both the $L$ and $\tilde{L}$ subbundles of the tangent space are integrable. Therefore there exist two folations $\mathcal{F}$ and $\tilde{\mathcal{F}}$ such that $T\mathcal{F} = L$ and $T\tilde{\mathcal{F}} = \tilde{L}$ [9] [13]. The two maximal isotropic bundle together are the tangent bundle $TM = T\mathcal{F} \oplus T\tilde{\mathcal{F}}$. Locally one can introduce coordinates $(x^a, \tilde{x}_a)$ and basis $\partial_a$, $\tilde{\partial}^a$ for the corresponding vector fields such that $K(\partial_a) = \partial_a$ and $K(\tilde{\partial}^a) = -\tilde{\partial}^a$. In these local coordinates, the leaves of $\mathcal{F}$ are given by $\tilde{x}_a = const$ and the leaves of $\tilde{\mathcal{F}}$ are given by $x^a = const$.

## 6.2 Connection Double Field theory and Generalized Geometry

We have shown that para-Hermitian manifolds are the right setting for Double field theory and metastring theory. It also allows us to see the connection between generalized geometry and double field theory. This will be done in this paragraph.

**Lemma 3.** *Let $(\mathcal{P}, \eta, K)$ be an almost para-Hermitian manifold. Also denote $T^*_\pm$ be the dual vectors to the eigenbundles $T_\pm$. The bundles $T_\pm$ and $T^*_\mp$ are isomorphic via $\eta$ which can be seen as*

$$\eta : T_\pm \to T^*_\mp$$
$$x_\pm \mapsto \eta(x_\pm, \cdot)$$

PROOF. The map $x_\pm \mapsto \eta(x_\pm, \cdot)$ is linear since $\eta$ is induced by a bilinear product. Now first look at the map

$$\eta : T\mathcal{P} \to T^*\mathcal{P}.$$

This is a non-degenerate linear map. The cotangent bundle is also split $T^*\mathcal{P} = T_+^* \oplus T_-^*$. Now without loss of generality, take an arbitrary 1-form $\xi \in T_\mp^*$. There exists $x \in T\mathcal{P}$ such that $x = x_+ + x_-$ and $\eta(x, \cdot) = \xi$. Now take an arbitrary $y_- \in T_-$ then,

$$\xi(y_-) = \eta(x_+ + x_-, y_-) = \eta(x_+, y_-) + \eta(x_-, y_-).$$

Since $T_-$ is maximal isotropic , $\eta(x_-, y_-) = 0$. Therefore we have that for all $y_- \in T_-$, $\xi(y_-) = \eta(x_+, y_-)$ hence , for all $\xi \in T_-^*$, there exists $x_+ \in T_+$ such that $\xi = \eta(x_+, \cdot)$. With this we can conclude that the map

$$\eta : T_\pm \to T_\mp^* x_\pm \mapsto \eta(x_\pm, \cdot)$$

is indeed an isomorphism. □

Now recall that for the integrable eigenbundles $T_\pm$ there exist foliations $\mathcal{F}_\pm$ such that $T_\pm = T\mathcal{F}_\pm$. Now we can define the following maps which are by the lemma above isomorphisms

$$\rho_\pm : T_+ \oplus T_\to T_\pm \oplus T_\pm^* X = x_+ + x_- \mapsto x_\pm + \eta(x_\mp).$$

If $T_+$ is integrable then, $T_+ \oplus T_+^* \cong T\mathcal{F} \oplus T^*\mathcal{F}_+$. Hence $\rho_+$ is the isomorphism

$$\rho_+ : T\mathcal{P} \to T\mathcal{F}_+ \oplus T^*\mathcal{F}_+.$$

A similar argument can be made for $T_-$ if it is integrable.

We started with a manifold $\mathcal{P}$ which was motivated by double field theory and metastring theory, in other words we have doubled dimension. If one defines on this manifold a suitable pair $(\eta, K)$ which is a metric and a real structure we get a para-Hermitian manifold. This real structure will decompose the tangent bundle $T\mathcal{P} = T_+ \oplus T_-$ into the eigenbundles $T_\pm$. Integrability of one of these eigenbundles corresponds to the fact that one can make a foliation $\mathcal{F}_\pm$ with respect to the eigenbundles such that $T\mathcal{F} = T_\pm$. Furthermore we showed that $T_\pm \cong T_\mp^*$ and therefore $T\mathcal{P} \cong T\mathcal{F}_\pm \oplus T^*\mathcal{F}_\pm$. We have established the relation between double field theory/metastring theory on the one hand and generalized geometry on the other.

# 7 Conclusion

At the end of chapter 2 we arrived at two seemingly different descriptions of the geometry of strings (37),(38). The target-space geometry is determined by the following set of parameters, leaving out the ghost oscillators

$$(X,\mathcal{P}) : \{x_0^\mu, p_\alpha, w^\nu\}$$
$$(\tilde{X},\mathcal{P}) : \{q_0^\mu, w_\alpha, p^\nu\}$$

With the connection to the last chapter we can think of $(x_0^\mu, q_0^\nu) \in M$ hence these coordinates live in the doubled manifold. If this manifold is para-Hermitian, it is split into foliations $\mathcal{F}, \tilde{\mathcal{F}}$ where, $x_0^\mu \in \mathcal{F}$ and $q_0^\mu \in \tilde{\mathcal{F}}$. The tangent bundle of $M$ is isomorphic to the generalized tangent bundles of $\mathcal{F}$ and $\tilde{\mathcal{F}}$. The momentum and winding coordinates can be seen as living in $TM$ or equivalently living in the generalized tangent bundles. We can therefore see the following identification of the two descriptions

$$(X,\tilde{X},\mathcal{P},\tilde{P}) \quad : \quad (M, TM) \tag{39}$$
$$(X,\mathcal{P}) \quad : \quad (\mathcal{F}, T\mathcal{F} \oplus T^*\mathcal{F}) \tag{40}$$
$$(\tilde{X},\tilde{\mathcal{P}}) \quad : \quad (\tilde{\mathcal{F}}, T\tilde{\mathcal{F}} \oplus T^*\tilde{\mathcal{F}}). \tag{41}$$

The description (39) is described in great detail by Double Field Theory and Metastring theory. The other descriptions (40),(41) are described by generalized geometry. We therefore have established the relation between the geometry starting from strings with compactified dimensions with Generalized Geometry.

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