Correlation functions in a CFT dual to a Black Hole

Computing correlation functions in a Conformal Field Theory which encode the spectrum for the Hawking Radiation

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CONTENT

CONTENTS

1

INTRODUCTION

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The physicist needs a facility
in looking at problems
from several points of view.
-Feynman-
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As Feynnman states so elegantly, in physics one of the greatest trump cards you can have, is being able to look at a problem from different perspectives, transforming a problem from one representation into another, or dualities between different theories. In physics we often find analogies or discover dualities between different subjects. Due to these dualities, techniques designed to solve one problem can often be used to aid in other fields of research. One of these dualities can be found between Quantum Field Theory and Statistical Mechanics. Although these fields of study originated from totally different roots, they are very closely linked [1, 2].

Another impact-full duality is the AdS/CFT correspondence often referred to as Holography[3, 4, 5, 6, 7]. The AdS/CFT correspondence is interesting as it's branches reach far and wide over the spectrum of physics. First of all it connects General Relativity(GR) with Quantum Field Theory(QFT), more specifically Conformal Field Theory. As QFT is closely linked to statistical mechanics it also grows it's branches in this direction; more specifically, in the case of a black hole in AdS a pure state behaves as a thermal one. As the correspondence links a String Theory on AdS to CFT, also string theory is incorporated. Also, Condensed matter physics uses the dictionary to find solutions to their problems[8]. Many aspects from Information Theory appear like the concept of information and entropy. And even some twigs spread into the research of networks as Tensor Networks can be used to express quantum wave functions in term of network diagrams[9].

The AdS/CFT correspondence has the great feature that it connects a gravity theory with a QFT. Therefore it has the possibility to shed light on problems arising in these theories. One of these problems is the Black Hole information paradox [10, 11], in 1975 Hawking calculated that a Black Hole emits Black Body radiation[12]; consequently, the radiation is only dependent on the temperature of the Black Hole and independent on the information that has entered the Black Hole. In other words, the Black Hole can destroy information that enters it. This is in contrast with the notion that Quantum Mechanics is a unitary theory and therefore quantum systems preserve information.

To better understand the relation between particles that fall into the Black Hole and the emitted radiation one can calculate correlation functions in a CFT that is dual to a Black Hole, for example, a four-point function with two heavy insertions and two light insertions which models a particle falling in a Black Hole and emitted at a later time. Or one can see if the Hawking radiation behaves chaotically by looking at the six-point function.

In this thesis we try to approximate this six-point function with $1/c$ contributions in a 2D CFT theory that is dual to an AdS Black Hole. Multiple attempts have been done to approximate higher point functions, but in general these attempts only take into account the leading contributions. We study this six-point function by setting up the theoretical framework for the AdS/CFT correspondence in chapter 2 where we discuss various concepts within this framework; i.e, Anti de Sitter space, Conformal Field Theory, Scrambling time, and finish with the idea of Quantum chaos which leads us to the six-point function. We see that to understand the chaoticness of a Black hole, a six-point function can be used to describe this. However, from discussing the CFT we know that correlation functions with more than three-points are difficult to compute, because they cannot be obtained immediately from the symmetry. The calculation of these higher point functions; specifically the four and six point function in the CFT framework are the main foci of this thesis.

In chapter 3 we study the paper [13], which is our main handle in exploring these functions. In this paper a four-point function with two heavy insertions and two light insertions is calculated. This four-point function describes light particle dropping into a black hole and retrieving it at later times. This correlation function is calculated by introducing a projection operator which splits the four-point function into a summation over Virasoro conformal blocks. Effectively a summation of terms consisting of a normalization in the denominator and two correlation functions in the numerator is obtained.

In chapter 4 we try to approximate the six-point function in a similar manner as in the previous chapter. In this case the projection operator has to be inserted twice; hence, we obtain three correlation functions in the numerator and a two normalization terms in the denominator. We start out by calculating the denominator and find clean results. For the numerator however, we haven't been able to find any clear result. The main problem we run into, is that, if we use the exact same approximation as in chapter 3 we need to have an expression for the four-point function.

2

THE ADS/CFT CORRESPONDENCE

To understand and solve a six-point function in a CFT theory that is dual to a Black Hole in Anti-de-Sitter space. We start out with building the theoretical framework of the AdS/CFT correspondence in which this problem is nested.

In this chapter we start with an introduction to Anti-de-Sitter Space; followed by an introduction to Conformal Field Theory, where we'll start with the underlying symmetry, introduce the commutation relations, and the Operator Product Expansion. These two tools will be used later to approximate the higher-point functions.

Subsequently we'll connect these fields and arrive at the AdS/CFT correspondence in section 2.3. Here we discuss the relation between the conformal dimension and mass, Entanglement entropy, and the Cardy formula.

Next we look into the idea of a pure system behaving as a mixed state in section 2.3.5 introducing the Thermal field Double state and touching on the ER=EPR paradox.

In section 2.3.7 we discuss concept of scrambling time, which reappears in the following section 2.3.8 on Quantum Chaos.

Next in section 2.3.9 we want to use this quantum chaos to calculate the chaoticness of a black hole. We see that a six-point function is needed to shed light on this concept.

2.1 ADS

Anti-de-Sitter is a space in General Relativity named after Willem de Sitter. It's a solution to Einstein's Field Equation[14] seen in eq.1 with negative Cosmological Constant Λ.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{1}$$

with G Newtons gravitational constant, c the speed of light. This space arises naturally from a gauged super-gravity theory. AdS also has a positive brother which arises naturally in inflation theory, called the De-Sitter space. It's similar but with a positive cosmological constant Λ. Both spaces are maximally symmetric Lorentzian manifolds[15].

AdS$_{d+1}$ is defined by the universal cover of the manifold with a Lorentzian signature[16]:

$$-x_0^2 - x_{d+1}^2 + \sum_{n=1}^{d} x_n^2 = -R^2 \tag{2}$$

Embedded in a pseudo-Riemannian manifold $R^{2,d-1}$ space with metric:

$$ds^2 = -dx_0^2 - dx_{d+1}^2 + \sum_{n=1}^{d} dx_n^2 \tag{3}$$

In Lorentzian signature AdS$_{d+1}$ space has a $SO(d,2)$ isometry. For convenience it is insight full to look at the Euclidean version of AdS, which is invariant under $SO(d+1,1)$, this signature is usually used for calculations. One can map the two spaces and solutions in found in the two spaces by using a Wick rotation, $x_{d+1} \rightarrow ix_{d+1}$. For example, an AdS$_2$ space is mapped to an H^2 when going from a Lorentzian to an Euclidean space. To illustrate the AdS space, we follow the general outline of [17].

$$-x_0^2 + x_{d+1}^2 + \sum_{n=1}^{d} x_n^2 = -R^2 \tag{4}$$

embedded in $R^{1,d+1}$. We can define Poincare coordinates as

$$\begin{aligned} X^0 &= \frac{R}{2}\frac{1+x^2+z^2}{z} \\ X^\mu &= R\frac{x^\mu}{z} \\ X^{d+1} &= \frac{R}{2}\frac{1-x^2-z^2}{z} \end{aligned} \tag{5}$$

and the metric

$$ds^2 = R^2\frac{dz^2 + \delta_{\mu\nu}dx^\mu dx^\nu}{z^2} \tag{6}$$

which diverges at $z = 0$. This metric can also be written as

$$ds^2 = \left(\frac{r}{R}\right)^2 dt^2 + \left(\frac{R}{r}\right)^2 dr^2 + \left(\frac{r}{R}\right)^2 \overrightarrow{dx}_3^2 \tag{7}$$

Or in the Lorentzian signature we obtain

$$ds^2 = -\left(\frac{r}{R}\right)^2 dt^2 + \left(\frac{R}{r}\right)^2 dr^2 + \left(\frac{r}{R}\right)^2 \overrightarrow{dx}_3^2 \tag{8}$$

Another useful coordinate system in Lorentzian signature is the Global coordinate system with universal cover $t \in R$

$$\begin{aligned} X^0 &= R\cos(t)\cosh(\rho) \\ X^\mu &= R\Omega^\mu \sinh(\rho) \\ X^{d+1} &= -R\sin(t)\cosh(\rho) \end{aligned} \tag{9}$$

where Ω^μ parametrizes a S^{d-1} with metric

$$ds^2 = R^2[-\cosh(\rho)^2 dt^2 + d\rho^2 + \sinh(\rho)^2 d\Omega_{d-1}^2] \tag{10}$$

In the limit $\rho \rightarrow \infty$ we approach the conformal boundary. To make this explicit we can compactify, by $\tanh(\rho) = \sin(r)$, the resulting metric is conformal to a solid cylinder with boundary $R \otimes S^{d-1}$ at $r = \frac{\pi}{2}$.

$$ds^2 = R^2 \left[-\cos(r)^2 dt^2 + dr^2 + \sin(r)^2 d\Omega_{d-1}^2 \right] \tag{11}$$

To obtain the trajectories of massive particles in AdS spacetime we can look at the geodesics in AdS. A simple example is a particle at $\rho = 0$ hence $X^\mu = 0$ in global coordinates, we can boost in X^1, X^{d+1} with an equivalent time-like geodesic $X^1 \cosh(\beta) = X^{d+1} \sinh(\beta)$.

$$\tanh(\rho) = \tanh(\beta) \sin(t) \tag{12}$$

It can be observed that timelike geodesics oscillate with period 2π in global time as depicted in fig.1.

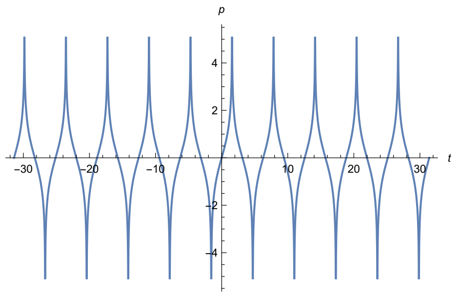


Figure 1.: Timelike geodesic: $\rho = \tanh^{-1}(\sin(t))$

In the case of light rays we want to have a look at null-geodesics. Looking at the null-ray

$$\begin{aligned} 0 &= X^{d+1} - X^1 = X^\mu = X^0 - R \\ &= R\cos(t)\cosh(\rho) - R \end{aligned} \tag{13}$$

We obtain the equality describing the orbit below

$$\cosh(\rho) = \frac{1}{\cos(t)} \tag{14}$$

Light rays start and end at the conformal boundary $\rho = \infty, t = \pm\pi/2$.

Instead of an empty Anti-de-sitter space, we can also throw in a heavy object, like a Black hole. In the case of a Schwarzschild Black Hole in AdS, often coined as Schwarzschild-AdS or SAdS, we get the metric seen below.

$$ds_5^2 = \left(\frac{r_0}{L}\right)^2 \frac{1}{u}(-hdt^2 + dx_3^2) + L^2 \frac{du^2}{hu^2} \tag{15}$$

with $h = 1 - u^4$. the boundary is at $u = 0$ and the horizon of the Black Hole at $u = 0$.

There also exist a solution for a Black Hole in AdS_3; i.e, the BTZ Black Hole[18]. With the metric

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2 \tag{16}$$

with $N^2 = -M + \frac{r^2}{l^2} + \frac{J^2}{4\pi r^2}$ and $N^\phi = -\frac{J}{2\pi r^2}$. This space is of specific interest, because the CFT on the boundary of this space, is a 2D CFT. In the next section we will see that a 2D CFT has some special properties.

2.1.1 *Information Paradox*

Although Black Holes were thought to only grow and eat all information that gets to close, Hawking proved in 1975[12] that they also radiate. More precisely that they emit thermal radiation. This radiation is better known has Hawking radiation. The spectrum of this radiation is only dependent on the temperature of the Black hole. This poses a problem, because it is independent of all the information the Black Hole absorbed. Therefore, when a Black Hole completely evaporates all the in initial information is lost. This is in opposition to the principle of unitarity in Quantum Mechanics. This paradox is also known as the Information Paradox.

2.2 CFT

This section we will look at second part of the duality, the CFT. Conformal Field Theory or CFT is a Quantum Field Theory that is invariant under conformal transformations. In Quantum Field Theories symmetries underlie the qualitative behaviour of the fields, hence the behaviour of the field can be understood by studying it's symmetry group; i.e, conformal symmetry[1].

We will follow the general outline of [19]. Conformal transformation acts on the metric as a Weyl transformation; namely, the metric is multiplied by a scalar $\Omega(x)$.

$$\begin{aligned} g'_{\mu\nu} &\to \Omega(x) g_{\mu\nu}(x) \\ g_{\mu\nu}(x) &\to g_{\mu\nu}(x) + w(x) g_{\mu\nu}(x) \end{aligned} \tag{17}$$

From this basic symmetry we can extract multiple transformations that leave the theory invariant; i.e,

- Translations: $x^\mu \to x^\mu + \epsilon^\mu$
- Lorentz Rotations: $x^\mu \to x^\mu + \omega^\mu_\nu x^\nu$
- Scale transformations: $x^\mu \to x^\mu + \sigma x^\mu$
- Special conformal transformation: $x^\mu \to x^\mu + b^\mu x^2 - 2x^\mu b \cdot x$

For each of these transformation one can identify a Differential Operator that transform a function $F'(x) \to F(x) + O(x, dx)F(x)$. Also called the generators of the conformal algebra.

$$\begin{aligned} P_\mu &= -i\partial_\mu \\ M_{\mu\nu} &= i(x_\mu \partial_\nu + x_\nu \partial_\mu) \\ D &= -i x_\mu \partial^\nu \\ K_\mu &= i(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu) \end{aligned} \tag{18}$$

Neother's theorem states that to each symmetry of a local Lagrangian, there corresponds a Conserved current; i.e, a charge that is conserved under its transformations[2]. For an infinitesimal conformal transformation ϵ^ν the change in $g_{\mu\nu}$ is

$$\delta g_{\mu\nu} = -\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu \tag{19}$$

According to eq.17 this has to be equal to $w(x) g_{\mu\nu}$. After taking the trace the condition for ϵ^ν seen below is obtained.

$$\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu = g_{\mu\nu} \frac{2}{d} \partial_\mu \epsilon_\mu \tag{20}$$

1 Although conformal invariance and scale invariance are often used interchangeably, they are not equal.

The resulting conserved current is $J(\epsilon)_\mu = T_{\mu\nu}\epsilon^\nu$ [19]. The energy-momentum tensor $T^{\mu\nu}$ can be defined in terms of the change in the action δS. Substitute $\delta g_{\mu\nu} \to w(x) g(x)_{\mu\nu}$ in the second line.

$$\begin{aligned} 0 = \delta S &= \frac{1}{2}\int d^d x \sqrt{g} T^{\mu\nu} \delta g_{\mu\nu} \\ &= \frac{1}{2}\int d^d x \sqrt{g} T^\mu_\mu w(x) \end{aligned} \tag{21}$$

Hence, $T^\mu_\mu = 0$ is the condition for Weyl invariance if this is to hold for any $w(x)$. Returning to the variation in the conserved current $\partial^\mu J(\epsilon)$ we see

$$\begin{aligned} \partial^\mu J(\epsilon)_\mu &= \partial^\mu (T_{\mu\nu}\epsilon^\nu) \\ &= (\partial^\mu T_{\mu\nu})\epsilon^\nu + T_{\mu\nu}(\partial^\mu \epsilon^\nu) \\ &= 0 + 0 \end{aligned} \tag{22}$$

$\partial^\mu T_{\mu\nu} = 0$ because the Energy-momentum tensor is conserved and $T_{\mu\nu}(\partial^\mu \epsilon^\nu) = 0$ due to eq.20.

In 2D the CFT has a special property, that is $\epsilon(x)$ is not restricted to be at most of second order in x. This can be understood by writing $\epsilon(x)$ in complex coordinates.

$$\begin{aligned} \partial_z \bar{\epsilon}(z, \bar{z}) = 0 \to \partial_z \bar{\epsilon}(\bar{z}) = 0 \\ \partial_{\bar{z}} \epsilon(z, \bar{z}) = 0 \to \partial_{\bar{z}} \epsilon(z) = 0 \end{aligned} \tag{23}$$

From this equality one can conclude that $\epsilon(z)$ can be an arbitrary function of z. The global transformation connecting all these results is $z \to f(z)$, infinitesimal transformation can be considered and a generating operator L_n can be defined.

$$L_n = -z^{n+1}\partial_z \tag{24}$$

This generating operator L_n are the normalized operators for the infinite series of currents J^n.

$$\begin{aligned} T(z) &= \sum_n z^{-n-2} L_n \\ L_n &= \oint \frac{dz}{2\pi i} z^{n+1} T(z) \end{aligned} \tag{25}$$

with raising operators

$$..., L_{-3}, L_{-2} \tag{26}$$

, lowering operators

$$L_2, L_3, ... \tag{27}$$

, and generators

$$L_{-1}, L_0, L_1 \tag{28}$$

These last three generate their own subalgebra, with commutation relations

$$\begin{aligned} [L_0, L_{-1}] &= L_{-1} \\ [L_0, L_1] &= -L_1 \\ [L_1, L_{-1}] &= 2L_0 \end{aligned} \tag{29}$$

Which is isomorphic $SU(2)$ algebra[2] when the identifications $L_0 = J$, $iL_1 = J^-$, and $iL_{-1} = J^+$ are made.

$L_{-1}, L_0 L_1$ are related to the generators of the conformal algebra seen in eq.18. In the case of 2D Euclidean coordinates $z = x_1 - ix_2$ we get the direct relations seen below.

$$\begin{aligned} P_1 &= -i(\partial_z + \partial_{\bar{z}}) &&= i(L_{-1} + \bar{L}_{-1}) \\ P_2 &= -i(\partial_z - \partial_{\bar{z}}) &&= i(L_{-1} - \bar{L}_{-1}) \\ M &= z\partial_z - \bar{z}\partial_{\bar{z}} &&= -L_0 + \bar{L}_0 \\ D &= -i(z\partial_z + \bar{z}\partial_{\bar{z}}) &&= i(L_0 + \bar{L}_0) \\ K_1 &= -i(z^2\partial_z - \bar{z}^2\partial_{\bar{z}}) &&= i(L_1 + \bar{L}_1) \\ K_2 &= -(z^2\partial_z - \bar{z}^2\partial_{\bar{z}}) &&= -(L_1 - \bar{L}_1) \end{aligned} \tag{30}$$

2.2.1 *Operator Product Expansion*

One nifty trick in QFT is the Operator Product Expansion or OPE. With this approach the product of two operators can be replaced with an effective vertex. Wilson and Zimmerman[20] proposed that the product could be replaced with a linear combination of operators.

$$\mathcal{O}_1(x)\mathcal{O}_2(0) \to \sum_n C_{12}^n(x)\mathcal{O}_n(0) \tag{31}$$

Consequently, a Greens function can be expended for small x in a sum over n of Greens functions G_n.

In Conformal Field theory we only deal with fields that are primaries or derivatives of primaries(descendants); therefore, in a CFT this sum will be over primaries or derivatives of primaries[21]. Hence, eq. 31 will be a sum over primaries $\mathcal{O}$ with $C_{12}^n(x) \to C_{12}^n(x, \partial_y)$ being a power series in ∂_y.

$$\mathcal{O}_1(x)\mathcal{O}_2(0) \to \sum_{\mathcal{O}} C_{\mathcal{O}}^n(x, \partial_y)\mathcal{O}(y)|_{y\to 0} \tag{32}$$

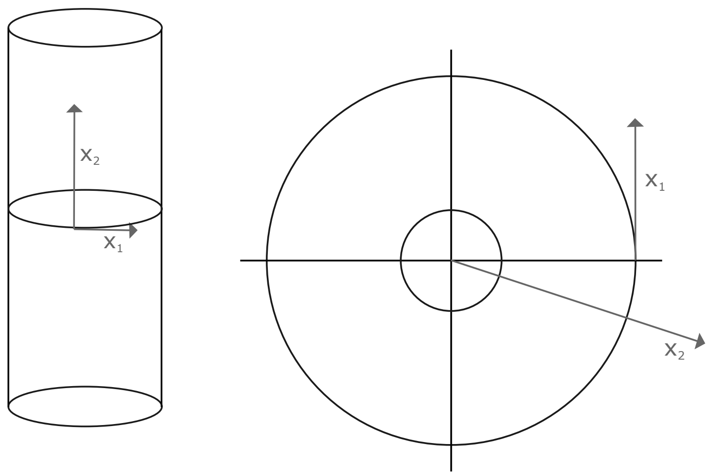
2 $SU(2)$ algebra: $[J, J^+] = J^+$, $[J, J^-] = -J^-$, and $[J^+, J^-] = 2J$

2.2.2 *Radial Quantization and the Virasoro Algebra*

The precise form of the OPE can be acquired by looking at the charge Q related to the conformal transformation on the complex plane. The charge Q is defined as

$$Q = \int d^{d-1}x J^0(x,t) \tag{33}$$

In 2D we only have an integral over one x, this can be simplified even further by using Radial Quantization. The space can be made finite by imposing periodic boundary conditions and as the theory is scale invariant we can put this at any value, hence we choose 2π. Therefore, we can write the coordinates as on the cylinder $(x^1, x^2) = (x^1, ix^0)$. Next we map the cylinder to the complex plane by introducing complex coordinates $z = x^1 - ix^2$ and performing a conformal transformation $w = e^{iz} = e^{x^2+ix^1}$ as illustrated in fig.2.



(a) On the Euclidian Cylinder

(b) On the complex plane

Figure 2.: Conformal transformation $w = e^{x^2+ix^1}$

By imposing periodic boundary conditions and mapping to the complex plane, the integral is expressed as a contour integral around z and $\bar{z}$ and $J_2 = -i(J_z - J_{\bar{z}})$.

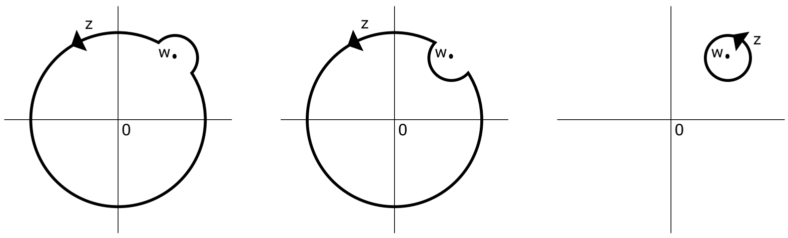
$$\begin{aligned} Q &= \frac{1}{2\pi}\int_0^{2\pi} dx^1 J^0 = \frac{1}{2\pi}\int_0^{2\pi} dx^1(-iJ_2) \\ &= -\frac{1}{2\pi}\left[\oint dz J_z^{cyl}(z,\bar{z}) - \oint d\bar{z} J_{\bar{z}}^{cyl}(z,\bar{z})\right] \end{aligned} \tag{34}$$

When we express the current in $\epsilon(z)T(z)$

$$Q = \frac{1}{2\pi}\left[\oint dz\epsilon(z)T(z) + \oint d\bar{z}\epsilon(\bar{z})\bar{T}(\bar{z})\right] \tag{35}$$

From conformal symmetry, we expect the infinitesimal form of the transformation generated by Q to look like the equation below, in the quantum world, this can be expressed as in the second line. However this only makes if $z > w$ in the first part and $w > z$ in the second part, that is, the operators have to be ordered radially $R\left(T(z)\phi(w,\bar{w})\right)$ just as operators have to be time ordered on the cylinder.

$$
\begin{aligned}
\delta_\epsilon \phi(w,\bar{w}) &= h\partial_w \epsilon(w)\phi(w,\bar{w}) + \epsilon(w)\partial\phi(w,\bar{w}) \\
&= [Q_\epsilon, \phi(w,\bar{w})] \\
&= \frac{1}{2\pi i}\left[\oint_{|z|>|w|} dz\epsilon(z)T(z)\phi(w,\bar{w}) - \oint_{|z|<|w|} dz\epsilon(z)\phi(w,\bar{w})T(z)\right] \\
&= \frac{1}{2\pi i}\left[\oint_{|z|>|w|} - \oint_{|z|<|w|}\right] dz\epsilon(z)R\left(T(z)\phi(w,\bar{w})\right) \\
&= \frac{1}{2\pi i}\oint dz\epsilon(z)R\left(T(z)\phi(w,\bar{w})\right)
\end{aligned}
\tag{36}
$$



(a) $z > w$ *(b) $z < w$* *(c) $z > w$ minus $z < w$*

Figure 3.: The different contour integrals

This product of operators can be expressed as a sum of descendants as seen in eq.32. The integration only makes sense if $R\left(T(z)\phi(w,\bar{w})\right)$ is analytic close to w, hence we can express this as a Laurent series. Now, the only way this Laurent series holds to the previous restriction is

$$
\begin{aligned}
R\left(T(z)\phi(w,\bar{w})\right) &= \sum_n (z-w)^n O_n(w,\bar{w}) \\
&= \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{(z-w)}\partial_w\phi(w,\bar{w})
\end{aligned}
\tag{37}
$$

where the power series in $(z-w)$ is excluded as they are free of poles and therefore not contribute to the contour integral.

