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Higher order derivative gravitational theories in the metric and Palatini formalisms

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Abstract

Lovelock and galileon theories are known extensions of general relativity that have actions containing second order derivatives, but do not produce field equation with third or higher order derivatives. We show that the dimensional reduction of Lovelock results in a generalized covariant galileon theory. Then we explain the Palatini formalism, a way to consider gravity in non-Riemannian geometry. It is known general relativity can be obtained in both the metric and Palatini formalism. This gives us a motivation to study other higher order derivative theories in the Palatini formalism as well. The possibility of using dimensional reduction as a tool to learn more about the Palatini formalism is explored in this thesis but does not lead to interesting results. Then Palatini formalism is applied to the cubic covariant galileon. The field equation for the connection can be solved and is found to be the Weyl connection. Furthermore, we show that the physics change in the Palatini formalism and point out the importance of projective invariance and some kind of duality between torsion and non-metricity in the cubic covariant galileon framework.

Introduction

1.1 Motivation

In physics, we usually consider theories that have actions with at most first order derivatives. Actions that depend on second or higher order derivatives give in general rise equations of motion containing higher than second order derivatives. Ostrogradsky proved that as a result, more initial conditions than degrees of freedom are needed and so-called *ghoststates* appear, see appendix A for an outline of his argument. Theories with ghosts are not stable and theoretically give infinite energy or particle production for example. However, there are some special second order derivative theories that give rise to at most second order derivative equations of motions. For this reason, these theories circumvent Ostrogradsky's theorem and are free of ghosts. Probably the best-known example of such a theory this is general relativity.

Since the beginning of the civilization, people have tried to explain why objects fall. The Greek philosopher Aristotle was one of the first people that we know of that tried to come up with a theory for this. His explanation was that all bodies move to their natural place, the center of the (geocentric) universe. Thousands of years after him, Newton wrote down a more mathematical description of gravity, the gravitational force that we all know as $F = G \frac{m_1 m_2}{r^2}$. His theory was a great success and able to predict a lot of phenomena, such as the movement of planets, very well. However, when astronomical observations improved, small deviations from this theory were found in the orbit of Mercury [1]. First, this problem in the discrepancy between theory and observations was tried to be solved by some ad-hoc solutions, such as the existence of a small planet closer to the sun. This would slightly modify the theoretical prediction to account for the observations. However, the hypothetical planet was never observed and it this puzzle wasn't solved by observing new matter.

Instead, a new theory solved the problem of Mercury. In the 20th century Einstein invented

general relativity (GR). He came with the idea that space-time is a dynamical entity that interacts with the particles living in it. So GR describes gravity as a distortion of space-time and this is a huge conceptual jump from Newtonian gravity. Einstein's theory predicted a lot of new physics, such as the existence of black holes and gravitational waves. These were indeed detected long after Einstein came up with GR. The predictions of GR are in agreement with experiments in scales that range from millimeters to astronomical units [2]. However, there are a few problems with general relativity. To explain the rotation curves of spiral galaxies for example, we need the existence of an enormous amount of dark matter. On the other hand, to explain the distribution of matter at large scales, we need another source of energy with repulsive gravitational properties, dark energy. In total, dark matter and dark energy should constitute 96% of the total amount of total matter in the universe and we have no idea what it is.

One thing that could solve this problem is the discovery of this matter, but so far, all the searches for dark matter and energy didn't help us any further. However, just as the orbit of Mercury could only be explained by a new theory, maybe this time we should investigate modifications of general relativity. Another problem of GR is that we can not make predictions with it on quantum scales. So it's a good idea to look at other gravitational theories. We went from Aristotle to Newton and Einstein, who knows what's next?

Modifying general relativity

If we want to stay as close to the original Einstein-Hilbert Lagrangian as we can without breaking any symmetry, there are a few ways to modify gravity with a Lagrangian formulation [1]. If we want to keep a single massless metric, we are forced to go to *higher dimensions*. Then we will end up with Lovelock theory, which we will explain in the next chapter. We can also stay in 4 dimensions and consider *extra fields*. If we consider scalar fields, these theories are called galileon theories.

These two modifications of gravity have in common that they are described by actions containing second derivatives, but their field equations do not produce third or higher order derivatives. We will discuss them and their relations in the first part of this thesis. Furthermore, we may think of *other geometric constructions* such as a different connection than Levi-Civita to modify gravity. If we don't assume the shortest path to be equal to the path of parallel transport, a whole new way of calculating things open. This formalism is called the *Palatini* or *first order formalism* and will form the main idea of the second part of the thesis.

Objectives of the thesis

In this thesis we are going to investigate relations between the different modifications shortly mentioned. It is known for GR it does not matter if we use the Palatini or metric formalism. So it is interesting to see what the effects of the Palatini formalism are on other higher derivative theories. In order to do this, we will split the thesis into two parts. The first half of the thesis (chapter 2-4) is dedicated to explaining different higher order derivative theories that give at most second order field equation and the relation between them. In this part we will try to answer the following research questions:

- We will explore modifications of General Relativity by considering Ostrogradsky-ghosts free theories. Which higher order derivative theories with second-order derivatives field equations do exist?
- Since Lovelock gives at most second-order field equations, its dimensionally reduced counterpart should do the same. How does dimensional reduction work and which specific scalar-tensor theory do we obtain by the dimensional reduction of Lovelock?

After this, we will introduce the Palatini formalism. We will apply this formalism to the theories considered in the first part. Specifically, the following subjects and research questions are explained in chapter 5-7:

- What is the Palatini formalism and what are its physical implications?
- We investigate the probability of dimensionally reducing a theory in the Palatini formalism. Can we learn more about the first order formalism of Lovelock and maybe even link it to the Palatini formalism of galileon theories?
- The last question that will be treated is what happens if we apply the Palatini formalism to galileons that live in curved space-time. How does it look like and to what extent is it different than the galileon in the metric formalism?

1.2 Outline

In the second chapter, a short summary of General Relativity is given. Its generalization to higher dimensions, Lovelock theory, is treated as well in this chapter and we will give an idea of how it avoids ghost states. In the third chapter, galileon theories are introduced we aim to give an overview of the differences, similarities and relations between different galileon theories. We will see that Lovelock and galileons have some similarities and in the fourth chapter we will link the Lovelock and galileon theories. Dimensional reduction is introduced and we will try to discover which galileon theory we obtain by the dimensional reduction of Lovelock.

In the fifth chapter, we explain the Palatini formalism. We will show how this formalism is equivalent to using the metric formalism for General Relativity. The consequences of using the Palatini formalism on Lovelock and galileon theory are discussed.

The possibilities to learn more about Palatini formalism for Lovelock and galileon theories are explored in chapter 6 and 7. In chapter 6 we start by calculating the dimensional reduction of the first two Lovelock terms in the Palatini formalism. In chapter 7 we will apply the Palatini formalism to the cubic galileon. The last chapter serves as a short summary and conclusion.

General relativity and beyond

One of the most elegant scientific theories ever developed is general relativity, Einstein's generalization of special relativity. In general relativity, gravity is a geometric phenomenon and the universe is described as an four-dimensional space-time. Let's review the basic principles of this theory.

2.1 The equivalence principle and its tests

General relativity is mainly based on some thought experiments of Einstein that lead to the equivalence principle. There are several forms of the equivalence principle [3][4]:

- The weak equivalence principle (WEP): The trajectory of a falling test body is independent on its internal structure and composition.
- The Einstein equivalence principle (EEP): (1) the WEP is valid, (2) the outcome of any local non-gravitational experiment is independent of the velocity of the freely-falling apparatus (Local Lorentz Invariance, LLI) and (3) the outcome of any local non-gravitational experiment is independent of where and when in the universe it is performed (Local Position Invariance, LPI).
- The strong equivalence principle (SEP): The EEP is not only valid for test bodies and non-gravitational experiments, but also for self-gravitating bodies and all other experiments.

With a 'test' body or particle we mean a particle that is not acted upon by forces such as electromagnetism and is too small to be affected by tidal gravitational forces. A local non-gravitational test is any experiment which is performed in a freely falling laboratory with negligible self-gravitating effects. We divided the equivalence principle into parts because when considering different theories than general relativity, we should look at how they function compared with the (different parts) of the equivalence principle. Let's specify

the differences between the equivalence principles.

The WEP implies a spacetime that has a family of preferred trajectories that are the worldlines of freely falling bodies. As far as the WEP is concerned, free-fall trajectories do not necessarily coincide with geodesics of the metric (so the WEP doesn't imply a metric) [4]. The EEP is more restrictive. The LLI and LPI together imply that in the local freely falling frame, which all observers in free-fall carry, the theory should reduce to Special Relativity. This implies a second rank tensor field that reduces in the local freely falling frame to the Minkowski metric $\eta_{\mu\nu}$. The EEP does not forbid theories with different gravitational fields than the metric, as long as they do not couple to matter. However, the SEP does forbid these theories. The only known theory that obeys the SEP is general relativity.

Since the equivalence principle is fundamental for GR, we should ask ourselves if there is any proof for it.

Tests of the equivalence principle

The weak equivalence principle, or the universality of free fall, has been tested lots of times and by far with the best accuracy of the three forms of the equivalence principle. The principle of all WEP-testing experiments is the same, let two objects with the same mass but with different composition fall freely in a gravitational field and measure the differences in acceleration. The equivalence principle tells us that the bodies fall at exactly the same rate, so any deviation in the compared accelerations is a violation of the equivalence principle.

The quantity in which the fractional difference in acceleration between two bodies is expressed is called the *Eötvös ratio*:

$$\eta \equiv 2 \frac{|a_1 - a_2|}{|a_1 + a_2|} \quad (2.1)$$

Where a_1 and a_2 are the different accelerations. Experimental limits on η place limits on the WEP-violation.

The first tests of the weak equivalence principles have been conducted in the 16th century, by Galileo and Stevin. They dropped different objects (with equal mass) from a tower and concluded that they landed at exactly the same time. Since then, the precision of the measurements has improved greatly.

One of the most recent experiments of WEP-violation is MICROSCOPE [5]. This is an experiment conducted with satellites and measures free-fall around the earth. It is expected that they can measure η up to an accuracy of 10^{-18} . In 2017 the first results of the MICROSCOPE experiment confirmed an accuracy of the weak equivalence principle

up to 10^{-15} and higher precision measurements are expected in 2018 [5].

The tests of local Lorentz invariance (LLI) consist simply put on finding deviations of the speed of light from one. This has been done in numerous different experiments and the current bound of the deviation is of the order of magnitude 10^{-20} [3]. Experiments of the (LPI) are done by comparing two clocks at different positions. For these experiments, the best bounds on the deviations between the transitions of isotopes hover around the one part per million.

The SEP has been tested to high precision in the solar system. Violation of the SEP implies that the earth should fall towards the sun at a slightly different rate than the moon. The Lunar Laser Ranging Experiment measures the distance between the Earth and the moon and put heavy constraints on the violation. Tests that put limits on the violation of the SEP outside the solar system can come for example from the measurement of gravitational waves. In the next chapter, we will come back to this.

2.2 General relativity

The consequences of the EEP are that spacetime must be endowed with a symmetric metric which determines geodesics and that locally in a freely falling frame one can use special relativity to describe all non-gravitational physics [6]. Gravity is caused by the curvature of spacetime.

The logical question to ask is what exactly curves spacetime. Newton described the mass of objects as the gravitational source, so at least mass should curve spacetime. In general, every type of energy and momentum of all types of fields present should have an influence. Einstein came up with the following equation:

$$G_{\mu\nu} = -\kappa T_{\mu\nu} \quad (2.2)$$

$G_{\mu\nu}$ is called the Einstein tensor and describes the curvature of space-time and $T_{\mu\nu}$ is the energy-momentum tensor. Conservation of energy implies that $\nabla_\mu T^{\mu\nu} = 0$ and therefore the divergence of the Einstein tensor should be zero as well. The Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \quad (2.3)$$

R is the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$, $R_{\mu\nu} = R_{\mu\gamma\nu}{}^\gamma$. The Riemann tensor is given by the following formula:

$$R_{\mu\nu\rho}{}^\gamma = \partial_\mu\Gamma_{\nu\rho}^\gamma - \partial_\nu\Gamma_{\mu\rho}^\gamma + \Gamma_{\mu\sigma}^\gamma\Gamma_{\nu\rho}^\sigma - \Gamma_{\nu\sigma}^\gamma\Gamma_{\mu\rho}^\sigma \quad (2.4)$$

Here we encounter for the first time the connection Γ , which tells us how to transport objects in spacetime in parallel. The connection defines affine geodesics, curves along which the tangent vector to the curve is parallel-transported (see picture 2.1):

$$\ddot{x}^\mu + \Gamma_{\nu\rho}^\mu \dot{x}^\nu \dot{x}^\rho = 0 \quad (2.5)$$

The covariant derivative is defined in terms of the connection:

$$\nabla_\rho T_\mu^\nu = \partial_\rho T_\mu^\nu - \Gamma_{\rho\nu}^\lambda T_\mu^\nu + \Gamma_{\rho\lambda}^\nu T_\mu^\lambda \quad (2.6)$$

When we assume the connection to be symmetric and metric compatible $\nabla_\mu g_{\rho\nu} = 0$, the connection takes a really simple form and is called the Levi-Civita connection:

$$\Gamma_{\nu\rho}^\mu = \frac{1}{2} g^{\mu\sigma} (\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}) \quad (2.7)$$

In this thesis, we use the notation $\partial_k = \frac{\partial}{\partial x^k}$ and work with a mostly minus metric $(+---)$. To obtain the correct Newtonian limit the constant κ is defined as $\kappa = 8\pi G_N$, where G_N is Newton's constant .

The Lagrangian formulation of general relativity

It is possible to construct an action that yields the Einstein equations as its Euler-Lagrange equations. The mathematician Hilbert introduced this action in 1915 and therefore it is called the Einstein-Hilbert action:

$$S = \int d^4x \sqrt{|g|} \left(\frac{1}{2\kappa} R + \mathcal{L}_M \right) \quad (2.8)$$

In this action $g = \det(g_{\mu\nu})$ and κ is the Einstein constant. Varying the action (2.8) with respect to the metric gives us the equations of motion as we have seen before if we define

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta(\sqrt{|g|} \mathcal{L}_{mat})}{\delta g^{\mu\nu}} \quad (2.9)$$

A characteristic of the Einstein-Hilbert action that is worth pointing out, is the fact the action depends on second derivatives of the metric $R \sim \partial\partial g$. However, the Einstein equation does not depend on third or higher order derivatives of the metric and is thus

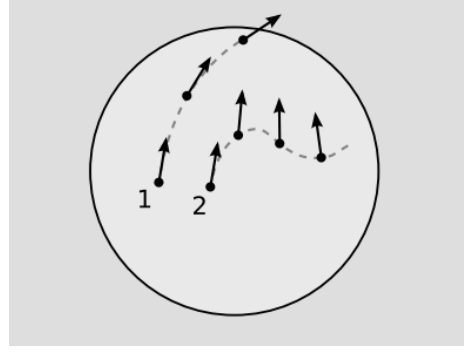


Figure 2.1: The affine geodesic, curve 1, preserves its tangency under parallel transport, whereas the non-geodesic curve 2 does not. A vector initially tangent to curve two is no longer tangent when parallel-transported around it [7].

free from unwanted extra degrees of freedom.

We can also write the action in terms of the reduced Planck mass $M_P = \sqrt{\frac{1}{8\pi G_N}}$. Note that we work with natural units, so $\hbar = c = 1$. With this, the Einstein-Hilbert Lagrangian is written as:

$$S = \int d^4x \sqrt{|g|} \left(\frac{M_P^2}{2} R + \mathcal{L}_M \right) \quad (2.10)$$

This notation has some conceptual advantages, as we will see later. That is because gravity is regarded as an effective field theory. Effective field theories only describe physics on energy scales below its cut-off scale. The cut-off scale of general relativity is M_P^2 . At last, let's take a look at the dimensions of the action because this will become important later on as well. If we write everything in dimensions of unit length (**L**), d^4x has a dimension of 4. The action has to be a scalar without units, therefore the total Lagrangian needs dimension -4 . All partial derivatives $\partial_\mu = \frac{\partial}{\partial x^\mu}$ have dimension -1 , so the Ricci scalar $R \sim \partial\partial g$ has dimension -2 . With all masses having dimension -1 as well, the combination $M_P^2 R$ exactly has dimension -4 . So sometimes the factor of M_P^2 is left out of the action, since we can always reconstruct by dimensional reduction later how many factors of M_P we must introduce.

The only way the Einstein-Hilbert action can be generalized (in four dimensions) without adding degrees of freedom, breaking the general covariance or generating higher than second order equations of motion is by introducing the *cosmological constant* Λ (with dimensions -4).

$$S_{EH} = \int d^4x \sqrt{|g|} \left(\frac{M_P^2}{2} R - \Lambda \right) \quad (2.11)$$

However, if we consider higher dimensional theories, we can add more terms to the action. We will see this in the next section.

2.3 More than four dimensions: Lovelock

Lovelock's theory is a modification of gravity that in four dimensions turns out to be exactly GR. In the Lovelock Lagrangian we stick to the field $g_{\mu\nu}$ with a Levi-Civita connection. As pointed out earlier, we want a theory to give at most second order differential field equations. If the field equations have higher order derivatives, we will get unstable 'ghosts' and there is no lower bound on the energy, see appendix A.

Lovelock formulated the most general Lagrangian with the field g and at highest second order derivative field equations in an arbitrary number of dimensions. In four dimensions there is only one modification that we can do, the addition of a cosmological constant term.

Other higher order curvature invariants will give us either a total derivative term (that doesn't contribute to the field equations) or add higher derivatives to the field equations. Lovelock showed that if we consider higher dimensions, we can add extra higher order derivative terms to the Lagrangian that will give us second order field equations. So we can see Lovelock theory as a generalization of GR in higher dimensions.

In order to construct a Lagrangian containing higher order curvature terms that nevertheless gives rise to second order field equation, Lovelock made the following assumptions:

- The generalization of the Einstein tensor should be a symmetric two-rank tensor:
 $A_{\mu\nu} = A_{\nu\mu}$
- The generalized Einstein tensor depends on the metric and its first two derivatives:
 $A_{\mu\nu}(g, \partial g, \partial^2 g)$
- The generalized Einstein tensor is divergence free, $\nabla^\mu A_{\mu\nu} = 0$

Lovelock's Lagrangian in arbitrary dimensions D is given by [8]:

$$S = \int d^D x \sqrt{|g|} L_{\text{lovelock}} \quad (2.12)$$

The Lovelock Lagrangian is described as

$$L_{\text{lovelock}} = \sum_{h=0}^{[\frac{D-1}{2}]} c_h \mathcal{L}_h \quad (2.13)$$

In this equation $[\frac{D-1}{2}]$ is the integer part of $\frac{D-1}{2}$, c_h are coupling constants and \mathcal{L}_h is given below. Consider for a moment the dimensionality of the coupling constants c_h . The dimensionality of R_h is $-2h$, whereas the dimension of $d^D x$ is D . Therefore, the constants c_h have dimensionality $2h - D$ and it is common to write them in terms of some higher dimensional fundamental cut-off scale M_D . [9]

In the case of compact dimensions, this scale can be linked with the Planck mass by the size of these extra dimensions. In general $M_P^2 \sim V_n M_D^{n+2}$ where n is the number of extra dimensions $D = 4 + n$ and V_n is the n dimensional volume of the n extra dimensions. Note that sometimes the higher dimensional cut-off scale M_D^{n+2} is denoted as \hat{M}_P^2 . An advantage of this notation is the action in the two formalisms looks the same, however in this notation the higher dimensional planck mass doesn't have dimension [-1].

Let's see what \mathcal{L}_h is:

$$\mathcal{L}_h = \frac{(2h)!}{2^h} \mathcal{R}_h \quad (2.14)$$

where

$$\mathcal{R}_h = \delta_{b_1 \dots b_{2h}}^{a_1 \dots a_{2h}} \prod_{i=1}^h R_{a_{2h-1} a_{2h}}^{b_{2i-1} b_{2i}} \quad (2.15)$$

and the first term $\mathcal{L}_0 = \Lambda$. From this last equation we can see that if $h > D/2$, \mathcal{R}_h will be zero because the delta tensor (see appendix C) has more indices than there are different indices available. The term with $h = D/2$ is always a topological term, so it only depends on the topology of the manifold. Therefore, it does not contribute to the equations of motions and is non-dynamical. We already discussed that in general terms like R^2 do give rise to higher than order two derivative equations of motion. However, the specific combinations of higher curvature terms R_h give rise to different higher order derivative terms in the equations of motion that exactly cancel each other.

For clarity the first few terms R_h are written down below.

$$\mathcal{R}_0 = \Lambda \quad (2.16)$$

$$\mathcal{R}_1 = R \quad (2.17)$$

$$\mathcal{R}_2 = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \quad (2.18)$$

The first two terms are the cosmological constant and the Einstein-Hilbert term. The last term is known as the Gauss-Bonnet term and was discovered before Lovelock came up with his generalization of gravity. In four dimensions, the Gauss-Bonnet term is non-dynamical one and hence does not contribute to the equations of motion. In higher dimensions though, the Gauss-Bonnet term is dynamical and should be included in the action, because it will modify the equations of motions.

Recap

In this chapter, we have introduced all actions that depend on at most second derivatives of the metric g , that give rise to second order differential field equations. The Einstein-Hilbert action is the only one relevant in four dimensions, but we have seen the Lovelock Lagrangians as an extension of general relativity to higher dimensions. In the next chapter, similar theories are introduced, however this time the starting point is not a tensor, but a scalar field.

Galileons

In the previous chapter, we have seen expressions for a purely tensorial theory with second order derivatives. In this chapter, we will discuss higher derivative scalar theories, the galileon theories. There are galileons that involve only scalars and theories that involve both tensors and scalars.

We know that GR is not complete and we should consider possible corrections. However, the equivalence principle strongly suggests that gravity should be described by a metric theory, so why even bother with scalars? Actually, the equivalence principle does not forbid a scalar term in the gravitational action. The weak equivalence principle only tells us how matter behaves. So we can introduce a scalar in the gravitational part of the action and as long as it doesn't appear in the matter part, it won't violate the equivalence principle:

$$S_{tot} = S_G(g_{\mu\nu}, \phi, \dots) + S_m(g_{\mu\nu}, \psi_m) \quad (3.1)$$

So the gravitational action may depend on other gravitational fields¹, while the weak equivalence principle tells us that the matter fields ψ_m only couple to the metric [1].

Of course, the matter might couple to the other gravitational fields as well. In that case, the weak equivalence principle is broken. As discussed in the previous chapter, the equivalence principle is tested extremely to extremely high precision, but there is still some room for breaking of the equivalence principle if the scalar does enter the matter Lagrangian.

