# Assigning finite values to divergent series 

## Bachelor's Project Mathematics

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"The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever."

Niels Henrik Abel

"If you know the restrictions, you can operate unrestrictedly within them."

Justus Anton (Jules) Deelder
"God is a mathematician of a very high order, and He used very advanced mathematics in constructing the universe."

Paul Adrien Maurice Dirac

"I find it difficult to be afraid of something I cannot imagine."

Justus Anton (Jules) Deelder

"May not music be described as the mathematics of the sense, mathematics as music of the reason? the soul of each is the same! Thus the musician feels mathematics, the mathematician thinks music, - music the dream, mathematics the working life - each to receive its consummation from the other when the human intelligence, elevated to its perfect type, shall shine forth glorified in some future Mozart-Dirichlet or Beethoven-Gauss - a union already not indistinctly foreshadowed in the genius and labours of a Helmholtz!"


#### Abstract

In order to assign a finite value to a divergent series in a useful and mathematically justified way, it turns out that we can 'extend' the notion of the convergence of an infinite series by introducing so-called summation methods. After we have explained the basic principles of summability theory (the mathematical subfield concerning summation methods and their interrelationships), we will focus on a particular type of summation methods: Cesàro summation. We will show various properties of Cesàro summation accompanied with examples in which we apply the methods for assigning finite values to divergent series. Thereafter, we will construct a full generalization of Cesàro summation which seems to be able to sum more 'wildly' divergent series. Furthermore, for the defining series of the Riemann zeta function, the generalized method corresponds with the analytic continuation of this function. In conclusion, we show that the assigned values are not completely picked out of thin air and we will see which insights summability theory can give us in (pure) mathematics as well as other sciences.


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## 1 Introduction

### 1.1 Motivation

Although no one expects it at first sight, the investigation of divergent series leads to many insightful and useful results in mathematics. According to the now general accepted definition given by Augustin-Louis Cauchy (1789-1857), an (infinite) series is called convergent if the sequence of partial sums converges to a finite limit; the series is divergent when this is not the case. Some examples of divergent series are:

$$
\begin{align*}
1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots & =\sum_{k=1}^{\infty} \frac{1}{k},  \tag{1.1}\\
1-1+1-1+1-\cdots & =\sum_{k=0}^{\infty}(-1)^{k}  \tag{1.2}\\
1+1+1+1+1+\cdots & =\sum_{k=0}^{\infty} 1,  \tag{1.3}\\
1+2+3+4+5+\cdots & =\sum_{k=1}^{\infty} k . \tag{1.4}
\end{align*}
$$

It turns out that by performing simple manipulations on a given divergent series, there can be assigned a (finite) limit to it. For the geometric series, $\sum_{k=0}^{\infty} c r^{k}$, one can for example say that

$$
\sum_{k=0}^{\infty} c r^{k}=c+\sum_{k=1}^{\infty} c r^{k}=c+\sum_{k=0}^{\infty} c r^{k+1}=c+r \sum_{k=0}^{\infty} c r^{k}
$$

in which one would say that $\sum_{n=0}^{\infty} c r^{k}=\frac{c}{1-r}$ whenever $r \neq 1$. Actually, this series only converges (to $\frac{c}{1-r}$ ) whenever $|r|<1$ and hence the performed manipulations and derivated result are only valid whenever $|r|<1$. It is however possible to extend or generalize the notion of ordinary summation to a more general notion so that, in this generalized framework, it holds that the 'sum' of $\sum_{k=0}^{\infty} c r^{k}$ does follow to be equal to $\frac{c}{1-r}$ whenever $r \neq 1$.

In this thesis we will explain the process of assigning sums by generalized notions of summation in more detail. After this we assess its veracity and state results which unexpectedly might lead to deep mathematical insights. In order to properly state research questions and formulate the essential theory, we will first give some historical background.

