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Knot Theory, The Jones Polynomial and Chern-Simons Theory

Bachelor’s Project Mathematics and Physics

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Abstract

In this bachelor thesis we first give a general overview of Knot Theory. In particular we describe the Jones Polynomial invariant of links. Then we introduce Gauge Field Theories, in particular those with gauge groups $U(1)$ and $SU(2)$. Finally we introduce a non-Abelian Chern-Simons Theory with gauge group $SU(2)$ and we show how we can use this theory to rediscover the Jones Polynomial using an intrinsically 3-dimensional approach.

Contents

1 Introduction

Knot Theory is a branch of low-dimensional topology and algebraic geometry concerned with the study of mathematical knots which are similar to the knots in our everyday lives. In particular most of the study of knots concerns ways in which they can be categorised into different kinds. To this aim, mathematicians try to discover knot invariants, which are functions that are constant over the equivalence class of every knot. One particular family of knot invariants are the so called polynomial knot invariants. These map knots to Laurent polynomials.

One would maybe be mistaken in assuming that such an abstract study was conceptualised by mathematicians. In truth, Knot Theory started out as a conjecture that was proposed by Sir William Thompson, better know as Lord Kelvin, a physicist famous for his work in thermodynamics. Together with Peter Guthrie Tait and later James Clerk Maxwell, they were the first to explore the connection between knots and physics. The conjecture put forward by Lord Kelvin was that atoms were vortices of aether swirling along knotted paths in space. The more complex the knot, the heavier the atom. The conjecture was ultimately fruitless, as we know well today. However, other attempts to find a connection between the topological properties of knots and topological properties in certain electromagnetic phenomena proved successful [6].

Knot Theory had a resurgence in the past few decades after Vaughan Jones discovered the Jones Polynomial invariant. This new knot invariant proved very powerful at differentiating between different equivalence classes of knots, while at the same time being relatively easy to compute. In 1988 Michael Atiyah proposed a research problem at the Hermann Weyl Symposium. The problem was to give a intrinsically 3-dimensional definition of the Jones Polynomial invariant. The algebraic definition given by Jones relied on 2-dimensional diagrams. Edward Witten published a paper in 1989 [5] where he devised a new kind of Quantum Field Theory, a so called Topological Quantum Field Theory. In particular he described a non-Abelian Chern-Simons Theory and he showed how the Jones Polynomial Invariant could be rediscovered within this theory.

2 Knot Theory

2.1 Preface

In Knot Theory we formalise the notion of 1-dimensional strings lying in ordinary 3-dimensional space. We restrict our study to the case where the two ends of the string are brought together, seamlessly. Then we have a loop of string that might or might not contain knots. We study the properties of these knotted strings. Which knots are different from others? Can we combine knots together or decompose a knot into simpler knots? A big part of Knot Theory is the study of knot invariants and the ongoing attempt at classifying knots. In this section we include various theorems, some of which are not proven in this text. Proofs for these theorems can be found in Cromwell [1] and Lickorish [2].

2.2 Basic Definitions

We start with the definition of a knot. Then we will introduce link diagrams, which are essential to our study of Knot Theory.

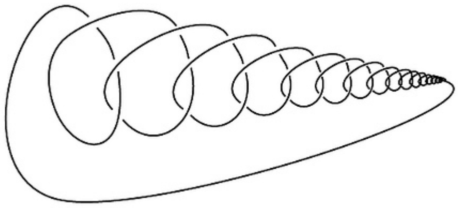


Figure 1: An example of a wild knot.

Definition 2.1 *A **knot** $K \subset \mathbb{R}^3$ is a subset of points homeomorphic to a circle.*

That is, a knot is a closed curve in $\mathbb{R}^3$ that does not intersect itself. Two examples of knots are depicted in Figure 2. 0_1 is the unknot or trivial knot. It is the simplest kind of knot, one lacking any kind of knotting. 3_1 is the trefoil, the most basic knot that we use in everyday life. The notation N_k means the knot has N crossings and order k. The order is a historical classification created by Dale Rolfsen [7]. Crossings are the points on the diagram of the knot where the knot passes over or under itself. We will explain more about this later on.

This definition of a knot, allows for so called *wild knots.* Wild knots have behaviour that make them unrepresentative of knots in everyday life. To eliminate knots with this pathological behaviour we will exclude them through the use of a stricter definition.

Definition 2.2 *A point p in a knot K is **locally flat** if there is some neighbourhood U of p such that the topological pair $(U, U \cap K)$ is homeomorphic to the unit ball, $B_0(1)$, plus a diameter. A knot K is locally flat if each point $p \in K$ is locally flat.*

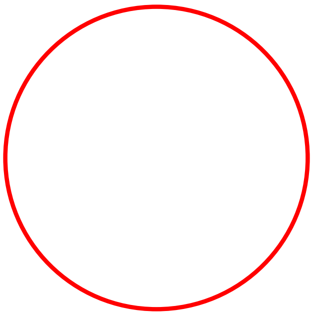
Wild knots are knots which are not locally flat at all points. An example of a wild knot is depicted in Figure 1. The pattern of knotting keeps going infinitely many times. We want to exclude these kinds of unconventional knots.

Definition 2.3 *A **tame knot** $K \subset \mathbb{R}^3$ is a locally flat subset of points homeomorphic to a circle.*

In everyday life, we distinguish between knots by their tying method. If we reposition a knot in $\mathbb{R}^3$ without untying it, we would say that it is the same knot. We will now formalise this notion by figuring out the kind of deformations that will not alter the knot type of a knot. For example in Figure 3 we have a deformed trefoil which is still the same type of knot as any other trefoil.

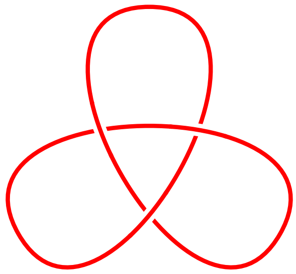
Definition 2.4 *A **homotopy** of a topological space $X \subset \mathbb{R}^3$ is a continuous map $h : X \times [0,1] \to \mathbb{R}^3$. The restriction of h to a level t is $h_t : X \times \{t\} \to \mathbb{R}^3$. We require that h_0 is the identity map and that h_t is continuous for all t.*

In our case, a homotopy is a continuous deformation of one knot into another. This is not enough because it allows curves to pass through themselves, which would allow one kind of knot to be deformed into another kind of knot. In fact, all knots are homotopic to the trivial knot.



0_1

(a) The Trivial Knot or Unknot, is the most trivial of knots.



3_1

(b) Trefoil

Figure 2: Two examples of knots.

Definition 2.5 *An **isotopy** is a homotopy for which h_t is injective for all $t \in [0, 1]$.*

The injectivity removes the problem of a knot passing through itself. This is still not enough. Because our knots have no thickness, one can deform a knot by tightening it to the point where it disappears. All knots are isotopic to the trivial knot.

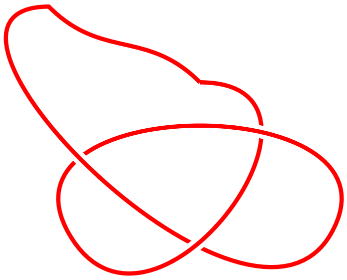


Figure 3: The previous trefoil knot with a slight deformation is still the trefoil knot.

Definition 2.6 *Two knots, K_1 and K_2 are **ambient isotopic** if there exists an isotopy $h : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$ such that $h(K_1, 0) = h_0(K_1) = K_1$ and $h(K_1, 1) = h_1(K_1) = K_2$. We say, K_1 is **equivalent** to K_2 if they are ambient isotopic.*

The difference between an isotopy and an ambient isotopy is that in the latter, we are deforming the whole of $\mathbb{R}^3$. An ambient isotopy allows use to stretch, compress or twist and rotate and region of the space containing the knot. But crucially it does not allow use to rip a hole in that space and pass the space through itself. This restriction is what forbids the knotting or unknotting of a knot embedded in that space. Ambient isotopy is an *equivalence relation.* In practice we will call an equivalence class of knots a *knot* as we do in real life, although rigorously speaking that is the *knot type* of the knot.

	e	r	s	t
e	e	r	s	t
r	r	e	t	s
s	s	t	e	r
t	t	s	r	e

Table 1: Group table of transformations of a knot.

Lemma 2.1 *Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ be an orientation preserving homeomorphism. Then for $K \subset \mathbb{R}^3$, $h(K)$ is ambient isotopic to K.*

Definition 2.7 *Let $r : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $r(x, y, z) = (-x, -y, -z)$. r is an orientation reversing homeomorphism and $K^* \equiv r(K)$ is the reflection or observe of K. If $K = K^*$ we say that K is acheiral. Otherwise K is cheiral and is either laevo or dextro. K can also be oriented. In this case $-K$ is the reverse orientation and $-K^*$ is called the inverse of K.*

Theorem 2.2 *Let s be the operation that sends K to $-K$ and let $t = rs$, such that $t(K) = -K^*$. Finally let $e(K) = K$ be the identity. The operations $\{e, r, s, t\}$ form a group. From this it follows that a knot can have either one of five types of topological symmetry:*

$$\begin{aligned}
&K = -K = K^* = -K^* && \textit{fully symmetric}\\
&K = -K && \textit{reversible (or invertible)}\\
&K = K^* && +\textit{amphicheiral}\\
&K = -K^* && -\textit{amphicheiral}\\
&K && \textit{asymmetric}
\end{aligned}$$

Definition 2.8 *A **link** is a finite disjoint union of knots: $L = K_1 \cup K_2 \cup \cdots \cup K_n$. Each knot K_i is called a **complement** of the link. The number of complements of a link L is called the **multiplicity** of the link, denoted $\mu(L)$. A subset of the components of the link is called a **sublink**.*

Note that it is not necessary for all sublinks of a link to be linked together. A **split link** is a link in which two or more sublinks are not linked together. A trivial link is an example of a split link.

Definition 2.9 *A **trivial link** of multiplicity n is a link made up of n trivial knots, such that the knots bound disjoint disks.*

An n-component link has 2^n ways to be oriented.

Definition 2.10 *Two links, L_1 and L_2 are ambient isotopic if there exists an isotopy $h : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$ such that $h(L_1, 0) = h_0(L_1) = L_1$ and $h(L_1, 1) = h_1(L_1) = L_2$.*

This definition is *weak equivalence* where we do not care about the orientation of the components of the links. A *strong equivalence* would require that the isotopy preserve the orientation and any labeling of the components. In a similar manner to knots, L is reversible if $L = -L$ and L is amphicheiral if $L = L^*$. A stronger condition still is ϵ*-amphicheiral*, where the equivalence preserves the orientation ϵ of L, with ϵ denoting a sign vector.

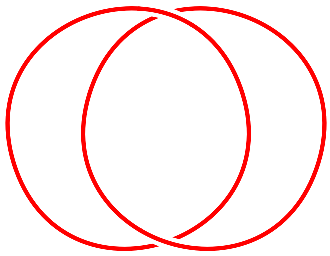


Figure 4: The Hopf link is the simplest non-trivial link.

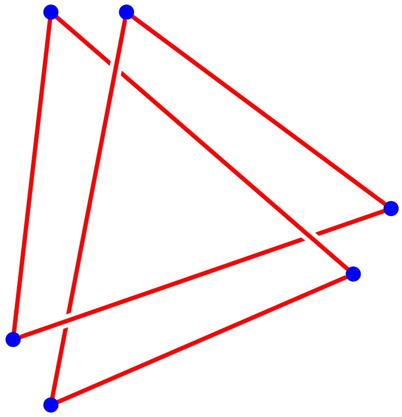


Figure 5: The trefoil knot, represented as a polygonal knot.

Definition 2.11 *A spatial polygon is a finite set of straight line segments in $\mathbb{R}^3$ that intersect only at their endpoints, such that intersections occur between exactly two endpoints. The straight line segments are called edges and their endpoints are called vertices.*

Knots can be smooth curves, but they can also be spatial polygons. Thus we have smooth knots and polygonal knots.

Definition 2.12 *A polygonal knot is a spatial polygon.*

Definition 2.13 *A smooth knot is described by a periodic curve in $\mathbb{R}^3$ such that the coordinate functions of the curve are all smooth.*

Theorem 2.3 *Polygonal knots are locally flat.*

Theorem 2.4 *A knot which is an image of a C^1 function is locally flat.*

Theorem 2.5 *A knot which is an image of a C^1 function is ambient isotopic to a polygonal knot.*

Theorem 2.6 *A polygonal knot is ambient isotopic to the image of a C^1 function.*

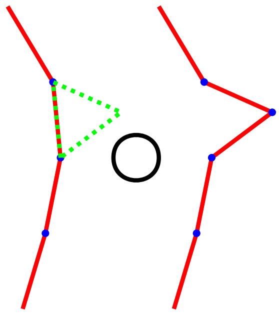


Figure 6: Δ-move

2.3 Combinatorial Approach

In this section we will introduce an approach to the study of links that has proven very effective in the study of tame links. The key insight is this: any smooth object can be approximated arbitrarily close by a piecewise linear object.

Definition 2.14 *(Δ-move) Let $L \in \mathbb{R}^3$ be a polygonal link. Let Δ be a triangle and let Δ_i, $i \in \{1, 2, 3\}$ be the sides of the triangle, further more let Δ be such that,*

1. $L \cap int(\Delta) = \emptyset$
2. $L \cap \partial\Delta = \Delta_i$ *for some* $i \in \{1, 2, 3\}$
3. *the vertices of L in $L \cap \Delta$ are also vertices of Δ*
4. *the vertices of Δ in $L \cap \Delta$ are also vertices of L*

*A Δ-**move** is then the following transformation,*

$$\Delta : L \to (L \setminus (L \cap \Delta)) \cup (\partial\Delta \setminus L). \tag{1}$$

A Δ-move is pictured in Figure 6. The basic idea is that you overlay a plane triangle on a polygonal link in such a way that one of the sides of the triangle coincides with an edge of the link. Then you remove the shared edge and you replace it with the other two edges of the triangle.

Definition 2.15 *(combinatorial equivalence) Two polygonal links L_1 and L_2 are combinatorially equivalent if there is a finite sequence of Δ-moves that transforms one into the other.*

Theorem 2.7 *It two links are ambient isotopic, they are also combinatorially equivalent. If two links are combinatorially equivalent, they are ambient isotopic.*

Hence our two equivalence definitions are themselves equivalent.

2.4 Link Diagrams

In this section we explore the conventions regarding link diagrams. These help us represent links on paper with ease and even allows us to do basic calculations on links such as calculating certain invariants.

Definition 2.16 *Let $L \subset \mathbb{R}^3$ be a link and let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be a projection map. A point $p \in \pi(L)$ is* ***regular*** *if $\pi^{-1}(p)$ is a single point. Otherwise p is* ***singular****. If $|\pi^{-1}(p)| = 2$, p is called a double point.*

A diagram of a link is a projection of a link onto $\mathbb{R}^2$. We want to choose a link within its equivalence class, such that the projection does not have any points through which the curve (or polygon) passes more than twice.

Definition 2.17 *A* ***regular projection*** *is one in which,*

1. *there is a finite number of singular points,*
2. *all singular points are transverse,*
3. *all singular points are double points.*

Theorem 2.8 *A tame link L has a regular projection.*

Definition 2.18 *A* ***link diagram*** *is a projection of a link equipped with crossing information at the double points. If a double point corresponds to points p_1 and p_2 on the link, and if p_1 is the point furthest away from the plane of projection, we say that p_1 is part of the* ***overcrossing*** *and p_2 is part of the* ***undercrossing****. An overcrossing is depicted as a continuous line, whereas an undercrossing is depicted as a break in the line.*

Note that diagrams are not unique. In fact there are infinitely many possible diagrams for every link.

Corollary 2.8.1 *Ever tame link has a diagram.*

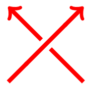
Corollary 2.8.2 *A link diagram represents a tame link.*

Definition 2.19 *Let $L \subset \mathbb{R}^3$ be a link, $\pi(L)$ its projection and D its diagram constructed from $\pi(L)$. A* ***subdiagram*** *is a diagram constructed from $L' \subset L$, using the same projection so that $\pi(L') \subset \pi(L)$. A component of a diagram is a subdiagram corresponding to a 1-component sublink of L.*

2.4.1 Properties of Diagrams

1. *Connected*: A diagram is connected if its projection is connected. A link with a disconnected diagram is called a *split link.*
2. *Oriented*: A diagram is oriented if each component is oriented.

