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Separability and Applications of the Helmholtz Equation

Bachelor Thesis Physics and Mathematics

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ABSTRACT

Partial differential equations have different forms for different coordinate systems. In this thesis the separability of two versions of the Helmholtz equation is studied. Conditions on the separability of the scalar Helmholtz equation in n dimensions are given and it is shown that the scalar Helmholtz equation in 3 dimensions separates in 11 coordinate systems. Furthermore, it is found that the vector Helmholtz equation has a more complex form and this equation separates only in rectangular coordinates. Finally, applications of both versions of the Helmholtz equation in quantum mechanics, electromagnetism and optics are treated.

Contents

Introduction

The Helmholtz equation arises in many problems in physics where waves are involved. Waves can be described by a wave function $\psi(x,t)$ which satisfies a differential equation, for example the wave equation or the Schrödinger equation. This is a differential equation in both space and time, and when separation of variables is used with $\psi(x,t) = X(x)T(t)$ the Helmholtz equation frequently arises. Namely, the space part often satisfies $(\nabla^2 + k^2)X(x) = 0$, where ∇^2 is the Laplace operator and k is a constant. This is the Helmholtz equation.

The Helmholtz equation has two forms, the scalar form and the vector form. The scalar form is given as $(\Delta + k^2)f = 0$, where Δ is the scalar Laplacian and f is a scalar function. The vector Helmholtz equation is given as $(\blacktriangle + k^2)\mathbf{f} = 0$, where $\blacktriangle$ is the vector Laplacian and $\mathbf{f}$ is a vector function. Both forms of the Helmholtz equation are partial differential equations, which are ideally split up into a set of coupled ordinary differential equations. When the Helmholtz equation can be written as a set of coupled ordinary differential equations, we say that it is *separable*.

Depending on the situation, a suitable coordinate system for a problem may be chosen. For example, when one attempts to calculate the electric field around an elliptically shaped charged body, elliptic coordinates might be useful. The Laplace operator has a specific expression for each coordinate system. In this thesis, the separability of the two forms of the Helmholtz equation in different coordinate systems will be studied.

The mathematical details of separability is studied in chapter 2. The coordinate systems considered are curvilinear orthogonal coordinate systems, which means that the coordinate systems are obtained from orthogonally intersecting surfaces. Using scale factors, one can give expressions for the (vector) Laplace operator in different coordinate systems. By looking at the (vector) Helmholtz equation in terms of scale factors, conditions can be given to split up the partial differential equation into a set of coupled ordinary differential equations. The scalar Helmholtz equation separates in 11 coordinates, which are degenerate forms of the confocal ellipsoidal coordinate system. The vector Helmholtz equation has a more complicated form than the scalar Helmholtz equation and it only separates in one of the 11 above mentioned coordinate systems, being the rectangular coordinate system.

Once the separability is put in a mathematical framework, in chapter 3 applications of the Helmholtz equation will be treated. The two forms of the Helmholtz equation appear in many fields of physics. The Schrödinger equation with zero potential energy reduces to the scalar Helmholtz equation, and we will see that quantum scattering and quantum billiard are closely related. Maxwell's equations and the vector Helmholtz equation are also related, but separability conditions for the vector Helmholtz equation do not necesarrily apply to Maxwell's equation. Furthermore, light rays obey the scalar Helmholtz equation and using conformal maps we can bend light around certain regions of space.

1 Preliminaries

1.1 Orthogonal Coordinate Systems

Depending on the situation, a suitable coordinate system may be chosen. Coordinate systems can be obtained from orthogonally intersecting surfaces and we will see that we can specify a point in space by the intersection of these surfaces.

In Euclidean 3-space, the position vector $\mathbf{r}$ of a point p can be defined by $\mathbf{r} = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z$ where $\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z$ are the standard basis vectors. These can be obtained by

$$\mathbf{i}_x = \frac{\partial \mathbf{r}}{\partial x}, \quad \mathbf{i}_y = \frac{\partial \mathbf{r}}{\partial y}, \quad \mathbf{i}_z = \frac{\partial \mathbf{r}}{\partial z}. \tag{1}$$

This point can also defined using orthogonal curvilinear coordinates. Curvilinear coordinate systems are obtained by looking at intersections of surfaces. We talk about orthogonal curvilinear coordinate systems when the surfaces intersect orhogonally. For example,the surfaces related to spherical coordinates are spheres, half planes and cones. Let $f(x, y, z) = \xi$ specify a surface characterized by a constant parameter ξ. Now define 3 invertible transformation functions which characterize 3 orthogonally intersecting surfaces.

$$\begin{aligned} f_1(x, y, z) &= \xi_1, \\ f_2(x, y, z) &= \xi_2, \\ f_3(x, y, z) &= \xi_3. \end{aligned}$$

A point of intersection can be defined by (ξ_1, ξ_2, ξ_3), which are called the orthogonal curvilinear coordinates. Since we will consider *regular* surfaces, we can guarentee the existence of inverse maps g_i, $i = 1, 2, 3$, such that $x = g_1(\xi_1, \xi_2, \xi_3)$ and so on. The space curves formed by the intersection of two surfaces are called the coordinate curves. The basis vectors of this coordinate system can be obtained using the same derivatives as above:

$$\mathbf{h}_1 = \frac{\partial \mathbf{r}}{\partial \xi_1}, \quad \mathbf{h}_2 = \frac{\partial \mathbf{r}}{\partial \xi_2}, \quad \mathbf{h}_3 = \frac{\partial \mathbf{r}}{\partial \xi_3}. \tag{2}$$

These vectors may not have unit length, so the curvilinear orthonormal basis vectors can be given by

$$\mathbf{e}_1 = \frac{1}{h_1}\mathbf{h}_1, \quad \mathbf{e}_2 = \frac{1}{h_2}\mathbf{h}_2, \quad \mathbf{e}_3 = \frac{1}{h_3}\mathbf{h}_3, \tag{3}$$

where $h_i = |\mathbf{h_i}|$. The total differential change in $\mathbf{r}$ is

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \xi_1} d\xi_1 + \frac{\partial \mathbf{r}}{\partial \xi_2} d\xi_2 + \frac{\partial \mathbf{r}}{\partial \xi_3} d\xi_3 = h_1 d\xi_1 \mathbf{e}_1 + h_2 d\xi_2 \mathbf{e}_2 + h_3 d\xi_3 \mathbf{e}_3, \tag{4}$$

and therefore h_i is the so called *scale factor for* ξ_i given by

$$h_i = \left|\frac{\partial \mathbf{r}}{\partial \xi_i}\right| = \sqrt{\left(\frac{\partial x}{\partial \xi_i}\right)^2 + \left(\frac{\partial y}{\partial \xi_i}\right)^2 + \left(\frac{\partial z}{\partial \xi_i}\right)^2}. \tag{5}$$

The six scalar products $g_{ij} = \mathbf{h}_i \cdot \mathbf{h}_j$ define the nine entries of the metric tensor (g_{ij}). Since the surfaces considered in this thesis are orthogonal, the metric tensor is diagonal and $g = g_{11}g_{22}g_{33}$.

The method above can be generalized to n dimensions and provides us with a way to describe operators in arbitrary curvilinear coordinate systems.

1.1.1 The Scalar and Vector Laplace Operator

The Laplacian plays a prominent role in the Helmholtz equation and we want to be able to give an expression for the Helmholtz equation in general coordinate systems. Therefore, we will now investigate the Laplacian in orthogonal coordinate systems. The Laplacian of a field gives us a quantitative measure of the "spreading out" of the change of the field in space. The Laplacian of any tensor field $\mathbf{T}$ is given as the divergence of the gradient of the tensor

$$\nabla^2 \mathbf{T} = \nabla \cdot (\nabla \mathbf{T}). \tag{6}$$

Whereas the scalar Laplacian and the vector Laplacian are two entirely different operators, generally the same symbol is used for both. To avoid confusion, we will denote the scalar Laplacian by Δ and the vector Laplacian by $\blacktriangle$. The scalar Laplacian for a scalar function f is then given by

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f. \tag{7}$$

If the scalar Laplacian gives us a large value, the field is rapidly going from not changing much at one point to changing a lot at another point. Using vector identities in three dimensions, one can obtain an expression for the vector Laplacian in terms of gradient, curl and divergence. Namely, the vector Laplacian of $\mathbf{f}$ is the gradient of the divergence of $\mathbf{f}$ minus the curl of the curl of $\mathbf{f}$, where $\mathbf{f}$ is a vector function:

$$\blacktriangle \mathbf{f} = \nabla^2 \mathbf{f} = \nabla(\nabla \cdot \mathbf{f}) - \nabla \times (\nabla \times \mathbf{f}). \tag{8}$$

In Cartesian coordinates, this reduces to the scalar Laplacian for each component of $\mathbf{f}$. Now, we will use scale factors to express these operators in terms of ξ_i coordinates. The scalar Laplacian for Euclidean n-space in an orthogonal coordinate system $(\xi_1, \xi_2, \ldots, \xi_n)$ can be written as

$$\Delta f = \frac{1}{\sqrt{g}} \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i}\left(\frac{\sqrt{g}}{g_{ii}} \frac{\partial f}{\partial \xi_i}\right). \tag{9}$$

So, for Euclidean 3-space the scalar Laplacian in terms of scale factors becomes

$$\Delta f = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial \xi_i} \left[\frac{h_1 h_2 h_3}{g_{ii}} \frac{\partial f}{\partial \xi_i} \right]. \tag{10}$$

We see that in rectangular coordinates, ($\xi_1 = x, \xi_2 = y, \xi_3 = z$), the scale factors become $h_1 = h_2 = h_3 = 1$ resulting in $\Delta f = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] f$, which is a familiar expression.