Galileons in every flavour

Arbitrary scalar Lagrangians with second derivatives will give higher order derivative equations of motions. However, similar to the way that we have seen in Lovelock theory, there is a way to add higher-order derivatives terms, resulting in (up to) second order

¹The terms *gravitational* and *non-gravitational* fields are quite ambiguous. With gravitational fields we refer to all extra fields that intervene in the generation of spacetime geometry by the matter fields.

derivative equations of motions. The Lagrangians that contain specific combinations of second derivatives of scalar fields and give rise to field equation with *up to* second order terms are called *generalized Galileons*. The subgroup that gives rise to equations of motion with *purely* second order field equations are called the *galileons*. These theories are invariant under the following Galilean symmetry:

$$\phi \rightarrow \phi + c + v_\mu x^\mu \quad (3.2)$$

Previously mentioned theories live in flat spacetime and contain only scalar fields. If we promote them to live in curved spacetime, we call them *covariant* theories and they will be combined scalar/tensor theories. While galileon theories have purely second order field equations, *covariant galileon* theories will have up to second order field equations as we will see. The biggest group is the *covariant generalized galileons*, which is the covariant version of the generalized galileons. By setting the curvature zero one finds the generalized galileons and by imposing more symmetries galileons.

History and motivation

The galileons were first introduced in [10] as the four-dimensional limit of the five-dimensional Dvali-Gabadadze-Porrati (DGP) model. This theory gives an alternative explanation for dark energy. In the DGP model, matter is confined to live on a 4-dimensional brane while gravity can propagate in five dimensions. The action at long distances is dominated by everything that lives in five dimensions. At short distances, the DGP model is well described by a 4D model of a scalar field mixed with the metric. Due to the galilean symmetry of this scalar field in Minkowski spacetime, in [10] they proposed to name the scalar field a galileon. One year later, it was shown that galileons naturally arise in the decoupling limit (the mass of the field approaches zero) of the most general massive gravity theory [11]. Later, inflations models based on covariant and generalized covariant galileons were investigated [12] [13] [14]. These cosmological models might produce an accelerated expansion of the universe without introducing any dark energy [15].

Purely scalar theories are interesting to study because they can give us better insight into how exactly higher derivative theories work. Physically these theories might arise from situations where the no-gravity limit is taken. Furthermore, they can be used as a pedagogical example because they could be regarded as a scalar analog to general relativity [16].

Some scalar-tensor theories have been developed years ago already. Horndeski theory is a prime example of this. Until a few years ago, people were thinking about this kind of theories as the addition of a scalar field to a gravitational field. However, we will see that

it is also possible to construct exactly the same theory by starting with scalar Lagrangian and adding gravity to it. This sounds similar but is fundamentally different. So a thorough understanding of these pure scalar field theories in flat space might give us more insight into existing tensor/scalar gravitational theories, such as Horndeski theory. Now we will give an overview of the different scalar-tensor theories with (up to) second order derivatives.

3.1 Galileons in flat spacetime

3.1.1 Galileons

Galileons are scalar Lagrangians that give purely second order equations of motion. They are invariant under the galilean symmetry mentioned earlier (3.2). The Lagrangian can be written as $\mathcal{L} = \sum_{N=1}^{D+1} c_N \mathcal{L}_N$. There are different notations found in the literature for \mathcal{L}_N that all differ by a total derivative. We will follow [17] and have a look at two of the notations. We call them galileon 1 ($G1$) and galileon 3 ($G3$):

$$\mathcal{L}_N^{G1} = (n+1)! \delta_{\nu_1 \dots \nu_{n+1}}^{\mu_1 \dots \mu_{n+1}} \phi_{\mu_{n+1}} \phi^{\nu_{n+1}} \phi_{\mu_1}^{\nu_1} \dots \phi_{\mu_n}^{\nu_n} \quad (3.3)$$

$$\mathcal{L}_N^{G3} = n! \delta_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} \phi_{\gamma} \phi^{\gamma} \phi_{\mu_1}^{\nu_1} \dots \phi_{\mu_n}^{\nu_n} \quad (3.4)$$

Where $\phi_{\mu} \equiv \partial_{\mu} \phi$ and $\phi_{\mu\nu} \equiv \partial_{\nu} \partial_{\mu} \phi$, N is the number of times that the scalar field occurs. $N \equiv n+2$, $N \leq D+1$ and the (trivial) $\mathcal{L}_1 = \phi$. Let's write down a few terms in both notations:

$$\begin{aligned} \mathcal{L}_2^{G1} &= \mathcal{L}_2^{G3} = (\partial\phi)^2 \\ \mathcal{L}_3^{G1} &= \square\phi\phi_{\mu}\phi^{\mu} + \phi^{\mu\nu}\phi_{\mu}\phi_{\nu} \\ \mathcal{L}_3^{G3} &= \square\phi\phi_{\mu}\phi^{\mu} \\ \mathcal{L}_4^{G1} &= (\square\phi)^2\phi_{\mu}\phi^{\mu} - 2\square\phi\phi_{\mu}\phi^{\mu\nu}\phi_{\nu} - \phi_{\mu\nu}\phi^{\mu\nu}\phi_{\rho}\phi^{\rho} + 2\phi_{\mu}\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho} \\ \mathcal{L}_4^{G3} &= \phi^{\gamma}\phi_{\gamma}((\square\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}) \end{aligned}$$

Here $\square\phi = g^{\mu\nu}\partial_{\mu}\partial_{\nu}\phi$. In literature, \mathcal{L}_3 is called the *cubic* galileon, \mathcal{L}_4 the *quartic* galileon and so on.

It is trivial that the first of these terms gives second order equations of motion. For Lagrangians \mathcal{L}_3 however, it isn't that obvious. The derivative of $\square\phi$ and second derivative of ϕ_{μ} give third order derivatives in the equation of motion. However, it turns out that all terms in the equations of motion with higher than second order derivatives will cancel

each other. The two different Lagrangians \mathcal{L}^{G1} and \mathcal{L}^{G3} can be connected to each other in the following way:

$$\mathcal{L}_N^{G1} = \frac{N}{2}\mathcal{L}_N^{G3} - \frac{N-2}{2}\partial_\mu J_N^\mu \quad (3.5)$$

$$J_N^\mu = n!\phi_\gamma\phi^\gamma\delta_{\nu_1,\nu_2,\dots,\nu_n}^{\mu,\mu_2,\dots,\mu_n}\phi^{\nu_1}\phi_{\mu_2}^{\nu_2}\dots\phi_{\mu_n}^{\nu_n} \quad (3.6)$$

$$(3.7)$$

Since total derivative terms never add to the equations of motion, the field equations of \mathcal{L}^{G1} and \mathcal{L}^{G3} are identical. The notation of $G3$ is more compact, so that is the one that is going to be used throughout this thesis.

3.1.2 Generalized galileons

By definition galileon theories have strictly second order derivative equations of motion on flat spacetime. From Ostrogradsky's theory we learn that we can not increase the order of derivatives in the field equations, however we can find the most general scalar theory with *at most* second order derivative field equations. It turns out that generalizing galileon theory on flat-space is really simple, we can just add some function f_N that depends on ϕ and $X \equiv \phi_\mu\phi^\mu$. With that, the generalized galileon lagrangian becomes:

$$\mathcal{L} = \sum_{N=2}^{D+1} \hat{\mathcal{L}}_N f_N \quad (3.8)$$

With

$$\hat{\mathcal{L}}_N f_N \equiv f_N(\phi, X)\mathcal{L}_N^{G3} \quad (3.9)$$

The equations of motion of this Lagrangian depend, for a non-constant f , not only on $\phi_{\mu\nu}$, but on ϕ_μ and ϕ as well. Naturally, the generalized galileons are no longer invariant under the galileon symmetry (3.2).

3.2 A recipe to add gravity to the galileon

So far, we have considered galileons in flat spacetime only. Extending the previous theories to a curved spacetime with metric $g_{\mu\nu}$ is called covariantization and the resulting theories are called *covariant galileons*. Covariantization means changing the partial derivatives ∂_μ to covariant derivatives ∇_μ . However, note that if we apply this change to \mathcal{L}_N , we might get higher than second order equations of motions. To see why, let's look at some terms of

the equation of motion (ε_3) of \mathcal{L}_3

$$\begin{aligned}\varepsilon_3 &\sim \nabla_\nu \nabla_\gamma \nabla_\mu \phi \nabla^\gamma \phi \nabla^\nu \nabla^\mu \phi - \nabla_\gamma \nabla_\nu \nabla_\mu \phi \nabla^\gamma \phi \nabla^\nu \nabla^\mu \phi \\ &= [\nabla_\nu, \nabla_\gamma] \nabla_\mu \phi \nabla^\gamma \phi \nabla^\nu \nabla^\mu \phi = -R_{\nu\gamma\mu}{}^\rho \nabla_\rho \phi \nabla^\gamma \phi \nabla^\nu \nabla^\mu \phi\end{aligned}$$

As we can see, in flat spacetime this terms disappears, but in curved spacetime it does not. Instead, the commutation of the derivatives introduces the Riemann tensor in the equation of motion. Therefore the equation of motion again does not contain higher than second order derivatives. Note that the field equation does involve first derivatives of ϕ and g now, so it's no longer *purely* second order.

In the case of \mathcal{L}_4 we are not so lucky that we obtain ghost-free equations of motion for free. Instead we get derivatives of the Ricci tensor and scalar $\varepsilon_4 \sim \nabla_\mu R + \nabla_\rho R^{\mu\nu}$. Since the Ricci scalar and tensor depend on second derivatives of the metric ∂ , previous terms will give us third-order derivatives of the metric. Hence we should add some terms to \mathcal{L}_4 that exactly cancel these terms in the equation of motion. Adding to \mathcal{L}_4^{G1} the term $\mathcal{L}_{4,1}^{G1} = -\phi^\gamma \phi_\gamma \phi_\mu G^{\mu\nu} \phi_\nu$ gives correct (ghost-free) field equations again. Note that we obtain a non-minimal coupling between the scalar and metric here.

Are we always able to find such counterterms that cancel the unwanted terms in the equation of motion? It turns out that we can. In article [15] it is shown that the "dangerous" terms in the equation of motion are first derivatives of the Riemann tensor and that we only need to add some finite number of counterterms to obtain correct equations of motions. With this information, we can summarize the covariantization process of galileons as follows:

1. Change all partial derivatives for covariant derivatives
2. Calculate the equations of motions
3. Determine the counterterms which remove all the higher than second order derivatives terms

In flat spacetime we can write the difference between two different formulations of galileons as a total derivative term. It turns out that in the covariantized version of both formalism, the difference is exactly the covariantized total derivative term. So you have to add different terms to the different formulations \mathcal{L}_4^{G1} and \mathcal{L}_4^{G3} , but the sum of the original and counterterms are again equal up to some integration by parts. The counterterm for \mathcal{L}_4^{G3} is $\mathcal{L}_{4,1}^{G3} = \frac{1}{4} \phi_\nu \phi^\nu \phi_\mu \phi^\mu R$.

The covariantization of all galileons is done in a systematic way by Deffayet and Steer [15]. They covariantized galileons in an arbitrary number of dimensions. Since we will

need the notation for the **covariant galileon** in chapter 7, here the first few terms of $\mathcal{L} = \sum_{N=1}^5 c_N \mathcal{L}_N$ in four dimensions will be written down:

$$\begin{aligned}\mathcal{L}_1 &= M^3 \phi \\ \mathcal{L}_2 &= X \\ \mathcal{L}_3 &= \frac{1}{M^3} X \nabla^\mu \nabla_\mu \phi \\ \mathcal{L}_4 &= \frac{1}{M^6} X (\nabla^\mu \nabla_\mu \phi \nabla^\nu \nabla_\nu \phi - \nabla^\mu \nabla_\nu \phi \nabla_\mu \nabla^\nu \phi + \frac{1}{4} R)\end{aligned}\tag{3.10}$$

Where M is a constant with dimensions of mass, and is seen as the cut off value of the galileon theory. For the reader interested in the full notation of covariantized galileons, the paper [17] is recommended.

3.2.1 The covariant generalized galileon

The covariantization process has been done for generalized galileons as well and the results are summarized in [17]. The procedure of the covariantization is similar to the procedure mentioned before. Let's see how these theories look like. As mentioned the notation of Deffayet and Steer is more or less followed. However, as their Riemann tensor is defined differently from ours, some changes have been made in the notation. The **covariant generalized galileons** are all described by the following Lagrangian.

$$\mathcal{L} = \sum_{N=2}^{D+1} \tilde{\mathcal{L}}_N^{cov} f_N(\phi, X)\tag{3.11}$$

where

$$\tilde{\mathcal{L}}_N^{cov} f_N(\phi, X) = \sum_{p=0}^{\frac{N-2}{2}} C_{N,p} \tilde{\mathcal{L}}_{N,p} f_N\tag{3.12}$$

The coefficients $C_{N,p}$ make sure all higher derivative terms exactly cancel each other and are given by the following expression in which $n = N - 2$:

$$C_{N,p} = \frac{1}{(-8)^p} \frac{n!}{(n-2p)!p!}\tag{3.13}$$

Furthermore, the part that depends on ρ and R are given as a product of three different functions

$$\tilde{\mathcal{L}}_{N,p} f_N = n! \delta_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} P_p R_p S_{q=n-2p}\tag{3.14}$$

with the following functions:

$$P_p = \int_{X_0}^X dX_1 \int_{X_0}^{X_1} dX_2 \dots \int_{X_0}^{X_{p-1}} dX_p f_N(\phi, X_p) X_p \quad (3.15)$$

$$R_p = \prod_{i=1}^p -R_{\mu_{2i-1}\mu_{2i}}^{\nu_{2i-1}\nu_{2i}} \quad (3.16)$$

$$S_q = \prod_{i=0}^{q-1} \partial^{\nu_{n-i}} \partial_{\mu_{n-i}} \phi \quad (3.17)$$

3.2.2 Equivalence of Horndeski's theory and covariantized generalized galileons

For the moment, let's take a look at a different scalar theory: Horndeski theory. Already in 1974 Horndeski came up with a theory of extended gravity. He formulated the most general scalar-tensor theory in 4 dimensions that yields second order field equations. His theory remained unnoticed until its recent discovery as generalized covariant galileons, in [17] it is proven that the two theories are actually equivalent in four dimensions.

This is not trivial since the construction of the two theories is totally different. Horndeski started by trying to find the *unique* set of scalar-tensor theories that give up to second order equations in curved four-dimensional spacetime. This is really similar to the procedure Lovelock followed. On the other hand, as we have seen, galileon theory starts with a pure scalar theory and constructs its covariantized generalization without knowing or imposing on forehand that it is unique. An open question remains if the equivalence between Horndeski and generalized galileons holds in dimensional larger than four, because at the time of writing the extension of Horndeski's construction for higher dimensions is still unknown [17].

After a few years of research, Horndeski changed his career to become a painter. His love for physics never disappeared completely and in some paintings he incorporated texts relating to physics. When Horndeski published his theory, it went by quite unnoticed. His paper published in 1974 got the first citation in 2010 and at the moment is cited more than 1100 times. That got him back to physics, as he says himself [18]: "It seems that the scalar-tensor equations that I developed in my Ph.D. thesis, and actually derived when I was 23, had become all the rage in Cosmology, and was



Figure 3.1: A painting by Horn-
deski about his own theory

being used to attempt to explain inflation in the early Universe as well as dark energy. So that got me interested in going back and doing some work on scalar-tensor field theories. So I am doing that stuff again now, as well as painting, and having a fantastic time doing both.”

3.3 The fate of covariant galileons after gravitational waves measurements

A thing one must always take in mind is how to check these models. When new fields are coupled to gravity, as in Horndeski and covariant galileon theories, the propagation speed of gravitational waves might be changed. Therefore, measurements of gravitational waves can be used to test theories of modified gravity.

As we have seen, the quadratic and cubic galileon couple minimal to gravity. Therefore, they are not expected to have an influence on the speed of gravitational waves. However, the quartic and quintic galileon do have a non-minimal coupling to gravity. Measurements of gravitational waves can only put constraints on these theories.

In 2017, a binary neutron star merger was observed with LIGO. The measurements of the produced gravitational waves placed strong bounds on scalar-tensor theories of gravity. In November 2017, a few different papers appeared on the bounds of scalar-tensor theories[19] [20] [21]. In [22] they showed that these measurements imply that the quartic and quintic galileons almost completely vanish, if you do not allow strong fine-tuning of the parameters. In this article they show that the deviation of the speed from the gravitational waves from 1 depends on $\partial_X f_4$, $\partial_\phi f_5$ and $\partial_X f_5$.

This fact combined with the measured limits of the speed of the gravitational waves, leads to the conclusion that the quartic Lagrangian reduces to $\bar{\mathcal{L}}_4^{cov} f_4 \sim f(\phi)R$ while the only surviving term of the quintic Lagrangian is $\bar{\mathcal{L}}_5^{cov} f_5 \sim G_{\mu\nu} \nabla^\mu \nabla^\nu \phi$.

Conclusion

In this chapter we introduced the scalar theories with equations of motions that have (up to) second order equations of motion. We have seen that they can be written in different notations, all related by a total derivative term. To promote such theories to curved spacetime, we change all partial derivatives for covariant derivatives and look for counterterms to add to the action to remove higher derivatives in the equation of

motion. It turns out that the difference between two covariantized galileons is exactly the covariantized version of the total derivative.

Horndeski theory is a scalar-tensor theory that was invented years ago as the most general scalar-tensor theory without ghost-like instabilities in four dimensions and turns out to be equivalent to the covariantized galileons. In principle, covariant galileons could yield interesting dark energy models, however cosmological observations of gravitational waves put super heavy constraints on the existence of them.

In the next chapter, we are going to relate the covariant galileons to Lovelock theory by dimensional reduction.

Dimensional reduction: from Lovelock to galileons

We have seen that the Lovelock and galileon theories are similar in the way that they both are the most general Lagrangians (tensor or scalar) that contain second order derivative terms, but do not give dangerous higher order derivative terms in their equations of motion. In this chapter we are going to investigate the relation of both extended gravities to each other, by using dimensional reduction to obtain a galileon theory from Lovelock gravity.

Even before string theory, the possibility of extra dimensions was considered. In 1919 Kaluza and Klein tried to unify electromagnetism and gravity by assuming a fifth dimension. Compactifying one of the coordinates, a four-dimensional theory combining both gravity and electromagnetism was obtained. However, the original theory failed to correctly match with experimental data, since it predicted the existence of a never observed scalar field as well. Nevertheless, the simplicity and beauty of the theory led to many unified field theories and the procedure for dimensional reduction by compactifying one or more of the coordinates still has their name.

In the Kaluza-Klein model, the compactification of higher dimensions gives rise to a scalar field. While this was an unwanted side effect for the original goal of Kaluza and Klein, it can give us a link between Lovelock and galileon theories. As we will see, Lovelock gravity might be a source of the scalar field in galileon theories. In [23] it was proven that you get galileon terms from Lovelock terms by dimensional reduction. In this analysis however they only took into account separate Lagrangian terms and didn't calculate the dimensional reduction of the sum of Lovelock Lagrangians. Furthermore, they only pointed out the terms they were interested in after the reduction, which does not give a very well-explained introduction to this topic.

The goal of this chapter is to introduce and explain the procedure for Kaluza-Klein dimen-

sional reduction and investigate the relation between galileons and dimensionally reduced Gauss-Bonnet. We will calculate the reduction of the Einstein-Hilbert and Gauss-Bonnet Lagrangians and check if we end up with a galileon theory. By explicitly calculating all terms, we hope to gain better insight into the relation between Lovelock and galileon theories and to find out exactly which galileon theory we obtain.

4.1 Kaluza-Klein theory

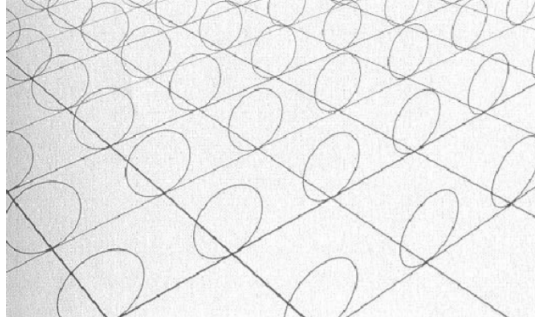


Figure 4.1: In every point of spacetime, there is one extra dimension rolled up. Here we show 2D space for simplicity [24].

Suppose that we have a theory in $N + 1$ dimensions and we want to obtain a N -dimensional theory. This can be done by compactifying one of the (spatial) coordinates, let's call it ω at the moment, on a circle S^1 of radius R_0 . To account for the unobservability of this extra dimension, we assume that this radius is very small (comparable to the Planck length). In figure 4.1, it is shown how we can imagine such a space with an extra dimension.

Low energy limit

From now on, we denote all objects living in the $N+1$ dimensional space with a hat ($\hat{g}, \hat{R}, \hat{x}$, etc), and objects living in a N -dimensional space without a hat (g, R, x, \dots). So the $N + 1$ higher dimensional coordinates, change into N coordinates plus the compactified coordinate $\hat{x} \rightarrow x, \omega$.

We could expand all components of the $(N + 1)$ -dimensional metric as a Fourier expansion:

$$\hat{g}_{\mu\nu}(x, \omega) = \sum_n g_{\mu\nu}^{(n)}(x) e^{in\omega/L} \quad (4.1)$$

If we do this, we see that we get an infinite number of fields in N dimensions. They are labeled by their Fourier mode number n . However, it turns out that only the fields with

mode number $n = 0$ are massless. To illustrate this, consider a toy model of a scalar field.

$$\hat{\phi}(s, \omega) = \sum_n \phi_n(x) e^{in\omega/R_0} \quad (4.2)$$

The Klein-Gordon equation in the higher dimension is:

$$\hat{\square}\hat{\phi} = 0 \quad (4.3)$$

Where $\hat{\square} = \partial^{\hat{\mu}}\partial_{\hat{\mu}}$. If we substitute our Fourier expansion in, we get:

$$\square\phi_n - \frac{n^2}{R_0^2}\phi_n = 0 \quad (4.4)$$

So all modes with $n \neq 0$ have a mass $\frac{|n|}{R_0}$. We assumed R_0 to be very small, so all modes with $n \neq 0$ would be really heavy. In the low energy limit, the mass-less mode is the only one that is relevant and it is safe to neglect all other modes. Therefore, we will take $\hat{g}_{\mu\nu}$ to be independent of ω and assume that none of the fields depend on the compact coordinate: $\partial_\omega = 0$.

