university of groningen

faculty of science and engineering

Output Regulation of Equilibrium Independent Passive Systems

Master Project Applied Mathematics

November 2018

Student: J.J. Koning

First supervisor: prof.dr. M.K. Camlibel

Second supervisor: prof.dr. A.J. van der Schaft

Abstract

In this thesis we discuss the problem of output regulation of nonlinear systems with respect to an equilibrium of the system. We present conditions on the interconnection of a plant and a controller under which output regulation is achieved, using a passivity-based approach. In particular, we use the notion of equilibrium independent passivity, which was defined by Hines et al. in [7]. We review this concept and find general classes of equilibrium independent passive systems, to gain insight in the type of systems that has this property. Based on these classes, we provide general forms of controllers that achieve output regulation.

Contents

Chapter 1

Introduction

One of the most important problems in control theory is that of controlling a plant in order to have its output tracking a reference signal [5], [8]. This problem is known as the problem of *output regulation* and has been studied extensively in the past three decades [4], [13], [9], [18]. The reference signal is assumed to be generated by an external dynamical model, the so-called exosystem. In the most general case, the exosystem generates a time-varying reference signal, but the special case where the reference signal is constant has received a lot of attention as well [18], [3], [17]. The output regulation problem has been studied for both linear systems [5] as well as for nonlinear systems [8].

Many of the existing results on output regulation require detailed analysis of the dynamical model describing the plant. Especially for high-dimensional nonlinear systems, the storage capacities and computational requirements that are needed to perform the analysis can be intractable. An approach to control theory that allows one to avoid analysis of the model is based on the notion of *passivity*. Using a passivity-based approach, system properties such as output regulation or stability can be verified, based on knowledge of the interconnection structure and passivity properties of the plant and the controller, see e.g. [1], [2]. Passivity of a component can then be viewed as an abstraction of the detailed dynamical model describing the dynamics.

Typically, passivity is referenced to a specific equilibrium of the system, and output regulation with respect to this particular equilibrium can be verified as a consequence of passivity. If a system is passive with respect to multiple or even all of its equilibria, this result can be applied to achieve output regulation with respect to an equilibrium of choice. In the literature, this form of passivity is also used to solve the output regulation problem while achieving other system specifications. For example, in the paper [9] it is used to find controllers that solve the output regulation problem while achieving disturbance rejection. Furthermore, the papers [1], [3], and [17] make the assumption that the system is passive with respect to *all* of its equilibria to achieve output regulation in cases where the equilibrium set is uncertain or difficult to compute. Indeed, the results from these papers do not rely on knowledge of the location of the equilibria. The paper [7] was the first to give this passivity property the name *equilibrium independent passivity*.

In this thesis we consider the output regulation problem for constant reference signals by using an approach that is based on equilibrium independent passivity. Specifically, for a given nonlinear plant and *any* of its equilibrium points, we want to control the plant in such a way that its output asymptotically converges to the corresponding equilibrium output. This design specification is very similar to the objective that the papers [3] and [17] pursue. The main difference is that we assume explicit knowledge of the equilibrium set of the plant, whereas these papers do not. As a result, the search for controllers in these papers is limited and only a specific form of controllers is provided. Shortly put, the aim of this thesis is to find much richer classes of output regulating controllers, by using a more general approach to the problem.

The outline of this thesis is as follows. In Chapter 2 we provide the mathematical preliminaries. In Chapter 3 we discuss the notion of equilibrium independent passivity, and we discuss certain classes of equilibrium independent passive systems. These classes will give insight in the type of systems that are equilibrium independent passive, and they will prove useful for the design of controllers later on. In Chapter 4 we perform analysis on the interconnection of a plant and a controller, in order to find sufficient conditions for output regulation. Chapter 5 will be devoted to the design of output regulating controllers, based on the theory from Chapter 3 and 4. Finally, we provide some concluding remarks in Chapter 6.

Chapter 2

Preliminaries

In this thesis we work with *set-valued maps*. A set-valued map is a mapping $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ that maps an element $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ to a set $F(x, u) \subseteq \mathbb{R}^n$. Set valued maps can be used to describe the dynamics of a system. This can be done using so-called *differential inclusions*, which are generalizations of differential equations [6]. Differential inclusions are inclusions of the form $\dot{x} \in F(x, u)$, where u is the input variable and $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is a set-valued map.

We also deal with convex sets and functions. A set $C \subset \mathbb{R}^n$ is called *convex* if for all $\alpha \in [0, 1]$ and $x, y \in C$ we have that $\alpha x + (1 - \alpha)y \in C$. To define the notion of a convex function, let $X \subseteq \mathbb{R}^n$ and consider a function $f : X \to [-\infty, \infty]$, which can take infinite values. The *effective domain* of f is defined by $\text{dom}(f) := \{x \in X \mid f(x) < \infty\}$, and the *epigraph* of f is defined by

$$\text{epi}(f) := \{(x, w) \in \mathbb{R}^{n+1} \mid x \in X,\ w \in \mathbb{R} \text{ and } f(x) \leq w\}.$$

The function f is called *convex* if the set $\text{epi}(f)$ is convex. If f is a finite-valued function, then f is convex if and only if the following inequality holds for all $\alpha \in [0, 1]$ and $x, y \in X$:

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y).$$

The *conjugate function* of f is the function $f^\star : \mathbb{R}^n \to (-\infty, \infty]$ defined by $f^\star(y) = \sup_{x \in \mathbb{R}^n}\{\langle x, y\rangle - f(x)\}$.

We say that a vector $g \in \mathbb{R}^m$ is a *subgradient* of f at a point $x \in \text{dom}(f)$ if

$$f(z) \geq f(x) + g^T(z - x) \quad \text{for all } z \in \mathbb{R}^n.$$

The set of all subgradients of f at x is called the *subdifferential* of f at x, and is denoted by $\partial f(x)$. Notice that the mapping $\partial f : x \mapsto \partial f(x)$ is a set-valued map. For differentiable functions, we have $\partial f(x) = \{\nabla f(x)\}$ for all $x \in \mathbb{R}^n$. The following proposition is a well-known result in convex optimization theory [14].

Proposition 1. *Let $f : \mathbb{R}^n \to [-\infty, \infty]$ be a proper convex function. Then f is minimized at a vector $x^* \in \mathbb{R}^n$ if and only if $0 \in \partial f(x^*)$.*

Proof. By definition of the subdifferential it is clear that $0 \in \partial f(x^*)$ if and only if $x^* \in \text{dom}\, f$ and $f(z) \geq f(x^*)$ for all $z \in \mathbb{R}^n$. Since the latter is equivalent to x^* being an optimizer of a proper function, this completes the proof. ■

Let $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map. The map M is called *monotone* if for any $x_1, x_2 \in \mathbb{R}^n$ and $y_1 \in M(x_1)$, $y_2 \in M(x_2)$ we have $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$. The following well-known result states that the subdifferential of a convex function is a monotone mapping [14].

Proposition 2. *Let $f : \mathbb{R}^n \to [-\infty, \infty]$ be a convex function. Then the operator $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is monotone.*

Proof. Let $x_1, x_2 \in \mathbb{R}^n$ and $g_1 \in \partial f(x_1), g_2 \in \partial f(x_2)$ be arbitrary. Since $g_1 \in \partial f(x_1)$, we have

$$f(x_2) \geq f(x_1) + \langle g_1, x_2 - x_1\rangle.$$

Similarly, $g_2 \in \partial f(x_2)$ implies that

$$f(x_1) \geq f(x_2) + \langle g_2, x_1 - x_2 \rangle.$$

Combining these two equations, we get $0 \geq \langle g_1, x_2 - x_1 \rangle + \langle g_2, x_1 - x_2 \rangle = \langle g_1 - g_2, x_2 - x_1 \rangle$. Hence $\langle g_2 - g_1, x_2 - x_1 \rangle \geq 0$, which completes the proof. ■

A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called *positive definite* if $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$. It is called *positive semidefinite* if $x^T Ax > 0$ for all nonzero $x \in \mathbb{R}^n$. For positive (semi)definite matrices we write $A \succ 0$ ($A \succeq 0$). We write $A \succ B$ ($A \succeq B$) if $A - B$ is positive (semi)definite. Negative (semi)definiteness is defined analogously.

Chapter 3

Equilibrium Independent Passive Systems

One of the very useful concepts in systems theory is the notion of a *dissipative system.* From the physics point of view, a dissipative system is one that never produces energy by itself. In other words, the increase of energy that is stored in a dissipative system is bounded by the amount of energy that is supplied to it through the input. This idea is formalized using so-called *dissipation inequalities.* Such inequalities involve two functions that need to be provided, namely a *supply rate* and a *storage function.* The supply rate is a function of the input and output variables and gives a measure for the amount of energy supplied to the system per unit time. The storage function is a nonnegative state function and gives a measure for the energy that is stored inside the system. Then a system is said to be dissipative with respect to a given supply rate if we can find a storage function for which the dissipation inequality is satisfied. By choosing different supply rates, many forms of dissipativity can be defined, see [12],[19].

A form of dissipativity that has proven very useful in particular is *passivity.* One of the problems where passivity is playing an important role is the problem of stabilization of nonlinear systems [19]. Passivity is referenced to an equilibrium point of the system, and we can stabilize that specific equilibrium under certain assumptions. It is possible for a system to be passive with respect to multiple or even all of its equilibrium points. A system that is passive with respect to all of its equilibrium points is said to be *equilibrium independent passive.* This notion was first defined in the paper [7], but it can also be found in earlier works, see e.g. [1], [9]. The advantage of equilibrium independent passivity is that it does not depend on a specific equilibrium of the system, but the system can be stabilized with respect to *any* of its equilibria. In the literature, this property is often used to prove stability of networks for which the equilibrium set is uncertain or difficult to compute. For example, the stability results from [1], [3] and [17] do not rely on knowledge of the location of the equilibria.

In this chapter we study equilibrium independent passive systems. We start by giving a formal definition together with an illustrative example. After that we provide three classes of equilibrium independent passive systems. One is a class of *port-Hamiltonian systems*, and the other two are classes of *gradient systems*, involving the subdifferential of a convex function.