Hence, the OPE of $T(z)O(z_i)$ can be expressed as

$$
T(z)O(z_i) = \frac{h_i}{(z-z_i)^2}O(z_i) + \frac{1}{z-z_i}\partial_{(z_i)}O(z_i) \tag{38}
$$

Now the commutation relation $[T(z), T(w)]$ can be obtained.

$$[T(z), T(w)] = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T(w) \tag{39}$$

With this the commutation relation $[L_n, L_m]$ is computed to be

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m} \tag{40}$$

With c being the central charge, depended on the particular theory[22]. This algebra is also known as the Virasoro Algebra. Having established the commutation relations in eq.40 for the CFT, they can be used to calculate correlation functions as will be done in section 3.5, and section 4.1 where we calculate normalization factors for the four- and six-point function.

2.2.3 *Correlation function*

The results of the previous section can be used to calculate some correlation functions in the CFT. We can start by applying L_{-1}, L_0, L_1 on the vacuum state $\langle 0|$ and combine this with the constraints

$$\langle 0|L_i = \langle 0|L_i^\dagger = \langle 0|L_{-i} = 0 \tag{41}$$

Inserting these in a correlation function, gives us the equality

$$\begin{aligned} 0 &= \langle 0|L_i\phi(z_1)...\phi(z_n)|0\rangle \\ &= \sum_j \langle 0|\phi(z_1)...\phi(z_{j-1})\left[L_i, \phi(z_j)\right]\phi(z_{j+1})...\phi(z_n)|0\rangle + \langle 0|\phi(z_1)...\phi(z_n)L_i|0\rangle \\ &= \sum_j \langle 0|\phi(z_1)...\phi(z_{j-1})\left[L_i, \phi(z_j)\right]\phi(z_{j+1})...\phi(z_n)|0\rangle \end{aligned} \tag{42}$$

Where we pull the L_i trough all the operators ϕ and therefore get the commutation relation. Also, the last term vanishes due to the relations seen above in eq.41. The commutation relations can be substituted with infinitesimal conformal transformations using eq.25 As the fields ϕ are conformal fields we get the general solutions for the commutation relation

$$\begin{aligned} \left[L_i, \phi(z_j)\right] &= \frac{1}{2\pi}\oint z^{i+1}T(w)\phi(z_j) \\ &= h(i+1)z_j\phi(z_j) + z_j^{i+1}\partial\phi(z_j) \\ &= h(\partial\epsilon_i(z_j))\phi(z_j) + \epsilon_i(z_j)\partial\phi(z_j) \end{aligned} \tag{43}$$

where epsilon is dependent on the choice of i in L_i, with $i = -1, 0, 1$ and $\epsilon = 1, z, z^2$ respectively. To calculate the propagator or two point function

$$G(z_1, z_2) = \langle O_1(z_1) O_2(z_2) \rangle \tag{44}$$

we can use the commutation relation from eq.43 substitute it in the general differential equation 42 to find that the propagator should obey the differential equation

$$[\epsilon_i(z_1)\partial_1 + h_1 \partial\epsilon_i(z_1) + \epsilon_i(z_2)\partial_2 + h_2 \partial\epsilon_i(z_2)] G(z_1, z_2) \tag{45}$$

Solving this for the different integers of i, makes it possible to distill the solution, as seen below.

$$\begin{aligned}
&\epsilon = 1 \rightarrow (\partial_1 + \partial_2) G(z_1, z_2) = 0 && \rightarrow G(x) && , x = (z_1 - z_2) \\
&\epsilon = x \rightarrow (x\partial_x + h_1 + h_2) G(x) = 0 && \rightarrow G(x) = x^{-h_1 - h_2} && \\
&\epsilon = 1 \rightarrow (h_1 - h_2)(z_1 - z_2) G(z_1, z_2) = 0 && \rightarrow G(x) = C x^{-2h} && , h = h_1 + h_2
\end{aligned} \tag{46}$$

Consequently, the final two-point function or propagator is obtained.

$$G(z_1, z_2) = C(z_1 - z_2)^{-2h} \tag{47}$$

2.2.4 *Higher point functions*

We just saw that the two-point function is nicely constrained by the symmetries of the CFT and therefore returns an explicit answer. For higher point function this is not always the case. Following a similar approach as for the two point functions the three-point function can be expressed as.

$$G_{ijk} = C_{ijk} (z_1 - z_2)^{h_3 - h_1 - h_2} (z_2 - z_3)^{h_1 - h_2 - h_3} (z_3 - z_1)^{h_2 - h_1 - h_3} \tag{48}$$

Applying this approach to the four-point function, one cannot get more specific than

$$G_(z_i \bar{z}_i) = f(x, \bar{x}) \prod_{i<j} z_{ij}^{-h_i - h_j + h/3} \bar{z}_{ij}^{-\bar{h}_i - \bar{h}_j + \bar{h}/3} \tag{49}$$

with $f(x)$ a function of $x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$. Correlation functions with more than three points cannot be obtained by just looking at the symmetry. For these higher point functions one actually has to do the calculations of the interactions. There are many ways to calculate these, for example the Monodromy method used in [23] and later in [24] for correlation functions with two heavy operators and any number of light operators. They show that for OPE channels with pairwise fusion of light operators an even number of light insertions reduces to a product of four-point functions and an odd number of light insertion results in a product of four-point functions and a three-point function. This approach is mainly useful in CFTs with a large central charge.

$$\langle O_H O_H \prod_i O_L(x_{i1}) O_L(x_{i2})) \rangle \approx \prod_i \langle O_H O_H O_L(x_{i1}) O_L(x_{i2})) \rangle \qquad (50)$$

The approach that we will use in this thesis is expressing the correlation function as a sum over exchanged states using the Virasoro conformal Blocks. In [13] it is used to calculate the four-point function modelling a light particle probing a Black Hole. This approach will be discussed in more detail in chapter 3 and in chapter 4 we will try to use the same approach to find the results for the six-point function in the CFT.

2.3 THE ADS/CFT CORRESPONDENCE

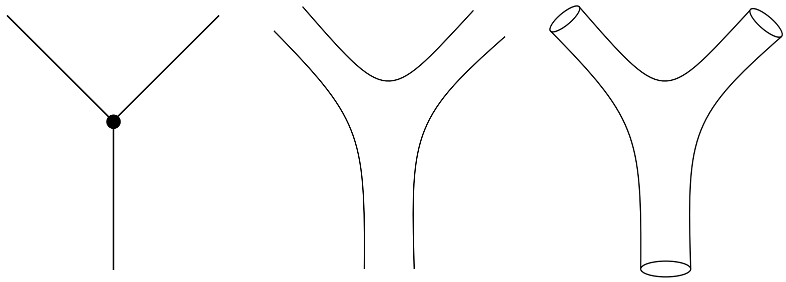
Next we will connect the two theories. The correspondence emerges from String theory, hence we'll start out with string theory and work our way up till the AdS/CFT correspondence emerges. We will see that the 5D AdS space can be encoded on it's boundary with a 4D CFT. All the information can be encoded in one dimension lower like a Hologram, therefore this principle is also known as the Holographic principle[25, 26].

2.3.1 *String theory*

String Theory first arose as a model to describe the Strong Interaction. Where a string connected the quarks and the oscillation of the strings gave a quantized mass spectrum. Qualitatively these results were interesting, but quantitatively they did not provide the right masses. Since QCD mimic the mass spectrum better it is accepted theory for Strong Interactions at this moment.

But String theory wouldn't die that easily as it has a remarkable property, it is the only quantum model which naturally gives rise to gravitons. In string theory one can have two different kind of strings, open and closed strings. Open strings have two directions to oscillate, which relate to two degrees of freedom which can be interpreted as the polarization of a Gauge field. Closed strings have $S = 2$, hence these can be interpreted as Gravitons. Gravitational and Gauge theories are fundamentally different, but in String Theory they reside in the same basic interactions. Hence, String theory unifies the Quantum World and gravity and therefore is called a Unified Theory.

When a string moves trough spacetime it draws a two-dimensional surface, consequently when there is an interaction, instead of drawing the usual Feynman diagram, one has to draw a surface. For closed strings the interactions look like connected tubes and for open strings as flat surfaces. The strength of the interaction is dependent on string coupling constant g_s. One can understand that interactions are defined by the topologies of the interactions.



(a) Feynman diagram *(b) Open string interaction* *(c) Closed string interaction*

Figure 4.: Feynman diagram versus String interactions

Besides moving trough space, strings can also oscillate, each oscillation corresponding to a different mass and hence to a different particle. As there are infinite number of possible oscillations on strings, you would get an infinite tower of particles. But higher oscillations require more energy, hence for practical purposes we can consider just the low oscillations. When $g_s \ll 1$ only the lowest topologies dominate, theories like these can be described by classical field theories and have local supersymmetry. These are also known as supergravity theories.

A parallel can be drawn with gauge theories, which can be understood by using Witten diagrams instead of Feynman diagrams. In large N_c gauge theories, vacuum amplitudes are also given by summations over topologies of two dimensional surfaces. Hence, large N_c gauge theory can be represented by classical gravitational string theory[3, 27]. From this their partition functions can be related.

$$Z_{\text{gauge}} = Z_{\text{String}} \tag{51}$$

$$\begin{aligned} \ln Z_{\text{gauge}} &= \sum_h N_c^{\chi} f_h(\lambda) \\ \ln Z_{\text{String}} &= \sum_h \left(\frac{1}{g_s}\right)^{\chi} \tilde{f}_h(l_s) \end{aligned} \tag{52}$$

At low energies we have a string theory with closed strings described by type-II string theory in 10D. This conflicts with the notion of four-dimensional spacetime. This can be solved by understanding that String Theory is a theory of gravity as well. Therefore it admits curved spacetimes. Such a curved spacetime can be used to describe the N_c gauge theory. More specifically by a 5D curved spacetime with ISO(1,3) with metric

$$ds^2 = \Omega(w)^2(-dt^2 + dx_3^2) + dw^2. \tag{53}$$

However, this does not tell us what gauge theory and which space time we have to use, for this another restriction has to be inserted. The one chosen is scale invariance. When we take the gauge group and go quantum, we introduce a scale, that is the renormalization scale. Hence we should choose as scale invariant gauge group for example the N=4 SYM[3]. When we apply scale invariance on our coordinates

$$\begin{aligned} x^\mu &\to a x^\mu \\ \Omega(w)^2 &\to a^{-2}\Omega(w)^2 \end{aligned} \tag{54}$$

the coordinate w has to transform non-linearly; i.e, $w \to w + L\ln(a)$ and our space-time metric in eq.53 changes to

$$\begin{aligned} ds^2 &= e^{2w/L}(-dt^2 + dx_3^2) + dw^2 \\ &= \left(\frac{r}{L}\right)^2(-dt^2 + dx_3^2) + \frac{L^2}{r^2}dr^2 \end{aligned} \tag{55}$$

Which can be identified as the 5D Anti-de-Sitter space seen for nD in eq.8. To be more precise the 10 dimensional string theory lives on 5D Anti-de-Sitter times the five-sphere; i.e, $\mathrm{AdS}_5 \times \mathrm{S}^5$[28]. Hence the complete space-time is described by the metric

$$ds^2 = \left(\frac{r}{L}\right)^2(-dt^2 + dx_3^2) + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5^2 \tag{56}$$

Both the N=4SYM and the 5D AdS are invariant under conformal invariance SO(2,4); consequently, the partition functions Z can be connected and the GKP-Witten relation is obtained[3, 4].

$$Z_{\mathrm{AdS}_5} = Z_{\mathrm{CFT}} \tag{57}$$

As we discussed before AdS can be empty, which is the $T = 0$ D3brane solutions which can be related to a CFT at zero temperature. But it also allows for Black Holes, these are $T \neq 0$ D3brane solutions. Naturally these can be related to CFT at finite temperature.

2.3.2 *The dictionary*

From the relation between the gauge and string partition function and knowing the explicit theories, a dictionary can be built between the different parameters in the theories. With on the left the 't Hooft

3 SYM=Super Yang Mills

coupling λ and number of colours N_c and on the right the radius of AdS L, string length l_s, and the 5D gravitational constant G_5.

$$\lambda = \left(\frac{L}{l_s}\right)^4, \quad N_c^2 = \frac{\pi}{2}\frac{L^3}{G_5} \tag{58}$$

The Stress-energy tensor on the boundary CFT can be identified with the gravitons in the bulk of AdS as seen below.

$$T_{\mu\nu} \rightarrow g_{\mu\nu} \tag{59}$$

Another relation that can be found is the relation between the mass in Anti-de-Sitter and the conformal dimension Δ in CFT. We follow the description of [29] and use the metric in eq.15 in the limit $u \rightarrow 0$. For a massive scalar field in $d+1$ AdS has the action

$$S_{\text{bulk}} = -\frac{1}{2}\int d^{d+1}x\sqrt{g}[(\nabla_M\phi)^2 + m^2\phi^2] \tag{60}$$

The equation of motion is given by

$$\frac{1}{\sqrt{g}}\partial_M(\sqrt{-g}g^{MN}\partial_N\phi) + m^2\phi = 0 \tag{61}$$

When we solve this asymptotically, we obtain the relation

$$\phi = u^\Delta, \quad \Delta(\Delta - d = m^2) \tag{62}$$

As AdS_{d+1} lives one dimension higher than the CFT_d, d is the dimension of the CFT.

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2} \tag{63}$$

2.3.3 *Entropy*

To calculate the thermodynamic quantities of a system it is very useful to know the entropy, because many of these quantities can be derived from it. The Entropy of a 2D CFT has been known for quite a while and is probably better known as the Cardy formula[30] seen in eq.64

$$S = 2\pi\sqrt{\frac{c}{6}(L_0 - \frac{c}{24})} \tag{64}$$

with c the central charge, $L_0 = ER$ the total energy times the radius of the system, and $\frac{c}{24}$ is a shift due to the Casimir energy[31]. In [32] Verlinde generalized the Cardy formula for $(n+1)$D CFTs. This was achieved by splitting the energy in an extensive E_E part and a Casimir contribution E_C, the subextensive part.

$$E = E_E + \frac{1}{2}E_C \tag{65}$$

By using conformal invariance the extensive and subextensive contribution to the energy should follow the general expressions seen below.

$$E_E = \frac{a}{4\pi R}S^{1+1/n} \quad E_C = \frac{b}{2\pi R}S^{1-1/n} \tag{66}$$

When these two expressions are combined to express the entropy squared S^2

$$\begin{aligned} S^2 &= S^{1+1/n}S^{1-1/n} \\ &= \frac{4\pi R E_E}{a}\frac{2\pi R E_C}{b} \\ &= \frac{8\pi^2 R^2}{ab}E_E E_C \\ &= \frac{4\pi^2 R^2}{ab}(2E - E_C)E_C \end{aligned} \tag{67}$$

Next the square root is taken and using conformal invariance one can identify $ER = L_0$ and $E_C = \frac{c}{12}$ as the Casimir contributiuon to the energy.

$$\begin{aligned} S &= \frac{2\pi R}{\sqrt{ab}}\sqrt{E_C(2E - E_C)} \\ &= \frac{2\pi}{\sqrt{ab}}\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)} \end{aligned} \tag{68}$$

By using the AdS/CFT correspondence one finds the relation $ab = n^2$ for a $D = n + 1$ dimensional CFT. Although this generalized result for the Cardy formula holds for strongly coupled theories in [33] these results were shown to not hold in weakly coupled CFTs.

2.3.4 *Entanglement Entropy*

A special version of entropy is the entanglement entropy. This entropy is the same as quantum notion of Von Neumann entropy, but for the reduced density matrix ρ_A. The density matrix ρ_A is often used in quantum Mechanics, because it can easily describe pure and mixed states. As seen below it is described by a mixture of states $|\psi_i^A$ with probability p_i and sum of probabilities $\sum_i p_i = 1$.

$$\rho_A = \sum_i p_i |\psi_i^A\rangle\langle\psi_i^A| \tag{69}$$

The entanglement entropy is defined as

$$S = -\text{Tr}[\rho_A \ln(\rho_A)]. \tag{70}$$

For a 2D CFT the entanglement entropy has been derived in [34] to be

$$S_A = \frac{c}{3}\log\left(\frac{L}{\pi a}\sin\left(\frac{\pi l}{L}\right)\right) \tag{71}$$

with central charge c ultra violet cut-off a, length l of subsystem A, and length L for the whole system.

In [35] Ryu and Takayanagi calculated the entanglement entropy for a 2D CFT from the AdS. They proposed that the entanglement entropy in AdS follows the area law similar to the area law of the Bekenstein-Hawking entropy.

$$S_A = \frac{A(\gamma_A)}{4G_N} \tag{72}$$

The AdS/CFT dictionary is used to find a relation between the central charge c and G_N and relate the UV-cutoff a in CFT to a bounded region ρ_0 in the AdS.

$$c = \frac{3R}{2G_N} \tag{73}$$

$$\frac{L}{a} \sim e^{\rho} \tag{74}$$

The minimal surface area γ in AdS_3 is obtained by calculating the geodesic length in global coordinates(eq.10), substituted for A in eq.72, and the Entanglement Entropy S_A is for AdS_3 found to be

$$\begin{aligned} S_A \simeq \frac{R}{4G_N}\log\left(e^{2\rho_0}\sin\frac{\pi l}{L}^2\right) &= \frac{c}{3}\log\left(e^{\rho_0}\sin\left(\frac{\pi l}{L}\right)\right) \\ &= \frac{c}{3}\log\left(\frac{L}{a}\sin\left(\frac{\pi l}{L}\right)\right) \end{aligned} \tag{75}$$

which is equal to the entanglement entropy eq.71 for a 2D CFT.

2.3.5 *Entanglement and mixed states*

A subsystem is not entangled when it can be described by a single pure state $|\psi^A\rangle \in \mathcal{H}_A$. In this case, the density matrix ρ_A of the subsystem A looks like

$$\rho_A = |\psi_i^A\rangle\langle\psi_i^A|. \tag{76}$$

All information of the subsystem is encoded in the subsystem itself.

Conversely, when the Quantum Entanglement is non-zero the system behaves thermally or mixed. That is to say, the ensemble representing the mixed state has $p_i \neq 1$, in general the probabilities in this thermal state look like

$$p_i = e^{-\beta E_i}/Z. \tag{77}$$

In chapter 3 we show how a correlation function with heavy operator $h_H > c/24$ in a pure microstate behaves thermally. This pure microstate behaves like a mixed state from a thermal ensemble[24].

Because an not entangled sub-system can be described by $|\psi^A\rangle \in \mathcal{H}_A$, the state of a system containing the pure subsystem can be written as the product $|\Psi\rangle = |\psi^A\rangle \otimes |\psi^{\bar{A}}\rangle$ [4]. Hence a direct relation is established between the uncertainty of the subsystems state and entanglement entropy of the sub-system.

One can also build a larger system for which a given ρ_A is the reduced density matrix of a pure state, this is called purification and can generally be build with states $|\psi_i^A\rangle$ in Hilbert space $\mathcal{H}_A$ and an orthogonal set of states $|\psi_i^B\rangle$ in Hilbert space $\mathcal{H}_B$. as seen below.

$$|\Psi\rangle = \sum_i \sqrt{p_i}|\psi_i^A\rangle \otimes |\psi_i^B\rangle \tag{79}$$

A special case of purification can be considered called a Thermal Field Double state. In this case the purifying system is a copy of the original system[7].

$$|\Psi_{TFD}\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_i e^{-\beta E_i/2}|E_i\rangle \otimes |E_i\rangle \tag{80}$$

2.3.6 *TFD*

This TFD has two fields living in different space times x_1 and x_2, resulting in doubled states $|E_n\rangle_1|E_n\rangle_2$. So we can tread a thermal state as a pure state in a bigger system.

$$|TFD\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2}|E_n\rangle_1|E_n\rangle_2 \tag{81}$$

One natural way to obtain a TFD is by considering the Eternal Black hole in AdS[36]. In the case of a two sided AdS Schwarzschild Black Hole, depicted in Figure 5, the two separate asymptotic AdS spacetimes can be related to the two non-interacting CFTs[6]. The

4 Tensorproduct $R = A \otimes B$ with tensors A and B of type (p, q) and (s, t), respectively. R will be of type $(p+s, q+t)$ and components.

$$R^{i_1,\dots,i_p,\alpha_1,\dots,\alpha_s}_{j_1,\dots,j_q,\beta_1,\dots,\beta_s} = A^{i_1,\dots,i_p}_{j_1,\dots,j_q} P^{\alpha_1,\dots,\alpha_s}_{\beta_1,\dots,\beta_s} \tag{78}$$

combined CFT from both sides; i,e. the product $|CFT\rangle \otimes |CFT\rangle$ is described by a TFD[37]. Defining the thermofield Hamiltonian as the difference between the two CFTs; i.e, $H_T = H_R - H_L$, it describes a single black hole in thermal equilibrium. A stationary observer sees a thermal spectrum in the vacuum, this is similar to the emergence of a thermal spectrum for a accelerating observer also called the Unruh effect.

One can also define the Hamiltonian $H = H_R + H_L$, this describes two Black Holes disconnected in space, but highly entangled[36].

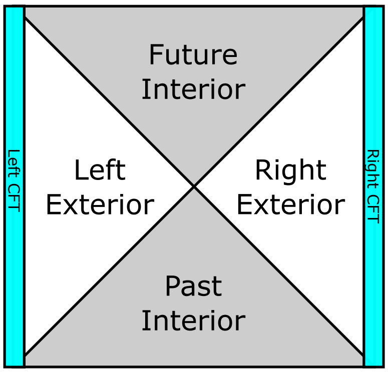


Figure 5.: Penrose diagram for a two sided Black hole in AdS

This brings us to the ER = EPR conjecture[38, 36, 39] . General Relativity (and Special Relativity) are build on the postulate that light, or information for that matter, can not travel faster than light. However, Einstein-Podolsky-Rosen correlations in QM and Einstein-Rosen bridges (Worm holes) in GR seem to violate this postulate of locality. In 2003[36] it was proposed that these two "violations"[5] could be understood as two sides of the same coin; i.e, identifying a wormhole connecting two black holes as a maximally entangled pair of black holes. This conjecture also shines new light on the Black Hole information paradox in relation to the firewall (AMPS) form.

2.3.7 *Scrambling time*

A system can be said to be thermalized when information has been encoded in the full system. The time it takes for information to be encoded in the full system is called scrambling time. In [40] Sekino and Susskind use this concept to define the idea of fast scramblers, these are systems that encode information particularly fast. To obtain a limit they start by approximating the time it takes for information from a given cluster to reach the most distant cluster with only a few

5 Both violations are understood not to hold. As Lorentzian worm holes are non-traversable and EPR can not be used to send information faster than the speed of light.

near neighbour interactions. The scrambling time in general can be expressed as

$$t_* = cN^{\frac{1}{d}} \tag{82}$$

where N denotes the scaling of the degrees of freedom in the Hamiltonian and d is the dimension of the system. As the rate of interaction is often dependent on the temperature and as thermalization if a case of diffusion which scales as $\sqrt{N}$ they obtain

$$\begin{aligned} \frac{t_*}{\beta} \equiv \tau &\geq C(\beta)N^{\frac{1}{d}} \\ \tau &\geq c(\beta)N^{\frac{2}{d}} \\ \tau &\geq c(\beta)\log N. \end{aligned} \tag{83}$$

Where in the last line they conjecture that when $d \to \infty$, this will scale as a log function. Fast scramblers are defined as systems that thermalize as $\tau \geq c(\beta)\log N$.

Next they conjecture that Black holes are the fastest scramblers in nature. They illustrate this with the following gedanken experiment. They drop a charged particle in a Black Hole in 10D Rindler space, calculate how long it takes for the charge to covers the whole horizon with radius R_s. And find that the scrambling time for such a system is given by

$$t_* = \beta \log\left(\frac{R_s}{l_s}\right). \tag{84}$$

Now assuming that l_s is of order Planck length and recalling that the entropy of a black hole is a power of $\frac{R_s}{l_p}$, the relation

$$\tau = C\log(S) \tag{85}$$

is obtained for the scrambling time of a black hole. As the entropy S of a Black Hole can be thought of as it's degrees of freedom, the Black Hole identifies as a Fast Scrambler. The scrambling time plays an interesting role in the chaoticnes of Black Holes.