In this thesis, the gradient, the divergence and the curl will only be used in 3 dimensions. We will now give the expressions for these operators in 3 dimensions. The gradient of a scalar function $f(\xi_1, \xi_2, \xi_3)$ is given as

$$\nabla f = \frac{1}{h_1} \frac{\partial f}{\partial \xi_1} \mathbf{e}_1 + \frac{1}{h_2} \frac{\partial f}{\partial \xi_2} \mathbf{e}_2 + \frac{1}{h_3} \frac{\partial f}{\partial \xi_3} \mathbf{e}_3, \tag{11}$$

where $\mathbf{e}_i$ is the unit vector corresponding to ξ_i and so on. The expression for the divergence of a vector function $\mathbf{f}(\xi_1, \xi_2, \xi_3)$ is

$$\nabla \cdot \mathbf{f} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial \xi_1} (h_2 h_3 f_1) + \frac{\partial}{\partial \xi_2} (h_3 h_1 f_2) + \frac{\partial}{\partial \xi_3} (h_1 h_2 f_3) \right], \tag{12}$$

where f_1 is the first component of $\mathbf{f}$ and so on. Finally, the expression for the curl of a vector function $\mathbf{f}(\xi_1, \xi_2, \xi_3)$ is

$$\nabla \times \mathbf{f} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \frac{\partial}{\partial \xi_1} & \frac{\partial}{\partial \xi_2} & \frac{\partial}{\partial \xi_3} \\ h_1 f_1 & h_2 f_2 & h_3 f_3 \end{vmatrix}. \tag{13}$$

Now we are in the position to obtain an expression for the vector Laplacian in 3-dimensions in terms of scale factors, using equation (8). We redirect interested readers to [1] for this expression.

1.2 The Quantum Mechanical Scattering Matrix S

In chapter 3 we will look at application of the scalar Helmholtz equation in quantum mechanical scattering. In the quantum theory of scattering, we imagine an incident wave $|\psi_{in}\rangle$ which encounters a scattering potential V producing an outgoing wave $|\psi_{out}\rangle$. This is realized by a fixed potential V in some region of space, called the scattering region. Outside the scattering region, the influence of the potential will be negligible. Therefore, the wave packets before and after scattering will be expected to behave freely. So we expect the existence of states $|\psi_{in}\rangle$ and $|\psi_{out}\rangle$ in the so called asymptotic past and future of the state $|\psi(t)\rangle$, respectively. Our goal in this section is to find an operator S such that

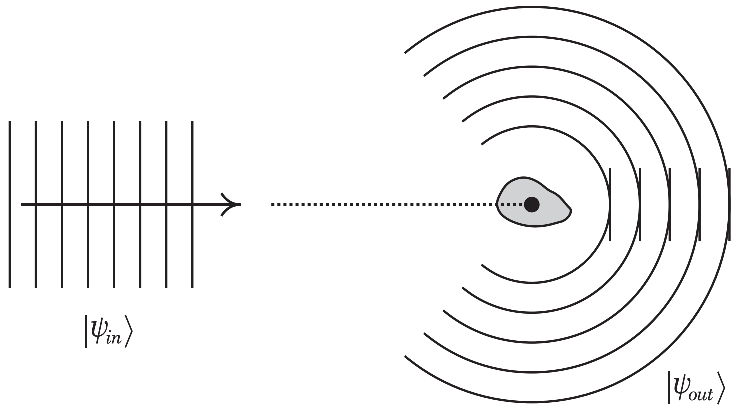

Figure 1: Schematic overview of quantum scattering where $|\psi_{out}\rangle = S\,|\psi_{in}\rangle$.

$$|\psi_{out}\rangle = S\,|\psi_{in}\rangle\,. \tag{14}$$

An important property of the Schrödinger equation is that during the evolution between two measurements, the norm of the state vector does not change. This leads to the *evolution operator* $U(t)$ which relates the initial state $|\psi(0)\rangle$ to the state at time t by

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle\,. \tag{15}$$

Here $U(t) = e^{-itH/\hbar}$ and $H = p^2/2M + V$. For $|\psi_{in}\rangle$ and $|\psi_{out}\rangle$ the potential is negligible so if we define $H_0 = p^2/2m$, then

$$\|\psi(t) - e^{-itH_0/\hbar}\psi_{\substack{in\\out}}\| \to 0, \quad t \to \mp\infty. \tag{16}$$

Substituting the expression from equation (15) gives

$$\|U(t)\psi(0) - e^{-itH_0/\hbar}\psi_{\substack{in\\out}}\| \to 0, \quad t \to \mp\infty. \tag{17}$$

Setting $t = 0$ in equation (15), we see that $U(0) = 1$ and since $\frac{d}{dt}\left[U(t)^\dagger U(t)\right] = 0$, we see that $U(t)$ is unitary:

$$U^\dagger(t)U(t) = I. \tag{18}$$

This gives us $U^{-1} = U^\dagger$ and therefore

$$\|\psi(0) - U(t)^\dagger e^{-itH_0/\hbar}\psi_{\substack{in\\out}}\| \to 0, \quad t \to \mp\infty. \tag{19}$$

So we see that

$$|\psi(0)\rangle = \lim_{t\to\mp\infty} e^{itH/\hbar}e^{-itH_0/\hbar}\left|\psi_{\substack{in\\out}}\right\rangle = \Omega_\pm \left|\psi_{\substack{in\\out}}\right\rangle, \tag{20}$$

where $\Omega_\pm$ are the Möller operators, which are also unitary:

$$\Omega_\pm \equiv \lim_{t\to\mp\infty} U^{-1}(t)e^{itH_0/\hbar}. \tag{21}$$

Using the equation above we see that

$$|\psi_{out}\rangle = \Omega_-^\dagger |\psi(0)\rangle = \Omega_-^\dagger\Omega_+ |\psi_{in}\rangle. \tag{22}$$

Now, if we define the Unitary Scattering Matrix S

$$S \equiv \Omega_-^\dagger\Omega_+, \tag{23}$$

we see that

$$|\psi_{out}\rangle = S|\psi_{in}\rangle. \tag{24}$$

So we defined an operator which relates the incoming state to the outgoing state.

1.3 Electromagnetism

Another application of the Helmholtz equation is found in electromagnetism. We will now look at the relation between Maxwell's equations and the Helmholtz equation. In chapter 3, we will take a look at the exact solutions of Maxwell's equations. Maxwell's equations in charge free vacuum are defined as

$$\begin{aligned} \nabla\cdot\mathbf{E} &= 0,\\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t},\\ \nabla\cdot\mathbf{B} &= 0,\\ \nabla\times\mathbf{B} &= \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}. \end{aligned} \tag{25}$$

Taking the curl of the curl equations gives

$$\begin{aligned} \nabla\times(\nabla\times\mathbf{E}) &= \nabla\times\left(-\frac{\partial\mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\nabla\times\mathbf{B} = -\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2},\\ \nabla\times(\nabla\times\mathbf{B}) &= \nabla\times\left(\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right) = \mu_0\varepsilon_0\frac{\partial}{\partial t}\nabla\times\mathbf{E} = -\mu_0\varepsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}. \end{aligned} \tag{26}$$

Now we use the expression for the vector Laplacian in equation (8) to get

$$\begin{aligned} \nabla\times(\nabla\times\mathbf{E}) &= \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2},\\ \nabla\times(\nabla\times\mathbf{B}) &= \nabla(\nabla\cdot\mathbf{B}) - \nabla^2\mathbf{B} = -\mu_0\varepsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}. \end{aligned} \tag{27}$$

If we use that $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$, we obtain the so called *vector Helmholtz wave equation* for both $\mathbf{E}$ and $\mathbf{B}$

$$\begin{aligned} \nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} &= 0, \\ \nabla^2 \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} &= 0. \end{aligned} \tag{28}$$

If we assume harmonic time dependence, i.e. $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r}) e^{i\omega t}$ and $\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0(\mathbf{r}) e^{i\omega t}$ both these equations are in the form of the so called *vector Helmholtz equation*

$$\begin{aligned} (\nabla^2 + k^2) \mathbf{E}_0 &= 0, \\ (\nabla^2 + k^2) \mathbf{B}_0 &= 0, \end{aligned} \tag{29}$$

where $k^2 = \omega^2 \mu_0 \varepsilon_0$.

2 Separability of the Helmholtz Equation

In chapter 1 we saw that we can give expressions for the scalar Laplacian and the vector Laplacian in terms of scale factors. Now we will go a step further and write both versions of the Helmholtz equation in terms of scale factors. Once this is done, we can give conditions for the partial differential equation to split in a set of ordinary differential equations. Using these conditions, we can determine which coordinate systems allow separability of the Helmholtz equation.

2.1 The Scalar Helmholtz Equation

Separability in Euclidean n-Space

Using the expression for the Laplacian in terms of scale factors as in equation (9), we see that the Helmholtz equation for an arbirtary orthogonal coordinate system $(\xi_1, \xi_2, \ldots, \xi_n)$ in Euclidean $n-$space can be written as

$$\nabla^2\psi + k^2\psi = \frac{1}{\sqrt{g}}\sum_{i=1}^{n}\frac{\partial}{\partial\xi_i}\left(\frac{\sqrt{g}}{h_i^2}\frac{\partial\psi}{\partial\xi_i}\right) + k^2\psi = 0, \tag{30}$$

where $\psi = \psi(\xi_1, \xi_2, \ldots, \xi_n)$. In this section we will first give conditions for the Helmholtz equation to separate. This means that the partial differential equation can be rewritten as a set of coupled ordinary differential equations. The necessary conditions for the Helmholtz equation to separate will turn out to be also sufficient. Therefore, one can obtain a theorem which deals with the separability of the Helmholtz equation [2] and the precise form of this theorem will be given in this section.