3.1 Equilibrium independent passivity

We consider state space systems defined by

$$\Sigma: \quad \begin{aligned} \dot{z}(t) &\in F\big(z(t), v(t)\big), \\ w(t) &= G\big(z(t), v(t)\big), \end{aligned} \tag{3.1}$$

where $F : \mathbb{R}^p \times \mathbb{R}^q \rightrightarrows \mathbb{R}^p$ is a set-valued map, G is a uniformly continuous mapping from $\mathbb{R}^p \times \mathbb{R}^q$ to $\mathbb{R}^q$, and for all t we have $z(t) \in \mathbb{R}^p$ and $v(t), w(t) \in \mathbb{R}^q$. An elaborate discussion on systems described by differential inclusions can be found in [6]. Note that it is a generalization of nonlinear state space systems in the classical form. All triples (v, z, w) that we regard as a solution of the system Σ are collected in a set, which we refer to as the *behavior* of Σ. The behavior of the system (3.1) is given by

$$\mathcal{B} = \big\{(v, z, w) \in \mathcal{L}_2^q \times \mathcal{L}_2^p \times \mathcal{L}_2^q \mid (3.1) \text{ holds for almost all } t \geq 0 \text{ and } v, z \text{ are Lipschitz continuous}\big\}. \quad (3.2)$$

Note that we assume that the input function must be Lipschitz continuous. In that case, the output w is the composition of two uniformly continuous functions, namely G and $(z(t), v(t))$. Hence the output w is uniformly continuous for all trajectories of the system. We will need this property in the next chapter. We define the *equilibrium input-output relation* as

$$\Gamma = \big\{(\bar{v}, \bar{w}) \in \mathbb{R}^q \times \mathbb{R}^q \mid \exists \bar{z} \in \mathbb{R}^p \text{ such that } 0 \in F(\bar{z}, \bar{v}) \text{ and } \bar{w} = G(\bar{z})\big\}. \quad (3.3)$$

The relation Γ represents all constant input-output pairs corresponding to an equilibrium solution of (3.1). We say that $\bar{w}$ is an *equilibrium output* of Σ if there exists a constant input $\bar{v}$ such that $(\bar{v}, \bar{w}) \in \Gamma$.

On the space of input and output variables, we can define a function $s : \mathcal{L}_2^q \times \mathcal{L}_2^q \to \mathcal{L}_2$ which we call the *supply rate*. Then we say that Σ is *dissipative* with respect to the supply rate s if there exists a function $V : \mathbb{R}^p \to \mathbb{R}_+$, called the *storage function*, such that for all trajectories $(v, z, w) \in \mathcal{B}$ the following dissipation inequality is satsfied:

$$V_{\bar{v},\bar{w}}(z(t_1)) \leq V_{\bar{v},\bar{w}}(z(t_0)) + \int_{t_0}^{t_1} s(v(t), w(t))\mathrm{d}t \qquad \text{for all } t_1 \geq t_0. \quad (3.4)$$

An important choice for the supply rate is $s(u, y) = \langle u, y\rangle$. If the system is dissipative with respect to this supply rate, then we call the system *passive*. We call the system *equilibrium independent passive* if for each equilibrium pair $(\bar{u}, \bar{y}) \in \Gamma$, the system is dissipative with respect to the supply rate $s(u, y) = \langle u - \bar{u}, y - \bar{y}\rangle$. Moreover, we say that the system is *input-strictly equilibrium independent passive* if there exists a constant $\rho > 0$ such that for all $(\bar{v}, \bar{w}) \in \Gamma$ the system is dissipative with respect to the supply rate $s(u, y) = \langle u - \bar{u}, y - \bar{y}\rangle - \rho|u - \bar{u}|^2$. Finally, Σ is said to be *output-strictly equilibrium independent passive* if there exists a constant $\rho > 0$ such that for all $(\bar{v}, \bar{w}) \in \Gamma$ the system is dissipative with respect to the supply rate $s(u, y) = \langle u - \bar{u}, y - \bar{y}\rangle - \rho|y - \bar{y}|^2$.

Example 1. [15] Consider n masses moving in one-dimensional space interconnected by m springs and p dampers. Associate the masses to the vertices of a graph, where the edges represent the springs. Denote the corresponding incidence matrix by E_s. Then associate the masses to a graph where the edges correspond to the dampers, and denote its incidence matrix by E_d. Let D, K and M denote the diagonal matrices containing the damping coefficients, the spring constants and the masses respectively. Furthermore, collect the momenta of the masses in a vector $p = \begin{bmatrix} p_1 & \dots & p_n \end{bmatrix}^T$, and the extensions of the springs in $q = \begin{bmatrix} q_1 & \dots & q_m \end{bmatrix}^T$. Then the dynamics of the total system is given by

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & E_s^T \\ -E_s & -E_d D E_d^T \end{bmatrix} \begin{bmatrix} Kq \\ M^{-1}p \end{bmatrix}.$$

We can control the system by applying forces to some of the masses, which we call the *boundary masses*. Let B be a matrix with as many columns as there are boundary masses; each column consists of zeros except for exactly one 1 in the row corresponding to the associated boundary mass. Then the dynamics of the system is given by

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & E_s^T \\ -E_s & -E_d D E_d^T \end{bmatrix} \begin{bmatrix} Kq \\ M^{-1}p \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u.$$

As output we take the velocities of the boundary masses, that is, we take $y = B^T M^{-1} p$. In that case, the supply rate $s(u, y) = \langle u, y\rangle$ equals the total power of all boundary masses. Hence, the term $\int_{t_0}^{t_1} \langle u(t), y(t)\rangle \mathrm{d}t$ in (3.4) is equal to the total externally supplied energy to the boundary masses during the time interval $[t_0, t_1]$. Therefore, an interpretation of passivity for this system is that the energy that is stored at time t_1 is bounded by the sum of the energy that was stored at t_0 and the total supplied energy. In the next section we will see that the mass-spring-damper system is indeed equilibrium independent passive.

3.2 Port-Hamiltonian systems

In the remainder of this chapter we study several classes of equilibrium independent passive systems. In this section we focus on *port-Hamiltonian systems.* For an overview of port-Hamiltonian systems we refer to the paper [15]. In the book [19] we find a chapter on control of port-Hamiltonian systems, where it becomes clear that port-Hamiltonian systems are equilibrium independent passive under certain assumptions. To put this observation into our framework, the following theorem discusses a general form of port-Hamiltonian systems and gives sufficient conditions under which this form is equilibrium independent passive. In this way, it provides a class of equilibrium independent passive systems.

Theorem 1. *Let J, $R \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{p\times p}$ be matrices with $J = -J^T$, $R \succeq 0$ and $S \succeq 0$, and let $G \in \mathbb{R}^{n\times p}$, and $d \in \mathbb{R}^p$ be arbitrary. Let $H : \mathbb{R}^n \to \mathbb{R}_+$ be a differentiable convex function. Then the system*

$$\begin{aligned} \dot{z}(t) &= (J-R)\frac{\partial H}{\partial z}(z(t)) + Gv(t) \\ w(t) &= G^T\frac{\partial H}{\partial z}(z(t)) + Sv(t) + d \end{aligned} \tag{3.5}$$

is equilibrium independent passive. Furthermore, if $S = 0$ and $\ker R \subseteq \ker G^T$, then the system is output-strictly equilibrium independent passive. If $S \succ 0$, then it is input-strictly equilibrium independent passive.

Proof. Let $(\bar{v}, \bar{w})$ be an arbitrary steady state input-output pair and let $\bar{x}$ denote the corresponding steady state, that is,

$$\begin{aligned} 0 &= (J-R)\frac{\partial H}{\partial z}(\bar{z}) + G\bar{v}, \\ \bar{w} &= G^T\frac{\partial H}{\partial z}(\bar{z}) + S\bar{v} + d. \end{aligned} \tag{3.6}$$

Define the function $V_{\bar{v},\bar{w}} : \mathbb{R}^n \to \mathbb{R}$ by

$$V_{\bar{v},\bar{w}}(z) = H(z) - \frac{\partial^T H}{\partial z}(\bar{z})(z-\bar{z}) - H(\bar{z}).$$

We claim that $V_{\bar{v},\bar{w}}$ is a storage function for the supply rate $\langle v-\bar{v}, w-\bar{w}\rangle$. Moreover, we will show that under the assumption $\ker R \subseteq \ker G^T$, we can find a constant $\rho > 0$ such that $V_{\bar{v},\bar{w}}$ is a storage function for the supply rate $\langle v-\bar{v}, w-\bar{w}\rangle - \rho|w-\bar{w}|$. Finally, if G has full column rank and $R \succ 0$, we will show that $V_{\bar{v},\bar{w}}$ is a storage function for the supply rate $\langle v-\bar{v}, w-\bar{w}\rangle - \rho|v-\bar{v}|$ for some $\rho > 0$.