Reduction of the metric

Now we want to decompose the metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ in N dimensional objects. We can write the different components of the metric as $\hat{g}_{\mu\nu}$, $\hat{g}_{\mu\omega}$ and $\hat{g}_{\omega\omega}$. In N dimensions these are a metric, a 1-form and a scalar field, so at first sight it looks like we can define them by $g_{\mu\nu}$, A_μ and ϕ . However, these objects do not behave correctly under infinitesimal coordinate transformations (see appendix D). Following equation (D.4) this is the infinitesimal coordinate transformation for a tensor:

$$\delta\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{\xi}^{\hat{\rho}}\partial_{\hat{\rho}}\hat{g}_{\hat{\mu}\hat{\nu}} + \hat{g}_{\hat{\rho}\hat{\nu}}\partial_{\hat{\mu}}\hat{\xi}^{\hat{\rho}} + \hat{g}_{\hat{\mu}\hat{\rho}}\partial_{\hat{\nu}}\hat{\xi}^{\hat{\rho}} \quad (4.5)$$

Now we can have a look at how the different components of the metric transform under a $N+1$ coordinate transformation.

$$\delta\hat{g}_{\omega\omega} = \xi^\rho\partial_\rho\hat{g}_{\omega\omega} = \delta\hat{g}_{\omega\omega} \quad (4.6)$$

$$\delta\hat{g}_{\mu\omega} = \xi^\rho\partial_\rho\hat{g}_{\mu\omega} + \hat{g}_{\rho\omega}\partial_\mu\xi^\rho + \hat{g}_{\omega\omega}\partial_\mu\xi^\omega \quad (4.7)$$

$$\delta\hat{g}_{\mu\nu} = \xi^\rho\partial_\rho\hat{g}_{\mu\nu} + \hat{g}_{\rho\nu}\partial_\mu\xi^\rho + \hat{g}_{\omega\nu}\partial_\mu\xi^\omega + \hat{g}_{\mu\rho}\partial_\nu\xi^\rho + \hat{g}_{\mu\omega}\partial_\nu\xi^\omega \quad (4.8)$$

We see that $\hat{g}_{\omega\omega}$ does transform as a scalar in both $N+1$ as N dimension, so there is no problem in using it. However, $\hat{g}_{\mu\omega}$ and $\hat{g}_{\mu\nu}$ do not transform as a vector and tensor respectively in lower dimensions, due to the presence of terms proportional to $\partial_\mu\xi^\omega$. Luckily

it is not so hard to define objects that transform in the right way, namely $A_\mu \equiv \frac{\hat{g}_{\mu\omega}}{\hat{g}_{\omega\omega}}$ (note: we know $\hat{g}_{\omega\omega} \neq 0$) and $g_{\mu\nu} = \hat{g}_{\mu\nu} - \frac{\hat{g}_{\mu\omega}\hat{g}_{\nu\omega}}{\hat{g}_{\omega\omega}}$. Now $g_{\mu\nu}$ transforms as a tensor and A_μ as a covector under the gauge potential $U(1)$ ¹:

$$\hat{\delta}A_\mu = \delta A_\mu + \partial_\mu \xi^\omega \quad (4.9)$$

For our reduced metric, we take the Ansatz [25]:

$$\hat{g}_{\mu\nu} = e^{2\alpha\phi} g_{\mu\nu} - e^{2\beta\phi} A_\mu A_\nu \quad (4.10)$$

$$\hat{g}_{\mu\omega} = -e^{2\beta\phi} A_\mu \quad (4.11)$$

$$\hat{g}_{\omega\omega} = -e^{2\beta\phi} \quad (4.12)$$

where α and β are arbitrary constants. The scalar field is called the dilaton and A_μ the Kaluza-Klein vector. We are only interested in a combined scalar gravity theory at the moment, so we will put all A_μ to zero. This is a consistent truncation, because we don't get solutions to the equations of motion that do not exist in the higher dimensional theory.

Reduction of the Riemann tensor

It turns out that for dimensional reduction, it is way easier to work with Vielbein coordinates than with curved coordinates. That's why we calculate the Vielbeins (see appendix B) of our N-dimensional metric in terms of the reduced metric.

$$\hat{e}_\mu^a = e^{\alpha\phi} e_\mu^a \quad \hat{e}_a^\mu = e^{-\alpha\phi} e_a^\mu \quad (4.13)$$

$$\hat{e}_\omega^z = e^{\beta\phi} \quad \hat{e}_z^\omega = e^{-\beta\phi} \quad (4.14)$$

The first object that we will have to calculate for the dimensional reduction of the action, is the square root of the determinant.

$$\sqrt{|\hat{g}|} = e^{\beta\phi} |e^{\alpha\phi} e_\mu^a| = e^{\phi(\beta+N\alpha)} \sqrt{|g|} \quad (4.15)$$

Now let's have a look at the flat derivatives:

$$\hat{\partial}_a = \hat{e}_a^\mu \partial_\mu \quad (4.16)$$

¹Remember that the electromagnetic tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is unvariant under the transformation of the potential $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$.

which gives us

$$\hat{\partial}_z = 0 \quad (4.17)$$

$$\hat{\partial}_a = e^{-\alpha\phi} \partial_a \quad (4.18)$$

In order to calculate the spin connection, we first will need to calculate all $\hat{\Omega}_{\hat{a}\hat{b}}^{\hat{c}}$. For that we will use equation (B.9). Since Ω is antisymmetric in its first two indices, we have five different contributions to $\hat{\Omega}_{\hat{a}\hat{b}}^{\hat{c}}$, namely $\hat{\Omega}_{ab}^c$, $\hat{\Omega}_{az}^c$, $\hat{\Omega}_{ab}^z$, $\hat{\Omega}_{az}^z$ and $\hat{\Omega}_{zz}^z$. Computing these will give us only two non-zero components:

$$\hat{\Omega}_{ab}^c = e^{-\alpha\phi - \Sigma_0} (\Omega_{ab}^c + \alpha \partial_a \phi \delta_b^c - \alpha \partial_b \phi \delta_a^c) \quad (4.19)$$

$$\hat{\Omega}_{az}^z = \beta e^{-\alpha\phi - \Omega_0} \partial_a \phi \quad (4.20)$$

By using formula (B.17), we can calculate the spin connections. The nonzero components are

$$\hat{\omega}_{ab}^c = e^{-\alpha\phi - \Omega_0} (\omega_{ab}^c - \alpha \partial^c \phi \eta_{ba} + \alpha \partial_b \phi \delta_a^c) \quad (4.21)$$

$$\hat{\omega}_{zb}^z = \beta e^{-\alpha\phi - \Omega_0} \partial_b \phi \quad (4.22)$$

$$\hat{\omega}_{zz}^c = \beta e^{-\alpha\phi - \Omega_0} \partial^c \phi \quad (4.23)$$

$$(4.24)$$

Now we have everything we need to finally compute our Riemann tensor by using equation (B.18). For Einstein-Hilbert, we could also just calculate the Ricci tensor, but we need Riemann when will calculate the dimensional reduction of the second Lovelock term (Gauss-Bonnett) next. After a long, but straightforward calculation we find:

$$\hat{R}_{abc}^d = (R_{abc}^d - \alpha^2 \partial_a \phi \partial_c \phi \delta_b^d + \alpha^2 \partial_a \phi \partial^d \phi \eta_{bc} + \alpha^2 \partial_b \phi \partial_c \phi \delta_a^d) \quad (4.25)$$

$$- \alpha^2 \partial_b \phi \partial^d \phi \eta_{ac} + \alpha^2 (\partial \phi)^2 \eta_{ac} \delta_b^d - \alpha^2 (\partial \phi)^2 \eta_{bc} \delta_a^d + \alpha D_a D_c \phi \delta_b^d - \alpha D_a D^d \phi \eta_{bc} - \alpha D_b D_c \phi \delta_a^d + \alpha D_b D^d \phi \eta_{ac})$$

$$\hat{R}_{azc}^z = \beta e^{-2\alpha\phi - 2\Omega_0} ((\beta - 2\alpha) \partial_a \phi \partial_c \phi + \alpha (\partial \phi)^2 \eta_{ac} + D_a D_c \phi) \quad (4.26)$$

Here, D_a is the covariant derivative for flat derivatives. We can write the expression in a more compact way by raising indices and using normalized antisymmetrization brackets ($[a, b] = \frac{1}{2}(ab - ba)$). Finally, we obtain:

$$\hat{R}_{ab}^{cd} = e^{-2\alpha\phi - 2\Omega_0} (R_{ab}^{cd} - 4\alpha^2 \partial_{[a} \phi \partial^{[c} \phi \delta_{b]}^d] + 2\alpha^2 (\partial \phi)^2 \delta_{ab}^{cd} + 4\alpha D_{[a} D^{[c} \phi \delta_{b]}^d]) \quad (4.27)$$

$$\hat{R}_{az}^{cz} = \beta e^{-2\alpha\phi - 2\Omega_0} ((\beta - 2\alpha) \partial_a \phi \partial^c \phi + \alpha (\partial \phi)^2 \delta_a^c + D_a D^c \phi) \quad (4.28)$$

Lovelock terms reduced by 1 dimension

Now we have almost all the ingredients that we need to start the dimensional reduction. We only need to know the form of the reduced Lovelock terms. By using symmetries of the Riemann tensor and the Lagrangian in the delta notation, it's not too hard to find an expression of this higher dimensional Lagrangian in terms of N-dimensional indices. We will try to find a general expression of the Lovelock terms dimensional reduction. For the Einstein-Hilbert terms this is quite trivial, so we show here how it works for the \mathcal{R}_2 , the Gauss-Bonnet term.

$$\hat{\mathcal{R}}_2 = \delta_{\hat{b}_1 \dots \hat{b}_4}^{\hat{a}_1 \dots \hat{a}_4} \hat{R}_{\hat{a}_1 \hat{a}_2}^{\hat{b}_1 \hat{b}_2} \hat{R}_{\hat{a}_3 \hat{a}_4}^{\hat{b}_3 \hat{b}_4} \quad (4.29)$$

$$= \delta_{b_1 \dots b_4}^{\hat{a}_1 \dots \hat{a}_4} \hat{R}_{\hat{a}_1 \hat{a}_2}^{b_1 b_2} \hat{R}_{\hat{a}_3 \hat{a}_4}^{b_3 b_4} + 4 \delta_{b_1 b_2 b_3 z}^{\hat{a}_1 \dots \hat{a}_4} \hat{R}_{\hat{a}_1 \hat{a}_2}^{b_1 b_2} \hat{R}_{\hat{a}_3 \hat{a}_4}^{b_3 z} \quad (4.30)$$

$$= \delta_{b_1 \dots b_4}^{a_1 \dots a_4} \hat{R}_{a_1 a_2}^{b_1 b_2} \hat{R}_{a_3 a_4}^{b_3 b_4} + 8 \delta_{b_1 b_2 b_3 z}^{a_1 a_2 a_3 z} \hat{R}_{a_1 a_2}^{b_1 b_2} \hat{R}_{a_3 a_4}^{b_3 z} \quad (4.31)$$

$$= \delta_{b_1 b_2 b_3 b_4}^{a_1 a_2 a_3 a_4} \hat{R}_{a_1 a_2}^{b_1 b_2} \hat{R}_{a_3 a_4}^{b_3 b_4} + 2 \delta_{b_1 b_2 b_3}^{a_1 a_2 a_2} \hat{R}_{a_1 a_2}^{b_1 b_2} \hat{R}_{a_3 z}^{b_3 z} \quad (4.32)$$

In general for \mathcal{R}_n , we find for a one-dimensional reduction

$$\begin{aligned} \delta_{\hat{b}_1 \dots \hat{b}_{2n}}^{a_1 \dots a_{2n}} \prod_{i=1}^n \hat{R}_{\hat{a}_{2i-1} \hat{a}_{2i}}^{\hat{b}_{2i-1} \hat{b}_{2i}} &= \delta_{b_1 \dots b_{2n}}^{a_1 \dots a_{2n}} \prod_{i=1}^n \hat{R}_{a_{2i-1} a_{2i}}^{b_{2i-1} b_{2i}} \\ &+ 2 \delta_{b_1 \dots b_{2n-1}}^{a_1 \dots a_{2n-1}} \hat{R}_{a_{2n-1} z}^{b_{2n-1} z} \prod_{i=1}^{n-1} \hat{R}_{a_{2i-1} a_{2i}}^{b_{2i-1} b_{2i}} \end{aligned} \quad (4.33)$$

The simplicity of this equation once again shows how useful it is to work with delta-tensors. Note that for a dimensional reduction over one dimension, we will never obtain terms containing multiplications of \hat{R}_{az}^{cz} . This is due to the fact that the delta tensor containing two compact coordinates, $\delta_{zz}^{a_1 a_2}$ will always give zero.

4.2 Dimensional reduction of Einstein-Hilbert

The first part that we are going to dimensionally reduce is the Einstein-Hilbert term. Using equation (4.33) we see that this is:

$$\hat{\mathcal{L}}_1 = \frac{\hat{M}^{N-1}}{2} \sqrt{|\hat{g}|} \left(\delta_{b_1 b_2}^{a_1 a_2} \hat{R}_{a_1 a_2}^{b_1 b_2} + 2 \delta_{b_1}^{a_1} \hat{R}_{a_1 z}^{b_1 z} \right) \quad (4.34)$$

Calculating this Lagrangian with the above formulas, we find:

$$S = \int \hat{\mathcal{L}}_1 d^{N+1}x = \frac{\hat{M}^{N-1}}{2} \int d^N x \sqrt{|g|} e^{(\beta+(N-2)\alpha)\phi} \left(R - (\alpha^2(N-1)(N-2) + 2\alpha\beta(N-2) + 2\beta^2)(\partial\phi)^2 + 2(\alpha(N-1) + \beta)D^2\phi \right) \quad (4.35)$$

Taking into account that $D^2\phi = \nabla^2\phi$ and using integration by parts gives us

$$S = \int d^{N+1}x \hat{\mathcal{L}}_1 = \frac{\hat{M}^{N-1}}{2} \int d^{N+1}x \sqrt{|g|} e^{(\beta+(N-2)\alpha)\phi} \left(R - \alpha(N-1)(\alpha(N-2) + 2\beta)(\partial\phi)^2 \right) \quad (4.36)$$

Now we can integrate over the extra dimension and find:

$$S = 2\pi R_0 \int d^N x \hat{\mathcal{L}}_1 = \frac{2\pi R_0 \hat{M}^{N-1}}{2} \int d^N x \sqrt{|g|} e^{(\beta+(N-2)\alpha)\phi} \left(R - \alpha(N-1)(\alpha(N-2) + 2\beta)(\partial\phi)^2 \right) \quad (4.37)$$

Note that this action isn't written as the common Einstein-Hilbert action, $\mathcal{L} \sim \sqrt{|g|}R$, because there is a factor $e^{\phi(x)}$ in front. We say that lagrangians of the form $\mathcal{L} \sim \sqrt{|g|}R$ are written in the *Jordan frame*, while lagrangians without this factor are written in the *Einstein frame*. The two metrics are related by a Weyl transformation $\hat{g}_{\mu\nu} = e^{\phi(x)}g_{\mu\nu}$. This transformation isn't a coordinate change, but fundamentally changes the geometric properties of space. It is not exactly clear yet which frame is the 'physical' frame, so with which metric you can compare experimental results [6]. Most of the time, the metric is considered in the Einstein frame and therefore it's important to write (4.37) in this frame. We can do this by choosing $\beta = -(N-2)\alpha$. Furthermore, we see that we can define $2\pi R_0 \hat{M}^{N-1} = M^{N-2}$ and $\alpha = \sqrt{\frac{1}{(N-2)(N-1)M^2}}$ to get to:

$$S = \int d^N x \sqrt{|g|} \left(\frac{M^{N-2}}{2} R + \frac{M^{N-4}}{2} (\partial\phi)^2 \right) \quad (4.38)$$

Which is in four dimensions:

$$S = \int d^4 x \sqrt{|g|} \left(\frac{M_P^2}{2} R + \frac{1}{2} (\partial\phi)^2 \right) \quad (4.39)$$

One last comment before going on to the Gauss-Bonnet term. Notice that we end up with a massless scalar field terms in the action, that, of course is a galileon term. It is a covariant galileon, however, note that equation (4.37) was written in the way of a generalized covariant galileon. Only after redefining our parameters, it becomes clear that this falls into the subgroup of covariant galileons.

4.3 Dimensional reduction of Gauss-Bonnet

We have all the ingredients to start with the dimensional reduction of the Gauss-Bonnet term in the Lagrangian. Let's start with formulating the reduced Gauss-Bonnet Lagrangian term by using formula (4.33):

$$\hat{\mathcal{L}}_2 = \sqrt{-\hat{g}}(\delta_{b_1 b_2 b_3 b_4}^{a_1 a_2 a_3 a_4} \hat{R}_{a_1 a_2}{}^{b_{12}} \hat{R}_{a_3 a_4}{}^{b_{34}} + 2\delta_{b_1 b_2 b_3}^{a_1 a_2 a_2} \hat{R}_{a_1 a_2}{}^{b_{12}} \hat{R}_{a_3 z}{}^{b_{3z}}) \quad (4.40)$$

So the action for this term becomes $S = \int d^D x \hat{\mathcal{L}}_2$. Just as we did for the Einstein Hilbert case, let's define the higher dimensional constant \hat{c}_2 as $\hat{c}_2 = \frac{c_2}{2\pi R_0}$. We know that c_2 has dimension $[4-N]$ whereas \hat{c}_2 has dimension $[4 - (N + 1)] = [3 - N]$, so dimensionally this fits perfectly.

It is clear to see that if we work this out, we will get terms proportional to the Riemann tensor, the Riemann tensor squared and a pure scalar term. For clarity, we will write them separately. After working out pages of calculations with a lot of care, this Lagrangian splits up in the following parts.

Terms that are proportional to R^2

The term in the Lagrangian that is proportional to the squared Riemann tensor is noted here. To shorten the notation, from now on we write $R_{a_1 a_2}{}^{b_1 b_2} = R_{a_1 a_2}^{b_1 b_2}$.

$$\mathcal{L}_{R^2} = \sqrt{|g|} e^{\phi(\beta + (N-4)\alpha)} \delta_{b_1 \dots b_4}^{a_1 \dots a_4} R_{a_1 a_2}^{b_1 b_2} R_{a_3 a_4}^{b_3 b_4} \quad (4.41)$$

We see that we get the original Gauss-Bonnet term again, with a conformal factor in front. Notice that if we choose the Einstein frame, $\beta = -(N - 2)\alpha$, we do not lose this factor.

All terms that are proportional to R

From now on, let's write $e^{\phi(\beta + (N-4)\alpha)} = e^{C\phi}$. The part of the Lagrangian that turns out to linear in the Riemann tensor is given by

$$\begin{aligned} \mathcal{L}_R = \sqrt{|g|} e^C R_{a_1 a_2}^{b_1 b_2} * \\ \left(2(\beta(\beta - 2\alpha) - (N - 3)\alpha^2) \partial_{a_3} \phi \partial^3 \phi \delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} + \right. \\ \frac{1}{3} \alpha(N - 2)(\alpha(N - 3) + 2\beta) (\partial \phi)^2 \delta_{b_1 b_2}^{a_1 a_2} + \\ \left. 2(\alpha(N - 3) + \beta) D_{a_3} D^{b_3} \phi \delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} \right) \end{aligned} \quad (4.42)$$

We can use integration by parts and the Bianchi identity to get rid of the first term in the previous expression. What we end up with in the end is the following expression:

$$\begin{aligned}\mathcal{L}_R = \sqrt{|g|}e^C R_{a_1 a_2}^{b_1 b_2} * \\ \left(\frac{1}{3} \alpha (N-2)(\alpha(N-3) + 2\beta)(\partial\phi)^2 \delta_{b_1 b_2}^{a_1 a_2} + \right. \\ \left. 2 \frac{\alpha^2 (N-3)^2 + \alpha\beta(2N-5)}{(N-4)\alpha + \beta} D_{a_3} D^{b_3} \phi \delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} \right)\end{aligned}\quad (4.43)$$

Scalar part of Lagrangian

For the scalar part of the Lagrangian we find at the end of our calculation some expression consisting of the terms $(\partial\phi)^4$, $(D^2\phi)^2$, $(DD\phi)^2$, $(\partial\phi)^2 D^2\phi$ and $(\partial\phi\partial\phi)(DD\phi)$. Here we used the notation $D^2\phi = D^a D_a \phi$ and $(DD\phi)^2 = D_a D_b \phi D^a D^b \phi$. By using integration by parts, we arrive at the following expression for the scalar part of the Lagrangian:

$$\begin{aligned}\mathcal{L}_\phi = \sqrt{|g|}e^C \frac{N-2}{3} \cdot \\ \left(\left(2\alpha\beta^3 + \frac{1}{2}\alpha^4(N-5)(N-4)(N-3) + 4(\alpha^2\beta^2(N-4) + 2\alpha^3\beta((N-4)^2 - (N-3))) \right) (\partial\phi)^4 \right. \\ + \left(2(N-3)(N-4)\alpha^3 + 6\alpha\beta^2 + 6(N-4)\alpha^2\beta \right) (\partial\phi)^2 D^2\phi \\ \left. + \left(2\alpha^2(N-3) + 4\alpha\beta \right) ((D^2\phi)^2 - (DD\phi)^2) \right)\end{aligned}\quad (4.44)$$

We have written the Lagrangian in the same form as the the generalized galileon/horndeski theory, in order to check the reduction. However, the last term can be removed by integration by parts to other scalar and mixed Ricci/scalar terms.

4.4 Checking the solutions

To check if our solution is a covariant generalized galileon, we will take a look at the mathematical notation of these theories. We will use the notation described by Deffayet and Steer [17]. We will check if our obtained Lagrangian indeed is a generalized covariant Galileon and fits eq (3.11). To do that, let's first explicitly write down some of the terms of equation (3.12).

$$\begin{aligned}
\tilde{\mathcal{L}}_2^{cov} f_2 &= f_0(\phi, X)X \\
\tilde{\mathcal{L}}_3^{cov} f_3 &= \nabla^2 \phi f_1(\phi, X)X \\
\tilde{\mathcal{L}}_4^{cov} f_4 &= \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} \nabla_{\mu_1} \nabla^{\nu_1} \phi \nabla_{\mu_2} \nabla^{\nu_2} \phi f_2(\phi, X)X + \frac{1}{4} \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} R_{\mu_1 \mu_2}^{\nu_1 \nu_2} \int_{X_0}^X dX_1 f_2(\phi, X_1)X_1 \\
\tilde{\mathcal{L}}_5^{cov} f_5 &= \delta_{\nu_1 \nu_2 \nu_3}^{\mu_1 \mu_2 \mu_3} \nabla_{\mu_1} \nabla^{\nu_1} \phi \nabla_{\mu_2} \nabla^{\nu_2} \phi \nabla_{\mu_3} \nabla^{\nu_3} \phi f_3(\phi, X)X \\
&\quad + \frac{3}{4} \delta_{\nu_1 \nu_2 \nu_3}^{\mu_1 \mu_2 \mu_3} R_{\mu_1 \mu_2}^{\nu_1 \nu_2} \nabla_{\mu_3} \nabla^{\nu_3} \phi \int_{X_0}^X dX_1 f_3(\phi, X_1)X_1 \\
\tilde{\mathcal{L}}_6^{cov} f_6 &= \delta_{\nu_1 \dots \nu_4}^{\mu_1 \dots \mu_4} \nabla_{\mu_1} \nabla^{\nu_1} \phi \cdot \dots \cdot \nabla_{\mu_4} \nabla^{\nu_4} \phi f_4(\phi, X)X \\
&\quad + \frac{3}{2} \delta_{\nu_1 \dots \nu_4}^{\mu_1 \dots \mu_4} R_{\mu_1 \mu_2}^{\nu_1 \nu_2} \nabla_{\mu_3} \nabla^{\nu_3} \phi \nabla_{\mu_4} \nabla^{\nu_4} \phi \int_{X_0}^X dX_1 f_4(\phi, X_1)X_1 \\
&\quad + \frac{3}{16} \delta_{\nu_1 \dots \nu_4}^{\mu_1 \dots \mu_4} R_{\mu_1 \mu_2}^{\nu_1 \nu_2} R_{\mu_3 \mu_4}^{\nu_3 \nu_4} \int_{X_0}^X dX_1 \int_{X_0}^{X_1} dX_2 f_4(\phi, X_2)X_2
\end{aligned}$$

Note that this is written in curved coordinates with a covariant derivative and our reduction in Vielbein coordinates. Since all indices are contracted, we can just change the Vielbein coordinates to curved coordinates and covariant Vielbein derivatives to curved covariant derivatives. A proof of this is given in appendix B.