2.3.8 *Quantum Chaos*

Before we go into the role of scrambling time in chaos theory, we first recap the notion of chaos and quantum chaos. Whereas in everyday language chaos is often associated with randomness. In physics and mathematics it is related to a specific kind of dynamic system. Dynamical systems that are deterministic but extremely sensitive to initial conditions are called chaotic. In the systems a slight change in the initial conditions will produce a high divergence in future states. To measure this chaos the Lyapunov exponent λ has been defined. This exponent quantifies the exponential divergence of a certain trajectory. In a continues one-dimensional system this is easily definable[41] as

$$\frac{\delta x(t)}{\delta x(0)} \sim e^{t\lambda}. \tag{86}$$

This can be generalized to quantum notion of Chaos[42]

$$\frac{\delta q(t)}{\delta q(0)} = \{q(t), p(0)\}. \tag{87}$$

Using canonical quantization this can be expressed as

$$\frac{1}{i\hbar}[\hat{q}(t), \hat{p}(0)]. \tag{88}$$

In general we are interested in expectation values, but these can have opposite since and therefore have the potential to cancel each other, we look at the the norm squared.

$$\frac{-1}{\hbar^2}[\hat{q}(t), \hat{p}(0)]^2 \tag{89}$$

Initially one would think that, because Quantum Mechanics is unitary, two states close together at the start will stay close together. However, orthogonal states which are physically similar can diverge. We can generalize eq.89 to an equation with two operators W and V.

$$\begin{aligned} C(t) &= -\langle [W(t), V(0)]^2 \rangle \\ &= -\langle W(t)V(0)W(t)V(0)\rangle - \langle V(0)W(t)V(0)W(t)\rangle \\ &+ \langle V(0)W(t)W(t)V(0)\rangle + \langle W(t)V(0)V(0)W(t)\rangle \end{aligned} \tag{90}$$

This expresses the influence of a perturbation V on W[43]. Looking at this equation we can expect the last two terms will tend to unity when t becomes large. This can be intuited by noticing the resemblance to the expectation value $\langle W(t)W(t)\rangle$ in a background V or vice versa; consequently, the expectation value factorizes to $\langle W(t)W(t)\rangle\langle VV\rangle$.

In [44] this factorization is contributed to the non-branch cut crossing of these orderings. On the other hand in the ordering $\langle W(t)V(0)W(t)V(0)\rangle$ a branch point is passed.

2.3.9 *The six-point function and Quantum Chaos*

One can use this definition of Quantum chaos to investigate how dependent the emitted Hawking radiation is on the initial infalling particles, in other words, investigate if the Hawking Radiation behaves chaotically. To probe this chaoticness of a black hole in a quantum framework we want to measure the quantum chaos of two states with a black hole background. When we make this background explicit, we obtain:

$$\langle [W(t), V(0)]^2 \rangle_\beta = \langle O_H | [W(t), V(0)]^2 | O_H \rangle \tag{91}$$

with O_H as the black hole or heavy state. One can observe that eq.91 consists of six-point functions describing the chaoticness. To probe the regions near the horizon, we want to look at photon interactions near the horizon.

Similar to eq.90, the out of time ordered Correlation function, eq. 92, is the most interesting to study, as the others will just factorize again.

In [44] this four-point function in a thermal background is approximated in a 2D CFT by using the monodromy method. With their method one start out by doing the coordinate transformations $z_i(x_i, t_i) = e^{\frac{2\pi}{\beta}(x+t)}$ and $\bar{z}_i(x_i, t_i) = e^{\frac{2\pi}{\beta}(x-t)}$.

$$\begin{aligned} \langle O_H | W(t) V(0) W(t) V(0) | O_H \rangle &= \langle W(z_1, \bar{z}_1) V(z_2, \bar{z}_2) W(z_3, \bar{z}_3) V(z_4, \bar{z}_4) \rangle_\beta \\ &= \frac{1}{z_{12}^{2h_W} z_{34}^{2h_V}} \frac{1}{\bar{z}_{12}^{-2\bar{h}_W} \bar{z}_{34}^{-2\bar{h}_V}} f(z, \bar{z}) \end{aligned} \tag{92}$$

The $f(z, \bar{z})$ can be expanded as a sum of global conformal blocks. Using the Virasoro conformal block of the identity operator, $f(z, \bar{z})$ can be replaced by $\mathcal{F}(z)\mathcal{F}(\bar{z})$ where $\mathcal{F}(z)$ can be approximated[23] by

$$\mathcal{F}(z) \approx \left(\frac{z}{1 - (1-z)^{1-12h_w/c}} \right)^{2h_v}. \tag{93}$$

This has a branch point at $z = 1$. Following contour around $z = 1$ and taking z is small

$$\mathcal{F}(z) \approx \left(\frac{1}{1 - \frac{24\pi i h_w}{cz}} \right)^{2h_v} \tag{94}$$

with

$$z \approx -e^{\frac{2\pi}{\beta}(x-t)} \epsilon_{12}^* \epsilon_{34}. \tag{95}$$

One obtains the following approximation for the normalized out of order 4-point function.

$$\frac{\langle W(t+i\epsilon_1) V(i\epsilon_3) W(t+i\epsilon_2) W(i\epsilon_4) \rangle_\beta}{\langle W(i\epsilon_1) W(i\epsilon_2) \rangle_\beta \langle V(i\epsilon_3) V(i\epsilon_4) \rangle_\beta} \approx \left(\frac{1}{1 + \frac{24\pi i h_w}{c_{12}^* c_{34}} e^{\frac{2\pi}{\beta}(t - t_* - x)}} \right)^{2h_v}. \tag{96}$$

The Lyapunov exponent λ is found to be $\lambda = \frac{2\pi}{\beta}$. It can also be observed that for $t > t_* + x$ the correlation function will start to decrease and for $t \gg t_* + x$ will even tend to zero; consequently a scrambling time $t_* = \frac{\beta}{2\pi} log(c)$ is defined.

$$-C(t) = \langle [W(t), V(0)]^2 \rangle_\beta$$
$$\approx 2\langle W(T)W(T)\rangle\langle V(0)V(0)\rangle \left[1 - \left(1 + \frac{24\pi i h_w}{c_{12}^* c_{34}} e^{\frac{2\pi}{\beta}(t - t_* - x)}\right)^{-2h_v}\right] \tag{97}$$

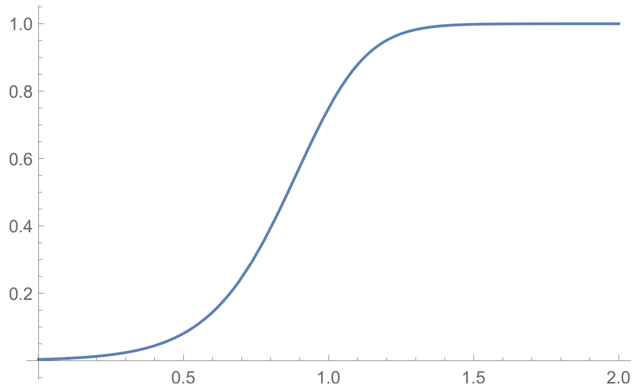


Figure 6.: Plot of $f(t) = 1 - \left(1 + e^{2\pi(t-1)}\right)^{-2h_v}$

3

COMPUTATION OF THE FOUR-POINT FUNCTION

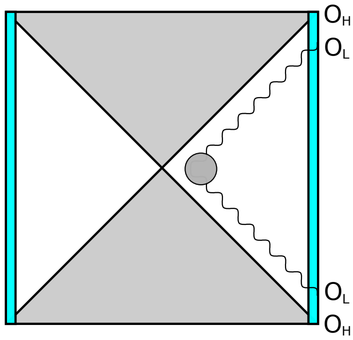


Figure 7.: four-point function

As discussed earlier in section 2.2.4 there is a limit in how exact correlation functions in a CFT with more than three points can be obtained by only using the symmetry. We also mentioned that different methods have been considered to calculate these. In this chapter we will review one of these approaches; that is, the approach taken by Fitzpatrick et al. in 2015[13]. In this paper they calculate the four-point function with two heavy insertions and two light insertions, which is dual to a BTZ Black hole with a light probe.

$$\langle O_{H_1}(\infty)O_{H_2}(1)O_{L_1}(w)O_{L_2}(0)\rangle \tag{98}$$

In section 2.2 we saw that the infinite series of currents J^n, can be generated by operators $L_n$and that these operators obey the Virasoro Algebra. Hence, different insertions can be glued together by connecting them with these current. This can be expressed by inserting a projection operator P_h in the correlation function. The operator can be expressed as a sum over Virasoro conformal blocks

$$P_h = \sum_i \frac{L^{k_1}_{-m_1}...L^{k_i}_{-m_i}|h_w\rangle\langle h_w|L^{k_i}_{m_i}...L^{k_1}_{m_1}}{\mathcal{N}_i} \tag{99}$$

with normalization factor

$$\mathcal{N}_i = \langle h|L^{k_i}_{m_i}...L^{k_1}_{m_1}L^{k_1}_{-m_1}...L^{k_i}_{-m_i}|h\rangle. \tag{100}$$

Practically, one sums over all possible states by acting on the primary state $|h\rangle$ with L_{m_i}. Inserting this projection operator in eq.98 one obtains a sum over the Virsasoro conformal blocks.

$$\langle O_H(\infty)O_H(1)P_hO_L(w)O_L(0)\rangle = \sum_i \langle OOL^{k_1}_{-m_1}...L^{k_i}_{-m_i}|h\rangle \frac{1}{\mathcal{N}_i}\langle h|L^{k_i}_{m_i}...L^{k_1}_{m_1}OO\rangle \tag{101}$$

In section 3.5 we will see that for $m_i > 1$ the correlation function will have contributions of order $\frac{1}{c}$. Hence, in the limit of large central charge $c \gg 1$ the four-point function can be expressed as

$$V_h(w) = \langle O_{H_1}(\infty) O_{H_2}(1) \left(\sum_k \frac{L_{-1}^k |h\rangle \langle h_w| L_1^k}{\langle h_w | L_1^k L_{-1}^k |h\rangle} \right) O_{L_1}(w) O_{L_2}(0) \rangle \quad (102)$$

The correlation function splits into three separate functions; i.e, the heavy sector, the light sector, and the normalization factor calculated in sections 3.3, 3.4, and 3.5 respectively.

3.1 TRANSFORMING TO A NEW BACKGROUND

However there is one complication, when the heavy insertions are to represent a Black hole, the conformal dimension h_H will be proportional to c. These can result in significant contributions from terms that would else be negligible. To counter this Fitzpatrick et al. propose a transformation to a non-trivial background metric. The transformation proposed is

$$1 - w = (1 - z)^{\alpha} \quad (103)$$

with $\alpha = \sqrt{1 - 24\frac{h_H}{c}}$. In the usual background the four-point function is equal to

$$\langle O_H(\infty) O_H(1) T(z) O_h(0) \rangle = C_{HHh} \left(\frac{h_H}{(z-1)^2} + \frac{h}{(1-z)z^2} \right). \quad (104)$$

It can be observed that when $h_H \propto c$ these contributions can be very significant even when the normalization factor is proportional to c. However in the new coordinate system this four point function can be expressed as[45]

$$\langle O_H(\infty) O_H(1) T(w) O_h(0) \rangle = C_{HHh} \left(\frac{h(1-z)}{z^2} \right). \quad (105)$$

Therefore taking $h_H \propto c$ does not complicate the situation anymore.

The resulting correlation function looks like

$$\lim_{c \to \infty} V(c, h, h_i, w) = (1-w)^{(h_L + \delta_L)(1 - \frac{1}{\alpha})} (\frac{w}{\alpha})^{h - 2h_L} {}_2F_1(h - \frac{\delta_H}{\alpha}, h + \delta_L, 2h, w) \quad (106)$$

with c the central charge, and h_i the conformal dimension of operator O_i. When O_H is an operator representing a black hole, $\delta_H = 0$;

similarly, when O_L is a light probe, $\delta_L = 0$. Hence, eq.106 simplifies to the equation below.

$$\lim_{c\to\infty} V(c,h,h_i,w) = (1-w)^{(h_L)(1-\frac{1}{\alpha})}(\frac{w}{\alpha})^{h-2h_L}{}_2F_1(h,h;2h;w) \quad (107)$$

3.2 OBTAINING THE SHAPE OF THE OPE

Next an expression has to be found for how the L_{-1} acts on the correlation function. This is done by taking the insertions of L_{-1}, and use eq.25 to substitute them with $\oint \frac{dz}{2\pi i}T(z)$. After that the contour integrals are taking and the expression is found. We start out with taking two insertions of L_{-1} operators as these might contain contributions of order c and the expression for one insertion can just be extracted from this result.

$$\begin{aligned}
&\langle O(x_1)O(x_2)L_{-1}L_{-1}|O(x_3)\rangle \\
&= \oint \frac{dz_1}{2\pi i}\oint \frac{dz_2}{2\pi i}\langle O(x_1)O(x_2)T(z_1)T(z_2)|O(x_3)\rangle \\
&= \oint \frac{dz_1}{2\pi i}\oint \frac{dz_2}{2\pi i}\Bigg[\frac{c/2}{(z_2-z_1)^4}\langle OOO\rangle \\
&\left(\frac{h_3}{(z_2-x_3)^2}+\frac{1}{z_2-x_3}\frac{\partial}{\partial x_3}\right)\langle O(x_1)O(x_2)T(z_1)|O(x_3)\rangle\Bigg] \\
&= \oint \frac{dz_1}{2\pi i}\oint \frac{dz_2}{2\pi i}\Bigg[\frac{c/2}{(z_2-z_1)^4}\langle O(x_1)O(x_2)O(x_3)\rangle \\
&\left(\frac{h_3}{(z_2-x_3)^2}+\frac{1}{z_2-x_3}\frac{\partial}{\partial x_3}\right)\langle O(x_1)|T(z_1)O(x_2)O(x_3)\rangle\Bigg] \\
&= \oint \frac{dz_1}{2\pi i}\oint \frac{dz_2}{2\pi i}\Bigg[\frac{c/2}{(z_2-z_1)^4} \\
&\left(\frac{h_3}{(z_2-x_3)^2}+\frac{1}{z_2-x_3}\frac{\partial}{\partial x_3}\right)\left(\frac{h_3}{(z_1-x_3)^2}+\frac{1}{z_1-x_3}\frac{\partial}{\partial x_3}\right)\Bigg]\langle O(x_1)O(x_2)O(x_3)\rangle \\
&= \oint \frac{dz_1}{2\pi i}\left[\frac{\partial}{\partial x_3}\left(\frac{h_3}{(z_1-x_3)^2}+\frac{1}{z_1-x_3}\frac{\partial}{\partial x_3}\right)\right]\langle O(x_1)O(x_2)O(x_3)\rangle \\
&= \left(\frac{\partial}{\partial x_3}\right)\left[\left(\frac{\partial}{\partial x_3}\right)\langle O(x_1)O(x_2)O(x_3)\rangle\right]
\end{aligned} \quad (108)$$

Next the 3-point function with the coordinates $\lim x_1 \to \infty$, $x_2 = x_2$, and $\lim_{x_3 \to 0}$ is inserted into the equation[1].

$$\begin{aligned}
&\left(\frac{\partial}{\partial x_3}\right)\left[\left(\frac{\partial}{\partial x_3}\right)\langle O_{L_1}(x_1)O_{L_2}(x_2)O_h(x_3)\rangle\right] \\
&= \lim_{x_3\to 0}\lim_{x_1\to\infty}\left(\frac{\partial}{\partial x_3}\right)\left[\left(\frac{\partial}{\partial x_3}\right)\langle O_{L_1}(x_1)O_{L_2}(x_2)O_h(x_3)\rangle\right] \\
&= \lim_{x_3\to 0}\left(\frac{\partial}{\partial x_3}\right)\left[\left(\frac{\partial}{\partial x_3}\right)(x_2-x_3)^{-L_2-h-L_1}\right] \\
&= \lim_{x_3\to 0}\frac{\partial^2}{\partial x_3^2}(x_2-x_3)^{-L_2-h+L_1} \\
&= \lim_{x_3\to 0}\frac{\partial^2}{\partial x_3^2}(x_2-x_3)^{-h+(L_1-L_2)}
\end{aligned} \tag{109}$$

We can extend this to k insertions of L_{-1}.

$$\begin{aligned}
&\langle O_H(\infty)O_H(1)L_{-1}^k|h_w\rangle \\
&= \oint\frac{dz}{2\pi i}...\oint\frac{dz_k}{2\pi i}\langle O_H(\infty)O_H(1)|T(z)...T(z_k)h_w\rangle \\
&= \prod_{n=0}^{k}\left[\oint\frac{dz_n}{2\pi i}\left(\frac{h_i}{(z_n-x_3)^2}+\frac{1}{z_n-x_3}\frac{\partial}{\partial x_3}\right)\right]\langle O_H(\infty)O_H(1)O_h(x_3)\rangle \\
&= \prod_{n}^{k}\left[\left(\frac{\partial}{\partial x_3}\right)\right]\langle O_H(\infty)O_H(1)O_h(x_3)\rangle \\
&= \frac{\partial^k}{\partial x_3^k}\langle O_H(\infty)O_H(1)O_h(x_3)\rangle
\end{aligned} \tag{110}$$

Hence, it can be observed that the L_{-1} operator acts on the three-point function as a derivative to x_3.

$$\langle O_H(\infty)O_H(1)L_{-1}^k|h_w\rangle = \frac{\partial^k}{\partial x_3^k}\langle O_H(\infty)O_H(1)O_h(x_3)\rangle \tag{111}$$

3.3 HEAVY SECTOR

For the heavy sector we start out with the correlation function seen below.

$$\langle O_H(\infty)O_H(1)\mathcal{L}_{-1}^k|h\rangle \tag{112}$$

1 See Appendix A.1 for the computation of this simplified three-point function.

First we express the correlation function as a function of w using the transformation $(1-w)^{\frac{1}{\alpha}} = 1-z$.

$$\begin{aligned}
\langle O_{H_1}(\infty)O_H(1)|h\rangle &\approx C_{HHh}(1-w_h)^{H_1-h-H_2} \\
&= \left(\frac{dz}{dw}\right)^h C_{HHh}(1-w_h)^{\frac{1}{\alpha}(H_1-h-H_2)} \\
&= \alpha^{-h}(1-w)^{h(\frac{1}{\alpha}-1)}C_{HHh}(1-w_h)^{\frac{1}{\alpha}(H_1-h-H_2)} \\
&= \alpha^{-h}C_{HHh}(1-w_h)^{h(\frac{1}{\alpha}-1)+\frac{1}{\alpha}(H_1-h-H_2)} \\
&= \alpha^{-h}C_{HHh}(1-w_h)^{-h+\frac{1}{\alpha}(\delta_H)}
\end{aligned} \tag{113}$$

Next the $\mathcal{L}_{-1}$ insertions are inserted, according to [13] these act similarly on $O(w)$ as the L_{-1} in a flat metric besides the $\delta_H \to \delta_H/\alpha$ with $\delta_H = H_1 - H_2$.

$$\begin{aligned}
\langle O_{H_1}(\infty)O_H(1)\mathcal{L}_{-1}^k|h\rangle &= \lim_{w\to 0}\partial_w^k \alpha^{-h}C_{HHh}(1-w_h)^{-h+\frac{1}{\alpha}(\delta_H)} \\
&= \lim_{w\to 0}\alpha^{-h}C_{HHh}(1-w_h)^{-h+\frac{\delta_H}{\alpha}-k}(h-\frac{\delta_H}{\alpha})^{(k)} \\
&= \alpha^{-h}C_{HHh}(h-\frac{\delta_H}{\alpha})^{(k)}
\end{aligned} \tag{114}$$

With Pochhammer symbol, or rising factorial $(h-\frac{\delta_H}{\alpha})^{(k)}$ defined as

$$x^{(n)} = x(x+1)(x+2)...(x+n-1) \tag{115}$$

These relate to the Hypergeometric function as seen below

$${}_2F_1(a,b;c;z) = \sum_{n=o^{\infty}} \frac{a^{(n)}b^{(n)}}{c^{(n)}}\frac{z^n}{n!} \tag{116}$$

Hence the same relation as Fitzpatrick et al. is obtained

$$\langle O_H(\infty)O_H(1)\mathcal{L}_{-1}^k|h\rangle = \alpha^{-h}C_{HHh}\left(h-\frac{\delta_H}{\alpha}\right)^{(k)} \tag{117}$$

3.4 LIGHT SECTOR

Next the light sector is computed. This is done a little different, it's calculated in the w coordinate system and later translated to the z coordinate system. The same expression for $\mathcal{L}_{-1}$ as seen in eq.111 can be used.

$$\begin{aligned}
\langle h_w|\mathcal{L}_1^k O_{L_1}(w)O_{L_2}(0)\rangle &= \langle O_{L_2}^\dagger(0)O_{L_1}^\dagger(w)\mathcal{L}_{-1}^k|h_w^\dagger\rangle \\
&= \lim_{v\to 0}\partial_v^k w^{-2L_1}\langle O_{L_2}(\infty)O_{L_1}(\frac{1}{w})O_h(v)\rangle
\end{aligned} \tag{118}$$

We use the same surviving OPE terms as in the heavy sector.

$$\begin{aligned}
\lim_{v\to 0}\partial_v^k w^{-2L_1}\langle O_{L_2}(\infty)O_{L_1}(\frac{1}{w})O_h(v)\rangle \\
\lim_{v\to 0}\partial_v^k w^{-2L_1}C_{LLh}(\frac{1}{w}-v)^{-h-\delta_L} \\
\lim_{v\to 0} w^{-2L_1}C_{LLh}(\frac{1}{w}-v)^{-h-\delta_L-k}(h+\delta_L)^{(k)} \\
w^{-2L_1}C_{LLh}(\frac{1}{w})^{-h-\delta_L-k}(h+\delta_L)^{(k)} \\
C_{LLh}(\frac{1}{w})^{-h-\delta_L+2L_1-k}(h+\delta_L)^{(k)}
\end{aligned} \tag{119}$$

Next the inverted Jacobian factor $\alpha^{-L_1-L_2}(1-w)^{-L_1(1-\frac{1}{\alpha})}$ is included.

$$\langle h_w|L_1^k O_{L_1}(z)O_{L_2}(0)\rangle = \alpha^{L_1+L_2}(1-w)^{L_1(1-\frac{1}{\alpha})}C_{LLh}w^{h-(L_1+L_2)}(h+\delta_L)^{(k)}w^k \tag{120}$$

3.5 NORMALIZATION

The last piece is the normalization factor.

$$\sum_k \langle h|L_1^k L_{-1}^k|h\rangle \tag{121}$$

These can be calculated using the commutator relations we saw before for operators L_i in eq.40.

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m} \tag{122}$$

$$\begin{aligned}
[L_1, L_{-1}] &= 2L_0 \\
[L_1, L_0] &= L_1
\end{aligned} \tag{123}$$

We can use that L_{-1} annihilates the state $\langle h|$ so $\langle h|L_{-1} = 0$ and $\langle h|L_0 = h\langle h|$. We can calculate the normalization factors for different k.

