university of groningen
faculty of science and engineering

Stability of Linear and Nonlinear Compartmental Systems

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Student: A.G. Wiersma

First supervisor: Dr. ir. B. Besselink

Second supervisor: Prof. dr. C. De Persis

Abstract

This thesis discusses properties that are related to the existence and stability of equilibria of linear and nonlinear compartmental systems. First, linear compartmental systems are considered and necessary and sufficient conditions for the system to be asymptotically stable will be given. Furthermore, for a class of nonlinear compartmental systems known as donor-controlled compartmental systems, sufficient conditions for boundedness of solutions and existence and uniqueness of an equilibrium are given. In addition a necessary condition for the existence of an equilibrium is given for compartmental systems with capacity constraints using the maximum flow of a network. Finally we show that the theory for compartmental systems can be applied to traffic networks and a necessary and sufficient condition for the existence of an equilibrium for traffic networks with demand functions is given.

Contents

1 Introduction

Dynamical systems that are characterized by conservation laws describing a flow between compartments are called compartmental systems. Compartmental systems have received significant attention due to the many applications such as, traffic networks [8], production networks [9], water distribution networks [10] and data networks [11].

In this thesis we will show a way of modeling linear and nonlinear compartmental systems. Furthermore we will look into the existence and uniqueness of equilibria and the stability of a compartmental system, and determine necessary and sufficient conditions on compartmental systems such that asymptotic stability is guaranteed. We will consider linear compartmental systems with constant non-negative input, where linear functions that describe the dynamics can depend on only the state of one compartment or the difference between the states of two compartments. In addition to linear compartmental systems we will also consider nonlinear compartmental systems with constant non-negative input. For nonlinear compartmental systems the functions that describe the dynamics can depend on the state of all compartments. A subcategory of nonlinear compartmental systems called donor-controlled systems where the functions of the dynamics of the system can only depend on the state of one compartment will also be discussed. Donor-controlled systems have additional properties that can be used to determine the uniqueness and stability of an equilibrium. Furthermore we consider static flow networks motivated by compartmental systems with capacity constraints on flow functions. Finally we will consider traffic networks as an application of a compartmental system.

In Section 2 we will formally define what we mean with a compartmental system, and give conditions on the linear and nonlinear compartmental systems such that stability is guaranteed. In Section 3 we will look at the max-flow min-cut theorem and show how this can be used to determine a necessary condition for the existence of an equilibrium for compartmental systems with capacity constraints. In Section 4 we give a way to model traffic networks and we show how the theory in the previous sections can be applied to traffic networks. In addition we give a necessary and sufficient condition for the existence of an equilibrium for traffic networks with demand functions.

2 Stability of compartmental systems

In this section we show how to model linear and nonlinear compartmental systems and analyze the existence and uniqueness of equilibria and the stability of a compartmental system.

2.1 Compartmental systems

By compartmental systems we mean dynamical systems that are characterized by conservation laws and that are partitioned into compartments such that material is transferred between these compartments. In the following definition we formally define what we mean with a compartmental system.

Definition 2.1 (Compartmental system)**.** *A compartmental system with* n *compartments is a system of the form*

$$\dot{x}_i = u_i + \sum_{i=1}^{n} (F_{ij} - F_{ji}) - F_{0i}, \tag{1}$$

where x_i *is the state of the i-th compartment. Figure 1 is the representation of the i-th compartment of a compartmental system. The inflow to a compartment* i *from another compartment* j*, with* $j \neq i$*, is given as* F_{ij}*. The outflow from compartment* i *to compartment* j *is given as* F_{ji}*. The inflow from outside the system to compartment* i *is given as* u_i*. The outflow from compartment* i *to outside the system is given as* F_{0i}*. In addition a compartmental system satisfies the following constraints:*

(i) $u_i \geq 0$ *for all* $t \geq 0$ *and* $i \in I_n$*;*

(ii) $F_{ij} \geq 0$ *and* $F_{0i} \geq 0$ *for all* $i, j \in I_n, i \neq j, x \geq 0$ *and* $t \geq 0$*;*

(iii) If $x_i = 0$ *then* $F_{0i} = 0$ *and* $F_{ji} = 0$ *for all* $i \in I_n$*,*

with $I_n = \{1, \ldots, n\}$*.*

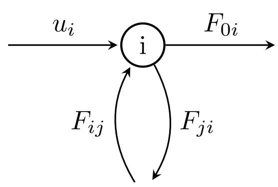

Figure 1: Representation of an i-th compartment of a compartmental system.

The inflows u_i are generally constant or functions of only time. In this section we only consider u_i to be constant. The functions F_{ji}, F_{ij} and F_{0i} can be functions of $x_1, \ldots, x_n$ and t.

An example of a compartmental system is a production network. Here the compartments represent the different stages of a production process and the state of a compartment determines the amount of the material in that process.

Before we go deeper into the theory of compartmental systems we first need some graph theory.

2.2 Graph theory

A directed graph, also called a digraph, is an ordered pair $G = (V, E)$, where V is the finite set of vertices or nodes and $E \subset V \times V$ is the set of edges. When $(i, j) \in E$ there is a directed edge from node i to j. We use a graph G to represent the compartmental system from Definition 2.1. We have $F_{ji} > 0$ for $x_i > 0$ if and only if $(i, j) \in E$. This means that if $F_{ji} = 0$ for $x_i > 0$ then $(i, j) \notin E$, i.e., there is no directed edge from node i to j. In the following definition we define what we mean by a directed path.

Definition 2.2 (Directed path)**.** *A directed path from node i to j is an ordered sequence $v_0 = i, v_1, v_2, v_3, \cdots, v_j = j$ of distinct nodes such that $(v_k, v_{k+1}) \in E$ for all k with $0 < k \leq j - 1$.*

An example of a directed path is the path from node 1 to node 4 in Figure 2. When there is a directed path from a node to outside the system we call such a node outflow connected which we define as follows.

Definition 2.3 (Outflow connected)**.** *A node $i \in V$ is called outflow connected (or externally connected) if there is an directed path from node $i \in V$ to outside the system, i.e., either $F_{0i} > 0$ for $x_i > 0$ or there is a directed path from node i to $j \in V$ with $F_{ji} > 0$ and $F_{0j} > 0$ for $x_i, x_j > 0$. A graph is called outflow connected when all the nodes $i \in V$ are outflow connected. When the graph of a compartmental system as in Definition 2.1 is outflow connected we also say that the system is outflow connected.*

An example of an node that is outflow connected is node 1 in Figure 2, as we have a directed path from node 1 to node 3 and for node 3 we have an outflow F_{03} to the outside world. Furthermore since all the nodes in the graph of Figure 2 are outflow connected we have that the graph is outflow connected. If we delete the edge $(4, 3) \in E$ then the graph in Figure 2 is not outflow connected as there is no path from node 2 and node 4 to node 3 which is the only node that has an outflow F_{03}.

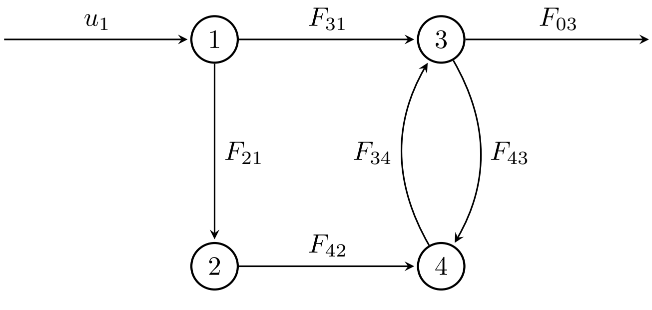

Figure 2: Graph which is outflow connected.

2.3 Linear compartmental systems

First we consider linear compartmental systems. We describe the linear compartmental system in the same manner as in Definition 2.1 with some additional constraints given as follows.

Definition 2.4 (Linear compartmental system)**.** *A linear compartmental system is a system of the form*

$$\dot{x}_i = u_i + \sum_{i=1}^{n} (f_{ij}x_j - f_{ji}x_i) - f_{0i}x_i, \tag{2}$$

in which the constants $f_{ij}, f_{0i} \in \mathbb{R}$ *satisfy*

(i) $f_{ij} \geq 0$ *for all* $i, j \in I_n$*;*

(ii) $f_{0i} \geq 0$ *for all* $i \in I_n$*;*

(iii) $u_i \geq 0$ *for all* $i \in I_n$*,*

with $I_n = \{1, \cdots, n\}$.

Definition 2.4 is a special case of Definition 2.1 with the flow functions $F_{ij} = f_{ij}x_j$. In order to analyze the stability of linear compartmental systems we will use some results from positive systems theory.

2.3.1 Positive systems

In this section we consider systems of the form

$$\dot{x}(t) = Ax(t) + b, \tag{3}$$

where $x \in \mathbb{R}^n$ is the state of a system and b is a non-negative vector $b \in \mathbb{R}^n$. When the state $x(t)$ is non-negative for all time $t \geq 0$ for a positive initial state $x(0)$ we call such a system a positive system. Before we give an exact definition of a positive system we first introduce some notation.
For a matrix $A = [a_{ij}]$ and a vector $x \in \mathbb{R}^n$ we write

(i) $x > 0$ if for all elements x_i of x we have $x_i > 0$;

(ii) $x \geq 0$ if for all elements x_i of x we have $x_i \geq 0$;

(iii) $A > 0$ if for all elements a_{ij} of A we have $a_{ij} > 0$;

(iv) $A \geq 0$ if for all elements a_{ij} of A we have $a_{ij} \geq 0$ and $a_{ij} > 0$ for at least one element.

This notation allows us to give the following definition of a positive system.

Definition 2.5 (Positive systems)**.** *A dynamical system as in* (3) *is positive if* $x(0) \geq 0$ *implies* $x(t) \geq 0$ *for all* $t \geq 0$.

There is a theorem that makes it easier to check if a system is positive. For this theorem we first need the following definition.

Definition 2.6 (Metzler matrix)**.** *A matrix* $A \in \mathbb{R}^{n \times n}, n \geq 2$*, is called Metzler if all its off-diagonal elements are non-negative.*

Now the following theorem from [4] can be used to check if a system as in (3) is positive.

Theorem 2.7 (Theorem 9.3 in [4])**.** *For the system* (3) *the following are equivalent:*

(i) the system is positive, that is, $x(t) \geq 0$ for all t and all $x(0) \geq 0$;

(ii) A is Metzler and $b \geq 0$.

Proof. The proof can be found in [4]. □

Before we give necessary and sufficient conditions for the stability of positive system we first need the following definition.

Definition 2.8 (Hurwitz matrix)**.** *A matrix A is called Hurwitz when all the eigenvalues of A have a negative real part, i.e, we have $Re(\lambda_i) < 0$ for all $\lambda_i \in \sigma(A)$.*

Now we can use the next theorem from [4] to determine if the system (3) has a unique equilibrium x^*.

Theorem 2.9 (Theorem 9.5 in [4])**.** *For a Metzler matrix A, the following statements are equivalent:*

(i) A is Hurwitz;

(ii) A is invertible and $-A^{-1} \geq 0$;

(iii) for all $b \geq 0$, there exists a unique equilibrium $x^* \geq 0$ solving $Ax^* + b = 0$.

Proof. The proof can be found in [4]. □

The previous theorem now leads to the following corollary, which we will use in the next section to check the stability of a compartmental system.

Corollary 2.9.1. *Consider a system as in* (3)*, where A is Metzler and $b \geq 0$. If the matrix A is Hurwitz, then the following statements hold*

(i) the system has a unique equilibrium point $x^* \in \mathbb{R}^n$, that is, a unique solution to $Ax^* + b = 0$;

(ii) the equilibrium point x^* is non-negative;

(iii) all trajectories converge asymptotically to x^*.