Since we reduced from R^2 , there can't appear more than four derivatives together. Two for every $R(\propto \partial\Gamma \propto \partial\partial g \propto \partial\partial\phi)$. Therefore, we expect the first term of $\tilde{\mathcal{L}}_5^{cov} f_5$ and the first two terms of $\tilde{\mathcal{L}}_6^{cov} f_6$ to be zero. As we can see, we indeed do not get this terms. Furthermore, the integrals of the last terms of $\tilde{\mathcal{L}}_5^{cov} f_5$ and $\tilde{\mathcal{L}}_6^{cov} f_6$ can only give us a function of ϕ , not of $X = (\partial\phi)^2$. We identify (4.41) with the last term of $\tilde{\mathcal{L}}_6^{cov} f_6$, and the last term of (4.43) with the last term of $\tilde{\mathcal{L}}_5^{cov} f_5$. The first term of (4.43) is exactly written in the form of the last term of $\tilde{\mathcal{L}}_4^{cov} f_4$.

Now we're just left with checking that the scalar part of the Lagrangian fits in this formalism as well. Let's take a look at equation (4.44). The first term is written in the form of $\tilde{\mathcal{L}}_2^{cov} f_2$, the second term in the form of $\tilde{\mathcal{L}}_3^{cov} f_3$ and the last term identifies with the first term of $\tilde{\mathcal{L}}_4^{cov} f_4$.

Since we have both terms of $\tilde{\mathcal{L}}_4^{cov} f_4$ in our reduced Lagrangian, we can check if we get twice the same function f_4 . From equation (4.44) we read off that:

$$f_4(\phi, X)X = e^C \frac{N-2}{3} (4\alpha^2(N-3) + 8\alpha\beta) \quad (4.45)$$

which gives us

$$\frac{1}{4} \int_{X_0}^X dX_1 f_4(\phi, X_1) X_1 = -e^C \frac{N-2}{3} (\alpha^2(N-3) + 2\alpha\beta) X \quad (4.46)$$

which is indeed what we have found as our first term in equation (4.43).

If we add the terms of the reduction of the Einstein-Hilbert Lagrangian, we are still in the generalized galileon framework. The terms from that reduction alter function $f_4(\phi, X)$, on which there are no further constraints and the integral $\int dX f_4 X$. However, since the term with the Ricci scalar from the Einstein-Hilbert reduction does not contain any dependence on $X = (\partial\phi)^2$, we can see this as an integration constant and therefore the theory is still valid.

So we checked our solution and see that all terms fit exactly into the generalized covariant galileon theory. Do they fit in the 'pure' covariant galileon theory as well? On first sight, the answer to this question is no, since they have a function $f(\phi) \propto e^\phi$ in front of them. This function of ϕ is not there in the 'pure' covariant theory. We could remove this by choosing α and β wisely, but then we gain a conformal factor in the reduced quantities of the Einstein-Hilbert term. Just by smartly choosing our parameters, it is impossible to remove all factors $e^\phi(x)$ and therefore we will always end up with a generalized covariant theory instead of the pure one.

Scalar Lagrangian in the weak field limit

We are also interested in the scalar Lagrangian in the weak field limit. Usually in four dimensions we expand the metric around a Minkowski metric ($g_{\mu\nu} \rightarrow \eta_{\mu\nu} + \frac{1}{M_P} h_{\mu\nu}$), change the covariant derivative for a partial one ($\nabla \rightarrow \partial$) and let the Planck mass go to infinity ($M_P \rightarrow \infty$). When the Ricci scalar is not coupled to anything, this becomes some function of $\partial^2 h$ that is not important for the kinematics of the other fields.

To see which fields are most suppressed, it is instructive to look at the four-dimensional example. For the Einstein-Hilbert part, we end up with $(\partial\phi)^2$, but the Gauss-Bonnet contributions all have at least a factor α^3 in front. We know from our Einstein-Hilbert action that α and β are proportional to $\frac{1}{M_P}$. Since the reduction of Gauss-Bonnet to four dimensions doesn't give us a multiplication with a dimensionful constant, these terms are much more heavily suppressed than the kinetic term from Einstein-Hilbert. So in the weak field limit, we obtain $S = \int \frac{1}{2} (\partial\phi)^2 d^4x$. A similar argument holds for all higher order reductions.

Is there some way in which we can obtain a different weak field limit, one in which a galileon contribution is retained? Actually, there are several. The first method would require extreme fine-tuning of the coupling constant of the Gauss-Bonnet term. Probably it's possible to keep some other scalar terms, but it makes the compactification of Gauss-Bonnet a rather unnatural way to physically derive the galileon terms. Furthermore, by fine-tuning of the coupling constant, we can not get rid of the conformal factor in front of the galileon terms, so we would end up with generalized galileons. We are not going to pursue this route further.

The second method is to consider separate Lovelock terms, instead of the sum. In that case, we are free to choose our parameters α and β and we can choose them in such a way that the conformal factor e^C cancels. Considering only separate Lovelock terms is done in the paper [23]. Based on dimensional analysis they state that the fields we would end up with in the weak field limit of the Lovelock reduction are pure galileon. Can we see that in our reduction as well? Following our previous argument, and the argument in [23] use as well, in the weak field limit only the terms that have the lowest factor of α and β remain. If we take a look at the scalar reduction of the Lovelock Lagrangian 4.44 we see this corresponds to the last terms. This is however a total derivative in flat space, so instead we end up with the $(\partial\phi)^2 D^2\phi$ term which indeed is a galileon term. In four dimensions this term would cancel however², so we would need to consider another dimension.

Conclusions and recap

We have seen that the sum of the reduction of the Einstein-Hilbert and Gauss-Bonnet terms gives us a scalar-tensor theory. Since Lovelock terms are known to give at most second order equations of motions, we expected to end up with a theory that fits the covariant generalized galileon framework. We have seen that this is indeed the case. Furthermore, we saw that it isn't possible to rewrite this as a covariant galileon theory and that in the weak field limit, all Gauss-Bonnet contributions disappear and we are only left with the kinetic term of Einstein-Hilbert.

Note that these results are only valid for a 1-dimensional reduction. For a higher dimensional reduction, we are not sure what happens and this could be a nice follow-up research question.

²To cancel for e^C , we would need to choose $\beta = -(N-4)\alpha$.

Palatini

In the previous chapters, we have introduced higher differential order scalar and tensor theories and explored the relation between them. This constituted the first part of the thesis, in this chapter we are going to introduce a new concept: the Palatini formalism. The Palatini (or first order) formalism gives us a new way to relate geometry to physics and to analyze the effects of gravitational theories.

We will start by explaining the Palatini formalism and its motivations. Then we will specifically look at the consequences of the Palatini formalism for GR, Lovelock and galileon theories in this chapter. There is not a lot of literature about the Palatini formalism of scalar-tensor theories as introduced in the previous chapters. So the next two chapters will consist of combining the knowledge of the Palatini formalism with everything we have seen in the first chapters to explore this largely unknown territory.

5.1 Introduction to the Palatini formalism

As we have seen in the introduction, General Relativity links gravity to geometric properties of space-time, namely to the metric (g) and the connection (Γ). The connection tells us how to parallelly transport vectors and tensors along curves, in other words, it learns us about the intrinsic curvature of our manifold. In contrary, the metric tells us how to measure distances between points in the manifold and it defines angles between vectors. They are two different objects, that specify two different trajectories. The metric defines geodesics, so curves that minimize or maximize the distance between two point. The connection defines the worldline of a free particle¹, so a curve along which there is no acceleration. In the standard formulation of General Relativity, the two curves are required to be identical and this fixes the connection to be the Levi-Civita (or metric) connection (2.7).

¹The affine geodesics we have seen in chapter 2

Assuming the Levi-Civita connections from the beginning is called the metric formalism and the geometry in which the connection is defined by the metric is called *Riemannian Geometry*. Only after Einstein came up with GR, the theory of non-Riemannian geometry was developed. So this gives new opportunities to consider gravity as a geometrical phenomenon [26].

Another motivation to study gravity in a non-Riemannian geometry² can be taken from condensed matter systems [27]. In these systems, the lattice structure gives rise to the emergence of a continuous geometry that can not be described in terms of the metric alone. Other geometric structures that can be related to the connection are necessary to account for physical characteristics (like plasticity) of these systems. If space-time would have some kind of "microstructure" (as expected in approaches to quantum gravity) we should explore the potential impact of geometric structures caused by the connection.

Properties that may arise when working with an affine connection are torsion $T_{\mu\nu}^{\rho} = \Gamma_{\mu\nu}^{\rho} - \Gamma_{\nu\mu}^{\rho}$ and non-metricity $\nabla_{\rho}g_{\mu\nu} \neq 0$. Geometrically we can describe them as the failure of a parallelogram to close (picture 5.1) and the change of the norm of a vector under parallel transport.

There are some reasons to prefer the Levi-Civita connection over a general one. We said that for arbitrary connections, the metric and affine geodesics do not coincide. Therefore, it is not clear what which would represent the trajectories of free particles [29]. Furthermore, since for the Levi-Civita connection both torsion and non-metricity are zero, we do not have to worry about their physical properties.

If and how exactly torsion and non-metricity are physical observable is still a topic about which there is a lot of uncertainty. Recently some articles discussing experiments to test torsion and non-metricity have appeared [27] [30]. In article [27] it is stated that the observable consequences of non-metricity still are largely unknown. It is known that the Dirac field (that describes fermions) couples to the torsion and non-metricity tensors. They use data

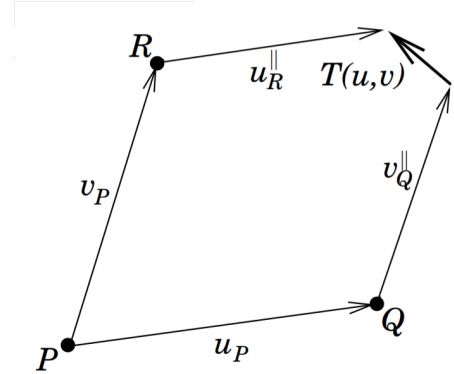


Figure 5.1: Two vector fields u and v are given. Starting at a point P , u and v are transported parallelly along v and u respectively. They become u_R^{\parallel} and v_Q^{\parallel} . We assume the point R and Q are infinitesimally near to P . If torsion is present, a closure failure $T(u, v)$ emerges [28].

²The study of gravity in a non-Riemannian geometry is also called *metric-affine gravity*.

from positron-electron scattering to give a lower bound on the energy scale at which non-metricity can be present of 1 TeV. In [30] they make a difference between torsion that is present in a purely gravitational theory and induced torsion by matter. The first type of torsion is constrained heavily while there are no strong constraints on torsion induced by matter.

We can conclude that there is no a priori reason to exclude torsion and non-metricity from a generic description of spacetime, it is left to matter to probe whether they have physical relevance and lead to observation [31].

Often it is assumed that the Equivalence Principle forces us to choose a symmetric, torsion-free connection. That it tells us that locally the gravitational force can be gauged away by a clever choice of coordinates. Mathematically this translates to the fact that at any point, coordinates can be found such that the connection is zero. Then one could argue that since the torsion is a tensor, this can only be the case if the torsion is zero everywhere [29]. However, because of the symmetry of the affine geodesic equation, the geodesics are only determined by the symmetric part of the connection [32]. Hence the torsion does not affect the geodesic equation.

In this chapter, the Levi-Civita connection will be denoted as $\overset{\circ}{\Gamma}_{\mu\nu}^{\rho}$ and an arbitrary connection as $\Gamma_{\mu\nu}^{\rho}$.

5.2 Palatini formalism for GR

We gave a lengthy introduction to Palatini formalism, now let's see how it works in practice. We start by taking the Einstein-Hilbert action and assume that the connection is independent of the metric. Instead of varying the action with respect to the metric only, we vary it with respect to $g^{\mu\nu}$ and $\Gamma_{\mu\nu}^{\rho}$. This is called the *Palatini* (or first order) formalism. However, what we nowadays call the Palatini formalism, wasn't formulated by Alletio Palatini. His paper is quoted incorrectly quite often, which might be due to the fact that it is written in Italian [33]. Palatini did not work in a nonmetric framework and didn't suggest a possible extension of his work to an arbitrary connection. At the end, it was Einstein that formulated the variational principle that is nowadays known as the Palatini principle.

The Einstein-Hilbert action in the Palatini formalism can be written as [26]:

$$S = \frac{M_P^2}{2} \int d^4x \sqrt{|g|} g^{\mu\nu} R_{\mu\nu}(\Gamma) \quad (5.1)$$

Note that at the moment, we only consider theories the existence of matter that does not have a coupling to the connection in its Lagrangian (such as scalars). We will discuss this point later on.

Varying the action with respect to both the metric and the connection, we obtain two equations of motion. The variation of the action with respect to the metric yields the Einstein equations. The field equation for the connection is [34]:

$$\nabla_\lambda g^{\mu\nu} - \nabla_\sigma g^{\sigma\nu} \delta_\lambda^\mu + \frac{1}{2} g^{\rho\tau} \nabla_\lambda g_{\rho\tau} g^{\mu\nu} - \frac{1}{2} g^{\rho\tau} \nabla_\sigma g_{\rho\tau} g^{\sigma\nu} \delta_\lambda^\mu - T_{\rho\lambda}^\rho g^{\mu\nu} + T_{\rho\sigma}^\rho g^{\sigma\nu} \delta_\lambda^\mu + T_{\sigma\lambda}^\mu g^{\sigma\nu} = 0$$

The first thing that should be pointed out, is that choosing the connection to be the Levi-Civita one, trivially solves this equation. Then the set of equations reduces to the set of equations in the metric formalism. Of course, that does not mean that the Levi-Civita connection is the most generic solution for the connection.

It turns out that the field equation of the connection can be solved to find the most generic solution of the connection. For the procedure, see [34] or a similar calculation in chapter 7. The solution for the connection in Einstein-Hilbert gravity is:

$$\Gamma_{\mu\nu}^\rho = \mathring{\Gamma}_{\mu\nu}^\rho + \frac{1}{D-1} T_{\mu\sigma}^\sigma \delta_\nu^\rho \equiv \mathring{\Gamma}_{\mu\nu}^\rho + \mathcal{A}_\mu \delta_\nu^\rho \quad (5.2)$$

The connection is solved in terms of the Levi-Civita connection and the trace of the torsion, that can be described by a D -dimensional vector $T_{\mu\sigma}^\sigma = (D-1)\mathcal{A}_\mu$. Note that the most generic solution for the connection is not just the Levi-Civita one and is non-metric compatible and non-symmetric.

What are the differences between using the Palatini connection (5.2) and the Levi-Civita connection? The first thing that deserves attention is the fact that the action does not depend on the connection that we use. The Ricci tensor in the Palatini formalism is

$$R_{\mu\nu} = \mathring{R}_{\mu\nu} + \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$$

So the symmetric part of the Ricci tensor is the same in both formalisms. This means that Ricci scalar is invariant $R = \mathring{R}$ and that the Einstein equations do not change using either the metric or Palatini formalism.

Actually, it has been known for some time that the Einstein-Hilbert action remains

invariant under what is called the *projective transformation*

$$\Gamma_{\mu\nu}^{\lambda} \rightarrow \Gamma_{\mu\nu}^{\lambda} + A_{\mu}\delta_{\nu}^{\lambda} \quad (5.3)$$

The fact that the EH-action is invariant under this transformation is called the *projective invariance*. We will see in chapter 7 that this has implications for the use of the Palatini formalism in extended theories of gravity.

The geometric consequences of the Palatini connection

In article [29] it is argued that for the full physical equivalence of the Palatini and metric formalism, the geometry of the Palatini connection must not give any observable differences from metric geometry. Because even though the Einstein equations do not change, a different connection will, in general, define a new affine geodesic. Therefore, the motion of test particles changes. Let's see what happens if we consider geodesics of the Palatini connection. The affine geodesics are defined by the following equation:

$$\dot{x}^{\rho}\nabla_{\rho}\dot{x}^{\mu} = 0 \quad (5.4)$$

The curve $x^{\mu}(\tau)$ is parametrized by an arbitrary parameter τ , which doesn't have to be the proper time. If the connection changes, we expect the geodesic equation to change as well. This would result in particles following different trajectories. Let's write (5.4) in terms of the metric geodesic and \mathcal{A}_{μ}

$$\begin{aligned} \dot{x}^{\rho}\nabla_{\rho}\dot{x}^{\mu} &= \dot{x}^{\rho}\overset{\circ}{\nabla}_{\rho}\dot{x}^{\mu} + \mathcal{A}_{\rho}\dot{x}^{\mu}\dot{x}^{\rho} = 0 \\ \Rightarrow \dot{x}^{\rho}\overset{\circ}{\nabla}_{\rho}\dot{x}^{\mu} &= -\mathcal{A}_{\rho}\dot{x}^{\mu}\dot{x}^{\rho} \end{aligned} \quad (5.5)$$

We know that if we use an arbitrary parameter λ , the metric pregeodesics (are reparametrized geodesics) are calculated by taking the extremum of the arc length

$$s(\lambda) = \int_0^{\lambda} \sqrt{|g_{\mu}\dot{x}^{\mu}\dot{x}^{\nu}|} d\lambda' \quad (5.6)$$

where a dot means the derivative with respect to λ' . In general, the extrema of this functional can be written in the form[32]:

$$\dot{x}^{\rho}\overset{\circ}{\nabla}_{\rho}\dot{x}^{\mu} = \frac{\ddot{S}}{\dot{S}}\dot{x}^{\mu} \quad (5.7)$$

The left side of this equation is purely geometric while the right hand side is dependent on parametrization of the curve. It is always possible to choose a linear reparametrization of the curve such that it vanishes. So we can see that (5.4) are not only the affine geodesics,

but can also be considered as reparametrizations of the Levi-Civita geodesics. We can transform (5.5) in equation (5.7) by the parameter choice [34]:

$$\begin{aligned}s(\lambda) &= \int_0^\lambda e^{-G(\lambda')} d\lambda' \\ G(\lambda) &= \int_0^\lambda \frac{dx^\rho}{d\lambda'} \mathcal{A}_\rho d\lambda'\end{aligned}\tag{5.8}$$

In particular the affine geodesic equation can be transformed in the geodesic equation of the Levi-Civita connection through the parameter change

$$\frac{d\tau}{d\lambda} = e^{-G(\lambda)}\tag{5.9}$$

So the affine geodesics described by the Levi-Civita connection and the Palatini solution are physically equivalent. In article [34] it is argued that besides this fact, also the non-metricity produced by the Palatini connection does not produce any physical observable effects. Although the norm of a vector changes under parallel transport, this is a mathematical construction rather than something that can be physically observed. So there are no observational differences between the geometries defined by the two different connections. However, note that this physical equivalence only holds if we are working with minimally coupled matter Lagrangians. If matter is allowed to couple to the connection, non-gravitational physics is affected and we cannot say that non-metricity has no observational effects [29]. Concluding we can say that the Palatini formalism for the Einstein-Hilbert points to the Levi-Civita connection as the most logical connection to use (instead of an a priori choice). We would like to stress that GR formalated in the metric formalism is equivalent to GR in the Palatini formalism since the equations of motion do not change and the motion of independent test particles (geodesics) coincide. We do not have to assume Riemannian geometry to obtain GR [35]. This gives us a good reason to consider other gravitational theories in non-Riemannian geometry as well.

5.3 Lovelock theory obtained by the Palatini formalism

Like we have seen in chapter 2, in 4 dimensions Einstein Hilbert is the natural choice for the action, but that in dimensions higher than four there is no reason to exclude quadratic or even higher order terms of the Riemann tensor. As long as you take not just some random combination of higher order terms, but specifically the Lovelock combination, we get a ghost-free theory. What happens if we consider Lovelock theory in the Palatini formalism instead of the metric formalism?

We start by considering the equivalence between the metric and Palatini formalisms. In the previous section, we have seen that choosing the Levi-Civita connection as an ansatz in the equations of motion obtained by the Palatini formalism, the equations reduced to the field equation obtained by the metric formalism. If that happens, we say the two formalisms are equivalent. This is not the case for all gravitational theories.

If we consider higher order theories (that are not per se Lovelock combinations) in the Palatini formalism, the Levi-Civita connection does not solve the equation of motion for the connection. In [36] it is proved that the Levi-Civita connection solves the connection field equation only if the Lagrangian is a Lovelock Lagrangian. This means that we now we have two different approaches to obtain Lovelock gravity: we can either demand at highest second order equations of motions, or the equivalence between the Palatini and metric formalism.

As opposed to what we have seen for the Einstein-Hilbert action, the most generic solution for the connection is not known yet for the field equation of the connection in Lovelock gravity. We do know that the Levi-Civita solution is a solution of the Palatini field equation and that the Lovelock Lagrangian is invariant under the same projective symmetry as the Einstein-Hilbert Lagrangian. However, it is unknown if the variational principle gives the Palatini connection for Lovelock.

An open question that remains is whether we are able to solve the equation of motion for the connection in Lovelock gravity and what the geometrical implications of this solution would be. For the Einstein-Hilbert action, the Palatini connection gives the same physics as the Levi-Civita connection. Should that be the case for Lovelock gravity as well? We already know that Levi-Civita is a solution of the field equation in the Palatini formalism. If the generic solution gives physics different than that of the Levi-Civita connection, it gives difficulties interpreting the Palatini formalism. Would it, for example, point out different classes of geodesics?

5.4 Palatini formalism galileons

There is surprisingly little literature about the Palatini formalism for covariant galileons. Probably, this is due to the facts that (i) the galileon theories are quite a new field of research and (ii) scalar gravitational fields make the geometric interpretation of gravity less clear. Articles that do study subjects regarding the Palatini formalism and scalar-tensor theories of gravity are mostly quite recent [35][6][37].

There exist another class of modified gravitational theories, where the Palatini formalism is used a lot. These are $f(R)$ theories, they are not studied in this thesis and are only mentioned because it is a good motivation for studying covariant galileons in the Palatini formalism. In short, $f(R)$ theories of gravity consider the gravitational actions that are functions of the Ricci scalar (so the Einstein-Hilbert action falls in this class as well). It is known that we can obtain scalar-tensor theories from $f(R)$ theories by a conformal transformation of the metric. We know this is quite abstract and refer the reader to [38] [31] [6] for more details. Studying $f(R)$ theories of gravity in the Palatini formalism showed that the use of the Palatini formalism and metric formalism is far from equivalent and gives interesting physics. Furthermore, cosmological predictions differ drastically. Since these theories can be linked to scalar-tensor theories of gravity, we think it is very useful to explore the implications of the Palatini formalism tot covariant galileons. So in chapter 6, we will try to link the Palatini formalism for Lovelock to galileons and in chapter 7 we write the covariant galileon in the Palatini formalism.

5.5 Does matter matter?

In the first part of this chapter, we have only considered the Palatini formalism in settings where matter doesn't couple to the connection. In other words, we only studied matter Lagrangians like a massless scalar field, the electromagnetic field or perfect fluid. This raises the question if we can consider all matter to be completely independent of the connection?