First note that $\frac{\partial V_{\bar{v},\bar{w}}}{\partial z}(z) = \frac{\partial H}{\partial z}(z) - \frac{\partial H}{\partial z}(\bar{x})$. Using this and the fact that $S \succeq 0$, we can write

$$\begin{aligned} \langle v(t)-\bar{v}, w(t)-\bar{w}\rangle &= \Big\langle v(t)-\bar{v}, G^T\big(\frac{\partial H}{\partial z}(z(t)) - \frac{\partial H}{\partial z}(\bar{x})\big) + S\big(v(t)-\bar{v}\big)\Big\rangle \\ &= \Big\langle v(t)-\bar{v}, G^T\frac{\partial V_{\bar{v},\bar{w}}}{\partial z}(z(t))\Big\rangle + (v(t)-\bar{v})^T S(v(t)-\bar{v}) \\ &= \Big\langle Gv(t)-G\bar{v}, \frac{\partial V_{\bar{v},\bar{w}}}{\partial z}(z(t))\Big\rangle + (v(t)-\bar{v})^T S(v(t)-\bar{v}) \end{aligned} \tag{3.7}$$

From (3.5) and (3.6) it follows that

$$Gv(t) - G\bar{v} = \dot{z}(t) + (R-J)\frac{\partial H}{\partial z}(z(t)) - (R-J)\frac{\partial H}{\partial z}(\bar{x}) = \dot{z}(t) + (R-J)\frac{\partial V_{\bar{v},\bar{w}}}{\partial z}(z(t)). \tag{3.8}$$

Substituting the above equality into (3.7) yields

$$\begin{aligned} \langle v(t)-\bar{v}, w(t)-\bar{w}\rangle &= \Big\langle \dot{z}(t) + (R-J)\frac{\partial V_{\bar{v},\bar{w}}}{\partial z}(z(t)), \frac{\partial V_{\bar{v},\bar{w}}}{\partial z}(z(t))\Big\rangle + (v(t)-\bar{v})^T S(v(t)-\bar{v}) \\ &= \Big\langle \dot{z}(t), \frac{\partial V_{\bar{v},\bar{w}}}{\partial z}(z(t))\Big\rangle + \frac{\partial^T V_{\bar{v},\bar{w}}}{\partial z}(z(t))R\frac{\partial V_{\bar{v},\bar{w}}}{\partial z}(z(t)) + (v(t)-\bar{v})^T S(v(t)-\bar{v}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}V_{\bar{v},\bar{w}}(z(t)) + \frac{\partial^T V_{\bar{v},\bar{w}}}{\partial z}(z(t))R\frac{\partial V_{\bar{v},\bar{w}}}{\partial z}(z(t)) + (v(t)-\bar{v})^T S(v(t)-\bar{v}), \end{aligned} \tag{3.9}$$

where the second equality follows from the fact that J is skew-symmetric. Since $R \succeq 0$ and $S \succeq 0$, this implies $\langle v(t)-\bar{v}, w(t)-\bar{w}\rangle \geq \frac{\mathrm{d}}{\mathrm{d}t}V_{\bar{v},\bar{w}}(z(t))$. Consequently,

$$\int_{t_0}^{t_1} \langle v(t)-\bar{v}, w(t)-\bar{w}\rangle \mathrm{d}t \geq V_{\bar{v},\bar{w}}(z(t_1)) - V_{\bar{v},\bar{w}}(z(t_0)) \qquad \text{for all } t_1 \geq t_0 \text{ and } (z,v,w) \in \mathcal{B}.$$

Hence, $V_{\bar{v},\bar{w}}$ is indeed a storage function for the pair $(\bar{v},\bar{w})$, which proves that the system is equilibrium independent passive.

To prove the second claim, we take $S = 0$ and we assume that $\ker R \subseteq \ker G^T$. Notice that

$$|w(t)-\bar{w}|^2 = \Big|G^T \frac{\partial H}{\partial z}(z(t)) - G^T \frac{\partial H}{\partial z}(\bar{z})\Big|^2 = \Big|G^T \frac{\partial V_{\bar{v},\bar{w}}}{\partial z}(z(t))\Big|^2 = \Big(\frac{\partial V_{\bar{v},\bar{w}}}{\partial z}(z(t))\Big)^T GG^T \Big(\frac{\partial V_{\bar{v},\bar{w}}}{\partial z}(z(t))\Big).$$

Combining this with (3.9), we get that for any $\rho \in \mathbb{R}$,

$$\langle v(t)-\bar{v}, w(t)-\bar{w}\rangle - \rho|w(t)-\bar{w}|^2 = \frac{\mathrm{d}}{\mathrm{d}t}V_{\bar{v},\bar{w}}(z(t)) + \Big(\frac{\partial V_{\bar{v},\bar{w}}}{\partial z}(z(t))\Big)^T \big(R - \rho GG^T\big)\Big(\frac{\partial V_{\bar{v},\bar{w}}}{\partial z}(z(t))\Big). \quad (3.10)$$

Since R is positive semidefinite, it has a square root $R^{\frac{1}{2}} \succeq 0$. Note that $\ker R^{\frac{1}{2}} = \ker R \subseteq \ker G^T$, so there exists a matrix X with $G^T = XR^{\frac{1}{2}}$. Hence

$$R - \rho GG^T = R^{\frac{1}{2}}(I - \rho X^T X)R^{\frac{1}{2}}.$$

Now define $\bar{\rho} > 0$ such that $I - \bar{\rho}X^T X \succ 0$. By the above equality, this implies $R - \bar{\rho}GG^T \succeq 0$. Then it follows from (3.10) that all trajectories of the system satisfy

$$\langle v(t)-\bar{v}, w(t)-\bar{w}\rangle - \bar{\rho}|w(t)-\bar{w}|^2 \geq \frac{\mathrm{d}}{\mathrm{d}t}V_{\bar{v},\bar{w}}(z(t)).$$

Therefore the system is dissipative with respect to the supply rate $s(v,w) = \langle v-\bar{v}, w-\bar{w}\rangle - \bar{\rho}|w-\bar{w}|^2$, that is, the system is output-strictly equilibrium independent passive.

Finally, suppose that S is positive definite. Since $R \succeq 0$, it follows from (3.9) that

$$\langle v(t)-\bar{v}, w(t)-\bar{w}\rangle = \frac{\mathrm{d}}{\mathrm{d}t}V_{\bar{v},\bar{w}}(z(t)) + (v(t)-\bar{v})^T S(v(t)-\bar{v}). \quad (3.11)$$

Since S is positive definite, all its eigenvalues are positive. Now define $\rho > 0$ as the smallest eigenvalue of S. Then we claim that all eigenvalues of the symmetric matrix $S - \rho I$ are nonnegative. To see this, notice that for any eigenvalue $\lambda \in \sigma(S-\rho I)$ we have $\lambda + \rho \in \sigma(S)$. Since ρ is the smallest eigenvalue of S, this implies $\lambda \geq 0$. Therefore, we have $S - \rho I \succeq 0$ and hence $S \succeq \rho I$. Together with (3.11) this implies

$$\langle v(t)-\bar{v}, w(t)-\bar{w}\rangle \geq \frac{\mathrm{d}}{\mathrm{d}t}V_{\bar{v},\bar{w}}(z(t)) + (v(t)-\bar{v})^T \rho I(v(t)-\bar{v}) = \frac{\mathrm{d}}{\mathrm{d}t}V_{\bar{v},\bar{w}}(z(t)) + \rho|v(t)-\bar{v}|^2.$$

It follows that the system is dissipative with respect to the supply rate $s(v(t),w(t)) = \langle v(t)-\bar{v}, w(t)-\bar{w}\rangle - \rho|v(t)-\bar{v}|^2$, which shows that the system is input-strictly equilibrium independent passive. ■

In the following example, we illustrate how the mass-spring-damper system from the previous section can be put into the form (3.5) using a coordinate transformation.

Example 2. Consider again the mass-spring-damper system from Example 1:

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0 & E_s^T \\ -E_s & -E_d D E_d^T \end{bmatrix} \begin{bmatrix} Kq \\ M^{-1}p \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u, \\ y &= B^T M^{-1} p. \end{aligned} \quad (3.12)$$

To use the previous theorem, we first have to determine whether we can write this system in the form (3.5). Consider the following coordinate transformation:

$$x = \begin{bmatrix} K^{\frac{1}{2}} & 0 \\ 0 & M^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}.$$

In these coordinates, the system (3.12) is represented by

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} K^{\frac{1}{2}} & 0 \\ 0 & M^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} \\
&= \begin{bmatrix} K^{\frac{1}{2}} & 0 \\ 0 & M^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 0 & E_s^T \\ -E_s & -E_d D E_d^T \end{bmatrix} \begin{bmatrix} K^{\frac{1}{2}} & 0 \\ 0 & M^{-\frac{1}{2}} \end{bmatrix} x + \begin{bmatrix} K^{\frac{1}{2}} & 0 \\ 0 & M^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 0 \\ B \end{bmatrix} u, \\
&= \begin{bmatrix} 0 & K^{\frac{1}{2}} E_s^T M^{-\frac{1}{2}} \\ -M^{-\frac{1}{2}} E_s K^{\frac{1}{2}} & -M^{-\frac{1}{2}} E_d D E_d^T M^{-\frac{1}{2}} \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-\frac{1}{2}} B \end{bmatrix} u, \\
y &= \begin{bmatrix} 0 & B^T M^{-\frac{1}{2}} \end{bmatrix} x.
\end{aligned}$$

Notice that the system is indeed of the form (3.5) with $H(x) = \frac{1}{2}|x|^2$ and

$$R = \begin{bmatrix} 0 & 0 \\ 0 & M^{-\frac{1}{2}} E_d D E_d^T M^{-\frac{1}{2}} \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & K^{\frac{1}{2}} E_s^T M^{-\frac{1}{2}} \\ -M^{-\frac{1}{2}} E_s K^{\frac{1}{2}} & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 \\ M^{-\frac{1}{2}} B \end{bmatrix}.$$

From Theorem 1 it follows that the system is equilibrium independent passive. Since $\ker R \subseteq \ker G^T$ if and only if $\ker E_d^T \subseteq \ker B^T$, it follows from Theorem 1 that the system is output-strictly equilibrium independent passive under the condition $\ker E_d^T \subseteq \ker B^T$.

3.3 Gradient systems

Another class of equilibrium independent passive systems has its origin in so-called *gradient systems*. In principle, a gradient system is a system whose dynamics is described by the gradient or subdifferential of a function, see e.g. [10], [11]. First we discuss the following form of gradient systems:

$$\begin{aligned} \dot{z} &\in -\partial g(z) + v, \\ w &= z + \rho v + d, \end{aligned} \tag{3.13}$$

where g is a convex function, ρ is a nonnegative constant, and $d \in \mathbb{R}^q$ is arbitrary. In the following theorem we show that such systems are always equilibrium independent passive. Here, monotonicity of the subdifferential is playing a crucial role.