2.3.2 Linear compartmental system and positive systems

We can rewrite a linear compartmental system which has the form (2) in the same matrix form as in (3) so that we can use Corollary 2.9.1 to prove the existence and asymptotic stability of the equilibrium. In order rewrite the system (2) in matrix form we first need the following definition.

Definition 2.10 (Compartmental matrix)**.** *A matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is called a compartmental matrix if*

(i) All the diagonal elements are non-positive, i.e., $a_{ii} \leq 0$ for all $i \in I_n$;

(ii) All off-diagonal elements are non-negative, i.e., $a_{ij} \geq 0$ for all $i \neq j$ and $i, j \in I_n$ (A is Metzler);

(iii) The column sum is non-positive, i.e., $\sum_{i=1}^{n} a_{ij} \leq 0$ for all $j \in I_n$ (A is column diagonally dominant $(|a_{jj}| \geq \sum_{i=1, i \neq j}^{n} a_{ij})$),

with $I_n = \{1, \ldots, n\}$.

We then consider a matrix $C = [c_{ij}] \in \mathbb{R}^{n \times n}$ which is defined as

$$c_{ij} = \begin{cases} -f_{0i} - \sum_{h=1, h \neq i}^{n} f_{hi} & \text{if } i = j, \\ f_{ij} & \text{if } i \neq j, \end{cases} \tag{4}$$

where f_{ij} are as in Definition 2.4. The matrix C is a compartmental matrix since by Definition 2.4 we have that $f_{ij}, f_{0i} \geq 0$ for all $i, j \in I_n$. Therefore we have that all the diagonal elements of C are non-positive since $f_{0i} + \sum_{h=1}^{n} f_{hi} \geq 0$. The off-diagonal elements of C are non-negative since $f_{ij} \geq 0$ for all $i, j \in I_n$. In addition for the $i - th$ column sum of C we have

$$-(f_{0i} + \sum_{h=1}^{n} f_{hi}) + \sum_{h=1}^{n} f_{hi} = -f_{0i} \leq 0.$$

Therefore the matrix C is compartmental. By using the matrix C we can now write (2) compactly as

$$\dot{x}(t) = Cx(t) + u, \tag{5}$$

where $u \in \mathbb{R}^n$ is the inflow vector. We can now use the following theorem from [4] which relates outflow connectivity of a graph of the compartmental system to the matrix C being Hurwitz.

Theorem 2.11 (Theorem 9.13 in [4])**.** *Consider the linear compartmental system as in Definition 2.4 written in the form as in* (5) *with compartmental matrix* C *and with graph* G*. Then the following statements are equivalent:*

(i) the system is outflow connected;

(ii) the compartmental matrix C *is Hurwitz.*

Proof. The proof can be found in [4] □

The above result leads to the main result in [4] about the asymptotic behavior of linear compartmental systems.

Theorem 2.12. *Consider the linear compartmental system as in Definition 2.4, with constant input* $u \geq 0$ *written in the form as in* (5) *with compartmental matrix* C *and with graph* G*. If the system is outflow-connected, then the compartmental matrix* C *is invertible and every solution tends exponentially to the unique equilibrium* $x^* = -C^{-1}u \geq 0$.

Proof. The linear system is in the form (3) and C is a compartmental matrix and therefore by definition also Metzler. Furthermore by assumption we have that $u \geq 0$. The graph is outflow connected therefore by Theorem 2.11 we have that C is Hurwitz. In addition, by Theorem 2.9 we have that the compartmental matrix C is invertible. Furthermore by Corollary 2.9.1 we have that every solution tends exponentially to the unique equilibrium $x^* = -C^{-1}u \geq 0$. □

Example 2.13. *Consider the following compartmental system whose graph is given in Figure 3.*

$$\dot{x} = \begin{bmatrix} -f_{31} - f_{21} & 0 & 0 & 0 \\ f_{21} & -f_{42} & 0 & 0 \\ f_{31} & 0 & -f_{43} & 0 \\ 0 & f_{42} & f_{43} & -f_{04} \end{bmatrix} x + \begin{bmatrix} u_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{6}$$

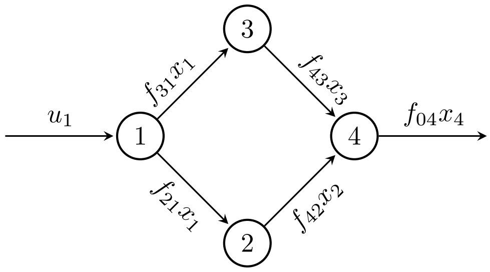

Figure 3: Graph of the system (6).

Let $f_{31} = f_{21} = 2, f_{43} = 3, f_{42} = 1$ *and* $f_{04} = 5$. *We then have*

$$C = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 2 & 0 & -3 & 0 \\ 0 & 1 & 3 & -5 \end{bmatrix}, \tag{7}$$

and since the graph in Figure 3 is outflow connected we use Theorem 2.12 to find that the system is asymptotically stable as every solutions tends to the unique equilibrium $x^* = \begin{bmatrix} \frac{3}{2} & 3 & 1 & \frac{6}{5} \end{bmatrix}^\top$ *for* $u_1 = 6$.

There are other ways to describe a linear compartmental system than the form as given in Definition 2.4. Another way, with different type of flows between compartments, is given in [2] which we will discuss in the next section.

2.3.3 Linear compartmental system with two types of flow

Another way of describing linear compartmental systems is given in [2], where nonlinear compartmental systems are considered. In this section the notions are translated to the linear case. There are two types of flow described, namely a so-called g-type flow and an h-type flow. An intuitive illustration of these types is given in Figure 4.

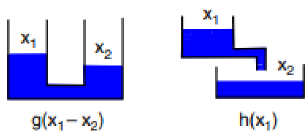

Figure 4: g-type (left) and h-type (right) flows in a fluid system.[2]

For the g-type flow the flow between node i and j can go in both directions and it depends on the difference of the states x_i and x_j. If there is a g-type flow from node i to node j the flow between the two nodes is given as $g_k(x_i - x_j)$, where g_k is the flow constant with $1 \le k \le r$ and r the number of g-type flows in the system. The linear functions $g_k(x_i - x_j)$ are combined in the vector function g^* as

$$g^* = \begin{bmatrix} g_1^* \\ \vdots \\ g_r^* \end{bmatrix}, \quad \text{with} \quad g_k^* = g_k(x_i - x_j). \tag{8}$$

For the h-type flow the flow can only go from node i to node j and it only depends on the state x_i. If there is h-type flow from node i to node j the flow is given as $h_l x_i$, where h_l is the flow constant with $1 \le l \le m$ and m the number of h-type flows. The linear functions $h_l x_i$ are combined in the vector function h^* as

$$h^* = \begin{bmatrix} h_1^* \\ \vdots \\ h_m^* \end{bmatrix}, \quad \text{with} \quad h_l^* = h_l x_i. \tag{9}$$

We can relate the g-type and the h-type flows to the flow constants f_{ij} which we used in Definition 2.4 in the the previous section as follows. For the g-type flow from node i to node j the flow is then described as two flows. Those are a flow $F_{ji} = f_{ji} x_i$ from node i to j and a flow $F_{ij} = f_{ij} x_j$ from node j to i. We then have that $f_{ij} = f_{ji} = g_k$ so that we get that the flow between node i to j is $f_{ji}(x_i - x_j)$. The h-type flow from node i to node j is then described as $F_{ji} = f_{ji} x_i$, and we have $f_{ji} = h_l$.

The inflow is as in Definition 2.4 described in a vector $u \in \mathbb{R}^n$, where u_i is the inflow from outside the system to node i. In addition, there is a control input $w(t) \in \mathbb{R}^p$, where p is the number of control inputs. The control input can be used to add more inflows like the inflow u if the flow does not depend on x. Or as will be done later in this section choose w such that the control input w becomes a g-type flow or a h-type flow. The dynamics are then described by

$$\dot{x}(t) = Sg^*(x(t)) + Rh^*(x(t)) + Bw(t) + u(t), \tag{10}$$

with g^* and h^* as in (8) and (9) respectively, constant inflow vector $u \in \mathbb{R}^n$ and the matrices S, R and B defined as follows:

$S_{ik} = -1, \quad S_{jk} = 1$ when there is a flow g-type flow g_k from node i to j;
$R_{il} = -1, \quad R_{jl} = 1$ when there is a h-type flow h_l from node i to j;
$B_{ip} = -1, \quad B_{jp} = 1$ when there is a control input w_p from node i to node j.

In Figure 5 an example of an compartmental system with g-type and h-type flows is shown. In Figure 6 the same compartmental system as in Figure 5 is shown with the flows as described in Definition 2.4. An example of a system with g-type and h-type flows is given next.

Example 2.14. *Consider the compartmental system whose graph is given in Figure 5. The dynamics of the system are then described by* (10) *with the following vector functions,*

$$g^*(x) = \begin{bmatrix} g_1(x_1 - x_3) \\ g_2(x_1 - x_2) \\ g_3(x_2 - x_4) \end{bmatrix}, \quad h^*(x) = \begin{bmatrix} h_1 x_3 \\ h_2 x_3 \end{bmatrix},$$

and matrices

$$S = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & -1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

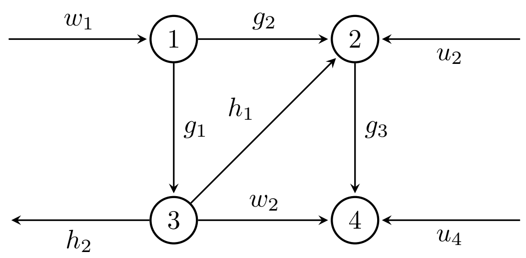

Figure 5: Graph of the system (10) with $g^*, h^* S, R$ and B as given in Example 2.14.

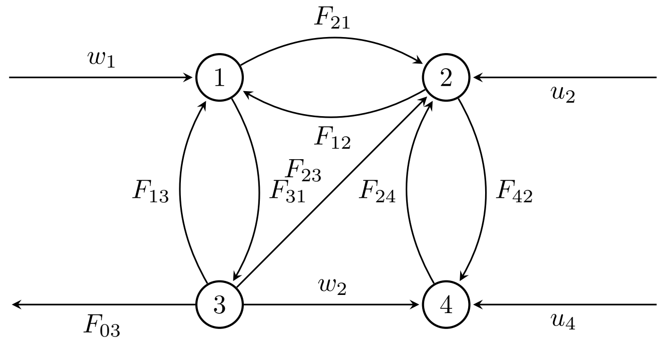

Figure 6: Graph of the system in Example 2.14 with notation as in Definition 2.4.

First lets look at the case were there is no added control input, i.e, $w = 0$. In this case we have

$$\dot{x}(t) = Sg^*(x(t)) + Rh^*(x(t)) + u. \tag{11}$$

We then define the matrices G, H and $\tilde{R}$ as

$$G = \operatorname{diag}(g_1, \cdots, g_r), \quad H = \operatorname{diag}(h_1, \cdots, h_m), \quad \tilde{R}_{ij} = \min\{R_{ij}, 0\},$$

where we again have that r is the number of g-type flows and m is the number of h-type flows. Then take $g^* = -GS^\top x$ and $h^* = -H\tilde{R}^\top x$ and we can rewrite the system (11) as

$$\begin{aligned} \dot{x}(t) &= -SGS^\top x(t) - RH\tilde{R}^\top x(t) + u, \\ &= -\left[SGS^\top + RH\tilde{R}^\top\right] x(t) + u. \end{aligned}$$

Now the system is in the form (5) which we had in the previous section. Since u is constant and positive and the matrix $-\left[SGS^\top + RH\tilde{R}^\top\right]$ is compartmental we can use Theorem 2.12. As a result we have that if the system is outflow connected every solution tends asymptotically to the unique equilibrium $x^* = \left[SGS^\top + RH\tilde{R}^\top\right]^{-1} u$.