### 1.2 Historical background

According to [3], till up to the 19th century divergent series were merely used. It is known that Newton and Leibniz used them in some of their computations and Euler used it as well. Mostly, divergent series were being treated as if they were convergent and this led to many contradictions and confusion. In the beginning of the 19th century, things became
to be rigorously defined by mathematicians such as Cauchy and Niels Henrik Abel (18021829) and the use of divergent series became somewhat 'forbidden territory'. Ironically though, Cauchy made one exception for the use of divergent series (the application of Stirling's formula to approximate the Gamma function) and Abel has given results to his name which are central in the study of divergent series.

At the end of the 19th century, when the concepts and results in mathematics became being defined in a rigorously and almost purely symbolic way, the act of performing mathematics matured. Mathematics became more and more being grounded on precisely stated definitions in which theorems were proposed as logical consequences of these definitions. People again started thinking about divergent series and posed new questions: In which ways can we alternatively define the 'sum of a series'? What are desirable properties for our 'sum'? What are the relations between the different approaches?

After 1890, Ernesto Cesàro (1859-1906) gave the first explicit definition of a summation method; shortly therafter, various other mathematicians begun to construct summation methods and built generalizations upon them. Also, people started to develop new methods for the investigation of divergent integrals (in some sense the uncountable version of the sum of a series), divergent products and multiplication of divergent series. Nowadays, the subject of assigning values to divergent series has been developed into a well-acknowledged field of mathematics (called summability theory) and can be stated in terms of functional analytic theory among other branches of modern mathematics. Also, it is widely applied within other subfields of mathematics (e.g. analytic number theory, differential equations and Fourier analysis) and in other sciences such as theoretical physics (e.g. regularization methods for vacuum energy calculations in quantum field theory, mathematical derivation of the Casimir effect and the calculation of the critical dimension in bosonic string theory).

According to [6], the usage of summation methods for divergent series can merely be seen as a 'tool' which sometimes just happens to work and there always have to be taken considerations regarding the context of the situation in order to apply the tool. That is to say, the use of summation methods which in general not comply with some of the most elementary properties of ordinary summation could possibly lead to inconsistent results and false interpretations. This is especially made clear when we reason in terms of extended number systems such as the hyperreal numbers, in which according to e.g. [4] we get a better picture of the nature of divergent series and conditions on when to use them. We can thus say that the veracity in the assigned value to some divergent series is greatly dependent on the particular summation method and the context in which one applies it.

### 1.3 Research questions

In order to properly conduct a meaningful and well-directed research, it may be useful to state a research question. Below, we will state the main research question which will be answered after giving the answers to three appropriate subquestions.

## Which insights does the assignment of finite values to divergent series give us?

1. How can we extend the notion of convergence of an infinite series?
2. How can we assign finite values to divergent series in a justified way?

3 . What is the veracity of the assigned values?

## 2 Summability theory

### 2.1 Desirable axioms

In order to give an answer to subquestion 1, we have to dive deeper into the theory of summability and develop a mathematical framework in which we can state alternative definitions of series' convergence. For this, we will mainly follow [3] and [2] and initially state our theory in the context of real analytic and linear algebraic theory so we restrict ourselves mainly to series with terms in the real numbers (although many of the statements hold valid when replacing $\mathbb{R}$ with $\mathbb{C}$ ).

Recall the classical notion of convergence of an infinite series $a=\sum_{k=0}^{\infty} a_{k}$ with partial sums $s_{n}=\sum_{k=0}^{n} a_{k}$ : The series $a$ converges to $L \in \mathbb{R}$ if its sequence of partial sums $s=\left(s_{n}\right)$ (with $n \in \mathbb{N}$, $\mathbb{N}$ will include 0 throughout this text unless stated otherwise) converges to $L$. We immediately see that it is valid to perform linear manipulations on convergent series and its limiting values and that both adding or subtracting terms from a convergent series will lead to the limiting value plus or minus those subtracted terms. We can thus regard these properties (from now on called linearity and stability) as quite essential and candidates for the axioms of the to be defined summation methods (also, following [6], we can regard these properties as essential in order to bear properly 'signification' in the interpretation of the summation method connected to the material world).