Assume that one can write ψ as a product of n functions X_i depending on coordinate ξ_i, so $\psi(\xi_1, \xi_2 \ldots \xi_n) = X_1(\xi_1)X_2(\xi_2)\ldots X_n(\xi_n)$. Here X_i depends on the coordinate system and on the separation constants $\alpha_1 = k^2, \alpha_2, \alpha_3, \ldots, \alpha_n$. Substituting this in equation (30) and multiplying with $\sqrt{g}$ gives

$$\sum_{i=1}^{n}\frac{1}{X(\xi_i)}\frac{\partial}{\partial\xi_i}\left(\frac{\sqrt{g}}{h_i^2}\frac{dX(\xi_i)}{d\xi_i}\right) + k^2\sqrt{g} = 0. \tag{31}$$

Here we see that some partial derivatives have disappeared. We now require that there exist functions f_i depending on ξ_i and F_i independent of ξ_i such that

$$\begin{cases} \frac{\sqrt{g}}{h_1^2} &= f_1(\xi_1)F_1(\xi_2, \xi_3, \ldots \xi_n) \\ \frac{\sqrt{g}}{h_2^2} &= f_2(\xi_2)F_2(\xi_1, \xi_3, \ldots \xi_n) \\ \vdots \\ \frac{\sqrt{g}}{h_n^2} &= f_n(\xi_n)F_n(\xi_1, \xi_2, \ldots \xi_{n-1}) \end{cases}, \tag{32}$$

which, in short notation, can be rewritten as $\frac{\sqrt{g}}{h_i^2} = f_i F_i$. Note that g and h_i depend on the chosen coordinate system. This implies that F_i and f_i are characteristics of the coordinate system and independent of k and boundary conditions. Now we see that equation (31) becomes

$$\sum_{i=1}^{n} \frac{F_i}{X_i} \frac{d}{d\xi_i} \left(f_i \frac{dX_i}{d\xi_i} \right) + k^2 \sqrt{g} = 0. \tag{33}$$

Note that there are no partial derivatives in this equation. We now differentiate equation (33) with respect to α_i to obtain n equations:

$$\begin{cases} \frac{\partial}{\partial \alpha_1} \left[\frac{F_1}{X_1} \frac{d}{d\xi_1} \left(f_1 \frac{dX_1}{d\xi_1} \right) + \frac{F_2}{X_2} \frac{d}{d\xi_2} \left(f_2 \frac{dX_2}{d\xi_2} \right) + \cdots + \frac{F_n}{X_n} \frac{d}{d\xi_n} \left(f_n \frac{dX_n}{d\xi_n} \right) \right] & = -\frac{\partial}{\partial \alpha_1} k^2 \sqrt{g} \\ \frac{\partial}{\partial \alpha_2} \left[\frac{F_1}{X_1} \frac{d}{d\xi_1} \left(f_1 \frac{dX_1}{d\xi_1} \right) + \frac{F_2}{X_2} \frac{d}{d\xi_2} \left(f_2 \frac{dX_2}{d\xi_2} \right) + \cdots + \frac{F_n}{X_n} \frac{d}{d\xi_n} \left(f_n \frac{dX_n}{d\xi_n} \right) \right] & = -\frac{\partial}{\partial \alpha_2} k^2 \sqrt{g} \\ & \vdots \\ \frac{\partial}{\partial \alpha_n} \left[\frac{F_1}{X_1} \frac{d}{d\xi_1} \left(f_1 \frac{dX_1}{d\xi_1} \right) + \frac{F_2}{X_2} \frac{d}{d\xi_2} \left(f_2 \frac{dX_2}{d\xi_2} \right) + \cdots + \frac{F_n}{X_n} \frac{d}{d\xi_n} \left(f_n \frac{dX_n}{d\xi_n} \right) \right] & = -\frac{\partial}{\partial \alpha_n} k^2 \sqrt{g}. \end{cases} \tag{34}$$

This can be written more conveniently if we introduce factors $\phi_{ij}(\xi_i)$ defined as

$$\phi_{ij}(\xi_i) = -\frac{1}{f_i(\xi_i)} \frac{\partial}{\partial \alpha_i} \left[\frac{1}{X_i} \frac{d}{d\xi_i} \left(f_i \frac{dX_i}{d\xi_i} \right) \right], \tag{35}$$

as equation (34) becomes

$$\begin{cases} f_1 F_1 \phi_{11}(\xi_1) + f_2 F_2 \phi_{21} + \cdots + f_n F_n \phi_{n1}(\xi_n) & = \sqrt{g} \\ f_1 F_1 \phi_{12}(\xi_1) + f_2 F_2 \phi_{22} + \cdots + f_n F_n \phi_{n2}(\xi_n) & = 0 \\ & \vdots \\ f_1 F_1 \phi_{1n}(\xi_1) + f_2 F_2 \phi_{2n} + \cdots + f_n F_n \phi_{nn}(\xi_n) & = 0. \end{cases} \tag{36}$$

We can solve this linear system of n equations and n unknowns $(f_i F_i)$ by use of the so called *Stäckel determinant* $\hat{S}$

$$\hat{S} = \begin{vmatrix} \phi_{11}(\xi_1) & \phi_{12}(\xi_1) & \cdots & \phi_{1n}(\xi_1) \\ \phi_{21}(\xi_2) & \phi_{22}(\xi_2) & \cdots & \phi_{2n}(\xi_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1}(\xi_n) & \phi_{n2}(\xi_n) & \cdots & \phi_{nn}(\xi_n) \end{vmatrix} \tag{37}$$

$$= \sum_{i=1}^{n} \phi_{ij} M_{ij}, \qquad j = 1, 2, \ldots, n. \tag{38}$$

To find the solution of equation (36), we take a look at the minors of the elements in the first column denoted by M_{i1}. Since a minor M_{i1} is obtained by deleting the *ith* row and first column, M_{i1} is independent of ξ_i. Next we use the orthogonality property of determinants [3]:

$$\sum_{i=1}^{n} M_{i1}\phi_{i1} = \hat{S}, \quad \sum_{i=1}^{n} M_{i1}\phi_{im} = 0; \quad m = 2, 3, \ldots n. \tag{39}$$

We see that, if $\hat{S} \neq 0$, equation (36) has solution

$$f_i F_i = \frac{\sqrt{g}}{\hat{S}} M_{i1}. \tag{40}$$

From equation (32) and equation (40) one can conclude that one condition for separability is that

$$g_{ii} = \frac{\hat{S}}{M_{i1}}. \tag{41}$$

Also from equation (40), we see that

$$\frac{\sqrt{g}}{\hat{S}} = f_i \frac{F_i}{M_{i1}}, \tag{42}$$

where $f_i = f_i(\xi_i)$ and F_i and M_{i1} are functions independent of ξ_i. So one can write, for $G_i = \frac{F_i}{M_{i1}}$,

$$\begin{aligned} \frac{\sqrt{g}}{\hat{S}} &= f_1(\xi_1)G_1(\xi_2, \xi_3, \ldots, \xi_n) \\ &= f_2(\xi_2)G_2(\xi_1, \xi_3, \ldots, \xi_n) \\ &\vdots \\ &= f_n(\xi_n)G_n(\xi_1, \xi_2, \ldots .\xi^{n-1}). \end{aligned}$$

Therefore it must be that

$$\frac{\sqrt{g}}{\hat{S}} = \prod_{i=1}^{n} f_i(\xi_i), \tag{43}$$

which is the second condition for separability. Now we have introduced the necessary terminology to state *Theorem 1*:

Theorem 1. *The Helmholtz equation* $\nabla^2\psi + k^2\psi = 0$ *in Euclidean n-space with orthogonal coordinates* $(\xi_1, \xi_2, \ldots, \xi_n)$ *and* $\psi = \psi(\xi_1, \xi_2, \ldots, \xi_n)$ *can be reduced to n ordinary differential equations if and only if the metric coefficients satisfy the two equations*

1. $h_i^2 = \frac{\hat{S}}{M_{i1}}$

2. $\frac{\sqrt{g}}{\hat{S}} = \prod_{i=1}^{n} f_i(\xi_i)$

We have already shown that the Helmholtz equation separates *only if* conditions 1 and 2 hold. What is left to show is that *if* conditions 1 and 2 hold, the Helmholtz equation separates.

Combining condition 1 and 2 gives

$$\frac{\sqrt{g}}{h_i^2} = \left[\prod_{i=1}^{n} f_i(\xi_i)\right] M_{i1}. \tag{44}$$

Substituting this in the Helmholtz equation (31) with $k^2 = \alpha_1$ and $\sqrt{g} = \hat{S}\prod f_i(\xi_i)$ gives

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{1}{X(\xi_i)} \frac{\partial}{\partial \xi_i}\left(\left[\prod_{i=1}^{n} f_i(\xi_i)\right] M_{i1} \frac{\partial X(\xi_i)}{\partial \xi_i}\right) + \alpha_1\sqrt{g} = 0 \\
\Rightarrow &\frac{1}{\hat{S}} \sum_{i=1}^{n} \frac{M_{i1}}{X(\xi_i) f_i(\xi_i)} \frac{\partial}{\partial \xi_i}\left(f_i(\xi_i) \frac{\partial X(\xi_i)}{\partial \xi_i}\right) + \alpha_1 = 0 \\
\Rightarrow &\sum_{i=1}^{n} \frac{1}{h_i^2 X(\xi_i) f_i(\xi_i)} \frac{\partial}{\partial \xi_i}\left(f_i(\xi_i) \frac{\partial X(\xi_i)}{\partial \xi_i}\right) + \alpha_1 = 0.
\end{aligned} \tag{45}$$

Combining the definition of the determinant with condition 1 gives us

$$\hat{S} = \sum_{i=1}^{n} \phi_{i1} M_{i1} \quad \Rightarrow \quad 1 = \sum_{i=1}^{n} \phi_{i1} \frac{M_{i1}}{\hat{S}} = \sum_{i=1}^{n} \frac{\phi_{i1}}{h_i^2}. \tag{46}$$

Using the orthogonality property from equation (39), we see that for $j \in \{2, \dots, n\}$

$$\sum_{i=1}^{n} \frac{\phi_{ij}}{h_i^2} = \frac{1}{\hat{S}} \sum_{i=1}^{n} \phi_{ij} M_{i1} = 0. \tag{47}$$

So we may write

$$\alpha_1 = \alpha_1 \sum_{i=1}^{n} \frac{\phi_{i1}}{h_i^2} + \alpha_2 \sum_{i=1}^{n} \frac{\phi_{i2}}{h_i^2} + \cdots + \alpha_n \sum_{i=1}^{n} \frac{\phi_{in}}{h_i^2}. \tag{48}$$

Substituting this in equation (45) gives

$$\sum_{i=1}^{n} \frac{1}{h_i^2} \left\{ \frac{1}{f_i X_i} \frac{d}{d\xi_i}\left(f_i \frac{dX_i}{d\xi_i}\right) + \sum_{j=1}^{n} \alpha_j \phi_{ij}(\xi_i) \right\} = 0, \tag{49}$$

which, in general, means that

$$\frac{1}{f_i X_i}\frac{d}{d\xi_i}\left(f_i\frac{dX_i}{d\xi_i}\right)+\sum_{j=1}^{n}\alpha_j\phi_{ij}(\xi_i)=0, \tag{50}$$

or, equivalently,

$$\frac{1}{f_i}\frac{d}{d\xi_i}\left(f_i\frac{dX_i}{d\xi_i}\right)+X_i\sum_{j=1}^{n}\alpha_j\phi_{ij}(\xi_i)=0. \tag{51}$$

The first term consists of derivatives of functions dependent on ξ_i with respect to ξ_i and the second term is only dependent on ξ_i and the separation constants. This means that, starting from conditions 1 and 2, we have obtained n coupled ordinary differential equations. Therefore conditions 1 and 2 are sufficient for separability.