Suppose for the moment that we have a gravitational action $S_g(g, \Gamma)$ depending on both the metric and connection and a matter action $S_m(g, \Psi)$ depending on the matter fields (Ψ), the metric but *not* on the connection. Actually, theories that have actions like this fall in a class of theories known as "metric theories of gravity" [3]. These theories assume a pseudo-Riemannian geometry *a priori* and test particles follow geodesics of the Levi-Civita connection.

Assuming an independent connection and a pseudo-Riemannian geometry sounds like a contradiction. However what happens is that the connection loses its geometrical meaning. Let's see it this way: since the matter Lagrangian is forbidden to couple to the connection, matter automatically follows metric geodesics. Since the connection no longer defines free fall of particles, it loses its geometric interpretation and reduces to some auxiliary field [38]. We can compare this to adding a gravitational scalar field to our theory of gravity, as we have seen before. Just like the scalar field influences the curvature of spacetime by matter without affecting the geometry, connections that are not coupled to matter work in

the same way. The connection field simply participates in the way matter curves spacetime.

So in order to keep the geometrical meaning of the connection, we have to let matter couple to the connection³ and specify if we take matter into account (and which kind of matter we consider). Only in this case, we can assume test particles to follow affine geodesics, even if we don't consider the specific forms of matter that couple to the connection. Furthermore, generally the field equation for the connection will depend on the matter action as well as the gravitational action. As a result, matter doesn't only tells spacetime how to curve but also how to twirl [38].

Let's end with a short overview of which types of matter do and which do not couple to the connection. The scalar and electromagnetic field do not couple to the connection and therefore will not introduce torsion or non-metricity. Another situation that is very interesting are matter configurations in which matter is treated macroscopically. The consideration of a universe that is filled with a perfect fluid is common in cosmology. Actually the matter action of a perfect fluid depends on three scalars (the energy density, pressure and velocity potential) and therefore does not depend on the connection [4]. A field that does have an explicit dependence on the connection is the Dirac field, that describes fermions. So a Dirac field will potentially produce torsion and non-metricity in the Palatini formalism of General Relativity.

Conclusions

In this chapter, we have introduced the Palatini formalism. The Palatini formalism gives us a strong argument for using the Levi-Civita connection in general relativity. Besides that the Palatini and metric formalism are actually equivalent for GR in vacuum, so that gives us a reason to consider other gravitational theories in the Palatini formalism as well. The only combinations of higher order derivative terms that are equivalent under the Palatini and metric formalism are exactly Lovelock combinations. For galileon theories, there isn't known a lot about the implications of using the Palatini formalism, but probably it will change the physics. Finally, we tried to emphasize the philosophical importance of stating if we allow matter to couple to the connection or not.

³Sometimes this is called the *metric-affine* formalism instead of the Palatini formalism [2] [38].

Dimensional reduction of the first order formalism

In the previous chapter we have seen that the Palatini formalism is an alternative to the metric formalism for analyzing gravitational theories. Furthermore in chapter 4 we showed that we can relate the Lovelock to galileons by Kaluza-Klein dimensional reduction. What would happen if we combine these two facts and try to find the dimensional reduction of Lovelock in the Palatini formalism?

Besides a purely theoretic interest in the possibility of reducing a theory in the first order formalism, doing this dimensional reduction might provide us with more information about two things. The first is that we possibly learn something about the Palatini formalism for Lovelock. We don't know the generic solution for the connection in Gauss-Bonnet gravity and the dimensional reduction might point certain aspects of this out. In the dimensionally reduced version of Lovelock-Palatini, we might see relations that were harder to discover in the full theory.

We know that using the Palatini formalism and substituting the Levi-Civita solution gives us an equivalent set of equations as the metric formalism in Lovelock gravity. In the reduced theory this must still be true and it might not be trivial, but instead happens because of a particular combination of scalar terms. So besides learning something about the Palatini formalism for Lovelock, dimensional reduction of the first order formalism might give us more information about the properties of scalar-tensor theories or even put constraints on it.

If we reduce the first order formalism for Einstein Hilbert gravity, we expect that the imposing Levi-Civita will solve the equations of motions in the reduced theory and that we can solve these equations of motion exactly. This should give us the same result as in the higher dimensional theory and therefore is a good check to see if we managed to

successfully reduce the first order formalism. So in the next section we will start by finding a way to reduce Einstein-Hilbert Lagrangian in the first order formalism. After that, we apply dimensional reduction on the first order formalism of Gauss-Bonnet.

6.1 Dimensional reduction of an arbitrary connection

How can we actually reduce a Lagrangian written in the first order formalism? From chapter 4 we know how to reduce g , but the whole idea is to take the connection independent of the metric. The connection itself is not a tensor, which makes it quite difficult to reduce dimensionally. We conclude that again is useful to write an arbitrary connection (Γ) as the sum of the Levi-Civita connection ($\overset{\circ}{\Gamma}$) and a tensor in the following way

$$\Gamma_{\mu\nu}{}^\rho = \overset{\circ}{\Gamma}_{\mu\nu}{}^\rho + K_{\mu\nu}{}^\rho \quad (6.1)$$

Since we already calculated dimensional reduction with the Levi-Civita connection, for our new theory we only have to calculate the part that corresponds to the tensor K . So let's figure out how we can calculate the reduction of a tensor.

This is really non-trivial. We can decompose a three-rank tensor into a new tensor, three matrices, three vectors and a scalar. However, just as with the metric, we see that if we just assume for example $\hat{K}_{xx}{}^x = \phi$, $K_{x\mu}{}^x = A_\mu$ etc, this doesn't work out because these properties don't transform like scalars, vectors or tensors. So we have to find a way to construct objects out of elements of $\hat{K}_{\mu\nu}{}^\rho$ that transform correctly. It turns out that the most natural way to decompose a three tensor is with all indices down. Let's look at how the different parts of the tensor transform using equation (D.4):

$$\delta \hat{K}_{xxx} = \xi^\gamma \partial_\gamma \hat{K}_{xxx} = \delta \hat{K}_{xxx} \quad (6.2)$$

$$\delta \hat{K}_{\mu xx} = \xi^\gamma \partial_\gamma \hat{K}_{\mu xx} + \partial_\mu \xi^\gamma \hat{K}_{\gamma xx} + \partial_\mu \xi^x \hat{K}_{xx} \quad (6.3)$$

$$= \delta \hat{K}_{\mu xx} + \partial_\mu \xi^x \hat{K}_{xx} \quad (6.4)$$

$$\delta \hat{K}_{\mu\nu x} = \xi^\gamma \partial_\gamma \hat{K}_{\mu\nu x} + \partial_\mu \xi^\gamma \hat{K}_{\gamma\nu x} + \partial_\mu \xi^x \hat{K}_{x\nu x} + \partial_\nu \xi^\gamma \hat{K}_{\mu\gamma x} + \partial_\nu \xi^x \hat{K}_{\mu x} \quad (6.5)$$

$$= \delta \hat{K}_{\mu\nu x} + \partial_\mu \xi^x \hat{K}_{x\nu x} + \partial_\nu \xi^x \hat{K}_{\mu x} \quad (6.6)$$

$$\delta \hat{K}_{\mu\nu\rho} = \xi^\gamma \partial_\gamma \hat{K}_{\mu\nu\rho} + \partial_\mu \xi^\gamma \hat{K}_{\gamma\nu\rho} + \partial_\mu \xi^x \hat{K}_{x\nu\rho} + \partial_\nu \xi^\gamma \hat{K}_{\mu\gamma\rho} \quad (6.7)$$

$$+ \partial_\nu \xi^x \hat{K}_{\mu x\rho} + \partial_\rho \xi^\gamma \hat{K}_{\mu\nu\gamma} + \partial_\rho \xi^x \hat{K}_{\mu\nu x} \quad (6.8)$$

$$= \delta \hat{K}_{\mu\nu\rho} + \partial_\mu \xi^x \hat{K}_{x\nu\rho} + \partial_\nu \xi^x \hat{K}_{\mu x\rho} + \partial_\rho \xi^x \hat{K}_{\mu\nu x} \quad (6.9)$$

$$(6.10)$$

As we can see \hat{K}_{xxx} transforms as a scalar in lower dimensions, but the other components of \hat{K} do not transform as vectors and tensors. The first thing that comes to mind, is to

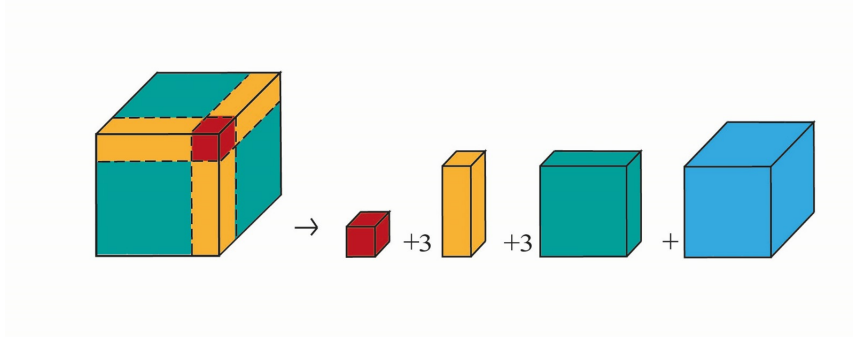


Figure 6.1: This is how we can imagine the dimensional reduction of a tensor. The big cube (\hat{K}) will reduce to one scalar Ψ (the small red cube), three vectors B, C, D (the yellow parts), three two-index tensors L, M, N (the green parts) and one three-index tensor K (the remaining cube).

construct objects in the same way as we did for the metric. Remember that we constructed the Kaluza-Klein vector as $\frac{\hat{g}_{\mu\omega}}{\hat{g}_{\omega\omega}}$. In the same way, we could define the object $\frac{\hat{K}_{\mu xx}}{\hat{K}_{xxx}}$ which transforms as covector under an $U(1)$ potential:

$$\begin{aligned}\hat{\delta}\left(\frac{\hat{K}_{\mu xx}}{\hat{K}_{xxx}}\right) &= \xi^\gamma \partial_\gamma \left(\frac{\hat{K}_{\mu xx}}{\hat{K}_{xxx}}\right) + \partial_\mu \xi^\gamma \frac{\hat{K}_{\gamma xx}}{\hat{K}_{xxx}} + \partial_\mu \xi^x \\ &= \delta\left(\frac{\hat{K}_{\mu xx}}{\hat{K}_{xxx}}\right) + \partial_\mu \xi^x\end{aligned}\tag{6.11}$$

However it is important to realize that where we knew that $\hat{g}_{\omega\omega}$ is never zero, \hat{K}_{xxx} can be zero. Using a vector composed in the way that we proposed gives us an inconsistent theory: in the lower dimensional theory we are able to find solutions that don't exist in the higher dimensional theory by putting $K_{xxx} = 0$ in the equations of motion. Instead, we take:

$$\hat{K}_{xxx} = e^{3\beta\phi}\Psi \quad (6.12)$$

$$\hat{K}_{\mu xx} = e^{\phi(\alpha+2\beta)}B_\mu + e^{3\beta\phi}\Psi A_\mu\gamma \quad (6.13)$$

$$\hat{K}_{x\nu x} = e^{\phi(\alpha+2\beta)}C_\nu + e^{3\beta\phi}\Psi A_\nu\gamma \quad (6.14)$$

$$\hat{K}_{xx\rho} = e^{\phi(\alpha+2\beta)}D_\rho + e^{3\beta\phi}\Psi A_\rho\gamma \quad (6.15)$$

$$\hat{K}_{x\nu\rho} = e^{\phi(2\alpha+\beta)}L_{\nu\rho} + e^{\phi(\alpha+2\beta)}A_\rho C_\nu\gamma + e^{\phi(\alpha+2\beta)}A_\nu D_\rho\gamma + e^{3\beta\phi}\Psi A_\nu A_\rho\gamma^2 \quad (6.16)$$

$$\hat{K}_{\mu x\rho} = e^{\phi(2\alpha+\beta)}M_{\mu\rho} + e^{\phi(\alpha+2\beta)}A_\mu D_\rho\gamma + e^{\phi(\alpha+2\beta)}A_\rho B_\mu\gamma + e^{3\beta\phi}\Psi A_\mu A_\rho\gamma^2 \quad (6.17)$$

$$\hat{K}_{\mu\nu x} = e^{\phi(2\alpha+\beta)}N_{\mu\nu} + e^{\phi(\alpha+2\beta)}A_\mu C_\nu\gamma + e^{\phi(\alpha+2\beta)}A_\nu B_\mu\gamma + e^{3\beta\phi}\Psi A_\mu A_\nu\gamma^2 \quad (6.18)$$

$$\hat{K}_{\mu\nu\rho} = e^{3\alpha\phi}K_{\mu\nu\rho} + e^{\phi(2\alpha+\beta)}\gamma A_\mu L_{\nu\rho} + e^{\phi(2\alpha+\beta)}\gamma A_\nu M_{\mu\rho} \quad (6.19)$$

$$+ e^{\phi(2\alpha+\beta)}\gamma A_\rho N_{\mu\nu} + e^{\phi(\alpha+2\beta)}\gamma^2 A_\mu A_\nu D_\rho + e^{\phi(\alpha+2\beta)}\gamma^2 A_\mu A_\rho C_\nu \quad (6.20)$$

$$+ e^{\phi(\alpha+2\beta)}A_\nu A_\rho B_m u\gamma^2 + e^{3\beta\phi}\Psi A_\mu A_\nu A_\rho\gamma^3 \quad (6.21)$$

$$(6.22)$$

In these equations A_μ is again the Kaluza-Klein vector as introduced in chapter 4. We can see that all our new objects $(\Psi, B, C, D, L, M, N, K)$ transform as proper scalars, 1-forms and tensors, without any gauge freedom. We chose the factors of e in such a way that the above expression simplifies enormously in Vielbein coordinates (even without setting A_μ to zero!):

$$\hat{K}_{zzz} = \Psi \quad \hat{K}_{azz} = B_a \quad (6.23)$$

$$\hat{K}_{zbz} = C_b \quad \hat{K}_{zzc} = D_c \quad (6.24)$$

$$\hat{K}_{abz} = N_{ab} \quad \hat{K}_{azc} = M_{ac} \quad (6.25)$$

$$\hat{K}_{zbc} = L_{ab} \quad \hat{K}_{abc} = K_{abc} \quad (6.26)$$

This are the objects that we will use for the dimensional reduction of the Einstein-Hilbert Lagrangian.

6.2 The reduction of the first order formalism in Einstein-Hilbert gravity

We can write the Ricci tensor with a general connection in terms of the Ricci tensor with a Levi-Civita connection and our previously defined tensor K , see also [32]:

$$R_{\mu\rho} = R_{\mu\gamma\rho}{}^\gamma = \mathring{R}_{\mu\rho} + \mathring{\nabla}_\mu K_{\gamma\rho}{}^\gamma - \mathring{\nabla}_\gamma K_{\mu\rho}{}^\gamma + K_{\mu\sigma}{}^\gamma K_{\gamma\rho}{}^\sigma - K_{\gamma\sigma}{}^\gamma K_{\mu\rho}{}^\sigma \quad (6.27)$$

By using partial integration, we can get rid of the derivative terms in the Einstein Hilbert actions. Therefore, the action reduces to

$$S = \int d^{N+1}x \sqrt{|g|} (\mathring{R} + g^{\mu\rho} (K_{\mu\sigma}{}^\gamma K_{\gamma\rho}{}^\sigma - K_{\gamma\sigma}{}^\gamma K_{\mu\rho}{}^\sigma)) \quad (6.28)$$

Note that the partial integration only was possible because (6.27) is written in terms of the Levi-Civita covariant derivative. The result that we obtained now can really easy be dimensionally reduced, because we already know the dimensional reduction of the 'ordinary' Ricci scalar. What we eventually end up with can be written in the Einstein frame in the following way (as before, we set $A_\mu = 0$):

$$\begin{aligned} S = \int d^N \sqrt{|g|} & \left(\mathring{R} + \frac{1}{2} (\partial\phi)^2 + K_{ab}{}^c K_c{}^{ab} - K_{ab}{}^a K_c{}^{cb} \right. \\ & - L_b{}^c M_c{}^b - M_a{}^c N_c{}^a - N_{ab} L^{ab} + M_a{}^a N_c{}^c + K_c{}^{cb} C_b + K_{ab}{}^a D^b \\ & \left. - \Psi (N_c{}^c + M_a{}^a) + D^c B_c + B_a C^a \right) \end{aligned} \quad (6.29)$$

The first thing to note is that if we take the equations of motion, every term in the field equations of the reduced quantities will at least contain one factor of \hat{K} . Therefore, if we take Levi-Civita as an ansatz the equation will be trivially solved and we recover the lower dimensional theory in metric formalism.

We can solve the equations of motion for our reduced quantities, which results do we expect? Since the solution for K for the Einstein-Hilbert action is $\hat{K}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \hat{\mathcal{A}}_{\hat{\mu}} \hat{g}_{\hat{\nu}\hat{\rho}}$. Therefore, we expect

$$\begin{aligned} L_{\nu\rho} &= -\Psi g_{\nu\rho} \\ K_{\mu\nu\rho} &= -B_\mu g_{\nu\rho} \end{aligned} \quad (6.30)$$

And that all other lower dimensional components (M, N, C, D) are zero.

Let's take the variation of the action (6.29) with respect to all the different tensors:

$$\begin{aligned}
0 &= N_\mu{}^\mu + M_\nu{}^\nu \\
0 &= D^\nu + C^\nu \\
0 &= B^\nu + K_\mu{}^{\mu\nu} \\
0 &= B^\nu + K_\mu{}^{\nu\mu} \\
0 &= M^{\mu\rho} + N^{\rho\mu} \\
0 &= -L^{\mu\nu} - N^{\mu\nu} + g^{\nu\mu} N_\rho{}^\rho - g^{\nu\mu} \Psi \\
0 &= -M^{\mu\rho} - L^{\rho\mu} + g^{\rho\mu} M_\nu{}^\nu - g^{\rho\mu} \Psi \\
0 &= K^{\mu\nu\rho} + K^{\rho\mu\nu} - g^{\nu\mu} K_\lambda{}^{\lambda\rho} - g^{\nu\rho} K_d + g^{\nu\rho} C^\mu + g^{\nu\mu} D^\rho
\end{aligned} \tag{6.31}$$

It is possible to solve this system of equations and we get: $C = D = M = N = 0$ are zero, $K_{\mu\nu}{}^\rho = -B_\mu \delta_\nu^\rho$ and $L_{\mu\nu} = -\Psi g_{\mu\nu}$. This is exactly what we expected. So we can conclude that we managed to successfully reduce a theory with arbitrary connection. Now we are going to reduce the next Lovelock term: Gauss-Bonnet.

If the reduced components have still some geometrical meaning in the lower dimension remains unclear.

6.3 The reduction of Gauss-Bonnet in the Palatini formalism

The first thing to notice is that with an arbitrary connection, we can define two independent Ricci tensors: $R_{\mu\nu} = R_{\mu\rho\nu}{}^\rho$ y $\bar{R}_{\mu\nu} = R_{\mu\nu\rho}{}^\rho$. Furthermore, while before we found $\hat{R}_{zbc}{}^d = 0$, this is not true anymore. Therefore, the formula for the reduction that we used previously (4.32) no longer is valid. Instead, we use:

$$\begin{aligned}
\hat{\mathcal{R}}_2 &= \delta_{\hat{b}_1 \dots \hat{b}_4}^{\hat{a}_1 \dots \hat{a}_4} \hat{R}_{\hat{a}_1 \hat{a}_2}{}^{\hat{b}_1 \hat{b}_2} \hat{R}_{\hat{a}_3 \hat{a}_4}{}^{\hat{b}_3 \hat{b}_4} \\
&= \delta_{b_1 b_2 b_3 b_4}^{a_1 a_2 a_3 a_4} \hat{R}_{a_1 a_2}{}^{b_1 b_2} \hat{R}_{a_3 a_4}{}^{b_3 b_4} + \delta_{b_1 b_2 b_3}^{a_1 a_2 a_2} \hat{R}_{a_1 a_2}{}^{b_1 b_2} \hat{R}_{a_3 z}{}^{b_3 z} \\
&+ \delta_{b_1 b_2 b_3}^{a_1 a_2 a_2} \hat{R}_{a_1 a_2}{}^{b_1 b_2} \hat{R}_{z a_3}{}^{z b_3} + \delta_{b_1 b_2 b_3}^{a_1 a_2 a_2} \hat{R}_{z a_1}{}^{b_1 b_2} \hat{R}_{a_2 a_3}{}^{z b_3} \\
&+ \delta_{b_1 b_2 b_3}^{a_1 a_2 a_2} \hat{R}_{a_1 z}{}^{b_1 b_2} \hat{R}_{a_2 a_3}{}^{b_3 z}
\end{aligned} \tag{6.32}$$

as before, we use

$$R_{\mu\nu}{}^{\rho\gamma} = \hat{R}_{\mu\nu}{}^{\rho\gamma} + \hat{\nabla}_\mu K_\nu{}^{\rho\gamma} - \hat{\nabla}_\nu K_\mu{}^{\rho\gamma} + K_\nu{}^{\rho\sigma} K_{\mu\sigma}{}^\gamma - K_\mu{}^{\rho\sigma} K_{\nu\sigma}{}^\gamma \tag{6.33}$$

What we find is an extremely long and complicated expression, that we cannot easily rewrite in a shorter form. Even if we set certain parts of the reduced tensor to zero, which

probably aren't even allowed truncations, the expression remains horrible.

Contrary to the reduction of the Einstein-Hilbert action, this time we obtain terms that contain a product of the scalar field and some component of \hat{K} . This is due to the product terms of $\hat{R}\hat{K}$ and products of the derivatives $\nabla\hat{R}$ or $\nabla\hat{K}$ with \hat{K} .

Our goal was to obtain more information about either the solution for the connection in Lovelock gravity or about the implementation of the Palatini formalism in galileon theories. If we have pure scalar theories, there is only room for a connection with one index¹. However, even if we set $K = L = M = N = 0$ and look what remains then we still can't find a relation between the scalar field and our vectors. So obviously this isn't a practical way to get more information or insight in a possible first-order formalism of galileon theories.

A thing that we should check as well is if Levi Civita ($K = 0$) still is a solution for the dimensional reduction of Lovelock. We know in the higher dimensional theory the Palatini and metric formalism are equivalent, let's see how this works out in the lower dimensional theory. As we have seen, for the Einstein Hilbert action it is trivial that $K = 0$ is a solution for the equations of motions of the connection tensor. There are only quadratic terms in the action, so every term in the equations of motions of the reduced Lagrangian will have at least one component of \hat{K} .