Up till now, we have only informally spoken about "summation methods". In order to make more precise statements regarding the axioms and properties of summation methods, we first have to give the mathematical definition of a summation method.

Definition 1. Let us consider the series $a$ with the corresponding sequence of partial sums $s$. On the set (or vector space) of sequences where it attains a finite value, $\mathcal{S}$, we define a summation method as a transformation $T: \mathcal{S} \rightarrow \mathbb{R}$. We say that $a$ is $T$-summable to $L \in \mathbb{R}$ if $T(s)=L$.

Remark 1. Often, $T$ can be written as a composition $T_{2} \circ T_{1}$ with $T_{1}: \mathcal{S} \rightarrow \mathcal{S}$ being a sequence-to-sequence transformation (or operator) and $T_{2}: \mathcal{S} \rightarrow \mathbb{R}$ being the 'limiting procedure'. For instance, in a method called Cesàro summation (which we will discuss more detailed in the next chapter) the operator $T_{1}$ can be seen as a matrix which maps the sequence vector $s=\left(s_{n}\right)$ to the sequence vector $\sigma=\left(\sigma_{m}\right)$ (in which $\sigma_{m}=\frac{s_{0}+s_{1}+\cdots+s_{m}}{m+1}$ ) and $T_{2}$ takes the limit of $\sigma_{m}$ for $m \rightarrow \infty$. In ordinary summation, $T_{2}$ is equally defined and $T_{1}$ can now in fact be seen as the identity matrix.
Remark 2. For some methods it is more convenient to state this in terms of the set of appropriate series ${ }^{1} \mathcal{A}$ and say that $T=T_{2} \circ T_{1}$ with $T_{1}: \mathcal{A} \rightarrow \mathcal{A}$ and $T_{2}: \mathcal{A} \rightarrow \mathbb{R}$. We now say that $a$ is $T$-summable to $L$ if $T(a)=L$. For instance, in a method called Abel summation the operator $T_{1}$ can be seen as a linear transformation which maps the series $a=\sum_{k=0}^{\infty} a_{k}$ to the series $\tilde{a}(x)=\sum_{k=0}^{\infty} a_{k} x^{k}$ and $T_{2}$ takes the limit of $\tilde{a}(x)$ for $x \rightarrow 1^{-}$.

Now that we have more precisely stated the notion of a summation method, we can state the as earlier described properties which could be regarded as desirable axioms in constructing our summation methods.

[^0]Definition 2 (Linearity). A summation method $T$ is linear if $T(\alpha s+t)=\alpha T(s)+$ $T(t)$, for all $\alpha \in \mathbb{R}$ and $s, t \in \mathcal{S}$. Note that this is also well-defined if we replace $\mathbb{R}$ with the complex numbers $\mathbb{C}$.

Remark. When $T$ is linear, $T_{1}$ it may be viewed as a linear operator and hence it can be represented by an (infinite) matrix (with respect to the standard basis by convention); with this in mind, we can apply linear algebraic and functional analytic arguments to it. The operator $T_{1}$ is sometimes also called the summability kernel.

The second desirable property for $T$ to have will be defined below. Sometimes, this is also called the 'shifting' property.

Definition 3 (Stability). Let us give a summation method $T$ and some $s=\left(s_{n}\right) \in \mathcal{S}$. Now define $s^{\prime}$ as the sequence with $s_{n}^{\prime}=s_{n+1}-s_{0}$; this can also be interpreted in the sense that the corresponding series $a$ is 'shifted' to the left (i.e. $a_{k}^{\prime}=a_{k+1}$ ). We now say that $T$ is stable if $s^{\prime} \in \mathcal{S}$ and $T\left(s^{\prime}\right)=T(s)-s_{0}$.

Remark. The method $T$ is also said to be stable if we can appropriately make a 'shift to the right' and 'plug in the term' $a_{0}^{\prime}=s_{0}^{\prime}$; i.e. $a_{k+1}^{\prime}=a_{k}, s_{n+1}^{\prime}=s_{n}+s_{0}^{\prime}$ and $T\left(s^{\prime}\right)=T(s)+s_{0}^{\prime}$.