Example 2.1. *Elliptic coordinates are defined as*

$$\begin{aligned} &x=\cosh\mu\cos\theta,\\ &y=\sinh\mu\sin\theta,\\ &\theta\in[0,2\pi),\quad \mu\in[0,\infty). \end{aligned} \tag{52}$$

These coordinates and an application of the scalar Helmholtz equation in these coordinates will be treated in chapter 3. For these coordinates, $h_1^2=h_2^2=\cosh^2\mu-\cos^2\theta$ *and* $\sqrt{g}=h_1h_2=\cosh^2\mu-\cos^2\theta$*. Some attempts for the Stäckel matrix show us that if we take*

$$\hat{S}=\begin{vmatrix}\cosh^2\mu & 1\\ \cos^2\theta & 1\end{vmatrix}$$

then conditions 1 and 2 are satisfied, namely:

$$\frac{\sqrt{g}}{\hat{S}}=1,\qquad h_1^2=\frac{\hat{S}}{M_{11}}=1,\qquad h_2^2=\frac{\hat{S}}{M_{21}}=1.$$

So we can use equation (51) *to find the separated equations:*

$$\frac{d^2M}{d\mu^2}+M(\alpha_1\cosh^2\mu-\alpha_2)=0, \tag{53}$$

$$\frac{d^2\phi}{d\theta^2}+\phi(\alpha_1\cos^2\theta-\alpha_2)=0, \tag{54}$$

where $X_1(\xi_1)=M(\mu)$ *and* $X_2(\xi_2)=\phi(\theta)$*. So the Helmholtz equation in elliptic coordinates can be separated.*

2.1.1 Separability in Euclidean 3-Space

We now have a theorem which deals with the separability of the scalar Helmholtz equation, so we can determine which coordinate systems actually do separate. In $\mathbb{R}^3$, i.e. when taking $n = 3$ in the discussion above, Eisenhart [4] showed that the Helmholtz equation separates only in eleven coordinate systems, which sometimes are called the *Eisenhart Coordinate Systems*:

1. Rectangular coordinates
2. Circular-cylinder coordinates
3. Elliptic-cylinder coordinates
4. Parabolic-cylinder coordinates
5. Spherical coordinates
6. Prolate spheroidal coordinates
7. Oblate spheroidal coordinates
8. Parabolic coordinates
9. Conical coordinates
10. Ellipsoidal coordinates
11. Paraboloidal coordinates

In fact, Morse and Feshbach [3] showed that the scalar Helmholtz equation separates for ellispoidal coordinates, and that the other coordinate systems are degenerate forms of this system. Namely, the equation

$$\frac{x^2}{\xi^2 - a^2} + \frac{y^2}{\xi^2 - b^2} + \frac{z^2}{\xi^2 - c^2} = 1; \quad a \geq b \geq c \geq 0, \tag{55}$$

for different values of the parameter ξ, represents three families of confocal quadric surfaces.

- $\xi > a$ Gives a complete family of confocal ellipsoids
- $a > \xi > b$ Gives a complete set of confocal hyperboloids of one sheet.
- $b > \xi > c$ Gives a complete set of confocal hyperboloids of two sheets.

The three families of surfaces are mutually orthogonal, so one can consider the three ranges for the parameter ξ to correspond to three families of coordinate surfaces. From this, coordinates (ξ_1, ξ_2, ξ_3) can be deduced. $\xi_1 > a$ corresponds to ellipsoids, $a > \xi_2 > b$ corresponds to hyperboloids of one sheet and $b > \xi_3 > c$ corresponds to hyperboloids of two sheets. When taking the ξ_i in the following way, this gives the so called ellipsoidal coordinates:

$$
\begin{aligned}
x &= \sqrt{\frac{(\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)}{a^2(a^2 - b^2)}}, \\
y &= \sqrt{\frac{(\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)}{b^2(b^2 - a^2)}}, \\
z &= \frac{\xi_1 \xi_2 \xi_3}{a}, \\
&\xi_1 > a > \xi_2 > b > \xi_3 > 0.
\end{aligned}
\tag{56}
$$

The scale factors h_1, h_2 and h_3 follow easily, and the conditions given in Theorem 1 can be satisfied, so the scalar Helmholtz equation separates. The 10 degenerate coordinate systems can be obtained by stretching, compressing and translating the coordinate surfaces. This is done by letting a, b, c go to zero or infinity. The coordinates (ξ_1, ξ_2, ξ_3) behave nicely with respect to these transformations, and the scalar Helmholtz equation separates for all coordinate systems. Therefore, the scalar Helmholtz equation separates also in these 10 coordinate systems.

2.2 The Vector Helmholtz Equation

We will now look at the seperability of the vector Helmholtz equation

$$
(\blacktriangle + k^2)\psi = 0, \tag{57}
$$

where $\blacktriangle$ is the vector Laplacian as discussed in chapter 1. The vector Helmholtz equation is most often used in $\mathbb{R}^3$ so we will take $n = 3$ in the following discussion. So we will take $\psi(\xi_1, \xi_2, \xi_3) = \psi_1 \mathbf{e}_1 + \psi_2 \mathbf{e}_2 + \psi_3 \mathbf{e}_3$, where $\mathbf{e_i}$ is the unit vector corresponding to coordinate ξ_i. Using the expression $\blacktriangle\psi = \nabla^2\psi = \nabla(\nabla \cdot \psi) - \nabla \times (\nabla \times \psi)$ and the expressions for the curl, the gradient and the divergence in terms of scale factors, we can write out the vector Helmholtz equation. Equating the components of this equation gives

$$
\begin{aligned}
&\left(\frac{1}{g_{ii}}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial\xi_i}\right)\left[\left(\frac{1}{g}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial\xi_i}\right)\left(\frac{g}{g_{ii}}\right)^{\frac{1}{2}}(\psi)_i\right]\\
&+\left(\frac{g_{ii}}{g}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial\xi_j}\right)\left[\left(\frac{g_{kk}}{g^{\frac{1}{2}}}\right)\left(\frac{\partial}{\partial\xi_j}\right)(g_{ii})^{\frac{1}{2}}(\psi)_i\right]\\
&+\left(\frac{g_{ii}}{g}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial\xi_k}\right)\left[\left(\frac{g_{jj}}{g^{\frac{1}{2}}}\right)\left(\frac{\partial}{\partial\xi_k}\right)(g_{ii})^{\frac{1}{2}}(\psi)_i\right]\\
&+\left(\frac{1}{g_{ii}}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial\xi_i}\right)\left[\left(\frac{1}{g}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial\xi_j}\right)\left(\frac{g}{g_{jj}}\right)^{\frac{1}{2}}(\psi)_j\right]\\
&+\left(\frac{1}{g_{ii}}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial\xi_i}\right)\left[\left(\frac{1}{g}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial\xi_k}\right)\left(\frac{g}{g_{kk}}\right)^{\frac{1}{2}}(\psi)_k\right]\\
&-\left(\frac{g_{ii}}{g}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial\xi_j}\right)\left[\left(\frac{g_{kk}}{g^{\frac{1}{2}}}\right)\left(\frac{\partial}{\partial\xi_i}\right)(g_{jj})^{\frac{1}{2}}(\psi)_j\right]\\
&-\left(\frac{g_{ii}}{g}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial\xi_k}\right)\left[\left(\frac{g_{jj}}{g^{\frac{1}{2}}}\right)\left(\frac{\partial}{\partial\xi_i}\right)(g_{kk})^{\frac{1}{2}}(\psi)_k\right]\\
&+k^2(\psi)_i=0,
\end{aligned}
\tag{58}
$$

where $i, j, k = 1, 2, 3$ and $i \neq j \neq k$. So, this is the Helmholtz equation for *one* component $(\psi)_i$. For the equation of component i to separate, we first note that it is necessary that components j and k drop out. So we require that

$$
\begin{aligned}
&\left(\frac{1}{g_{ii}}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial\xi_i}\right)\left[\left(\frac{1}{g}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial\xi_j}\right)\left(\frac{g}{g_{jj}}\right)^{\frac{1}{2}}(\psi)_j\right]\\
&-\left(\frac{g_{ii}}{g}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial\xi_j}\right)\left[\left(\frac{g_{kk}}{g^{\frac{1}{2}}}\right)\left(\frac{\partial}{\partial\xi_i}\right)(g_{jj})^{\frac{1}{2}}(\psi)_j\right]=0,
\end{aligned}
\tag{59}
$$

because then equation (58) becomes

$$
\begin{aligned}
&\left(\frac{1}{g_{ii}}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial\xi_i}\right)\left[\left(\frac{1}{g}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial\xi_i}\right)\left(\frac{g}{g_{ii}}\right)^{\frac{1}{2}}(\psi)_i\right]\\
&+\left(\frac{g_{ii}}{g}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial\xi_j}\right)\left[\left(\frac{g_{kk}}{g^{\frac{1}{2}}}\right)\left(\frac{\partial}{\partial\xi_j}\right)(g_{ii})^{\frac{1}{2}}(\psi)_i\right]\\
&+\left(\frac{g_{ii}}{g}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial\xi_k}\right)\left[\left(\frac{g_{jj}}{g^{\frac{1}{2}}}\right)\left(\frac{\partial}{\partial\xi_k}\right)(g_{ii})^{\frac{1}{2}}(\psi)_i\right]\\
&+k^2(\psi)_i=0.
\end{aligned}
\tag{60}
$$

We are now able to make a comparison with the *scalar* Helmholtz equation, which in three dimensions can be written as

$$
\begin{aligned}
&(\Delta + k^2)\psi = \\
&\frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial \xi_1}\right)\left[\frac{\sqrt{g}}{g_{11}}\frac{\partial \psi}{\partial \xi_1}\right] \\
&+\frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial \xi_2}\right)\left[\frac{\sqrt{g}}{g_{22}}\frac{\partial \psi}{\partial \xi_2}\right] \\
&+\frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial \xi_3}\right)\left[\frac{\sqrt{g}}{g_{33}}\frac{\partial \psi}{\partial \xi_3}\right] \\
&+k^2\psi = 0.
\end{aligned} \tag{61}
$$