For the control input we now take $w = -\gamma B^\top x$ where $\gamma \in \mathbb{R}_+$ is a positive gain. In the following example we can see what happens when we take this control input. When the control input $w = -\gamma B^\top x$ is used for the system as in (10) the dynamics for the system become

$$\begin{aligned} \dot{x}(t) &= -SGS^\top x(t) - RH\tilde{R}^\top x(t) - \gamma BB^\top x + u, \\ &= -\left[SGS^\top + RH\tilde{R}^\top + \gamma BB^\top\right] x(t) + u. \end{aligned} \tag{12}$$

If the compartmental system (12) is outflow connected, has a constant non-negative inflow vector u and the matrix

$$-\left[SGS^\top + RH\tilde{R}^\top + \gamma BB^\top\right],$$

is a compartmental matrix. Then by Theorem 2.12 we have that every solution tends exponentially to the unique equilibrium

$$x^* = \left[SGS^\top + RH\tilde{R}^\top + \gamma BB^\top\right]^{-1} u.$$

Example 2.15. *Consider the system in Example 2.14 with control input $w = -\gamma B^\top x$. Then the dynamics for the system become*

$$\dot{x}(t) = Sg^*(x(t)) + Rh^*(x(t)) - \gamma BB^\top x + u,$$

with

$$-\gamma BB^\top x = -\gamma \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\gamma x_1 \\ 0 \\ -\gamma(x_3 - x_4) \\ \gamma(x_3 - x_4) \end{bmatrix}.$$

So for node 1 we now have that the control input w_1 is a flow $-\gamma x_1$ which is an h-type flow from node 1 to outside the system. Here we have that the flow constant $h_3 = \gamma$. The control input w_3, which is between node 3 and 4, is now an g-type flow $\gamma(x_3 - x_4)$ between node 3 and 4, where the flow constant $g_4 = \gamma$. In Figure 7 the graph of the new system can be seen.

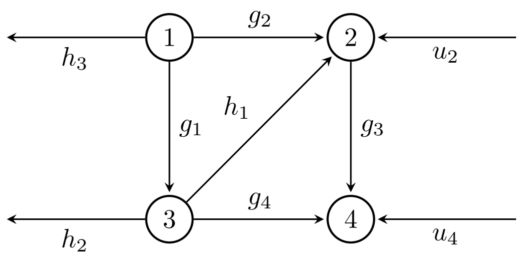

Figure 7: Graph of the system in Example 2.15 with $w = -\gamma B^\top x$ and $h_3 = g_4 = \gamma$.

We now consider a shifted system for which we have a similar theorem as Theorem 2.12 for the system of the form (13). For this theorem we need the following assumption.

Assumption 2.16. *An unique equilibrium vector $\bar{x}$ exists corresponding to a control $\bar{w}$:*

$$0 = Sg^*(\bar{x}) + Rh^*(\bar{x}) + B\bar{w} + u.$$

Now let $v = w - \bar{w}$ and $z = x - \bar{x}$ then without restrictions we can consider the stabilization problem with the shifted system

$$\dot{z}(t) = Sg^*(z(t)) + Rh^*(z(t)) + Bv(t). \tag{13}$$

For this shifted system we can say something about the stability using a theorem from [2], for this we first need the following definition.

Definition 2.17 (Strongly connected)**.** *A graph $G = (V, E)$ of a compartmental system* (13) *is called strongly connected if there is an directed path from node i to node j for all $i, j \in V$.*

Remark 2.18. *Note that from Definition 2.2 we had that a directed path from node i to j is an ordered sequence $v_0 = i, v_1, v_2, v_3, \cdots, v_j = j$ of distinct nodes such that $(v_k, v_{k+1}) \in E$ for all k with $0 < k \le j - 1$. If there is a g-type flow from node i to node j then we have $(i, j), (j, i) \in E$ (although we only draw one direction in the associated graph). As the flow of a g-type flow can go in both directions. For an h-type flow from node i to node j we have $(i, j) \in E$.*

This leads us to the following theorem.

Theorem 2.19 (Theorem 1 in [2])**.** *If the graph is strongly connected, the following statements are equivalent;*

(i) *Matrix $\begin{bmatrix} S & R & B \end{bmatrix}$ has row rank n;*

(ii) *The system graph is outflow connected;*

(iii) *System* (13) *can be globally asymptotically stabilized to $z = 0$ by the control $v = -\gamma B^\top z$.*

Proof. The proof can be found in [2]. □

This theorem is similar to Theorem 2.12 from the previous section. The added assumption of the graph being strongly connected is there so that statement (i) implies that statement (ii) holds. We can rewrite the shifted system (13) in the same manner as we did for the system (11), where we take $v = -\gamma B^\top z$. We then have

$$\dot{z}(t) = -\left[SGS^\top + RH\tilde{R}^\top + \gamma BB^\top\right] z(t). \tag{14}$$

This system has an equilibrium at $z = 0$ and by Theorem 2.12 we have that this equilibrium of the system (13) is asymptotically stable if the system graph is outflow connected.

So in conclusion for linear compartmental systems with constant non-negative input we have found a sufficient condition such that the system asymptotically stable. This sufficient condition is that the system should be outflow connected. In addition we can specify a linear compartmental system with a control input that has two types of flow. One that is dependent on the difference between states and the other which only depends the state of the donor. For this system we found that the necessary and sufficient conditions for the system to be asymptotically stable are that the system should be outflow connected and strongly connected.

2.4 Nonlinear compartmental system

In this section we look at the existence and uniqueness of equilibria and the stability of nonlinear compartmental systems. We describe the nonlinear compartmental system extending Definition 2.1 in the following definition.

Definition 2.20 (Nonlinear compartmental system). *A nonlinear compartmental system is a system of the form*

$$\dot{x}_i = u_i + \sum_{i=1}^{n} (F_{ij} - F_{ji}) - F_{0i}. \tag{15}$$

Where the functions F_{ij}, F_{ji} and F_{0i} are functions of $x_1, \ldots, x_n$ and t that satisfy

(i) $u_i \geq 0$ for all $t \geq 0$ and $i \in I_n$;

(ii) $F_{ij} \geq 0$ and $F_{0i} \geq 0$ for all $i, j \in I_n, i \neq j$, $t \geq 0$ and $x \geq 0$;

(iii) If $x_i = 0$ then $F_{0i} = 0$ and $F_{ji} = 0$ for all $i \in I_n$,

with $I_n = \{1, \ldots, n\}$.

In order to analyze the stability of nonlinear compartmental systems we first use the following theorem from Jacquez [3].

Theorem 2.21 (Theorem 5 in [3]). *Let $\dot{x} = F(x)$ be a C^1 differential equation on a convex subset Ω of $\mathbb{R}^n$. Suppose that the Jacobian matrix $DF(x) = (\partial F_i / \partial x_j)$ is a compartmental matrix for all $x \in \Omega$. Then,*

(i) $V(x) = \sum_i |F_i(x)|$ is a monotone decreasing function on solutions $x(t)$ of $\dot{x} = F(x)$;

(ii) Every solution of $\dot{x} = F(x)$ is either unbounded or tends to the equilibrium set.

Remark 2.22. *Note that in order to use Theorem 2.21 for the compartmental system (15) we take*

$$F(x) = \begin{bmatrix} \sum_{j \neq 1} (F_{1j} - F_{j1}) - F_{01} \\ \sum_{j \neq 2} (F_{2j} - F_{j2}) - F_{02} \\ \vdots \\ \sum_{j \neq n} (F_{nj} - F_{jn}) - F_{0n} \end{bmatrix} + u. \tag{16}$$

Since u is constant we then need that the Jacobian of

$$\begin{bmatrix} \sum_{j \neq 1} (F_{1j} - F_{j1}) - F_{01} \\ \sum_{j \neq 2} (F_{2j} - F_{j2}) - F_{02} \\ \vdots \\ \sum_{j \neq n} (F_{nj} - F_{jn}) - F_{0n} \end{bmatrix},$$

is a compartmental matrix for all $t \geq 0$. This is a similar requirement as we had for the linear compartmental system $\dot{x} = Cx + u$ in Theorem 2.12, namely that the matrix C is compartmental.

For linear compartmental systems the stability of the equilibrium is related to the outflow connectedness of the graph of the compartmental system, recall Theorem 2.12. This is also the case of nonlinear compartmental systems as we can see in the following theorem from Jacquez [3]. This theorem gives necessary and sufficient conditions on the system as in (15) such that an equilibrium exists and every solution is bounded. This theorem is a generalized theorem from Maeda et al. [1].

Theorem 2.23 (Theorem 9 in [3]). *Consider a system as in* (15) *which has constant input vector* $u \geq 0$ *and satisfies the monotonicity hypothesis*

$$\frac{\partial F_{ij}}{\partial x_k} \geq 0 \text{ for all } i, j, k, i \neq j. \tag{17}$$

Then,

(i) For any fixed input vector u, (15) *has an equilibrium if and only if every solution of* (15) *is bounded;*

(ii) System (15) *has an equilibrium for every fixed* $u \geq 0$ *if and only if* (15) *is outflow connected and the functions* F_{ij} *that are part of the path to the outside world are unbounded.*

So we now have conditions on the system (15) such that an equilibrium exists and the solutions are bounded. Note that the theorem does not guarantee that the equilibrium is unique. If we combine the conclusions from the previous two theorems we can conclude the following.

Theorem 2.24. *Suppose that compartmental system* (15) *has constant input* $u \geq 0$ *and let* $F_i(x) = \sum_{j \neq i}(-F_{ji} + F_{ij}) - F_{0i} + u_i$. *Assume that*

(i) $\frac{\partial F_{ij}}{\partial x_k} \geq 0$ *for all* $i, j, k, i \neq j$;

(ii) $\frac{\partial F_i}{\partial x_i} \leq 0$ *for all* i;

(iii) $\frac{\partial F_i}{\partial x_k} \geq 0$ *for all* $i, k, i \neq k$;

(iv) The compartmental system is outflow connected and the functions F_{ij} *that are part of the path to the outside world are unbounded.*

Then all the solutions are bounded and tend to the equilibrium set.

Proof. We will first prove that the system (15) has an equilibrium and that the solutions of the system (15) are bounded. As an assumption we have that the input u is a non-negative constant and by condition (i) that the system satisfies the monotonicity hypothesis. Therefore all the assumptions of Theorem 2.23 are satisfied and we find that the system (15) has an equilibrium for every fixed $u \geq 0$ if and only if the system is outflow connected and the functions F_{ij} that are part of a path to the outside world are unbounded. Therefore by condition (iv) in the statement of Theorem 2.24 we have that an equilibrium exists for every fixed $u \geq 0$, which means that by Theorem 2.23 we also have that the solutions of the system (15) are bounded. We will now prove that all the solutions tend to the equilibrium set. Consider $F(x)$ as in (16), we can then write the system (15) (for fixed $u \geq 0$) as $\dot{x} = F(x)$, which has the following Jacobian.

$$DF(x) = \begin{bmatrix} \frac{\partial F_1(x)}{\partial x_1} & \cdots & \frac{\partial F_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n(x)}{\partial x_1} & \cdots & \frac{\partial F_n(x)}{\partial x_n} \end{bmatrix}, \tag{18}$$

with $F_i(x)$ as defined in Theorem 2.24. From Definition 2.10 we have that this is a compartmental matrix if the following conditions hold:

(i) $\frac{\partial F_i(x)}{\partial x_i} \leq 0$ for all i;

(ii) $\frac{\partial F_i(x)}{\partial x_k} \geq 0$ for all $i, k, i \neq k$;

(iii) $\sum_{i=1}^{n} \frac{\partial F_i(x)}{\partial x_k} \leq 0$ for all k.