3. *Positive crossing*:



4. *Negative crossing*:

5. *Positive*: A diagram is positive when all its crossings have the same sign. A link is called *positive* if it has a positive diagram.

6. *Alternating*: A diagram is alternating if each component has overcrossings and undercrossings in an alternating manner. A link is *alternating* if it has an alternating diagram.

7. *Descending*: A diagram is descending if following through each components, first one encounters all the overcrossings and then all the undercrossings. The complementary diagram where one first encounters all undercrossings and then all overcrossings is called an *ascending* diagram. Both descending and ascending diagrams represent trivial links.

Definition 2.20 *Let D be a diagram of an oriented polygonal knot. Each vertex in the diagram has an exterior angle. Each angle is signed: counter-clockwise is positive. The sum of these signed exterior angles divided by $360°$ is the* ***winding number****. The winding number of a link is the sum of the winding numbers of its components.*

The winding number is not a link invariant.

Definition 2.21 *Let D be an oriented diagram. Let $c \in D$ mean that c is a crossing in D. Let*

$$\epsilon(c) = \begin{cases} +1 & c \textit{ is a positive crossing} \\ -1 & c \textit{ is a negative crossing} \end{cases}, \tag{2}$$

then the ***writhe*** *of D is defined as*

$$w(D) = \sum_{c \in D} \epsilon(c). \tag{3}$$

The writhe is also not a link invariant.

2.4.2 Moves on Diagrams

A move on a diagram is a localised change to a diagram. One could conceive of many such moves but we will list a few that are historically relevant or that will be used throughout this text.

The **flype move** is pictured in Figure 7. You have to imagine that there is a part of the link hidden behind the blue square. This hidden segment is called a tangle. The move involves reflecting the tangle in the horizontal axis, before shifting it to the other side of the crossing.

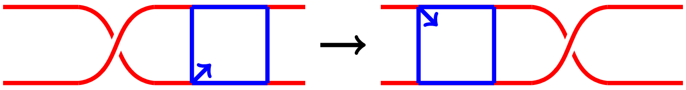


Figure 7: Flype move

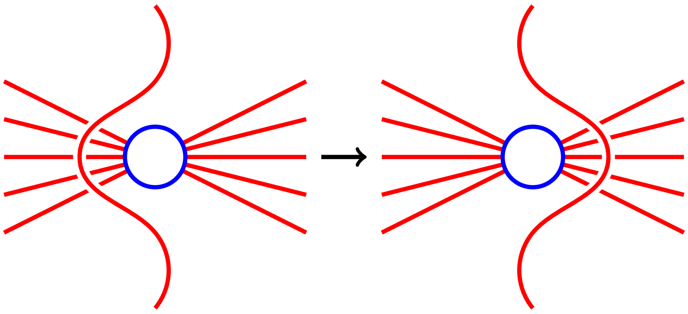
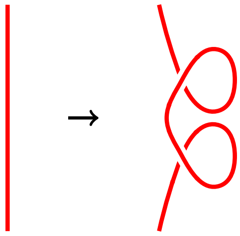
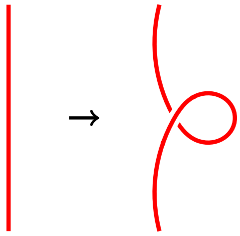


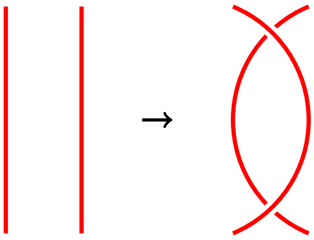
Figure 8: 5-pass move



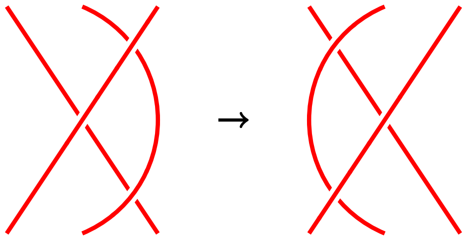
(a) Ω_0



(b) Ω_1



(c) Ω_2



(d) Ω_3

Figure 9: Reidemeister Moves

The **pass move** is depicted in Figure 8. Again, behind the circle lies a tangle, that in this case remains unchanged. Any number of strands can go in and come out of the tangle. In the figure we have a 5-pass move.

The flype and pass moves are historically relevant but are not used anymore.

In Figure 9, we have pictured the four Reidemeister moves. The inverse of these moves are intuitively the transformations from right to left. These moves are alternatively denoted by Type I, Type II, Type III, instead of Ω_1, Ω_2, Ω_3.

Theorem 2.9 *Two diagrams representing two equivalent links in $\mathbb{R}^3$, are related by a finite sequence of $\Omega_1^{\pm 1}, \Omega_2^{\pm 1}, \Omega_3^{\pm 1}$ Reidemeister moves.*

This theorem is very important for us. As we will see later on, it is very useful in proving the invariance of knot polynomials.

Theorem 2.10 *Two diagrams of an oriented link are related by a finite sequence of* $\Omega_2^{\pm 1}$ *and* $\Omega_3^{\pm 1}$ *Reidemeister moves if and only if they have the same writhe and winding number.*

The writhe of a diagram is changed by $\Omega_1^{\pm 1}$ moves but not by $\Omega_2^{\pm 1}$ and $\Omega_3^{\pm 1}$ moves. This is why the writhe is not a link invariant.

2.5 Invariants

Definition 2.22 *A link invariant is a function from the set of links to some other set such that the value of the function is constant over the equivalence class of the link.*

Conceptually the idea of using a link invariant to classify knots is easy to understand. The link invariant is always constant over the equivalence classes of links. If two links return different values for some link invariant, they cannot belong to the same equivalence class. However, we should not be mistaken into thinking that if two links return the same value, then they necessarily are in the same equivalence class. A link invariant can indeed return the same value for many different equivalence classes. Indeed, we say a link invariant is powerful if it can differentiate between many different equivalence classes. The other important feature of a link invariant is its ease of computation. Below we define the following numerical link invariants: the multiplicity, the unknotting number, the polygon index, the crossing number and the linking number.

Definition 2.23 *The* ***multiplicity*** *of a link,* $\mu(L)$ *is the number of components of the link.*

$\mu(L)$ is one of the simplest examples of a link invariant.

Definition 2.24 *The* ***unknotting number*** $u(L)$ *is the minimum number of times that a link must pass through itself to be transformed into a trivial link.*

The unknotting number of the trefoil is $u(3_1) = 1$ since you can imagine switching the sign of any crossing on a diagram of the trefoil and the result will be a diagram of the unknot.

We will see that quite a few of these numerical link invariants are defined in terms of minimums. These definitions are easy to state but hard to compute, since one has to use bruteforce in most cases. This makes them rather unwieldy.

Definition 2.25 *The* ***polygon index*** $p(L)$ *is the minimum number of edges (or equivalently vertices) needed to represent a link in polygonal form.*

For example a trivial link has $p(L) = 3\mu(L)$. That is, each component of the trivial link can be represented by a triangle with 3 edges, and there are $\mu(L)$ components.

Definition 2.26 *Let* $L \subset \mathbb{R}^3$ *be a link. Let* $c(D)$ *be the number of crossings in a diagram* D*. The* ***crossing number*** *of* L*,* $c(L)$ *is the minimum number of crossings over the set of diagrams* D *of* L*,*

$$c(L) = \min\{c(D) : D \in \{\textit{diagrams of } L\}\}. \tag{4}$$

Definition 2.27 *Let D be an oriented diagram of a 2-component link $K_1 \cup K_2$ with D_i being the subdiagram of D corresponding to K_i. Consider the crossings of $c \in D_1 \cap D_2$ The* ***linking number*** *is then defined as,*

$$lk(D_1, D_2) = \frac{1}{2} \sum_{c \in D_1 \cap D_2} \epsilon(c), \tag{5}$$

where $\epsilon(c)$ is the sign of the crossing as defined previously in Equation (2).

Definition 2.28 *For a link $L = K_1 \cup K_2 \cup \cdots \cup K_n$ the* ***linking number*** *is defined by,*

$$lk(L) = \sum_{i<j} lk(K_i, K_j). \tag{6}$$

Theorem 2.11 *The linking number is an invariant of oriented links.*

Proof. *To show that a function is a link invariant we need to show that it is not affected by any of the following three Reidemeister: $\Omega_1^{\pm 1}, \Omega_2^{\pm 1}, \Omega_3^{\pm 1}$. We refer the reader back to Figure (9). The linking number ignores crossings of a link component with itself so a $\Omega_1^{\pm 1}$ move do not affect its value. A $\Omega_2^{\pm 1}$ move where the two segments are from different link components will induce both a positive and a negative crossing which nullify the affect of each other. Again $lk(L)$ is unaffected. Finally a $\Omega_3^{\pm 1}$ move only alters the position, but not the sign of the crossings. Nor does it alter which components are involved in forming each crossing. Therefore it too does not affect $lk(L)$.* ∎

All the properties mentioned in this section are numerical link invariants. For a long time, these were the only invariants known to knot theorists. Their unwieldy nature is evident. It is hard to prove that any property of a diagram is indeed a minimum, when the set of diagrams of any link is infinite.

2.6 Framing

For any link L embedded in $\mathbb{R}^3$ we have that the tangent space of the link, at any point $p \in L$ is a subspace of the tangent space of $\mathbb{R}^3$,

$$T_pL \subseteq T_p\mathbb{R}^3 \cong \mathbb{R}^3. \tag{7}$$

Definition 2.29 *The framing of a link $L \subset \mathbb{R}^3$ is a vector field v defined on L, such that $v(p) \notin T_pL$ for all $p \in L$. A link equipped with a framing is called a* ***framed link****. The* ***standard framing****, also called the blackboard framing, is the unit vector field that is orthogonal to the plane of projection of the link, oriented towards the point of projection.*

Definition 2.30 *Equivalently, the framing of a link L is another link L_f, in the tubular neighbourhood of L. The framing is then the linking number $lk(L, L_f)$.* ***Standard framing*** *is then the framing for which $lk(L, L_f) = 0$.*

Thus framing can be thought of as an embedding in $\mathbb{R}^3$ homeomorphic to a solid torus. The added information here is the twisting of the torus around itself. Since a normal knot has no thickness, it cannot have any twisting. Another way to think of a framed link, is to think of having a link tied out of a ribbon. One edge of the ribbon is the link and the other edge is

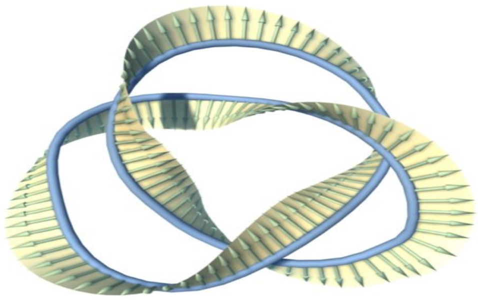


Figure 10: Pictured here is a framed trefoil knot. You can see how the framing can be thought of as both a vector field or as a knot tied out of a ribbon. You can see the twists in the ribbon and you can imagine a trefoil with more or fewer twists, corresponding to a different framing.

its framing. The projective diagram of this link will then have crossings between the link and its framing. These crossing correspond to the twists in the ribbon itself. Therefore framing can be thought of intuitively as the twists of a ribbon around itself after it is tied into a knot. Any link can have any kind of framing.

Definition 2.31 *Two framed links, L_1 and L_2 are equivalent, if there exists an ambient isotopy that takes L_1 to L_2 and moreover the ambient isotopy also takes the framing of L_1 to the framing of L_2.*

Theorem 2.12 *Two diagrams representing two framed links in $\mathbb{R}^3$, from the same ambient isotopy class, are related by a finite sequence of $\Omega_0^{\pm 1}, \Omega_2^{\pm 1}, \Omega_3^{\pm 1}$ Reidemeister moves.*

Here we are saying that a $\Omega_1^{\pm 1}$ move does not preserve the framing. Indeed it induces a twist that changes the framing by a magnitude of 1. Instead we use a $\Omega_0^{\pm 1}$ move. This move induces two twists that nullify other, thus preserving the framing.

The convention in Knot Theory is to assume that a link is in its standard framing as defined above, unless explicitly stated otherwise. To use a link invariant to assess whether two links are equivalent, you have to first make sure that they are both in the same framing, because link invariants are not guaranteed to be constant over a change of framing. However, this is in most cases not a problem. These complications arise when we do surgery on links, for example in satellite constructions in Section 2.7 and in Chern-Simons Theory in Section 5.1.

2.7 Compositions and Decompositions of Links

We start with a more rigorous definition of a split link.

Definition 2.32 *A link L is a* ***split link*** *if there exists 2-sphere, S^2 embedded in $\mathbb{R}^3 \setminus L$ such that,*

1. *$R^3 \setminus S^2 = U_1 \sqcup U_2$ (disjoint union),*
2. *$L_i := U_i \cap L \neq \emptyset$,*
3. *$L = L_1 \sqcup L_2$ (we say L is a distant union),*

where U_1 and U_2 are the two disjoint subsets of $\mathbb{R}^3$ bounded by S^2.

Definition 2.33 *A loop is **simple** if it contains no self-intersections.*

Definition 2.34 *Let W be a solid torus. A **meridional disk** D is a disk that is properly embedded in W in such a way that its boundary ∂D does not bound a disk in ∂W.*

Definition 2.35 *(satellite construction) Let W be a solid torus. A (possibly knotted) simple loop $\lambda \subset W$ is said to be essential if it meets every meridional disk in W. Let $P \subset W$ be a link embedded in an unknotted solid torus in such a way that at least one component of P is an essential loop in W. Let C be a knot in $\mathbb{R}^3$. Let V be the tubular neighbourhood of C. Let $h : W \to V$ be any homeomorphism. Then,*

1. *$S = h(P)$ is a new link, which is called the **satellite link**.*
2. *C is the so called **companion knot**.*
3. *P is the so called **pattern link**.*

In Figure 11 we can see the construction of a cable knot trefoil.

To clarify matters let us run through a physical method in which such a link S could be constructed from a link P and a knot C.

1. Construct the link P out of wire rope.
2. Embed the link in a solid torus made out of rubber maybe by using a mold. Make sure that at least one component of the link touches each meridional disk of the rubber torus.
3. Cut the torus in such a way as to produce one solid tube of rubber.
4. Label the protruding ends of the steel wire on one end with the same marking as their corresponding protruding ends on the other end.
5. Tie the knot C with the rubber tube and bring the two ends back together, being careful to match the labels. Weld the steel wire back together accordingly.
6. Dissolve the rubber in some solution.

The end result is the new link S made out of steel wire. An example of

Definition 2.36 *The **wrapping number** is the minimum absolute intersection number of P with the meridional disks.*

Theorem 2.13 *The trivial knot has no non-trivial companions.*

What this means is that if you think of the trivial knot as a satellite knot, then there is no way of constructing it if you use a non-trivial companion knot.