Equation (60) contains terms $\frac{\partial}{\partial \xi_i}(g_{jj})^{\frac{1}{2}}\psi_j$ and and $\frac{\partial}{\partial \xi_i}\left(\frac{g}{g_{ii}}\right)^{\frac{1}{2}}(\psi)_i$ in the places where equation (61) has terms $\frac{\partial \psi}{\partial \xi_i}$. Therefore, we see that there are less coordinate systems in which the vector Helmholtz equation separates.
In order that equation (60) separates we need that

$$
\left\{
\begin{array}{ll}
(g_{11})^{\frac{1}{2}} & = f_1(\xi_1)f_2(\xi_2)f_3(\xi_3) \\
(g_{22})^{\frac{1}{2}} & = g_1(\xi_1)g_2(\xi_2)g_3(\xi_3) \\
(g_{33})^{\frac{1}{2}} & = h_1(\xi_1)h_2(\xi_2)h_3(\xi_3)
\end{array}
\right. \tag{62}
$$

such that the terms with the partial derivatives with respect to ξ_i do not contain the coordinates ξ_j and ξ_k. Equation (60) then becomes

$$
\begin{aligned}
&\left(\frac{1}{f_i f_j^2 f_k^2}\right)\left(\frac{\partial}{\partial \xi_i}\right)\left[\left(\frac{1}{f_i}\right)\left(\frac{\partial}{\partial \xi_i}\right) g_i h_i (\psi)_i\right] \\
&+\left(\frac{1}{g_i^2 g_j g_k^2 h_j}\right)\left(\frac{\partial}{\partial \xi_j}\right)\left[\frac{h_j}{f_j g_j}\left(\frac{\partial}{\partial \xi_j}\right) f_j (\psi)_i\right] \\
&+\left(\frac{1}{g_k h_i^2 h_j^2 h_k}\right)\left(\frac{\partial}{\partial \xi_k}\right)\left[\frac{g_k}{f_k h_k}\left(\frac{\partial}{\partial \xi_k}\right) f_k (\psi)_i\right] \\
&+k^2(\psi)_i = 0,
\end{aligned} \tag{63}
$$

which in some cases can be separated. Now we obtained two conditions for separability of the vector Helmholtz equation. One can calculate the scale factors h_i for the Eisenhart coordinate systems and compare it with the conditions given above. The condition imposed in equation (59) is satisfied in only three of the eleven Eisenhart coordinate systems: rectangular, circular-cylinder and spherical coordinates. The condition imposed in equation (62) is satisfied for rectangular coordinates only. Therefore, the rectangular coordinate system is the only coordinate system in which all equations for all components separate. In circular-cylinder and spherical coordinates equation (62) is satisfied only if the components ψ_i, ψ_j, ψ_k satisfy certain conditions [5].

3 Applications of the Helmholtz Equation

If the Helmholtz equation separates, we can solve the ordinary differential equations individually and combine the results to find the exact solution of the Helmholtz equation. So for a given problem involving the Helmholtz equation, we can find a suitable coordinate system which allows separability and therefore we can find the exact solution the the problem. We will now use this methodology in problems arising in quantum mechanics, electromagnetism and optics.

3.1 Quantum Billiard

3.1.1 The Inside-Outside Duality

The separability of the scalar Helmholtz equation can be used in the inside-outside duality for planar billiards. The duality is about the strong link between the so called *interior* and *exterior* problem. In the interior problem, we consider a quantum particle inside a convex region in the plane. The particle is trapped inside this region, which means that the potential is zero inside and infinite outside this region. The interior problem is also called the billiard problem because of the similarities with the game going with same name. On the other hand, in the exterior problem we scatter waves on the same region of the plane. Now, the potential is constant outside and infinity in the scattering region. The scattering is characterized by the scattering matrix S, and it turns out that the eigenvalues of this matrix are related to the eigenenergies E of the particles trapped inside the billiard. This was stated by Pillet [6], which comes down to the following (leaving aside some details regarding the boundary):

Theorem 2. *E is a Dirichlet eigenvalue of the interior problem if and only if the on-shell scattering matrix S has an eigenvalue equal to* 1*.*

Here, a Dirichlet eigenvalue refers to Dirichlet boundary conditions for the billiard, which means that the wavefunction of the particle vanishes on the boundary of the billiard. In this section we will show that this statement holds for an ellipse. We will do this by first giving the wavefunction corresponding to an eigenvalue E for ellipsoidal billiard with Dirichlet boundary conditions. Then, we will use the asymptotic behavior of the wave function to determine the scattering matrix S and take a look at its spectral properties. We will look at eigenvalues of $S-I$ being equal to 0, which correspond to S having an eigenvalue equal to one, namely:

$$\begin{aligned}
&0 \quad \text{is an eigenvalue of} \quad S-I \\
&\Leftrightarrow det((S-I)-0I)=0 \\
&\Leftrightarrow det(S-I)=0 \\
&\Leftrightarrow 1 \quad \text{is an eigenvalue of} \quad S.
\end{aligned}$$

3.1.2 Elliptic Billiard

We will now solve the billiard problem for the case of an ellipse, and therefore it is convenient to introduce elliptic coordinates (θ, μ) given by

$$\begin{aligned} x &= \cosh\mu\cos\theta, \\ y &= \sinh\mu\sin\theta, \\ \theta &\in [0, 2\pi), \quad \mu \in [0, \infty). \end{aligned} \tag{64}$$

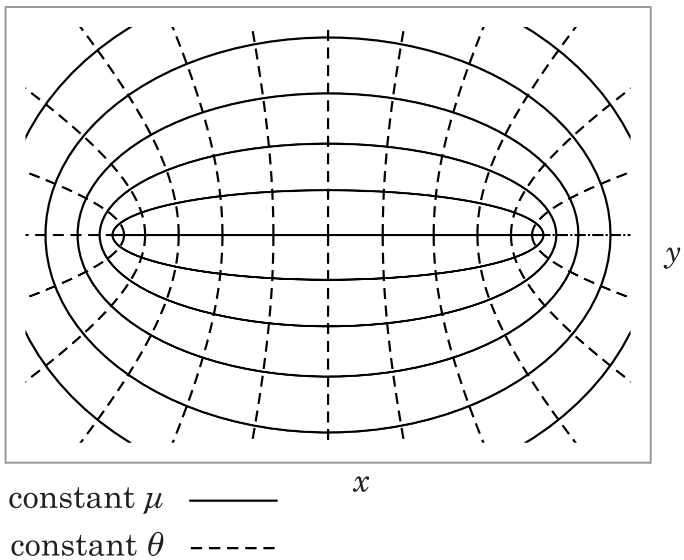

Figure 2: Coordinate lines for elliptic coordinates.

The coordinate lines are ellipses for constant μ and hyperbolae for constant θ, both with focal points on the y-axis at $x = 1$ and $x = -1$. This is illustraded in figure 2.

To find the eigenstates of the elliptic billiard, we need to solve the stationary Schrödinger equation in elliptic, coordinates

$$\hat{H}(\mu, \theta)\Psi(\mu, \theta) = E\Psi(\mu, \theta). \tag{65}$$

Here $\hat{H}$ is the Hamiltonian, which in this case is equal to the sum of the kinetic and potential energy,

$$\hat{H}(\mu, \theta) = -\frac{\hbar^2}{2m}\nabla^2 + V(\mu, \theta). \tag{66}$$

If we denote the boundary of the ellipse by a positive constant μ_0, the potential energy $V(\mu, \theta)$ is given by

$$V(\mu,\theta) = \begin{cases} 0 & \text{if } \mu \leq \mu_0 \\ \infty & \text{if } \mu > \mu_0 \end{cases}, \tag{67}$$

and the Dirichlet boundary condition is given by $\Psi(\mu_0,\theta) = 0$. In the billiard, the Schrödinger equation reduces to the scalar Helmholtz equation:

$$(\nabla^2 + k^2)\Psi(\mu,\theta) = 0, \tag{68}$$

where $k^2 = \frac{2m}{\hbar^2}E$ is the rescaled energy. In section 2 we saw that the Helmholtz equation separates in elliptic coordinates. In this case, it can also easily be seen by using the expression for the Laplacian in elliptic coordinates:

$$\nabla^2\Psi = \frac{1}{\cosh^2\mu - \cos^2\theta}\left(\frac{\partial^2\Psi}{\partial\mu^2} + \frac{\partial^2\Psi}{\partial\theta^2}\right). \tag{69}$$

Then, equation (68) becomes

$$\frac{\partial^2\Psi}{\partial\mu^2} + \frac{\partial^2\Psi}{\partial\theta^2} + k^2(\cosh^2\mu - \cos^2\theta)\Psi = 0. \tag{70}$$

Rewriting gives

$$\frac{\partial^2\Psi}{\partial\mu^2} + k^2\cosh^2\mu\Psi = -\frac{\partial^2\Psi}{\partial\theta^2} + k^2\cos^2\theta\Psi, \tag{71}$$

and using the ansatz $\Psi = \Phi(\theta)M(\mu)$ gives

$$\frac{1}{M}\frac{\partial^2 M}{\partial\mu^2} + k^2\cosh^2\mu = -\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\theta^2} + k^2\cos^2\theta. \tag{72}$$

We are now in the position to obtain two ordinary differential equations with separation constant b. The right hand side of equation (72) gives us the so called *standard Mathieu equation*:

$$\frac{\partial^2}{\partial\theta^2}\Phi(\theta) + (b - k^2\cos^2\theta)\Phi(\theta) = 0, \tag{73}$$

and the left hand side of equation (72) gives us the so called *modified Mathieu equation*.

$$-\frac{\partial^2}{\partial\mu^2}M(\mu) + (b - k^2\cosh^2\mu)M(\mu) = 0. \tag{74}$$

These ordinary differential equations have solutions which are widely discussed in the literature. Equation (73) has solutions $\Phi^{e(o)}_{2n(+1)}(k;\theta)$ [7], which are even or odd about $\theta = 0$:

$$\Phi^e_{2n}(k;\theta) = \sum_{r=0}^{\infty} A^{2n}_{2r}(k)\cos(2r\theta), \tag{75}$$

$$\Phi^e_{2n+1}(k;\theta) = \sum_{r=0}^{\infty} A^{2n+1}_{2r+1}(k)\cos[(2r+1)\theta], \tag{76}$$

$$\Phi^o_{2n}(k;\theta) = \sum_{r=0}^{\infty} B^{2n}_{2r}(k)\sin(2r\theta), \tag{77}$$

$$\Phi^o_{2n+1}(k;\theta) = \sum_{r=0}^{\infty} B^{2n+1}_{2r+1}(k)\sin[(2r+1)\theta], \tag{78}$$

where the index $2n$ or $2n+1$ indicates the periodicity of the cosine or sine in the respectively even or odd solution. The coefficients $A^{2n(+1)}$ and $B^{2n(+1)}$ follow from recurrence relations involving b and k. The obtained functions all satisfy the following orthogonality relation