Condition (i) and (ii) hold by our assumption. In addition the monotonicity hypothesis implies that condition (iii) also holds as we have

$$\sum_{i=1}^{n} \frac{\partial F_i(x)}{\partial x_k} = -\sum_{i=1}^{n} \frac{\partial F_{0i}}{\partial x_k} \leq 0.$$

We have proven that the Jacobian of $F(x)$ is a compartmental matrix, therefore by Theorem 2.21 we have that the solutions of (15) are either unbounded or tend to the equilibrium set. We already had that by Theorem 2.23 that the solutions were bounded in addition to that an equilibrium exist for every fixed input $u \geq 0$. Therefore we have that all solutions of (15) are bounded and tend to the equilibrium set, which completes the proof. □

Remark 2.25. *Note that condition (ii) and (iii) in Theorem 2.24 are the first two conditions in Definition 2.10 of a compartmental matrix. So condition (ii) and (iii) are there so that the Jacobian of $F(x)$, where $F(x)$ is defined as in* (16)*, is a compartmental matrix for all $t \geq 0$. The third condition in Definition 2.10 is then replaced by condition (i) of Theorem 2.24 since condition (i) of Theorem 2.24 implies that condition (iii) of Definition 2.10 holds. Therefore we have that conditions (i),(ii) and (iii) are also sufficient conditions in order for the Jacobian of $F(x)$ to be a compartmental matrix.*

In the following examples we show why outflow connectedness is needed for an equilibrium to be stable and the unboundedness of the flow functions guarantees that an equilibrium exists for any fixed input vector u.

Example 2.26. *Consider the system which has a graph as in Figure 8.*

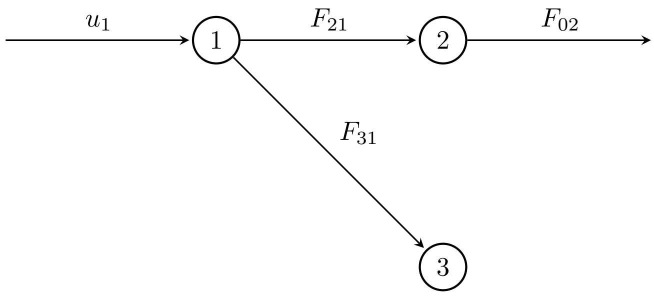

Figure 8: Compartment system which is not outflow connected.

It is clear that the system is not outflow connected since there is no path to the outside world from compartment 3. The dynamics for the compartmental system is given as

$$\begin{aligned}\dot{x}_1 &= u_1 - F_{21} - F_{31},\\ \dot{x}_2 &= F_{21} - F_{02},\\ \dot{x}_3 &= F_{31},\end{aligned} \tag{19}$$

where it is recalled that the functions F_{ij} are as in Definition 2.20. There is an equilibrium when we have $\dot{x} = 0$. So in this case we need that

$$\begin{aligned}F_{21} &= u_1,\\ F_{21} &= F_{02},\\ F_{31} &= 0.\end{aligned}$$

Note that in order for the equilibrium to be stable we need that $F_{31} = 0$ for any $x > \bar{x}$. If this is the case the state x_3 only increases which means that the equilibrium is unstable. Furthermore if we have $F_{31} = 0$ for $x_1 > 0$ then there is no edge $(1,3) \in E$ in the graph. The reduced system obtained by removing the edge $(1,3)$ is outflow connected.

Example 2.27. *Consider the system which has a graph as in Figure 9.*

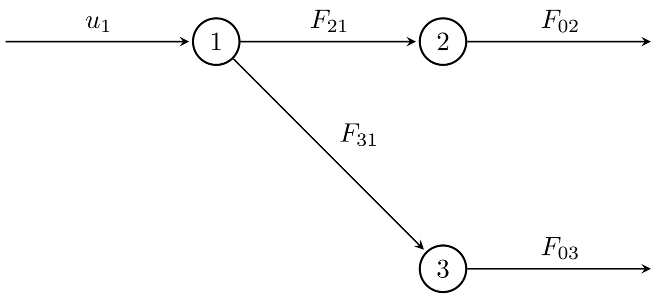

Figure 9: Compartment system which is outflow connected.

The dynamics for the compartmental system is given as

$$\begin{aligned}\dot{x}_1 &= u_1 - F_{21} - F_{31},\\ \dot{x}_1 &= F_{21} - F_{02},\\ \dot{x}_3 &= F_{31} - F_{03}.\end{aligned} \tag{20}$$

Suppose that the flow functions F_{21} and F_{31} are bounded, i.e., there exist $a, b \in \mathbb{R}^n$ such that $F_{21} \leq a$ and $F_{31} \leq b$. Then we can no longer guarantee that an equilibrium exists for any input vector u since if we have $u_1 > a+b$ no equilibrium exists. Namely, we need $\dot{x}_1 = -F_{21} - F_{31} + u = 0$. So in order to guarantee that an equilibrium exists for any constant input vector u we need that the flow functions are unbounded.

2.4.1 Donor-controlled systems

Donor-controlled systems are compartmental systems for which the flow functions F_{ij} only depend on the state of the donor x_j. We describe the donor-controlled systems by extending Definition 2.1 with some additional constraints in the following definition.

Definition 2.28. *A donor-controlled system is a system of the form*

$$\dot{x}_i = u_i + \sum_{i=1}^{n} (F_{ij}(x_j) - F_{ji}(x_i)) - F_{0i}(x_i), \tag{21}$$

where the functions F_{ji} and F_{0i} are functions of x_i and t that satisfy

(i) $u_i \geq 0$ for all $t \geq 0$ and $i \in I_n$;

(ii) $F_{ij} \geq 0$ and $F_{0i} \geq 0$ for all $i, j \in I_n, i \neq j$ and $t \geq 0$;

(iii) If $x_i = 0$ then $F_{0i} = 0$ and $F_{ji} = 0$ for all $j \in I_n$,

with $I_n = \{1, \ldots, n\}$.

Note that the donor-controlled system as in Definition 2.28 is a positive system. We can show this as follows, anytime that there exists an i such that $x_i(t) = 0$, the conditions for the function F_{ji} in Definition 2.28 imply that $F_{ji}(x_i) = F_{0i}(x_i) = 0$, therefore the dynamics for the system (21) become

$$\dot{x}_i = \sum_{j \neq i} F_{ij}(x_j) + u_i. \tag{22}$$

Which means that $\dot{x}_i \geq 0$, whenever $x_i(t) = 0$ since $F_{ij}(x_j) \geq 0$ and $u_i \geq 0$. Therefore we have that the system (21) is non-negative for every $x \geq 0$ and non-negative initial condition $x(0) \geq 0$. Which is by Definition 2.5 a positive system.

Due to a property of the donor-controlled system (21) with $u_i \geq 0$ we can find sufficient conditions for the system to be asymptotically stable, namely the property that

$$\frac{\partial F_{ij}}{\partial x_k} = 0, \quad \text{for } j \neq k.$$

Before we state the theorem that gives these sufficient conditions we first need the following theorem from [1].

Theorem 2.29 (Theorem 5 in [1])**.** *Consider a donor-controlled system as in Definition 2.28 with a constant input $u \geq 0$ and let the flow functions F_{ij} satisfy*

$$\frac{\partial F_{ij}}{\partial x_j} \geq 0 \textit{ for all } i, j, i \neq j.$$

Then, the equilibrium is independent of the initial state and uniquely determined by u if the system is outflow connected and the flow functions F_{ij} that are part of a path to the outside world satisfy

$$F_{ij}(x_j) - F_{ij}(y_j) > 0, \quad \textit{for any } x_j > y_j \geq 0.$$

Proof. The proof can be found in [1]. □

We can now state the main theorem of this section.

Theorem 2.30. *Consider a donor-controlled system as in Definition 2.28 with a constant input $u \geq 0$ and let the system satisfy*

(i) $\frac{\partial F_{ij}}{\partial x_j} \geq 0$ *for all $i, j \in I_n$ with $i \neq j$;*

(ii) The system (21) *is outflow connected and the functions that are part of a path to the outside world are unbounded;*

(iii) The F_{ij} that are part of a path to the outside world satisfy $F_{ij}(x_j) - F_{ij}(y_j) > 0$, for any $x_j > y_j \geq 0$ and $i, j \in In$,

with $I_n = \{1, \ldots, n\}$ Then, the equilibrium is globally asymptotically stable for $x \geq 0$.

Proof. We will first show that the equilibrium of the system (21) is unique. By assumption we have that the system (21) has a constant input $u \geq 0$, the functions F_{ij} satisfy the monotonicity hypothesis (condition (i)), the system is outflow connected and the functions F_{ij} that are part of the path to the outside world are unbounded. In addition we have that condition (iii) is satisfied, which means that by Theorem 2.29 we have that the equilibrium is unique. We will now show that every solution of (21) tends to the equilibrium by showing that the conditions of Theorem 2.24 are satisfied. As in Theorem 2.24 let

$$F_i(x) = \sum_{j \neq i} (F_{ij} - F_{ji}) - F_{0i} + u_i,$$

then for $i \neq j$ we have

$$\frac{\partial F_i(x)}{\partial x_i} = \sum_{j \neq i} \left(\frac{\partial F_{ij}}{\partial x_i} - \frac{\partial F_{ji}}{\partial x_i} \right) - \frac{\partial F_{0i}}{\partial x_i} = \sum_{j \neq i} \left(-\frac{\partial F_{ji}}{\partial x_i} \right) - \frac{\partial F_{0i}}{\partial x_i},$$

since F_{ij} of a donor-controlled system only depends on the state x_i, i.e., we have $\frac{\partial F_{ij}}{\partial x_i} = 0$. In addition by the monotonicity hypothesis we have

$$\frac{\partial F_i(x)}{\partial x_i} = \sum_{j \neq i} \left(-\frac{\partial F_{ji}}{\partial x_i} \right) - \frac{\partial F_{0i}}{\partial x_i} \leq 0,$$

so condition (ii) of Theorem 2.24 is satisfied. Furthermore for $i \neq k$ we have

$$\frac{\partial F_i(x)}{\partial x_k} = \sum_{j \neq i} \left(\frac{\partial F_{ij}}{\partial x_k} - \frac{\partial F_{ji}}{\partial x_k} \right) - \frac{\partial F_{0i}}{\partial x_k} = \frac{\partial F_{ik}}{\partial x_k} \geq 0,$$

which means that condition (iii) of Theorem 2.24 is also satisfied. By assumption we have that condition (i) and (iv) of Theorem 2.24 are satisfied. So by Theorem 2.24 we have that all the solutions are bounded and tend to the equilibrium set. Furthermore by Theorem 2.29 we had that the equilibrium is unique therefore we have that the system is asymptotically stable, which completes the proof. □

Remark 2.31. *The unboundedness of the functions F_{ji} that are part of the outside world is not an necessary condition for the donor-controlled system to be asymptotically stable. The unbounded of the functions F_{ij} only guarantees that an equilibrium exists for any input $u \geq 0$. If an equilibrium for the system exists and all the other conditions are satisfied then the system is still asymptotically stable.*

Example 2.32. *Consider the system which has a graph as in Figure 10.*

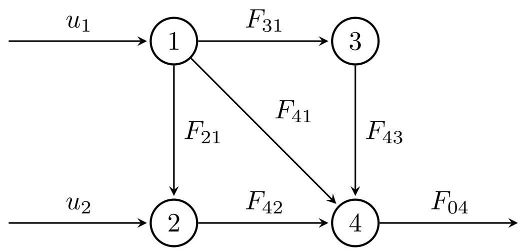

Figure 10: Graph of the system in Example 2.32

The dynamics of the compartmental system is given as

$$\begin{aligned}
\dot{x}_1 &= -F_{21}(x_1) - F_{31}(x_1) - F_{41}(x_1) + u_1, \\
\dot{x}_2 &= -F_{42}(x_2) + F_{21}(x_1) + u_2, \\
\dot{x}_3 &= -F_{43}(x_3) + F_{31}(x_1), \\
\dot{x}_4 &= F_{41}(x_1) + F_{42}(x_2) + F_{43}(x_3) - F_{04}(x_4),
\end{aligned} \tag{23}$$

with the functions given as

$$\begin{aligned}
F_{21}(x_1) &= 4(e^{x_1} - 1), \\
F_{31}(x_1) &= 2(1 - e^{-x_1}), \\
F_{41}(x_1) &= 3(1 - e^{-x_1}), \\
F_{42}(x_2) &= 2(e^{x_2} - 1), \\
F_{43}(x_3) &= 1(e^{x_3} - 1), \\
F_{04}(x_4) &= 3(e^{x_4} - 1).
\end{aligned} \tag{24}$$