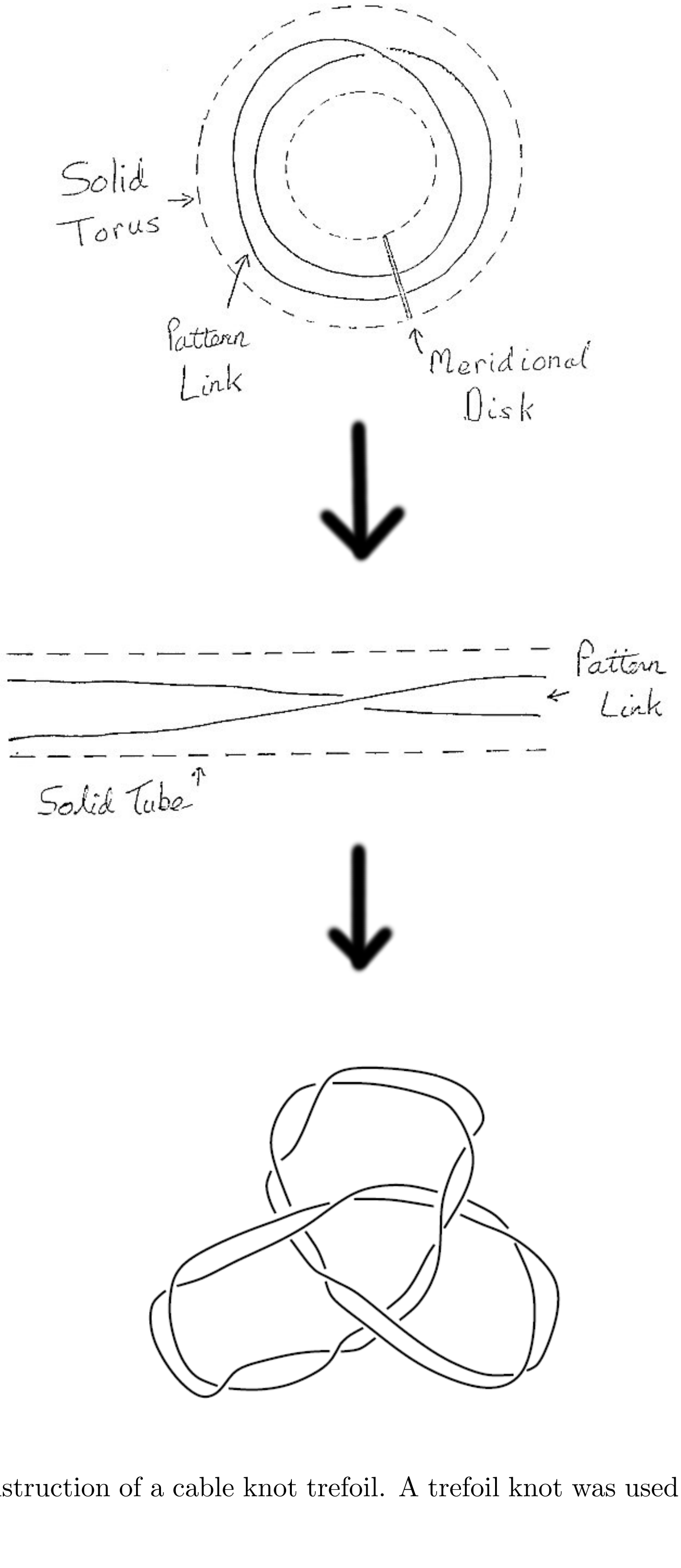


Figure 11: The construction of a cable knot trefoil. A trefoil knot was used as the companion knot.

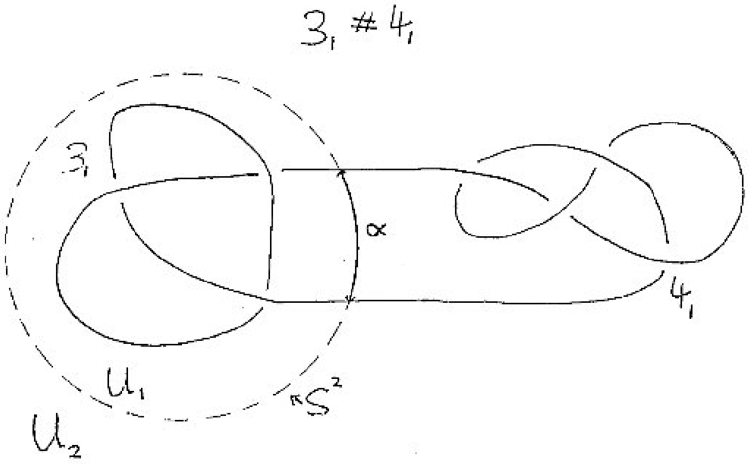


Figure 12: The process of factoring a link is depicted. We have pictured the product link $3_1\#4_1$.

Definition 2.37 *Let L be a link and let S^2 be such that it meets L transversely in exactly two points: a and b. Let $\alpha \subset S^2$ be an arc that connects a to b. Let $R^3 \setminus S^2 = U_1 \sqcup U_2$ where U_1 and U_2 are the two disjoint subset of $\mathbb{R}^3$ that are bounded by S^2. Then,*

$$L_i = (L \cap U_i) \cup \alpha \qquad for \quad i = \{1, 2\}, \tag{8}$$

*are the **factor links** of L and $L = L_1\#L_2$ is the **product link**.*

Note that each factor of a product link is also a companion of it. To understand this this, consider the tabular neighbourhood of the link outside of the factorising sphere. Then include the interior of the sphere as part of the tabular region. The factor link inside the sphere is P and the other factor link is C.

Definition 2.38 *A **proper factor** of a link is one that is neither the trivial knot, nor the link itself. A link with proper factors is **composite** and a link without proper factors is **locally trivial**. A link is **prime** if it non-trivial, non-split and locally trivial.*

Definition 2.39 *A companion of a non-trivial link is a **proper companion** if it is not the trivial link and also not equal to any of the link's components.*

Note, any knot with no proper companions is prime.

Definition 2.40 *A link is **simple** if it is prime and has no proper companions.*

Definition 2.41 *A **proper pattern** P is a pattern link with wrapping number greater or equal to 2.*

Definition 2.42 *A **proper satellite** is a satellite link with a **proper pattern** and a non-trivial companion knot.*

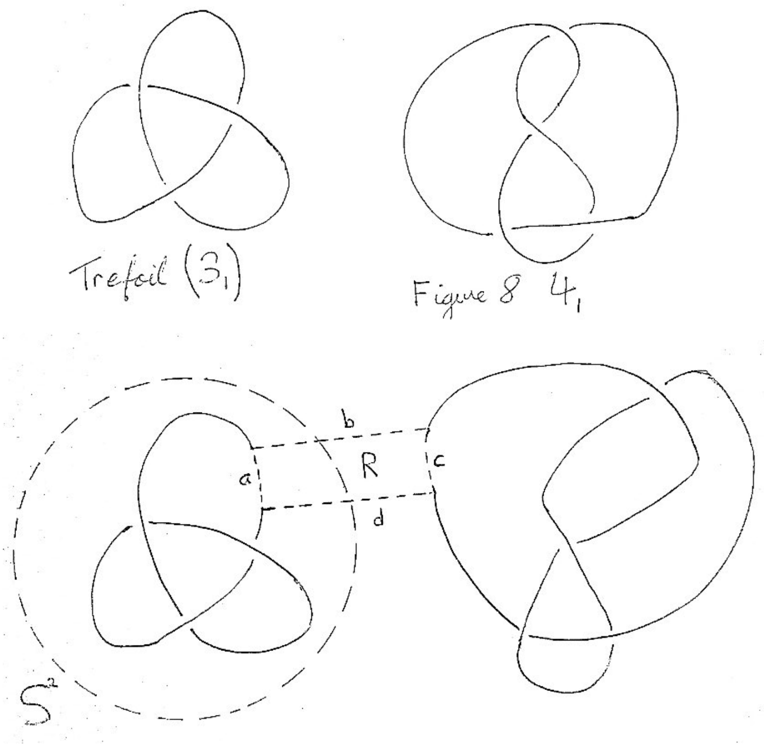


Figure 13: The construction of the product link $3_1\#4_1$.

Theorem 2.14 *A proper satellite is prime if its pattern is a prime knot or the trivial knot.*

Theorem 2.15 *A knot has a finite number of factors.*

Theorem 2.16 *Let $K = K_A\#K_B$ Let K_P be a prime knot, such that $K = K_P\#K_Q$. Then there exists K_C such that one of the following holds:*

1.
$$K_A = K_P\#K_C \quad \text{and} \quad K_Q = K_C\#K_B, \tag{9}$$

2.
$$K_B = K_P\#K_C \quad \text{and} \quad K_Q = K_C\#K_A. \tag{10}$$

Theorem 2.17 *Let K_P be a prime knot and suppose,*

$$K_P\#K_Q = K_A\#K_B. \tag{11}$$

If $K_P = K_A$ then $K_Q = K_B$.

Theorem 2.18 *The factors of a knot are uniquely determined up to order.*

Definition 2.43 *(product of links) Let L_1 and L_2 be links. Let S^2 be a sphere in $\mathbb{R}^3$. That is, Let U_1 and U_2 be the two disjoint subsets of $\mathbb{R}^3$ bounded by S^2. We position S^2 in such a way that it encloses only L_1. That is, we want,*

$$U_1 \cap L_1 = L_1 \quad U_1 \cap L_2 = \emptyset \quad U_2 \cap L_1 = \emptyset \quad U_2 \cap L_2 = L_2. \tag{12}$$

Let R be a rectangular disk, a surface with boundary composed of four arcs $\partial R = a \cup b \cup c \cup d$. Let arc a be such that $L_1 \cap R = a$ and let arc c be such that $L_2 \cap R = c$. Let R be such that $R \cap S^2$ is a single simple arc. Then the product link can be constructed as follows,

$$L = L_1 \# L_2 = (L_1 \setminus a) \cup (L_2 \setminus c) \cup b \cup d. \tag{13}$$

The process is depicted in Figure 13.

In simple words, we cut L_1 to end up with two ends e_1 and e_2. We cut L_2 to end up with two ends e_3 and e_4. We glue e_1 and e_3 and e_2 to e_4 to produce the product link. One will immediately wonder whether it matters where the cuts are done. It does not. In general this operation is however not well defined. We consider some cases:

- If the factors are two oriented knots, the oriented product is well defined. It is independent of where we decide to cut and glue.
- If the factors are two unoriented knots and either one of the factors is reversible, the product is well defined. Otherwise if both are non-reversible then there is a choice between two distinct product links.
- When the factors are links there is a choice of which component from one link to connect to which other component from the other link.

Theorem 2.19 *For any non-trivial knot K there exists the **anti-knot** K^{-1} such that $K \# K^{-1}$ is the trivial knot.*

Theorem 2.20 *Let $\mathbb{K}$ be the set of oriented knots. $(\mathbb{K}, \#)$ is an abelian semigroup with unit, and unique factorisation.*

Note that,

$$\mu(L_1 \# L_2) = \mu(L_1) + \mu(L_2) - 1, \tag{14}$$

that is, the multiplicity of the product will be one less than the addition of the multiplicity of the factors, which is trivial since you take one component from each factor and merge them into one component.

Note also that,

$$u(K_1 \# K_2) \leq u(K_1) + u(K_2), \tag{15}$$

which can be seen easily since, keeping the two attached knots far apart, you can still unknot them separately using the same sequence of operations as you would if they were just factors. Then you end up with a trivial knot which is unknotted. It is conjectured that this is actually an equality.

2.8 Surfaces

We can think of a link as being the boundary of some surface. To explore this further, we need some prerequisites from Topology which I have included in the Appendix I.

Definition 2.44 *(link genus) The **genus** of an oriented link L is the minimum genus of any connected orientable surface that spans L. The genus of an unoriented link is the minimum taken over all possible choices of orientation. We denote the genus of L by $g(L)$.*

Theorem 2.21 *$g(L)$ is a link invariant.*

An orientable surface which spans a link is often called a **Seifert surface**.

2.8.1 Seifert's Algorithm

Seifert's Algorithm generates an orientable surface, called a **projection surface**, out of a link.

1. Choose an orientation and a link diagram for the link.
2. At each crossing, take the other path that preserves the orientation. The resulting loops are called Seifert circles.
3. For each Seifert circle λ, assign the index $h(\lambda)$
$$h(\lambda) = \text{number of Seifert circles that contain } \lambda \tag{16}$$
4. For every Seifert circle λ, put a disk Δ in the plane $z = h(\lambda)$ such that $\partial\Delta$ projects onto λ. The disks inherit the orientation of λ.
5. Insert a half-twisted rectangle at the site of each crossing, in such a way that it produces the right crossing. To be rigorous, let C_1 and C_2 be the two Seifert circles. Let the rectangle have sides a, b, c, d. Side a merges with ∂C_1. Now the rectangle gains its orientation. Since we can twist the rectangle, there are two ways in which we can attach side c to ∂C_2 but only one of them is correct. We need to twist the rectangle in such a way that the orientation of the rectangle matches with the orientation of C_2. In doing so we notice that the projection of b and the projection of d together, form the original crossing that we had on the diagram.

Figure 14 is an example of this algorithm applied to the Hopf Link.

Theorem 2.22 *Every link bounds an orientable surface.*

Proof. *Let L be a link. Using Seifert's Algorithm, we generate a surface with boundary L. It remains to be shown that the whole surface is two-sided, and thus orientable. Let Δ be a disk generated by the algorithm. Taking a closed path from this disk, along the surface and back to this disk, we want to ensure that the path goes over an even number of half-twists, in such a way that if we leave from the counterclockwise face of Δ, we will always end up back on the counterclockwise face of Δ and the clockwise face of Δ is unreachable through any path from this counterclockwise face. To this end we note that it is not possible for Δ to be connected though an odd number of half-twists, because this would induce a conflict in the definition of the orientation of Δ. Thus the generated surface is two-sided.* ∎

The crucial part of the proof is depicted in Figure 15 for 3 half-twists.

Theorem 2.23 *The Euler characteristic of a projection surface F, constructed from a diagram D with $s(D)$ Seifert circles and $c(D)$ crossings is,*

$$\chi = s(D) - c(D). \tag{17}$$

Proof. *The projection surface is built from disks and half-twisted bands. We triangulate a band by drawing a diagonal across a rectangle. For a disk which meets n bands, we construct a regular n-polygon and take a line from each vertex to the center, thus resulting into $2n$*

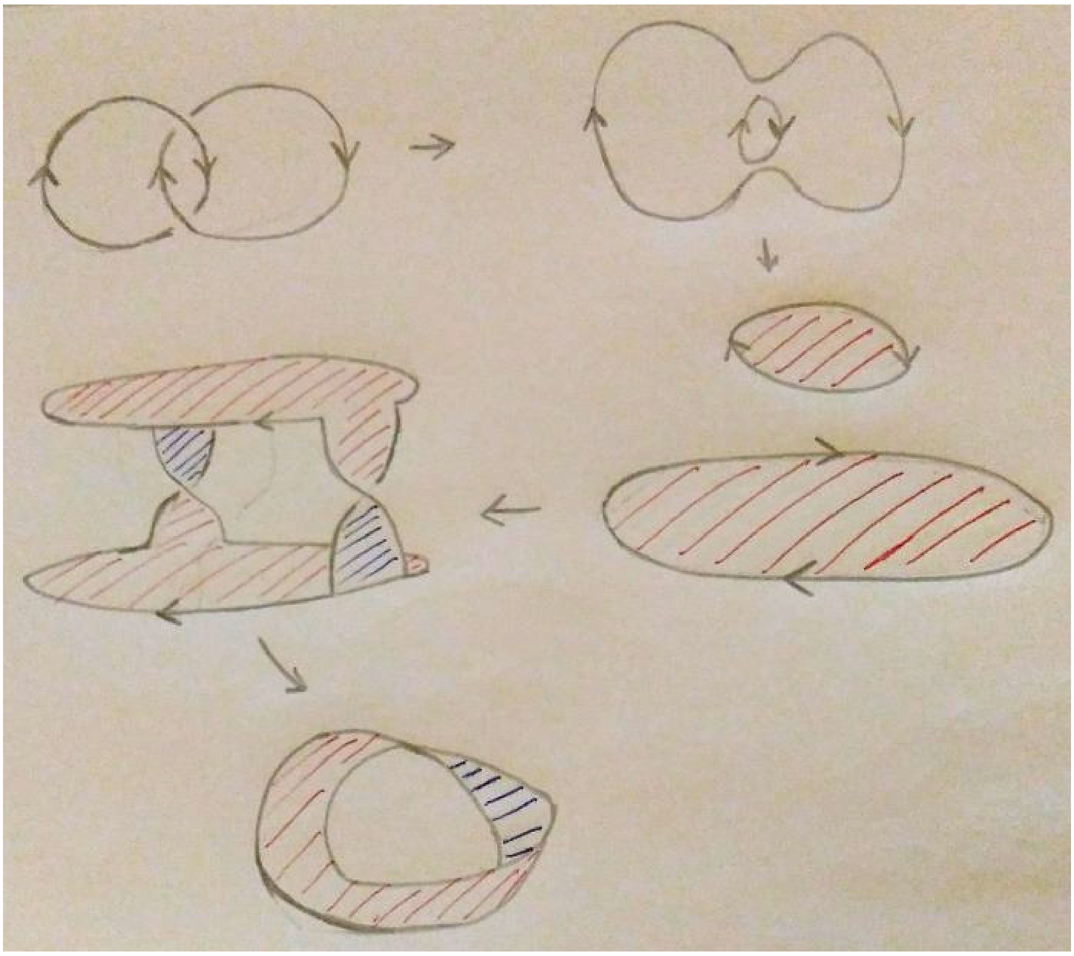


Figure 14: Seifert Surface of a Hopf Link

triangles. Let J be the total number of joins between bands and disks. Each band is attached at both sides and each band corresponds to one crossing so $J = 2c(D)$. There are $2J$ triangles in total in the disks. So the number of triangles is $T = 2J + 2c(D) = 3J$, taking into account also the two triangles in each band. Each join corresponds to two vertices and the remaining vertices are those in the center of disks, so $V = 2J + s(D)$. J joins corresponds to $2J$ edges on the boundary of disks and $2J$ edges between the boundary and the centres of disks. Each crossing corresponds to three extra edges, so in total the number of edges is $E = 4J + 3c(D)$. Finally,

$$\chi(F) = V - E + T = 2J + s(D) - 4J - 3c(D) + 3J = s(D) - c(D). \tag{18}$$

∎

Theorem 2.24 *The genus of a projection surface F constructed from a connected diagram D satisfies,*

$$2g(F) = [1 - s(D) + c(D)] + [1 - \mu(D)]. \tag{19}$$

Proof. *A bounded oriented surface F is homeomorphic to a closed surface S, with a set of disks removed. We know furthermore that $g(F) = g(S)$ and since S is closed, $|\partial S| = 0$ so*