Theorem 2. *Let g be a convex function and let $\rho \geq 0$ be arbitrary. Then the system* (3.13) *is equilibrium independent passive. Moreover, if $\rho > 0$, then it is input-strictly equilibrium independent passive.*

Proof. Let $t_1 \geq t_0$ be arbitrary and let (v, z, w) be any solution of the system on the interval $[t_0, t_1]$. Let $(\bar{v}, \bar{w}) \in \Gamma_c$ be any steady-state input-output pair. According to the dynamics of the system we have $v(t) - \dot{z}(t) \in \partial g(z(t))$ for all $t \in [t_0, t_1]$. Also, $(\bar{v}, \bar{w}) \in \Gamma_c$ implies that there exists a steady state $\bar{z}$ such that $\bar{u} \in \partial g(\bar{z})$. By Proposition 2, we know that ∂g is monotone, so it follows that

$$\langle v(t) - \dot{z}(t) - \bar{v}, z(t) - \bar{z} \rangle \geq 0.$$

Consequently, we have

$$\langle v(t) - \bar{v}, z(t) - \bar{z} \rangle \geq \langle \dot{z}(t), z(t) - \bar{z} \rangle = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |z(t) - \bar{z}|^2. \tag{3.14}$$

From (3.13) it follows that $w(t)-\bar{w}=z(t)-\bar{z}+\rho(v(t)-\bar{v})$, and hence

$$\begin{aligned}\langle v(t)-\bar{v},w(t)-\bar{w}\rangle &= \langle v(t)-\bar{v},z(t)-\bar{z}\rangle+\rho|v(t)-\bar{v}|^2\\ &\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z(t)-\bar{z}|^2+\rho|v(t)-\bar{v}|^2,\end{aligned}$$

as follows from (3.14). Therefore we have

$$\int_{t_0}^{t_1}\langle v(t)-\bar{v},w(t)-\bar{w}\rangle\mathrm{d}t \geq \frac{1}{2}|z(t_1)-\bar{z}|^2-\frac{1}{2}|z(t_0)-\bar{z}|^2+\int_{t_0}^{t_1}\rho|v(t)-\bar{v}|^2\mathrm{d}t. \tag{3.15}$$

Since $\rho|v(t)-\bar{v}|^2\geq 0$ for all t, this implies that the function $W_{\bar{v},\bar{w}}(z(t))=\frac{1}{2}|z(t)-\bar{z}|^2$ is a storage function for the supply rate $\langle v(t)-\bar{v},w(t)-\bar{w}\rangle$. Hence the system is equilibrium independent passive. Moreover, notice that (3.15) implies that $W_{\bar{v},\bar{w}}(z(t))$ is a storage function for the supply rate $\langle v(t)-\bar{v},w(t)-\bar{w}\rangle-\rho|v(t)-\bar{v}|^2$ as well, which shows that the system is input-strictly equilibrium independent passive if $\rho>0$.

■

Next, we consider a class of systems in which the gradient shows up in the expression for the output:

$$\begin{aligned}\dot{z} &= -z+v,\\ w &= \nabla g(z)+\rho v+d,\end{aligned} \tag{3.16}$$

where g is a differentiable convex function, ρ is a nonnegative constant and $d\in\mathbb{R}^q$ is arbitrary. We show that also these systems are equilibrium independent passive, using similar ideas as before.

Theorem 3. *Let g be a differentiable convex function and let $\rho\geq 0$ and $d\in\mathbb{R}^q$ be arbitrary. Then the system* (3.16) *is equilibrium independent passive. Moreover, if $\rho>0$, then it is input-strictly equilibrium independent passive.*

Proof. Let $(\bar{v},\bar{w})$ be an arbitrary equilibrium input-output pair with corresponding steady state $\bar{z}$, and let (v,z,w) denote an arbitrary trajectory of the system. By (3.16) we have $w(t)-\bar{w}=\nabla g(z(t))-\nabla g(\bar{z})+\rho(v(t)-\bar{v})$ and hence

$$\begin{aligned}\langle w(t)-\bar{w},v(t)-\bar{v}\rangle &= \langle\nabla g(z(t))-\nabla g(\bar{z}),v(t)-\bar{v}\rangle+\rho|v(t)-\bar{v}|^2\\ &= \langle\nabla g(z(t))-\nabla g(\bar{z}),\dot{z}(t)+z(t)-\bar{z}\rangle+\rho|v(t)-\bar{v}|^2.\end{aligned}$$

Since g is convex, it follows from Proposition 2 that $\langle\nabla g(z(t))-\nabla g(\bar{z}),z(t)-\bar{z}\rangle\geq 0$. Consequently, the above equation implies

$$\begin{aligned}\langle w(t)-\bar{w},v(t)-\bar{v}\rangle &\geq \langle\nabla g(z(t))-\nabla g(\bar{z}),\dot{z}(t)\rangle+\rho|v(t)-\bar{v}|^2\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big[g(z(t))-\langle\nabla g(\bar{z}),z(t)\rangle\big]+\rho|v(t)-\bar{v}|^2\end{aligned} \tag{3.17}$$

Now we take the integral on both sides of (3.17) over an arbitrary interval $[t_0,t_1]$ use the above equality to obtain

$$\int_{t_0}^{t_1}\langle w(t)-\bar{w},v(t)-\bar{v}\rangle\mathrm{d}t = g(z(t_1))-g(z(t_0))+\langle\nabla g(\bar{z}),z(t_1)-z(t_0)\rangle+\rho\int_{t_0}^{t_1}|v(t)-\bar{v}|^2\mathrm{d}t. \tag{3.18}$$

Finally, consider the function $V_{\bar{v},\bar{w}}(z(t))=g(z(t))-\langle\nabla g(\bar{z}),z(t)-\bar{z}\rangle-g(\bar{z})$, and notice that

$$\begin{aligned}V_{\bar{v},\bar{w}}(z(t_1))-V_{\bar{v},\bar{w}}(z(t_0)) &= g(z(t_1))-g(z(t_0))+\langle\nabla g(\bar{z}),z(t_1)-z(t_0)\rangle\\ &\leq \int_{t_0}^{t_1}\langle w(t)-\bar{w},v(t)-\bar{v}\rangle-\rho|v(t)-\bar{v}|^2\mathrm{d}t,\end{aligned} \tag{3.19}$$

where the inequality follows from (3.18). Since $V_{\bar{v},\bar{w}}(z(t))$ is positive semidefinite, it follows that it is a storage function for the supply rate $\langle v(t)-\bar{v}, w(t)-\bar{w}\rangle - \rho|v(t)-\bar{v}|^2$, and hence that the system is input-strictly equilibrium independent passive if $\rho > 0$. Furthermore, since $\rho|v(t)-\bar{v}|^2 \geq 0$ for all t, the inequality (3.19) implies that $V_{\bar{v},\bar{w}}(z(t))$ is a storage function for the supply rate $\langle v(t)-\bar{v}, w(t)-\bar{w}\rangle$ as well, and hence that the system is equilibrium independent passive for any $\rho \geq 0$. ■

Chapter 4

Passivity-Based Output Regulation

A problem in control theory where a passivity-based approach has proven advantageous is the problem of *output regulation* of nonlinear systems [3], [17], [12]. The idea of output regulation is to control a system in such a way that its output exhibits a desired behavior. The particular behavior that we are interested in is convergence of the output to an equilibrium output of the system. The papers [3] and [17] deal with the same problem, but these papers deal with systems for which the equilibrium is uncertain. This leads to a very particular structure for controllers. Assuming knowledge of the equilibrium set allows us to develop a more general theory on output regulating controllers, which is precisely the aim of the remainder of this thesis. The current chapter focuses on finding sufficient conditions on the closed loop system under which output regulation is achieved. In the next chapter we apply this theory to find general forms of output regulating controllers.

4.1 Problem formulation

In this section we introduce the output regulation problem. Throughout the remainder of this thesis, we denote a plant that we want to control by Σ_p, and a controller by Σ_c. The corresponding equilibrium input-output sets, defined by (3.3), will be denoted by Γ_p and Γ_c respectively.

Consider a plant Σ_p with state variable $x \in \mathbb{R}^p$, input variable $u \in \mathbb{R}^q$ and output variable $y \in \mathbb{R}^q$, with dynamics of the form (3.1), given by

$$\Sigma_\mathrm{p}: \quad \begin{aligned} \dot{x}(t) &\in F\big(x(t), u(t)\big), \\ y(t) &= G\big(x(t), u(t)\big), \end{aligned}$$

where $F : \mathbb{R}^p \times \mathbb{R}^q \rightrightarrows \mathbb{R}^p$ is a set-valued map, and G is a uniformly continuous mapping from $\mathbb{R}^p \times \mathbb{R}^q$ to $\mathbb{R}^q$. We define the behavior $\mathcal{B}_\mathrm{p}$ of Σ_p as in (3.2). We assume that Σ_p is equilibrium independent passive. For a given equilibrium output $\bar{y}$, we want to control the plant in such a way that the output $y(t)$ converges to the constant vector $\bar{y}$ as time tends to infinity. For this, we first specify the form of controllers that we are interested in, and define the interconnection structure. We consider controllers of the form (3.1) as well. Specifically, we consider controllers of the form

$$\Sigma_\mathrm{c}: \quad \begin{aligned} \dot{\eta}(t) &\in H\big(\eta(t), \zeta(t)\big), \\ \mu(t) &= J\big(\eta(t), \zeta(t)\big), \end{aligned} \tag{4.1}$$

where H and J are set-valued maps, $\eta \in \mathbb{R}^m$ and $\zeta, \mu \in \mathbb{R}^q$. Define the behavior $\mathcal{B}_\mathrm{c}$ of Σ_c as in (3.2). Given an *interconnection matrix* $E \in \mathbb{R}^{p \times q}$, the interconnection of Σ_p and Σ_c, denoted by $\Sigma_\mathrm{p} \times \Sigma_\mathrm{c}$, is given by

$$\begin{aligned} u(t) &= -E\mu(t), \\ \zeta(t) &= E^T y(t). \end{aligned} \tag{4.2}$$

We define the equilibrium input-output relation of the interconnection $\Sigma_\mathrm{p} \times \Sigma_\mathrm{c}$ by

$$\Gamma_{\mathrm{p}\times\mathrm{c}} = \big\{(\bar{u}, \bar{y}, \bar{\zeta}, \bar{\mu}) \in \mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^p \mid \exists \bar{x} \in \mathbb{R}^p \text{ such that } 0 \in F(\bar{z}, \bar{v}) \text{ and } \bar{w} = G(\bar{z})\big\}.$$

If such a controller indeed achieves the objective $\lim_{t\to\infty} y(t) = \bar{y}$ for any initial condition on the closed-loop system, we say that it achieves *output regulation with respect to* $\bar{y}$. In the current and the next chapter we investigate how we can construct controllers that achieve output regulation with respect to any equilibrium output $\bar{y}$. Our aim is captured in the following problem statement.