It is clear that the system is outflow connected and that the functions F_{ij} satisfy the monotonicity hypothesis. Furthermore the functions F_{31}, F_{42}, F_{43} and F_{04}, which are part of a path to the outside world are unbounded and satisfy condition (iii) of Theorem 2.30. So by Theorem 2.30 we have that the system is asymptotically stable. A plot of the trajectories for zero initial conditions is given in Figure 11 , which shows that the solutions of the system (23) *indeed tend to an equilibrium and that the equilibrium is stable.*

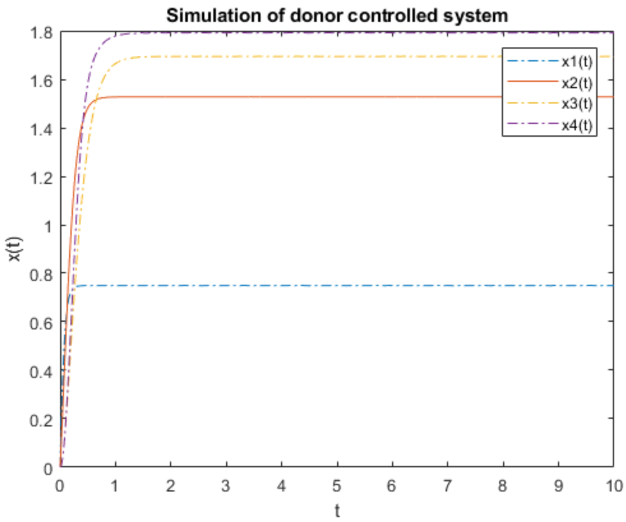

Figure 11: The Matlab simulation of the system (23), with functions F_{ij} as given in (24) and $u_1 = 10, u_2 = 5$.

So in conclusion for nonlinear compartmental systems with constant non-negative input and with flows F_{ji} that are functions of $x_1, \ldots, x_n$ and t we found necessary and sufficient conditions such that the solutions of the system are all bounded and tend to the equilibrium set. Necessary and sufficient conditions for this where that the system is outflow connected, the Jacobian of the system is compartmental and that the functions F_{ji} satisfy the monotonicity hypothesis and that the functions that are part of a path to the outside world are unbounded. For the nonlinear donor-controlled system we had the additional constraint that the functions F_{ji} should be strictly increasing which resulted in a stronger theorem, namely that the equilibrium is unique and asymptotically stable. In addition the requirement that the functions are unbounded is not necessary, if there exists a equilibrium then the system is asymptotically stable when the other requirements are satisfied.

3 Flow networks with capacity constraints

In this section we look into the existence of an equilibrium of an compartmental system with capacity constraints, i.e., the flow F_{ij} is bounded. When the flow functions F_{ij} are bounded then Theorem 2.24 for nonlinear compartmental systems and Theorem 2.30 for donor-controlled nonlinear compartmental systems can't be used to guarantee that an equilibrium exists. However as the unboundedness of the flow F_{ij} is not a necessary condition for the existence of an equilibrium an equilibrium can still exist when F_{ij} is bounded. In this section we will look into necessary conditions for the existence of an equilibrium for a compartmental system with capacity constraints.

First, contrary to Section 2 we do not consider the dynamics of a compartmental system. Instead we focus on the structure of the graph and the capacity of the edges of the graph of the compartmental system. Consider a flow network, which consists of a directed graph $G = (V, E)$, where V is the set of nodes and $E \subset V \times V$ is the set of edges. As we had in Section 2, when $(i, j) \in E$, with $i, j \in V$, there is a directed edge from node i to j and a flow F_{ji} from node i to j. In addition a flow network consists of a source and sink. A source $s \in V$ is a node which has only outgoing flow, i.e., for $i, s \in V$ with $i \neq s$ we have $(s, i) \in E$ for some i and $(i, s) \notin E$ for all i. Dually, a sink $t \in V$ only has incoming flow, i.e., for $i, t \in V$ with $i \neq t$ we have $(i, t) \in E$ for some i and $(t, i) \notin E$ for all i.

A flow network is a graph where there is a flow going to the sink from the source, where the maximum flow through an edge is limited by the capacity of an edge. As in Definition 2.1 F_{ji} is the flow from node i to j through edge (i, j). Let $c(i, j)$ be the capacity of the edge (i, j) then we have $F_{ji} \leq c(i, j)$ for all $i, j \in V$. This leads us to the following definition of a flow network.

Definition 3.1 (Flow network)**.** *A flow network (G, s, t, c) consists of a directed graph $G = (V, E)$, where V is the set of nodes and $E \subset V \times V$ is the set of edges, a source $s \in V$, a sink $t \in V$ and a capacity function $c : V \times V \to \mathbb{R}_+$. Furthermore the flow F_{ji} from node i to j satisfies the following constraints:*

(i) For any non-source and non-sink node, the input flow is equal to output flow, i.e., for all $i \in V \backslash \{s, t\}$ we have $\sum_{j=1} F_{ij} = \sum_{j=1} F_{ji}$;

(ii) For any flow in an edge we have $0 \leq F_{ji} \leq c(i, j)$.

Remark 3.2. *Note that Definition 3.1 assumes a single source s and a single sink t. If a network has more than one source or sink we can translate this network to the flow network from Definition 3.1 by adding two nodes which are the main source and sink. From the main source we then add edges with an infinite capacity to all the source nodes and from the sinks we add edges with an infinity capacity to the main sink.*

We can translate the notion of a source and a sink of a flow network for the graph of a compartmental system as in Definition 2.1 as follows. First add the nodes s, t to the set of nodes V. Let the source s be the source of all the flows u_i, i.e., if we have an inflow u_i to node i then $(s, i) \in E$ with capacity $c(i, s) = u_i$. In addition let the sink be the receiver of the flows F_{0i} from node i, i.e., we if we have an outflow F_{0i} from node i then $(i,t) \in E$ with capacity $c(i,t)$ and we have $F_{0i} \leq c(i,t)$. An illustrative example is shown in Figure 12 and Figure 13.

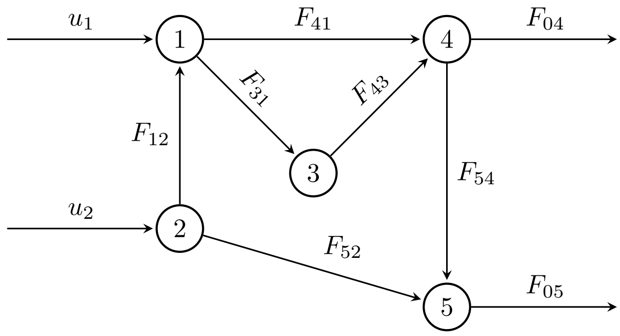

Figure 12: The graph of a compartmental system.

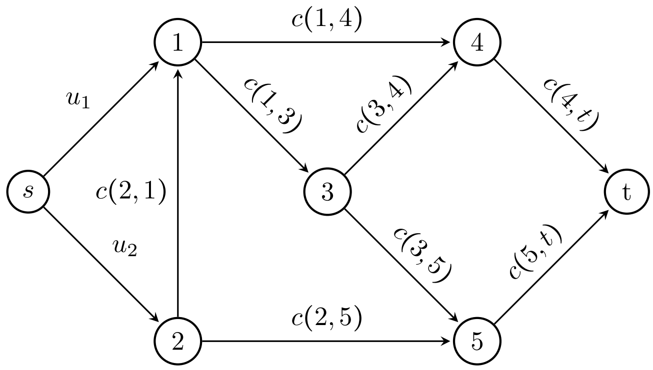

Figure 13: The flow network of the compartmental system as in Figure 12 with added source and sink node. The capacity $c(i,j)$ is given along the edges.

Next we define what we mean with the maximum flow.

Definition 3.3 (Maximum flow)**.** *The maximum flow F_{max} is the maximum amount of flow that can go through the graph from the source to the sink while the constraints $F_{ji} \leq c(i,j)$ are satisfied for all $(i,j) \in E$.*

An example of a flow network with the maximum flow is given next.

Example 3.4 (Flow network)**.** *Consider the following flow network which has a graph as in Figure 14, where the capacities $c(i,j)$ of the edges are given along the edges. The ordered sets V and E of the graph are given as*

$$V = \{s, 1, 2, 3, 4, 5, t\},$$
$$E = \{(s,1), (s,2), (1,4), (1,3), (2,3), (2,5), (3,4), (4,t), (5,t)\}.$$

The maximum flow of the flow network is 5, since it is not possible to increase the flow as otherwise we would get either $F_{1s} > c(s,1)$ or $F_{2s} > c(s,2)$.

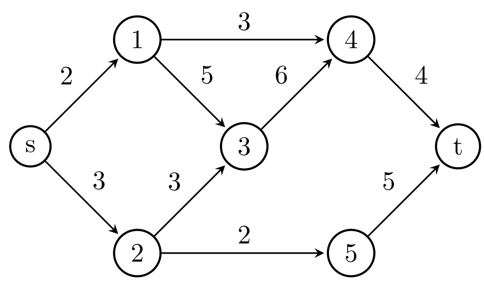

Figure 14: Glow network of Example 3.4. The capacities $c(i,j)$ of the edges are given along the edges.

We now introduce a way of dividing a flow network into two sets which separates the source and the sink.

Definition 3.5 ($s-t$ cut)**.** *Consider the flow network (G,s,t,c), an $s-t$ cut of a flow network is a set of edges whose endpoints belong to the different subsets S and T that satisfy*

(i) $S \cup T = V$*;*

(ii) $S \cap T = \varnothing$*;*

(iii) $s \in S$ *and* $t \in T$*.*

An edge $(i,j) \in E$ is in an $s-t$ cut if $i \in S$ and $j \in T$.

Consider again the flow network of Figure 14. Then, a possible $s-t$ cut is formed by the edges $(s,1)$ and $(s,2)$, where we have $S = \{s\}$ and $T = \{1,2,3,4,5,t\}$. Another example of an s–t cut is formed by the edges $(s,1),(3,4)$ and $(2,5)$, where we have $S = \{s,2,3\}$ and $T = \{1,4,5,t\}$. In the following definition we determine the capacity of an $s-t$ cut.

Definition 3.6 (Capacity of an $s-t$ cut)**.** *The capacity $C[S,T]$ of an $s-t$ cut is given by the sum of the capacities of the edges that are in the $s-t$ cut, i.e., we have*

$$C[S,T] = \sum_{(i,j)\in(S,T)} c(i,j). \tag{25}$$

Consider the flow network of Figure 14 again. For the $s-t$ cut formed by the edges $(s,1)$ and $(s,2)$ we have $C[S,T] = 5$. The $s-t$ cut formed by the edges $(s,1),(3,4)$ and $(2,5)$ has an capacity of $C[S,T] = 10$. We now need one more definition before we go go the main theorem in this section, which is the following definition.