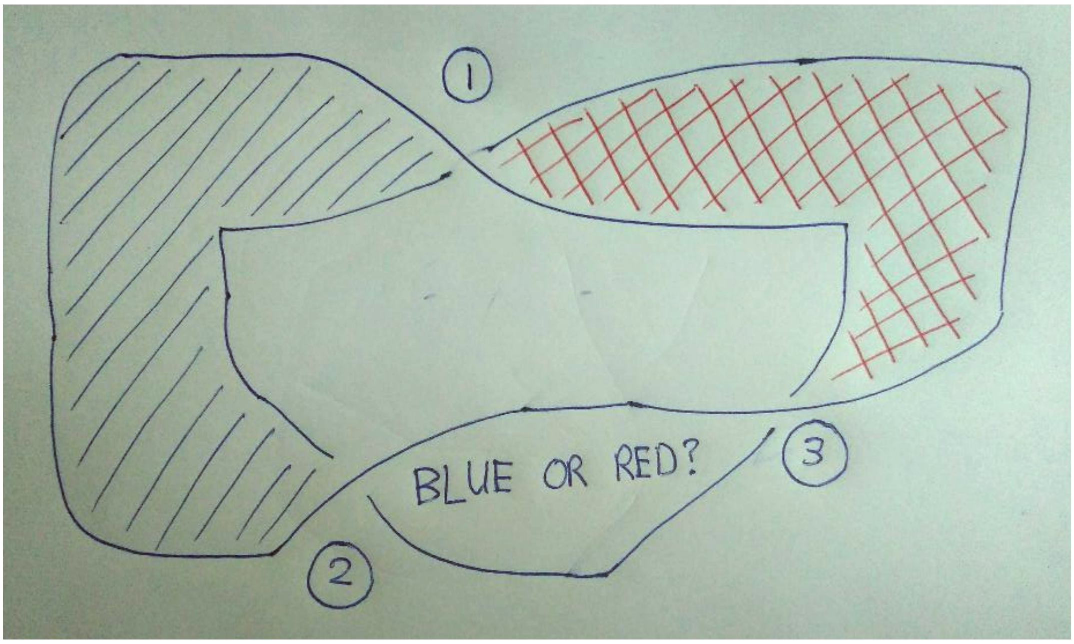


Figure 15: There cannot be a closed path starting and ending at disk Δ and going through exactly 3 (or any odd number) of half-twists. This contradicts the orientation of Δ.

indeed we have,

$$\begin{aligned} 2g(F) &= 2g(S) \\ &= 2 - \chi(S) \\ &= 2 - (\chi(F) + |\partial F|) \\ &= 1 - s(D) + c(D) + 1 - \mu(D). \end{aligned}$$

∎

Definition 2.45 *Two surfaces are S-equivalent if they are related by a sequence of tubing and compressing operations.*

Theorem 2.25 *If F_1 and F_2 are two surfaces such that ∂F_1 and ∂F_2 are equivalent links, then F_1 and F_2 are S-equivalent.*

Theorem 2.26 *Suppose $L = L_1 \# L_2$ has a connected incompressible spanning surface of minimal genus. Then,*

$$g(L_1 \# L_2) = g(L_1) + g(L_2). \tag{20}$$

Theorem 2.27 *The genus of a satellite knot S constructed from pattern P with companion C, framing zero, and winding number n is bounded below,*

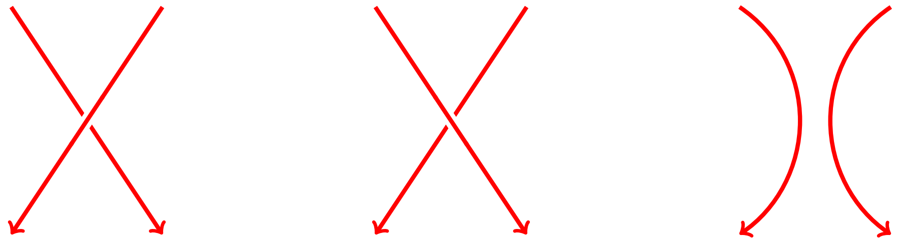
$$g(S) \geq g(P) + ng(C). \tag{21}$$

2.9 Conway Polynomial

Now we get our first taste of polynomial link invariants.

Definition 2.46 *The **Conway polynomial** of an oriented link L, denoted by $\nabla(L)$ or $\nabla_L(z)$, is defined by the following axioms:*

1. *Invariance: $\nabla_L(z)$ is invariant under ambient isotopy of L.*
2. *Normalisation: if K is the trivial knot, then $\nabla_K(z) = 1$.*
3. *Skein Relation: $\nabla(L_+) - \nabla(L_-) = z\nabla(L_0)$ where L_+, L_-, L_0 have diagrams D_+, D_-, D_0, respectively.*



(a) crossing modification D_+ (b) crossing modification D_- (c) crossing modification D_0

Figure 16: Conway triple

The skein relation is the main point of interest. Let us explain how it works. Let D be a diagram of a link L. Pick a crossing $c \in D$. Then L_+ corresponds to D_+ which is D with c replaced by a positive crossing. D_- is D with c replaced by a negative crossing. And D_0 is D with c replaced by a smoothing. Alternatively, we can pick a point $p \in D$ that looks like the smoothing D_0 and modify the neighbourhood of p in such a way as to create a positive or negative crossing, corresponding to D_+ and D_-, respectively. The diagrams D_+, D_- and D_0 represent modifications of the link L into L_+, L_- and L_0 respectively.

By definition, the Conway polynomial is a link invariant. It is a polynomial in $\mathbb{Z}[z]$. The skein relation is used to compute the polynomial recursively. In Figure 17 an example walkthrough of the calculation of the Conway polynomial for the 4_1 knot is shown. The way to calculate polynomial invariants is to apply them recursively. Each time you rewrite expand a link as two modifications of it, you end up with two simpler links. Applying the skein relation recursively leads to a polynomial in terms of the Conway polynomial invariant for very basic knots which will be known. For example, the trivial knot evaluates to 1. We see that for the 4_1 knot we have,

$$\nabla(4_1) = 1 + z(0 - z \cdot 1) = 1 - z^2. \tag{22}$$

The reason for the zero for the split link follows.

Theorem 2.28 *If L is a split link, then $\nabla(L) = 0$.*

Proof. *Let $L = L_1 \sqcup L_2$ and consider a disconnected diagram of D of L, such that the projections of L_1 and L_2 are disjoint on the diagram. Consider a circular neighbourhood, U of some point on D in such a way that $U \cap L_1$ is an arc and $U \cap L_2$ is an arc. Then we can consider this to be a D_0 diagram of L. We can then construct D_+ and D_- and we note that*

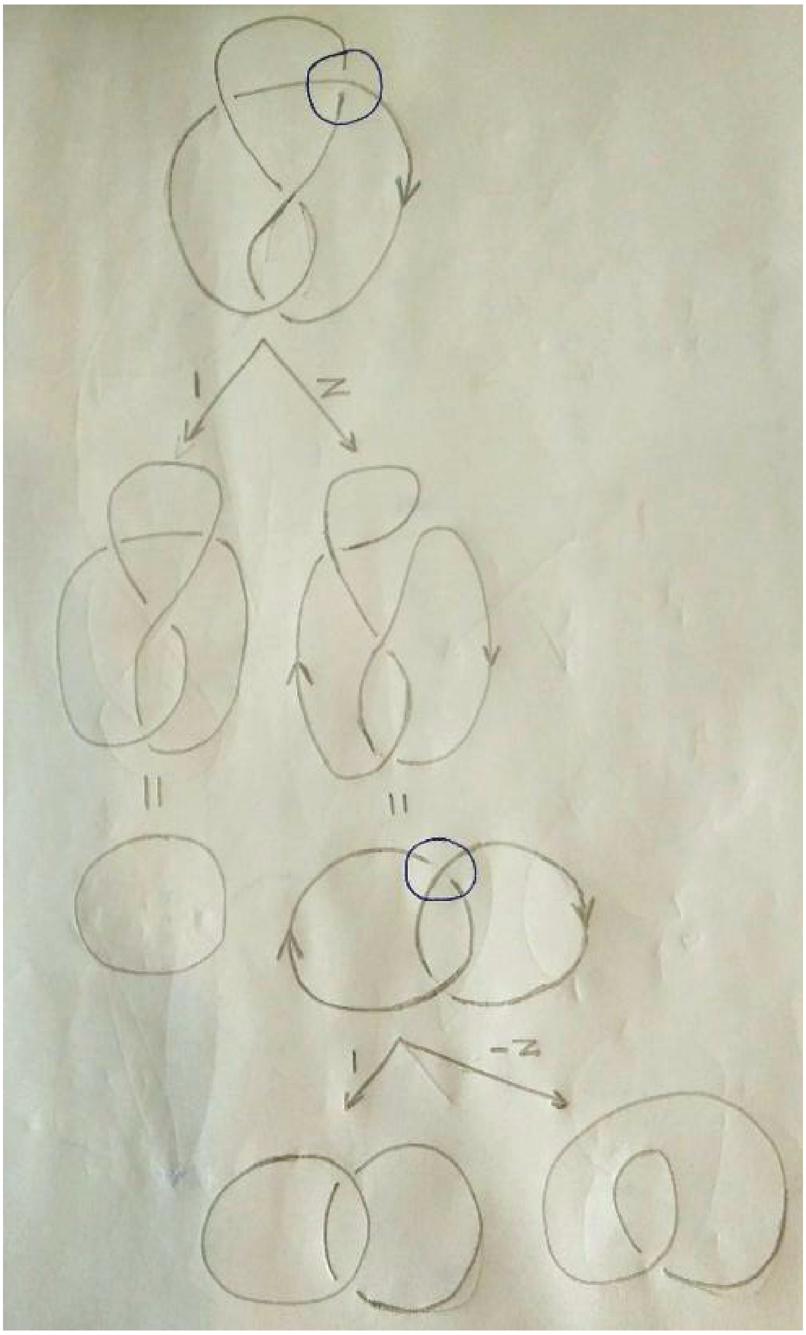


Figure 17: Working out the Conway polynomial for the knot 4_1.

D_+ can be ambient isotopically deformed into D_- by twisting the space in the neighbourhood of L_2. Hence, by the first axiom,

$$\nabla(D_+) = \nabla(D_-). \tag{23}$$

By the third axiom,

$$z\nabla(D_0) = \nabla(D_+) - \nabla(D_-) = 0. \tag{24}$$

∎

The steps of the proof above are illustrated in Figure 18. Only the arcs protruding from both links are visible. The ambient isotopy that takes L_1 to L_2 is pictured in the bottom of the diagram.

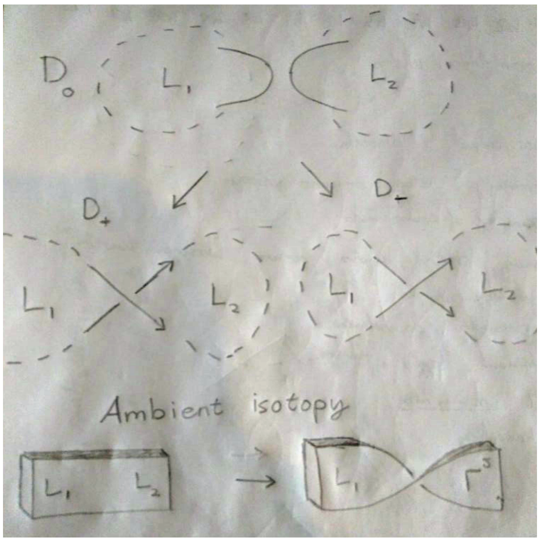


Figure 18: Steps in proof of Theorem 2.28

Theorem 2.29 *If $L = L_1 \# L_2$ is a composite link then $\nabla(L_1 \# L_2) = \nabla(L_1)\nabla(L_2)$.*

Proof. *Start by unknotting the L_1 factor link. Now you have a polynomial expression that is only a function of L_2,*

$$c_1(z)\nabla(L_2) + c_2(z)\nabla(L_2) + ... + c_n(z)\nabla(L_2) = (c_1(z) + c_2(z) + ... + c_n(z))\nabla(L_2). \quad (25)$$

Hence you have $\nabla(L_1 \# L_2) = \nabla(L_1)\nabla(L_2)$. If the unknotting of L_1 results in a split link, then both sides of the equation are zero so the equality holds. ∎

2.10 The Jones Polynomial

The **Laurent polynomial** over a field $\mathbb{F}$ is defined as,

$$p = \sum_{k \in \mathbb{Z}} p_k X^k \quad p_k \in \mathbb{F}, \quad (26)$$

where X is an indeterminate and p_k is non-zero for finitely many k.

Definition 2.47 *The Kauffman bracket is a function from unoriented link diagrams in the oriented plane to Laurent polynomials with integer coefficients in an indeterminate A. It maps a diagram D to $\langle D \rangle \in \mathbb{Z}[A^{-1}, A]$ and is characterised by,*

1. $\langle \bigcirc \rangle = 1$,
2. $\langle D \sqcup \bigcirc \rangle = (-A^{-2} - A^2) \langle D \rangle$,
3. $\langle D_+ \rangle = A \langle D_0 \rangle + A^{-1} \langle D_\infty \rangle$.

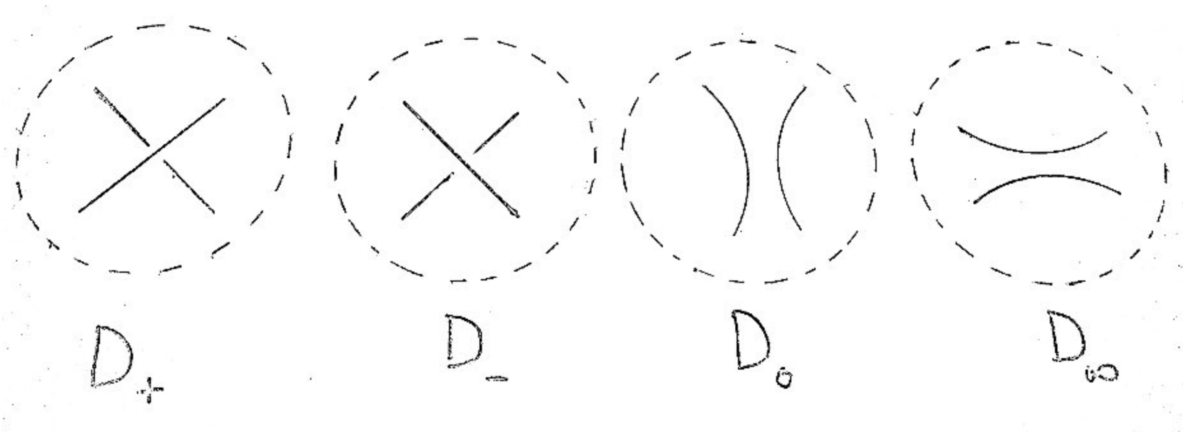
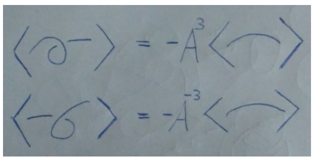


Figure 19: D_+, D_-, D_0, D_∞ for the definition of the bracket polynomial.

