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Proofs of Transcendence

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Abstract

A transcendental number is a number which is not a root of a non-zero polynomial equation with integer or rational coefficients. This is a particularly interesting field of study as there are many unsolved problems, therefore we aim to understand some of the intricate proofs intrinsical to transcendental number theory. We look at methods involving auxiliary functions and rational approximation, specifically by BBP-type formulas.

1 Introduction

The aim of this thesis is to give a short intro to transcendental theory. Then we will explain some basic principles for proving transcendence and provide examples. Specifically we will recall and discuss in detail classical transcendence proofs of e and π.
We look at the history and results of two main techniques and how they work. Firstly proofs involving auxiliary functions, which is to say explicitly constructed functions with properties useful for transcendence proofs. Initial proofs of transcendence of e and π are based on this method and it has resulted in more general theorems such as the Gelfond-Schneider theorem and Baker's theorem. Secondly methods that rely on approximation by rational numbers as it is known that algebraic numbers cannot be very well approximated by rational numbers. This path makes it interesting to look at approximation methods such as BBP-type formulas.

Transcendental number theory started when Joseph Liouville proved for the first time in the 1840's the existence of numbers that are not algebraic. Liouville constructed a criterion that proved transcendence of a class of numbers now known as Liouville numbers. This criterion is not a necessary condition for transcendence as it for example doesn't prove the transcendence of e. The criterion says that if α is an algebraic number of degree $d \geq 2$ and $\epsilon > 0$ then the inequality

$$|\alpha - \frac{p}{q}| < \frac{1}{q^{d+\epsilon}}$$

can be hold for only finitely many rational numbers $\frac{p}{q}$. This criterion says, means that algebraic numbers cannot be well approximated by rational numbers. The exponent in the criterion was later improved from $d+\epsilon$ to $\frac{d}{2}+1+\epsilon$ and most recently in 1955, to $2+\epsilon$ by Thue, Siegel and Roth. Thus this last one is called the Thue-Siegel-Roth theorem. For the exponent 2 the theorem is no longer true so it seems we cannot improve it any further in this way.
These results can be used to construct transcendental numbers as well as prove the transcendence of many numbers.
Specifically we will look at its uses for numbers with well converging continued fraction expansions and BBP-type numbers.A BBP-type number is

of the form

$$\sum_{k=0}^{\infty} \frac{1}{b^k} \frac{p(k)}{q(k)}$$

where p and q are polynomials in integer coefficients and $b \geq 2$ is an integer. For example we have that

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right).$$

Different proofs for transcendence involve auxiliary functions. These are functions that typically have many zeros, possibly with high multiplicity. For example Charles Hermite proved in 1873 the transcendence of e using the function

$$f_k(x) = x^k (x-1)^{k+1} \dots (x-n)^{k+1}.$$

Generalizing this proof in 1882 Lindemann proved that e^α is transcendental for nonzero algebraic numbers α. This proved that π is transcendental as $e^{\pi i} = -1$ is algebraic. In 1885 Weierstrass expanded this result into what is known as the Lindemann-Weierstrass theorem.

The next result in this field was discovered independently by Gelfond and Schneider in the 1930's. The Gelfond-Schneider theorem states the transcendence of numbers of the form a^b where a and b are algebraic, a is not zero or one, and b is irrational.

Finally in 1966 Baker's theorem was proved which gives a lower bound for the absolute value of linear combinations of logarithms of algebraic numbers. This can be seen as a generalisation of the Gelfond-Schneider theorem.

2 The auxiliary function method

We will look at the general method of Lindemann and Weierstrass using auxiliary functions, as described in [4]. This was used firstly, as Hermite did, to prove the transcendence of e and after that the transcendence of π and the Lindemann-Weierstrass theorem. We will show the proofs for e and π.