Definition 3.7 (Minimum cut)**.** *An $s-t$ cut is a minimum cut if the sum of the capacities of the $s-t$ cut is the minimum sum of all the possible $s-t$ cuts. Let C_{mincut} be the capacity of the minimum cut then we have*

$$C_{mincut} = \min_{(S,T)} \left\{ \sum_{(i,j)\in(S,T)} c(i,j) \right\}.$$

The minimum cut of the flow network in Figure 14 is the $s-t$ cut formed by the edges $(s,1)$ and $(s,2)$, which has a capacity of $C[S,T]=5$. So the minimum cut of the flow network in Figure 14 is the same as the maximum flow. This is not a coincide as we can see from the following theorem.

Theorem 3.8 (Max flow/min cut)**.** *Let (G,s,t,c) be a flow network where F_{max} is the maximum flow on the network and C_{mincut} is the capacity of the minimum cut. Then we have that the maximum flow on the network is equal to the minimum cut, i.e., we have*

$$F_{max} = C_{mincut}. \tag{26}$$

Proof. The proof of can be found in [6]. □

An illustrative example on how to determine the minimum cut and the maximum flow by using Theorem 3.8 is given next.

Example 3.9. *Consider the flow network which has a graph as in Figure 15, where the capacities of the edges are given along the edges.*

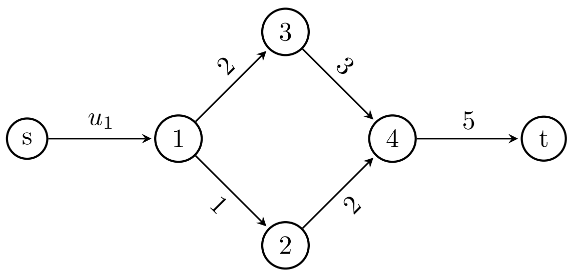

Figure 15: Flow network of Example 3.9 where the capacities of the edges are given along the edges.

There are six possible $s-t$ cuts for the graph in Figure 15, which are

1. $(s,1)$ *with* $C[S,T]=u_1$*;*
2. $(1,2),(1,3)$ *with* $C[S,T]=3$*;*
3. $(2,4),(3,4)$ *with* $C[S,T]=5$*;*
4. $(1,2),(3,4)$ *with* $C[S,T]=4$*;*
5. $(1,3),(2,4)$ *with* $C[S,T]=4$*;*
6. $(4,t)$ *with* $C[S,T]=5$*.*

In this case we want to see what the maximum possible inflow u_1 is so we consider that the capacity of the edge $(s,1)$ to be infinite. The minimum cut of this network is then the $s-t$ cut formed by the edges $(1,3)$ and $(1,2)$, which has capacity $C[S,T]=3$ so we also have $F_{max}=3$.

The max-flow min-cut theorem is a result for the static flow network from Definition 3.1. However this result can be used for the dynamic compartmental system with capacity constraints. In the case of a compartmental system with capacity constraints we have a compartmental system as in Definition 2.1 where the flow functions F_{ij} are bounded, i.e., we have $0 \leq F_{ji}(x) \leq c(i,j)$, where $c(i,j)$ is the capacity of the edge (i,j) and F_{ji} the flow from node i to node j. In the next example we show how the compartmental system with capacity constraints can be modelled and how the maximum flow can be used to analyze the system.

Example 3.10. *Consider the compartmental system which has a graph as in Figure 16.*

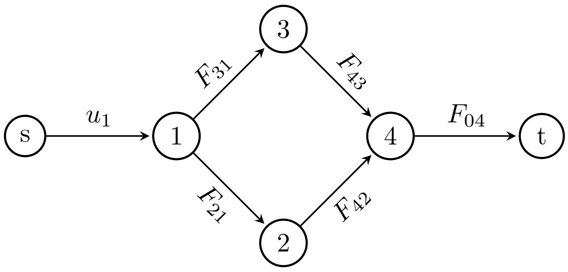

Figure 16: Graph of the compartmental system of Example 3.10.

The dynamics of the compartmental system is given as

$$\begin{aligned}
\dot{x}_1 &= u_1 - F_{21}(x_1) - F_{31}(x_1), \\
\dot{x}_2 &= F_{21}(x_1) - F_{42}(x_2), \\
\dot{x}_3 &= F_{31}(x_1) - F_{43}(x_3), \\
\dot{x}_4 &= F_{42}(x_2) + F_{43}(x_3) - F_{04}(x_4),
\end{aligned} \tag{27}$$

with the flow functions F_{ij} given as

$$\begin{aligned}
F_{21}(x_1) &= 1(1 - e^{-x_1}), \\
F_{31}(x_1) &= 2(1 - e^{-x_1}), \\
F_{42}(x_2) &= 2(1 - e^{-x_2}), \\
F_{43}(x_3) &= 3(1 - e^{-x_3}), \\
F_{04}(x_4) &= 5(1 - e^{-x_4}).
\end{aligned} \tag{28}$$

As the edges have the same capacity constraints as in Example 3.9 we have that the maximum flow of this network is 3, and an equilibrium exists if $u < F_{max} = 3$. A plot of the trajectories for zero initial conditions is given in Figure 17 with $u_1 = 2.9$. From this figure we can see that an equilibrium indeed exists and that the equilibrium is stable. When we increase the input u_1 so that we have $u_1 > F_{max}$ an equilibrium does not exists which can be seen in Figure 18.

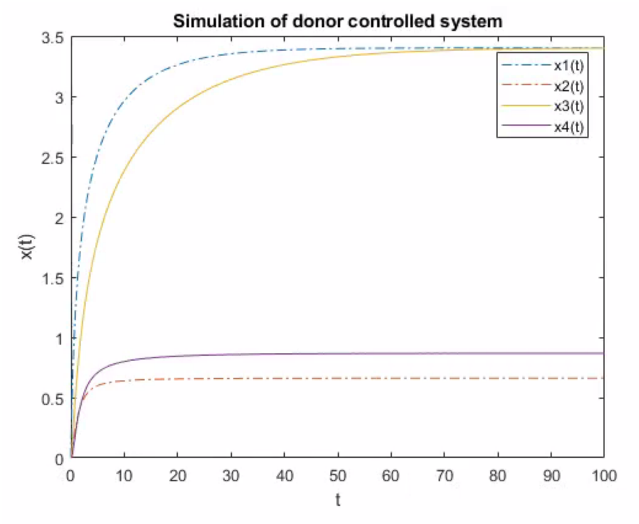

Figure 17: The Matlab simulation of the system (27) with $u_1 = 2.9$.

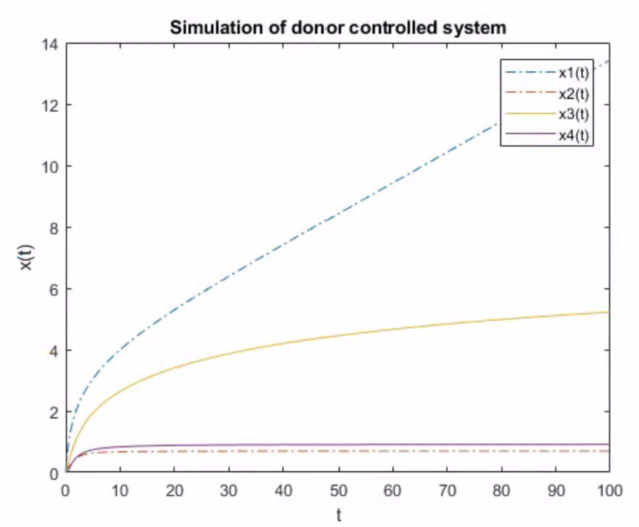

Figure 18: The Matlab simulation of the system (27) with $u_1 = 3.1$.

We can now state a necessary condition for the existence of an equilibrium of the compartmental system as in Definition 2.1 with capacity constraints, provided that the compartmental system is outflow connected, which is that

$$\sum_{i=1}^{n} u_i < F_{max}. \tag{29}$$

If $\sum u_i > F_{max}$ then an equilibrium for that compartmental system does not exist. Note that it is not a sufficient condition it can still be the case that an equilibrium does not exist even when we have $\sum u_i < F_{max}$, we show this in the following example.

Example 3.11. *Consider the compartmental system which has a graph as in Figure 19.*

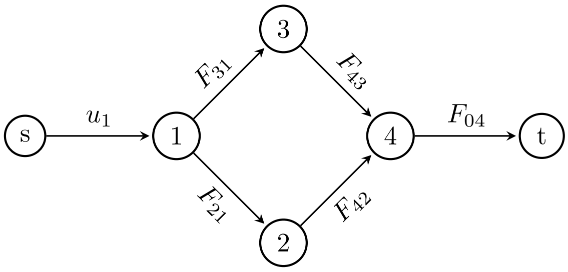

Figure 19: Graph of the compartmental system in Example 3.11.

The dynamics of the compartmental system is given as

$$\begin{aligned} \dot{x}_1 &= u_1 - F_{21}(x_1) - F_{31}(x_1), \\ \dot{x}_2 &= F_{21}(x_1) - F_{42}(x_2), \\ \dot{x}_3 &= F_{31}(x_1) - F_{43}(x_3), \\ \dot{x}_4 &= F_{42}(x_2) + F_{43}(x_3) - F_{04}(x_4), \end{aligned} \tag{30}$$

with the functions given as

$$\begin{aligned} F_{21}(x_1) &= 2(1 - e^{-x_1}), \\ F_{31}(x_1) &= 2(1 - e^{-x_1}), \\ F_{42}(x_2) &= 1(1 - e^{-x_2}), \\ F_{43}(x_3) &= 3(1 - e^{-x_3}), \\ F_{04}(x_4) &= 5(1 - e^{-x_4}). \end{aligned} \tag{31}$$

As in Example 3.10 we have that the maximum flow of this network is 3, however when we choose $u_1 = 2.1 < F_{max}$ *an equilibrium does not exist. A plot of the trajectories for zero initial conditions is given in Figure 20 with* $u_1 = 2.1$*, which shows that an equilibrium does not exist for this* u_1*. The reason an equilibrium does not exist is that from node 1 the flow to node 2 and 3 is equal as they have the same flow functions. Therefore around half of the inflow* u_1 *is going to node 2, we then have that* x_2 *only increases as the capacity of the edge* $(2, 4)$ *is not enough to transport this inflow as* $\frac{1}{2}u_1 = 1.05 > c(2, 4) = 1$.

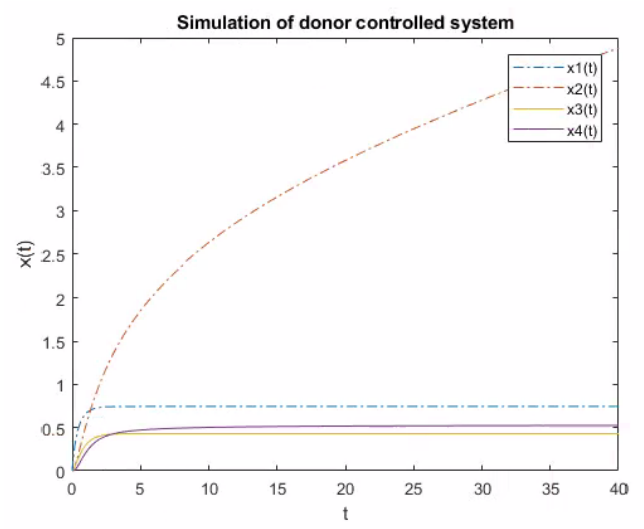

Figure 20: The Matlab simulation of system (30) with $u_1 = 2.1$

.

So in conclusion we have found that by analyzing the static flow network we can find an necessary condition for the existence of an equilibrium for the compartmental system with capacity constraints. This necessary condition is that the inflow u should be less then the maximum flow of the compartmental system. The maximum flow of a compartmental system can be determined by considering the upper-bound of the flow functions F_{ji} as the capacity of the edge (i, j) and then use the max-flow min-cut theorem to determine the maximum flow.

4 Traffic networks as compartmental systems

In this section we give a way to model traffic networks and show that traffic networks are compartmental system so that the theorems from the previous sections can be used to analyze the system. Furthermore we analyze a traffic network with demand functions and give a necessary and sufficient condition for the existence of an equilibrium.