Let's see how this functions with the Lovelock Lagrangian. First look at the Lagrangian in higher dimensions. In the next expression, we omitted the hats for clarity, but every object and index lives in the higher dimension:

$$\begin{aligned}
\mathcal{L} &= \sqrt{|g|} (\delta_{b_1 b_2 b_3 b_4}^{a_1 a_2 a_3 a_4} R_{a_1 a_2}{}^{b_1 b_2} R_{a_3 a_4}{}^{b_3 b_4}) \\
&= \delta_{b_1 \dots b_4}^{a_1 \dots a_4} (\mathring{R}_{a_1 a_2}{}^{b_1 b_2} + 2\mathring{\nabla}_{a_1} K_{a_2}{}^{b_1 b_2} + 2K_{a_1 c}{}^{b_2} K_{a_2}{}^{b_1 c}) \\
&\quad (\mathring{R}_{a_3 a_4}{}^{b_3 b_4} + 2\mathring{\nabla}_{a_3} K_{a_4}{}^{b_3 b_4} + 2K_{a_3 d}{}^{b_4} K_{a_4}{}^{b_3 d}) \\
&= \delta_{b_1 \dots b_4}^{a_1 \dots a_4} (\mathring{R}_{a_1 a_2}{}^{b_1 b_2} \mathring{R}_{a_3 a_4}{}^{b_3 b_4} + 4\mathring{R}_{a_1 a_2}{}^{b_1 b_2} \mathring{\nabla}_{a_3} K_{a_4}{}^{b_3 b_4} + 4\mathring{R}_{a_1 a_2}{}^{b_1 b_2} K_{a_3 d}{}^{b_4} K_{a_4}{}^{b_3 d} \\
&\quad + 4\mathring{\nabla}_{a_1} K_{a_2}{}^{b_1 b_2} \mathring{\nabla}_{a_3} K_{a_4}{}^{b_3 b_4} + 8\mathring{\nabla}_{a_1} K_{a_2}{}^{b_1 b_2} K_{a_3 d}{}^{b_4} K_{a_4}{}^{b_3 d} + 4K_{a_3 d}{}^{b_4} K_{a_4}{}^{b_3 d} K_{a_1 c}{}^{b_2} K_{a_2}{}^{b_1 c})
\end{aligned} \tag{6.34}$$

Looking at this equation, it isn't directly obvious that $K = 0$ will solve the equations of motions, since we now have the term $R\nabla K$ and $\nabla K\nabla K$ which won't per se produce terms with more than one factor of K . However, we can do some smart manipulations:

$$\begin{aligned}
\int \sqrt{|g|} \delta_{b_1 \dots b_4}^{a_1 \dots a_4} \mathring{R}_{a_1 a_2}{}^{b_1 b_2} \mathring{\nabla}_{a_3} K_{a_4}{}^{b_3 b_4} &= - \int \sqrt{|g|} \delta_{b_1 \dots b_4}^{a_1 \dots a_4} \mathring{\nabla}_{a_3} \mathring{R}_{a_1 a_2}{}^{b_1 b_2} K_{a_4}{}^{b_3 b_4} \\
&= - \int \sqrt{|g|} \delta_{b_1 \dots b_4}^{a_1 \dots a_4} \mathring{\nabla}_{[a_3} \mathring{R}_{a_1 a_2]}{}^{b_1 b_2} K_{a_4}{}^{b_3 b_4}
\end{aligned} \tag{6.35}$$

¹We will not further investigate the Palatini formalism for flat galileons in this thesis and for coherence did not explain it in detail. However, for the reader interested in how we could implement some kind of Palatini formalisms in flat galileon theories, the paper [16] is recommended.

This gives us zero by the Bianchi identity! So we don't have to worry about this term.

$$\begin{aligned} \int \sqrt{|g|} \delta_{b_1 \dots b_4}^{a_1 \dots a_4} \overset{\circ}{\nabla}_{a_1} K_{a_2}{}^{b_1 b_2} \overset{\circ}{\nabla}_{a_3} K_{a_4}{}^{b_3 b_4} &= - \int \sqrt{|g|} \delta_{b_1 \dots b_4}^{a_1 \dots a_4} K_{a_2}{}^{b_1 b_2} \overset{\circ}{\nabla}_{a_1} \overset{\circ}{\nabla}_{a_3} K_{a_4}{}^{b_3 b_4} \\ &= - \int \sqrt{|g|} \delta_{b_1 \dots b_4}^{a_1 \dots a_4} K_{a_2}{}^{b_1 b_2} \overset{\circ}{\nabla}_{[a_1} \overset{\circ}{\nabla}_{a_3]} K_{a_4}{}^{b_3 b_4} \end{aligned} \quad (6.36)$$

The anticommutator of two derivatives, gives us the Riemann tensor. So we can write this as a product of the Riemann tensor and two K tensors. So in the end, every term in the Lagrangian will be a product with a least two factors of K in it. Therefore, every term in the equations of motion will contain at least one part of the reduced tensor K and setting them all to zero will trivially solve the equations of motion.

Conclusions

Concluding we can say that we have succeeded in the finding the dimensional reduction of a Lagrangian in the Palatini formalism. Doing this with the Einstein-Hilbert action didn't teach us anything new, as expected. For Gauss-Bonnet, the reduction soon turned out to be too complicated to use it as a way to learn more about the Palatini formalism in the higher dimension. Furthermore, we did not succeed in finding a link between the first order formalism of Gauss-Bonnet and galileons in any way. However, we wouldn't have known if we hadn't tried.

The Palatini formalism for the cubic covariant galileon

In the previous chapter, unfortunately we did not get much wiser about the Palatini formalism for galileons by using dimensional reduction as a tool to relate Lovelock in the Palatini formalism with galileons. In this chapter we take a different approach and apply the Palatini formalism to the cubic covariant galileon (\mathcal{L}_3 , see chapter 3). The first reason to look at this variant of the galileon, is simplicity. The cubic galileon is the first covariant galileon with a second derivative scalar field, and therefore the first galileon theory that has a connection in its action. Therefore it is the simplest example galileon term to which we can apply the Palatini formalism. Starting here will help us to gain insight in how the first order formalism works for gravitational theories with more than one gravitational field.

Furthermore, in recent years research interest in the cosmological predictions of the cubic galileon has gotten renewed interest [14], and treating the galileon in a first order formalism might have an influence on the cosmological predictions of the cubic galileon.

7.1 A problem arises for the cubic galileon in the Palatini formalism

The simplest¹ form of this galileon in metric formalism is $\mathcal{L}_3 = -\alpha X g^{\mu\nu} \nabla_\nu \nabla_\mu \phi$, where $X = (\partial\phi)^2 = \partial^\mu \phi \partial_\mu \phi$. For clarity, we omit the kinetic term $((\partial\phi)^2)$ at the moment. We can do this since it doesn't add to the equation of motion for the connection and can always be added later. We consider the following action:

¹Remember from chapter 3 that we can write the Lagrangian in more than one way, differing by a total derivative.

$$S = \int d^4x \sqrt{|g|} \left(\frac{M_P^2}{2} \mathring{R} - \alpha (\partial\phi)^2 g_{\mu\nu} \mathring{\nabla}_\nu \mathring{\nabla}_\mu \phi \right) \quad (7.1)$$

In the first order formalism, we change the Levi-Civita derivatives to covariant derivatives defined by an arbitrary connection ($\mathring{\nabla}_\nu \mathring{\nabla}_\mu \phi \rightarrow \nabla_\nu \nabla_\mu \phi$). If we do this however, quite a big problem arises. To see why the notation of the galileon in the first order formalism isn't straightforward, let's calculate the equation of motion with respect to an arbitrary connection for the above Lagrangian:

$$\begin{aligned} \frac{M_P^2}{2} (\nabla_\lambda g^{\mu\nu} - \nabla_\rho g^{\rho\nu} \delta_\lambda^\mu + \frac{1}{2} g^{\mu\nu} g^{\sigma\tau} \nabla_\lambda g_{\sigma\tau} - \frac{1}{2} g^{\rho\nu} g^{\sigma\tau} \nabla_\rho g_{\sigma\tau} \delta_\lambda^\mu - g^{\mu\nu} T_{\sigma\lambda}^\sigma + g^{\rho\nu} T_{\rho\lambda}^\mu \\ + g^{\rho\nu} T_{\sigma\rho}^\sigma \delta_\lambda^\mu) + \alpha (\partial\phi)^2 \partial_\lambda \phi g^{\mu\nu} = 0 \end{aligned} \quad (7.2)$$

When we take the δ_ν^λ trace of this equation, we end up with $X \partial^\mu \phi = 0$. This requirement puts a huge constraint on the scalar field and in fact eliminates our whole galileon from the action. Does this mean that there is no such a thing as a galileon theory in first order formalism or is there a deeper lying reason for this inconsistency and can we do something about it?

This inconsistency stems from the fact that, as we have seen earlier, the Einstein-Hilbert action is invariant under the projective transformation

$$\Gamma_{\mu\nu}^\lambda \rightarrow \Gamma_{\mu\nu}^\lambda + A_\mu \delta_\nu^\lambda \quad (7.3)$$

The galileon Lagrangian however is obviously not invariant under this transformation, since it only contains one connection term. In order to remove the strong constraint on the theory we found, it is necessary to construct a projectively invariant galileon [35]. So our goal is to find an action that **(i)** is invariant under (7.3) and **(ii)** simplifies to (7.1) for the Levi-Civita connection.

Note that we can formulate different terms in the Palatini formalism that reduce to $g^{\mu\nu} \mathring{\nabla}_\nu \mathring{\nabla}_\mu \phi$ in the metric formalism:

$$g^{\mu\nu} \nabla_\nu \nabla_\mu \phi \qquad \nabla_\mu (g^{\mu\nu} \nabla_\nu \phi)$$

This is due to the fact that covariant derivative of the metric tensor isn't zero when using an arbitrary connection. Taking some combination of $\nabla^\mu \nabla_\mu \phi$ and $\nabla_\mu \nabla^\mu \phi$ looks as a promising way to construct a projectively invariant galileon Lagrangian.

The projectively invariant cubic galileon

Consider action (7.1) in the Palatini formalism with a linear combination of the previous mentioned derivative terms:

$$S = \int d^D x \sqrt{g} \left(\frac{M_P^2}{2} R - \alpha \gamma g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \alpha(1-\gamma) g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi \nabla_\mu (g^{\mu\nu} \nabla_\nu \phi) \right) \quad (7.4)$$

To easiest way to see if this Lagrangian solves the problem stated above is to calculate the equation of motion with respect to the connection:

$$\begin{aligned} \frac{M_P^2}{2} \left(\nabla_\lambda g^{\mu\nu} - \nabla_\rho g^{\rho\nu} \delta_\lambda^\mu + \frac{1}{2} g^{\mu\nu} g^{\sigma\tau} \nabla_\lambda g_{\sigma\tau} - \frac{1}{2} g^{\rho\nu} g^{\sigma\tau} \nabla_\rho g_{\sigma\tau} \delta_\lambda^\mu - g^{\mu\nu} T_{\sigma\lambda}^\sigma + g^{\rho\nu} T_{\rho\lambda}^\mu \right. \\ \left. + g^{\rho\nu} T_{\sigma\rho}^\sigma \delta_\lambda^\mu \right) + \alpha \gamma (\partial\phi)^2 \partial_\lambda \phi g^{\mu\nu} - \alpha(1-\gamma) (\partial\phi)^2 \partial^\nu \phi \delta_\lambda^\mu = 0 \end{aligned} \quad (7.5)$$

If we now again take the δ_ν^λ trace we get

$$\gamma (\partial\phi)^2 \partial^\mu \phi - (1-\gamma) (\partial\phi)^2 \partial^\mu \phi = 0 \quad (7.6)$$

This time the equation can be solved without constraining the galileon theory, by choosing $\gamma = \frac{1}{2}$. Note that we cannot freely choose γ between one and zero. Our projectively invariant galileon Lagrangian is:

$$\mathcal{L}_3^{(pi)} = -\alpha \frac{1}{2} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \alpha \frac{1}{2} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi \nabla_\mu (g^{\mu\nu}) \nabla_\nu \phi \quad (7.7)$$

It is an easy check to see that this Lagrangian indeed is invariant under the transformation (7.3) and reduces to the 'normal' cubic covariant galileon for the Levi-Civita connection.

7.2 Solving the field equation of the connection

Now we are ready to investigate the cubic galileon in the Palatini formalism. First, we will solve the equation of the equation of motion for $\Gamma_{\mu\nu}^\lambda$. We will follow the procedure of [34] where this is done for the Einstein-Hilbert Lagrangian. We start with equation (7.5) with $\gamma = \frac{1}{2}$ and calculate the equation of motion with respect to the connection:

$$\begin{aligned} \nabla_\lambda g^{\mu\nu} - \nabla_\rho g^{\rho\nu} \delta_\lambda^\mu + \frac{1}{2} g^{\mu\nu} g^{\sigma\tau} \nabla_\lambda g_{\sigma\tau} - \frac{1}{2} g^{\rho\nu} g^{\sigma\tau} \nabla_\rho g_{\sigma\tau} \delta_\lambda^\mu - g^{\mu\nu} T_{\sigma\lambda}^\sigma + g^{\rho\nu} T_{\rho\lambda}^\mu \\ + g^{\rho\nu} T_{\sigma\rho}^\sigma \delta_\lambda^\mu + \frac{\alpha}{M_P^2} (\partial\phi)^2 \partial_\lambda \phi g^{\mu\nu} - \frac{\alpha}{M_P^2} (\partial\phi)^2 \partial^\nu \phi \delta_\lambda^\mu = 0 \end{aligned} \quad (7.8)$$

To simplify this equation we take the δ^μ_λ trace:

$$\nabla_\rho g^{\rho\nu} = -\frac{1}{2}g^{\sigma\tau}\nabla^\nu g_{\sigma\tau} + \frac{D-2}{D-1}g^{\rho\nu}T^\sigma_{\sigma\rho} - \frac{\alpha}{M_P^2}(\partial\phi)^2\partial^\nu\phi \quad (7.9)$$

Where D is the dimension. Substituting this trace in the first equation gives:

$$\nabla_\lambda g^{\mu\nu} - g^{\mu\nu}T^\sigma_{\sigma\lambda} + g^{\rho\nu}T^\mu_{\rho\lambda} + \frac{1}{D-1}g^{\rho\nu}T^\sigma_{\sigma\rho}\delta^\mu_\lambda + \frac{1}{2}g^{\mu\nu}g^{\sigma\tau}\nabla_\lambda g_{\sigma\tau} + \frac{\alpha D}{2}(\partial\phi)^2\partial_\lambda\phi g^{\mu\nu} = 0 \quad (7.10)$$

We should check if this equation is still equivalent to the previous one to make sure no information is lost. We do this by taking the trace of this equation and we see that it gives exactly (7.9) again. So we get back the condition that we used and we can treat our equation as equivalent. From this new equation, we take the $g_{\mu\nu}$ trace:

$$\frac{1}{2}g^{\sigma\tau}\nabla_\lambda g_{\sigma\tau} = \frac{D}{D-1}T^\sigma_{\sigma\lambda} - \frac{\alpha D}{M_P^2(D-2)}(\partial\phi)^2\partial_\lambda\phi \quad (7.11)$$

Substituting this in (7.10)

$$\nabla_\lambda g^{\mu\nu} + \frac{1}{D-1}g^{\mu\nu}T^\sigma_{\sigma\lambda} + \frac{1}{D-1}g^{\rho\nu}T^\sigma_{\sigma\rho}\delta^\mu_\lambda + g^{\rho\nu}T^\mu_{\rho\lambda} + \frac{2\alpha}{M_P^2(D-2)}(\partial\phi)^2\partial_\lambda\phi g_{\mu\nu} = 0 \quad (7.12)$$

Lowering the indices:

$$-\nabla_\lambda g_{\mu\nu} + \frac{1}{D-1}g_{\mu\nu}T^\sigma_{\sigma\lambda} + \frac{1}{D-1}g_\mu T^\sigma_{\sigma\nu} + g_{\sigma\mu}T^\sigma_{\nu\lambda} + \frac{2\alpha}{M_P^2(D-2)}(\partial\phi)^2\partial_\lambda\phi g_{\mu\nu} = 0 \quad (7.13)$$

Writing this in terms of the connection a:

$$\partial_\lambda g_{\mu\nu} - \Gamma^\sigma_{\lambda\mu}g_{\nu\sigma} - \Gamma^\sigma_{\nu\lambda}g_{\mu\sigma} - \frac{1}{D-1}T^\sigma_{\sigma\lambda}g_{\mu\nu} - \frac{1}{D-1}T^\sigma_{\sigma\nu}g_{\lambda\mu} + \frac{2\alpha}{M_P^2(D-2)}(\partial\phi)^2\partial_\lambda\phi g_{\mu\nu} = 0 \quad (7.14)$$

To solve this equation, we permute the indices twice

$$\partial_\mu g_{\nu\lambda} - \Gamma^\sigma_{\mu\nu}g_{\lambda\sigma} - \Gamma^\sigma_{\lambda\mu}g_{\nu\sigma} - \frac{1}{D-1}T^\sigma_{\sigma\mu}g_{\nu\lambda} - \frac{1}{D-1}T^\sigma_{\sigma\lambda}g_{\mu\nu} + \frac{2\alpha}{M_P^2(D-2)}(\partial\phi)^2\partial_\mu\phi g_{\nu\lambda} = 0 \quad (7.15)$$

$$\partial_\nu g_{\lambda\mu} - \Gamma^\sigma_{\nu\lambda}g_{\mu\sigma} - \Gamma^\sigma_{\mu\nu}g_{\lambda\sigma} - \frac{1}{D-1}T^\sigma_{\sigma\nu}g_{\lambda\mu} - \frac{1}{D-1}T^\sigma_{\sigma\mu}g_{\nu\lambda} + \frac{2\alpha}{M_P^2(D-2)}(\partial\phi)^2\partial_\nu\phi g_{\lambda\mu} = 0 \quad (7.16)$$

Adding the last two equation and subtracting the first:

$$\begin{aligned} &\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} - 2\Gamma^\sigma_{\mu\nu}g_{\lambda\sigma} - \frac{2}{D-1}T^\sigma_{\sigma\mu}g_{\nu\lambda} - \frac{2\alpha}{M_P^2(D-2)}(\partial\phi)^2\partial_\lambda\phi g_{\mu\nu} \\ &+ \frac{2\alpha}{M_P^2(D-2)}(\partial\phi)^2\partial_\mu\phi g_{\nu\lambda} + \frac{2\alpha}{M_P^2(D-2)}(\partial\phi)^2\partial_\nu\phi g_{\lambda\mu} = 0 \end{aligned} \quad (7.17)$$

Which ultimately can be written as

$$\begin{aligned} \mathring{\Gamma}_{\mu\nu}^\rho - \Gamma_{\mu\nu}^\rho - \frac{1}{D-1} T_{\sigma\mu}^\sigma \delta_\nu^\rho - \frac{\alpha}{M_P^2(D-2)} (\partial\phi)^2 \partial^\rho \phi g_{\mu\lambda} + \frac{\alpha}{M_P^2(D-2)} (\partial\phi)^2 \partial_\mu \phi \delta_\nu^\rho \\ + \frac{\alpha}{M_P^2(D-2)} (\partial\phi)^2 \partial_\nu \phi \delta_\mu^\rho = 0 \end{aligned} \quad (7.18)$$

We see that we can solve the equation of motion for the connection. Let's write the connection in terms of the Levi-Civita connection, $\mathcal{A}_\mu = \frac{1}{D-1} T_{\sigma\mu}^\sigma$ and $W_\mu = \frac{\alpha}{M_P^2(D-2)} (\partial\phi)^2 \partial_\mu \phi$:

$$\Gamma_{\mu\nu}^\rho = \mathring{\Gamma}_{\mu\nu}^\rho + \mathcal{A}_\mu \delta_\nu^\rho - W^\rho g_{\mu\nu} + W_\mu \delta_\nu^\rho + W_\nu \delta_\mu^\rho \quad (7.19)$$

We found the most generic solution for the connection, without assuming anything about symmetry or torsion in the beginning. As expected, we get the same term for projective invariance as in the Einstein-Hilbert Palatini formalism, namely $\mathcal{A}_\mu \delta_\nu^\rho$. Without loss of generality, we can set this term to zero. Notice that the connection depends on the galileon field. This means that, even the scalar field doesn't enter the matter action explicitly, a coupling of the scalar field with matter may arise through the connection.

The connection that we found is actually a well-studied connection in both mathematics and physics, it is known as the Weyl connection. In a bit, we will give the history of this connection and will comment on its geometrical properties. However, let's first check if and how the action and equation of motion change using this connection instead of the Levi-Civita connection.

7.3 The effective action and equations of motion

In chapter 5, we saw that the Palatini solution for the connection did not change the Einstein-Hilbert action and its field equations. That implies that the physics of the theory in the metric and Palatini formalism is equivalent. We will see that things are really different for the cubic galileon.

The effective action

First, let's check if the action changes by substituting the solution for the connection (7.19) in it. This is quite easy to calculate, the calculation is included in appendix E and we obtain:

$$S = \int d^4x \sqrt{|g|} \left(\frac{M_P^2}{2} \mathring{R} - \alpha X \mathring{\nabla}^\mu \mathring{\nabla}_\mu \phi - \frac{3\alpha^2 X^3}{4M_P^2} \right) \quad (7.20)$$

It is obvious that this action is different than (7.1). An extra factor $X^3 = (\partial\phi)^6$ appears in the action, so this isn't a covariant galileon theory anymore. However, note that the

new term is suppressed by a factor of M_P^2 . Therefore, if we turn gravity off (by setting $M_P^2 \rightarrow \infty$) we recover the galileon in the weak field limit.

So we can say that we found another type of galileon theory, the Palatini covariant galileon. It is a subclass of the generalized covariant galileon theory, but simplifies to pure galileon theory in the limit of no gravity.