For $k = 1$

$$\langle h|L_1 L_{-1}|h\rangle = \langle h|2L_0|h\rangle = 2h \tag{124}$$

For $k = 2$

$$\begin{aligned}
\langle h|L_1 L_1 L_{-1} L_{-1}|h\rangle &= \langle h|L_1(2L_0 + L_{-1}L_1)L_{-1}|h\rangle \\
&= \langle h|(2L_0 + 2 + 2L_0)L_1 L_{-1}|h\rangle \\
&= 2(2h+1)\langle h|L_1 L_{-1}|h\rangle \\
&= 2h(2h+1)2
\end{aligned} \tag{125}$$

For $k = 3$

$$\begin{aligned}
\langle h|L_1^3 L_{-1}^3|h\rangle &= \langle h|L_1^2(2L_0 + L_{-1}L_1)L_{-1}^2|h\rangle \\
&= \langle h|L_1(4L_0 + 2 + L_{-1}L_1)L_1 L_{-1}^2|h\rangle \\
&= \langle h|(4L_0 + 4 + 2 + 2L_0)L_1^2 L_{-1}^2|h\rangle \\
&= \langle h|(6L_0 + 6)L_1^2 L_{-1}^2|h\rangle \\
&= 3(2h+2)\langle h|L_1^2 L_{-1}^2|h\rangle \\
&= 2h(2h+1)(2h+2)3!
\end{aligned} \tag{126}$$

Calculating this for general k

$$\begin{aligned}
C_{1^k} &= \langle h|L_1^k L_{-1}^k|h\rangle \\
&= 2\langle h|L_1^{k-1} L_0 L_{-1}^{k-1}|h\rangle + \langle h|L_1^{k-1} L_{-1} L_1 L_{-1}^{k-1}|h\rangle \\
&= 2\langle h|L_1^{k-2}(L_0+1)L_1 L_{-1}^{k-1}|h\rangle + \langle h|L_1^{k-2}(2L_0 + L_{-1}L_1)L_1 L_{-1}^{k-1}|h\rangle \\
&= 2\langle h|L_1^{k-2}(2L_0+1)L_1 L_{-1}^{k-1}|h\rangle + \langle h|L_1^{k-2} L_{-1} L_1^2 L_{-1}^{k-1}|h\rangle \\
&= 2\langle h|L_1^{k-3}(3L_0+1+2)L_1^2 L_{-1}^{k-1}|h\rangle + \langle h|L_1^{k-3} L_{-1} L_1^3 L_{-1}^{k-1}|h\rangle \\
&= 2\langle h|L_1^{k-i}(iL_0 + \sum_{j=1}^{i}(j-1))L_1^{i-1} L_{-1}^{k-1}|h\rangle + \langle h|L_1^{k-i} L_{-1} L_1^i L_{-1}^{k-1}|h\rangle \\
&= \langle h|L_1^{k-i}(2iL_0 + i(i-1))L_1^{i-1} L_{-1}^{k-1}|h\rangle + \langle h|L_1^{k-i} L_{-1} L_1^i L_{-1}^{k-1}|h\rangle \\
&= \langle h|L_1^{k-k}(2kL_0 + k(k-1))L_1^{k-1} L_{-1}^{k-1}|h\rangle + \langle h|L_1^{k-k} L_{-1} L_1^k L_{-1}^{k-1}|h\rangle \\
&= \langle h|(2kL_0 + k(k-1))L_1^{k-1} L_{-1}^{k-1}|h\rangle \\
&= k(2h + (k-1))\langle h|L_1^{k-1} L_{-1}^{k-1}|h\rangle \\
&= \prod_{j=1}^{k} j(2h + j - 1) \\
&= (2h)^{(k)} k!
\end{aligned} \tag{127}$$

Hence we obtain the solution for the normalization factor $\mathcal{N}_k$.

$$\langle h|L_1^k L_{-1}^k|h\rangle = (2h)^{(k)} k! \tag{128}$$

3.6 PUTTING THINGS TOGETHER

Having obtained the building blocks, we combine the results for the Heavy sector in eq.117,

$$\langle O_H(\infty) O_H(1) L_{-1}^k|h\rangle = \alpha^{-h}(h - \frac{\delta_H}{\alpha})^{(k)} \tag{129}$$

; for the Light sector in eq.120,

$$\langle h_w|L_1^k O_{L_1}(z) O_{L_2}(0)\rangle = \alpha^{L_1+L_2}(1-w)^{L_1(1-\frac{1}{\alpha})}(w^{h+\delta_L})(h+\delta_L)^{(k)} w^k \tag{130}$$

; and the normalization function in eq.128.

$$\langle h|L_1^k L_{-1}^k|h\rangle = (2h)^{(k)} k! \tag{131}$$

We have set $C_{HHh} = C_{LLh} = 1$. We obtain the solution

$$\begin{aligned}\lim_{c\to\infty} V(c,h,h_i,w) &= \sum_{k=0} \langle O_H(\infty)O_H(1)L_{-1}^k|h\rangle \frac{1}{\langle h|L_1^k L_{-1}^k|h\rangle} \langle h_w|L_1^k O_{L_1}(z) O_{L_2}(0)\rangle \\ &= (1-w)^{(1-\frac{1}{\alpha})L_1} \left(\frac{w}{\alpha}\right)^{h-(L_1+L_2)} {}_2F_1(h-\frac{\delta_H}{\alpha}, h+\delta_L, 2h, w)\end{aligned} \tag{132}$$

with

$${}_2F_1(h-\frac{\delta_H}{\alpha}, h+\delta_L, 2h, w) = \sum_{k=0} \frac{(h-\frac{\delta_H}{\alpha})^{(k)}(h+\delta_L)^{(k)}}{(2h)^{(k)}} \frac{w^k}{k!} \tag{133}$$

If we take the O_H as the black hole with the same conformal dimension initially as in the the end, $\delta_H = 0$. Similarly, Taking the ingoing and outgoing particle O_L with the same conformal dimensions. $L_1 = L_2 = L$ and $\delta_L = 0$. We obtain the simplified version

$$\lim_{c\to\infty} V(c,h,h_i,w) = (1-w)^{(1-\frac{1}{\alpha})L} \left(\frac{w}{\alpha}\right)^{h-2L} {}_2F_1(h,h,2h,w) \tag{134}$$

3.7 EXPONENTIAL DECAY

To improve our understanding of this result one can simplify it by assuming the interactions are dominated by graviton interactions[24]. Practically this means taking the Identity block $h = 0$, these blocks include the Stress Tensor and it's descendents[46]. In this limit the hypergeometric function ${}_2F_1$ reduces to ${}_2F_1(0,0;0;w) = 1$.

$$\lim_{c\to\infty} V(c,h=0,h_i,w) = (1-w)^{h_L(1-\frac{1}{\alpha})} \left(\frac{w}{\alpha}\right)^{-2h_L} \tag{135}$$

Using the AdS/CFT relation in eq.63, the conformal dimension $\Delta = h_H$ can be related to the mass of the black hole.

$$h = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2} \tag{136}$$

where d is the dimension of the CFT, 2 in this case and m the mass of the black hole. So $d \ll m \approx h_H$. Also we can split the h_H into two sectors; i.e, $h_H > \frac{c}{24}$ and $h_H < \frac{c}{24}$ for AdS geometries that are above and below the BTZ black hole threshold respectively[23].

$$\begin{aligned} h_H > \tfrac{c}{24} \quad & \alpha = i\sqrt{24\tfrac{h_H}{c} - 1} \\ h_H < \tfrac{c}{24} \quad & \alpha = \sqrt{1 - 24\tfrac{h_H}{c}} \end{aligned} \tag{137}$$

We can relate this back to the cylinder with $1 - w \to (1-z)^{\alpha} \to e^{-\alpha t_E}$[45] and doing a Wick rotation $t_E \to -it_L$ taking us from Euclidean to Lorentzian time.

$$\begin{aligned}\lim_{c\to\infty} V(c,0,h_i,t) &= e^{-t_E h_L(\alpha-1)}(1-e^{-\alpha t_E})^{-2h_L}\alpha^{2h_L} \\ &= e^{it_L h_L(\alpha-1)}(1-e^{\alpha i t_L})^{-2h_L}\alpha^{2h_L}\end{aligned} \tag{138}$$

Taking the BTZ limit we can substitute $\alpha = i\sqrt{24\frac{h_H}{c}-1}$ and obtain the following expression for a BTZ Black Hole.

$$\begin{aligned}\lim_{c\to\infty} V(c,0,h_i,t) &= e^{-it_L h_L} e^{t h_L\sqrt{24\frac{h_H}{c}-1}}(1-e^{-t_L\sqrt{24\frac{h_H}{c}-1}})^{-2h_L}\alpha^{2h_L}\alpha^{2h_L} \\ &= \frac{e^{-it_L h_L}}{2}\sinh\left(\frac{t_L\sqrt{24\frac{h_H}{c}-1}}{2}\right)^{-2h_L}\alpha^{2h_L} \\ &\sim e^{-t_L h_L\sqrt{24\frac{h_H}{c}-1}}\end{aligned} \tag{139}$$

It can be observed from eq.139 that the correlation function decays exponentially for large t_L. This exponential decaying correlation function conflicts with the notion of unitarity in quantum mechanics. In a finite Quantum system there should be a lower bound on the decay equal to e^{-S} else information has to be destroyed. This contradiction makes us believe that higher order c contributions could result in a lower bound on the correlation function and cancel the exponential decay at a certain point.

3.8 ONLY $k = 0$ CONTRIBUTIONS

When the limit $h \to 0$ is taken in the hypergeometric functions ${}_2F_1(h,h,2h,w)$, we notice that the hypergeometric function becomes one. In other words the summation in k which leads to the hypergeometric function (seen in 133) goes to one. This leads to the suspicion that this four-point function has only contributions from $k = 0$. Hence if we take eq.102 and take $k = 0$ as seen below, we should get the same result as eq.134.

$$\langle O_H(\infty)O_H(1)L^0_{-1}|h\rangle \frac{1}{\langle h|L^0_1 L^0_{-1}|h\rangle}\langle h_w|L^0_1 O_{L_1}(z)O_{L_2}(0)\rangle \tag{140}$$

To calculate this, we take the same steps as before, but we take $k = 0$. First we calculate the normalization, next the light sector, and finally the heavy sector.

For $k = 0$ the normalization is given by

$$\langle h|L^0_1 L^0_{-1}|h\rangle = \langle h||h\rangle = 1. \tag{141}$$

The light sector is calculated similarly to sec.3.4 but with $k = 0$.

$$\begin{aligned} \langle h_v | L_1^0 O_{L_1}(w) O_{\mathcal{L}_2}(0) \rangle &= \langle O_{L_2}^\dagger(0) O_{L_1}^\dagger(w) \mathcal{L}_{-1}^0 | h_v^\dagger \rangle \\ &= \lim_{v \to 0} \partial_v^0 w^{-2L_1} \langle O_{L_2}(\infty) O_{L_1}(\frac{1}{w}) O_h(v) \rangle \end{aligned} \tag{142}$$

Therefore no derivatives in v remain and the correlation function is calculated.

$$\begin{aligned} &\lim_{v \to 0} w^{-2L_1} \langle O_{L_2}(\infty) O_{L_1}(\frac{1}{w}) O_h(v) \rangle \\ &\lim_{v \to 0} w^{-2L_1} C_{LLh}(\frac{1}{w} - v)^{-h-\delta_L} \\ &C_{LLh}(\frac{1}{w})^{-h-L_1-L_2} \end{aligned} \tag{143}$$

Next the Jacobian factor $\alpha^{-L_1-L_2}(1-w)^{-L_1(1-\frac{1}{\alpha})}$ is included and the light sector is found to be

$$\begin{aligned} \langle h_w | L_1^0 O_{L_1}(z) O_{L_2}(0) \rangle &= \alpha^{L_1+L_2}(1-w)^{L_1(1-\frac{1}{\alpha})} C_{LLh}(\frac{1}{w})^{-h-\delta_L+2L_1} \\ &= \alpha^{L_1+L_2}(1-w)^{L_1(1-\frac{1}{\alpha})} C_{LLh} w^{h-(L_1+L_2)}. \end{aligned} \tag{144}$$

Finally for the heavy sector the approach is the same as in sec.3.3, but again with $k = 0$.

$$\begin{aligned} \langle \mathcal{O}_{H_1}(\infty) O_H(1) \mathcal{L}_{-1}^0 | h \rangle &= \lim_{w \to 0} \partial_w^0 \langle O_{H_1}(\infty) O_{H_2}(1) O_h(w) \rangle \\ &\approx \lim_{w \to 0} C_{HHh}(1-w_h)^{H_1-h-H_2} \end{aligned} \tag{145}$$

Again we translate to z coordinates

$$\begin{aligned} &\lim_{w \to 0} \left(\frac{dz}{dw}\right)^h C_{HHh}(1-w_h)^{\frac{1}{\alpha}(H_1-h-H_2)} \\ &\lim_{w \to 0} \alpha^{-h}(1-w)^{h(\frac{1}{\alpha}-1)} C_{HHh}(1-w_h)^{\frac{1}{\alpha}(H_1-h-H_2)} \\ &\lim_{w \to 0} \alpha^{-h} C_{HHh}(1-w_h)^{h(\frac{1}{\alpha}-1)+\frac{1}{\alpha}(H_1-h-H_2)} \\ &\lim_{w \to 0} \alpha^{-h} C_{HHh}(1-w_h)^{-h+\frac{\delta_H}{\alpha}} \\ &\alpha^{-h} C_{HHh} \end{aligned} \tag{146}$$

Combining the solutions for the normalization in eq.141, the light sector in eq.144, and the heavy sector in eq.146, setting $C_{HHh} = C_{LLh} = 1$, and $L_1 = L_2 = L$ we obtain the solution

$$\begin{aligned}\lim_{c\to\infty} V(c, h = 0, h_i, w) &= \alpha^{L_1+L_2}\alpha^{-h} C_{HHh} C_{LLh} (1-w)^{L_1(1-\frac{1}{\alpha})} w^{h-(L_1+L_2)} \\ &= (1-w)^{(1-\frac{1}{\alpha})L} \left(\frac{w}{\alpha}\right)^{h-2L}.\end{aligned} \tag{147}$$

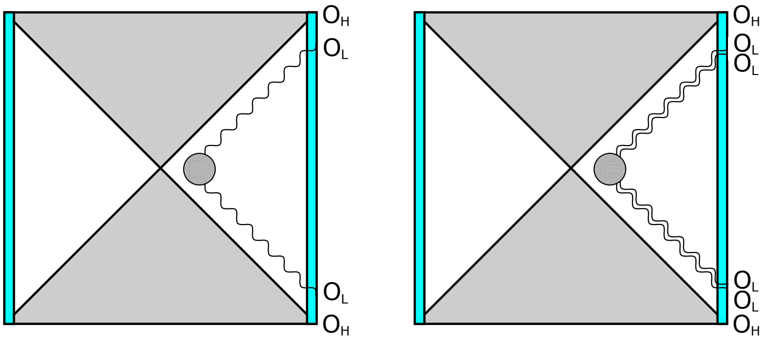
When we take $h = 0$ we get the same solution as in 134. Hence, when in the limit $h = 0$ only $k = 0$ terms contribute in this correlation function. From this we can conclude that the four-point function in this limit is simply the product of the heavy correlator and the light correlator as seen below.

$$\langle O_H(\infty) O_H(1) O_{L_1}(z) O_{L_2}(0)\rangle \approx \langle O_H(\infty) O_H(1)\rangle \langle O_{L_1}(z) O_{L_2}(0)\rangle \tag{148}$$

We emphasize that this is true in the coordinate system discussed in section 3.1.

4

COMPUTATION OF THE SIX-POINT FUNCTION



(a) Four-point function: $\langle O_H O_H O_L O_L \rangle$

(b) Six-point function: $\langle O_H O_H O_L O_L O_L O_L \rangle$

Instead of looking at the 4-point function we studied before which probed a heavy state with a light state and which can be interpreted as the interactions of a light probe in the neighbourhood of the Black hole. We now have a look at the 6-point function. In section 2.3.9 we argued that the 6-point function can tell us something about the chaotic behaviour of Hawking radiation and therefore the scrambling time of a black hole. We will try to approximate this 6-point function using the method used in Fitzpatrick et al.

$$\langle O_H O_H O_L O_L O_L O_L \rangle \tag{149}$$

We can try to decompose the 6-point function in a similar manner as we did with the 4-point function in the previous section. Where we split the correlation function by inserting P_h. Previously we only took the leading terms and hence only considered $m = 1$ and $k \geq 0$ terms.

$$\begin{aligned} \langle OOPOO \rangle &= \sum_i \langle OOL^{k_1}_{-m_1} ... L^{k_i}_{-m_i} |h\rangle \mathcal{N}^{-1} \langle h| L^{k_i}_{m_i} ... L^{k_1}_{m_1} OO \rangle \\ &= \sum_k \langle O_{H_1}(\infty) O_{H_2}(1) \left(\frac{L^k_{-1}|h\rangle \langle h| L^k_1}{\langle h| L^k_1 K^k_{-1} |h\rangle} \right) O_{L_1}(w) O_{L_2}(0) \rangle \end{aligned} \tag{150}$$

For the 6-point function this results in the general decomposition:

$$\begin{aligned} &\langle OOPOOPOO \rangle \\ &= \sum_{ij} \langle OOL^{k_1}_{-m_1} ... L^{k_i}_{-m_i} |h\rangle \mathcal{N}^{-1} \langle h| L^{k_i}_{m_i} ... L^{k_1}_{m_1} OOL^{l_1}_{-n_1} ... L^{l_j}_{-n_j} |h\rangle \mathcal{N}^{-1} \langle h| L^{l_j}_{n_j} ... L^{l_1}_{n_1} OO \rangle \end{aligned} \tag{151}$$

Effectively we obtain four separate correlation functions, a 3-point heavy sector $C_{m_i k_i HH} = \langle OOL^{k_1}_{-m_1} ... L^{k_i}_{-m_i} |h\rangle$, a 4-point light sector $C_{m_i k_i LL n_j l_j} = \langle h| L^{k_i}_{m_i} ... L^{k_1}_{m_1} OOL^{l_1}_{-n_1} ... L^{l_j}_{-n_j} h| \rangle$, a 3-point light sector $C_{n_j l_j LL} = \langle h| L^{l_j}_{n_j} ... L^{l_1}_{n_1} OO \rangle$, and twice the normalization $\mathcal{N}$.

Because the numerator includes a four-point function which cannot be obtained immediately from the conformal symmetry we took only the identity blocks $h = 0$. This reduces the four-point function to a two-point function. But as a consequence one cannot take the OPE of L_m with h_w anymore. Hence an alternative for the approach taken in [13] had to be found. We attempted alternative ways of applying the OPE, but no satisfactory results have been found for the numerator. However we include our attempts in appendixA.3 for future reference.

On the other hand in the next section we show that results have been found for the denominator. These can tell us which modes contribute to the $\frac{1}{c}$ expansion. We will find that only terms with one L_m with $m \neq 1, 0$ and with L_1^k for any $k \geq 0$ will contribute to this expansion. Earlier we discussed the exponential decay of the four-point function and how this decay might be stopped at a certain point if $1/c$ terms are included; therefore, our approach does include the $1/c$ terms.

4.1 DENOMINATOR

To calculate the normalization $\mathcal{N}_{mnk} = C_{mn1^k}$ previously mentioned commutator relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta(n+m) \tag{152}$$

$$\langle h|L_n L_{-n}|h\rangle = 2nh + \frac{c}{12}n(n^2-1) \tag{153}$$

will be used. It can be observed that when $n > 1$ we obtain terms that are suppressed by c. For example the normalization for $n = 2$ goes as

$$\langle h|L_2 L_{-2}|h\rangle = 4h + \frac{c}{12}. \tag{154}$$

Also, for terms with $k > 1$ and $n > 1$ we obtain term suppressed by C^k.

4.1.1 *For k operators of n*

Next we calculate the normalization function $\mathcal{N} = C_n^k$ for different values of k.

For $k = 0$ we define

$$\begin{aligned} C &= \langle h||h\rangle \\ &= 1 \end{aligned} \tag{155}$$

For $k = 1$

$$\begin{aligned} C_n &= \langle h|L_n L_{-n}|h\rangle \\ &= 2nh + \frac{c}{12}n(n^2-1)C \end{aligned} \tag{156}$$

For $k = 2$

$$\begin{aligned} C_{nn} &= \langle h|L_n L_n L_{-n} L_{-n}|h\rangle \\ &= \langle h|L_n L_{-n} L_n L_{-n}|h\rangle + 2n\langle h|L_n L_0 L_{-n}|h\rangle + \frac{c}{12}n(n^2-1)C_n \\ &= 4n\langle h|L_0 L_n L_{-n}|h\rangle + 2n^2 C_n + 2\frac{c}{12}n(n^2-1)C_n \\ &= (4nh + 2n^2 + 2\frac{c}{12}n(n^2-1))C_n \end{aligned} \tag{157}$$

For $k = 3$

$$\begin{aligned} C_{nnn} &= \langle h|L_n L_n L_n L_{-n} L_{-n} L_{-n}|h\rangle \\ &= \langle h|L_n^2 L_{-n} L_n L_{-n}^2|h\rangle + 2n\langle h|L_n^2 L_0 L_{-n}^2|h\rangle + \frac{c}{12}n(n^2-1)C_{nn} \\ &= \langle h|L_n L_{-n} L_n^2 L_{-n}^2|h\rangle + 4n\langle h|L_n L_0 L_n L_{-n}^2|h\rangle + 2n^2 C_{nn} + 2\frac{c}{12}n(n^2-1)C_{nn} \\ &= (6nh + 6n^2 + 3\frac{c}{12}n(n^2-1))C_{nn} \end{aligned} \tag{158}$$

Every time we pull L_{-n} trough a L_n we obtain: one $\frac{c}{12}n(n^2-1)$ term, one $2nh$ terms, and $2n^2 m$ where m is the number of times it has been pulled trough before.

From this we can obtain the general rule

$$\begin{aligned} C_{n^k} &= \langle h|L_n^k L_{-n}^k|h\rangle \\ &= k!n^{2k}\left(\frac{2h}{n} + \frac{c}{12}(n - \frac{1}{n}\right)^{(k)} \end{aligned} \tag{159}$$

Taking the limit of a large central charge c this can be approximated by

$$C_{n^k} \approx k!\left(\frac{c}{12}n(n^2-1)\right)^k \tag{160}$$

As can be seen, if $n > 1$ $C_{n^k} \sim c^k$; hence, in the $\frac{1}{c}$ expansion only $k = 1$ for $n > 1$ should be included. That is only one operator L_n with $n > 1$ has to be included in this expansion.

4.1.2 *Operators of m and n*

Next the Normalization for the term that includes L_n and L_m with $n \neq m$ is calculated.

$$
\begin{aligned}
C_{mn} &= \langle h|L_m L_n L_{-n} L_{-m}|h\rangle \\
&= \langle h|L_m L_{-n} L_n L_{-m}|h\rangle + 2n\langle h|L_m L_0 L_{-m}|h\rangle + \frac{c}{12}n(n^2-1)C_m \\
&= (m+n)\langle h|L_{m-n} L_n L_{-m}|h\rangle + \left(2nh + 2mn + \frac{c}{12}n(n^2-1)\right) C_m
\end{aligned}
\tag{161}
$$

If $n > m$ we obtain

$$
\begin{aligned}
C_{mn} &= \left(2nh + 2mn + \frac{c}{12}n(n^2-1)\right) C_m \\
&\approx \frac{c}{12}n(n^2-1)C_m.
\end{aligned}
\tag{162}
$$

Only have $m = 1$ contributions in the $\frac{1}{c}$ expansion
If $n < m$ we get the following

$$
\begin{aligned}
C_{mn} &= (m+n)\langle h|L_\Delta L_n L_{-m}|h\rangle + \left(2nh + 2mn + \frac{c}{12}n(n^2-1)\right) C_m \\
&= (m+n)^2\langle h|L_\Delta L_{-\Delta}|h\rangle + \left(2nh + 2mn + \frac{c}{12}n(n^2-1)\right) C_m \\
&= (m+n)^2 \left(2\Delta h + \frac{c}{12}\Delta(\Delta^2-1)\right) C + \left(2nh + 2mn + \frac{c}{12}n(n^2-1)\right) C_m \\
&\approx \frac{c}{12}\left((m+n)^2\Delta(\Delta^2-1)C + n(n^2-1)C_m\right)
\end{aligned}
\tag{163}
$$

Where $\Delta = m - n$. Hence for $m, n > 1$ we obtain the following

$$
\begin{aligned}
C_{mn} &\approx \frac{c}{12}n(n^2-1)C_m \\
&= \left(\frac{c}{12}\right)^2 n(n^2-1)m(m^2-1).
\end{aligned}
\tag{164}
$$

We observe that inserting an L_m and L_n with $m, n > 1$ provide terms going as c^2. Knowing which terms go as c, the six-point function $\langle OOPOOPOO\rangle$ (eq.151) can be simplified to

$$
\sum_{k,l,p,q;m,n} \langle OOL_{-1}^l L_{-m} L_{-1}^k\rangle \mathcal{N}_{k,l,m}^{-1} \langle L_1^k L_m L_1^l OOL_{-1}^q L_{-n} L_{-1}^p\rangle \mathcal{N}_{p,q,n}^{-1} \langle L_1^p L_n L_1^q OO\rangle
\tag{165}
$$

where k, l, p, q, m, n are positive integers, and m and n can not be larger than one at the same time. Hence, we can simplify the summation.

$$\sum_{k,l,p;m} \mathcal{N}^{-1}_{k,l,m}\mathcal{N}^{-1}_{p}\Big[\langle OOL^{l}_{-1}L_{-m}L^{k}_{-1}\rangle\langle L^{k}_{1}L_{m}L^{l}_{1}OOL^{p}_{-1}\rangle\langle L^{p}_{1}OO\rangle + \langle OOL^{p}_{-1}\rangle\langle L^{p}_{1}OOL^{l}_{-1}L_{-m}L^{k}_{-1}\rangle\langle L^{k}_{1}L_{m}L^{l}_{1}OO\rangle\Big] \tag{166}$$

However, there is a caveat to this last simplification; that is, it might turn out that certain terms in the numerator are of order c[1].