Note that as a consequence of this we have $\langle D_- \rangle = A^{-1} \langle D_0 \rangle + A \langle D_\infty \rangle$.

In the above definition the notation D_+, D_-, D_0, D_∞ refers to any diagram D of the link, which has been modified in some neighbourhood of a point $p \in D$, to look like one of the diagrams in Figure 19, respectively.

Lemma 2.30 *If a diagram is changed by a Type I Reidemeister move, its bracket polynomial changes as follows,*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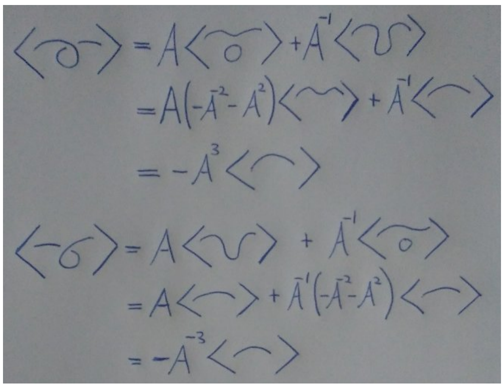
Proof.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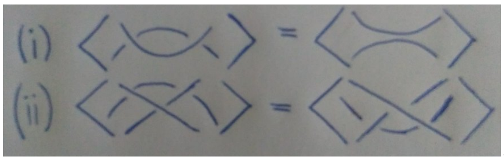
∎

Lemma 2.31 *If a diagram D is changed by a Type II or Type III Reidemeister move, $\langle D \rangle$ remains unchanged. That is,*



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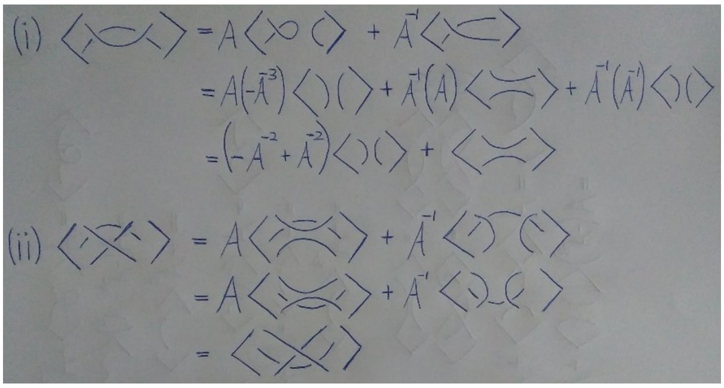
Proof.



.

∎

Theorem 2.32 *Let D be a diagram of an oriented link L. Then the expression*

$$(-A)^{-3\omega(D)} \langle D \rangle \tag{27}$$

is an invariant of the oriented link L.

Proof. *$\langle D \rangle$ is constant under Type II and Type III Reidemeister moves so the above expression is constant under such moves. $\langle D \rangle$ changes by a factor of $-A^3$ or $-A^{-3}$ under a Type I Reidemeister move, and you can observe that $\omega(D)$ changes according by a $+1$ or -1 sign to ensure that the above expression remains constant also, under a Type I Reidemeister move. Any two equivalent links are related by a sequence of Type I, Type II and Type III Reidemeister moves, so indeed the above expression is an invariant of oriented links.* ∎

Definition 2.48 *The Jones polynomial $V(L)$ of an oriented link L is the Laurent polynomial in $t^{1/2}$, with integer coefficients, defined by,*

$$V(L) = (-A)^{-3\omega(D)} \langle D \rangle \Big|_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}], \tag{28}$$

where D is any oriented diagram for L.

The following preposition is often used as an alternative definition of the Jones polynomial invariant.

Proposition 2.1 *The Jones polynomial invariant is a function,*

$$V : \{\textit{oriented links in } \mathbb{S}^3\} \to \mathbb{Z}[t^{-1/2}, t^{1/2}], \tag{29}$$

such that,

1. $V(unknot) = 1$,
2. *Whenever three oriented links L_+, L_-, L_0 are the same except in the neighbourhood of a point where they differ as in Figure 16, then we have*

$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0. \tag{30}$$

Proof. *From the Kauffman bracket we have,*

$$\langle D_+ \rangle = A \langle D_0 \rangle + A^{-1} \langle D_\infty \rangle, \tag{31}$$

$$\langle D_- \rangle = A \langle D_\infty \rangle + A^{-1} \langle D_0 \rangle. \tag{32}$$

Multiplying the first equation by A and the second by A^{-1} and subtracting one from the other we end up with,

$$A \langle D_+ \rangle - A^{-1} \langle D_- \rangle = (A^2 - A^{-2}) \langle D_0 \rangle. \tag{33}$$

Now we note that,

$$V(L) = (-A)^{-3\omega(D)} \langle D \rangle \implies (-A)^{3\omega(D)}V(L) = \langle D \rangle. \tag{34}$$

Applying this substitution we then get,

$$AV(D_+)(-A)^{3\omega(D_+)} - A^{-1}V(D_-)(-A)^{3\omega(D_-)} - = (A^2 - A^{-2})(-A)^{3\omega(D_0)}. \tag{35}$$

Then we note that,

$$\omega(D_+) - 1 = \omega(D_0) = \omega(D_-) + 1. \tag{36}$$

Which leads to,

$$AV(D_+)(-A)^{3\omega(D_0)+3} - A^{-1}V(D_-)(-A)^{3\omega(D_0)-3} = (A^2 - A^{-2})V(D_0)(-A)^{3\omega(D_0)}, \tag{37}$$

$$AV(D_+)(-A)^3 - A^{-1}V(D_-)(-A)^{-3} = (A^2 - A^{-2})V(D_0), \tag{38}$$

and after simplifying we get,

$$-A^4V(L_+) + A^{-4}V(L_-) = (A^2 - A^{-2})V(L_0). \tag{39}$$

Finally, make the substitution $t^{1/2} = A^{-2}$ and multiply by -1 to get the skein relation (30). ∎

Lemma 2.33 *Let L' be a link L together with an additional unknotted unlinked component, then,*

$$V(L') = (-t^{-1/2} - t^{1/2})V(L). \tag{40}$$

Lemma 2.34 *For two factor knots K_1 and K_2,*

$$V(K_1 \# K_2) = V(K_1)V(K_2). \tag{41}$$

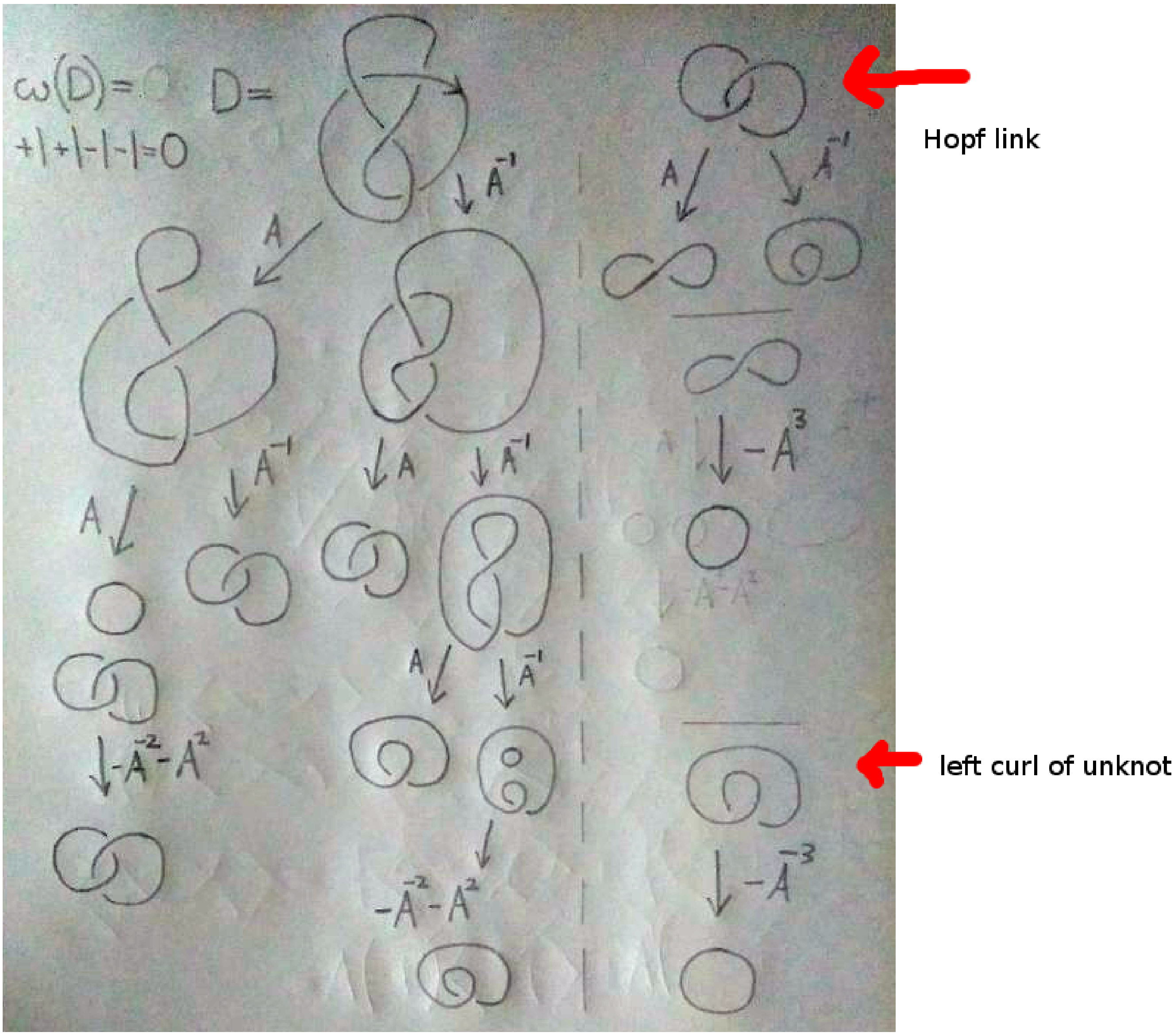


Figure 20: Working out the Jones polynomial for the knot 4_1.

Finally we work out an example of the Jones Polynomial for the 4_1 knot, as shown in Figure 20. We see that $\omega(4_1) = 0$. Then we see that the Hopf link has Kauffman bracket,

$$-A^4 - A^{-4}, \tag{42}$$

and the left curl of the unknot has Kauffman bracket,

$$-A^{-3}. \tag{43}$$

Finally we work out the resolving tree until everything is in terms of the Hopf link and the left curl of the unknot. We end up with the expression:

$$(-A^4 - A^{-4})(-1 - A^4 + 1 + 1) + (-A^{-3})(A^{-1} - A^{-5} - A^{-1}), \tag{44}$$

which we simply to,

$$A^8 - A^4 + 1 - A^{-4} + A^{-8}, \tag{45}$$

and substituting $A = t^{-1/4}$ we get,

$$V(4_1) = t^2 + t^{-2} - t - t^{-1} + 1. \tag{46}$$

In Appendix II, we give also provide definitions, properties and worked examples for the Kauffman and HOMFLY polynomial invariants.

3 The Feynman Path Integral and non-Abelian Gauge Field Theories

3.1 Gauge Field Theories

In Gauge Field Theories the action is invariant under spacetime dependent unitary gauge transformations. This means that for every field configuration there are infinitely many corresponding field configurations that can be obtained by some gauge transformation of the original one. Between all these corresponding field configurations the physics remains unchanged. We want our Lagrangian to be invariant under $U(1)$ transformations,

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \qquad e^{i\alpha(x)} \in U(1), \tag{47}$$

where $\psi(x)$ is a Dirac spinor [3], that is, a field that describes spin-1/2 fermions. We call this a gauge transformation. We note that $\alpha(x)$ is not constant throughout spacetime. The derivative of $\psi(x)$ along some vector n^μ,

$$n^\mu \partial_\mu \psi(x) = \lim_{\epsilon\to 0} \frac{\psi(x+\epsilon n) - \psi(x)}{\epsilon}, \tag{48}$$

is no longer well defined because $\psi(x)$ transforms differently from $\psi(x+\epsilon n)$, since $\alpha(x) \neq \alpha(x+\epsilon n)$. Hence if we want such terms in our Lagrangian, we must come up with a new derivative that accounts for this, the so called covariant derivative. We introduce the comparator $U(y,x)$ that is a scalar valued map that transforms as follows,

$$U(y,x) \to e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)}, \tag{49}$$

with $U(y,y) = 1$ and $|U(y,x)| = 1$. It is easy to see that $U(y,x)\psi(x)$ transforms like $\psi(y)$,

$$U(y,x)\psi(x) \to e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)}e^{i\alpha(x)}\psi(x) = e^{i\alpha(y)}U(y,x)\psi(x). \tag{50}$$

We can now define the covariant derivative $D_\mu\psi$ along a vector n^μ,

$$n^\mu D_\mu \psi(x) := \lim_{\epsilon\to 0} \frac{\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)}{\epsilon}, \tag{51}$$

for $y = x + \epsilon n$. To determine $U(y,x)$ we assume that it is continuous and we assume an infinitesimal displacement, which allows us to expand it as follows,

$$U(x+\epsilon n, x) = 1 - ie\epsilon n^\mu A_\mu(x) + \mathcal{O}(\epsilon^2), \tag{52}$$

where $A_\mu(x)$ is an arbitrary vector field called the gauge field or gauge connection and e is just a scalar constant. We can now rewrite the covariant derivative,

$$D_\mu\psi(x) = \lim_{\epsilon\to 0}\frac{\psi(x+\epsilon n) - (1 - ie\epsilon n^\mu A_\mu(x))\psi(x)}{\epsilon} = (\partial_\mu + ieA_\mu)\psi(x). \tag{53}$$

The covariant derivative transforms as,

$$D_\mu\psi(x) = e^{i\alpha(x)}D_\mu\psi(x), \tag{54}$$

that is, it transforms like $\psi(x)$. This is what allows us to compare $\psi(x)$ to $\psi(x+\epsilon n)$. Meanwhile from the transformation law of the comparator and its expansion above, we can deduce that the connection must therefore transform as,

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x) \tag{55}$$

For a finite transformation one has,

$$U(y,x) := \exp\left[ie\int_x^y dz^\mu A_\mu(z)\right], \tag{56}$$

which is called a Wilson line. As evident from the definition, Wilson lines can be segmented and concatenated as follows

$$U(z,x) = U(z,y)U(y,x). \tag{57}$$

Wilson lines depend on the path taken from x to y. A Wilson line on a closed path is called a Wilson loop. A Wilson loop is gauge invariant since $U(y,y) = 1$. We therefore consider the Wilson loop around an infinitesimal square as pictured in Figure 21 and we find that,

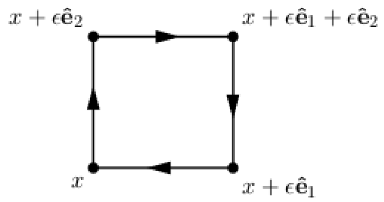


Figure 21: Infinitesimal square in the plane.

$$U(x+\epsilon\hat{\mathbf{e}}_1, x)U(x+\epsilon\hat{\mathbf{e}}_1+\epsilon\hat{\mathbf{e}}_2, x+\epsilon\hat{\mathbf{e}}_1)U(x+\epsilon\hat{\mathbf{e}}_2, x+\epsilon\hat{\mathbf{e}}_1+\epsilon\hat{\mathbf{e}}_2)U(x, x+\epsilon\hat{\mathbf{e}}_2) \tag{58}$$

$$\approx 1 + ie\epsilon^2\hat{\mathbf{e}}_1^\mu\hat{\mathbf{e}}_2^\nu(-\partial_\mu A_\nu(x) + \partial_\nu A_\mu(x)), \tag{59}$$

from which we deduce that

$$F_{\mu\nu} := \partial_\mu A_\nu + \partial_\nu A_\mu \tag{60}$$

is gauge invariant. This is called the curvature of the connection or the field strength tensor. These scalars will transform under a unitary transformation,

$$F_{\mu\nu} \to UF_{\mu\nu}U^\dagger = UU^\dagger F = UU^{-1}F = F \qquad U = e^{i\alpha(x)} \in U(1), \tag{61}$$

and again we see how in the case of an Abelian gauge group the curvature is gauge invariant. We note that by construction we have,

$$
\begin{aligned}
[D_\mu, D_\nu] &= D_\mu D_\nu - D_\nu D_\mu && (62)\\
&= (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu) - (\partial_\nu + ieA_\nu)(\partial_\mu + ieA_\mu) && (63)\\
&= \partial_\mu \partial_\nu - \partial_\nu \partial_\mu && (64)\\
&\quad + ieA_\mu \partial_\nu + \partial_\mu ieA_\nu - ieA_\nu \partial_\mu - \partial_\nu ieA_\mu && (65)\\
&\quad + e^2 A_\nu A_\mu - e^2 A_\mu A_\nu && (66)\\
&= [\partial_\mu, \partial_\nu] + ie([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu]) - e^2[A_\mu, A_\nu] && (67)\\
&= ie(\partial_\mu A_\nu - \partial_\mu A_\nu) = ieF_{\mu\nu}. && (68)
\end{aligned}
$$

Finally we can conceive of a Lagrangian that is gauge invariant. As an example, we provide here the Lagrangian for Quantum Electrodynamics which describes our fermion field ψ interacting with the electromagnetic field where in this case $F_{\mu\nu}$ is the electromagnetic field tensor.