The Lindemann-Weierstrass theorem generalizes these two results. Its proof follows the same structure and is explained well in Bakers book: [4, p. 6] The theorem is stated as follows

Theorem 1. *If $\alpha_1, \dots, \alpha_n$ are algebraic and distinct, and if $\beta_1, \dots, \beta_n$ are algebraic and nonzero, then*

$$\beta_1 e^{\alpha_1} + \dots + \beta_n e^{\beta_n} \neq 0$$

Or in other words the e^{α_i} are linearly independent over the algebraic numbers. This ofcourse implies the transcendence of e and π. If they were algebraic there would be such an expression which is 0, as they would be the root of a polynomial of that form.
The proofs follows a specific method every time and are based on a contradiction. We write an equation in exponents that shows that the number is algebraic. For e we simply assume e is algebraic. Now we define an appropiate polynomial f, the auxiliary function, with which we construct a number J which we prove to be integral and bigger than 0 and at the same time smaller than 1, the contradiction.
The trick in the proof lies in finding the right f and corresponding J and then there is some work in proving J is integral. Deriving an upper bound turns out to be easier by taking the following standard construction for J. So for all 3 proofs define the following for a real polynomial f

$$I(t) = \int_0^t e^{t-x} f(x) dx$$

Integrating by parts gives

$$I(t) = (-e^{t-x} f(x))|_0^t + \int_0^t e^{t-x} f'(x) dx = -f(t) + e^t f(0) + \int_0^t e^{t-x} f'(x) dx$$

Now to have the last term vanish we integrate by parts $n = \deg f$ times to get

$$I(t) = e^t \sum_{j=0}^{n} f^{(j)}(0) - \sum_{j=0}^{n} f^{(j)}(t)$$

Now define $F(x) = \sum |a_i| x^i$ to be the polynomial whose coefficients are the absolute values of those for $f(x) = \sum a_i x^i$. Then by simple bounds

$$|I(t)| \leq \int_0^t |e^{t-x} f(x)| dx \leq |t| e^{|t|} F(|t|)$$

Where the last inequality holds because e^{t-x} is a decreasing function so it will be highest at $x = 0$ and f is a polynomial with all positive coefficients, thus it will be biggest at $x = |t|$. Therefore the integrand is bounded by $e^{|t|}|F(|t|)|$ and thus follows the final bound.

2.1 Transcendence of e

Now using this construction we will prove that e is transcendent. The proof presented here follow the exposition in [8] and [4, p. 4] with added details for clarity.

Theorem 2. *e is transcendental*

Assume e is algebraic. Then e satisfies some integer polynomial $\sum a_i x^i = 0$ with $a_0 \neq 0$. So for some m

$$a_0 + a_1 e + a_2 e^2 + ... + a_m e^m = 0$$

Let p be a prime, define the auxiliary function as a polynomial of degree $n = (m+1)p - 1$ by

$$f(x) = \frac{x^{p-1}(x-1)^p \dots (x-m)^p}{(p-1)!}$$

Now we define J for which we will show that is an integer not equal to 0 and smaller in absolute value than 1

$$J = a_0 I(0) + a_1 I(1) + \cdots + a_m I(m) = \sum_{k=0}^{m} a_k I(k)$$

Then by definition of I

$$= \sum_{k=0}^{m} a_k (e^k \sum_{j=0}^{n} f^{(j)}(0) - \sum_{j=0}^{n} f^{(j)}(k))$$

$$= \sum_{k=0}^{m} a_k e^k \sum_{j=0}^{n} f^{(j)}(0) - \sum_{k=0}^{m} a_k \sum_{j=0}^{n} f^{(j)}(k)$$

Where the first term vanishes because it is the polynomial which vanishes by our assumption that e is algebraic

$$= -\sum_{k=0}^{m} \sum_{j=0}^{n} a_k f^{(j)}(k)$$

Now we look at the values of the derivatives $f^{(j)}(k)$. These derivatives consist by the product rule of a sum of products of the factors $(x-i)$ of f that are respectively derived a number of times, together adding up to j. Deriving one of the factors of f once by the chain rule simply decreases its power by 1 and multiplies by the exponent. If $j < p-1$ then none of the factors in f vanish, thus for every k there is a factor 0 in $f^{(j)}(k)$. Therefore these terms vanish.
If $j = p-1$, only the factor x^{p-1} vanishes in one term of $f^{(j)}(k)$ and thus $f^{(j)}(k) = 0$ for $k > 0$. The one term that does not vanish is where x^{p-1} is differentiated all $j = p-1$ times. This term gives

$$f^{(p-1)}(0) = \frac{(p-1)!(-1)^p(-2)^p \dots (-m)^p}{(p-1)!} = (-1)^{mp}(m!)^p$$

Lastly, if $j \geq p$ then the terms in $f^{(j)}(k)$ that are nonzero are those in which $(x-k)^p$ has been differentiated away. In these cases the term is a multiple of p as one factor has been differentiated p times to get $p!$ divided by $(p-1)!$.