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \Phi^{e(o)}_m(k;\theta)\Phi^{e(o)}_n(k;\theta)d\theta = \delta_{nm}. \tag{79}$$

Equation (74) has even solutions $M^{e\pm}_{2n(+1)}$ and odd solutions $M^{o\pm}_{2n(+1)}$. The expressions for $M^{e\pm}_{2n}$ are [7]:

$$\epsilon_m M^{e+}_{2n}(k,\mu) = \sum_{r=0}^{\infty} i^{2n}(-1)^r \frac{A^{2n}_{2r}(k)}{A^{2n}_{2m}(k)}\left(H^+_{r+m}(\frac{k}{2}e^{\mu})J_{r-m}(\frac{k}{2}e^{-\mu}) + H^+_{r-m}(\frac{k}{2}e^{\mu})J_{r+m}(\frac{k}{2}e^{-\mu})\right),$$

$$M^{e-}_{2n}(k,\mu) = \sum_{r=0}^{\infty} i^{2n}(-1)^r \frac{A^{2n}_{2r}(k)}{A^{2n}_{2m}(k)}\left(H^-_{r+m}(\frac{k}{2}e^{\mu})J_{r-m}(\frac{k}{2}e^{-\mu}) + H^-_{r-m}(\frac{k}{2}e^{\mu})J_{r+m}(\frac{k}{2}e^{-\mu})\right),$$

where

$$\epsilon_m = \begin{cases} 2 & \text{if } m = 0 \\ 1 & \text{if } m \neq 0. \end{cases} \tag{80}$$

Here $H^{\pm}_{\ell}(x)$ and $J_{\ell}(x)$ are Bessel functions of integer order. The solutions $M^{e\pm}_{2n+1}$, $M^{o\pm}_{2n(+1)}$ have similar forms. We will use these solutions to obtain the scattering matrix S. The total wave function at large distances is given as the sum of a plane wave and a scattered wave

$$\Psi = e^{ik\cosh\mu\cos(\theta-\theta_i)} + \Psi_{scat}. \tag{81}$$

The plane incoming wave can be given in terms of the solutions Φ_m and M_m [3]

$$e^{ik\cosh\mu\cos(\theta-\theta_i)} = \sqrt{8\pi}\sum_{n=0}^{\infty}\left\{\Phi^e_m(k;\theta_i)\,\Phi^e_m(k;\theta)MJ^e_m(k;\mu) + \Phi^o_m(k;\theta_i)\,\Phi^o_m(k;\theta)MJ^o_m(k;\mu)\right\}, \tag{82}$$

where

$$MJ_m^{e(o)}(k;\mu) = \frac{M_m^{e(o)+}(k;\mu) + M_m^{e(o)-}(k,\mu)}{2}. \tag{83}$$

Since we are dealing with a second order differential equation, we will have two linearly independent solutions which will be denoted as the even and odd solutions. The even solution will be denoted as ψ^e and the odd solution as ψ^o. Using the expansion of the Bessel functions for large arguments one can obtain

$$\begin{aligned}\Psi^{e(o)}(\theta,\mu) =& \sqrt{8\pi}\sum_{m=0}^{\infty} i^m \Phi_m^{e(o)}(k;\theta_i)\,\Phi_m^{e(o)}(k;\theta) MJ_m^{e(o)}(k;\mu)\\ &+ \sqrt{2\pi}\sum_{m,m'=0}^{\infty}\left(S_{mm'}^{e(o)} - \delta_{mm'}\right) i^{m'}\Phi_{m'}^{e(o)}(k;\theta_i)\,\Phi_{m'}^{e(o)}(k;\theta) M_{m'}^{e(o)+}(k;\mu).\end{aligned} \tag{84}$$

Now we are in the position to obtain expressions for the scattering matrix. Imposing the Dirichlet boundary condition gives us

$$\begin{aligned}0 =& \sqrt{8\pi}\sum_{m=0}^{\infty} i^m \Phi_m^{e(o)}(k;\theta_i)\,\Phi_m^{e(o)}(k;\theta) MJ_m^{e(o)}(k;\mu_0)\\ &+ \sqrt{2\pi}\sum_{m,m'=0}^{\infty}\left(S_{mm'}^{e(o)} - \delta_{mm'}\right) i^{m'}\Phi_{m'}^{e(o)}(k;\theta_i)\,\Phi_{m'}^{e(o)}(k;\theta) M_{m'}^{e(o)+}(k;\mu_0).\end{aligned} \tag{85}$$

Multiplying by $\Phi_\ell^{e(o)}(k;\mu_0)$, integrating over θ and using the orthogonality property gives $m = m' = \ell$ and therefore

$$0 = 2MJ_\ell^{e(o)}(k;\mu_0) + \left(S_{\ell\ell}^{e(o)} - 1\right) M_\ell^{e(o)+}(k;\mu_0). \tag{86}$$

Now we see that the scattering matrix is diagonal and is equal to

$$S_{mm'}^{e(o)} = -\delta_{mm'}\frac{M_m^{o(o)-}(k;\mu_0)}{M_m^{e(o)+}(k;\mu_0)}. \tag{87}$$

Because S is diagonal, $S - I$ is also diagonal. $S - I$ has an eigenvalue equal to zero if and only if one of its diagonals is equal to zero:

$$-\frac{M_m^{e(o)-}(k;\mu_0)}{M_m^{e(o)+}(k;\mu_0)} - 1 = 0 \quad \text{for some } m, m' \tag{88}$$

$$\Leftrightarrow MJ_m(k;\mu_0) = 0. \tag{89}$$

So 1 is in the spectrum of S if and only if $MJ_m(k;\mu_0) = 0$. Actually, the quantization condition for the elliptic billiard is that $MJ_m(k;\mu_0) = 0$, so we can conclude that E is an eigenvalue for the billiard problem if and only if 1 is an eigenvalue of S.

3.2 Exact Solution of the Vector Helmholtz Equation

In this section, we will see that the constraints placed on the separability of the vector Helmholtz equation do not necessarily limit the separability of Maxwell's equations. If we use the expressions $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0(\mathbf{r})e^{i\omega t}$ and $\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r})e^{i\omega t}$, Maxwell's equations read

$$\begin{aligned} \nabla \times H &= i\omega\epsilon_0 E, \\ \nabla \times E &= -i\omega\mu_0 H, \end{aligned} \tag{90}$$

where, for readability we wrote E instead of $\mathbf{E}_0(\mathbf{r})$, H instead of $\mathbf{H}_0(\mathbf{r})$ and we used $B = \mu_0 H$. In prolate spheroidal coordinates, we saw that the vector Helmholtz equation does not but the scalar Helmholtz equation does separate. However, we will see that the set of equations given above allows separabilty under certain transformations.

Prolate spheroidal coordinates are obtained by rotating the two dimensional elliptic coordinates about the focal axis of the ellipse. Prolate spheroidal coordinates(ξ, η, ϕ) and focal distance f are given by:

$$\begin{aligned} x &= f\sqrt{(\xi^2-1)(1-\eta^2)}\cos\phi, \\ y &= f\sqrt{(\xi^2-1)(1-\eta^2)}\sin\phi, \\ z &= f\xi\eta, \\ \xi &\in [1,\infty), \quad \eta \in [-1,1], \quad \phi \in [0,2\pi]. \end{aligned} \tag{91}$$

Surfaces of constant ξ are prolate spheroids and surfaces of constant η are hyperboloids of revolutions. The scale factors are given by

$$h_\xi = f\left(\frac{\xi^2-\eta^2}{\xi^2-1}\right)^{1/2}, \quad h_\eta = f\left(\frac{\xi^2-\eta^2}{1-\eta^2}\right)^{1/2}, \quad \text{and} \quad h_\phi = f\left[\left(\xi^2-1\right)\left(1-\eta^2\right)\right]^{1/2}. \tag{92}$$

Using the expression for the vector Laplacian in terms of scale facors gives

$$\begin{aligned} &\frac{1}{h_\eta h_\phi}\left[\frac{\partial}{\partial\eta}(h_\phi H_\phi) - \frac{\partial}{\partial\phi}(h_\eta H_\eta)\right]\hat{i}_\xi + \frac{1}{h_\phi h_\xi}\left[\frac{\partial}{\partial\phi}(h_\xi H_\xi) - \frac{\partial}{\partial\xi}(h_\phi H_\phi)\right]\hat{i}_\eta \\ &\qquad + \frac{1}{h_\xi h_\eta}\left[\frac{\partial}{\partial\xi}(h_\eta H_\eta) - \frac{\partial}{\partial\eta}(h_\xi H_\xi)\right]\hat{i}_\phi = i\omega\epsilon_0\left(\hat{i}_\xi E_\xi + \hat{i}_\eta E_\eta + \hat{i}_\phi E_\phi\right) \end{aligned} \tag{93}$$

and

$$\begin{aligned} &\frac{1}{h_\eta h_\phi}\left[\frac{\partial}{\partial\eta}(h_\phi E_\phi) - \frac{\partial}{\partial\phi}(h_\eta E_\eta)\right]\hat{i}_\xi + \frac{1}{h_\phi h_\xi}\left[\frac{\partial}{\partial\phi}(h_\xi E_\xi) - \frac{\partial}{\partial\xi}(h_\phi E_\phi)\right]\hat{i}_\eta \\ &\qquad + \frac{1}{h_\xi h_n}\left[\frac{\partial}{\partial\xi}(h_\eta E_\eta) - \frac{\partial}{\partial\eta}(h_\xi E_\xi)\right]\hat{i}_\phi = -i\omega\mu_0\left(\hat{i}_\xi H_\xi + \hat{i}_\eta H_\eta + \hat{i}_\phi H_\phi\right), \end{aligned} \tag{94}$$