Problem 1. *Consider an equilibrium independent passive plant* Σ_{p} *of the form* (3.1) *and an interconnection matrix* E. *For a given equilibrium output* $\bar{y}$ *of* Σ_{p}, *find controllers of the form* (4.1) *that achieve output regulation with respect to* $\bar{y}$ *in the interconnection defined by* (4.2).

4.2 Conditions for output regulation

As mentioned before, the aim of this chapter is to come up with conditions on the closed-loop system under which output regulation is achieved with respect to the given equilibrium output $\bar{y}$. This will then pave the way for designing output regulating controllers. The results on the closed-loop system are presented in two separate theorems, stated below. Note that the first theorem can be applied only if the plant Σ_{p} is *output-strictly* equilibrium independent passive, whereas the second theorem can be applied to any equilibrium independent passive plant.

Theorem 4. *Consider the interconnection given by* (3.1), (4.1) *and* (4.2). *Suppose that* Σ_{p} *is output-strictly equilibrium independent passive and* Σ_{c} *is equilibrium independent passive. If there exists a vector* $\bar{\mu}$ *such that* $(-E\bar{\mu}, \bar{y}) \in \Gamma_{\mathrm{p}}$ *and* $(E^T\bar{y}, \bar{\mu}) \in \Gamma_{\mathrm{c}}$, *then* $\lim_{t\to\infty} y(t) = \bar{y}$ *for any initial condition on* Σ_{p} *and* Σ_{c}.

Proof. Let $(u, x, y) \in \mathcal{B}_{\mathrm{p}}$ and $(\zeta, \eta, \mu) \in \mathcal{B}_{\mathrm{c}}$ be arbitrary. We will show that $\lim_{t\to\infty} y(t) = \bar{y}$. By assumption on Σ_{c} and Σ_{p}, there exist positive semidefinite functions $W_{\bar{\zeta},\bar{\mu}}$ and $V_{\bar{u},\bar{y}}$ such that for all $t \geq 0$ we have

$$W_{\bar{\zeta},\bar{\mu}}(\eta(t)) \leq W_{\bar{\zeta},\bar{\mu}}(\eta(0)) + \int_0^t \langle \zeta(t) - \bar{\zeta}, \mu(t) - \bar{\mu} \rangle \mathrm{d}t \tag{4.3}$$

$$V_{\bar{u},\bar{y}}(x(t)) \leq V_{\bar{u},\bar{y}}(x(0)) + \int_0^t \langle u(t) - \bar{u}, y(t) - \bar{y} \rangle - \rho |y(t) - \bar{y}|^2 \mathrm{d}t. \tag{4.4}$$

From the equalities $u = -E\mu$ and $\zeta = E^T y$ it follows that $\langle u(t) - \bar{u}, y(t) - \bar{y} \rangle + \langle \zeta(t) - \bar{\zeta}, \mu(t) - \bar{\mu} \rangle = 0$, so adding (4.3) and (4.4) yields

$$W_{\bar{\zeta},\bar{\mu}}(\eta(t)) + V_{\bar{u},\bar{y}}(x(t)) \leq W_{\bar{\zeta},\bar{\mu}}(\eta(0)) + V_{\bar{v},\bar{w}}(x(0)) - \rho \int_0^t |y(t) - \bar{y}|^2 \mathrm{d}t,$$

for all $t \geq 0$. Since $W_{\bar{\zeta},\bar{\mu}}$ and $V_{\bar{u},\bar{y}}$ are nonnegative, it follows that

$$\rho \int_0^t |y(t) - \bar{y}|^2 \mathrm{d}t \leq W_{\bar{\zeta},\bar{\mu}}(\eta(0)) + V_{\bar{v},\bar{w}}(x(0)),$$

for all $t \geq 0$. Taking the limit $t \to \infty$ yields

$$\rho \lim_{t\to\infty} \int_0^t |y(t) - \bar{y}|^2 \mathrm{d}t \leq W_{\bar{\zeta},\bar{\mu}}(\eta(0)) + V_{\bar{v},\bar{w}}(x(0)) < \infty.$$

Since y is uniformly continuous for all trajectories of the closed-loop system, it follows from Barbalat's lemma that $\lim_{t\to\infty} |y(t) - \bar{y}| = 0$. Since we took an arbitrary trajectory of the interconnection, this completes the proof. ■

Theorem 5. *Consider the interconnection given by* (3.1), (4.1) *and* (4.2). *Suppose that* Σ_{p} *is equilibrium independent passive,* Σ_{c} *is input-strictly equilibrium independent passive, and* E *has full row rank. If there exists a vector* $\bar{\mu}$ *such that* $(-E\bar{\mu}, \bar{y}) \in \Gamma_{\mathrm{p}}$ *and* $(E^T\bar{y}, \bar{\mu}) \in \Gamma_{\mathrm{c}}$, *then* $\lim_{t\to\infty} y(t) = \bar{y}$ *for any initial condition on* Σ_{p} *and* Σ_{c}.

Proof. Let $(u, x, y) \in \mathcal{B}_\mathrm{p}$ and $(\zeta, \eta, \mu) \in \mathcal{B}_\mathrm{p}$ be arbitrary. We will show that $\lim_{t\to\infty} \zeta(t) = \bar{\zeta}$. By assumption on Σ_c and Σ_p, there exist positive semidefinite functions $W_{\bar{\zeta},\bar{\mu}}$ and $V_{\bar{u},\bar{y}}$ such that for all $t \geq 0$ we have

$$W_{\bar{\zeta},\bar{\mu}}(\eta(t)) \leq W_{\bar{\zeta},\bar{\mu}}(\eta(0)) + \int_0^t \langle \zeta(t) - \bar{\zeta}, \mu(t) - \bar{\mu} \rangle - \rho|\zeta(t) - \bar{\zeta}|^2 \mathrm{d}t \tag{4.5}$$

$$V_{\bar{u},\bar{y}}(x(t)) \leq V_{\bar{u},\bar{y}}(x(0)) + \int_0^t \langle u(t) - \bar{u}, y(t) - \bar{y} \rangle \mathrm{d}t. \tag{4.6}$$

From the equalities $u = -E\mu$ and $\zeta = E^T y$ it follows that $\langle u(t) - \bar{u}, y(t) - \bar{y} \rangle + \langle \zeta(t) - \bar{\zeta}, \mu(t) - \bar{\mu} \rangle = 0$, so adding (4.5) and (4.6) yields

$$W_{\bar{\zeta},\bar{\mu}}(\eta(t)) + V_{\bar{u},\bar{y}}(x(t)) \leq W_{\bar{\zeta},\bar{\mu}}(\eta(0)) + V_{\bar{v},\bar{w}}(x(0)) - \rho \int_0^t |\zeta(t) - \bar{\zeta}|^2 \mathrm{d}t,$$

for all $t \geq 0$. Since $W_{\bar{\zeta},\bar{\mu}}$ and $V_{\bar{u},\bar{y}}$ are nonnegative, it follows that

$$\rho \int_0^t |\zeta(t) - \bar{\zeta}|^2 \mathrm{d}t \leq W_{\bar{\zeta},\bar{\mu}}(\eta(0)) + V_{\bar{v},\bar{w}}(x(0)),$$

for all $t \geq 0$. Taking the limit $t \to \infty$ yields

$$\rho \lim_{t\to\infty} \int_0^t |\zeta(t) - \bar{\zeta}|^2 \mathrm{d}t \leq W_{\bar{\zeta},\bar{\mu}}(\eta(0)) + V_{\bar{v},\bar{w}}(x(0)) < \infty.$$

Since ζ is uniformly continuous for all trajectories of the closed-loop system, it follows from Barbalat's lemma that $\lim_{t\to\infty} |\zeta(t) - \bar{\zeta}| = 0$. Considering that $\zeta(t) - \bar{\zeta} = E^T(y(t) - \bar{y})$ and that the matrix E has full row rank, we conclude that $\lim_{t\to\infty} y(t) = \bar{y}$. Since we took an arbitrary trajectory of the interconnection, this completes the proof. ■

Remark 1. For a given vector $\bar{y}$, the existence of a vector $\bar{\mu}$ such that $(-E\bar{\mu}, \bar{y}) \in \Gamma_\mathrm{p}$ and $(E^T\bar{y}, \bar{\mu}) \in \Gamma_\mathrm{c}$ is in fact equivalent with the existence of a steady-state solution $(\bar{u}, \bar{y}, \bar{\zeta}, \bar{\mu})$ of the interconnection $\Sigma_\mathrm{p} \times \Sigma_\mathrm{c}$. To see this, notice that if $(-E\bar{\mu}, \bar{y}) \in \Gamma_\mathrm{p}$ and $(E^T\bar{y}, \bar{\mu}) \in \Gamma_\mathrm{c}$ for some vector $\bar{\mu}$, then a steady-state solution of the interconnection is given by $(-E\bar{\mu}, \bar{y}, E^T\bar{y}, \bar{\mu})$. Conversely, any steady-state solution $(\bar{u}, \bar{y}, \bar{\zeta}, \bar{\mu})$ satisfies $(\bar{u}, \bar{y}) \in \Gamma_\mathrm{p}$ and $(\bar{\zeta}, \bar{\mu}) \in \Gamma_\mathrm{c}$. Furthermore, according to the definition of the interconnection in (4.2), we must have $\bar{u} = -E\bar{\mu}$ and $\bar{\zeta} = E^T\bar{y}$ and hence $(-E\bar{\mu}, \bar{y}) \in \Gamma_\mathrm{p}$ and $(E^T\bar{y}, \bar{\mu}) \in \Gamma_\mathrm{c}$.