Now if $p > m$ the terms in $m!$ will all be lower than p and therefore all have a prime factorization with primes lower than p and multiplying and taking the pth power will then only do the same operations to all these lower prime factors. Therefore $f^{(p-1)}(0)$ does not have a prime factor p. Thus

$$J = Np + a_0 M$$

with M, N integers and $p \nmid M$. Thus J must be non-zero and an integer.
Now using the general bound for $I(t)$ we found earlier:

$$|I(t)| \leq |t|e^{|t|}|F(|t|)|,$$

we get that

$$|J| \leq |a_1|eF(1) + \ldots + |a_m|me^m F(m).$$

Now our f consist of m factors to the power p for every $0 < x < m$, after taking the absolute value of the coefficients each of these factors can contribute at most $2m$ to the size of this product. This is because if we look at the last $(x - m)$ for $x = m$ and taking absolute values in the worst case made us add up the two m's to get $2m$ and all of the other factors at worst are going to be two lower numbers added up. Thus we have

$$F(x) \leq \frac{(2m)^{p-1}(2m)^{mp}}{(p-1)!} = \frac{(2m)^n}{(p-1)!}$$

.
Let a be the maximum of the $(|a_1|, \ldots, |a_n|)$ then

$$|J| \leq \frac{ame^m(2m)^{(m+1)p-1}}{(p-1)!} = \frac{\frac{ame^m}{2m}((2m)^{m+1})^p}{(p-1)!} \leq \frac{c^p}{(p-1)!}$$

.
Where we can make the righthandside arbitrarily small by taking p large enough, specifically we can make it less than 1. Note we can take p arbitrarily large as there are infinitely many primes. Now we have constructed J such that is an integer that is in absolute value bigger than 0 as well as lower than 1 and as such an integer does not exist we have a contradiction. This proves our assumption that e is algebraic cannot hold and thus e is transcendental. □

2.2 Symmetric polynomials

The proof of transcendence of π and the proof Lindemann-Weierstrass are very similar to this. The difficult part will now be proving that a similar J construction is an integer. For this we will need the fundamental theorem of symmetric polynomials.
Consider a polynomial f in n variables, so for example $f \in \mathbb{Q}[x_1, \ldots x_n]$, the polynomial ring in n variables x_i over $\mathbb{Q}$. A polynomial is called symmetric if

we can exchange the position of these n variables without changing the polynomial. Formally $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$ for every permutation σ of $\{1, 2, \ldots, n\}$. For example

$$x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2$$

is a symmetric polynomial in 3 variables.
The fundamental theorem of symmetric polynomials says that all such polynomials are built out of so called elementary symmetric functions. Let

$$(T - x_1)(T - x_2) \ldots (T - x_n) = T^n - s_1 T^{n-1} + s_2 T^{n-2} - \ldots + (-1)^n s_n$$

.
The coefficients s_i of this function are called elementary symmetric functions. So they are:

$$\begin{aligned} s_1 &= x_1 + x_2 + \ldots + x_n \\ s_2 &= x_1 x_2 + x_1 x_3 + \ldots + x_1 x_n + x_2 x_3 + \ldots + x_{n-1} x_n \\ s_3 &= x_1 x_2 x_3 + \ldots = \sum_{1 \leq i < j < k \leq n} x_i x_j x_k. \end{aligned}$$

The fundamental theorem of symmetric polynomials says that any symmetric polynomial can be written as a polynomial in the elementary symmetric functions $s_1, s_2, \ldots, s_n$ For example

$$x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 = s_3 + s_1^2 - 2s_2$$

2.3 Transcendence of π

Theorem 3. *π is transcendental*

Assume π is not transcendental. Then $i\pi$ is not either. So it has a minimal polynomial, let

$$f(x) = \sum_{i=0}^{d} a_i x^i$$

We call the roots of this polynomial $\theta_1 = i\pi, \theta_2, \ldots, \theta_d$. Then we have

$$(1 + e^{\theta_1})(1 + e^{\theta_2}) \ldots (1 + e^{\theta_d}) = 0$$

as $e^{i\pi} = -1$. If we expand this product we get a summation of powers of e with exponents

$$\beta_{\epsilon_1, \ldots, \epsilon_d} = \epsilon_1 \theta_1 + \epsilon_2 \theta_2 + \ldots + \epsilon_d \theta_d$$

where each ϵ_i is either 0 or 1. Every ϵ has 2 options so there are 2^d such terms. These terms are 0 when all ϵ are 0 but some combination of θ could