4.1.3 *Inserting L^l_1, and one m operator*

Next we calculate the expression for one insertion L_m and l insertions of L_1.

$$\begin{aligned} C_{1^l m} &= \langle h|L^l_1 L_m L_{-m} L^l_{-1}|h\rangle \\ &= \frac{c}{12}m(m^2-1)C_{1^l} + \langle h|L^l_1 L_0 L^l_{-1}|h\rangle + \langle h|L^l_1 L_{-m} L_m L^l_{-1}|h\rangle \end{aligned} \tag{167}$$

Calculating the factor we obtain from the L_0 component.

$$\begin{aligned} C_{1^l L_0} &= \langle h|L^l_1 L_0 L^l_{-1}|h\rangle \\ &= \langle h|L^{l-1}_1 (L_0+1) L_1 L^l_{-1}|h\rangle \\ &= \langle h|L^{l-1}_1 L_0 L_1 L^l_{-1}|h\rangle + \langle h|L^l_1 L^l_{-1}|h\rangle \\ &= \langle h|L^{l-2}_1 L_0 L^2_1 L^l_{-1}|h\rangle + 2\langle h|L^l_1 L^l_{-1}|h\rangle \\ &= \langle h|L^{l-i}_1 L_0 L^i_1 L^l_{-1}|h\rangle + i\langle h|L^l_1 L^l_{-1}|h\rangle \\ &= \langle h|L_0 L^l_1 L^l_{-1}|h\rangle + l\langle h|L^l_1 L^l_{-1}|h\rangle \\ &= (h+l)\langle h|L^l_1 L^l_{-1}|h\rangle \\ &= (h+l)C_{1^l} \end{aligned} \tag{168}$$

1 One of our attempts for the numerator had terms of order c. This result can be found in eq.265

Calculating the factor we obtain from the $L_{-m}L_m$ component.

$$
\begin{aligned}
C_{1^l L_{-m}} &= \langle h|L_1^l L_{-m} L_m L_{-1}^l|h\rangle \\
&= \langle h|L_1^{l-1} L_{-m} L_1 L_m L_{-1}^l|h\rangle + (m+1)\langle h|L_1^{l-1} L_{-m+1} L_m L_{-1}^l|h\rangle \\
&= \langle h|L_1^{l-2} L_{-m} L_1^2 L_m L_{-1}^l|h\rangle + (m+1)\langle h|L_1^{l-2} L_{-m+1} L_m L_1 L_{-1}^l|h\rangle \\
&+ (m+1)\langle h|L_1^{l-2} L_{-m+1} L_1 L_m L_{-1}^l|h\rangle + (m+1)m\langle h|L_1^{l-2} L_{-m+2} L_m L_{-1}^l|h\rangle \\
&= \langle h|L_1^{l-2} L_{-m} L_1^2 L_m L_{-1}^l|h\rangle + 2(m+1)\langle h|L_1^{l-2} L_{-m+1} L_m L_1 L_{-1}^l|h\rangle \\
&+ (m+1)m\langle h|L_1^{l-2} L_{-m+2} L_m L_{-1}^l|h\rangle \\
&= \langle h|L_1^{l-3} L_{-m} L_1^3 L_m L_{-1}^l|h\rangle + 3(m+1)\langle h|L_1^{l-3} L_{-m+1} L_1^2 L_m L_{-1}^l|h\rangle \\
&+ 3(m+1)m\langle h|L_1^{l-3} L_{-m+2} L_1 L_m L_{-1}^l|h\rangle + (m+1)m(m-1)\langle h|L_1^{l-3} L_{-m+3} L_m L_{-1}^l|h\rangle \\
&= B_{3,0} + 3B_{3,1} + 3B_{3,2} + B_{3,3} \\
&= B_{4,0} + 4B_{4,1} + 6B_{4,2} + 4B_{4,3} + B_{4,4}
\end{aligned}
\tag{169}
$$

We defined $B_{a,b} = (m+1)_{(b)} \times \langle h|L_1^{l-a} L_{-m+b} L_1^{a-b} L_m L_{-1}^l|h\rangle$. We can see that the prefactor $c(a,b)$ for $B_{a,b}$ can be calculated by $c(a,b) = c(a-1,b-1) + c(a-1,b)$ which of course reminds us of Pascal's triangle giving us binomial coefficients $\binom{k}{i} = \frac{k!}{i!(k-i)!}$. Hence

$$
\begin{aligned}
C_{1^l L_{-m}} &= \sum_{i=0}^{n} \binom{n}{i} B_{n,i} \\
&= \sum_{i=0}^{l} \binom{l}{i} B_{l,i} \\
&= \sum_{i=0}^{l} \binom{l}{i} (m+1)_{(i)} \langle h|L_1^{l-l} L_{-m+i} L_1^{l-i} L_m L_{-1}^l|h\rangle \\
&= \sum_{i=0}^{l} \binom{l}{i} (m+1)_{(i)} \langle h|L_{-m+i} L_1^{l-i} L_m L_{-1}^l|h\rangle
\end{aligned}
\tag{170}
$$

As terms with $i < m$ annihilate, we obtain the equation below for $l < m$.

$$
\begin{aligned}
C_{1^l m} &= \langle h|L_1^l L_m L_{-m} L_{-1}^l|h\rangle \\
&= \frac{c}{12} m(m^2-1) C_{1^l} + (h+l) C_{1^l} + \sum_{i=0}^{l} \binom{l}{i} (m+1)_{(i)} \langle h|L_{-m+i} L_1^{l-i} L_m L_{-1}^l|h\rangle \\
&= \left[\frac{c}{12} m(m^2-1) + h + l\right] C_{1^l} + 0 \\
&\approx \frac{c}{12} m(m^2-1) C_{1^l}
\end{aligned}
\tag{171}
$$

When $l > m$ we don't get such a great simplification

$$
\begin{aligned}
C_{1^l m} &= \frac{c}{12} m(m^2-1) C_{1^l} + (h+l) C_{1^l} \\
&+ \binom{l}{m} (m+1)_{(m)} h \langle h | L_1^{l-m} L_m L_{-1}^l | h \rangle + \sum_{i=m+1}^{l} \binom{l}{i} (m+1)_{(i)} \langle h | L_{-m+i} L_1^{l-i} L_m L_{-1}^l | h \rangle \\
&\approx \frac{c}{12} m(m^2-1) C_{1^l} + \sum_{i=m+1}^{l} \binom{l}{i} (m+1)_{(i)} \langle h | L_{-m+i} L_1^{l-i} L_m L_{-1}^l | h \rangle
\end{aligned}
\tag{172}
$$

No terms of order c appear in the first term of the second row; however, in the second term this is not that obvious. To calculate in case of $l > m$, we start with the summation up till $a = m$ in $B_{a,b} = (m+1)_{(b)} \times \langle h | L_1^{l-a} L_{-m+b} L_1^{a-b} L_m L_{-1}^l | h \rangle$.

$$
\begin{aligned}
C_{1^l L_{-m}} &= \langle h | L_1^l L_{-m} L_m L_{-1}^l | h \rangle \\
&= \langle h | L_1^{l-1} L_{-m} L_1 L_m L_{-1}^l | h \rangle + (m+1) \langle h | L_1^{l-1} L_{-m+1} L_m L_{-1}^l | h \rangle \\
&= \langle h | L_1^{l-2} L_{-m} L_1^2 L_m L_{-1}^l | h \rangle + (m+1) \langle h | L_1^{l-2} L_{-m+1} L_1 L_m L_{-1}^l | h \rangle \\
&+ (m+1) \langle h | L_1^{l-2} L_{-m+1} L_1 L_m L_{-1}^l | h \rangle + (m+1) m \langle h | L_1^{l-2} L_{-m+2} L_m L_{-1}^l | h \rangle \\
&= \langle h | L_1^{l-2} L_{-m} L_1^2 L_m L_{-1}^l | h \rangle + 2(m+1) \langle h | L_1^{l-2} L_{-m+1} L_1 L_m L_{-1}^l | h \rangle \\
&+ (m+1) m \langle h | L_1^{l-2} L_{-m+2} L_m L_{-1}^l | h \rangle \\
&= \langle h | L_1^{l-3} L_{-m} L_1^3 L_m L_{-1}^l | h \rangle + 3(m+1) \langle h | L_1^{l-3} L_{-m+1} L_1^2 L_m L_{-1}^l | h \rangle \\
&+ 3(m+1) m \langle h | L_1^{l-3} L_{-m+2} L_1 L_m L_{-1}^l | h \rangle + (m+1) m (m-1) \langle h | L_1^{l-3} L_{-m+3} L_m L_{-1}^l | h \rangle \\
&= B_{3,0} + 3B_{3,1} + 3B_{3,2} + B_{3,3} \\
&= B_{4,0} + 4B_{4,1} + 6B_{4,2} + 4B_{4,3} + B_{4,4} \\
&= \sum_{i=0}^{n} \binom{n}{i} B_{n,i} \\
&= \sum_{i=0}^{m} \binom{m}{i} B_{m,i}
\end{aligned}
\tag{173}
$$

Now the last term is proportional to

$$
\langle h | L_1^{l-m} L_{-m+m} L_m L_{-1}^l | h \rangle. \tag{174}
$$

From this point, it's no use pulling $L_{m-m=0}$ trough as the next term will obtain a L_1 and doesn't annihilate the $\langle h|$. Let's look at an explicit example. We take $l = 4, m = 2$.

$$\begin{aligned}
C_{1^4 L_{-2}} &= \sum_{i=0}^{m=2} \binom{n}{i} B_{m,i} \\
&= \sum_{i=0}^{m=2} \binom{n}{i} B_{2,i} \\
&= B_{2,0} + 2B_{2,1} + B_{2,2} \\
&= B_{3,0} + 3B_{3,1} + 2B_{3,2} + B_{2,2} \\
&= B_{4,0} + 4B_{4,1} + 3B_{4,2} + 2B_{3,2} + B_{2,2} \\
&= 3B_{4,2} + 2B_{3,2} + B_{2,2}
\end{aligned} \tag{175}$$

First two terms annihilate nicely; however, the last three terms are more troublesome. In general $B_{n,i}$ terms annihilate with $n = l, i < m$, what remains is a summation with an overall factor of $(m+1)_{(m)}$(which we'll leave out for simplicity).

$$\begin{aligned}
&= 3\langle h|L_0 L_1^2 L_2 L_{-1}^4|h\rangle + 2\langle h|L_1 L_0 L_1 L_2 L_{-1}^4|h\rangle + \langle h|L_1^2 L_0 L_2 L_{-1}^4|h\rangle \\
&= 3\langle h|L_0 L_1^2 L_2 L_{-1}^4|h\rangle + 2\langle h|L_0 L_1^2 L_2 L_{-1}^4|h\rangle + 2\langle h|L_1^2 L_2 L_{-1}^4|h\rangle \\
&+ \langle h|L_1 L_0 L_1 L_2 L_{-1}^4|h\rangle + \langle h|L_1^2 L_2 L_{-1}^4|h\rangle \\
&= 5\langle h|L_0 L_1^2 L_2 L_{-1}^4|h\rangle + +2\langle h|L_1^2 L_2 L_{-1}^4|h\rangle \\
&+ \langle h|L_0 L_1 L_1 L_2 L_{-1}^4|h\rangle + \langle h|L_1^2 L_2 L_{-1}^4|h\rangle + \langle h|L_1^2 L_2 L_{-1}^4|h\rangle \\
&= 6\langle h|L_0 L_1^2 L_2 L_{-1}^4|h\rangle + +4\langle h|L_1^2 L_2 L_{-1}^4|h\rangle \\
&= (6h+4)\langle h|L_1^2 L_2 L_{-1}^4|h\rangle
\end{aligned} \tag{176}$$

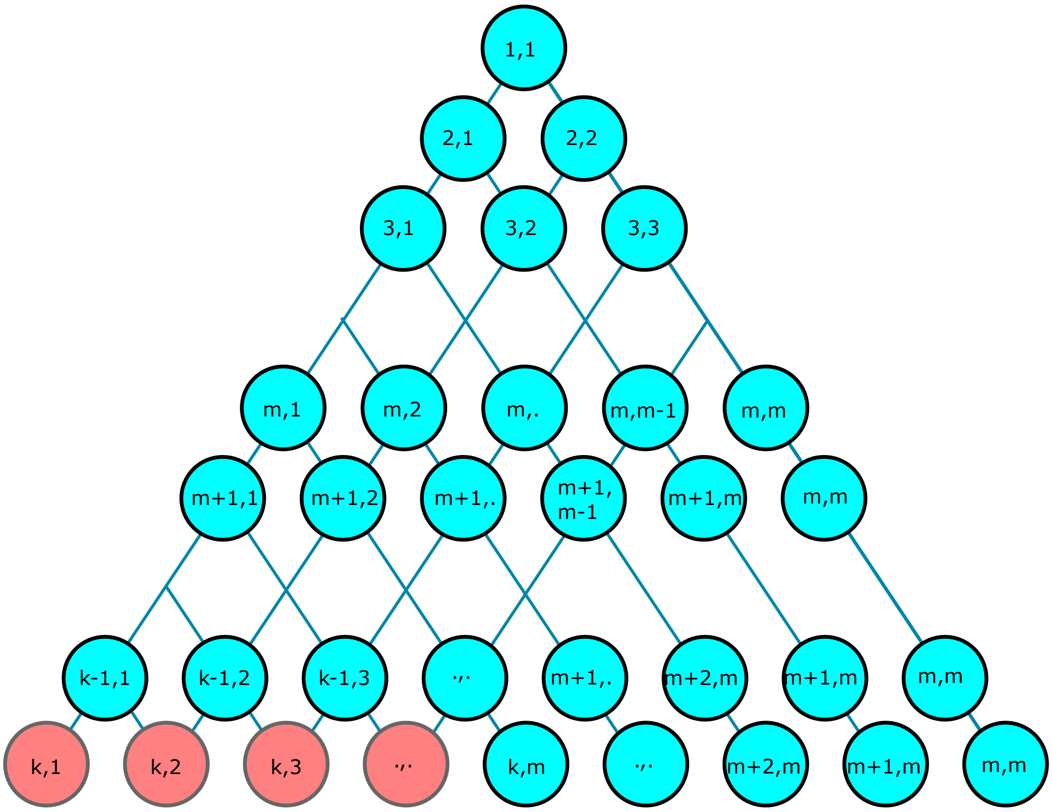


Figure 9.: Commutation: All terms with k being the first index annihilate. And we can stop commutating when the second index is m. So we end up with a summation from m to k

To solve this we stop commutating the terms $L_{-m+l=0}$ and notice that terms with $L_1^{l-a=0} L_{-m+b<0}$ annihilate. This is illustrated above.

$$
\begin{aligned}
C_{1^k L_{-m}} &= \sum_{i=0}^{m} \binom{m}{i} B_{m,i} \\
&= \sum_{i=0}^{m-1} \binom{m+1}{i} B_{m+1,i} + \binom{m}{m-1} B_{m+1,m} + \binom{m}{m} B_{m,m} \\
&= \sum_{i=0}^{m-2} \binom{m+2}{i} B_{m+2,i} + \binom{m+1}{m-1} B_{m+2,m} + \binom{m}{m-1} B_{m+1,m} + \binom{m}{m} B_{m,m} \\
&= \sum_{n=1}^{k-m} \binom{m+n-1}{m-1} B_{m+n,m} + \binom{m}{m} B_{m,m} \\
&= \sum_{n=0}^{k-m} \binom{m+n-1}{m-1} B_{m+n,m} \\
&= \sum_{n=m}^{k} \binom{n-1}{m-1} B_{n,m} \\
&= \sum_{n=m}^{k} \binom{n-1}{m-1} (m+1)_{(m)} \langle h | L_1^{k-n} L_0 L_1^{n-m} L_m L_{-1}^k | h \rangle
\end{aligned}
\tag{177}
$$

In line 3 to 4 we switch the summation from the second indice in $B_{m,i}$ to the first. And in the second to last line we notice that $\binom{m}{m} = \binom{m-1}{m-1}$.

Next pulling L_0 to the front (from eq.168)

$$\begin{aligned} \langle h|L_1^{k-n} L_0 L_1^{n-m} L_m L_{-1}^k |h\rangle &= (h+k-n)\langle h|L_1^{k-n} L_1^{n-m} L_m L_{-1}^k |h\rangle \\ &= (h+k-n)\langle h|L_1^{k-m} L_m L_{-1}^k |h\rangle \end{aligned} \tag{178}$$

We see the correlator is independent of n in this case; hence, we obtain

$$\begin{aligned} C_{1^k L_{-m}} &= \sum_{n=m}^{k} \binom{n-1}{m-1} (m+1)_{(m)} \langle h|L_1^{k-n} L_0 L_1^{n-m} L_m L_{-1}^k |h\rangle \\ &= (m+1)_{(m)} \langle h|L_1^{k-m} L_m L_{-1}^k |h\rangle \left[\sum_{n=m}^{k} \binom{n-1}{m-1} (h+k-n) \right] \\ &= (m+1)_{(m)} \left[\frac{(1+k-m)(h+k+-m+hm)k!}{m(1+m)(1+k-m)!(m-1)!} \right] \langle h|L_1^{k-m} L_m L_{-1}^k |h\rangle. \end{aligned} \tag{179}$$

So we just have to solve this last correlation function. If we consider the similarity

$$\begin{aligned} [L_m, L_{-1}] &= (m+1)L_{m-1} \\ [L_1, L_{-m}] &= (m+1)L_{1-m} \\ &= (m+1)L_{-(m-1)} \end{aligned} \tag{180}$$

Again defining a $F_{a,b} = (m+1)_{(b)} \times \langle h|L_1^{k-m} L_{-1}^{a-b} L_{m-b} L_{-1}^{k-a} |h\rangle$ also the sum will make sense only for $m \geq b$.

$$\begin{aligned} \langle h|L_1^{k-m} L_m L_{-1}^k |h\rangle &= \sum_{i=0}^{m} \binom{m}{j} F_{m,j} \\ &= \sum_{i=m}^{k} \binom{i-1}{m-1} F_{i,m} \\ &= \sum_{i=m}^{k} \binom{i-1}{m-1} (m+1)_{(m)} \langle h|L_1^{k-m} L_1^{i-m} L_0 L_{-1}^{k-i} |h\rangle \\ &= \sum_{i=m}^{k} \binom{i-1}{m-1} (m+1)_{(m)} (h+k-i) \langle h|L_1^{k-m} L_{-1}^{k-m} |h\rangle \\ &= (m+1)_{(m)} \langle h|L_1^{k-m} L_{-1}^{k-m} |h\rangle \left[\sum_{i=m}^{k} \binom{i-1}{m-1} (h+k-i) \right] \\ &= (m+1)_{(m)} \left[\frac{(1+k-m)(h+k-m+hm)k!}{m(1+m)(1+k-m)!(m-1)!} \right] C_{1^{k-m}} \end{aligned} \tag{181}$$

Combining equations 167, 179, and 181.

$$\begin{aligned}
C_{1^l m} &= \langle h|L_1^l L_m L_{-m} L_{-1}^l|h\rangle \\
&= \left(\frac{c}{12}m(m^2-1)+h+l\right)C_{1^l} + \langle h|L_1^l L_{-m} L_m L_{-1}^l|h\rangle \\
&= \left(\frac{c}{12}m(m^2-1)+h+l\right)C_{1^l} + (m+1)^2_{(m)}\left[\frac{(1+l-m)(h+l-m+hm)l!}{m(1+m)(1+l-m)!(m-1)!}\right]^2 C_{1^{l-m}} \\
&\approx \frac{c}{12}m(m^2-1)C_{1^l}
\end{aligned} \tag{182}$$

4.1.4 *Inserting one m operator, and* L_1^k

$$\begin{aligned}
C_{m1^k} &= \langle h|L_m L_1^k L_{-1}^k L_{-m}|h\rangle \\
&= \langle h|L_m L_1^k L_{-1}^{k-1} L_{-m} L_{-1}|h\rangle + (m-1)\langle h|L_m L_1^k L_{-1}^{k-1} L_{-m-1}|h\rangle \\
&= \langle h|L_m L_1^k L_{-1}^{k-2} L_{-m} L_{-1}^2|h\rangle + 2(m-1)\langle h|L_m L_1^k L_{-1}^{k-2} L_{-m-1} L_{-1}^1|h\rangle \\
&+ (m-1)m\langle h|L_m L_1^k L_{-1}^{k-2} L_{-m-2}|h\rangle \\
&= \langle h|L_m L_1^k L_{-1}^{k-3} L_{-m} L_{-1}^3|h\rangle + 3(m-1)\langle h|L_m L_1^k L_{-1}^{k-3} L_{-m-1} L_{-1}^2|h\rangle \\
&+ 3(m-1)m\langle h|L_m L_1^k L_{-1}^{k-3} L_{-m-2} L_{-1}|h\rangle + (m-1)m(m+1)\langle h|L_m L_1^k L_{-1}^{k-3} L_{-m-3}|h\rangle
\end{aligned} \tag{183}$$

Define $A_{a,b} = (m-1)^{(b)} \times \langle h|L_m L_1^k L_{-1}^{k-a} L_{-m-b} L_{-1}^{a-b}|h\rangle$. Similar to $B_{a,b} = (m+1)_{(b)} \times \langle h|L_1^{l-a} L_{-m+b} L_1^{a-b} L_m L_{-1}^l|h\rangle$. Again we recognize binomial coefficients $\binom{n}{i} = \frac{n!}{i!(n-i)!}$. We write the expansion as n terms and take $n \to k$ in line 6. Because $m+i \geq 2$ if $m \geq 2$ we don't have to worry about the case where they go to 1 and halt the summation at that point.

$$\begin{aligned}
C_{m1^k} &= \langle h|L_m L_1^k L_{-1}^{k-3} L_{-m} L_{-1}^3|h\rangle + 3(m-1)\langle h|L_m L_1^k L_{-1}^{k-3} L_{-m-1} L_{-1}^2|h\rangle \\
&+ 3(m-1)m\langle h|L_m L_1^k L_{-1}^{k-3} L_{-m-2} L_{-1}|h\rangle + (m-1)m(m+1)\langle h|L_m L_1^k L_{-1}^{k-3} L_{-m-3}|h\rangle \\
&= A_{3,0} + 3A_{3,1} + 3A_{3,2} + A_{3,3} \\
&= A_{4,0} + 4A_{4,1} + 6A_{4,2} + 4A_{4,3} + A_{4,4} \\
&= \sum_{i=0}^{n} \binom{n}{i} A_{n,i} \\
&= \sum_{i=0}^{k} \binom{k}{i} A_{k,i} \\
&= \sum_{i=0}^{k} \binom{k}{i} (m-1)^{(i)} \langle h|L_m L_1^k L_{-m-i} L_{-1}^{k-i}|h\rangle
\end{aligned} \tag{184}$$

Next we move L_m to the middle to get an expression in the $\langle h|L_1^l L_m L_{-m} L_{-1}^l|h\rangle$ form. If we again consider the similarity in commutator relationship from eq.180.

$$\begin{aligned} C_{m1^k} &= \sum_{i=0}^{k} \binom{k}{i} (m-1)^{(i)} \langle h|L_m L_1^k L_{-m-i} L_{-1}^{k-i}|h\rangle \\ &= \sum_{i=0}^{k} \binom{k}{i} (m-1)^{(i)} \left[\sum_{j=0}^{k} \binom{k}{j} (m-1)^{(j)} \langle h|L_1^{k-j} L_{m+j} L_{-m-i} L_{-1}^{k-i}|h\rangle \right] \\ &\approx \sum_{i=0}^{k} \binom{k}{i}^2 \left((m-1)^{(i)} \right)^2 \langle h|L_1^{k-i} L_{m+i} L_{-m-i} L_{-1}^{k-i}|h\rangle \end{aligned} \tag{185}$$

Because we look at the large c limit, using $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta(n+m)$ we see terms with $j = i$ will mainly contribute when $m > 1$.

$$\begin{aligned} C_{m1^k} &= \langle h|L_m L_1^k L_{-1}^k L_{-m}|h\rangle \\ &\approx \sum_{i=0}^{k} \binom{k}{i}^2 \left((m-1)^{(i)} \right)^2 \langle h|L_1^{k-i} L_{m+i} L_{-m-i} L_{-1}^{k-i}|h\rangle \\ &= \sum_{i=0}^{k} \binom{k}{i}^2 \left((m-1)^{(i)} \right)^2 C_{1^{k-i}m+i} \\ &\approx \sum_{i=0}^{k} \binom{k}{i}^2 \left((m-1)^{(i)} \right)^2 \left[\frac{c}{12} \left((m+i)^3 - (m+i) \right) \right] C_{1^{k-i}} \end{aligned} \tag{186}$$

One can make the relation more intuitive, but also more cluttering by using the following identities.

$$\begin{aligned} \binom{k}{i} &= \frac{k!}{i!(k-i)!} \\ (m-1)^{(i)} &= \frac{(m+i-2)!}{(m-2)!} \end{aligned} \tag{187}$$

$$\begin{aligned} \binom{k}{i} (m-1)^{(i)} &= \frac{k!}{i!(k-i)!} \frac{(m+i-2)!}{(m-2)!} \\ &= \frac{k!(m+i-2)!}{i!(k-i)!(m-2)!} \end{aligned} \tag{188}$$

One downfall of this expression is that when $m = 1$, we should not get any c dependence. But this is not the result of this expression, hence we can only use this expression when $m \geq 2$.