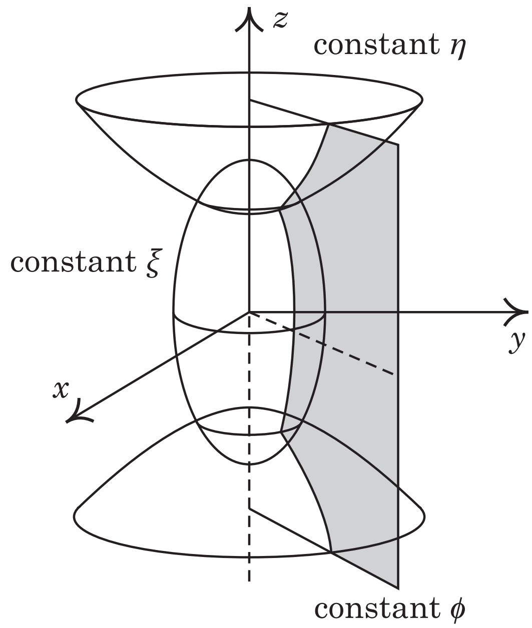

Figure 3: Prolate spheroidal coordinates with the coordinate surfaces given corresponding to constant ξ, η and ϕ.

where $\hat{i}_\eta$ is the unit vector in the η direction and so on. Now, if we let

$$\begin{aligned} &\left\{ \begin{array}{c} \overline{E}_\xi \\ \overline{H}_\xi \end{array} \right\} = h_\eta h_\phi \left\{ \begin{array}{c} E_\xi \\ H_\xi \end{array} \right\}, \quad \left\{ \begin{array}{c} \overline{E}_\eta \\ \overline{H}_\eta \end{array} \right\} = h_\xi h_\phi \left\{ \begin{array}{c} E_\eta \\ H_\eta \end{array} \right\}, \\ \text{and} \quad &\left\{ \begin{array}{c} \overline{E}_\phi \\ \overline{H}_\phi \end{array} \right\} = h_\phi \left\{ \begin{array}{c} E_\phi \\ H_\phi \end{array} \right\}, \end{aligned} \tag{95}$$

then equating the vector components gives us the following six equations:

$$\frac{\partial}{\partial \eta} \overline{H}_\phi - \frac{\partial}{\partial \phi} \left(\frac{h_\eta}{h_\xi h_\psi} \overline{H}_\eta \right) = i\omega\epsilon_0 \overline{E}_\xi, \tag{96}$$

$$\frac{\partial}{\partial \phi} \left(\frac{h_\xi}{h_\eta h_\phi} \overline{H}_\xi \right) - \frac{\partial}{\partial \xi} \overline{H}_\phi = i\omega\epsilon_0 \overline{E}_\eta, \tag{97}$$

$$\frac{\partial}{\partial \xi} \left(\frac{h_\eta}{h_\xi h_\phi} \overline{H}_\eta \right) - \frac{\partial}{\partial \eta} \left(\frac{h_\xi}{h_\eta h_\phi} \overline{H}_\xi \right) = i\omega\epsilon_0 \frac{h_\xi h_\eta}{h_\phi} \overline{E}_\phi, \tag{98}$$

$$\frac{\partial}{\partial \eta} \overline{E}_\phi - \frac{\partial}{\partial \phi} \left(\frac{h_\eta}{h_\xi h_\phi} \overline{E}_\eta \right) = -i\omega\mu_0 \overline{H}_\xi, \tag{99}$$

$$\frac{\partial}{\partial \phi} \left(\frac{h_\xi}{h_\eta h_\phi} \overline{E}_\xi \right) - \frac{\partial}{\partial \xi} \overline{E}_\phi = -i\omega\mu_0 \overline{H}_\eta, \tag{100}$$

$$\frac{\partial}{\partial \xi}\left(\frac{h_\eta}{h_\xi h_\phi}\overline{E}_\eta\right) - \frac{\partial}{\partial \eta}\left(\frac{h_\xi}{h_\eta h_\phi}\overline{E}_\phi\right) = -i\omega\mu_0 \frac{h_\xi h_\eta}{h_\phi}\overline{H}_\phi. \tag{101}$$

Now we use equation (99),(100) and (101) to get expressions for $\overline{H}_\xi, \overline{H}_\eta, \overline{H}_\phi$:

$$\overline{H}_\xi = \frac{i}{\omega\mu_0}\left[\frac{\partial}{\partial \eta}\overline{E}_\phi - \frac{\partial}{\partial \phi}\left(\frac{h_\eta}{h_\xi h_\phi}\overline{E}_\eta\right)\right], \tag{102}$$

$$\overline{H}_\eta = \frac{i}{\omega\mu_0}\left[\frac{\partial}{\partial \phi}\left(\frac{h_\xi}{h_\eta h_\phi}\overline{E}_\xi\right) - \frac{\partial}{\partial \xi}\overline{E}_\phi\right], \tag{103}$$

$$\overline{H}_\phi = \frac{i}{\omega\mu_0}\frac{h_\phi}{h\xi h_\eta}\left[\frac{\partial}{\partial \xi}\left(\frac{h_\eta}{h_\xi h_\phi}\overline{E}_\eta\right) - \frac{\partial}{\partial \eta}\left(\frac{h_\xi}{h_\eta h_\phi}\overline{E}_\phi\right)\right]. \tag{104}$$

Adding equation (96) and equation (97), substituting $\overline{H}_\xi, \overline{H}_\eta, \overline{H}_\phi$ and using $k_0^2 = \omega^2\epsilon_0\mu_0$ gives

$$\begin{aligned}
&\frac{\partial}{\partial \eta}\frac{h_\phi}{h_\xi h_\eta}\left[\frac{\partial}{\partial \xi}\left(\frac{h_\eta}{h_\xi h_\phi}\overline{E}_\eta\right) - \frac{\partial}{\partial \eta}\left(\frac{h_\xi}{h_\eta h_\phi}\overline{E}_\phi\right)\right] \\
&\quad -\frac{\partial}{\partial \phi}\left(\frac{h_\eta}{h_\xi h_\psi}\left[\frac{\partial}{\partial \phi}\left(\frac{h_\xi}{h_\eta h_\phi}\overline{E}_\xi\right) - \frac{\partial}{\partial \xi}\overline{E}_\phi\right]\right) \\
&\quad +\frac{\partial}{\partial \phi}\left(\frac{h_\xi}{h_\eta h_\phi}\left[\frac{\partial}{\partial \eta}\overline{E}_\phi - \frac{\partial}{\partial \phi}\left(\frac{h_\eta}{h_\xi h_\phi}\overline{E}_\eta\right)\right]\right) \\
&-\frac{\partial}{\partial \xi}\frac{h_\phi}{h_\xi h_\eta}\left[\frac{\partial}{\partial \xi}\left(\frac{h_\eta}{h_\xi h_\phi}\overline{E}_\eta\right) - \frac{\partial}{\partial \eta}\left(\frac{h_\xi}{h_\eta h_\phi}\overline{E}_\phi\right)\right] = \\
&\qquad\qquad k_0^2(\overline{E}_\xi + \overline{E}_\eta).
\end{aligned}$$

Using that $\frac{h_\eta}{h_\xi h_\phi} = M(\eta)$, $\frac{h_\xi}{h_\eta h_\phi} = N(\xi)$, we can take some terms out of the derivatives. Furthermore, noting that

$$\frac{1}{h_\xi^2} + \frac{1}{h_\eta^2} = \frac{1}{f^2}, \quad \text{and} \quad \frac{\partial}{\partial \eta}\left[\frac{1}{h_\eta^2}\right] = \frac{\partial}{\partial \eta}\left[\frac{1}{h_\xi^2}\right], \tag{105}$$

one can obtain

$$\frac{1}{h_\xi^2}\left(\frac{\partial^2 F}{\partial \xi^2}\right) + \frac{1}{h_\eta^2}\left(\frac{\partial^2 F}{\partial \eta^2}\right) + \frac{1}{h_\phi^2}\left(\frac{\partial^2 F}{\partial \phi^2}\right) + \left[\frac{\partial}{\partial \xi}\left(\frac{1}{h_\xi^2}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{h_\eta^2}\right)\right]\left[\frac{\partial F}{\partial \xi} + \frac{\partial F}{\partial \eta}\right] + k_0^2 F = 0, \tag{106}$$

where $F = \overline{E}_\xi + \overline{E}_\eta$. In a similar manner, one can obtain for $G = \overline{E}_\xi - \overline{E}_\eta$

$$\frac{1}{h_\xi^2}\left(\frac{\partial^2 G}{\partial \xi^2}\right)+\frac{1}{h_\eta^2}\left(\frac{\partial^2 G}{\partial \eta^2}\right)+\frac{1}{h_\phi^2}\left(\frac{\partial^2 G}{\partial \phi^2}\right)+\left[\frac{\partial}{\partial \xi}\left(\frac{1}{h_\xi^2}\right)-\frac{\partial}{\partial \eta}\left(\frac{1}{h_\eta^2}\right)\right]\left[\frac{\partial G}{\partial \xi}-\frac{\partial G}{\partial \eta}\right]+k_0^2 G=0. \tag{107}$$

Now as ansatz for F and G we take $F=(\xi-\eta)K(\xi,\eta,\phi)$ and $G=(\xi+\eta)L(\xi,\eta,\phi)$. This gives

$$\frac{1}{h_\xi^2}\left(\frac{\partial^2 F}{\partial \xi^2}\right)=\frac{1}{f^2}\frac{\xi^2-1}{\xi^2-\eta^2}\left(2(1-\eta)\frac{\partial K}{\partial \xi}+(\xi-\mu)\frac{\partial^2 K}{\partial \xi^2}\right), \tag{108}$$

$$\frac{1}{h_\eta^2}\left(\frac{\partial^2 F}{\partial \eta^2}\right)=\frac{1}{f^2}\frac{1-\eta^2}{\xi^2-\eta^2}\left(2(\xi-1)K+(\xi-\eta)\frac{\partial^2 K}{\partial \eta^2}\right), \tag{109}$$

$$\frac{1}{h_\phi^2}\left(\frac{\partial^2 F}{\partial \phi^2}\right)=\frac{1}{f^2}\frac{1}{(\xi^2-1)(1-\eta^2)}(\xi-\eta)\frac{\partial^2 K}{\partial \phi^2}. \tag{110}$$