also cancel eachother. The exponents that are not zero we call $\alpha_1, \dots \alpha_n$ and we know $n < 2^d$ because at least one is 0.
We have

$$(2^d - n) + e^{\alpha_1} + \dots + e^{\alpha_n} = 0$$

Now as our auxiliary function we wil take

$$f(x) = \frac{a_d^{mp} x^{p-1}(x-\alpha_1)^p \dots (x-\alpha_m)^p}{(p-1)!}$$

and we will, similarly to before, prove that

$$J = I(\alpha_1) + \dots + I(\alpha_m)$$

is both an integer and between 0 and 1.
Let $n = degf = (m+1)p - 1$. We have by definition of I

$$J = \sum_{k=0}^{m} I(\alpha_k)$$
$$= \sum_{k=1}^{m} \left(e^{\alpha_k} \sum_{j=0}^{n} f^{(j)}(0) - \sum_{j=0}^{n} f^{(j)}(\alpha_k) \right)$$
$$= \sum_{k=1}^{m} e^{\alpha_k} \sum_{j=0}^{n} f^{(j)}(0) - \sum_{k=1}^{m} \sum_{j=0}^{n} f^{(j)}(\alpha_k)$$

And since $(2^d - n) + \sum_{k=1}^{m} e^{\alpha_k} = 0$ we have

$$= (n - 2^d) \sum_{j=0}^{n} f^{(j)}(0) - \sum_{j=0}^{n} \sum_{k=1}^{m} f^{(j)}(\alpha_k)$$

To finish the proof we have to show that this number is an integer and between 0 and 1.
Firstly we use the fundamental theorem of symmetric polynomials to show its an integer. Note that the first sum is clearly an integer as before.
Similarly to before $f^{(j)}(\alpha_k)$ is a sum of all possible combinations of the factors of f being derived a different number of times but j times in total. Thus the sum over k is a symmetric polynomial in $a_d\alpha_1, \dots, a_d\alpha_n$. As the α_i are simply θ_i added together the $a_d\alpha_i$ are also algebraic integers like the $a_d\theta_i$ are. So this inner sum is by the fundamental theorem of symmetric polynomials, a polynomial in elementary symmetric functions on the $a_d\alpha_i$. α_i are nonzero β, which are sums of θ_i thus the sum is of elementary symmetric functions of $a_d\theta_i$. These in turn are the coefficients of the minimal polynomial of $a_d i\pi$ and therefore integers. thus the inner sum is an integer.

Finally we should make a contradicting estimate for J. By the general bound found earlier

$$|I(t)| \leq |t|e^{|t|}|F(|t|)|,$$

we get that

$$|J| \leq |\alpha_1|e^{|\alpha_1|}F(|\alpha_1|) + \ldots + |\alpha_m|e^{|\alpha_m|}F(|\alpha_m|)$$

and by the same reasoning as before we have that $f^{(j)}(\alpha_k) = 0$ when $j < p$ is an integral multiple of p and that $f^{(j)}(0)$ is an integral multiple of p when $j \neq p - 1$. Furthermore we have that

$$f^{(p-1)}(0) = \frac{(p-1)!}{(p-1)!}(-a_d)^{np}(\alpha_1 \ldots \alpha_m)^p =$$

$$f^{(p-1)}(0) = (-a_d)^{np}(\alpha_1 \ldots \alpha_m)^p$$

which is not divisible by p if p is chosen to be bigger than $a_d\alpha_1 \ldots \alpha_m$. Thus J is non-zero as the terms can't cancel each other out. Similarly to the previous proof we can also make $|J|$ arbitrarily small by taking p large enough though. Therefore, we make $|J|$ such that it is an integer smaller than 1 and bigger than 0. Thus we have a contradiction, so $i\pi$ and therefore π must be transcendental. □

2.4 The Gelfond-Schneider theorem and Baker's theorem

In this sections we will state these two theorems which expand upon the auxiliary function method. Using a notation using logarithms for this, we show that Baker's theorem is a generalization of the Gelfond-Schneider theorem

Gelfond-Schneider theorem was independently proved by Alexander Gelfond and Theodor Schneider in the 1934.

Theorem 4. *If $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$ are algebraic numbers, then any value of α^β is transcendental.*

The proof of this theorem involves non-explicit auxiliary functions. How this works can be found on [3, p. 25]. This proves transcendence of numbers such as $\sqrt{2}^{\sqrt{2}}$ and e^π as $e^\pi = (e^{i\pi})^{-i} = (-1)^{-i}$.