Example 3. An example where we encounter the output regulation problem for constant reference signals can be found in [16]. This paper tries to regulate the output of a certain state space system, in such a way that the input function converges to the minimizer of a certain cost function. However, we are only interested in regulating the output, without bothering properties of the corresponding input function. This gives room for a much simpler and more general approach than the method of Scholting et al. [16]. Here we show that the theory from the current and previous chapter can be applied to the system that is discussed in [16]. The system is given by

$$\begin{aligned} \dot{x} &= -B\lambda + u \\ y &= x, \end{aligned}$$

where λ and u are the input functions. The interconnection matrix E is chosen to be the identity matrix. Now consider an additional output $w = B^T x$, and define $z := \begin{bmatrix} w^T & y^T \end{bmatrix}^T$. With this notation and the additional output, the system can be written as

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -B & I \end{bmatrix} \begin{bmatrix} \lambda \\ u \end{bmatrix} \\ z &= \begin{bmatrix} -B & I \end{bmatrix}^T x. \end{aligned}$$

Notice that this system is of the form (3.5), with $R = J = 0$, $G = \begin{bmatrix} -B & I \end{bmatrix}$ and $H(x) = \frac{1}{2}|x|^2$. Hence we know from Theorem 1 that the system is equilibrium independent passive. Therefore, Theorem 5 can be used to solve the output regulation problem for this plant. In the next chapter we discuss how controllers that meet the demands of this theorem can be designed.

Chapter 5

Controller Design

Thus far, we have discussed equilibrium independent passive systems, and we have investigated the behavior of the interconnection of two such systems, namely a plant and a controller. In particular, we have derived conditions on the closed-loop system under which output regulation is achieved with respect to a given equilibrium output of the plant. The aim of this chapter is to actually solve Problem 1 for a given plant and an equilibrium output. In particular, we aim to design controllers in such a way that the conditions of either Theorem 4 or Theorem 5 are met. Since both theorems require the controller to be equilibrium independent passive, we will base our controller design on Chapter 3, where we introduced three classes of equilibrium independent passive systems, namely a class of Port-Hamiltonian systems and two classes of gradient systems. In Section 5.1 we investigate how systems from the port-Hamiltonian class can serve as controllers, and in Section 5.2 we will do the same for the two classes of gradient systems.

5.1 Port-Hamiltonian controllers

In this section we investigate controllers from the class of port-Hamiltonian systems given by (3.5). An issue that we need to address is that the interconnection of the plant and the controller has no equilibrium solution in general. Making an appropriate choice for the vector d in (3.5), we can shift the equilibria of the controller in order to make sure such an equilibrium solution exists. This idea is worked out in the following theorems. Theorem 6 provides controllers that can regulate the output of *output-strictly* equilibrium independent passive systems. Theorem 7 solves the problem for equilibrium independent passive systems that are not necessarily output-strictly equilibrium independent passive.

Theorem 6. *Let Σ_{p} be an output-strictly equilibrium independent passive plant with a given equilibrium output $\bar{y}$. Suppose that there exists a vector $\bar{\mu}$ such that $(-E\bar{\mu}, \bar{y}) \in \Gamma_{\mathrm{p}}$. Let J, $R \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times p}$ be matrices with $J = -J^T$ and $R \succeq 0$. Suppose that $J - R$ is invertible. Let $H : \mathbb{R}^n \to \mathbb{R}_+$ be a differentiable nonnegative strongly convex function. Then a controller that solves Problem 1 is given by*

$$\begin{aligned}\dot{\eta}(t) &= (J-R)\frac{\partial H}{\partial \eta}(\eta(t)) + G\zeta(t)\\ \mu(t) &= G^T\frac{\partial H}{\partial \eta}(\eta(t)) + \bar{\mu} + G^T(J-R)^{-1}GE^T\bar{y}.\end{aligned} \tag{5.1}$$

Proof. First we note that the controller (5.1) is of the form (3.5) with $S = 0$ and $d = \bar{\mu} + G^T(J-R)^{-1}GE^T\bar{y}$, so we know from Theorem 1 that it is equilibrium independent passive. Also, the plant is assumed to be output-strictly equilibrium independent passive and there exists a vector $\bar{\mu}$ such that $(-E\bar{\mu}, \bar{y}) \in \Gamma_{\mathrm{p}}$. Hence to apply Theorem 4, it remains to show that $(E^T\bar{y}, \bar{\mu})$ is an equilibrium input-output pair of the controller

(5.1). Note that this is equivalent to the existence of a vector $\bar{\eta} \in \mathbb{R}^n$ that satisfies

$$0 = (J - R)\frac{\partial H}{\partial \eta}(\bar{\eta}) + GE^T\bar{y}, \tag{5.2a}$$

$$\bar{\mu} = G^T\frac{\partial H}{\partial \eta}(\bar{\eta}) + \bar{\mu} + G^T(J - R)^{-1}GE^T\bar{y}. \tag{5.2b}$$

Since $J - R$ is invertible and H is strongly convex, there indeed exists a unique $\bar{\eta} \in \mathbb{R}^n$ that satisfies (5.2a). To see that this $\bar{\eta}$ solves (5.2b) as well, note that (5.2a) implies $GE^T\bar{y} = -(J - R)\frac{\partial H}{\partial \eta}(\bar{\eta})$. Hence,

$$G^T\frac{\partial H}{\partial \eta}(\bar{\eta}) + \bar{\mu} + G^T(J - R)^{-1}GE^T\bar{y} = G^T\frac{\partial H}{\partial \eta}(\bar{\eta}) + \bar{\mu} - G^T\frac{\partial H}{\partial \eta}(\bar{\eta}) = \bar{\mu},$$

which shows that $\bar{\eta}$ satisfies (5.2b) as well. Since $\bar{\eta}$ satisfies (5.2), we have that $(E^T\bar{y}, \bar{\mu}) \in \Gamma_{\mathrm{c}}$. Hence, we conclude from Theorem 4 that $\lim_{t\to\infty} y(t) = \bar{y}$. ■

Theorem 7. *Let Σ_{p} be an equilibrium independent passive plant with a given equilibrium output $\bar{y}$. Suppose that the interconnection matrix E has full row rank, and let $\bar{\mu}$ be such that $(-E\bar{\mu}, \bar{y}) \in \Gamma_{\mathrm{p}}$. Let J, $R \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times p}$ and $S \in \mathbb{R}^{p\times p}$ be matrices with $J = -J^T$, $R \succeq 0$ and $S \succ 0$, and suppose that $J - R$ is invertible. Let $H : \mathbb{R}^n \to \mathbb{R}_+$ be a differentiable nonnegative strongly convex function. Then a controller that solves Problem 1 is given by*

$$\begin{aligned}\dot{\eta}(t) &= (J - R)\frac{\partial H}{\partial \eta}(\eta(t)) + G\zeta(t) \\ \mu(t) &= G^T\frac{\partial H}{\partial \eta}(\eta(t)) + \bar{\mu} + G^T(J - R)^{-1}GE^T\bar{y} + S(\zeta(t) - E^T\bar{y}).\end{aligned} \tag{5.3}$$

Proof. First we note that the controller (5.3) is of the form (3.5) with $d = \bar{\mu} + G^T(J-R)^{-1}GE^T\bar{y} - SE^T\bar{y}$ and that it satisfies the necessary conditions of Theorem 1. Since $S \succ 0$, the theorem tells us that the controller is input-strictly equilibrium independent passive. Furthermore, notice that $(E^T\bar{y}, \bar{\mu})$ is an equilibrium input-output pair of (5.3) if and only if there exists an $\bar{\eta}$ solving (5.2). We already saw that under our assumptions such $\bar{\eta}$ indeed exists. Hence we can apply Theorem 5 to conclude that (5.3) achieves output regulation. ■

Remark 2. By choosing $G = 0$ in the above theorems, the state η of the controller has no influence on the output μ, so we can leave it out. In that case, the controller (5.1) can be written as a static controller $\mu(t) = \bar{\mu}$, and (5.3) simply becomes $\mu(t) = \bar{\mu} + S(\zeta(t) - E^T\bar{y})$.

5.2 Gradient controllers

As one would expect, the ideas from the previous section can also be applied to the two classes of gradient systems we found in Section 3.3. Again using either Theorem 4 or Theorem 5, it can be shown that both classes can serve as controllers to solve Problem 1. In this section we present this result in the form of four theorems. For these controllers, it requires a bit more work to make sure the interconnection has an equilibrium solution.