Substituting these expressions in equation (106) and equation (107) gives us the same partial differential equation for K and L

$$\left(\xi^2-1\right)\frac{\partial^2 K}{\partial \xi^2}+\left(1-\eta^2\right)\frac{\partial^2 K}{\partial \eta^2}+\frac{\xi^2-\eta^2}{\left(\xi^2-1\right)\left(1-\eta^2\right)}\frac{\partial^2 K}{\partial \phi^2}+ \tag{111}$$

$$2\xi\frac{\partial K}{\partial \xi}-2\eta\frac{\partial K}{\partial \eta}+k_o^2 f^2\left(\xi^2-\eta^2\right)K=0. \tag{112}$$

The equation for L is identical but with K replaced by L. Now if we use the ansatz $K=\left(\xi^2-1\right)^{m/2}U(\xi)\left(1-\eta^2\right)^{m/2}V(\eta)\psi(\phi)$, we see that equation (111) separates into 3 coupled ordinary differential equations:

$$\frac{d^2\psi}{d\phi^2}+m^2\psi=0, \tag{113}$$

$$\left(\xi^2-1\right)\frac{d^2 U}{d\xi^2}+2\xi(m+1)\frac{dU}{d\xi}-\left(b-k_o^2 f^2\xi^2\right)U=0, \tag{114}$$

$$\left(1-\eta^2\right)\frac{d^2 V}{d\eta^2}-2\eta(m+1)\frac{dV}{d\eta}+\left(b-k_0^2 f^2\eta^2\right)V=0. \tag{115}$$

The equation for L separates in the exact same manner. So we see that using a couple of suitable transformations, Maxwell's equations in prolate spheroidal coordinates do separate. One can compute the exact solutions of the vector Helmholtz once it has separated. For example, one can solve (113), (114) and (115) individually and do reverse transformations. Then, one has to satisfy the periodicity condition, make the solution regular, impose boundary conditions and add the time factor $e^{i\omega t}$ to obtain expressions for $\mathbf{E}=(E_\xi,E_\eta,E_\phi)e^{i\omega t}$ and $\mathbf{H}=(H_\xi,H_\eta,H_\phi)e^{i\omega t}$ [8]. Maxwell's equations and the vector Helmholtz equation are closely related, but the separability conditions for Maxwell's equations are not the same as those for the vector Helmholtz equation.

3.2.1 Lightning Inception by Ice Particles

For a physical problem, we can find a suitable coordinate system and write down the equations in this coordinate system. If the equations separate, we may find exact solutions for the problem and give accurate results for the physical problem. An example of this is found in lightning research. In lightning research, one of the main points of focus is to find out how lightning is originated. Observations of lightning show that theoretically the electric field in thunderclouds is too small for lightning to kick-off. The presence of ice particles in this electric field might be the solution to this problem.

A so called *hydrometeor* in a background electric field can enhance this electric field due to its high dielectric permittivity. These airborne particles, at an altitude of about 5.5 kilometer, can for example be droplets, snowflakes, graupel or hail. The enhanced electric field can accelerate electrons and the accelerated electrons strike air molecules and knock off electrons and other particles. These knocked off secondaries strike more air molecules, creating a chain reaction and an ionized path in space. This mechanism is believed to be one of the candidates for lightning inception.

Hydrometeors can come in lots of shapes, but their shape in the direction perpendicular to the background electric field does not contribute much to the enhanced electric field. However, the shape of the tip parallel to the background electric field is what is believed to determine the enhanced electric field strength. Therefore we consider a tip as a prolate ellipsoid of revolution with length ℓ and radius of curvature R. For a prolate ellipsoid, the enhanced electric field can be calculated exactly [9]. This field depends on the size of the hydrometeor, and therefore we can calculate the size needed to accelerate electrons fast enough to initiate lightning [10].

3.3 Invisibility Devices

The Helmholtz equation is encountered in optics where it can be used to describe the trajectory of light rays. We will now globally give the methodology that can be used to create invisibility devices. These devices are obtained by creating a refractive index profile that can bend light around regions of space, making it invisible within the accuracy of geometrical optics.

In the regime of geometrical optics, light propagation can be described by light rays. Light rays behave according to Fermat's principle, which states that light follows the shortest optical path in a medium. The optical path length is given as an integral over the refractive index n, and therefore light can be bend in media by manipulating the refractive index profile. We will consider a two-dimensional situation with refractive index profile given by $n = n(x, y)$.

Consider a dielectric medium that is uniform in one direction and light of wavenumber k that propagates orthogonal to that direction. Both amplitudes ψ of the polarizations satisfy the two-dimensional Helmholtz equation with constant n^2k^2:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + n^2k^2\right)\psi = 0. \tag{116}$$

To describe the behaviour of light we will use complex numbers $z = x + iy$ with its conjugate $\overline{z} = x - iy$. The partial derivatives are $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}$ and $\frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right)$. In these coordinates equation (116) becomes

$$\left(4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} + n^2k^2\right)\psi = 0. \tag{117}$$

Now suppose that we introduce an analytic function $w(z)$ that is independent of $\overline{z}$. This function is a conformal map, so it preserves angles. In w space with refractive index n' the Helmholtz equation becomes

$$\left(4\frac{\partial}{\partial w}\frac{\partial}{\partial \overline{w}} + n'^2k^2\right)\psi = 0. \tag{118}$$

It is not difficult to see that $\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = \left|\frac{dw}{dz}\right|^2 \frac{\partial}{\partial w}\frac{\partial}{\partial \overline{w}}$, so by equation (117) and equation (118) we can find the relation between n and n':

$$n = n'\left|\frac{dw}{dz}\right|. \tag{119}$$

Now as an example, we take the refractive index profile to be $n(x, y) = \left|1 - \frac{a^2}{z^2}\right|$, where a is a constant. If we then define the analytic map w to be

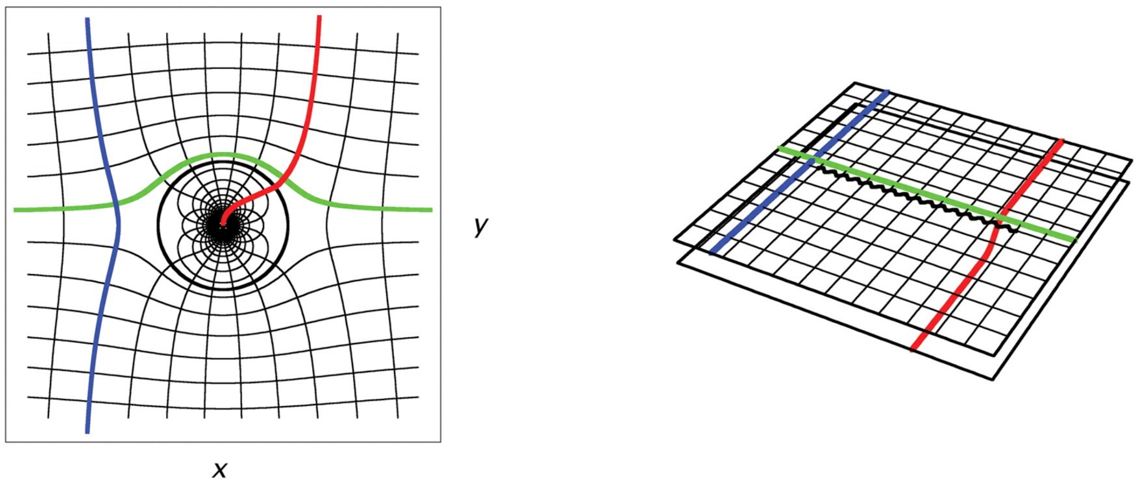

Figure 4: The rays in physical space are mapped to straight lines in w space. The exterior of a circle is mapped to the upper Riemann sheet, the interior is mapped to the lower Riemann sheet and the boundary of the circle is represented as a branch cut in w space. (Figure taken from [11])

$$
\begin{aligned}
w &= z + \frac{a^2}{z}, \quad \text{which implies} \\
z &= \frac{1}{2}\left(w \pm \sqrt{w^2 - 4a^2}\right).
\end{aligned} \tag{120}
$$

We see that $\left|\frac{dw}{dz}\right| = \left|1 - \frac{a^2}{z^2}\right|$, and by equation (119) we see that $n' = 1$, which implies that the light rays are mapped to straight lines in w space. The map w maps to two Riemann sheets [11] , which is illustrated in figure 4.

Now, we will give a method to make a region of the circle with radius a inpenetrable for light rays. The green and blue lines in figure 4 do not enter the circle, so we only need to consider lines such as the red one. In w space, these lines are the ones that go from the exterior sheet to the interior sheet. This is possible because the lines go through the branch cut between the two branch points $-2a$ and $2a$.

Inside the circle of radius a, we will impose a new refractive index profile. This is done to guide the lines back through the branch cut to the exterior sheet. Therefore we require a closed trajectory in w space such that the lines return to the same location and in the same direction. This is realized by either a so called Harmonic oscillator profile or a Kepler profile, respectively given by

$$
n'^2 = 1 - \frac{|w - w_1|^2}{r_0^2} \qquad \text{or} \qquad n'^2 = \frac{r_0}{|w - w_1|} - 1. \tag{121}
$$

Now, in both cases r_0 defines a circle on the interior w sheet in which lines cannot enter. This means that light rays in physical space are not able to access this region.

So, using a conformal map w which is related to the refractive index profile through its derivative, we were able to "steer" light through a medium. This can also be done for other maps w, as well, as for other types of waves. However, one should keep in mind the conditions for optical geometry as well as the imperfections in the refractive index profile for the medium.

4 Conclusion

In this thesis, literature regarding the separability and application of the Helmholtz equation has been reviewed. The Helmholtz equation appears frequently when solving problems in physics involving waves. Depending on the problem, a suitable coordinate system can be chosen. The Helmholtz equation has a different form for each coordinate system. Using scale factors, one can give conditions on the separability of the Helmholtz equation. In Euclidean 3-space the scalar Helmholtz equation separates in 11 coordinate systems and the vector Helmholtz equation separates only in rectangular coordinates.

The separability of the Helmholtz equation allows us to find exact solutions to problems. Using the exact solutions of the scalar Helmholtz equation in 2 dimensions we can solve the Schrödinger equation for elliptic quantum billiards. It can be shown that Maxwell's equations and the vector Helmholtz equation in Euclidean 3-space have similarities, but that the separability conditions are not the same. Finally, using the separability of the scalar Helmholtz equation with complex variables we can construct a refractive index profile such that a design for invisibility devices can be made.

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