Now we prove that this theorem is equivalent to the following:

Theorem 5. *If α and β are nonzero algebraic numbers with $\log\alpha$ and $\log\beta$ linearly independent over $\mathbb{Q}$, then $\log\alpha$ and $\log\beta$ are linearly independent over the algebraic numbers.*

This statement means that if α and β are nonzero algebraic numbers and $\beta \neq 1$ and then the quotient $\frac{\log \alpha}{\log \beta}$ is either a rational number or transcendental.

First we will proof that theorem 4 implies theoreom 5. Let $l \neq 0$, $\alpha = e^l$ and $\beta \notin \mathbb{Q}$. Note that $\alpha \neq 0, 1$. Now at least one of e^l, β and $e^{\beta l}$ is transcendental, because if $\alpha = e^l$ and β are both algebraic then by theorem 4 $e^{\beta l}$ is transcendental.

Let $l = \log \beta$ and $\beta' = \frac{\log \alpha}{\log \beta}$. We have $l \neq 0$ and if we assume $\log \alpha$ and $\log \beta$ are linearly independent over the rationals, then $\beta' = \frac{\log \alpha}{\log \beta} \notin \mathbb{Q}$ thus by what we showed above at least one of $e^l =, \beta'$ and $e^{\beta' l}$ is transcendental. $e^l = \beta$ and $e^{\beta' l} = \alpha$ so if α and β are algebraic, then β' is trancendental which means $\log \alpha$ and $\log \beta$ are linearly independent over the algebraic numbers. Therefore theorem 4 implies theorem 5.

For the other direction we will use a proof by contradiction. Assume $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$ are algebraic numbers and that α^β is **not** transcendental. Note that $\log \alpha^\beta = \beta \log \alpha$. So as β, α^β and α are all algebraic, $\log \alpha^\beta$ and $\log \alpha$ are linearly dependent over the algebraic numbers. Thus by theorem 5 they are linearly dependent over the rational numbers therefore $\beta \in \mathbb{Q}$. But we assumed that $\beta \notin \mathbb{Q}$ and we have a contradiction. $\square$

The new formulation of the Gelfond-Schneider theorem works for only two algebraic numbers. This raises the question whether or not it also holds for an arbitrary number of algebraic numbers. This was proven in 1966 by Alan Baker and was called Baker's theorem.

Theorem 6. *If $\alpha_1, \ldots, \alpha_n$ are nonzero algebraic numbers with $\log \alpha_1, \ldots \log \alpha_n$ linearly independent over $\mathbb{Q}$, then $\log \alpha_1, \ldots \log \alpha_n$ are linearly independent over the algebraic numbers.*

A proof of this theorem can be found on [4, p. 12]. Like the Gelfond-Schneider theorem this is can also be stated in way that more directly shows the transcendence of a particular type of number. The following theorem can be proved using theorem 6 as demonstrated on [4, p. 11] :

Theorem 7. *$\alpha_1^{\beta_1} \ldots \alpha_n^{\beta_n}$ is transcendental for any algebraic numbers $\alpha_1, \ldots, \alpha_n$ other than 0 or 1, and any algebraic numbers $\beta_1, \ldots, \beta_n$ with $1, \beta_1, \ldots, \beta_n$ linearly independent over the rationals.*

3 Rational approximations

The previous method initially set out to prove the transcendence of constants like e and π and proves transcendence of a lot of numbers involving e or algebraic numbers raised to the power of irrational algebraic numbers and their products. This list of transcendental numbers however, is far from exhausitive. The following method unfortunately has the same limitation and fails to give a necesarry condition on transcendence, in fact it fails to even prove the transcendence of e at this point in time. It does allow the construction and proves transcendence of very different types of numbers, namely numbers that are well-approximable.