For describing the dynamics of a traffic network a routing matrix R is often used. We define the routing matrix R for a compartmental system as in Definition 2.1 as follows.

Definition 4.1 (Routing matrix). *The routing matrix $R = [R_{ij}] \in \mathbb{R}^n$ is a matrix whose entries R_{ij} represent the fraction of outflow from node i that is directed towards node j, i.e., we have*

$$R_{ij} = \begin{cases} 0 & \text{if } i = j, \\ \frac{F_{ji}}{z_i} & \text{if } i \neq j, \end{cases} \tag{32}$$

where $z_i = \sum_{j=1} F_{ji} + F_{0i}$ and represents the total outflow from node i. Furthermore R is subject to the following constraints:

(i) $R_{ij} = 0$ if $(i,j) \notin E$;

(ii) $\sum R_{ij} \leq 1$ with strict inequality only if $F_{0i} > 0$ for $x_i > 0$.

As an example consider the following routing matrix R

$$R = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{33}$$

We have $R_{12} = R_{13} = \frac{1}{2}$, which means that half of the outflow of node 1 is going to node 2 and the other half to node 3. Furthermore we have $R_{23} = 1$, which means that the only outflow of node 2 is the flow from node 2 to node 3. In addition $R_{31} = R_{32} = R_{33} = 0$, which means that the only outflow from node 3 is the outflow F_{03} to outside the system. In the following definition we formally define what we mean by a traffic network.

Definition 4.2 (Traffic network). *A traffic network is a system of the form*

$$\dot{x} = u - (I - R^\top)z, \tag{34}$$

with constant inflow vector $u \in \mathbb{R}^n_+$, constant routing matrix $R \in \mathbb{R}^{n\times n}_+$ and non-negative n-dimensional vector z. Where $z_i = \sum_{j\neq i, j=1} F_{ji}(x) + F_{0i}(x)$ and the the functions $F_{ji}(x)$ and $F_{0i}(x)$ are functions that depend on x and t as in Definition 2.1.

The system in the definition above is of the same form as the system in Definition 2.1. To see this we first rewrite the system (34) in the following form

$$\dot{x_i} = u_i + \sum_{j=1}^{n} R_{ji} z_j - z_i. \tag{35}$$

We had $R_{ji} = F_{ij}/z_j$ therefore we have

$$\dot{x_i} = u_i + \sum_{j=1}^{n} F_{ij} - z_i.$$

Furthermore $z_i = \sum_{j \neq i, j=1} F_{ji} + F_{0i}$ so

$$\dot{x_i} = u_i + \sum_{j=1}^{n} F_{ij} - \sum_{j \neq i, j=1} F_{ji} - F_{0i}, \tag{36}$$

which is the same form as we had in Definition 2.1.

In traffic networks a digraph $N = (V, I)$ is often used to represent a road network, where $V = \{v_0, v_1, v_2, \ldots, v_m\}$ is the node set. Here, the nodes $v_1, \ldots, v_m$ represent the junctions between the roads and the node v_0 represents the outside world. In addition, the set I consists of the edges that represent the roads in the traffic network. An example of a traffic network is given in Figure 27. Contrary to the previous section we consider the roads, represented as the edges, as the compartments of the system instead of the nodes as they contain the traffic.

In the previous section we used a different graph to represent the compartmental system. The graph $G = (V, E)$ as we used in Section 2 and 3 is the so-called line-digraph of $N = (V, I)$ [7]. The edges of the graph $N = (V, I)$ represent the nodes of the graph $G = (V, E)$. When there is an edge $(i, j) \in E$ in a graph G that means that there is an edge $i \in I$ going to a node $v_k \in V$ and an edge $j \in I$ starting in node v_k, i.e., we have

$$i = (v_h, v_k) \text{ and } j = (v_k, v_l), \quad \text{for some } 0 \leq h, l \leq m \text{ and } 1 \leq k \leq m. \tag{37}$$

In addition for the graph N there is an edge $i = (v_0, v_k) \in I$ when there is an inflow u_i and an edge $i = (v_k, v_0) \in I$ when there is an outflow F_{0i}, with $0 \leq k \leq n$. As an example consider the simple graph $N = (V, I)$ in Figure 21. In this road network, there is an edge $i \in I$ directed towards node $v_k \in V$ and a edge $j \in I$ starting in node v_k, which means that for the line-digraph G there is a edge $(i, j) \in E$, which can be seen in Figure 22. The element R_{ij} in the routing matrix R is now considered to be the fraction of flow that is going through edge i which is now going through edge j. So when $R_{ij} = \frac{1}{2}$ that means that half of the traffic on road i is going to road j after the junction.

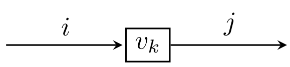

Figure 21: A simple graph $N = (V, I)$ representing a road traffic network.

The line-graph associated with the network in Figure 21 is given in Figure 22.

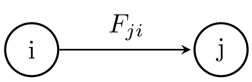

Figure 22: The corresponding line-graph $G = (V, E)$ of the road traffic network in Figure 21.

Using the above relation between traffic networks and compartmental systems, we can use the theorems from Section 2 to determine to stability and existence of an equilibrium. If the traffic network consists of linear functions, i.e., we have

$$z_i = \sum f_{ji} x_i + f_{0i} x_i, \tag{38}$$

where f_{ji} and f_{0i} are non-negative constants as in Definition 2.4. Then Theorem 2.12 can be used to determine that the system is asymptotically stable. If the traffic network in Definition 34 is nonlinear and donor-controlled as in Definition 2.28, i.e., we have

$$z_i = \sum F_{ji}(x_i) + F_{0i}(x_i), \tag{39}$$

where F_{ji} is a function which only depends on x_i as in Definition 2.28. Then Theorem 2.30 can be used to determine that the system is asymptotically stable. If the traffic network is not donor-controlled, i.e., we have

$$z_i = \sum F_{ji}(x) + F_{0i}(x), \tag{40}$$

where F_{ji} is a function which depends on x and t as in Definition 2.20. Then Theorem 2.24 can be used to determine that all the solutions of the system are bounded and tend to the equilibrium set. If the functions that are part of a path to the outside world are not unbounded we can still use both theorems if we can determine that an equilibrium exists. We show how to determine that an equilibrium exists in the next section.

4.1 Traffic network with demand functions

We can also consider traffic networks with demand functions, where a demand function characterizes the maximum possible flow on an edge. In this case the edges that represent the roads have a certain capacity. This leads us the the following definition.

Definition 4.3 (Demand functions). *The demand function $\phi_i(x_i)$ are functions which represents the maximum possible flow through edge $i \in I$ given the current mass x_i, so that $0 \geq z_i \geq \phi_i(x_i)$. The demand functions $\phi_i(x_i)$ are assumed to be continuous, non-decreasing, concave, with $\phi_i(0) = 0$ and have a flow capacity C_i, where*

$$C_i = \sup_{x_i \geq 0} \phi_i(x_i),$$

In order to give a necessary and sufficient conditions such that an equilibrium for the traffic network exists we choose $z_i = \phi_i(x_i)$ and we can rewrite the system (34) as

$$\dot{x} = u - (I - R^\top)\phi(x), \tag{41}$$

where $\phi(x) = (\phi_i(x_i))_{i \in V}$ is an n-dimensional vector. Provided that the system is outflow connected, a necessary and sufficient condition for the existence of an equilibrium for the system (41) is that the vector

$$z^* = (I - R^\top)^{-1} u, \tag{42}$$

satisfies

$$z_i^* < C_i, \quad i \in V. \tag{43}$$

In the next example we show how this condition can be used to find an inflow u such that a nonlinear donor-controlled traffic network has an equilibrium and therefore by Theorem 2.30 we have, provided that the traffic network satisfies the remaining condition of the theorem (with the exception of the unboundedness of the functions F_{ij} that are part of a path to the outside world), that the equilibrium is asymptotically stable.

Example 4.4. *Consider the traffic network in Figure 27. The edges represent the roads and the nodes represent the junctions between roads. The associated graph $G = (V, E)$ of the traffic network in Figure 23 is given in Figure 24. The dynamics of the traffic network is given as*

$$\dot{x} = u - (I - R^{\top})\phi(x), \tag{44}$$

with R and $\phi(x)$ as

$$R = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \phi(x) = \begin{bmatrix} 3(1 - e^{-x_1}) \\ 5(1 - e^{-x_2}) \\ 4(1 - e^{-x_3}) \\ 2(1 - e^{-x_4}) \\ 3(1 - e^{-x_5}) \\ 1(1 - e^{-x_6}) \\ 5(1 - e^{-x_7}) \\ 4(1 - e^{-x_8}) \end{bmatrix}. \tag{45}$$

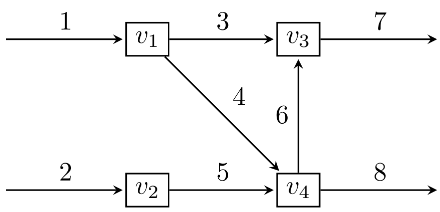

Figure 23: A digraph $N = (V, I)$ representing the road traffic network of Example 4.4.

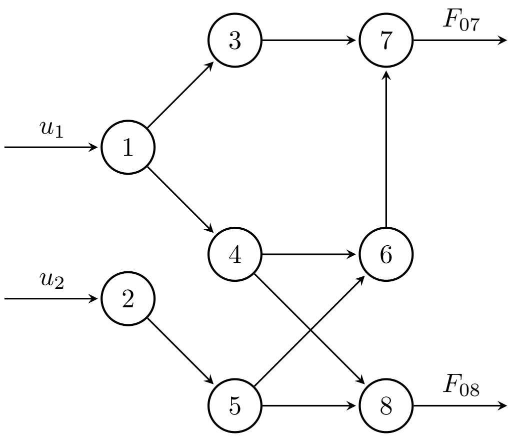

Figure 24: The corresponding graph $G = (V, E)$ of the road traffic network in Figure 23.

As the traffic network is outflow connected an necessary and sufficient condition for the existence of an equilibrium is that $z^* = (I - R^\top)^{-1} < C$*. So for an equilibrium to exists for the traffic network* (44) *we need that*

$$z^* = \begin{bmatrix} u_1 \\ u_2 \\ \frac{1}{2}u_1 \\ \frac{1}{2}u_2 \\ u_2\frac{1}{8}u_1 + \frac{1}{5}u_2 \\ \frac{5}{8}u_1 + \frac{1}{5}u_2 \\ \frac{3}{8}u_1 + \frac{4}{5}u_2 \end{bmatrix} < \begin{bmatrix} 3 \\ 5 \\ 4 \\ 2 \\ 3 \\ 1 \\ 5 \\ 4 \end{bmatrix}. \tag{46}$$

When $u_1 < 3$ *and* $u_2 < 3$ *the constraints above are all satisfied. Therefore if we choose an input* u *with* $u_1 < 3$ *and* $u_2 < 3$ *an equilibrium exists and as all the other requirement of Theorem 2.30 are satisfied we have that this equilibrium is unique and asymptotically stable. A plot of the trajectories for zero initial conditions is given in Figure 25 with* $u_1 = u_2 = 2.9$*. In Figure 26 a plot of the trajectories for zero initial conditions is given with* $u_1 = 3.01$ *and* $u_2 = 2.9$ *which shows that an equilibrium does not exist for this* u*.*

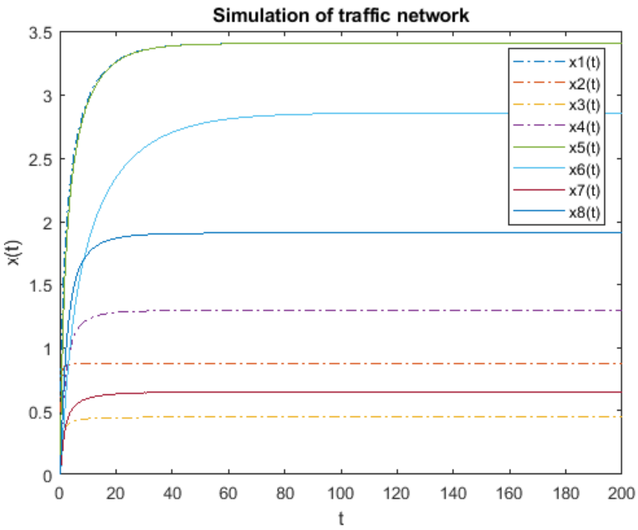

Figure 25: The Matlab simulation of the traffic network (44) with R and $\phi(x)$ as in (45) and $u_1 = 2.9$ and $u_2 = 2.9$.