4.1.5 *Inserting* $L_1^l L_m L_1^k$

As we just evaluated C_{m1^k} by putting it in the same form as $C_{1^l m}$ (eq.186), we can easily generalize this expression by merging the two to get the form $C_{1^l m 1^k}$ with $m > 1$.

$$\begin{aligned}
C_{1^l m 1^k} &= \langle h|L_1^l L_m L_1^k L_{-1}^k L_{-m} L_{-1}^l|h\rangle \\
&\approx \sum_{i=0}^{k} \binom{k}{i}^2 \left((m-1)^{(i)}\right)^2 \langle h|L_1^l L_1^{k-i} L_{m+i} L_{-m-i} L_{-1}^{k-i} L_{-1}^l|h\rangle \\
&= \sum_{i=0}^{k} \binom{k}{i}^2 \left((m-1)^{(i)}\right)^2 \langle h|L_1^{k+l-i} L_{m+i} L_{-m-i} L_{-1}^{k+l-i}|h\rangle \\
&= \sum_{i=0}^{k} \binom{k}{i}^2 \left((m-1)^{(i)}\right)^2 C_{1^{k+l-i} m+i} \\
&\approx \sum_{i=0}^{k} \binom{k}{i}^2 \left((m-1)^{(i)}\right)^2 \left[\frac{c}{12}\left((m+i)^3 - (m+i)\right)\right] C_{1^{k+l-i}}
\end{aligned} \tag{189}$$

4.1.6 *Mapping* $C_{1^l m 1^k}$ *to the normalization factor* $\mathcal{N}$

We get the following expression for $\mathcal{N}_{k,l,m}$ with $m > 1$.

$$\begin{aligned}
\mathcal{N}_p &= C_{1^p} = (2h)^{(p)} p! \\
\mathcal{N}_{k,l,m} &= C_{1^l m 1^k} \approx \sum_{i=0}^{k} \binom{k}{i}^2 \left((m-1)^{(i)}\right)^2 \left[\frac{c}{12}\left((m+i)^3 - (m+i)\right)\right] C_{1^{k+l-i}}
\end{aligned} \tag{190}$$

$$\begin{aligned}
\mathcal{N}_{k,l,m}^{-1} \mathcal{N}_p^{-1} &= \frac{1}{C_{1^l m 1^k} C_{1^p}} \\
&= \left[\sum_{i=0}^{k} \binom{k}{i}^2 \left((m-1)^{(i)}\right)^2 \left[\frac{c}{12}\left((m+i)^3 - (m+i)\right)\right] C_{1^{k+l-i}} C_{1^p}\right]^{-1} \\
&= \left[\sum_{i=0}^{k} \binom{k}{i}^2 \left((m-1)^{(i)}\right)^2 \left[\frac{c}{12}\left((m+i)^3 - (m+i)\right)\right]\right. \\
&\quad \left. \times (2h)^{(k+l-i)} (k+l-i)! (2h)^{(p)} p! \right]^{-1} \\
&= \left[\sum_{i=0}^{k} \left(\frac{k!(m+i-2)!}{i!(k-i)!(m-2)!}\right)^2 \left[\frac{c}{12}\left((m+i)^3 - (m+i)\right)\right]\right. \\
&\quad \left. \times (2h)^{(k+l-i)} (k+l-i)! (2h)^{(p)} p! \right]^{-1}
\end{aligned} \tag{191}$$

4.1.7 *Summary*

To summarize we obtain the following results for the denominator taking the large c limit.

For $C_{1^k} = \langle h|L_1^k L_{-1}^k|h\rangle$ in eq.128 we obtain the result:

$$C_{1^k} = \langle h|L_1^k L_{-1}^k|h\rangle = (2h)^{(k)}k! \tag{192}$$

When we take $h = 0$, the identity blocks, this is equal to zero, unless $k = 0$. This is troublesome as these terms are inserted in the denominator, as we divide by zero. Hopefully, these get cancelled by zeros in the numerator.

Inserting k operators L_n with $n > 1$, i.e, $C_{n^k} = \langle h|L_n^k L_{-n}^k|h\rangle$ in eq.160 we obtain a c to the power k dependence as seen below.

$$C_{n^k} \approx k!\left(\frac{c}{12}n(n^2-1)\right)^k \tag{193}$$

Also, for inserting operators $L_m L_n$ with $n, m > 1$ in C_{mn} (eq.164) we also obtain a dependence on c^2

$$C_{mn} \approx \left(\frac{c}{12}\right)^2 n(n^2-1)m(m^2-1) \tag{194}$$

From these results we can conclude that only terms having one L_m term with $m > 1$ in P should be included in the $\frac{1}{c}$ expansion. As insertions of L_1 don't provide any c dependence, multiple can be inserted in combination with a L_m. Hence, eq.151 can be simplified to

$$\begin{aligned} &\langle OOPOOPOO\rangle \\ &= \sum_{k,l,p,q;m,n} \langle OOL_{-1}^l L_{-m} L_{-1}^k\rangle \mathcal{N}_{k,l,m}^{-1} \langle L_1^k L_m L_1^l OOL_{-1}^q L_{-n} L_{-1}^p\rangle \mathcal{N}_{p,q,n}^{-1} \langle L_1^p L_n L_1^q OO\rangle \end{aligned} \tag{195}$$

where k, l, p, q, m, n are positive integers, and m and n can not be larger than one at the same time. Hence this "simplifies" too

$$\begin{aligned} \sum_{k,l,p;m} \mathcal{N}_{k,l,m}^{-1}\mathcal{N}_p^{-1}\Big[&\langle OOL_{-1}^l L_{-m} L_{-1}^k\rangle\langle L_1^k L_m L_1^l OOL_{-1}^p\rangle\langle L_1^p OO\rangle \\ &+ \langle OOL_{-1}^p\rangle\langle L_1^p OOL_{-1}^l L_{-m} L_{-1}^k\rangle\langle L_1^k L_m L_1^l OO\rangle\Big] \end{aligned} \tag{196}$$

Because the expression for $C_{1^l m 1^k}$ does not have a satisfying result for $m = -1, 0, 1$, we should split the previous result in a c and non-c expansion.

$$\sum_{q,p} \mathcal{N}_q^{-1} \mathcal{N}_p^{-1} \Big[\langle OOL_{-1}^q \rangle \langle L_1^q OOL_{-1}^p \rangle \langle L_1^p OO \rangle \Big]$$
$$+ \sum_{k,l,p;m} \mathcal{N}_{k,l,m}^{-1} \mathcal{N}_p^{-1} \Big[\langle OOL_{-1}^l L_{-m} L_{-1}^k \rangle \langle L_1^k L_m L_1^l OOL_{-1}^p \rangle \langle L_1^p OO \rangle$$
$$+ \langle OOL_{-1}^p \rangle \langle L_1^p OOL_{-1}^l L_{-m} L_{-1}^k \rangle \langle L_1^k L_m L_1^l OO \rangle \Big] \tag{197}$$

Expressed in C_{hhm}

$$\sum_{q,p} \frac{1}{C_{1^q} C_{1^p}} \Big[C_{HH(-1)^q} C_{LL1^p(-1)^q} C_{1^p LL} \Big]$$
$$+ \sum_{k,l,p;m} \frac{1}{C_{1^l m 1^k} C_{1^p}} \Big[C_{HH(-1)^l(-m)(-1)^k} C_{1^k m 1^l LL(-1)^p} C_{1^p LL}$$
$$+ C_{LL(-1)^p} C_{1^p LL(-1)^l m(-1)^k} C_{1^k m 1^l LL} \Big] \tag{198}$$

The correlator $C_{1^l m}$ with $l < m$ (eq.171) and with $l \geq m$ while $m > 1$ (eq.182) gives us:

$$C_{1^l m} \approx \frac{c}{12} m(m^2 - 1) C_{1^l} \tag{199}$$

a very nice result when $l = 0$, but when $l > 0$ and taking the identity blocks, we get $C_{1^l m} \propto c0$. Zero's of first order.

Inserting the L_1 in the middle as in C_{m1^k} (eq.186)

$$C_{m1^k} \approx \sum_{i=0}^{k} \binom{k}{i}^2 \left((m-1)^{(i)} \right)^2 \left[\frac{c}{12} \left((m+i)^3 - (m+i) \right) \right] C_{1^{k-i}} \tag{200}$$

The previous two results can be combined to get an approximation for $C_{1^l m 1^k}$ (eq.189)

$$C_{1^l m 1^k} \approx \sum_{i=0}^{k} \binom{k}{i}^2 \left((m-1)^{(i)} \right)^2 \left[\frac{c}{12} \left((m+i)^3 - (m+i) \right) \right] C_{1^{k+l-i}} \tag{201}$$

For $k = 0$ this reduces to eq.199 and with $l = 0$ to eq.200; consequently, this expression captures these two solutions and therefore can replace them. Still $C_{1^{k+l-i}}$ has zeros order one.

5

CONCLUSION

In physics we have two theories that provide great results and good predictions; namely, General Relativity and Quantum Field Theory. GR is our model for gravity and works on very large scales and QFT is our model for interactions between fundamental particles and works on very small scales. Therefore, when we work in one theory we can in general ignore the other. But what we would like is to unite these theories to obtain a Theory Of Everything.

Although, we have not been able to unify gravity and the quantum world for our universe, we have a model that does connect these fields; i.e, Anti-de-Sitter/Conformal-Field-Theory or the AdS/CFT correspondence. This correspondence is interesting as it connects a Quantum field theory, CFT, with gravity, AdS. This correspondence was discussed in this thesis and an introduction was given in chapter 1.

When we want to understand how a large scale gravity theory and a small scale QFT interact with each other, it is instructive to look at Black holes. Because the interactions in gravity and Quantum field Theory appear on the same length scale near Black holes. One big problem with Black Holes is the consequence of Hawking Radiation, a Black Hole has a thermal, it's only dependent on it's temperature and independent on the information that the Black Hole has eaten during it's lifetime. Therefore, when a Black Hole evaporates all information about the initial particles is lost. This paradox is also known as the Information Paradox.

To shed light on this problem we took a closer look at the Hawking radiation and investigate if it behaves chaotically. To understand how sensitive the Hawking radiation is to initial conditions, it becomes clear that we need to have an expression for the six-point function. However, four-point functions and higher cannot be obtained by just using the conformal symmetry; hence, the six-point function has to be calculated from the interactions involved. For this we turned to the paper [13] by Fitzpatrick et al.

In chapter 2 we studied the four-point function with two heavy insertions and two light insertions. We followed the approach taken by Fitzpatrick et al. to calculate the four-point function in an alternative coordinate system. The four-point function for a light particle probing a Black Hole looks like

$$(1-w)^{(1-\frac{1}{\alpha})L}\left(\frac{w}{\alpha}\right)^{h-2L}{}_2F_1(h,h,2h,w) \tag{202}$$

When the limit $h \to 0$ is taken, we noticed that the geometric function becomes 1. In other words only contributions from $k = 0$. This was confirmed when we calculated the four-point function with only the $k = 0$ contributions seen below.

$$(1-w)^{(1-\frac{1}{\alpha})L}\left(\frac{w}{\alpha}\right)^{h-2L} \tag{203}$$

From this we could conclude that in the limit of $h = 0$, the four point factorizes in two two-point functions, one with both heavy operators and one with both light operators, in this particular coordinate system as seen below

$$\langle O_H(\infty)O_H(1)O_{L_1}(z)O_{L_2}(0)\rangle \approx \langle O_H(\infty)O_H(1)\rangle\langle O_{L_1}(z)O_{L_2}(0)\rangle \tag{204}$$

We also observed that the four-point function decays exponentially even at late times, as mentioned before this is should not happen in a unitary theory. Therefore, it is very possible that at large t the $1/c$ contributions that have not been included in this approach might contribute enough to stop the exponential decay.

In Chapter 4 we dove into the problem of the six-point function. We used a similar approach as used in the previous section to look at the four-point function. Again we insert an operator P, but now twice, to decompose the correlator in smaller correlators. The correlator was split into a numerator which was split again into three smaller correlation functions and a denominator which consist of two normalization factors. Subsequently by first considering the denominator or normalization $\mathcal{N}_{klm} = C_{1^k m 1^l}$ we found which terms contribute to the $1/c$ expansion in section 4.1. Again solutions were found by using the commutator relations. As can be seen below we found that the order of c in the normalization is equal to the number of operators L_m with $m > 1$ that are included.

$$C_{m^k} = \langle h|L_m^k L_{-m}^k|h\rangle \approx k!\left(\frac{c}{12}m(m^2-1)\right)^k \tag{205}$$

The explicit results for C_{1^k} (eq.128), C_{n^k} (eq.160), and C_{mn} (eq.164) can be found in section 4.1.

When trying to compute the terms in the numerator we ran into difficulties. By inserting the projection operator P twice one obtains a four-point function in the numerator. One can try to resolve this by taking only contributions from the Identity blocks by taking $h = 0$. But in that situation it is not clear how the Operator Product Expansion should be done. Multiple attempts have been done during the research, which can be found in the Appendix, but no satisfactory results have been found.

6

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A

APPENDIX

A.1 THREE POINT

By taking the limit of $z_1 \to \infty$ and $z_3 \to 0$ a simplified version of the three-point function can be obtained. Starting from the definition of the three point function in section 2.2.4 eq.48

$$G_{ijk} = C_{ijk}(z_1 - z_2)^{h_3-h_1-h_2}(z_2 - z_3)^{h_1-h_2-h_3}(z_3 - z_1)^{h_2-h_1-h_3} \quad (206)$$

Inserting $h_1 = h_2 = h_H$ and $h_3 = h$ and taking the limits $z_1 \to \infty$ and $z_3 \to 0$. To take the limit to infinity in line four we transform the coordinate $z_1 \to 1/w$ to map infinity to zero.

$$\begin{aligned}
G_{HHh} &= \lim_{z_1\to\infty}\lim_{z_3\to 0} C_{HHh}(z_1 - z_2)^{h-h_H-h_H}(z_2 - z_3)^{h_H-h_H-h}(z_3 - z_1)^{h_H-h_H-h} \\
&= C_{HHh}\lim_{z_1\to\infty}\lim_{z_3\to 0}\left[(z_1 - z_2)^{h-2h_H}(z_2 - z_3)^{-h}(z_3 - z_1)^{-h}\right] \\
&= C_{HHh}\lim_{z_1\to\infty}\left[(z_1 - z_2)^{h-2h_H}(z_2)^{-h}(-z_1)^{-h}\right] \\
&= C_{HHh}\lim_{w\to 0} w^{-2h_H}\left[(1/w - z_2)^{h-2h_H}(z_2)^{-h}(w)^{h}\right] \\
&= C_{HHh}\lim_{w\to 0}\left[(1/w - z_2)^{h-2h_H}(z_2)^{-h}(w)^{h-2h_H}\right] \\
&= C_{HHh} z_2^{-h}
\end{aligned} \quad (207)$$

A.2 ABBREVIATIONS USED

For correlation functions we use the abbreviations

$$\begin{aligned}
C_{1^k} &= \langle h|L_1^k L_{-1}^k|h\rangle \\
C_n &= \langle h|L_n L_{-n}|h\rangle \\
C_{n^k} &= \langle h|L_n^k L_{-n}^k|h\rangle \\
C_{m1^k} &= \langle h|L_m L_1^k L_{-1}^k L_{-m}|h\rangle \\
C_{1^l m} &= \langle h|L_1^l L_m L_{-m} L_{-1}^l|h\rangle \\
C_{1^l m 1^k} &= \langle h|L_1^l L_m L_1^k L_{-1}^k L_{-m} L_{-1}^l|h\rangle
\end{aligned} \quad (208)$$

$$
\begin{aligned}
B_{a,b} &= (m+1)_{(b)} \times \langle h|L_1^{l-a}L_{-m+b}L_1^{a-b}L_m L_{-1}^{l}|h\rangle \\
A_{a,b} &= (m-1)^{(b)} \times \langle h|L_m L_1^{k} L_{-1}^{k-a} L_{-m-b} L_{-1}^{a-b}|h\rangle \\
D_{a,b} &= (m+i-1)_{(b)} \times \langle h|L_m L_1^{k-a} L_{-m-i+b} L_1^{a-b} L_{-1}^{k-i}|h\rangle \\
F_{a,b} &= (m+1)_{(b)} \times \langle h|L_1^{k-m} L_{-1}^{a-b} L_{m-b} L_{-1}^{k-a}|h\rangle
\end{aligned}
\tag{209}
$$

Binomial coefficients, elements of pascal's triangle.

$$
\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{210}
$$

And falling and rising factorials

$$
\begin{aligned}
n^{(k)} &= n(n+1)(n+2)...(n+k-1) \\
n^{(k)} &= (n+k-1)_{(k)} \\
n_{(k)} &= n(n-1)(n-2)...(n-k+1) \\
\frac{n_{(k)}}{k!} &= \binom{n}{k} \\
\frac{n^{(k)}}{k!} &= \binom{n+k-1}{k}
\end{aligned}
\tag{211}
$$

$$
\begin{aligned}
\binom{k}{i} &= \frac{k!}{i!(k-i)!} \\
(m-1)^{(i)} &= \binom{m-2+i}{i} i! \\
&= \frac{(m+i-2)!}{i!(m-2)!} i! \\
&= \frac{(m+i-2)!}{(m-2)!}
\end{aligned}
\tag{212}
$$

A.3 NUMERATOR

While working on this thesis we could not obtain satisfactory results for numerator terms. We did not include them in the main thesis, but for documentation purposes and future reference we include them here in the Appendix.

A.3.1 *Operator Product Expansion*

The four-point function can be expressed as

$$
f(z_1, z_2) = \langle T(z_1) O_L O_L T(z_2) \rangle \tag{213}
$$

and we know that

$$\frac{\partial}{\partial \bar{z}} T(z) = 0 \rightarrow \frac{\partial}{\partial \bar{z}} f(z) = 0 \tag{214}$$

Also, $f(z \rightarrow \infty) = 0$, so $f(z)$ is a Holomorphic functions except for it's 3 poles with corresponding singularities. Therefore this $f(z)$ is a Meromorphic function; consequently, we can solve this function explicitly. Usually, we would have to do an OPE with an infinite terms, but due to being a meromorphic function, the fucntion is fully defined by it's singular parts. Hence, using the conformal ward identities, we get the following OPE terms for $T(z)O(z_i)$ calculated in eq.38.

$$T(z)O(z_i) = \frac{h_i O(z_i)}{(z - z_i)^2} + \frac{1}{z - z_i} \frac{\partial O(z_i)}{\partial_{z_i}} \tag{215}$$

$$T(z)T(z_i) = \frac{c/2}{(z - z_i)^4} + \frac{h_i T(z_i)}{(z - z_i)^2} + \frac{1}{z - z_i} \frac{\partial T(z_i)}{\partial_{z_i}} \tag{216}$$

Using these OPE terms we tried to obtain results for the terms in the numerator.

A.3.2 *Numerator current tries and problems*

In section A.3.3 we calculate the separate OPE terms for C_{mLL} and C_{mnLL}.

Taking OPE of $T(z)$ with $O_h(w_i)$ in C_{mLL} eq.233

$$\begin{aligned} C_{mLL} &= \langle L_m O_L(w_1) O_L(w_2) \rangle \\ &= \sum_i \left(h_L(m+1) w_i^m + w_i^{m+1} \partial_{w_i} \right) C_{LL} \end{aligned} \tag{217}$$

Taking the OPE of $T(z_1)$ with $T(z_2)$ followed by $T(z_2)$ with $O(w_i)$ eq.254. Here the c term only survives when $m = -n$, hence only in the $\langle L_m O O L_{-m} \rangle$. These appear in expansions like $\langle O O L_m \rangle \langle L_m O O L_{-m} \rangle \langle L_m O O \rangle$. The first OPE term containing the c contribution from 253

$$C_{mnLL}^{T \leftrightarrow T} = \frac{c}{12}(m-1)m(m+1) \tag{218}$$

Here the c term only survives when $n = -m$, hence only in the $\langle L_m O O L_{-m} \rangle$. These appear in expansions like $\langle O O L_m \rangle \langle L_m O O L_{-m} \rangle \langle L_m O O \rangle$ and any combination with L_1 as these are included in the $C_L L$ and therefore don't influence the order or dissapearence of c. The second term containing no c contributions eq.254

$$C_{mnLL}^{T \leftrightarrow T} = \sum_i w_i^{m+n} \left[(m-n)(m+n+1) h_L + (m-n) w_i \partial_{w_i} \right] C_{LL} \tag{219}$$

Taking the OPE of $T(z_1)$ with $O(w_i)$ followed by $T(z_2)$ with $O(w_i)$ eq.263.

$$C_{mnLL}^{T\leftrightarrow O} = \sum_i w_i^m \left(w_i^{n+1} \left(\frac{h_L(n+1)}{w_i} + \partial_{w_i} \right) + \sum_j \right.$$
$$\left. w_j^n \left[h_L(m+1) \left(h_L(n+1) + w_j \partial_{w_j} \right) + w_i \left(h_L(n+1) + w_j \partial_{w_j} \right) \partial_{w_i} \right] \right) C_{LL} \tag{220}$$

Which are combined and summarized in section A.3.4.

In section A.3.5.1, $L_m O_L O_L$, no $O(w)$ is inserted, so we just work with the two-point function. Also the OPE of $T(z)$ is taken with both $O(w)$. We obtain the result in eq.269

$$\begin{aligned} C_{mLL} &= \langle L_m O_L(w_1) O_L(w_2) \rangle \\ &= \left((m+1)(w_2^m w_1 - w_1^m w_2) + (m-1)(w_1^{1+m} - w_2^{1+m}) \right) \\ &\times h_L (w_1 - w_2)^{-1-2h_L} \end{aligned} \tag{221}$$

For $m = -1, 0, 1$ this result goes to zero, for $m > 2$ and $m \leq -2$ we obtain

$$\begin{aligned} m > -2 &\rightarrow h_L(w_1 - w_2)^{-1-2h_L} \left((m+1)(w_2^m w_1 - w_1^m w_2) + (m-1)(w_1^{1+m} - w_2^{1+m}) \right) \\ m \leq -2 &\rightarrow h_L(w_1 - w_2)^{-1-2h_L} \left[((m+1)(w_2^m w_1 - w_1^m w_2) + (m-1)(w_1^{1+m} - w_2^{1+m})) \right. \\ &\left. + (w_1 - w_2)^3 \sum_{i=m}^{0} i(m-i)(w_1^{-1+i} w_2^{m-i-1}) \right] \end{aligned} \tag{222}$$

In the subsequent section A.3.5.2, $L_m L_n O_L O_L$, we use the same approach but inserting two operators $L_{m,n}$. This solution (eq.273) was solved using mathematica, all c dependence drops out in this case. While the normalization $\mathcal{N} \propto c^2$.

$$\begin{aligned}
C_{mnLL} &= \langle L_m L_n O_L(w_1) O_L(w_2) \rangle \\
&= \big[(n-1)(h_L(m-1)-n)(w_1^{2+m+n} + w_2^{2+m+n}) \\
&- 2\left(-1+n^2+h_L(-1+mn)\right)\left(w_1^{1+m+n}w_2 + w_2^{1+m+n}w_1\right) \\
&+ (1+n)(h_L + h_L m + n)(w_1^{m+n}w_2^2 + w_2^{m+n}w_1^2) \\
&+ h_L(1+m)(-1+n)(w_1^{2+n}w_2^m + w_2^{2+n}w_1^m) \\
&+ h_L(-1+m)(1+n)(w_1^{2+m}w_2^n + w_2^{2+m}w_1^n) \\
&- 2(1+h_L+h_L mn)(w_1^{1+n}w_2^{1+m} + w_1^{1+m}w_2^{1+n})\big] \\
&\times h_L(w_1-w_2)^{-2-2h_L}
\end{aligned} \tag{223}$$

When we take the limit $w_1 \to 0$ the same results as in Fitzpatrick are found; i.e, $C_{mnLL} = h_L(h_L(m-1)(n-1) + n(n-1))$.

In section A.3.5.3 we first use the result from section A.3.5.1 and set $m =, -1, 0, 1$ and obtain the result (eq.276)

$$C_{-1LL} = C_{0LL} = C_{1LL} = 0 \tag{224}$$

all zeros of first order.