Equations of motion

In order to compare the equations of motion in the Palatini formalism to those found in the metric formalism, we start by calculating the latter ones. The equation of motion with respect to ϕ of (7.1) is:

$$\begin{aligned} 0 &= 2\alpha\partial^\sigma\phi\overset{\circ}{\nabla}_\sigma\overset{\circ}{\nabla}^\mu\overset{\circ}{\nabla}_\mu\phi - 2\alpha\partial^\sigma\phi\overset{\circ}{\nabla}^\mu\overset{\circ}{\nabla}_\mu\overset{\circ}{\nabla}_\sigma\phi + 2\alpha\overset{\circ}{\nabla}_\sigma\overset{\circ}{\nabla}^\sigma\phi\overset{\circ}{\nabla}^\mu\overset{\circ}{\nabla}_\mu\phi - 2\alpha\overset{\circ}{\nabla}^\nu\overset{\circ}{\nabla}^\rho\phi\overset{\circ}{\nabla}_\nu\overset{\circ}{\nabla}_\rho\phi \\ &= \overset{\circ}{\nabla}_\sigma\overset{\circ}{\nabla}^\sigma\phi\overset{\circ}{\nabla}^\mu\overset{\circ}{\nabla}_\mu\phi - \overset{\circ}{\nabla}^\nu\overset{\circ}{\nabla}^\rho\phi\overset{\circ}{\nabla}_\nu\overset{\circ}{\nabla}_\rho\phi + \overset{\circ}{R}_\sigma{}^\rho\partial^\sigma\phi\partial_\rho\phi \end{aligned} \quad (7.21)$$

The equation of motion for $g^{\mu\nu}$ gives us

$$\begin{aligned} M_P^2(\overset{\circ}{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\overset{\circ}{R}) &= 2\alpha\overset{\circ}{\nabla}_\rho\overset{\circ}{\nabla}_\sigma\phi\partial^\rho\phi\partial^\sigma\phi g_{\mu\nu} + 2\alpha\overset{\circ}{\nabla}^\rho\overset{\circ}{\nabla}_\rho\phi\partial_\mu\phi\partial_\nu\phi \\ &\quad - 2\alpha(\overset{\circ}{\nabla}_\mu\overset{\circ}{\nabla}_\rho\phi\partial^\rho\phi\partial_\nu\phi + \overset{\circ}{\nabla}_\nu\overset{\circ}{\nabla}_\rho\phi\partial^\rho\phi\partial_\mu\phi) \end{aligned} \quad (7.22)$$

In order to be able to compare the equations of motion, we have substituted the solution for the connection in the equations of motions. So what we did is calculating the equations of motion of the action (7.5) with $\gamma = \frac{1}{2}$ and then a posteriori substituting the Weyl connection². The equation of motion with respect to ϕ in the Palatini formalism is

$$\begin{aligned} &2\overset{\circ}{\nabla}_\sigma\overset{\circ}{\nabla}^\sigma\phi\overset{\circ}{\nabla}^\mu\overset{\circ}{\nabla}_\mu\phi - 2\overset{\circ}{\nabla}^\nu\overset{\circ}{\nabla}^\rho\phi\overset{\circ}{\nabla}_\nu\overset{\circ}{\nabla}_\rho\phi + 2\overset{\circ}{R}_\sigma{}^\rho\partial^\sigma\phi\partial_\rho\phi + \frac{6\alpha(D-1)}{2M_P^2(D-2)}\overset{\circ}{\nabla}^\rho\overset{\circ}{\nabla}_\rho\phi(\partial\phi)^4 \\ &+ \frac{24\alpha(D-1)}{2M_P^2(D-2)}\overset{\circ}{\nabla}_\rho\overset{\circ}{\nabla}_\nu\phi\partial^\rho\phi\partial_\nu\phi(\partial\phi)^2 = 0 \end{aligned} \quad (7.23)$$

and the variation of the action with respect to $g^{\mu\nu}$ gives us the following equation:

²Actually this is equivalent to calculating the equations of motion for the effective action, which is way easier because then we only work with Levi-Civita connections. However, it is a good check doing it the way mentioned in the main text. We refer the reader interested in the equations of motions written in terms of the arbitrary connection to appendix E

$$\begin{aligned}
\frac{M_P^2}{2}(\dot{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\dot{R}) &= \alpha\dot{\nabla}_\rho\dot{\nabla}_\sigma\phi\partial^\rho\phi\partial^\sigma\phi g_{\mu\nu} + \alpha\dot{\nabla}^\rho\dot{\nabla}_\rho\phi\partial_\mu\phi\partial_\nu\phi - \frac{\alpha^2}{4M_P^2}\frac{D-1}{D-2}(\partial\phi)^6g_{\mu\nu} \\
&+ \frac{3\alpha^2}{2M_P^2}\frac{D-1}{D-2}(\partial\phi)^4\partial_\mu\phi\partial_\nu\phi - \alpha(\dot{\nabla}_\mu\dot{\nabla}_\rho\phi\partial^\rho\phi\partial_\nu\phi + \dot{\nabla}_\nu\dot{\nabla}_\rho\phi\partial^\rho\phi\partial_\mu\phi)
\end{aligned}
\tag{7.24}$$

Comparing this with the Einstein equation in the metric formalism, we note that we have two extra terms. So physics in the metric and Palatini formalism is certainly not equivalent for scalar tensor-theories. However, the two new terms are heavily suppressed so only become important at very high energy scales.

Since the equations of motion change, the physical and cosmological implication of the theory will change as well. The inequivalence of the equations of motion in the Palatini and metric formalism for the cubic galileon, is a motivation to study the cosmological consequences of the theory not only in metric [14] but also in the Palatini formalism. Now we will see what implications for the geometry are if we would consider not the Levi-Civita, but the Weyl connection.

7.4 The Weyl connection

The generic solution for the connection is the Weyl connection. We will give its physical properties and a short history of this connection.

A short history: the first gauge theory ever by Weyl

In 1918, Weyl was one of the first to try to incorporate electromagnetism with gravity. He assumed that general relativity should be scale invariant [32]. He "recalibrated" the metric:

$$\bar{g}_{\mu\nu} = e^{\Lambda(x)}g_{\mu\nu} \tag{7.25}$$

and introduced a new, non-metric, connection:

$$\hat{\Gamma}_{\mu\nu}^\rho = \overset{\circ}{\Gamma}_{\mu\nu}^\rho - A_\mu\delta_\nu^\rho - A_\nu\delta_\mu^\rho + A^\rho g_{\mu\nu} \tag{7.26}$$

Now, under the transformation (7.25) this connection is invariant, if the vector A_μ transforms as $A_\mu \rightarrow \bar{A}_\mu = A_\mu + \partial_\mu\Lambda$. Weyl identified this vector with the electromagnetic potential and concluded that one could change "gauge" distances if compensated by a gauge transformation of A_μ .

However, soon it was pointed out by Einstein that this theory was inconsistent and it would

lead to observational deviations in the lines of atomic absorption spectra [32]. A solution that would solve this problem is if there would exist a scalar ϕ , such that $A_\mu = \partial_\mu \phi$ ³ (see [39] for a more elaborate explanation of the problems of Weyl's theory). In that case, the electromagnetic field would disappear completely and therefore Weyl had to abandon the assumption that A_μ could serve as the electromagnetic potential⁴. However, Weyl's failed attempt to unify electromagnetism with gravity laid the foundations for both gauge theory as well as physics in non-Riemannian geometry.

In integrable Weyl geometry, it is a well known fact that null geodesics are preserved, so spacetime has the same causal structure as Riemannian geometry. To see why this happens, let's take a look at the geodesic equation

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\nu\rho}^\mu \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} = \frac{d^2 x^\mu}{ds^2} + \mathring{\Gamma}_{\nu\rho}^\mu \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} + 2\partial_\nu \phi \frac{dx^\nu}{ds} \frac{dx^\mu}{ds} - \partial^\mu \phi g_{\nu\rho} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} = 0 \quad (7.27)$$

Treating null geodesics, the last term vanishes. Then we take the reparametrization:

$$\frac{d\tau}{ds} = e^{-2\phi(s)} \quad (7.28)$$

And we obtain:

$$\begin{aligned} \frac{d^2 x^\mu}{ds^2} + \Gamma_{\nu\rho}^\mu \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} &= e^{-4\phi} \left(\frac{d^2 x^\mu}{d\tau^2} - 2 \frac{d\phi}{d\tau} \frac{dx^\mu}{d\tau} \right) + e^{-4\phi} \mathring{\Gamma}_{\nu\rho}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} + 2e^{-4\phi} \partial_\nu \phi \frac{dx^\nu}{d\tau} \frac{dx^\mu}{d\tau} \\ &= e^{-4\phi} \left(\frac{d^2 x^\mu}{d\tau^2} + \mathring{\Gamma}_{\nu\rho}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} \right) \end{aligned} \quad (7.29)$$

On first sight, it looks like this argument doesn't hold for non-integrable Weyl geometry. However by performing a coordinate transform as an integral over a worldline, like in section 5.1, we can obtain a similar result for non-integrable Weyl geometry. So null geodesics (and thereby the causal structure) are preserved in all Weyl-geometries.

So the connection that solves the field equation for the connection in the Palatini defines in general different geodesics than in Riemannian geometry, but the null geodesics remain invariant.

³A geometry that is described by a Weyl connection with $A_\mu = \partial_\mu \phi$ is called an *integrable* Weyl geometry.

⁴Note that in this thesis this is already the second theory that was found by a failed attempt to combine gravity with electromagnetism. In chapter 4 we have seen that Kaluza-Klein theory was developed with the same goal in mind. It's quite nice that trying to combine gravity and electromagnetism led to various new theories and formulations that did not succeed in what they intended to do, but nevertheless turned out to be very useful and insightful on the foundations of GR.

7.4.1 A duality between torsion and nonmetricity

The torsion ($T_{\mu\nu}^\rho = \Gamma_{\mu\nu}^\rho - \Gamma_{\nu\mu}^\rho$) of the Weyl connection is zero, because the connection is symmetric in its two lower indices. Furthermore, the non-metricity of the metric is proportional to the Weyl tensor $\nabla_\mu g^{\sigma\rho} = 2W_\mu g^{\sigma\rho}$.

We know that the action of the cubic galileon is invariant under the transformation $\Gamma_{\mu\nu}^\rho \rightarrow \bar{\Gamma}_{\mu\nu}^\rho + A_\mu \delta_\nu^\rho$. Therefore, the field equations do not change under this transformation and in chapter five we discussed that geodesics motion is invariant under the projective transformation as well (see the discussion under the Einstein-Hilbert Palatini formalism in chapter 5). When we perform a projective transformation on our Weyl connection, we obtain the following connection:

$$\bar{\Gamma}_{\mu\nu}^\rho = \overset{\circ}{\Gamma}_{\mu\nu}^\rho - W^\rho g_{\mu\nu} + W_\nu \delta_\mu^\rho \quad (7.30)$$

Now the connection is metric compatible $\nabla_\rho g^{\mu\nu} = 0$, but not torsion-free anymore ($T_{\mu\nu}^\rho = W_\nu \delta_\mu^\rho - W_\mu \delta_\nu^\rho$). However, since we have seen in chapter 5 that the physical properties of a projectively invariant theory do not change under this information, we can conclude that torsion and nonmetricity are physically equivalent properties for pure cubic covariant galileon theory.

In a paper that appeared in October 2018 [31], the same duality is observed for another extended theory of gravity, where the gravitational action is $S_g \int d^4x \sqrt{|g|} R^2$. They extended the analysis to cosmology and showed that both torsion and non-metricity may be considered as a source of cosmological acceleration in their gravitational theory.

However, it has to be pointed out that this duality probably doesn't hold in the presence of matter that couples to the connection. An interesting study would be to find out how this duality of torsion and non-metricity is broken (or preserved) by particular forms of matter.

One last thing that we would like to point out is that since the obtained connection is not equal to the Levi-Civita connection, bosons (minimally coupled fields) and fermions (the Dirac-field, non-minimally coupled) probably will have different couplings to the galileon field ϕ in the Palatini formalism.

A question that arises naturally is if the Weyl connection is a solution for the higher order galileons as well or if this is a peculiarity of the cubic galileon. We will address this later on, but first we are going to look at how the equations of motion for the cubic galileon change in the metric formalism.

7.5 First order formalism galileon \mathcal{L}_4

For the cubic galileon the first order formalism is solvable and gives us a Weyl connection. Furthermore, because of the lack of projective invariance of this galileon, we had to adjust our starting Lagrangian. How are these two properties for the next galileon, \mathcal{L}_4 ? In the Levi-Civita notation this covariant Lagrangian is:

$$\mathcal{L}_4 = \beta(\partial\phi)^2(\overset{\circ}{\nabla}^\mu\overset{\circ}{\nabla}_\mu\phi\overset{\circ}{\nabla}^\nu\overset{\circ}{\nabla}_\nu\phi - \overset{\circ}{\nabla}_\mu\overset{\circ}{\nabla}_\nu\phi\overset{\circ}{\nabla}^\mu\overset{\circ}{\nabla}^\nu\phi + \frac{1}{4}(\partial\phi)^2R) \quad (7.31)$$

Again, to see how the Lagrangian should look like in Palatini formalism, we start by taking the most general combination of second derivatives terms that gives us (7.31) in the metric formalism. After that we calculate the equation of motion for $\Gamma_{\mu\nu}^\gamma$ and set the δ_γ^ν trace to zero. Then the total action in the first order formalism reads:

$$\begin{aligned} S = \int d^4x \sqrt{|g|} \Bigg(& \frac{M_P^2}{2}R + \beta(\partial\phi)^2 \Big(\frac{1}{4}\nabla^\mu\nabla_\mu\phi\nabla^\nu\nabla_\nu\phi + \frac{1}{2}\nabla^\mu\nabla_\mu\phi\nabla_\nu\nabla^\nu\phi \\ & + \frac{1}{4}\nabla_\mu\nabla^\mu\phi\nabla_\nu\nabla^\nu\phi - \frac{1}{2}\nabla_\mu\nabla_\nu\phi\nabla^\nu\nabla^\mu\phi \\ & - \frac{1}{4}\nabla_\mu\nabla^\nu\phi\nabla_\nu\nabla^\mu\phi - \frac{1}{4}\nabla^\mu\nabla_\nu\phi\nabla^\nu\nabla_\mu\phi + \frac{1}{4}(\partial\phi)^2R \Big) \Bigg) \end{aligned} \quad (7.32)$$

Note how different this Lagrangian looks than the one in the Palatini formalism: in the metric formalism we have the $\overset{\circ}{\nabla}_\mu\overset{\circ}{\nabla}_\nu\phi\overset{\circ}{\nabla}^\mu\overset{\circ}{\nabla}^\nu\phi$ term, but we don't have the equivalent of that $\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi$ in the first order formalism.

We are not able to solve the field equation for the connection in the same way as we did for the cubic covariant galileon. However, we are mostly interested in the question if the Weyl connection is again a solution for this galileon. We will check this and to make our calculation as general as possible, we take the combination of \mathcal{L}_3 and \mathcal{L}_4 . So we start with the following action:

$$S = \int d^4x \sqrt{|g|} \Big(\frac{M_P^2}{2}R + \mathcal{L}_3^{(pj)} + \mathcal{L}_4^{(pj)} \Big) \quad (7.33)$$

Where $\mathcal{L}^{(pj)}$ means the projectively invariant galileon Lagrangian in Palatini formalism. The easiest way to check if the Weyl connection solves the equation of motion for the connection is to use $\Gamma_{\mu\nu}^\rho = \overset{\circ}{\Gamma}_{\mu\nu}^\rho + K_{\mu\nu}^\rho$ again. Then equation of motion with respect to

$K_{\mu\nu}{}^\lambda$ and raising all indices gives us:

$$\begin{aligned}
& \epsilon \left(K^{\lambda\mu\nu} + K^{\nu\lambda\mu} - g^{\mu\lambda} K_\rho{}^{\rho\nu} - g^{\mu\nu} K_\rho{}^{\lambda\rho} \right) + \alpha X g^{\mu[\nu} \partial^{\lambda]} \phi \\
& + \frac{\beta X}{2} \left(4 \mathring{\nabla}^2 \phi g^{\mu[\lambda} \partial^{\nu]} \phi + g^{\mu\nu} \partial^\lambda \phi \partial_\gamma \phi (K_\sigma{}^{\sigma\gamma} - K_\sigma{}^{\gamma\sigma}) \right. \\
& \left. - g^{\mu\lambda} \partial^\nu \phi \partial_\gamma \phi (K_\sigma{}^{\sigma\gamma} - K_\sigma{}^{\gamma\sigma}) + 4 \mathring{\nabla}^\mu \mathring{\nabla}^{[\nu} \phi \partial^{\lambda]} + 2 K^{\lambda[\mu\gamma]} \partial_\gamma \phi \partial^\nu \phi + 2 K^{\nu[\gamma\mu]} \partial^\lambda \phi \partial_\gamma \phi \right) = 0
\end{aligned} \tag{7.34}$$

Where $\epsilon = (\frac{M_P^2}{2} + \frac{1}{4}\beta X^2)$. Note that in this notation the projective invariance is much easier to see as well. Now, if we plug in as an ansatz the Weyl connection $K_{\mu\nu}{}^\rho = -W^\rho g_{\mu\nu} + W_\mu \delta_\nu^\rho + W_\nu \delta_\mu^\rho$ we only keep a few terms:

$$\begin{aligned}
& g^{\mu\nu} \left[\epsilon(2-D)W^\lambda + \left(\frac{\alpha X}{2} - \beta X \mathring{\nabla}^2 \phi \right) \partial^\lambda \phi + \beta X(2-D)W^\gamma \partial^\lambda \phi \partial_\gamma \phi \right] \\
& + g^{\mu\lambda} \left[\epsilon(D-2)W^\nu - \left(\frac{\alpha X}{2} - \beta X \mathring{\nabla}^2 \phi \right) \partial^\nu \phi + \beta X(D-2)W^\gamma \partial^\nu \phi \partial_\gamma \phi \right] \\
& + 2\beta X \mathring{\nabla}^\mu \mathring{\nabla}^{[\nu} \phi \partial^{\lambda]} \phi = 0
\end{aligned} \tag{7.35}$$

We could solve the first two lines for W , however, we cannot get rid of the last terms. The last line can never be written in terms of $g^{\mu\nu}$ or $g^{\mu\lambda}$, so this equation can not be solved for W . So the solution of the first order formalism for the fourth Galileon isn't a Weyl connection, but probably a Weyl connection plus some other terms.

Final comment on the quartic covariant galileon

In the previous discussion, we started with the quartic galileon in metric formalism and found a way to write it in a projectively invariant way that reduces to this form. However, recall from the discussion on covariantizing galileon in section 3.3, that we had to add $+\frac{1}{4}XR$ to the Lagrangian in order to avoid derivatives of the Ricci scalar and tensor in the equations of motion. They would produce third order derivatives of g .

In the first-order formalism having derivatives of the Ricci scalar and tensor doesn't give problems regarding higher derivatives and unwanted ghosts, because the Ricci scalar only depends on first order derivatives of Γ . So that would give an argument for removing $\frac{1}{4}XR$ from the Lagrangian, but other physical effects should be considered as well before we can say that it gives a correct theory.

Conclusions

We applied the Palatini formalism to covariant galileons and discovered a lot of interesting properties of scalar-tensor Lagrangians in the Palatini formalism. First we saw that when applying the Palatini formalism, it is crucial to notate the theory in a projective-invariant way. We proceeded in doing this for the cubic and quartic galileon, one open question is if this could be extended to all covariant galileons.

We solved the equation of the connection for the cubic galileon and found it to be the Weyl connection. In the effective action an extra term appeared, which clearly indicates that the physics is different when the theory is regarded in the Palatini or metric formalism. This could change (cosmological) predictions of the theory, so we conclude that the Palatini formalism for covariant galileons does deserve more consideration.

Furthermore the Palatini formalism indicates some kind of duality between the physical meaning of torsion and non-metricity and a different coupling of the scalar field to the Boson and Dirac fields. It would be very interesting to investigate the precise meaning of this duality.

This section can be extended in the following ways: firstly one could look at the possibility of formulating projectively-invariant galileon theories. Besides that, implications of the Palatini formalism for the cosmological prediction of the cubic galileon deserves attention. The exact meaning of the duality between torsion and non-metricity should be studied in order to better understand the relation between geometry and physics. The inclusion of matter in this theory would be interesting to investigate and finally the extension of this chapter to generalized or higher order galileons is worth exploring.

Conclusions and outlook

We started by motivating the search for extended theories of gravity. Since there exist observations that do not match with the predictions of GR and we still do not know how to combine gravity with quantum physics, extensions to GR are worth studying. Two characteristics of GR are that (i) it is a higher order derivative theory without ghosts and (ii) that we obtain the same theory if we formulate it in the metric or Palatini formalism. These two subjects have been studied throughout this thesis in order to better understand their implications and to find which other gravitational theories they point out.

In the second chapter, we gave a recap of GR and its founding principles and introduced Lovelock theory as its extension to higher dimensions. Just as GR, Lovelock Lagrangians depend on second order derivatives of the metric but give only up to second order derivative field equations. In four dimensions GR and Lovelock are equivalent. In the third chapter we have seen if we require second order derivative equations of motion for *scalar* field, we end up with theories that are called galileons. So in a way, galileons can be regarded as a scalar portrayal of Lovelock theory. In this chapter we discussed the different types of galileon theories that exist and tried to clarify the distinction between galileons, generalized galileons and covariant galileons. After that, in chapter four, we introduced Kaluza-Klein reduction and demonstrated that the dimensional reduction of the Lovelock Lagrangian gives rise to *generalized* covariant galileons. A remarkable result was that in the flat space limit, the dimensional reduction of the sum of the Lovelock Lagrangians only gives rise to the kinetic scalar term. It could be interesting to study what extra terms appear if we consider dimensional reduction over more than one dimension.

After the introduction and study of higher order derivative theories, in chapter five we introduced another way to describe GR: in the Palatini formalism. In the Palatini formalism the connection is taken as an independent quantity as opposed to related to the metric in the metric formalism. We saw that in Einstein-Hilbert gravity, the first-order formalism

and metric formalism give rise to exactly the same physics in vacuum. We also discussed the importance of including matter for the geometrical meaning of the connection. For Lovelock gravity, the first-order formalism is equivalent to the metric formalism in the way that the different sets of equations of motion give the same information when demanding the connection to be Levi-Civita. However, the general solution for the connection is not found yet. The Palatini formalism for galileon theories is largely an unknown domain. The possibility to learn more about both the first-order formalism for Lovelock as the formalism for galileons are explored in chapter 6 and 7. In chapter 6 we started by calculating the dimensional reduction of the first two Lovelock terms in the Palatini formalism. We succeeded in finding a way to reduce a theory in the first-order formalism, by regarding that having an independent connection is equivalent to a Levi-Civita connection and tensor. For Einstein-Hilbert, we could again solve the equations of motions but did not gain any physical insight on the Palatini formalism. In the case of Gauss-Bonnet, we couldn't find a way to extract useful information of the formalism. Even equivalence between the metric and Palatini formalism turned out to be trivial in the reduced theory and did not give any relations between the scalar field and the connection.

Then finally in chapter 7 the Palatini formalism was applied to the cubic covariant galileon. We noticed a few different interesting concepts. Firstly we have seen that in order to apply the Palatini formalism on covariant galileons, it is crucial that they have the same projective invariance as the Ricci scalar. We found a projectively invariant notation for both the cubic and quartic covariant galileon. In contrast to the case of Einstein-Hilbert and Lovelock, for the covariant galileon the Palatini formalism is certainly not equivalent to the metric formalism. Therefore, a study of the physical and cosmological implications of the cubic galileon in the Palatini formalism would be a nice follow-up. Furthermore, in the cubic galileon framework, there seemed to be some duality between torsion and non-metricity. This duality has appeared in different theories as well. The extension to which this duality is preserved in situations where matter is included or the exact physical meaning remains unknown and deserves to be studied in more detail to really understand the relation between geometry and physics.

Ostrogradsky's theorem

Ostrogradsky was a 19th century mathematician and came up with a theory that restricts the inclusion of non-degenerate higher (time) derivative terms in Lagrangians [40]. Non-degeneracy means that the higher derivative terms can not be written to as a total derivative and contribute to the field equations. Ostrogradsky's theorem is important as it puts severe constraints on higher derivative theory and the idea behind the theory is quite elegant and simple.

In order to explain the theorem, we will consider two different Lagrangians. The first Lagrangian doesn't depend on second time derivatives: $\mathcal{L} = \mathcal{L}(x, \dot{x})$. So from this Lagrangian we obtain the Euler Lagrange equation $\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$. Non-degeneracy implies that $\frac{\partial \mathcal{L}}{\partial \dot{x}}$ depends on \dot{x} , so the equation of motion depends on second order derivatives of x , \ddot{x} .