Theorem 8. *Let Σ_{p} be an output-strictly equilibrium independent passive plant with a given equilibrium output $\bar{y}$. Suppose that there exists a vector $\bar{\mu}$ such that $(-E\bar{\mu}, \bar{y}) \in \Gamma_{\mathrm{p}}$. Let g be a convex function that is minimized at $E^T\bar{y}$. Then a controller that solves Problem 1 is given by*

$$\begin{aligned}\dot{\eta} &\in -\partial g^\star(\eta) + \{\zeta\} \\ \mu &= \eta + \bar{\mu}\end{aligned}$$

Proof. Since g is a convex function, the function $g^\star$ is convex as well, so the controller is of the form (3.13) with $\rho = 0$ and $v = \bar{\mu}$. Hence we know from Theorem 2 that the controller is equilibrium independent passive. Also, the plant is output-strictly equilibrium independent passive and we have $(-E\bar{\mu}, \bar{y}) \in \Gamma_{\mathrm{p}}$. Hence, in order to use Theorem 4, it remains to show that $(E^T\bar{y}, \bar{\mu})$ is an equilibrium input-output pair of the controller. To prove this, we try to find a vector $\bar{\eta}$ such that

$$\begin{aligned} &0 \in -\partial g^\star(\bar{\eta}) + \{E^T\bar{y}\} \\ &\bar{\mu} = \bar{\eta} + \bar{\mu}. \end{aligned} \tag{5.4}$$

Since the function g is minimized at $E^T\bar{y}$, we have $0 \in \partial g(E^T\bar{y})$, from which it follows that $E^T\bar{y} \in \partial g^\star(0)$. Hence, $0 \in -\partial g^\star(0) + \{E^T\bar{y}\}$, from which it becomes clear that $\bar{\eta} = 0$ satisfies (5.4), so we indeed have $(E^T\bar{y}, \bar{\mu}) \in \Gamma_{\mathrm{c}}$. By Theorem 4 this implies that $\lim_{t\to\infty} y(t) = \bar{y}$ for any initial condition on Σ_{p} and Σ_{c}, as desired. ■

Theorem 9. *Let Σ_{p} be an output-strictly equilibrium independent passive plant with a given equilibrium output $\bar{y}$. Suppose that there exists a vector $\bar{\mu}$ such that $(-E\bar{\mu}, \bar{y}) \in \Gamma_{\mathrm{p}}$. Let g be a differentiable convex function that is minimized at $E^T\bar{y}$. Then a controller that solves Problem 1 is given by*

$$\begin{aligned} \dot{\eta} &= -\eta + \zeta \\ \mu &= \nabla g(\eta) + \bar{\mu}. \end{aligned}$$

Proof. Since the controller is of the form (3.16) with $\rho = 0$ and $v = \bar{\mu}$, we know from Theorem 3 that it is equilibrium independent passive. Also, the plant is output-strictly equilibrium independent passive and we have $(-E\bar{\mu}, \bar{y}) \in \Gamma_{\mathrm{p}}$. Hence, in order to use Theorem 4, it remains to show that $(E^T\bar{y}, \bar{\mu})$ is an equilibrium input-output pair of the controller. To prove this, we try to find a vector $\bar{\eta}$ such that

$$\begin{aligned} 0 &= -\bar{\eta} + E^T\bar{y} \\ \bar{\mu} &= \nabla g(\bar{\eta}) + \bar{\mu}. \end{aligned} \tag{5.5}$$

Since the function g is minimized at $E^T\bar{y}$, we know that its derivative vanishes at that point, that is, $\nabla g(E^T\bar{y}) = 0$. Therefore it is clear that $\bar{\eta} = E^T\bar{y}$ satisfies (5.5), so we indeed have $(E^T\bar{y}, \bar{\mu}) \in \Gamma_{\mathrm{c}}$. From Theorem 4 it follows that $\lim_{t\to\infty} y(t) = \bar{y}$ for any initial condition on Σ_{p} and Σ_{c}, as desired. ■

Theorem 10. *Let Σ_{p} be an equilibrium independent passive plant with a given equilibrium output $\bar{y}$. Suppose that the interconnection matrix E has full row rank, and let $\bar{\mu}$ be such that $(-E\bar{\mu}, \bar{y}) \in \Gamma_{\mathrm{p}}$. Let g be a convex function that is minimized at $E^T\bar{y}$ and let $\rho > 0$ be arbitrary. Then a controller that solves Problem 1 is given by*

$$\begin{aligned} &\dot{\eta} \in -\partial g^\star(\eta) + \{\zeta\} \\ &\mu = \eta + \bar{\mu} + \rho(\zeta - E^T\bar{y}) \end{aligned}$$

Proof. First we note that the controller is of the form (3.13) with $v = \bar{\mu} - \rho E^T\bar{y}$. Since $\rho > 0$, we know from Theorem 2 that the controller is input-stricly equilibrium independent passive. Also, the plant is equilibrium independent passive and the interconnection matrix has full row rank. Hence, in order to use Theorem 5, it remains to show that $(E^T\bar{y}, \bar{\mu})$ is an equilibrium input-output pair of the controller. To prove this, we try to find a vector $\bar{\eta}$ such that

$$\begin{aligned} &0 \in -\partial g^\star(\bar{\eta}) + \{E^T\bar{y}\} \\ &\bar{\mu} = \bar{\eta} + \bar{\mu} + \rho(E^T\bar{y} - E^T\bar{y}) = \bar{\eta} + \bar{\mu}. \end{aligned} \tag{5.6}$$

Since the function g is minimized at $E^T\bar{y}$, we have $0 \in \partial g(E^T\bar{y})$, from which it follows that $E^T\bar{y} \in \partial g^\star(0)$. Hence, $0 \in -\partial g^\star(0) + \{E^T\bar{y}\}$, from which it becomes clear that $\bar{\eta} = 0$ satisfies (5.6), so we indeed have $(E^T\bar{y}, \bar{\mu}) \in \Gamma_{\mathrm{c}}$. ■

Theorem 11. *Let $\Sigma_{\rm p}$ be an equilibrium independent passive plant with a given equilibrium output $\bar{y}$. Suppose that the interconnection matrix E has full row rank, and let $\bar{\mu}$ be such that $(-E\bar{\mu}, \bar{y}) \in \Gamma_{\rm p}$. Let g be a differential convex function that is minimized at $E^T\bar{y}$ and let $\rho > 0$ be arbitrary. Then a controller that solves Problem 1 is given by*

$$\begin{aligned} \dot{\eta} &= -\eta + \zeta \\ \mu &= \nabla g(\eta) + \bar{\mu} + \rho(\zeta - E^T\bar{y}). \end{aligned}$$

Proof. Since the controller is of the form (3.16) with $v = \bar{\mu} - \rho E^T\bar{y}$ and $\rho > 0$, we know from Theorem 3 that it is input-stricly equilibrium independent passive. Also, the plant is equilibrium independent passive and the interconnection matrix has full row rank. Hence, in order to use Theorem 5, it remains to show that $(E^T\bar{y}, \bar{\mu})$ is an equilibrium input-output pair of the controller. To prove this, we try to find a vector $\bar{\eta}$ such that

$$\begin{aligned} 0 &= -\bar{\eta} + E^T\bar{y} \\ \bar{\mu} &= \nabla g(\bar{\eta}) + \bar{\mu} + \rho(E^T\bar{y} - E^T\bar{y}) = \nabla g(\bar{\eta}) + \bar{\mu}. \end{aligned} \tag{5.7}$$

Since the function g is minimized at $E^T\bar{y}$, we know that its derivative vanishes at that point, that is, $\nabla g(E^T\bar{y}) = 0$. Therefore it is clear that $\bar{\eta} = E^T\bar{y}$ satisfies (5.4), so we indeed have $(E^T\bar{y}, \bar{\mu}) \in \Gamma_{\rm c}$. From Theorem 5 we know that $\lim_{t\to\infty} y(t) = \bar{y}$ for any initial condition on $\Sigma_{\rm p}$ and $\Sigma_{\rm c}$, as desired. ∎

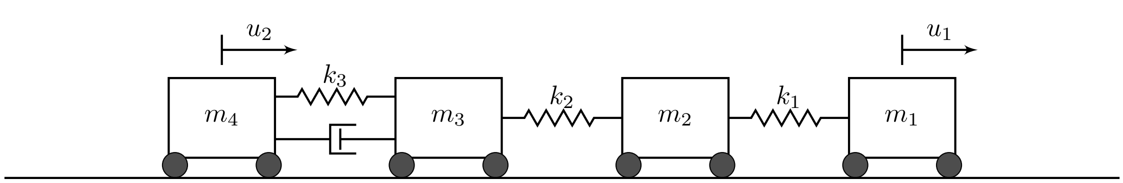
Remark 3. Recall from Example 3 that the class of systems discussed in the paper [16] is equilibrium independent passive. In that paper, the interconnection matrix E is given by the identity matrix. Hence to solve Problem 1 for this class of systems, the controllers from Theorem 7, Theorem 10 and Theorem 11 suffice. Furthermore, recall that the mass-spring-damper system from Example 1 and 2 is equilibrium independent passive as well. Hence, under the condition that the interconnection matrix has full row rank, the controllers from Theorem 7, Theorem 10 and Theorem 11 also solve Problem 1 for the mass-spring-damper system. In Example 2 we noted that the mass-spring-damper system is output-strictly equilibrium independent passive under the assumption $\ker E_d^T \subseteq \ker B^T$. Hence in that case we can also apply Theorem 6, Theorem 8 or Theorem 9 to solve Problem 1.

5.3 Example

Consider the one-dimensional mass-spring-damper system that is depicted below. Suppose that we can control the system by applying inputs to the first and the last cart, and that the measured output is given by the velocities of these two carts. In this example we aim at constructing a controller that achieves output regulation with respect to a given equilibrium of the system.

First, let us define the dynamical model. Let $p \in \mathbb{R}^4$ denote the vector of momenta of the masses, and let $q \in \mathbb{R}^3$ denote the vector of elongations of the springs. Denote the damping coefficient by D, and let $K \in \mathbb{R}^{3\times3}$ and $M \in \mathbb{R}^{4\times4}$ denote the diagonal matrices with the spring constants and the masses on the diagonal. In this example we consider the following values:

$$D = 2, \qquad K = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad M = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Furthermore, let E_s and E_d denote the incidence matrices corresponding to the springs and the dampers respectively, i.e.,

$$E_s = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \qquad E_d = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

Since we apply inputs to the first and last cart, the input matrix B is given by

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Recall from Example 1 that the dynamics of the system is then given by

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & E_s^T \\ -E_s & -E_d D E_d^T \end{bmatrix} \begin{bmatrix} Kq \\ M^{-1}p \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u.$$

As outputs we choose the velocities of the masses to which we apply the inputs, that is, $y = B^T M^{-1} p$. For the interconnection matrix we choose the identity matrix. Then the interconnection of the system with a controller with input ζ and output μ is given by

$$u = -\mu, \qquad \zeta = y.$$

The equilibrium output that we are interested in is the one where the first and last cart have an equal velocity of 1. It is easily verified that for any constant $\alpha \in \mathbb{R}$, an equilibrium with this equilibrium output is given by

$$\bar{u} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \bar{q} = \alpha \begin{bmatrix} 1/4 \\ 1 \\ 1 \end{bmatrix}, \qquad \bar{p} = \begin{bmatrix} 9 \\ 16 \\ 4 \\ 1 \end{bmatrix} \quad \bar{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{5.8}$$

The physical meaning of this equilibrium is as follows. The velocity of the carts are given by $M^{-1}p$, so in the above equilibrium configuration they are all constant and equal to 1. The elongations of the springs are distributed according to the spring constants, and are a consequence of the constant inputs that are applied to the first and last cart. The parameter α gives an indication how much the springs are elongated. To make sure the velocities of the carts are constant, the inputs are of equal magnitude and act in opposite directions.