Subsequently we set $m, n = 1$ for the results in section A.3.5.2 and also obtain a zero of first order (eq.278).

$$C_{11LL} = 0 \tag{225}$$

In section A.3.6 we insert an $|h_w\rangle$ and take the OPE of $T(z)$ only with this operator. Hence we obtain C_{LLhn} we try different order of taken the contour integral around $z = o, w$ and taking the limit of $w \to 0$ all resulting in zeros.

Finally in section A.3.7 we insert an $|h_w\rangle$ just like in the previous section, but now we take the OPE with all operators. We try two approaches the first one we take the contour integral only around the coordinates of the three-point function and taking $h \to 0$ without taking $w \to 0$. This approach gives nice results(eq.285) and zeros for $m = -1, 0, 1$

$$\begin{aligned}
m < -1 &\to C_{mLL} = h_L(w_1-w_2)^2(w_1w_2)^m \sum_i^m \left(i(m-i)w_1^{m-1-i}w_2 i - 1\right)(w_1-w_2)^{-2h_L} \\
m > 1 \quad &\to C_{mLL} = h_L(w_1-w_2)^2 \sum_i^m \left(i(m-i)w_1^{m-1-i}w_2 i - 1\right)(w_1-w_2)^{-2h_L}
\end{aligned} \tag{226}$$

A.3.3 *Calculation of OPE terms*

In this section we calculate only the OPE terms and we keep the two-point functions complete. Therefore these results can also be extended to three-point functions.

A.3.3.1 C_{mLL}

$$\begin{aligned} C_{mLL} &= \langle L_m O_L(w_1) O_L(w_2) \rangle \\ &= \oint \frac{dz}{2\pi i} z^{m+1} \langle T(z) O_L(w_1) O_L(w_2) \rangle \\ &= \oint \frac{dz}{2\pi i} z^{m+1} \sum_i \left(\frac{h_L}{(z-w_i)^2} + \frac{1}{z-w_i} \partial_{w_i} \right) C_{LL} \\ &= \sum_i \left(\oint \frac{dz}{2\pi i} z^{m+1} \frac{h_L}{(z-w_i)^2} + \oint \frac{dz}{2\pi i} z^{m+1} \frac{1}{z-w_i} \partial_{w_i} \right) C_{LL} \end{aligned} \tag{227}$$

As C_{LL} is independent of z we can take these contour integrals without looking at the derivative. Contour integrals are taken in Contour Integrals.nb sec.**??**(TOone and TOtwo). From the first term we get the separate terms:

$$\oint_{z=w_i} \frac{dz}{2\pi i} z^{m+1} \frac{h_L}{(z-w_i)^2} = h_L(m+1) w_i^m \tag{228}$$

Contour integral around $z = 0$ and $z = \infty$

$$\begin{aligned} \oint_{z=0} \frac{dz}{2\pi i} z^{m+1} \frac{h_L}{(z-w_i)^2} &= -h_L(m+1) w_i^m \quad \text{For } m \leq 0 \\ \oint_{z=\infty} \frac{dz}{2\pi i} z^{m+1} \frac{h_L}{(z-w_i)^2} &= \oint_{x=0} \frac{dx}{2\pi i} (-x^{-2}) x^{-m-1} \frac{h_L}{(1/x - w_i)^2} \\ &= -h_L(m+1) w_i^m \quad \text{For } m \geq 0 \\ \left(\oint_{z=0} \frac{dz}{2\pi i} + \oint_{z=\infty} \frac{dz}{2\pi i} \right) z^{m+1} \frac{h_L}{(z-w_i)^2} &= -h_L(m+1) w_i^m \end{aligned} \tag{229}$$

Combining the poles $z = 0$, $z = w_i$, and $z = \infty$

$$\left(\oint_{z=w_i} \frac{dz}{2\pi i} + \oint_{z=0} \frac{dz}{2\pi i} + \oint_{z=\infty} \frac{dz}{2\pi i} \right) z^{m+1} \frac{h_L}{(z-w_i)^2} = 0 \tag{230}$$

Second term gives

$$\oint_{z=w_i} \frac{dz}{2\pi i} z^{m+1} \frac{1}{z-w_i} \partial_{w_i} = w_i^{m+1} \partial_{w_i} \tag{231}$$

Contour integral around $z = 0$ and $z = \infty$

$$
\begin{aligned}
\oint_{z=0} \frac{dz}{2\pi i} z^{m+1} \frac{1}{z - w_i} \partial_{w_i} &= -w_1^{m+1} \partial_{w_i} \quad \text{For } m < -1 \\
\oint_{z=\infty} \frac{dz}{2\pi i} z^{m+1} \frac{h_L}{z - w_i} \partial_{w_i} &= \oint_{x=0} \frac{dx}{2\pi i} (-x^{-2}) x^{-m-1} \frac{h_L}{1/x - w_i} \partial_{w_i} \\
&= -w_i^{m+1} \partial_{w_i} \quad \text{For } m \geq -1 \\
\left(\oint_{z=0} \frac{dz}{2\pi i} + \oint_{z=\infty} \frac{dz}{2\pi i} \right) z^{m+1} \frac{1}{z - w_i} \partial_{w_i} &= -w_i^{m+1} \partial_{w_i}
\end{aligned}
\tag{232}
$$

For both contributions we get zeros when m is outside the indicated domain. These solutions are indepenedent of the correlation function $\langle O_L O_L \rangle$ accept for the integer i we have to sum over. So C_{LL} ($i = 1, 2$)could also be the three-point functions C_{LLL} ($i = 1, 2, 3$) or heavy solution C_{HH} ($h_L \to h_H$).

$$
\begin{aligned}
C_{mLL} &= \langle L_m O_L(w_1) O_L(w_2) \rangle \\
&= \sum_i \left(h_L(m+1) w_i^m + w_i^{m+1} \partial_{w_i} \right) C_{LL}
\end{aligned}
\tag{233}
$$

When we look at the pole $z = \infty$ we notice that it cancels the contributions from the poles $z = w_i$. Hence $C_{mLL} = 0$ for all values of m

$$
\left(\oint_{z=w_i} \frac{dz}{2\pi i} + \oint_{z=0} \frac{dz}{2\pi i} + \oint_{z=\infty} \frac{dz}{2\pi i} \right) z^{m+1} \frac{1}{z - w_i} \partial_{w_i} = 0 \tag{234}
$$

A.3.3.2 C_{mnLL} *OPE of* T *with* T

Inserting two operators L_m, first we take the OPE of $T(z_1)$ with $T(z_2)$ followed by $T(z_2)$ with O_L. Next we use the previous results to take the OPE of $T(z_1)$ with O_L.

$$\begin{aligned} C^{T\leftrightarrow T}_{mnLL} &= \langle L_m L_n O_L(w_1) O_L(w_2)\rangle \\ &= \oint \frac{dz_1}{2\pi i} z_1^{m+1} \oint \frac{dz_2}{2\pi i} z_2^{n+1} \langle T(z_1)T(z_2)O_L(w_1)O_L(w_2)\rangle \\ &= \oint \frac{dz_1}{2\pi i} z_1^{m+1} \oint \frac{dz_2}{2\pi i} z_2^{n+1} \left(\frac{c/2}{(z_1-z_2)^4} C_{LL} + \frac{h_2}{(z_1-z_2)^2} C_{TLL}(z_2) + \frac{1}{z_1-z_2}\frac{\partial C_{TLL}(z_2)}{\partial_{z_2}} \right) \end{aligned} \tag{235}$$

starting with the contour integral around $z_1 = 0$ and $z_1 = z_2$.

$$\oint \frac{dz_1}{2\pi i} z_1^{m+1} \left(\frac{c/2}{(z_1-z_1)^4} \right) C_{LL} = \left(\oint_{z_1=0} \frac{dz_1}{2\pi i} + \oint_{z_1=z_2} \frac{dz_1}{2\pi i} \right) z_1^{m+1} \frac{c/2}{(z_1-z_1)^4} C_{LL} \tag{236}$$

$$\begin{aligned} \oint_{z_1=0} \frac{dz_1}{2\pi i} z_1^{m+1} \frac{c/2}{(z_1-z_2)^4} &= -\frac{c}{12}(m-1)m(m+1) z_2^{m+n-1} \quad \text{For } m<0 \\ \oint_{z_1=z_2} \frac{dz_1}{2\pi i} z_1^{m+1} \frac{c/2}{(z_1-z_2)^4} &= \frac{c}{12}(m-1)m(m+1) z_2^{m+n-1} \\ \left(\oint_{z_1=z_2} + \oint_{z_1=0} \right) \frac{dz_1}{2\pi i} z_1^{m+1} \frac{c/2}{(z_1-z_2)^4} &= \frac{c}{12}(m-1)m(m+1) z_2^{m+n-1} \quad \text{For } m>0 \end{aligned} \tag{237}$$

Next taking the contour integral around $z_2 = 0$ we get non-zero contributions from $m + n = 0$ with $m > 1$. (TTone)

$$\oint_{z_2=0} \frac{dz_2}{2\pi i} \frac{c}{12}(m-1)m(m+1) z_2^{m+n-1} C_{LL} = \frac{c}{12}(m-1)m(m+1) C_{LL} \qquad \text{For } m+n=0 \tag{238}$$

Hence this term only contributes in the middle $C_{mLL-n} = \langle L_m O_L O_L L_{(n=-m)}\rangle$. When we include L_1 or L_{-1} terms, they are included in the C_{LL} part and not in the T with T OPE.

When we account for the pole $z = \infty$, we get a similar result as we saw in the C_{mLL} correlator.

$$\begin{aligned} \oint_{z_1=\infty} \frac{dz_1}{2\pi i} z_1^{m+1} \frac{c/2}{(z_1-z_2)^4} &= \oint_{x=0} \frac{dx}{2\pi i} (-x^{-2}) x^{-m-1} \frac{c/2}{(1/x - z_2)^4} \\ &= -\frac{c}{12}(m-1)m(m+1) z_2^{m+n-1} \qquad \text{For } m>1 \end{aligned} \tag{239}$$

$$\left(\oint_{z_1=z_2} \frac{dz_1}{2\pi i} + \oint_{z_1=0} \frac{dz_1}{2\pi i} + \oint_{z_1=\infty} \frac{dz_1}{2\pi i} \right) z_1^{m+1} \frac{c/2}{(z_1-z_2)^4} = 0 \tag{240}$$

Second term in eq.235 , we again start with contour integral around $z_1 = 0, z_2$.(TTtwo)

$$\oint \frac{dz_1}{2\pi i} z_1^{m+1} \oint \frac{dz_2}{2\pi i} z_2^{n+1} \frac{h_2}{(z_1 - z_2)^2} C_{TLL}(z_2) \tag{241}$$

$$\begin{aligned}
\oint_{z_1=0} \frac{dz_1}{2\pi i} z_1^{m+1} z_2^{n+1} \frac{h_2}{(z_1 - z_2)^2} &= -h_2(m+1) z_2^{m+n+1} \qquad \text{For } m \leq -1 \\
\oint_{z_1=z_2} \frac{dz_1}{2\pi i} z_1^{m+1} z_2^{n+1} \frac{h_2}{(z_1 - z_2)^2} &= (m+1) h_2 z_2^{m+n+1} \\
\oint_{z_1=\infty} \frac{dz_1}{2\pi i} z_1^{m+1} z_2^{n+1} \frac{h_2}{(z_1 - z_2)^2} &= -(m+1) h_2 z_2^{m+n+1} \qquad \text{For } m \geq -1
\end{aligned} \tag{242}$$

Again, we see that contributions from $z_1 = 0, \infty$ cancel the terms from $z_1 = z_w$ Because $C_{TLL}(z_2)$ is dependent on z_2 we have to include the OPE of $T(z_2)$ with O_L.

$$\oint_{z_2} \frac{dz_2}{2\pi i} (m+1) h_2 z_2^{m+n+1} C_{TLL} = \oint_{z_2} \frac{dz_2}{2\pi i} (m+1) h_2 z_2^{m+n+1} \sum_i \left(\frac{h_L}{(z_2 - w_i)^2} + \frac{1}{z_2 - w_i} \partial_{w_i} \right) C_{LL} \tag{243}$$

First part of the OPE

$$\begin{aligned}
\oint_{z_2=w_i} \frac{dz_2}{2\pi i} (m+1) h_2 z_2^{m+n+1} \frac{h_L}{(z_2 - w_i)^2} &= h_2 h_L (m+1)(m+n+1) w_i^{m+n} \\
\oint_{z_2=0} \frac{dz_2}{2\pi i} (m+1) h_2 z_2^{m+n+1} \frac{h_L}{(z_2 - w_i)^2} &= h_2 h_L (m+1)(m+n+1) w_i^{m+n} \\
&\text{For } m+n < 0 \\
\left(\oint_{z_2=w_i} + \oint_{z_2=0} \right) \frac{dz_2}{2\pi i} (m+1) h_2 z_2^{m+n+1} \frac{h_L}{(z_2 - w_i)^2} &= h_2 h_L (m+1)(m+n+1) w_i^{m+n} \\
&\text{For } m+n > -2
\end{aligned} \tag{244}$$

Second part of the OPE

$$\begin{aligned}
\oint_{z_2=w_i} \frac{dz_2}{2\pi i} (m+1) h_2 z_2^{m+n+1} \frac{1}{(z_2 - w_i)} &= h_2(m+1) w_i^{m+n+1} \\
\oint_{z_2=0} \frac{dz_2}{2\pi i} (m+1) h_2 z_2^{m+n+1} \frac{1}{(z_2 - w_i)} &= -h_2(m+1) w_i^{m+n+1} \\
&\text{For } m+n < -1 \\
\left(\oint_{z_2=w_i} + \oint_{z_2=0} \right) \frac{dz_2}{2\pi i} (m+1) h_2 z_2^{m+n+1} \frac{1}{(z_2 - w_i)} &= h_2(m+1) w_i^{m+n+1} \\
&\text{For } m+n > -2
\end{aligned} \tag{245}$$

Combined for $m+n>-2$ (OTTtwo)

$$\begin{aligned}
&\oint_{z_2}\frac{dz_2}{2\pi i}(m+1)h_2 z_2^{m+n+1}C_{TLL}\\
=&\sum_i\left(h_2h_L(m+1)(m+n+1)w_i^{m+n}+h_2(m+1)w_i^{m+n+1}\partial_{w_i}\right)C_{LL}\\
=&\,h_2(m+1)\sum_i\left[w_i^{m+n}\left((m+n+1)h_L+w_i\partial_{w_i}\right)\right]C_{LL}
\end{aligned} \tag{246}$$

Third term in eq.235, we again start with contour integral around $z_1=0,z_2$. (TTthree)

$$\oint\frac{dz_1}{2\pi i}z_1^{m+1}\oint\frac{dz_2}{2\pi i}z_2^{n+1}\left(\frac{1}{z_1-z_2}\frac{\partial C_{TLL}(z_2)}{\partial_{z_2}}\right) \tag{247}$$

$$\begin{aligned}
\oint_{z_1=0}\frac{dz_1}{2\pi i}z_1^{m+1}z_2^{n+1}\frac{1}{z_1-z_2}&=-z_2^{2+m+n} &&\text{For } m<-1\\
\oint_{z_1=z_2}\frac{dz_1}{2\pi i}z_1^{m+1}z_2^{n+1}\frac{1}{z_1-z_2}&=z_2^{2+m+n}\\
\left(\oint_{z_1=0}+\oint_{z_1=z_2}\right)\frac{dz_1}{2\pi i}z_1^{m+1}z_2^{n+1}\frac{1}{z_1-z_2}&=z_2^{2+m+n} &&\text{For } m>-2
\end{aligned} \tag{248}$$

Next the contour integral around $z_2=0,w_i$, like before C_{TLL} is dependent on z_2, consequently we have to take the different OPE parts into consideration.

$$\begin{aligned}
\oint_{z_2}\frac{dz_2}{2\pi i}z_2^{2+m+n}\partial_{z_2}C_{TLL}&=\oint_{z_2}\frac{dz_1}{2\pi i}z_2^{2+m+n}\sum_i\partial_{z_2}\left(\frac{h_L}{(z_2-w_i)^2}+\frac{1}{z_2-w_i}\partial_{w_i}\right)C_{LL}\\
&=\oint_{z_2}\frac{dz_2}{2\pi i}z_2^{2+m+n}\sum_i\left(-2\frac{h_L}{(z_2-w_i)^3}-\frac{1}{(z_2-w_i)^2}\partial_{w_i}\right)C_{LL}
\end{aligned} \tag{249}$$

First part of the OPE (TDTone)

$$\begin{aligned}
\oint_{z_2=w_i}\frac{dz_2}{2\pi i}z_2^{2+m+n}\left(-2\frac{h_L}{(z_2-w_i)^3}\right)&=-h_L(m+n+1)(m+n+2)w_i^{m+n}\\
\oint_{z_2=0}\frac{dz_2}{2\pi i}z_2^{2+m+n}\left(-2\frac{h_L}{(z_2-w_i)^3}\right)&=h_L(m+n+1)(m+n+2)w_i^{m+n}\\
&\quad\text{For } m+n<0\\
\left(\oint_{z_2=0}+\oint_{z_2=w_i}\right)\frac{dz_2}{2\pi i}z_2^{2+m+n}\left(-2\frac{h_L}{(z_2-w_i)^3}\right)&=-h_L(m+n+1)(m+n+2)w_i^{m+n}\\
&\quad\text{For } m+n>-3
\end{aligned} \tag{250}$$

Second part of the OPE

$$\begin{aligned}
\oint_{z_2=w_i} \frac{dz_1}{2\pi i} z_2^{2+m+n}\left(-\frac{1}{(z_2-w_i)^3}\right) &= -(m+n+2)w_i^{m+n+1} \\
\oint_{z_2=0} \frac{dz_1}{2\pi i} z_2^{2+m+n}\left(-\frac{1}{(z_2-w_i)^3}\right) &= (m+n+2)w_i^{m+n+1} \qquad \text{For } m+n<-1 \\
\left(\oint_{z_2=w_i}+\oint_{z_2=0}\right) \frac{dz_1}{2\pi i} z_2^{2+m+n}\left(-\frac{1}{(z_2-w_i)^3}\right) &= -(m+n+2)w_i^{m+n+1} \qquad \text{For } m+n>-3
\end{aligned} \tag{251}$$

Combined

$$\begin{aligned}
\oint_{z_2} \frac{dz_2}{2\pi i} z_2^{2+m+n}\partial_{z_2} C_{TLL} &= \sum_i \left[-h_L(m+n+1)(m+n+2)w_i^{m+n} - (m+n+2)w_i^{m+n+1}\right] C_{LL} \\
&= -(m+n+2)\sum_i w_i^{m+n}\left(h_L(m+n+1)+w_i\partial_{w_i}\right) C_{LL}(w_i)
\end{aligned} \tag{252}$$

Hence reviewing previous results eq.238; taking only the $z_1 = z_2$ (The contributions from $z_1 = 0, \infty$ are the same but negative; they cancel the $z_1 = z_2$ terms) pole and $z_2 = 0$

$$\oint_{z_2=0} \frac{dz_2}{2\pi i}\frac{c}{12}(m-1)m(m+1)z_2^{m+n-1}C_{LL} = \frac{c}{12}(m-1)m(m+1)C_{LL} \qquad \text{For } m+n=0 \tag{253}$$

Combining the results from eq.246 and eq.252. Again we only take the results from $z_1 = z_2$[1].

$$\begin{aligned}
C_{mnLL}^{T\leftrightarrow T} =& h_2(m+1)\sum_i \left[w_i^{m+n}\left((m+n+1)h_L + w_i\partial_{w_i}\right)\right] C_{LL} \\
& -(m+n+2)\sum_i \left[w_i^{m+n}\left(h_L(m+n+1)+w_i\partial_{w_i}\right)\right] C_{LL} \\
=& \Bigg[\sum_i \Big(h_2(m+1)w_i^{m+n}\left((m+n+1)h_L + w_i\partial_{w_i}\right) \\
& -(m+n+2)w_i^{m+n}\left(h_L(m+n+1)+w_i\partial_{w_i}\right)\Big)\Bigg] C_{LL} \\
=& \Bigg[\sum_i w_i^{m+n}\Big[\left(h_2(m+1)-(m+n+2)\right)(m+n+1)h_L \\
& + \left(h_2(m+1)-(m+n+2)\right)w_i\partial_{w_i}\Big]\Bigg] C_{LL} \\
=& \Bigg[\sum_i w_i^{m+n}\Big[(m-n)(m+n+1)h_L + (m-n)w_i\partial_{w_i}\Big]\Bigg] C_{LL}
\end{aligned} \tag{254}$$

1 I have not yet caclulated all the contributiosn from ∞ , but I expect this trend to continue

A.3.3.3 C_{mnLL} OPE of T with O

Next we take previous results to calculate the OPE of $T(z_1)$ with O_L.

$$C^{T\leftrightarrow O}_{mnLL} = \oint \frac{dz_2}{2\pi i} z_2^{n+1} \sum_i \left(\oint \frac{dz}{2\pi i} z^{m+1} \frac{h_L}{(z-w_i)^2} + \oint \frac{dz}{2\pi i} z^{m+1} \frac{1}{z-w_i} \partial_{w_i} \right) C_{TLL}$$
$$= \oint \frac{dz_2}{2\pi i} z_2^{n+1} \sum_i \left(h_L(m+1) w_i^m + w_i^{m+1} \partial_{w_i} \right) C_{TLL} \tag{255}$$

Next taking the OPE $T(z_2)$ with O_L, this is also similar to the previous result but in the second term we have to take the differential to w_i into account.

First term

$$\oint \frac{dz_2}{2\pi i} z_2^{n+1} h_L(m+1) w_i^m C_{TLL} = h_L(m+1) w_i^m \oint \frac{dz_2}{2\pi i} z_2^{n+1} \sum_j \left(\frac{h_L}{(z_2-w_j)^2} + \frac{1}{z_2-w_j} \partial_{w_j} \right) C_{LL}$$
$$= h_L(m+1) w_i^m \sum_j \left(h_L(n+1) w_j^n + w_j^{n+1} \partial_{w_j} \right) C_{LL}$$
$$= h_L(m+1) w_i^m \sum_j w_j^n \left(h_L(n+1) + w_j \partial_{w_j} \right) C_{LL} \tag{256}$$

Second term

$$\oint \frac{dz_2}{2\pi i} z_2^{n+1} w_i^{m+1} \partial_{w_i} C_{TLL} = \oint \frac{dz_2}{2\pi i} z_2^{n+1} \sum_j w_i^{m+1} \partial_{w_i} \left(\frac{h_L}{(z_2-w_j)^2} + \frac{1}{z_2-w_j} \partial_{w_j} \right) C_{LL} \tag{257}$$

Using "Leibniz rule" we can sum the two differentials.

$$w_i^{m+1} \oint \frac{dz_2}{2\pi i} z_2^{n+1} \partial_{w_i} C_{TLL} = w_i^{m+1} \oint \frac{dz_2}{2\pi i} z_2^{n+1} \sum_j \left(\partial_{w_i} \frac{h_L}{(z_2-w_j)^2} + \partial_{w_i} \frac{1}{z_2-w_j} \partial_{w_j} \right) C_{LL}$$
$$+ w_i^{m+1} \oint \frac{dz_2}{2\pi i} z_2^{n+1} \sum_j \left(\frac{h_L}{(z_2-w_j)^2} + \frac{1}{z_2-w_j} \partial_{w_j} \right) \partial_{w_i} C_{LL} \tag{258}$$

First term gives non-zero results only when $i = j$.

$$\oint \frac{dz_2}{2\pi i} z_2^{n+1} \left(\partial_{w_i} \frac{h_L}{(z_2-w_i)^2} + \partial_{w_i} \frac{1}{z_2-w_i} \partial_{w_i} \right) = \oint \frac{dz_2}{2\pi i} z_2^{n+1} \left(\frac{2h_L}{(z_2-w_i)^3} + \frac{1}{(z_2-w_i)^2} \partial_{w_i} \right) \tag{259}$$

Taking the contour integral

$$\oint \frac{dz_2}{2\pi i} z_2^{n+1} \left(\frac{2h_L}{(z_2-w_i)^3} + \frac{1}{(z_2-w_i)^2} \partial_{w_i} \right) = w_i^n \left(\frac{h_L(n+1)}{w_i} + \partial_{w_i} \right) \tag{260}$$

Second term is the usual OPE

$$\oint \frac{dz_2}{2\pi i} z_2^{n+1} \left(\frac{h_L}{(z_2 - w_j)^2} + \frac{1}{z_2 - w_j} \partial_{w_j} \right) = h_L(n+1) w_j^n + w_j^{n+1} \partial_{w_j} \tag{261}$$

Combining results from eq.260 and eq.261.

$$w_i^{m+1} \oint \frac{dz_2}{2\pi i} z_2^{n+1} \partial_{w_i} C_{TLL} = w_i^{m+1} \left[w_i^n \left(\frac{h_L(n+1)}{w_i} + \partial_{w_i} \right) + \sum_j \left(h_L(n+1) w_j^n + w_j^{n+1} \partial_{w_j} \right) \partial_{w_i} \right] C_{LL} \tag{262}$$

Substituting eq.256 and eq.262 in eq.255.

$$\begin{aligned} C_{mnLL}^{T \leftrightarrow O} &= \oint \frac{dz_2}{2\pi i} z_2^{n+1} \sum_i \left(h_L(m+1) w_i^m + w_i^{m+1} \partial_{w_i} \right) C_{TLL} \\ &= \sum_i \Bigg(h_L(m+1) w_i^m \sum_j w_j^n \left(h_L(n+1) + w_j \partial_{w_j} \right) \\ &+ w_i^{m+1} \left[w_i^n \left(\frac{h_L(n+1)}{w_i} + \partial_{w_i} \right) + \sum_j \left(h_L(n+1) w_j^n + w_j^{n+1} \partial_{w_j} \right) \partial_{w_i} \right] \Bigg) C_{LL} \\ &= \sum_i w_i^m \Bigg(w_i^{n+1} \left(\frac{h_L(n+1)}{w_i} + \partial_{w_i} \right) + \sum_j \\ & w_j^n \left[h_L(m+1) \left(h_L(n+1) + w_j \partial_{w_j} \right) + w_i \left(h_L(n+1) + w_j \partial_{w_j} \right) \partial_{w_i} \right] \Bigg) C_{LL} \end{aligned} \tag{263}$$

A.3.4 *Summary of OPE terms*

Taking OPE of $T(z)$ with $O_h(w_i)$ in C_{mLL} eq.233 (only contributions from $z = w_i$ poles)

$$\begin{aligned} C_{mLL} &= \langle L_m O_L(w_1) O_L(w_2) \rangle \\ &= \sum_i \left(h_L(m+1) w_i^m + w_i^{m+1} \partial_{w_i} \right) C_{LL} \end{aligned} \tag{264}$$

Taking the OPE of $T(z_1)$ with $T(z_2)$ followed by $T(z_2)$ with $O(w_i)$. Here the c term only survives when $m = -n$, hence only in the $\langle L_m O O L_{-m} \rangle$. These appear in expansions like $\langle O O L_m \rangle \langle L_m O O L_{-m} \rangle \langle L_m O O \rangle$.