3.1 Liouville's criterion

In the 1850s Liouville found that algebraic numbers cannot be well approximated by rationals. Now what does well approximated mean in this context? An irrational number α is called well-approximable if for all positive integer n there is a rational number $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}.$$

Constructing a number which is well-approximable and therefore transcendental can for example be done in the following way. Let

$$\alpha = 0.1\underbrace{0\ldots0}_{m_1-1}1\underbrace{0\ldots0}_{m_2-1}1\underbrace{0\ldots0}_{m_3-1}10\ldots$$

where (m_i) is an increasing sequence of integer greater than or equal to 1. For example

$$\sum_{n=1}^{\infty} 10^{-n!} = 0.110001000000000000000001\ldots$$

where this sequence is $(m_i)_{i\in\mathrm{N}} = (i+1)! - i! = (n+1)n! - n! = nn!$, because this number has a 1 at every decimal place corresponding to a factorial. Now let

$$\alpha_k = 0.1\underbrace{0\ldots0}_{m_1-1}1\underbrace{0\ldots0}_{m_2-1}1\ldots1\underbrace{0\ldots0}_{m_k-1}1$$

Now note that α_k is a rational number with denominator

$$q_k = 10^{m_1+\ldots+m_k+1}$$

so that

$$\frac{1}{q_k^n} = 0.\underbrace{0\quad\ldots\quad0}_{n(m_1+\cdots+m_k+1)}1$$

and

$$|\alpha - \alpha_k| = 0.\underbrace{0 \quad \ldots \quad 0}_{m_1 + \ldots + m_k + m_{k+1}} 10\ldots$$

And because $n(kk!+1) < (k+1)(k+1)!$ clearly holds for $n < k$ this means that for every n we can find and α_k such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}.$$

Therefore α is transcendental by Liouville's criterion.

Liouville's criterion was improved over the years by improving the bound on the expression by reducing the exponent. This resulted in the Thue-Siegel-Roth theorem:

Theorem 8. *Let α be an algebraic number with $\alpha \notin \mathbb{Q}$. Let $\epsilon > 0$. Then there are at most finitely many rational numbers $\frac{p}{q}$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}.$$

A proof for this can be found on [9, p. 21].

3.2 BBP formula

The Bailey-Borwein-Plouffe formula for π was discovered by Simon Plouffe in 1995. A BBP-type number is of the form

$$\sum_{k=0}^{\infty} \frac{1}{b^k} \frac{p(k)}{q(k)}$$

where p and q are polynomials in integer coefficients and $b \geq 2$ is an integer. For example we have that

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right).$$

For more information on how to prove this formula see [5].
Transcendence of certain types of such numbers can be shown using the Thue-Siegel-Roth theorem, as seen in [2].
We will show how this formula might be used to compute the n'th digit of transcendental numbers without having to compute the previous $n-1$.

3.2.1 Digits

We will look at ln(10/9) for which we can get the digits in regular base 10. To find the BBP formula we look at the following taylor expansion.

$$(\log(1-x))' = \frac{-1}{1-x} = -(1+x+x^2+\dots),$$

thus

$$-\log(1-x) = (x + \frac{1}{2}x^2 + \frac{1}{3}x^3 \dots) = \log\frac{1}{1-x}.$$

Then when we take $x = 1/10$ we get

$$\ln(10/9) = \sum_{k=1}^{\infty} \frac{1}{k10^k}$$

Now we multiply by 10^n to get the decimal point in the nth place:

$$10^n \sum_{k=1}^{\infty} \frac{1}{k10^k} = \sum_{k=1}^{\infty} \frac{1}{k} 10^{n-k}$$

Now if we take this number modulo 10 we are left with exactly the digit before the decimal point which we made the nth digit. This is useful as there is an economic way of calculating the exponentation modulo k for small integers using an algorithm as described in [1]. Furthermore it is clear that the part of the sum where $k > n-1$ will give a contribution less than 1 and can also be left out if we are only interested in the nth digit. So in fact we calculate

$$\sum_{k=1}^{n-1} \frac{10^{n-k} \mod 10}{k}$$

A similar algorithm was found for the BBP formula for π which finds the n'th hexadecimal digits of π efficiently [1].

4 Conclusion

We have looked at some proof techniques for theorems on transcendence of numbers. These techniques can be expanded and generalized, as in the refered to literature, to find some more results. Firstly we gave a general overview of main results in the field. Then we looked closer at the method using auxiliary functions. To illustrate this method we looked at the classical proofs of transcendence of e and π. We stated the Gelfond-Schneider theorem and Baker's theorem. We rewrote them in forms that show they are statements about algebraic independence of logarithms, illustrating how they are related. Another proof technique is given, using the fact that algebraic numbers can't be well approximated by rationals. The main results, Liouville's criterion and the Thue-Siegel-Roth theorem, are discussed. An example is given how to use these theorems to construct transcendental numbers. An application of BBP-type formulas is given to calculate digits of ln(10/9). Lastly we note that none of the discussed methods can find all transcendental numbers yet.

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