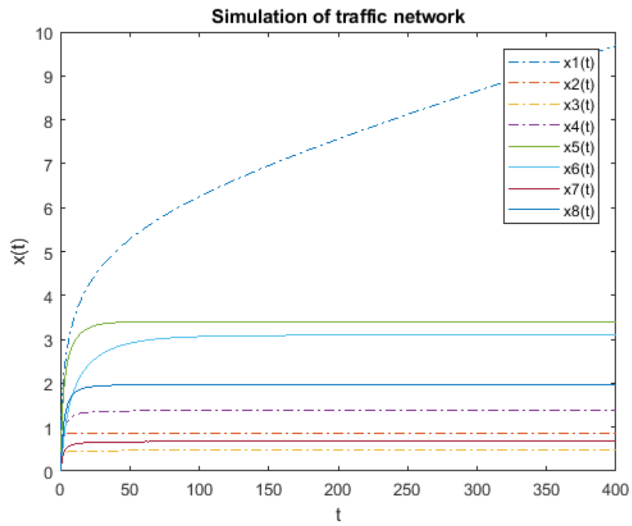

Figure 26: The Matlab simulation of the traffic network (44) with R and $\phi(x)$ and $u_1 = 3.01$ and $u_2 = 2.9$.

In the previous section we noted that an necessary condition for an equilibrium to exist is that $\sum u_i < F_{max}$ and that this was not a sufficient condition. However if we do not consider that the routing matrix R of the traffic network (41) is fixed then we can find a routing matrix R such that $\sum u_i < F_{max}$ is a sufficient condition. So in this case we change the routing matrix R such that the maximum flow through the traffic network can be achieved and an equilibrium exists and therefore the traffic network is asymptotically stable. We show how this can be done in the next example.

Example 4.5. *Consider the traffic network in Figure 27. The edges represent the roads and the nodes represent the junctions between roads.*

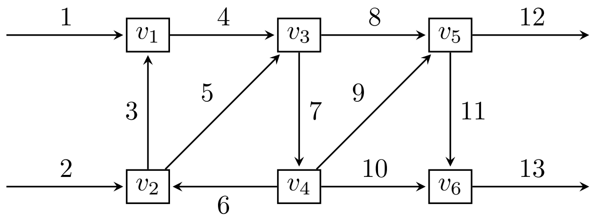

Figure 27: A digraph $N = (V, I)$ representing the road traffic network of Example 4.5.

The associated graph $G = (V, E)$ of the traffic network in Figure 27 is given in Figure 28.

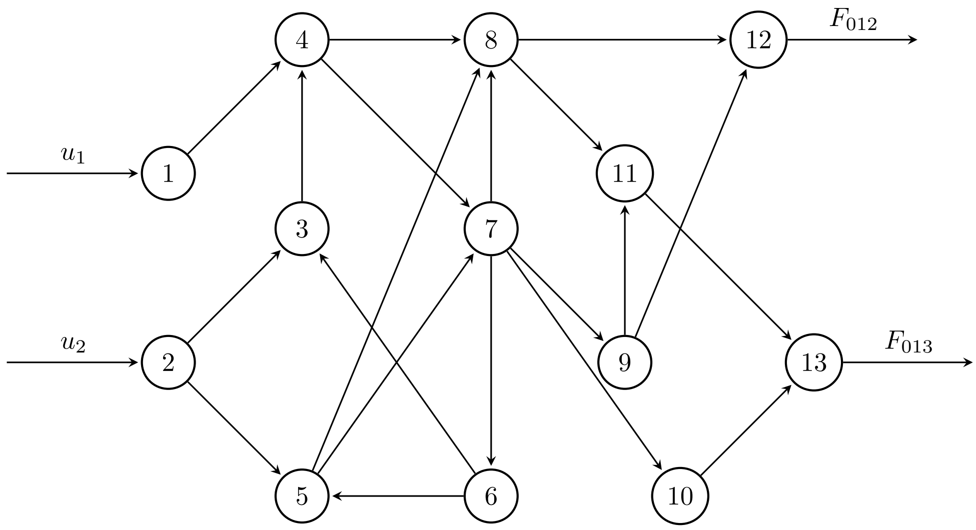

Figure 28: The corresponding graph $G = (V, E)$ of the road traffic network in Figure 27.

The dynamics of the traffic network is given as

$$\dot{x} = u - (I - R^{\top})\phi(x), \tag{47}$$

where $\phi(x) = a \arctan(x)$ *and* $a \in \mathbb{R}^n$ *is given as*

$$a = \begin{bmatrix} 2 & 2 & 1 & 3 & 2 & 1 & 2 & 1 & 3 & 2 & 1 & 3 & 3 \end{bmatrix}^{\top}.$$

The functions $\phi_i(x)$ are bounded above by the constant a_i, which we now consider as the capacities of the edge, i.e., the capacity of edge i is a_i. The flow network with the capacities listed along the edges is given in Figure 29.

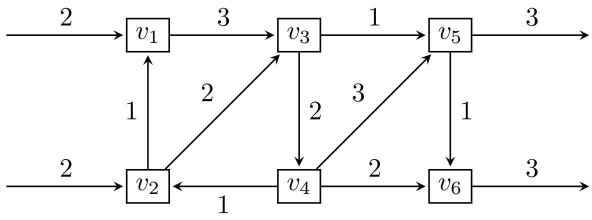

Figure 29: A digraph $N = (V, I)$ representing the road traffic network of Example 4.5, where the capacities of the edges are given along the edge

The minimum cut of the network in Figure 29 is the s-t cut formed by the edges 7 and 8, which has a capacity of $C[S,T] = 3$*. Therefore we have that the maximum flow* $F_{max} = 3$*. There are several possible directed paths that can be used to achieve this maximum flow. A routing matrix to achieve the maximum flow is by setting the non-zero entries of* R *as*

$$\begin{aligned} R_{14} &= 1, \\ R_{25} &= 1, \\ R_{47} &= 3/4, \\ R_{48} &= 1/4, \\ R_{57} &= 1/2, \\ R_{58} &= 1/2, \\ R_{7,10} &= 1, \\ R_{8,12} &= 1, \\ R_{10,13} &= 1. \end{aligned} \tag{48}$$

In addition we choose $u_1 = 2$ *and* $u_2 = 1$*. The corresponding possible flow network is given in Figure 30.*

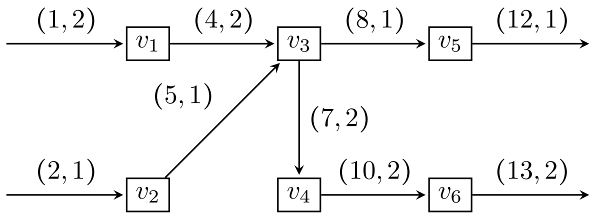

Figure 30: A digraph $N = (V, I)$ representing a road traffic network, where the (i, j) above the edge is given as, $i\ inV$ the number of the edge and j the flow through that edge.

A plot of the trajectories for zero initial conditions of the traffic network can be seen in Figure 31, which shows that the solutions of the system (47) *indeed tend to an equilibrium. Therefore even though we do not have functions* F_{ji}*, which are part of the outside world that are unbounded, we can still use Theorem 2.30 to find that this equilibrium is stable.*

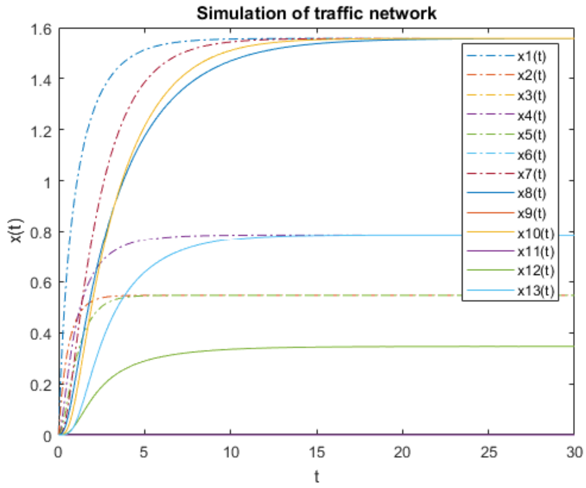

Figure 31: The Matlab simulation of the traffic network (47) with R and u as given above.

So in conclusion we have found that traffic networks are indeed compartmental systems and as such the theorems from linear and nonlinear compartmental systems of Section 2 can be used to analyze the stability of traffic networks. Furthermore we have found that a necessary and sufficient conditions for an equilibrium to exists is that

$$z^* = (I - R^\top)^{-1} u < C,$$

where z is the vector that contains the total outflow of all the cells and C the vector that contains the maximum outflow of all the cells. Finally we can use the maximum flow, which was shown how to obtain in the previous section, to determine a new routing matrix for the traffic network such that the maximum flow can be achieved and an equilibrium exists.

5 Conclusion

We were looking into the existence and uniqueness of equilibria and the stability of a compartmental system, and wanted to determine necessary and sufficient conditions on compartmental systems such that asymptotic stability is guaranteed.

For linear compartmental systems with a constant non-negative input we found that a sufficient condition for the system to be asymptotically stable is that the system is outflow connected. Furthermore we have shown another way of modeling a linear compartmental system where we specify two different types of flow. For this system we found that the necessary and sufficient conditions for asymptotic stability are that the system should be outflow connected and strongly connected.

For nonlinear compartmental system additional conditions were found such that all solutions tend to the equilibrium set. Sufficient conditions for this is that the system is outflow connected as in the linear case, the functions of the system need to satisfy the monotonicity hypothesis, the Jacobian of the system needs to be compartmental, and the functions that are part of a path to the outside world are unbounded. These additional constraints were not necessary for the linear compartmental system as these are already properties of a linear function.

For donor controlled system the sufficient conditions of the functions of the system that are part of a path to the outside word, being strictly increasing is enough for the system to be asymptotically stable. In addition the requirement for the Jacobian of the compartmental system is not necessary as the monotonicity hypothesis already implies that the Jacobian is compartmental. Furthermore the unboundedness of the functions is not a necessary condition for the donor controlled system to be asymptotically stable but only a condition that guarantees that an equilibrium exist for any constant non-negative input. If it can be found that an equilibrium exits then the Theorem for asymptotic stability for donor controlled system can still be used to proof asymptotic stability.

In addition, motivated by compartmental systems with capacity constraints we looked at a static flow network. Where by using the max-flow min-cut theorem a necessary condition for the existence of an equilibrium for a compartmental system with capacity constraints was found. This condition was that the total inflow should not exceed the maximum flow of the compartmental system.

Furthermore a way to model a traffic network was given and it was shown that a traffic network is compartmental system so that the theorems from the previous section can be used to analyze the stability of the traffic network. In addition an necessary and sufficient condition, provided that the system is outflow connected, for the existence of an equilibrium for traffic network with demand functions was found. Finally, when the routing matrix of the traffic network is not fixed an new routing matrix can be found such that the maximum flow is achieved and an equilibrium for the traffic network exists.

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