Taking only the $z_1 = z_2$ (The contributions from $z_1 = 0, \infty$ are the same but negative; they cancel the $z_1 = z_2$ terms) pole and $z_2 = 0$

$$\oint_{z_2=0} \frac{dz_2}{2\pi i} \frac{c}{12}(m-1)m(m+1)z_2^{m+n-1}C_{LL} = \frac{c}{12}(m-1)m(m+1)C_{LL} \qquad \text{For } m+n=0 \tag{265}$$

Here the c term only survives when $n = -m$, hence only in the $\langle L_m O O L_{-m} \rangle$. These appear in expansions like $\langle O O L_m \rangle \langle L_m O O L_{-m} \rangle \langle L_m O O \rangle$ and any combination with L_1 as these are included in the $C_L L$ and therefore don't influence the order or disappearance of c. When we take the limit to large c.

$$\langle O_1^i O_m O_1^j O_L O_L O_{-1}^k O_{-m} O_{-1}^l \rangle \propto \frac{c}{12}(m-1)m(m+1)\langle O_1^{i+j} O_L O_L O_{-1}^{k+l} \rangle \tag{266}$$

And result from eq.254, also taking OPE of $T(z_1)$ with $T(z_2)$ followed by $T(z_2)$ with $O(w_i)$. With only terms from $z_1 = z_2$ and $z_2 = w_i$.

$$C_{mnLL}^{T\leftrightarrow T} = \left[\sum_i w_i^{m+n} \left[(m-n)(m+n+1)h_L + (m-n) w_i \partial_{w_i} \right] \right] C_{LL} \tag{267}$$

Taking the OPE of $T(z_1)$ with $O(w_i)$ followed by $T(z_2)$ with $O(w_i)$ eq.263.

$$\begin{aligned} C_{mnLL}^{T\leftrightarrow O} = \sum_i w_i^m \Bigg(& w_i^{n+1} \left(\frac{h_L(n+1)}{w_i} + \partial_{w_i} \right) + \sum_j \\ & w_j^n \left[h_L(m+1) \left(h_L(n+1) + w_j \partial_{w_j} \right) + w_i \left(h_L(n+1) + w_j \partial_{w_j} \right) \partial_{w_i} \right] \Bigg) C_{LL} \end{aligned} \tag{268}$$

A.3.5 *Two-point function with L_m insertions*

In this section we calculate the two-point functions with L_m insertions in contrast to the previous section the two-point function is written completely and derivatives are taken.

A.3.5.1 $L_m O_L O_L$

The OPE is taken with all $T(z_i)$ and $O(w_i)$.

$$\begin{aligned}
C_{mLL} &= \langle L_m O_L(w_1) O_L(w_2) \rangle \\
&= \oint \frac{dz}{2\pi i} z^{m+1} \langle T(z) O_L(w_1) O_L(w_2) \rangle \\
&= \oint \frac{dz}{2\pi i} z^{m+1} \sum_{i=1,2} \left(\frac{h_L}{(z-w_i)^2} + \frac{1}{z-w_i} \partial_{w_i} \right) (w_1 - w_2)^{-2h_L} \\
&= \oint \frac{dz}{2\pi i} z^{m+1} \sum_{i=1,2} \left(\frac{h_L}{(z-w_i)^2} + (-1)^i \frac{2h_L (w_1 - w_2)^{-1}}{(z-w_i)} \right) (w_1 - w_2)^{-2h_L} \\
&= \oint \frac{dz}{2\pi i} z^{m+1} h_L \frac{(w_1 - w_2)^{2-2h_L}}{(w_1 - z)^2 (w_2 - z)^2} \\
&= \Big((m+1)(w_2^m w_1 - w_1^m w_2) + (m-1)(w_1^{1+m} - w_2^{1+m}) \Big) \\
&\times h_L (w_1 - w_2)^{-1-2h_L}
\end{aligned} \tag{269}$$

With $w_{i+1}(i=2) = w_1$. Notice that $C_{mLL(w_2=0)} = h_L(m-1)w_1^{m-2h_L}$. Solving this full correlation function in mathematica results in the same answer. For $m = -1, 0, 1$ this correlation goes to 0. This is problematic at first sight, but recall that $C_{1^k} = 0$ to, so we might be able to solve this using L'hospital. Maybe, first take this inserting operator $\langle h|$. Also, the singularity at $z = 0$ has not been taken into account here. For $m > -2$ these contributions are 0, but for $m \leq -2$ these contributions are not equal to 0.

$$\begin{aligned}
&\text{Res}\left[z^{m+1} h_L \frac{(w_1 - w_2)^{2-2h_L}}{(w_1 - z)^2 (w_2 - z)^2}, z = 0 \right] \\
&= h_L \, (w_1 - w_2)^{\, 2-2h_L} \sum_{i=m}^{0} i(m-i)(w_1^{-1+i} w_2^{m-i-1})
\end{aligned} \tag{270}$$

Combining these results we obtain the solutions

$$\begin{aligned}
m > -2 &\rightarrow h_L (w_1 - w_2)^{-1-2h_L} \Big((m+1)(w_2^m w_1 - w_1^m w_2) + (m-1)(w_1^{1+m} - w_2^{1+m}) \Big) \\
m \leq -2 &\rightarrow h_L (w_1 - w_2)^{-1-2h_L} \Big[((m+1)(w_2^m w_1 - w_1^m w_2) + (m-1)(w_1^{1+m} - w_2^{1+m})) \\
&\quad + (w_1 - w_2)^{\, 3} \sum_{i=m}^{0} i(m-i)(w_1^{-1+i} w_2^{m-i-1}) \Big]
\end{aligned} \tag{271}$$

A.3.5.2 $L_m L_n O_L O_L$

Next, inserting two operators L_m and L_n and doing the OPE with all light operators O_L. As we have a normalization $\mathcal{N}$ of order c. More specificly, these terms go as $\frac{1}{c^2}$.

$$
\begin{aligned}
C_{mnLL} &= \langle L_m L_n O_L(w_1) O_L(w_2) \rangle \\
&= \oint \oint \frac{dz_1}{2\pi i} z_1^{m+1} \frac{dz_2}{2\pi i} z_2^{n+1} \langle T_(z_1) T(z_2) O_L(w_1) O_L(w_2) \rangle \\
&= \oint \oint \frac{dz_1}{2\pi i} z_1^{m+1} \frac{dz_2}{2\pi i} z_2^{n+1} \langle T_(z_1) T(z_2) O_L(w_1) O_L(w_2) \rangle \\
&= \oint \oint \frac{dz_1}{2\pi i} z_1^{m+1} \frac{dz_2}{2\pi i} z_2^{n+1} \left[\frac{c/2}{(z_1 - z_2)^4} \langle OO \rangle \right. \\
&+ \left. \sum_{x=z_2,w_1,w_2} \left(\frac{h_x}{(z_1-x)^2} + \frac{1}{z_1 - x} \partial_x \right) \langle TOO \rangle \right] \\
&= \oint \oint \frac{dz_1}{2\pi i} z_1^{m+1} \frac{dz_2}{2\pi i} z_2^{n+1} \left[\frac{c/2}{(z_1 - z_2)^4} \right. \\
&+ \left. \sum_{x=z_2,w_1,w_2} \left(\frac{h_x}{(z_1-x)^2} + \frac{1}{z_1 - x} \partial_x \right) \sum_{y=w_1,w_2} \left(\frac{h_x}{(z_2-y)^2} + \frac{1}{z_2 - y} \partial_y \right) \right] \langle OO \rangle \\
&= \oint \oint \frac{dz_1}{2\pi i} z_1^{m+1} \frac{dz_2}{2\pi i} z_2^{n+1} \left[\frac{c/2}{(z_1 - z_2)^4} + \sum_{x=z_2,w_1,w_2} \left(\frac{h_x}{(z_1-x)^2} + \frac{1}{z_1 - x} \partial_x \right) \times \right. \\
& \left. \sum_{y=w_1,w_2} \left(\frac{h_x}{(z_2-y)^2} + \frac{(-1)^{\delta(w_1-y)} 2h_L}{(z_2-y)(w_1-w_2)} \right) \right] \langle OO \rangle
\end{aligned} \tag{272}
$$

Solved using mathematica, script can be found in section **??** for $m, n > -1$. When $m, n \leq -1$ we also have singularities at $z = 0$, however the c dependence does also drop out in this limit.

$$
\begin{aligned}
C_{mnLL} = &\left[(n-1)(h_L(m-1) - n)(w_1^{2+m+n} + w_2^{2+m+n}) \right. \\
&- 2\left(-1 + n^2 + h_L(-1+mn)\right)\left(w_1^{1+m+n} w_2 + w_2^{1+m+n} w_1\right) \\
&+ (1+n)(h_L + h_L m + n)(w_1^{m+n} w_2^2 + w_2^{m+n} w_1^2) \\
&+ h_L(1+m)(-1+n)(w_1^{2+n} w_2^m + w_2^{2+n} w_1^m) \\
&+ h_L(-1+m)(1+n)(w_1^{2+m} w_2^n + w_2^{2+m} w_1^n) \\
&\left. - 2(1 + h_L + h_L mn)(w_1^{1+n} w_2^{1+m} + w_1^{1+m} w_2^{1+n}) \right] \\
&\times h_L (w_1 - w_2)^{-2-2h_L}
\end{aligned} \tag{273}
$$

Doing this contour integral all c dependence will drop out of the equation. Therefore, in the non-transformed correlation function, no $\frac{1}{c}$ corrections will arise from $L_m L_n$ with $m, n \geq 2$ or $m, n \leq 2$ terms. As an extra note, when taking $w_1 \to 0$ we obtain $C_{mnLL} = h_L(h_L(m-1)(n-1) + n(n-1))$, this is the same result as [45].

In contrast, this might not happen in the transformed correlation function. In the heavy sector of this correlation function, non-integer powers of w exist, which make the contour integral harder. Terms that should be explicitly looked at that might go as c in the transformed case are

$$\oint\oint \frac{dz_1}{2\pi i} z_1^{m+1} \frac{dz_2}{2\pi i} z_2^{n+1} \frac{c/2}{(z_1-z_2)^4} \times \left[\sum_{y=w_1,w_2} \left(\frac{h_x}{(z_2-y)^2} + \frac{(-1)^{\delta(w_1-y)} 2h_L}{(z_2-y)(w_1-w_2)} \right) \right] \langle OO \rangle \tag{274}$$

A.3.5.3 *Inserting operators L_{-1}, L_0, or L_1*

Now we get on murky terrain. When we choose $m = -1, 0, 1$ all all go to zero. So we use

$$\begin{aligned} C_{mLL} &= \langle L_m O_L(w_1) O_L(w_2) \rangle \\ &= \Big((m+1)(w_2^m w_1 - w_1^m w_2) + (m-1)(w_1^{1+m} - w_2^{1+m}) \Big) \\ &\times h_L (w_1 - w_2)^{-1-2h_L} \end{aligned} \tag{275}$$

Now simply substitute $m = -1, 0, 1$

$$\begin{aligned} C_{-1LL} &= \Big((-1+1)(w_2^{-1} w_1 - w_1^{-1} w_2) + (-1-1)(w_1^{1-1} - w_2^{1-1}) \Big) \\ &\times h_L (w_1 - w_2)^{-1-2h_L} \\ &= \Big(0(w_2^{-1} w_1 - w_1^{-1} w_2) - 2(0) \Big) h_L (w_1 - w_2)^{-1-2h_L} \\ C_{0LL} &= \Big((0+1)(w_2^0 w_1 - w_1^0 w_2) + (0-1)(w_1^{1+0} - w_2^{1+0}) \Big) \\ &\times h_L (w_1 - w_2)^{-1-2h_L} \\ &= (w_1 - w_2) - (w_1 - w_2) \times h_L (w_1 - w_2)^{-1-2h_L} \\ C_{1LL} &= \Big((1+1)(w_2^1 w_1 - w_1^1 w_2) + (1-1)(w_1^{1+1} - w_2^{1+1}) \Big) \\ &\times h_L (w_1 - w_2)^{-1-2h_L} \\ &= \Big(2(0) + (0)(w_1^2 - w_2^2) \Big) \\ &\times h_L (w_1 - w_2)^{-1-2h_L} \end{aligned} \tag{276}$$

And all three results go to zero. All zeros of order one.

$$C_{-1LL} = C_{0LL} = C_{1LL} = 0 \tag{277}$$

Using the result for C_{mnLL} and setting $m, n = 1$

$$\begin{aligned}
C_{mnLL} = & \big[(1-1)(h_L(1-1)-1)(w_1^{2+1+1} + w_2^{2+1+1}) \\
& - 2\left(-1+1^2+h_L(-1+1*1)\right)(w_1^{1+1+1}w_2 + w_2^{1+1+1}w_1) \\
& + (1+1)(h_L + h_L 1 + 1)(w_1^{1+1}w_2^2 + w_2^{1+1}w_1^2) \\
& + h_L(1+1)(-1+1)(w_1^{2+1}w_2^1 + w_2^{2+1}w_1^1) \\
& + h_L(-1+1)(1+1)(w_1^{2+1}w_2^1 + w_2^{2+1}w_1^1) \\
& - 2(1+h_L+h_L 1*1)(w_1^{1+1}w_2^{1+1} + w_1^{1+1}w_2^{1+1})\big] \\
& \times h_L(w_1 - w_2)^{-2-2h_L} \\
& = \big[(0)(h_L 0 - 1)(w_1^4) + w_2^4) \\
& - 2(0 + h_L 0)(w_1^3 w_2 + w_2^3 w_1) \\
& + 2(2h_L+1)(w_1^2 w_2^2)2 \\
& + h_L 2(0) w_1^3 w_2 + w_2^3 w_1)2 \\
& - 2(1+2h_L)(w_1^2 w_2^2)2\big] \\
& \times h_L(w_1 - w_2)^{-2-2h_L} \\
& = \big[0 - 0 + 2(2h_L+1)(w_1^2 w_2^2)2 + 0 - 2(1+2h_L)(w_1^2 w_2^2)2\big] h_L (w_1 - w_2)^{-2-2h_L} \\
& = 0
\end{aligned} \tag{278}$$

We also obtain a first order zero. Hence, these can compensate for the zero's of larger order in the denominator.

A.3.6 *Three-point function with the OPEtaken with $|h_w\rangle$*

In this section an operator $O_h(w)$ is inserted with a L_n acting on it and OPE with T with $O(w)$ is taken. This is the same approach as taken in section 3.4. As the heavy operator O_H is inserted at inifnity, contributions from these terms go to zero, consequently only the OPE with $O_(w)$ is taken.

First we will look at the heavy sector $\langle O_H(\infty) O_H(1) L_n O_h(w)\rangle$ Numerator for $n \leq -2$. As before, taking the OPE of T with all the operators will go to zero. So instead we look again at the OPE of T with $O(w)$. This can also be supported by understanding that we use L_n as a raising operator on $O(w)$ to get all the possible interactions.

$$\begin{aligned}
C_{HHn} &= \langle OOL_n O(w)\rangle \\
&= \oint \frac{dz}{2\pi i} z^{n+1} \langle OOT(z)O(w)\rangle \\
&= \oint \frac{dz}{2\pi i} z^{n+1} \left(\frac{h}{(z-w)^2}\langle OOO\rangle + \frac{1}{z-w}\partial_w \langle OOO(w)\rangle\right) \\
&= \oint \frac{dz}{2\pi i} z^{n+1} \left(\frac{h}{(z-w)^2} + \frac{1}{z-w}\partial_w\right)\langle OOO(w)\rangle
\end{aligned} \tag{279}$$

Some different approaches can be taken from here.

1. Taking the limit of $w \to 0$ first, derivative vanishes, then taking the contour integral around $z = 0$
2. Taking the derivative to w, followed by $w \to 0$, successfully taking the contour integral around $z = 0$
3. First taking the contour integral around $z = 0$ and $z = w$, taking the derivative, and afterwards taking $w \to 0$

We will try the different approaches and compare the answers, we will see that all correlation functions will go to zero using this method when $n \neq 0$.

Approach 1: Taking limit $w \to 0$, followed by the contour integral around $z = 0$ for $n \neq 1$

$$\begin{aligned} C_{HHn} &= \oint \frac{dz}{2\pi i} z^{n+1} \left(\frac{h}{(z-w)^2} + \frac{1}{z-w}\partial_w \right) \langle OOO(w) \rangle \\ &= \oint \frac{dz}{2\pi i} z^{n+1} \frac{h}{z^2} \langle OOO(0) \rangle \\ &= 0 \end{aligned} \tag{280}$$

For $n = -1$ we obtain $C_{HH-1} = 1 \times \langle OOO(0) \rangle$.

Approach 2: Starting with the derivative to w, followed by taking $w \to 0$, $x2 = 1$, and $x1 \to \infty$.

$$\begin{aligned} C_{HHn} &= \oint \frac{dz}{2\pi i} z^{n+1} \left(\frac{h}{(z-w)^2} + \frac{1}{z-w}\partial_w \right) \langle O(x1)O(x2)O(w) \rangle \\ &= \oint \frac{dz}{2\pi i} z^{n+1} \left(\frac{h}{(z-w)^2} + \frac{-h}{z-w}((-1)^{-h}(x2-w)^{-1}) + (w-x1)^{-1} \right) \\ &\times \langle OOO(w) \rangle \end{aligned} \tag{281}$$

Next, we take $w \to 0$, $x2 = 1$, and $x1 \to \infty$.

$$\begin{aligned} &= \oint \frac{dz}{2\pi i} z^{n+1} \left(\frac{h}{z^2} + \frac{-h}{z}((-1)^{-h}(x2)^{-1}) + (-x1)^{-1} \right) \langle O(x1)O(x2)O(0) \rangle \\ &= \oint \frac{dz}{2\pi i} z^{n+1} \left(\frac{h}{z^2} + \frac{-h}{z}(-1)^{-h} \right) \langle O(\infty)O(1)O(0) \rangle \\ &= \oint \frac{dz}{2\pi i} z^{n-1} h \left(1 - z(-1)^{-h} \right) \langle O(\infty)O(1)O(0) \rangle \\ &= \oint \frac{dz}{2\pi i} h \left(z^{n-1} - z^{n}(-1)^{-h} \right) \langle O(\infty)O(1)O(0) \rangle \\ &= 0 \end{aligned} \tag{282}$$

Approach 3: Here the contour integral is first taken around $z = 0$ and $z = w$ and afterwards we take the limit $w \to 0$.

$$\begin{aligned} C_{HHn} &= \oint \frac{dz}{2\pi i} z^{n+1} \left(\frac{h}{(z-w)^2} + \frac{1}{z-w}\partial_w \right) \langle O(x1)O(x2)O(w) \rangle \\ &= \left(\frac{(-n-1)h}{w^{-n}} - \frac{1}{w^{-n-1}}\partial_w - \frac{(-n-1)h}{w^{-n}} + \frac{1}{w^{-n-1}}\partial_w \right) \langle O(x1)O(x2)O(w) \rangle \\ &= 0 \end{aligned} \tag{283}$$

Approach 4: Not a new approach but trying the same thing again, forgetting I tried this berfore. If we look at TOOO3, this helps remove the zero of second order for $n = -1$, but $n = 1$ still gives us a second order zero.

A.3.7 *Three-point function and the OPE taken with all insertions*

Here the results for the correlation function $\langle L_m O_{hl}(w_1) O_{hl(}w_2) O_h(w_3) \rangle$ are given. The OPE is taken with all $O(w_i)$ insertions.

1. taking the contour only around $w_{i=1,2,3}$. This gives nice zeros for $-2 < m < 2$. If we subsequently take $h \to 0$ we get clean solutions with no w_3 dependence. Results can be found in eq.284
2. taking the contour also around w gives zeros for $m < 1$, so we can probably throw this one overboard.

$$\begin{aligned} m = -5 &\to \frac{2\mathrm{hl}\,(w_1 - w_2)^{2-2\mathrm{hl}} \left(2w_1^3 + 3w_1^2 w_2 + 3w_1 w_2^2 + 2w_2^3\right)}{w_1^5 w_2^5} \\ m = -4 &\to \frac{\mathrm{hl}\,(w_1 - w_2)^{2-2\mathrm{hl}} \left(3w_1^2 + 4w_1 w_2 + 3w_2^2\right)}{w_1^4 w_2^4} \\ m = -3 &\to \frac{2\mathrm{hl}\,(w_1 - w_2)^{2-2\mathrm{hl}} (w_1 + w_2)}{w_1^3 w_2^3} \\ m = -2 &\to \frac{\mathrm{hl}\,(w_1 - w_2)^{2-2\mathrm{hl}}}{w_1^2 w_2^2} \\ m = -1 &\to 0 \\ m = 0 &\to 0 \\ m = 1 &\to 0 \\ m = 2 &\to \mathrm{hl}\,(w_1 - w_2)^{2-2\mathrm{hl}} \\ m = 3 &\to \mathrm{hl}\,(w_1 - w_2)^{2-2\mathrm{hl}} (2w_1 + 2w_2) \\ m = 4 &\to \mathrm{hl}\,(w_1 - w_2)^{2-2\mathrm{hl}} \left(3w_1^2 + 4w_1 w_2 + 3w_2^2\right) \\ m = 5 &\to \mathrm{hl}\,(w_1 - w_2)^{2-2\mathrm{hl}} \left(4w_1^3 + 6w_1^2 w_2 + 6w_1 w_2^2 + 4w_2^3\right) \end{aligned} \tag{284}$$

Hence

$$
\begin{aligned}
m < -1 &\rightarrow C_{mLL} = h_L(w_1 - w_2)^2 (w_1 w_2)^m \sum_i^m \left(i(m-i) w_1^{m-1-i} w_2 i - 1 \right) (w_1 - w_2)^{-2h_L} \\
m > 1 \quad &\rightarrow C_{mLL} = h_L(w_1 - w_2)^2 \sum_i^m \left(i(m-i) w_1^{m-1-i} w_2 i - 1 \right) (w_1 - w_2)^{-2h_L}
\end{aligned}
\tag{285}
$$

Problem is, they already go to 0 for $m = -1, 0, 1$ without taking the limit of $h \rightarrow 0$.