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{69}$$

γ^μ are the Dirac matrices and $\overline{\psi}$ is the Dirac adjoint [3].

We now generalise all this theory to the gauge group $SU(2)$ which is non-Abelian. Note that $SU(2)$ is isomorphic to $SO(3)$ and it is therefore helpful to think of it as the group of rotations in 3-space. Let σ^i with $i \in \{1, 2, 3\}$, be the Pauli matrices. These are Hermitian as is required since $T^i = \frac{\sigma^i}{2}$ are the three generators of our special unitary group, $SU(2)$. Let $\psi(x) = \begin{bmatrix}\psi_1(x)\\ \psi_2(x)\end{bmatrix}$ be a doublet of Dirac fields. The condition of local symmetry we now need is,

$$\psi(x) \to V(x)\psi(x) \qquad V(x) = e^{i\alpha^i(x)\frac{\sigma^i}{2}} \tag{70}$$

where $V(x) \in SU(2)$. Because of the non-Abelian nature of the gauge theory the three orthogonal transformations do not commute with each other. This causes some additional complications. The comparator must now transform in the following more general manner,

$$U(y, x) \to V(y)U(y, x)V^\dagger(x) \qquad U(y, y) = 1 \qquad |\det(U(y, y))| = 1. \tag{71}$$

For $x \neq y$ we restrict the comparator to a special unitary matrix since we restricted ourselves to working with a special unitary group. Near $U = 1$ we can then expand the comparator in terms of the aforementioned Hermitian generators. The covariant derivative is then,

$$D_\mu = \partial_\mu - igA^i_\mu \frac{\sigma^i}{2}, \tag{72}$$

where g is some constant. The new connection obeys the transformation law,

$$A^i_\mu(x)\frac{\sigma^i}{2} \to V(x)\left(A^i_\mu(x)\frac{\sigma^i}{2} + \frac{i}{g}\partial_\mu\right)V^\dagger(x). \tag{73}$$

We can then consider the infinitesimal transformation of the connection by expanding $V(x)$ to first order in α to get,

$$A^i_\mu(x)\frac{\sigma^i}{2} \to A^i_\mu(x)\frac{\sigma^i}{2} + \frac{1}{g}(\partial_\mu\alpha^i)\frac{\sigma^i}{2} + i[\alpha^i\frac{\sigma^i}{2}, A^j_\mu(x)\frac{\sigma^j}{2}] + \dots. \tag{74}$$

Following from this, the infinitesimal transformation of the covariant derivative is then,

$$D_\mu\psi \to \left(1 + i\alpha^i \frac{\sigma^i}{2}\right) D_\mu\psi, \tag{75}$$

up to terms of order α^2. The curvature is then,

$$[D_\mu, D_\nu] = -igF^i_{\mu\nu}\frac{\sigma^i}{2}, \tag{76}$$

with,

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk}A^j_\mu A^k_\nu, \tag{77}$$

where the Levi-Cevita symbols ϵ^{ijk}, are the structure constants in our particular case. The curvature has the transformation,

$$F^i_{\mu\nu}\frac{\sigma^i}{2} \to V(x)F^j_{\mu\nu}\frac{\sigma^j}{2}V^\dagger(x). \tag{78}$$

The infinitesimal transformation is then,

$$F^i_{\mu\nu} \to F^i_{\mu\nu} - \epsilon^{ijk}\alpha^j F^k_{\mu\nu}, \tag{79}$$

and we note that the curvature is now no longer gauge invariant since we now have three separate components each corresponding to an axis of rotation. A Lagrangian that is invariant under this new gauge group is the Yang-Mills Lagrangian [3],

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}(F^i_{\mu\nu})^2. \tag{80}$$

We now examine in more detail the comparator $U(y,x)$ in the case where x and y are separated by a finite distance. $U(y,x)$ depends on the path taken from x to y. The Wilson line is defined as,

$$U_P(z,y) := P\left\{\exp\left[ig\int_0^1 ds\frac{dx^\mu}{ds}A^a_\mu(x(s))T^a\right]\right\}, \tag{81}$$

where we have a path $P := x(s)$ from y to z and the notation $P\ldots$ is the path ordering. The path ordering works as follows. Let $x(s=0) = y$ and $x(s=1) = z$. Then $P\{\exp[\ldots]\}$ is the power series expansion of the exponential, where in each term of the expansion, the matrices with higher values of s go to the left. This is necessary because these matrices do not commute. The Wilson line is not gauge invariant. The Wilson loop,

$$\operatorname{tr} U_P(y,y), \tag{82}$$

defined as the trace of the Wilson line over a closed path, is instead gauge invariant.

3.2 The Feynman Path Integral for Fields

The Feynman Path Integral is an approach to Quantum Mechanics and Quantum Field Theory (QFT) that takes inspiration from Hamilton's action principle, Lagrangian mechanics, and the calculus of variation. Here we will consider the path integral in QFT. What makes the path integral daunting in this case is the fact that we have both non-physical degrees of freedom due to gauge invariance as well as physical degrees of freedom. We then need to count all the field configurations that are gauge transformations of each other, only once, while taking care to integrate over all physical degrees of freedom.[4] The action functional,

$$S[\Phi] = \int_{t_0}^{t} L[\Phi]\, dt, \tag{83}$$

is classical and real valued. The Lagrangian is defined over a Lagrangian density,

$$L[\Phi] = \int d^n x \mathcal{L}(\Phi, \partial_\mu \Phi), \tag{84}$$

where Φ is a field. The action functional obeys Hamilton's principle of least action $\delta S[\Phi] = 0$. The so called vacuum state, is the state with the lowest possible energy. The transition amplitude from the vacuum state at $t = -\infty$ to $t = +\infty$

$$Z[0] = \left\langle \mathbf{0}(t = +\infty) \middle| \exp\left(-i\hat{H}t\right) \middle| \mathbf{0}(t = -\infty) \right\rangle = \int \mathcal{D}\Phi e^{iS[\Phi]} \tag{85}$$

is often called the partition function. $\mathcal{D}\Phi$ is some Lebesgue type measure that needs to be chosen depending on the theory, to address the problem of counting only the physical degrees of freedom. The square of the absolute value of the transition amplitude is then the probability of such a transition occurring. In the path integral formalism of QFT, the states are trajectories in the configuration space. In the free theory $Z = 1$, because it starts and remains in the vacuum state. In the interaction theory we add a source term,

$$\mathcal{L} \to \mathcal{L} + J\Phi, \tag{86}$$

and then we have the generating functional,

$$Z[J] = \int \mathcal{D}\Phi e^{i \int d^n x [\mathcal{L}(\Phi, \partial_\mu \Phi) + J\Phi]}. \tag{87}$$

Finally, the Vacuum Expectation Value (VEV) of an observable is then,

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}\Phi \mathcal{O}[\Phi] e^{i \int d^n x [\mathcal{L}(\Phi, \partial_\mu \Phi) + J\Phi]}}{\int \mathcal{D}\Phi e^{i \int d^n x [\mathcal{L}(\Phi, \partial_\mu \Phi) + J\Phi]}}. \tag{88}$$

4 Non-Abelian Anyons

Chern-Simons Theory describes the behaviour of anyons. In this section we briefly explain what these quasiparticles are and where they actually appear in our physical world.

Consider N particles in $(3+1)$ spacetime, in initial positions $R_1, \ldots, R_N$ at initial time t_i. After reaching time t_f, they end up in final positions $R_1, \ldots, R_N$, that is, they end up in the same spatial positions. If the particles are distinguishable then the worldlines of the particles are topologically equivalent to straight lines in the direction of time. However, if the particles are indistinguishable we cannot tell which particle ended up in which position. In this case the worldlines of the particles can be classified in separate topological equivalence classes corresponding to the elements of the permutation group S_N.

In $(2+1)$ spacetime we have an altogether different situation. Consider two identical particles. One has constant position in space. The other wraps around the first before returning to its original spatial position. Whereas in $(3+1)$ spacetime, one could continuously deform the worldline of the second particle to one in which the particle is constant in space, in $(2+1)$ spacetime this is not possible. Hence, we have a well defined understanding of what it means for one particle to wind around another. Consider two identical particles in $(2+1)$ spacetime. When the particles are exchanged in a counterclockwise manner with the worldines twisting around each other, the wave function gains a phase,

$$\psi(\mathbf{r}_1, \mathbf{r}_2) \to e^{i\theta}\psi(\mathbf{r}_1, \mathbf{r}_2). \tag{89}$$

When $\theta = 0$ we have bosons. When $\theta = \pi$ we have fermions. When θ is otherwise, we have anyons with particle statistics θ. In this latter case the topological equivalence classes correspond to the elements of the braid group $\mathcal{B}_N$. The major difference between S_N and $\mathcal{B}_N$ is that the latter is infinite and non-Abelian. Non-Abelian anyons are then anyons whose wave function is trajectory dependent in a non-commutative manner. We will not go in further detail regarding the properties and behaviour of anyons. Further exploration on this topic can be found in [8].

We have not yet said anything about where to find anyons in our physical world. In effect, these quasiparticles are an emergent property of a many-particle system when it is confined to 2-dimensional plane, while at the same time being in a topological phase of matter. A system is in a topological phase of matter if at low temperatures and low energies and long wavelengths, all observable properties are invariant under smooth deformations of the spacetime manifold in which it lives. In this case there is an energy gap separating the degenerate ground state from the lowest excited states and the topological phase only holds when the system is in the ground states. Equivalently, one can say that a system is in a topological phase of matter if its low energy effective field theory is a topological quantum field theory (TQFT). A quantum field theory is topological if the vacuum expectation value (VEV) of any product of the operators of the theory is constant under a change of the metric of spacetime,

$$\frac{\delta}{\delta g^{\mu\nu}} \left\langle \mathcal{O}_{\sigma(1)} \mathcal{O}_{\sigma(2)} \dots \mathcal{O}_{\sigma(p)} \right\rangle = 0, \tag{90}$$

in which case they are the observables of the theory. One way to construct such a Topological Quantum Field Theory (TQFT), is to construct an action that is independent of the metric $g_{\mu\nu}$. These are so called Schwarz-type TQFTs.

We can now imagine the classical spacetime trajectory of an anyon-hole pair forming a loop, which we can think of as analogous to a knot from Knot Theory. The topological phase of matter in which the whole system resides, is then responsible for the topological properties analogous to those in Knot Theory.

5 Chern-Simons Theory

In this section we will introduce Chern-Simons Theory which is an example of a Schwarz-type TQFT, and we will see how the Jones Polynomial invariant defined in Section 2.10 (2.48) , can be rediscovered within it. In particular we will deal with a non-Abelian Chern-Simons theory, which describes non-Abelian anyons. In Chern-Simons Theory, we have an oriented 3-manifold M, and a compact simple gauge group G. Here we will work with $SU(2)$. The connection A_i has the infinitesimal gauge transformation law,

$$A_i \to A_i - D_i\varepsilon, \tag{91}$$

where ε is a generator of the gauge group and the covariant derivative is defined as,

$$D_i\varepsilon = \partial_i\varepsilon + [A_i, \varepsilon]. \tag{92}$$

Note that both A_i and ε are elements of the Lie algebra. The curvature is defined as

$$F_{ij} = [D_i, D_j] = \partial_i A_j - \partial_j A_i + [A_i, A_j]. \tag{93}$$

We then pick the Chern-Simons 3-form to form our Lagrangian, noting how no choice of metric is required anywhere in this definition. The action is given by,

$$S_{CS}[A] = \frac{k}{4\pi}\int_M \mathrm{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right), \tag{94}$$

where k is the coupling constant which determines the strength of the interaction. This action has the property that under a gauge transformation of winding number m, the action transforms as,

$$S_{CS}[A] \to S_{CS}[A] + 2\pi k m. \tag{95}$$

$SU(2)$ is the group of rotation transformations around the origin and the winding number of $SU(2)$ is the number of revolutions of the rotation around the origin, with a +ve sign if counterclockwise and −ve sign otherwise. We only require that $\exp(iS_{CS}[A])$ be gauge invariant. This means that the coupling constant k must then necessarily be an integer. For observables of the theory, we pick the Wilson loop operators,

$$W_R(K) = \mathrm{tr}_R\, P \exp \oint_K A_i dx^i, \tag{96}$$

where K is some oriented closed curve in M and R is the representation of the gauge group while as before P is the path ordering. Again this observable is independent of the metric. As one would have probably guessed, the oriented closed curve K is a knot. Let $L = K_i \sqcup \cdots \sqcup K_r$ be a link in M. Let R_i be a representation associated to each K_i. The Feynman path integral,

$$Z(M; L) = \int D\mathcal{A} \exp(iS[A]) \prod_{i=1}^{r} W_{R_i}(K_i), \tag{97}$$

is the partition function of the link L. It is an integral over all gauge orbits, which are just the equivalence classes of gauge fields, where a class contains all the equivalent field configurations due to gauge symmetry.

5.1 Calculation of Invariants

We will now explicitly show how the skein relation of the Jones polynomial invariant (30) emerges from Chern-Simons theory with gauge group $SU(2)$.

Consider an arbitrary 3-manifold M with some link L embedded in it. Let the Wilson loops forming this link have the fundamental representation of $SU(2)$ denoted here as R.

We identify a crossing of L and we surround it with a sphere S^2. This situation is depicted in Figure 22 for the trefoil knot. We can then think of M as the connected sum: $M = M_L \# M_R$ where M_L and M_R are the two disjoint subsets of M with boundary S^2, corresponding to the outside and inside of the S^2 respectively. That means that $M_R = S^2 \cup B_R$ where B_R is the 3-ball bounded by S^2, and $M_L = M \setminus B_R$. Note that the boundary of M_R is S^2 and the

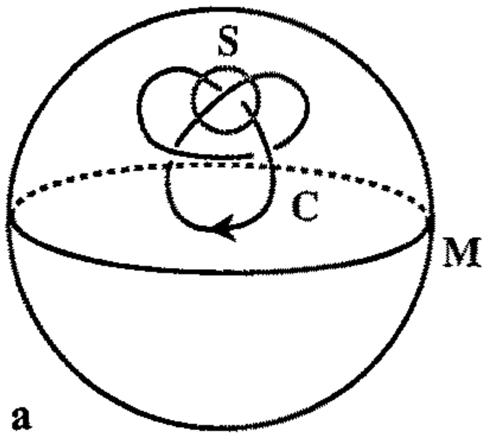


Figure 22: Manifold M containing the trefoil knot. A sphere encapsulates a crossing.