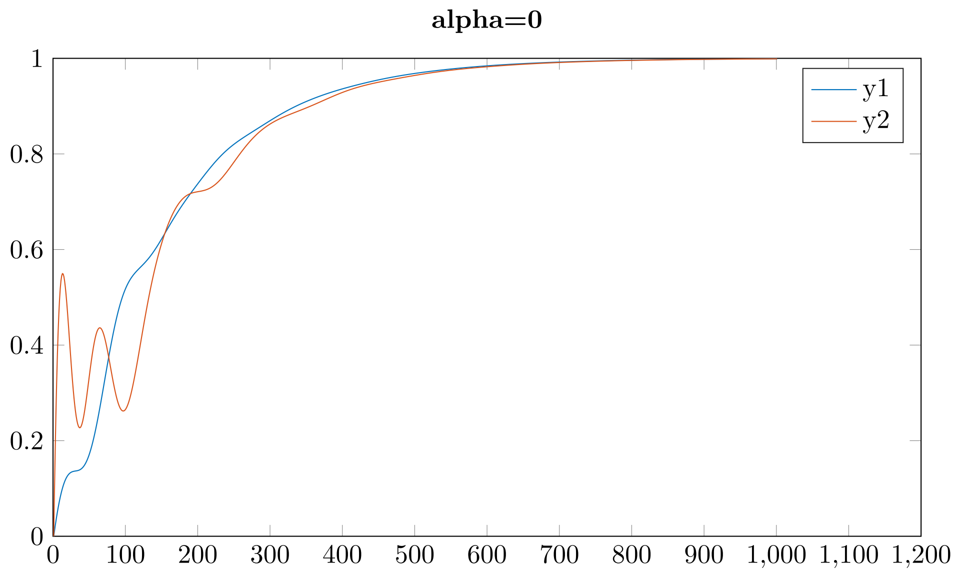
As we know from Example 2, mass-spring-damper systems are equilibrium independent passive, which allows us to use the input-strictly equilibrium independent passive controllers that were presented in this chapter. As an example, consider a port-Hamiltonian controller of the form (5.3), and let $G = 0$. As explained in Remark 2, this yields the static controller

$$\mu(t) = \bar{\mu} + S(\zeta(t) - \bar{y}), \tag{5.9}$$

where $\bar{\mu} = -\bar{u}$. Theorem 7 tells us that the above controller achieves output regulation with respect to $\bar{y}$ if S is positive definite. Hence we take S to be the 4×4 identity matrix. First, consider the equilibrium (5.8) with $\alpha = 0$. Filling in the values for $\bar{\mu}$ and $\bar{y}$ yields the following controller:

$$\mu(t) = y(t) - \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Using Matlab to plot the resulting output trajectory starting from the initial state $x_0 = 0$, we obtain the plot that is shown below. We indeed achieve output regulation with respect to $\bar{y}$.

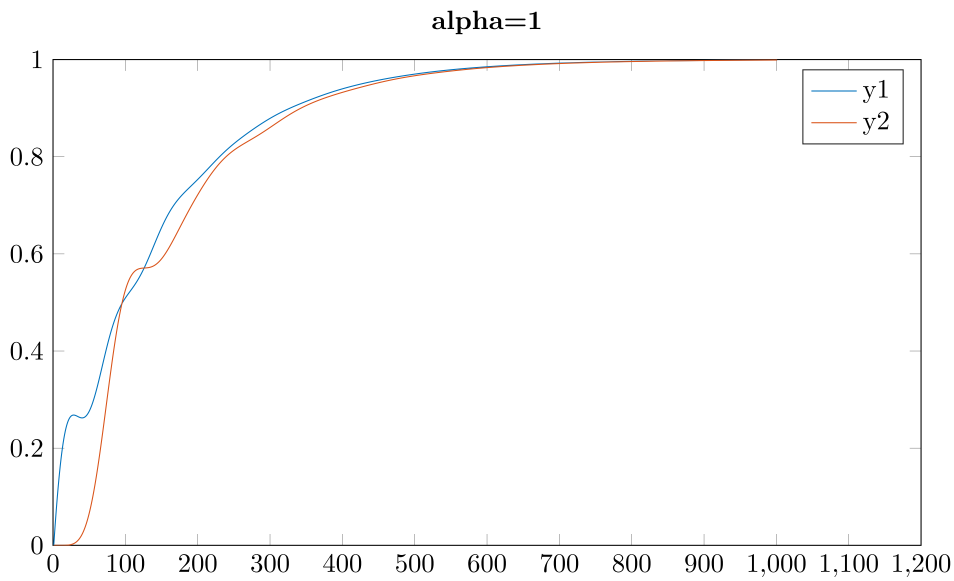


Recall that achieving output regulation does not guarantee convergence of the input and the state to the equilibrium configuration in general. However, in this example the physical meaning of the equilibria is rather clear. When the carts start driving, the strings will be stretched out in the first place. The controller that we constructed focuses on the equilibrium point where the strings return to their initial configuration, viz. $\alpha = 0$. Focusing on an equilibrium where the strings are stretched out, i.e. $\alpha > 0$, might result in a smoother curve. Here we show the effect of choosing a different equilibrium with the same equilibrium output.

Consider the equilibrium (5.8) with $\alpha = 1$. The controller (5.9) corresponding to this choice can be found by filling in the values for $\bar{\mu} = -\bar{u}$ and $\bar{y}$ again. This gives the following controller.

$$\mu(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + y(t) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = y(t) - \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

The plot of the output is shown below. We see that the curve is indeed smoother than the previous one. However, note that the time the controller takes to reach the equilibrium output is not improved.



Chapter 6

Conclusions

In this thesis we discussed the problem of output regulation with respect to an equilibrium of the system. The approach that we used was based on the notion of equilibrium independent passivity. First we introduced equilibrium independent passivity, and we provided some classes of equilibrium independent passive systems. After that, we investigated the interconnection of a plant and a controller, and derived conditions under which output regulation is achieved. These conditions included equilibrium independent passivity of both the plant and the controller. Finally, we designed controllers according to these conditions. We presented three different classes of controllers, based on the classes of equilibrium independent passive systems that we found in Chapter 3.

The controllers that we designed rely on knowledge of the equilibrium set of the plant. An interesting problem is to find controllers that do not require this information. Such controllers were already presented in the papers [3] and [17]. Hence, a topic for further research would be to see if their ideas can be applied to the controllers that were presented in this thesis.

Another idea for further research is to *design* the interconnection matrix E to obtain stability results. In that case, the problem would be to design a controller *and* an interconnection matrix for a given plant such that output regulation is achieved.

Bibliography

[1] M. Arcak and E. D. Sontag. A passivity-based stability criterion for a class of biochemical reaction networks. *Mathematical Biosciences and Engineering*, 5(1):1–19, 2008.

[2] H. Bai, M. Arcak, and J. Wen. *Cooperative Control Design: A Systematic, Passivity-Based Approach*. Communications and Control Engineering Series. Springer, New York, 2011.

[3] M. Bürger, D. Zelazo, and F. Allgöwer. Duality and network theory in passivity-based cooperative control. *Automatica*, 50, 01 2013.

[4] C. A. Desoer and C. Lin. Tracking and disturbance rejection of MIMO nonlinear systems with PI controller. *1985 American Control Conference*, pages 1596–1599, 1985.

[5] B. Francis and W. Wonham. The internal model principle for linear multivariable regulators. *Applied Mathematics and Optimization*, 2:170–194, 01 1975.

[6] Z. Han, X. Cai, and J. Huang. *Theory of Control Systems Described by Differential Inclusions*. Princeton University Press, 1972.

[7] G. H. Hines, M. Arcak, and A. K. Packard. Equilibrium-independent passivity: a new definition and implications. *Automatica J. IFAC*, 47(9):1949–1956, 2011.

[8] A. Isidori and C. I. Byrnes. Output regulation of nonlinear systems. *IEEE Transactions on Automatic Control*, 35(2):131–140, 02 1990.

[9] B. Jayawardhana. Tracking and disturbance rejection for passive nonlinear systems. *Proceedings of the 44th IEEE Conference on Decision and Control*, pages 3333–3338, 2005.

[10] E. Lavretsky, C. Cao, and N. Hovakimyan. Exponential stability of gradient systems with applications to nonlinear-in-control design methods. In *2007 American Control Conference*, pages 6003–6008, 07 2007.

[11] W. Lu and J. Wang. Convergence analysis of a class of nonsmooth gradient systems. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 55(11):3514–3527, 12 2008.

[12] A. Pavlov and L. Marconi. Incremental passivity and output regulation. In *Proceedings of the 45th IEEE Conference on Decision and Control*, pages 5263–5268, 12 2006.

[13] F. D. Priscoli. Output regulation with nonlinear internal models. *Systems & Control Letters*, 53:177–185, 2004.

[14] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, 1972.

[15] A. J. van der Schaft and B. M. Maschke. Port-Hamiltonian systems on graphs. *SIAM Journal on Control and Optimization*, 51(2):906–937, 2013.

[16] T. Scholten, C. de Persis, and P. Tesi. Quasi-optimal regulation of flownetworks with input constraints. *CoRR*, abs/1606.08220, 2016.

[17] M. Sharf and D. Zelazo. Analysis and synthesis of MIMO multi-agent systems using network optimization. *IEEE Transactions on Automatic Control*, 2018.

[18] V. Sundarapandian. Output regulation of the Pan System. *ISRN Applied Mathematics*, 2011, 06 2011.

[19] Arjan van der Schaft. *L2-Gain and Passivity Techniques in Nonlinear Control*. Springer, 3rd edition, 2016.