We can express the second derivative of x as a function containing the first derivative and x : $\ddot{x} = F(x, \dot{x})$. Therefore, we can solve for x as $x(t) = X(t, x_0, \dot{x}_0)$ and it becomes apparent that the solution depends upon two pieces of initial data, which means that we can take two canonical coordinates. We call them $Q \equiv x$ and $P \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}}$. We can write the first derivative of x as a function of both canonical coordinates $\dot{x} \equiv v(Q, P)$ and find for the Hamiltonian:

$$H = P\dot{Q} - \mathcal{L} = Pv(Q, P) - \mathcal{L}(Q, v(Q, P)) \quad (\text{A.1})$$

So far so good. Now let's consider a second Lagrangian $\mathcal{L}(x, \dot{x}, \ddot{x})$ that depends on second derivatives of x as well. Its Euler-Lagrange equation is:

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} + \frac{d^2}{dt^2} \frac{\partial \mathcal{L}}{\partial \ddot{x}} = 0 \quad (\text{A.2})$$

Again the assumption of nondegeneracy implies that $\frac{\partial \mathcal{L}}{\partial \ddot{x}}$ depends on \ddot{x} and the EL-equations contains up to fourth order derivatives of x . This time we can express the fourth order derivative as a function of the lower derivatives $x^{(4)} = F(x, \dot{x}, \ddot{x}, x^{(3)})$ and see that there

are four initial conditions. So we need to choose four canonical coordinates, following Ostrogradsky we call them $Q_1 \equiv x$, $Q_2 \equiv \dot{x}$, $P_1 \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \ddot{x}}$ and $P_2 \equiv \frac{\partial \mathcal{L}}{\partial \ddot{x}}$. The acceleration can be written as $\ddot{x} = a(Q_1, Q_2, P_2)$, so it doesn't depend on P_1 . The Hamiltonian is:

$$H = P_1 Q_2 + P_2 a(Q_1, Q_2, P_2) - \mathcal{L}(Q_1, Q_2, a(Q_1, Q_2, P_2)) \quad (\text{A.3})$$

In contrast to the Hamiltonian considered before, this one is linear in the canonical momentum P_1 . This implies that no system having this Hamiltonian can be stable and since P_1 can get infinitely negative, the Hamiltonian is not bound from below. That means that some degrees of freedom of the system can have arbitrarily negative energies.

In higher than second order derivatives theories, the Hamiltonian gets even more linear instabilities [40]. This analysis is based on classical mechanics, but a similar argument holds for field theories.

The Vielbein formalism of General Relativity

The equivalence principle tells us that we can find, at any point P, a local inertial frame where the physical laws become those from Minkowski space. We use in a point P locally free-falling coordinates $\xi^a(P)$. We can choose these coordinates to be orthonormal. Then the metric is given by:

$$ds^2 = \eta_{ab} d\xi^a d\xi^b \quad (\text{B.1})$$

For clarity, it is common to label the inertial coordinates by latin indices and the arbitrary coordinates by greek sign. We define the vielbein fields e_α^a and the inverse vielbeins e_a^α as:

$$d\xi^a = \frac{\delta \xi^a}{\delta x^\alpha} dx^\alpha \equiv e_\alpha^a(x) dx^\alpha \quad dx^\alpha = \frac{\delta x^\alpha}{\delta \xi^a} d\xi^a \equiv e_a^\alpha(x) d\xi^a \quad (\text{B.2})$$

So we can see the vielbein as the transformation matrix between arbitrary coordinates x and inertial coordinates ξ . So the vielbeins permit the transformations of tensors between a different base:

$$V^a = e_\alpha^a V^\alpha \quad V^\alpha = e_a^\alpha V^a \quad (\text{B.3})$$

The vielbein and inverse vielbein are related by

$$e_\alpha^a e_a^\beta = \delta_\alpha^\beta \quad e_a^\alpha e_\alpha^b = \delta_a^b \quad (\text{B.4})$$

Vielbein means "many legs" in German, in four dimension the vielbein is also called Vierbein or tetrad. Let's look at the metric

$$ds^2 = \eta_{ab} d\xi^a d\xi^b = \eta_{ab} e_\mu^a e_\nu^b dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu \quad (\text{B.5})$$

So we see that we can see the vielbein as kind of the square root of the metric

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b \quad (\text{B.6})$$

Taking the determinant, it follows that

$$|e_\mu^a| = \sqrt{|g|} \quad (\text{B.7})$$

One important thing to note is that, because the vielbeins depend on the coordinates of the manifold, it's partial derivatives $\partial_a \equiv e_a^\mu \partial_\mu$ do not commute. If we consider a scalar field ϕ :

$$[\partial_a, \partial_b]\phi = -\Omega_{ab}^c \partial_c \phi \quad (\text{B.8})$$

where

$$\Omega_{ab}^c = e_a^\mu e_b^\nu (\partial_\mu e_\nu^c - \partial_\nu e_\mu^c) \quad (\text{B.9})$$

Just as in the curved formalism, we can define covariant derivatives in the vielbein formalism as well. We denote with ∇ the covariant derivative with respect to curved coordinates, D the covariant derivative with respect to flat coordinates and \mathcal{D} the total covariant derivative.

$$D_\mu \phi = \partial_\mu \phi \quad (\text{B.10})$$

$$D_\mu T_a^b = \partial_\mu T_a^b + \omega_{\mu c}^b T_a^c - \omega_{\mu a}^c T_c^b \quad (\text{B.11})$$

where *omega* is called the spin connection. Just like the connection Γ the spin connection does not transform as a tensor, but the difference between two spin connections does. The complete covariant derivative acts on tensors in the following way:

$$\mathcal{D}_\mu T_\nu^b = \partial_\mu T_\nu^b - \Gamma_{\mu\nu}^\rho T_\rho^b + \omega_{\mu a}^b T_\nu^a \quad (\text{B.12})$$

The vielbein postulates

The *first vielbein postulate* is that the total covariant derivative of the vielbein field is zero

$$\mathcal{D}_\mu e_\nu^a = 0 \quad (\text{B.13})$$

This postulate gives us a relation between the spin and the affine connection:

$$\omega_{\mu a}^b = e_\rho^b e_a^\nu \Gamma_{\mu\nu}^\rho - e_a^\nu \partial_\mu e_\nu^b \quad (\text{B.14})$$

The *second vielbein postulate* is the compatible of the metric with the covariant derivative $\nabla_\mu g_{\nu\rho} = 0$, which gives us

$$\nabla_\mu (e_\nu^a e_\rho^b \eta_{ab}) = 0 \quad (\text{B.15})$$

If we combine this with the first postulate, we see that the minkowski metric is compatible with the spin connection

$$D_\mu \eta_{ab} = 0 \quad (\text{B.16})$$

If we assume the Levi-Civita connection, Γ is completely determined in terms of the metric. This allows us to write the spin connection with all flat indices in a very beautiful way

$$\omega_{ab}{}^c \equiv e_a^\mu \omega_{\mu b}{}^c = \frac{1}{2} (\Omega_{ad}{}^e \eta^{cd} \eta_{be} + \Omega_{bd}{}^e \eta^{cd} \eta_{ae} - \Omega_{ab}{}^c) \quad (\text{B.17})$$

Now we can write the Riemann curvature tensor with flat indices $R_{abc}{}^d$ as:

$$R_{abc}{}^d = \delta_a \omega_{bc}{}^d - \delta_b \omega_{ac}{}^d - g a_{ac}{}^f \omega_{bf}{}^d + \omega_{bc}{}^f \omega_{af}{}^d - \Omega_{ab}{}^f \omega_{fc}{}^d \quad (\text{B.18})$$

Since we have all geometric objects expressed with curved and flat indices, both formalisms carry the same information about the gravitational action.

Proof of equivalence flat and covariant derivatives

Here we want to proof that using $\delta_{cd}^{ab} D_a V_b{}^{cd}$, where V is a unspecified tensor, is equivalent to $\delta_{\rho\gamma}^{\mu\nu} \nabla_\mu V_\nu{}^{\rho\gamma}$. Since we only have flat indices, we can write

$$\delta_{cd}^{ab} D_a V_b{}^{cd} = \delta_{cd}^{ab} \mathcal{D}_a V_b{}^{cd}$$

Now going to curved coordinates and using the first Vielbein postulate:

$$\begin{aligned} \delta_{cd}^{ab} \mathcal{D}_a V_b{}^{cd} &= \delta_{cd}^{ab} e_a^\mu \mathcal{D}_\mu (e_b^\nu e_\rho^c e_\gamma^d V_\nu{}^{\rho\gamma}) \\ &= \delta_{cd}^{ab} e_a^\mu e_b^\nu e_\rho^c e_\gamma^d \mathcal{D}_\mu V_\nu{}^{\rho\gamma} \\ &= \delta_{\rho\gamma}^{\mu\nu} \mathcal{D}_\mu V_\nu{}^{\rho\gamma} \end{aligned}$$

We have everything written in curved coordinates so we can change the derivative

$$\delta_{\rho\gamma}^{\mu\nu} \mathcal{D}_\mu V_\nu{}^{\rho\gamma} = \delta_{\rho\gamma}^{\mu\nu} \nabla_\mu V_\nu{}^{\rho\gamma}$$

With that we have proven that

$$\delta_{cd}^{ab} D_a V_b{}^{cd} = \delta_{\rho\gamma}^{\mu\nu} \nabla_\mu V_\nu{}^{\rho\gamma}$$

Kronecker delta properties

Working with Kronecker delta tensor's and (anti)symmetrization brackets simplifies our calculations a lot. In this section of the appendix, the most frequently used expressions will be given.

We define the Kronecker delta function as follows:

$$\delta_{b_1 \dots b_n}^{a_1 \dots a_n} = \delta_{[b_1}^{a_1} \delta_{b_2}^{a_2} \dots \delta_{b_n]}^{a_n} \quad (\text{C.1})$$

$[]$ denote normalized antisymmetrization brackets, for example $[ab] = \frac{1}{2}(ab - ba)$. Now we have a look what happens when we contract the Kronecker delta tensor with Kronecker delta symbols.

$$\delta_{b_1 \dots b_n}^{a_1 \dots a_n} \delta_{a_1}^{b_1} = \frac{D - (n - 1)}{n} \delta_{b_2 \dots b_n}^{a_2 \dots a_n} \quad (\text{C.2})$$

Where D denotes the dimension where the indices live in. Another really useful property is that antisymmetry brackets will cancel when contracted with the Kronecker delta tensor.

$$\delta_{b_1 \dots b_n}^{a_1 \dots a_n} T_{[a_1}^{b_1} X_{a_2]}^{b_2} = \delta_{b_1 \dots b_n}^{a_1 \dots a_n} \frac{1}{2} (T_{a_1}^{b_1} X_{a_2}^{b_2} - T_{a_2}^{b_1} X_{a_1}^{b_2}) \quad (\text{C.3})$$

$$\delta_{b_1 b_2 \dots b_n}^{a_1 a_2 \dots a_n} T_{a_2}^{b_1} X_{a_1}^{b_2} = \delta_{b_1 b_2 \dots b_n}^{a_2 a_1 \dots a_n} T_{a_1}^{b_1} X_{a_2}^{b_2} = -\delta_{b_1 b_2 \dots b_n}^{a_1 a_2 \dots a_n} T_{a_1}^{b_1} X_{a_2}^{b_2} \quad (\text{C.4})$$

$$\delta_{b_1 \dots b_n}^{a_1 \dots a_n} T_{[a_1}^{b_1} X_{a_2]}^{b_2} = \delta_{b_1 \dots b_n}^{a_1 \dots a_n} T_{a_1}^{b_1} X_{a_2}^{b_2} \quad (\text{C.5})$$

Infinitesimal coordinate transformations

Usually it is much easier to use infinitesimal transformation to see if an action is invariant under some kind of transformation, than to calculate the change of the whole Lagrangian under that transformation. We know that the action is invariant if its infinitesimal variation gives zero, $\delta S = 0$.

Since vectors and tensor are defined by their coordinate transformation rules, we would like to know the infinitesimal changes of coordinate transformations. Let's consider the following coordinate change: $x'^\mu = x^\mu - \xi^\mu(x)$ (see picture D.1) which gives us the variation $\delta x^\mu = -\xi^\mu(x)$. Here $\xi^\mu(x)$ is an arbitrary vector function.

From this follows that:

$$\frac{x'^\mu}{x^\nu} = \delta^\mu_\nu - \partial_\nu \xi^\mu \quad (D.1)$$

$$\frac{x^\mu}{x'^\nu} = \delta^\mu_\nu + \partial_\nu \xi^\mu \quad (D.2)$$

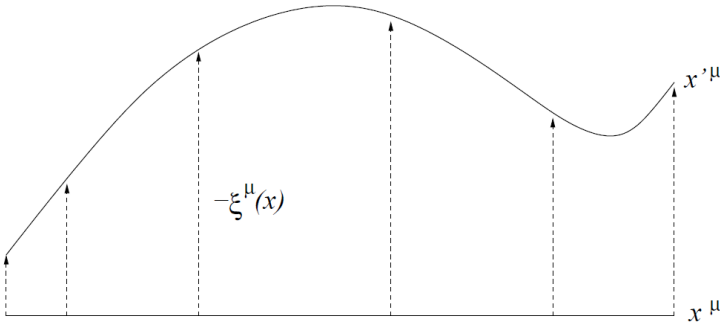


Figure D.1: An infinitesimal coordinate transformation: by adding an $\xi^\mu(x)$ to the coordinate x^μ we obtain a new coordinate x'^μ [32].

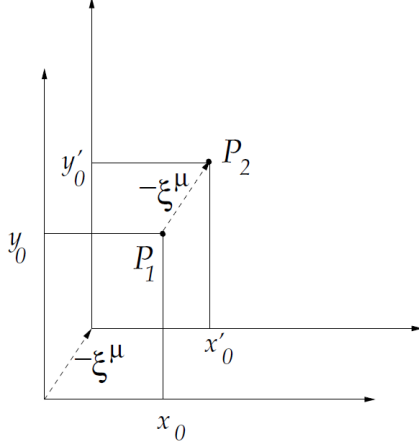


Figure D.2: The infinitesimal transformation of a scalar field Ψ under a coordinate transformation [32].

For the functional variation of an object under a coordinate transformation, we compare the value of the object at a point P_1 with coordinate system O_1 with the value at another point P_2 in coordinate system O_2 , see picture D.2. The coordinates in both system are the same, but we look at different points. For a scalar[32]:

$$\delta\phi = \phi'(x) - \phi(x) = \phi'(x') + \xi^\mu \partial_\mu \phi'(x') - \phi(x) = \xi^\mu \partial_\mu \phi(x) \quad (\text{D.3})$$

Where we used that $\phi'(x') = \phi(x)$, because this is the numerical value of a scalar at the same point and a Tayler expansion. For vectors and tensors we get

$$\delta A^\mu = \xi^\rho \partial_\rho A^\mu(x) - \partial_\rho \xi^\mu A^\rho(x) \quad (\text{D.4})$$

$$\delta B_\mu = \xi^\rho \partial_\rho B_\mu(x) + \partial_\mu \xi^\rho B_\rho \quad (\text{D.5})$$

$$\delta S_\mu^\nu = \xi^\rho \partial_\rho S_\mu^\nu + \partial_\mu \xi^\rho S_\rho^\nu - \partial_\rho \xi^\nu S_\mu^\rho \quad (\text{D.6})$$

The functional variation δ_ξ form a closed algebra and are the infinitesimal generators of the group of general coordinate transformations.

Weyl connection properties

Under the Weyl connection $\Gamma_{\mu\nu}^\rho = \mathring{\Gamma}_{\mu\nu}^\rho - W^\rho g_{\mu\nu} + W_\mu \delta_\nu^\rho + W_\nu \delta_\mu^\rho$ the Ricci tensor changes as:

$$\begin{aligned} R_{\mu\rho} &= \mathring{R}_{\mu\rho} + (D-1)\mathring{\nabla}_\mu W_\rho - \mathring{\nabla}_\rho W_\mu + \mathring{\nabla}_\nu (W^\nu g_{\mu\rho}) \\ &\quad + (D-2)W^\nu W_\nu g_{\mu\rho} - (D-2)W_\rho W_\mu \end{aligned} \quad (\text{E.1})$$

And the Ricci scalar:

$$R = \mathring{R} + (2D-2)\mathring{\nabla}_\rho W^\rho + (D-2)(D-1)W^\rho W_\rho \quad (\text{E.2})$$

The resultant action of the cubic galileon

We started with the following action:

$$S = \int d^4x \sqrt{g} \left(\frac{M_P^2}{2} R - \frac{1}{2} X g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \alpha \frac{1}{2} X \phi \nabla_\mu (g^{\mu\nu} \nabla_\nu \phi) \right) \quad (\text{E.3})$$

Using our relation for R we found earlier, the first terms gives us:

$$\begin{aligned} \frac{M_P^2}{2} R &\rightarrow \frac{M_P^2}{2} \mathring{R} + \frac{M_P^2}{2} (D-2)(D-1) W^\mu W_\mu \\ &\rightarrow \frac{M_P^2}{2} \mathring{R} + \frac{\alpha^2 X^3 (D-1)}{2M_P^2 (D-2)} \end{aligned} \quad (\text{E.4})$$

The second two terms will give us (up to a total derivative):

$$\begin{aligned} -\frac{1}{2} \alpha X g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \alpha X \nabla_\mu (g^{\mu\nu} \nabla_\nu \phi) &\rightarrow -\alpha X \mathring{\nabla}^\mu \mathring{\nabla}_\mu \phi + \frac{\alpha}{2} X g^{\mu\nu} K_{\mu\nu}{}^\lambda \partial_\lambda \phi - \frac{\alpha}{2} X K_{\nu\lambda}{}^\nu \partial^\lambda \phi \\ &\rightarrow -\alpha X \mathring{\nabla}^\mu \mathring{\nabla}_\mu \phi - \frac{\alpha^2 X^3 (D-1)}{M_P^2 (D-2)} \end{aligned} \quad (\text{E.5})$$

So the total resultant action becomes

$$\begin{aligned}
S &= \int d^4x \sqrt{|g|} \left(\frac{M_P^2}{2} \dot{R} - \alpha X \dot{\nabla}^\mu \dot{\nabla}_\mu \phi - \frac{\alpha^2 X^3 (D-1)}{2M_P^2 (D-2)} \right) \\
&= \int d^4x \sqrt{|g|} \left(\frac{M_P^2}{2} \dot{R} - \alpha X \dot{\nabla}^\mu \dot{\nabla}_\mu \phi - \frac{3\alpha^2 X^3}{4M_P^2} \right)
\end{aligned} \tag{E.6}$$

Equations of motions of the cubic galileon in the Palatini formalism

The equation of motion for ϕ in the metric affine formalism gives us:

$$\begin{aligned}
&g^{\lambda\gamma} \nabla_\sigma g_{\lambda\gamma} \partial^\sigma \phi \nabla^\nu \nabla_\nu \phi + 3 \nabla_\sigma g^{\sigma\rho} \partial_\rho \nabla^\nu \nabla_\nu \phi + 2 \nabla^\rho \nabla_\rho \phi \nabla^\nu \nabla_\nu \phi + \partial^\sigma \phi \nabla_\sigma g^{\mu\nu} \nabla_\mu \nabla_\nu \phi \\
&+ \frac{1}{2} g^{\lambda\gamma} \nabla_\sigma g_{\lambda\gamma} \partial^\sigma \phi \partial_\mu \phi \nabla_\nu g^{\mu\nu} + \nabla_\sigma g^{\sigma\rho} \nabla_\nu g^{\mu\nu} \partial_\mu \phi \partial_\rho \phi - 2 \partial^\sigma \phi \nabla_\sigma \nabla_\nu g^{\mu\nu} \\
&- \frac{3}{4} g^{\lambda\gamma} \nabla_\mu g_{\lambda\gamma} (\partial\phi)^2 \nabla_\nu g^{\mu\nu} - \frac{3}{2} \nabla_\nu g^{\mu\nu} \nabla_\mu g^{\sigma\rho} \partial_\sigma \phi \partial_\rho \phi - \frac{1}{2} (\partial\phi)^2 \nabla_\mu \nabla_\nu g^{\mu\nu} - \frac{1}{2} (\partial\phi)^2 \nabla^\nu g^{\lambda\gamma} \nabla_\nu g_{\lambda\gamma} \\
&- \frac{1}{2} g^{\lambda\gamma} \nabla^\mu \nabla_\mu g_\gamma (\partial\phi)^2 - g^{\lambda\gamma} \nabla^\nu g_{\lambda\gamma} \nabla_\nu g^{\sigma\rho} \partial_\sigma \phi \partial_\rho \phi - 2 g^{\lambda\gamma} \nabla_\nu g_{\lambda\gamma} \nabla^\nu \nabla_\sigma \partial^\sigma \phi \\
&- \frac{1}{4} g^{\lambda\gamma} \nabla^\nu g_{\lambda\gamma} \nabla_\nu g_{\sigma\rho} (\partial\phi)^2 g^{\sigma\rho} - g^{\mu\nu} \nabla_\nu \nabla_\mu g^{\sigma\rho} \partial_\sigma \phi \partial_\rho \phi - 4 \nabla^\nu \nabla_\sigma \phi \partial_\rho \phi \nabla_\nu g^{\sigma\rho} - 2 \nabla^\nu \nabla_\sigma \phi \nabla_\nu \nabla_\rho \phi g^{\sigma\rho} \\
&+ 2 \partial^\sigma g^{\mu\nu} (\nabla_\sigma \nabla_\mu \nabla_\nu \phi - \nabla_\nu \nabla_\mu \nabla_\sigma \phi) = 0
\end{aligned} \tag{E.7}$$

And the field equation for the metric is:

$$\begin{aligned}
&R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{\alpha}{2} (\partial\phi)^2 \nabla^\rho \nabla_\rho \phi g_{\mu\nu} + \frac{\alpha}{4} (\partial\phi)^2 \partial_\sigma \phi g_{\mu\nu} \nabla_\rho g^{\sigma\rho} - \alpha \partial_\mu \phi \partial_\nu \phi \nabla^\rho \nabla_\rho \phi \\
&- \frac{\alpha}{2} (\partial\phi)^2 \nabla_\mu \nabla_\nu \phi - \frac{\alpha}{2} \partial_\mu \phi \partial_\nu \phi \partial_\sigma \phi \nabla_\rho g^{\rho\sigma} + \frac{\alpha}{2} \partial_\rho \phi \partial_\nu \phi \partial_\sigma \phi \nabla_\mu g^{\rho\sigma} + \alpha \nabla_\mu \nabla_\sigma \phi \partial^\sigma \phi \partial_\nu \phi \\
&+ \frac{\alpha}{4} g^{\sigma\rho} \nabla_\mu g_{\sigma\rho} (\partial\phi)^2 \partial_\nu \phi = 0
\end{aligned} \tag{E.8}$$

By using the the expressions for the first and second derivatives listed below, we will find the same equations of motions as written down in chapter 7.

Non-metricity of the Weyl connection

$$\begin{aligned}
\nabla_\mu g_{\sigma\rho} &= -2W_\mu g_{\sigma\rho} \\
\nabla_\mu g^{\sigma\rho} &= 2W_\mu g^{\sigma\rho} \\
\nabla_\nu \nabla_\mu g_{\sigma\rho} &= -2g_{\sigma\rho} \dot{\nabla}_\nu W_\mu - 2g_{\sigma\rho} g_{\mu\nu} W^\gamma W_\gamma + 8g_{\sigma\rho} W_\mu W_\nu \\
\nabla_\nu \nabla_\mu g^{\sigma\rho} &= 2g^{\sigma\rho} \dot{\nabla}_\nu W_\mu + 2g^{\sigma\rho} g_{\mu\nu} W^\gamma W_\gamma
\end{aligned}$$

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