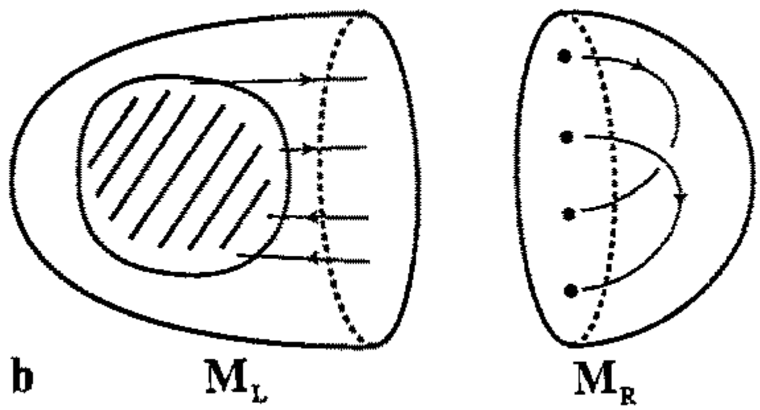


Figure 23: Manifold M split into M_R, which is the interior of S^2, and M_L which is the rest of M. On ∂M_R we see the 4 marked points.

boundary of M_L is also S^2, but ∂M_L has opposite orientation from ∂M_R. This situation is depicted in Figure 23 where the disk face of each manifold represents the boundary S^2.

The intersections of L with S^2 are then the so called marked points. If you surround just one crossing there will always be 4 of them, as depicted in Figure 23 on the boundary of M_R.

The path integral quantisation over M_L assigns a 2-dimensional physical Hilbert space $\mathcal{H}$ to $\partial M_L = S^2$. The dimensionality of the Hilbert space is determined by the number of marked points on S^2 in a non-trivial manner. Here we just quote this fact that is well known in Conformal Field Theory [5]. Then, the partition function on M_L, $Z(M_L)$, determines some vector χ in $\mathcal{H}$. In a very similar fashion, the path integral quantisation over M_R assigns a physical Hilbert space $\mathcal{H}'$ to $\partial M_R = S^2$ where $\mathcal{H}'$ is the dual space of $\mathcal{H}$. $Z(M_R)$ then determines some vector ψ in $\mathcal{H}'$.

Now, suppose we somehow apply a diffeomorphism to M_R in such a way that we swap the marked points around. Let us call the modified manifold M_R'. Then when we put $M = M_L \# M_R'$ back together, we would have changed the equivalence class of the knot, because

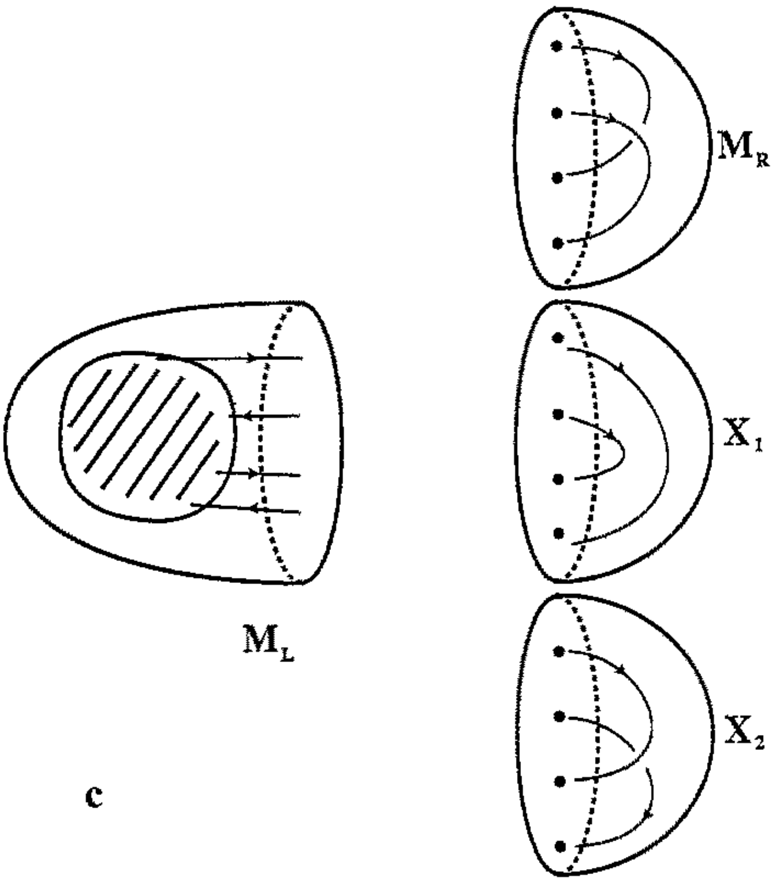


Figure 24: Manifold M_R and two possible diffeomorphisms of it, X_1 and X_2.

we would have modified the crossing. This is what we aim to do since this is analogous to the way in which the polynomial invariants in Knot Theory work. The skein relation (30) is the expansion of a link in terms of two variations of that same link. The two variations are obtained by taking any crossing and considering its two other variations from the set of the Conway triple depicted in Figure 16. We can write the partition function of L as,

$$Z(M;L) = (\chi, \psi), \tag{98}$$

where the RHS is the natural pairing of the vectors, meaning that for any nonzero vector $\chi \in \mathcal{H}$ there is a vector $\psi \in \mathcal{H}'$ such that $(\chi, \psi) \neq 0$, and vice-versa. This is a result from TQFT [9]. We can then consider two diffeomorphisms of M_R, namely X_1 and X_2 as pictured in Figure 24 that swap the marked points in such a way that they result in the other two crossings from the Conway triple. Let ψ_1 and ψ_2 be the corresponding vectors of X_1 and X_2 in $\mathcal{H}_R$. Since $\mathcal{H}_R$ is a 2-dimensional vector space, we can immediately conclude that there must be a linear dependence,

$$\alpha\psi + \beta\psi_1 + \gamma\psi_2 = 0, \tag{99}$$

from which it follows immediately that,

$$\alpha(\chi, \psi) + \beta(\chi, \psi_1) + \gamma(\chi, \psi_2) = 0, \tag{100}$$

and thus

$$\alpha Z(L) + \beta Z(L_1) + \gamma Z(L_2) = 0. \tag{101}$$

This is effectively already the skein relation analogous to Equation (30) of the Jones polynomial invariant.

However we strive to show how we can arrive to the Jones polynomial from this theory. We require first and foremost that the link L is embedded in S^3 (or equivalently $\mathbb{R}^3$) since links in Knot Theory are embedded in this space unless explicitly stated otherwise. We require that the gauge group is $SU(2)$ and that the Wilson loops forming the link components, have the fundamental representation, R as already stated above. Finally we require that the link L has standard framing as defined in Section 2.6.

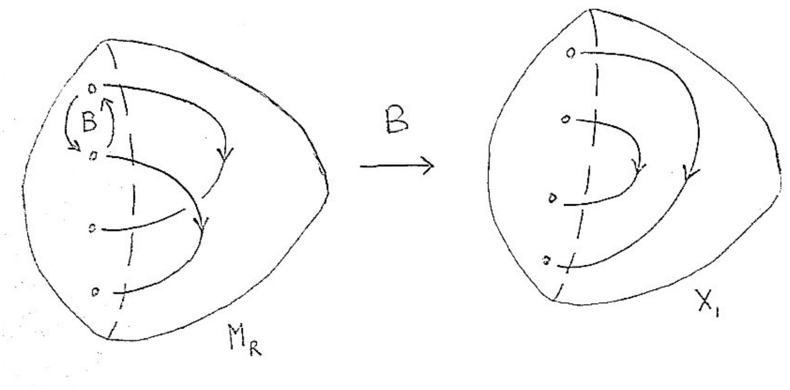


Figure 25: Diffeomorphism B.

To continue we have to compute the coefficients: α, β and γ. The diffeomorphisms of M_R that are useful in our case are called half-monodromies. Again these are concepts in Conformal Field Theory [5]. Here we just explain their behaviour. We will represent these half-monodromies as a linear transformation B such that,

$$\psi_1 = B\psi \qquad \psi_2 = B^2\psi. \tag{102}$$

The diffeomorphism B is pictured in Figure 25. Figure 26 shows the same diffeomorphism of M_R, with the standard framing of L made visible. We see that after the diffeomorphism the framing is not the standard framing anymore. A twist is induced in the framed link segments. To fix this we need to apply another kind of diffeomorphism, called a Dehn twist [5], pictured in Figure 27, to reverse this unwanted twisting. The effect of this operation on the vectors ψ, ψ_1 and ψ_2, is to multiply them by,

$$D = \exp\left(\frac{-3\pi i}{2(2+k)}\right). \tag{103}$$

Since B is 2 dimensional, the characteristic polynomial of B is,

$$B^2 - \mathrm{tr}(B)B + \det(B) = 0, \tag{104}$$

and then if we multiply everywhere by ψ we get,

$$B^2\psi - \mathrm{tr}(B)B\psi + \det(B)\psi = 0, \tag{105}$$

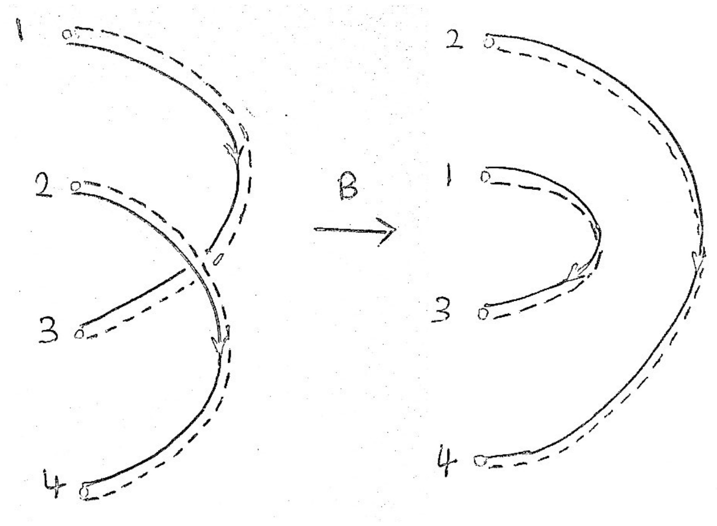


Figure 26: A twist is induced in the framing.

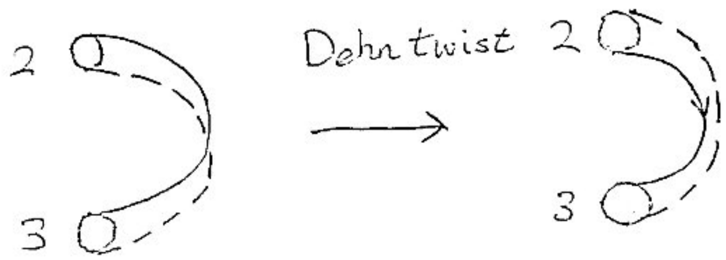


Figure 27: Dehn twist to fix framing.

which is equal to,

$$\psi_2 - \mathrm{tr}(B)\psi_1 + \det(B)\psi = 0. \tag{106}$$

The two eigenvalues of B for $SU(2)$ are

$$\lambda_1 = \exp\left(\frac{-i\pi}{2(2+k)}\right) \quad \lambda_2 = -\exp\left(\frac{3\pi i}{2(2+k)}\right), \tag{107}$$

which we then use together with Equation (99) and Equation (106) to determine that,

$$\begin{aligned} \alpha &= -\exp\left(\frac{\pi i}{2+k}\right), \\ \beta &= -\,\mathrm{tr}(B) = \exp\left(\frac{-\pi i}{2(2+k)}\right) - \exp\left(\frac{3\pi i}{2(2+k)}\right), \\ \gamma &= 1. \end{aligned} \tag{108}$$

But now we must consider that due to $\psi_1 = B\psi$, ψ has a twist in its framing. We want to revert back to standard framing. We do this by applying a Dehn diffeomorphism that untwists

ψ_1, which is equivalent to multiplying the coefficient of ψ_1 by D, defined in (103). Similarly, because $\psi_2 = B^2\psi$, ψ_2 needs to be untwisted by applying the Dehn diffeomorphism twice. Finally the real coefficients are then,

$$\begin{aligned}
\alpha &= -\exp\left(\frac{\pi i}{2+k}\right) \\
\beta &= \exp\left(\frac{-3\pi i}{2(2+k)}\right)\left[\exp\left(\frac{-\pi i}{2(2+k)}\right) - \exp\left(\frac{3\pi i}{2(2+k)}\right)\right] = \exp\left(\frac{-2\pi i}{2+k}\right) - 1 \\
\gamma &= \exp\left(\frac{-3\pi i}{2+k}\right)
\end{aligned} \tag{109}$$

We can simplify by multiply everything by $\exp\left(\frac{\pi i}{2+k}\right)$ and doing the substitution $t = \exp\left(\frac{2\pi i}{2+k}\right)$. We end up with the Jones polynomial skein relation,

$$-tZ(L) + (t^{1/2} - t^{-1/2})Z(L_1) + t^{-1}Z(L_2) = 0. \tag{110}$$

which agrees with (30). Finally we have that,

$$V_t(L) = \frac{Z(L)}{q} \qquad Z(\textit{unknotted Wilson loop}) = q. \tag{111}$$

That is, we have found a way to compute Jones polynomial invariants using the partition function of non-Abelian Chern Simons theory with gauge group $SU(2)$. In a very similar fashion we can also apply this theory to computing the HOMFLY and Kauffman polynomial invariants, defined in Appendix II. These correspond to Chern-Simons theory with gauge groups $U(N)$ and $SO(N)$, respectively.[10][5]

6 Conclusion

In this thesis we have retraced the steps that Edward Witten and others took in 1989. The original intent of Edward Witten and Michael Atiyah was to find an intrinsically 3-dimensional description of the Jones polynomial invariant. However the implications of this theory are much more significant than that.

We have just had a glimpse of the birth of a new field of mathematical physics. The study of Topological Quantum Field Theories has flourished in the last three decades. People are very excited about the idea of using a physical manifestation of Chern-Simons Theory to build a new kind of quantum computer, a so called topological quantum computer. Without going into detail, the main problem with conventional quantum computers is noise. Background noise leads to decoherence which causes data loss and inhibits computation. A topological quantum computer could store data and do computations using the worldlines of anyons. Any small perturbations will not change the topological equivalence class of these quantum states meaning that such a device could be much more tolerant to noise.[8]

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7 Appendix I: Topology of Surfaces

Definition 7.1 *(embedding) h is an embedding of X in Y, if h is a homeomorphism such that,*

$$h : X \to h(X) \subset Y. \tag{112}$$

Moreover, if X and Y are bounded and,

$$h(X) \cap \partial Y = h(\partial X), \tag{113}$$

h is a proper embedding.

Definition 7.2 *(immersion) $h : X \to h(X) \subset Y$ is an immersion if for all $x \in X$ there exists a neighbourhood U of x, such that $h : U \to h(U)$ is an embedding.*

An arc is the immersion of $[0, 1]$, and a loop is an immersed circle.

Theorem 7.1 *1. Let λ be a loop embedded in $\mathbb{R}^2$. It follows that,*

   (a) λ separates $\mathbb{R}^2$ into two connected components, each bounded by λ.

   (b) There exists a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(\lambda)$ is the unit circle.

   (c) For $\mathbb{R}^2 \setminus \lambda = C_1 \sqcup C_2$, either $\overline{C_1}$ or $\overline{C_2}$ are homeomorphic to a disk.

   (d) There are no wild knots in the plane.

   (e) All knots in the plane are trivial.

2. A loop λ embedded in $\mathbb{S}^2$ separates the sphere into two connected components, each of which is a disk bounded by λ.

3. A sphere embedded in $\mathbb{R}^3$ separates $\mathbb{R}^3$ into two connected components, each of which is bounded by the sphere.

4. If S is a tame embedding of a sphere in $\mathbb{R}^3$, then there is a homoemorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(S)$ is the unit sphere.

5. A tame sphere S, embedded in $\mathbb{R}^3$ separates the space into two components, and bounds a ball on one side.

6. A tame sphere S, embedded in $\mathbb{S}^3$ separates the glome into two components, and bounds a ball on both sides.

A surface F, has two intrinsic topological properties that are enough to distinguish one class of surfaces from another:

1. Boundary components, denoted $|\partial F|$: the cardinality of the set of loops that together form the boundary of F,

$$\partial F = \{l_1, l_2, ..., l_n\} \quad |\partial F| = n \tag{114}$$

   .

F	Sphere	Torus	Disc	Annulus
$\|\partial F\|$	0	0	1	2
$g(F)$	0	2	0	0

Table 2: Boundary components and genus of the some common surfaces.

2. Genus, denoted $g(F)$: This is a measure of the connectedness of a surface. Consider the set
$$C = \{\text{loops embedded in } F\} \cup \{\text{arcs properly embedded in } F\} \tag{115}$$
for a connected surface F. The genus is the maximum number of cuts along elements of C, that will keep the surface F connected. For a disconnected surface $g(F_1 \cup F_2) = g(F_1) + g(F_2)$.

Theorem 7.2 *The genus is a topological invariant of surfaces.*

We provide some examples of these properties for the most common surfaces in Table 2.

Theorem 7.3 *Two connected orientable surfaces are homeomorphic if and only if they have the same number of boundary components and the same genus.*

Theorem 7.4 *1. The genus of an orientable surface is even.*

2. A closed connected orientable surface of genus g is homeomorphic to a sphere with g handles attached. A handle is just a tubular part of a surface.

3. An orientable surface with boundary is homeomorphic to one of the closed surfaces with a set of disks removed.

4. A connected orientable surface with boundary is homeomorphic to a surface constructed by attaching bands to a disk.

Theorem 7.5 *An orientable surface can be triangulated.*

Definition 7.3 *The Euler characteristic of a triangulated surface F is*
$$\chi(F) = V - E + T \tag{116}$$
where

1. *V : number of vertices*
2. *E : number of edges*
3. *T : number of triangles*

Theorem 7.6 *The Euler characteristic is independent of the triangulation and is a topological invariant of the surface.*

Theorem 7.7 *For a connected surface F,*
$$2g(F) = 2 - \chi(F) - |\partial F| \tag{117}$$

Theorem 7.8 *Let F_1 and F_2 be two connected orientable surfaces, α be an arc and λ be a loop. Moreover,*

- $F_1 \sqcup F_2$ *: the distant union of F_1 and F_2,*
- $F_1 \cup_\alpha F_2$ *: the intersection of F_1 and F_2 is the arc: $F_1 \cap F_2 = \alpha$,*
- $F_1 \cup_\lambda F_2$ *: the intersection of F_1 and F_2 is the loop: $F_1 \cap F_2 = \lambda$.*

Then we have that,

F	$\chi(F)$	$g(F)$
$F_1 \sqcup F_2$	$\chi(F_1) + \chi(F_2)$	$g(F_1) + g(F_2)$
$F_1 \cup_\alpha F_2$	$\chi(F_1) + \chi(F_2) - 1$	$g(F_1) + g(F_2)$
$F_1 \cup_\lambda F_2$	$\chi(F_1) + \chi(F_2)$	$g(F_1) + g(F_2)$

8 Appendix II: Kauffman and HOMFLY Polynomials

Definition 8.1 *The Kauffman Polynomial is defined as*

$$F(D)(a,z) = a^{-\omega(L)} L(D) \tag{118}$$

where D is a link diagram, $\omega(D)$ is the writhe of the link diagram, and $L(D)$ is a polynomial in a and z defined on link diagrams by the following axioms:

1. *$L(\bigcirc) = 1$, where $\bigcirc$ is the unknot.*
2. *$L(s_r) = aL(s)$, where s is a strand, and s_r is the result of applying a right handed Type I Reidemeister move on s and s_l is the result of applying a left handed Type I Reidemeister move.*
3. *$L(s_l) = a^{-1}L(s)$.*
4. *L is constant under Type II and Type III Reidemeister moves.*
5. *Skein relation: $L(D_+) + L(D_-) = zL(D_0) + zL(D_\infty)$.*

We provide an example of a Kauffman polynomial calculation in Figure 28 and Figure 29. We find out that the polynomial for the knot 4_1 is,

$$a^2z^2 + z^2a^{-2} - a^2 - a^{-2} + az^3 + a^{-1}z^3 - az - za^{-1} + 2z^2 - 1. \tag{119}$$

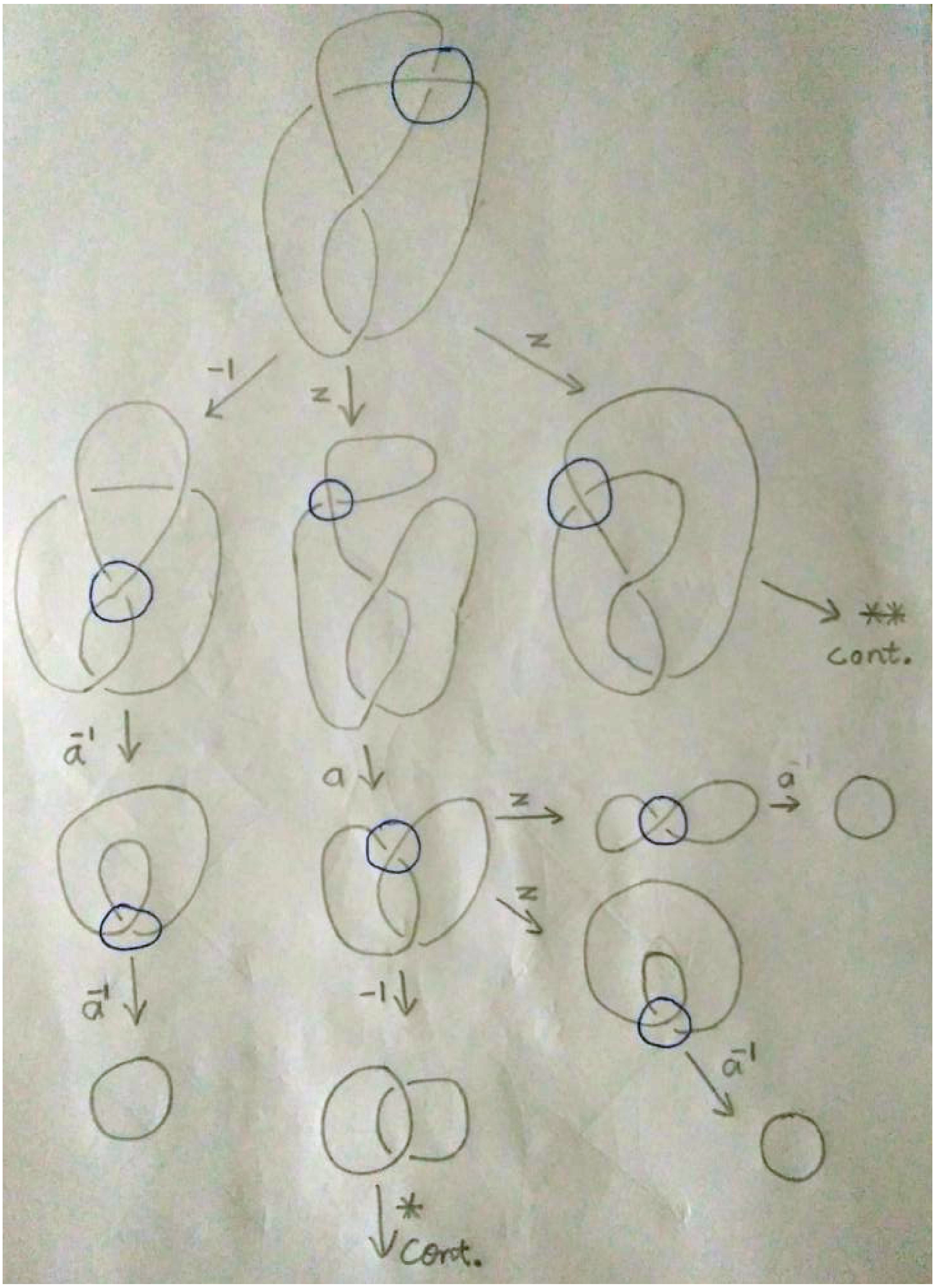

Figure 28: Working out the Kauffman polynomial for the knot 4_1.

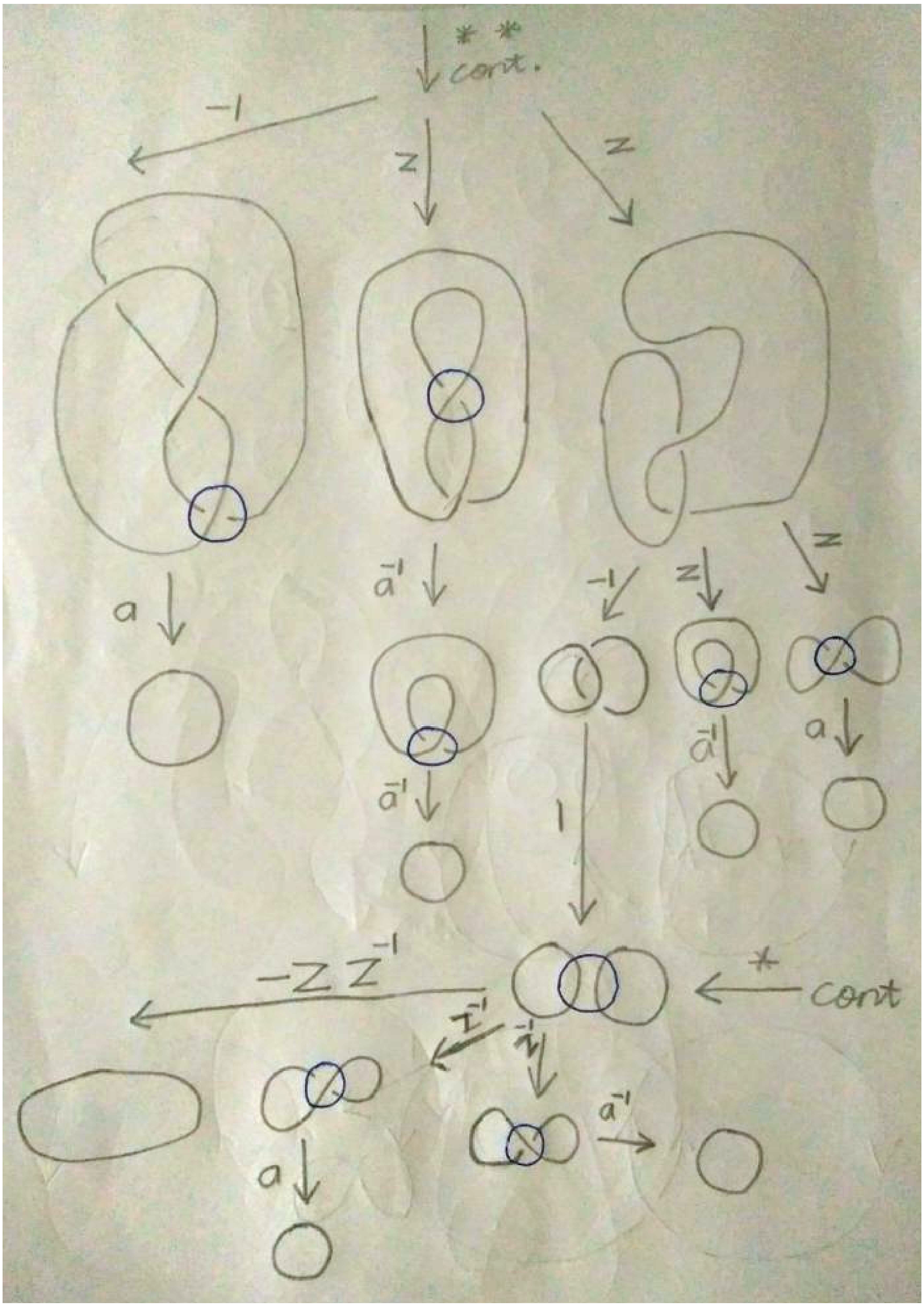

Figure 29: Continued example of Kauffman polynomial for 4_1.

Definition 8.2 *The HOMFLY polynomial of an oriented link L, denoted $P(L)$ or $P_L(v, z)$ is defined by the following three axioms:*

1. *Invariance: $P_L(v, z)$ is invariant under ambient isotopy of L.*
2. *Normalisation: if K is the trivial knot, $P_K(v, z) = 1$.*
3. *Skein relation: $v^{-1}P(L_+) - vP(L_-) = zP(L_0)$ where L_+, L_-, L_0, have diagrams D_+, D_-, D_0 as in Figure 16, in the neighbourhood of the relevant crossing.*

We provide an example of a HOMFLY polynomial calculation in Figure 30. We end up with the following polynomial for the 4_1 knot,

$$a^2 + a^{-2} - z^2 - 1. \tag{120}$$

Some properties of the Kauffman and HOMFLY polynomials:

1. $P(L_1 \# L_2) = P(L_1)P(L_2)$
2. $F(L_1 \# L_2) = F(L_1)F(L_2)$
3. $P(L_1 \sqcup L_2) = z^{-1}(v^{-1} - v)P(L_1)P(L_2)$
4. $F(L_1 \sqcup L_2) = ((a + a^{-1})z^{-1} - 1)F(L_1)F(L_2)$

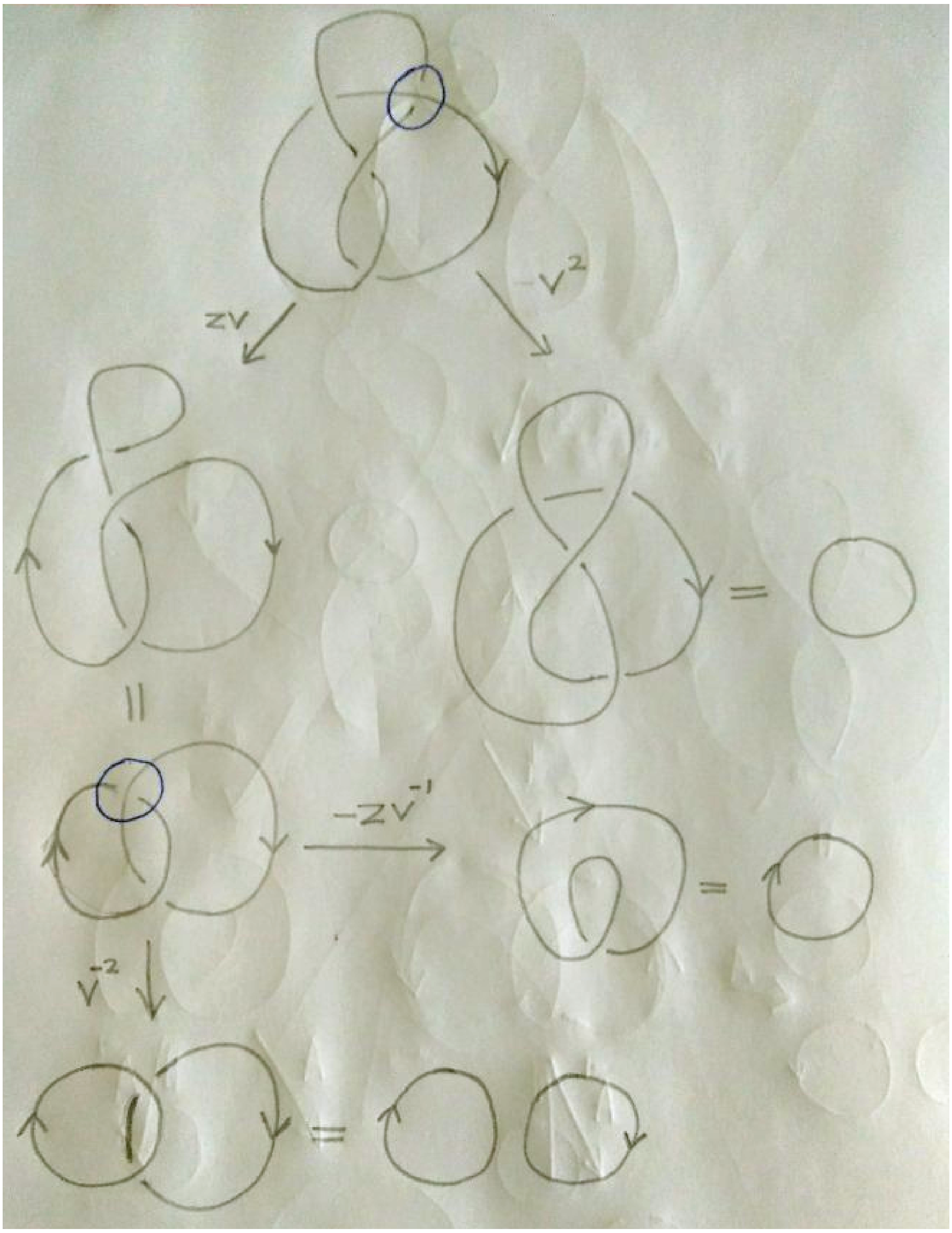

Figure 30: Working out the HOMFLY polynomial for the knot 4_1.