Of course, a summation method has preferably be as compatible possible to 'ordinary' summation. For this reason we define the notion of regularity of a summation method; when we argument in the perspective of an e.g. hyperreal or extended real codomain of $T$ (where $\pm \infty$ are included as well-defined objects) and the method is still regular, we will say it is totally regular.

Definition 4 (Regularity \& Total Regularity). A summation method $T$ is called regular if, for all $s=\left(s_{n}\right) \in \mathbb{R}^{\mathbb{N}}$, it holds that $\lim _{n \rightarrow \infty} s_{n}=L \in \mathbb{R}$ implies $s \in \mathcal{S}$ and $T(s)=L$. We can furthermore say that $T$ is totally regular whenever, for all $s \in \mathbb{R}^{\mathbb{N}}$, it holds that $\lim _{n \rightarrow \infty} s_{n}= \pm \infty$ implies $T(s)= \pm \infty$.

The regularity of a method is (by historical reasons) called an Abelian theorem; in general, Abelian theorems will state the 'compatibility' between different summation methods. Abelian theorems also have converse statements (Tauberian theorems); such statements assert that whenever the summation method assigns a (finite) value and some case-specific side condition holds (often regarding the asymptotic nature of the terms in the series), the series was summable in its 'ordinary' (or initially defined) manner anyway and the corresponding limits coincide. When one considers the negation of a Tauberian theorem, it can be in some way regarded as a 'limitation theorem'; it turns out that all summation methods in principle fail to sum series which diverge too rapidly or too slow. In the next section we will talk more about the compatibility between summation methods and sufficient and necessary conditions for $T$ to be (totally) regular.

### 2.2 Compatibility of methods

Let us give two methods $T$ (with domain $\mathcal{S}$ ) and $T^{\prime}$ (with domain $\mathcal{S}^{\prime}$ ). The methods are said to be consistent if $T(s)=L$ and $T^{\prime}(s)=L^{\prime}$ implies $L=L^{\prime}$ for each $s \in \mathcal{S} \cap \mathcal{S}^{\prime}$, this in fact means that the two methods 'sum' the same series to the same value. In the following definition, we will give a more precise meaning to the relationships between $\mathcal{S}$ and $\mathcal{S}^{\prime}$.

Definition 5 (Compatibility, Strength \& Equivalence). Let us give two consistent methods $T$ and $T^{\prime}$ with domains $\mathcal{S}$ and $\mathcal{S}^{\prime}$ respectively. We say $T$ is compatible to $T^{\prime}$ if $\mathcal{S} \supseteq \mathcal{S}^{\prime}, T$ is stronger than $T^{\prime}$ if $\mathcal{S} \supset \mathcal{S}^{\prime}$ and $T$ is equivalent to $T^{\prime}$ if $\mathcal{S}=\mathcal{S}^{\prime}$.

Since we can also regard 'ordinary summation' as a summation method, we can derive the needed properties for $T$ in order to be regular (or compatible/stronger than ordinary summation). As stated earlier we prefer that $T$ is linear; in this case the following theorem holds (considering the scope of this thesis, only the sufficiency will be proven here).

Theorem 1 (Silverman-Toeplitz, one-sided proof). The linear summation method $T$ with operator $T_{1}=\left(t_{m n}\right)$ is regular if and only if the following holds:
(i) $\sup \left\{\sum_{n=0}^{\infty}\left|t_{m n}\right|: m \in \mathbb{N}\right\}<\infty$,
(ii) $\lim _{m \rightarrow \infty} t_{m n}=0$ for all $n \in \mathbb{N}$,
(iii) $\lim _{m \rightarrow \infty} \sum_{n=0}^{\infty} t_{m n}=1$.

Proof. Let us give the convergent series $a$ with its sequence of partial sums $\left(s_{n}\right)$ converging to $L$. By the convergence of $\left(s_{n}\right)$ we know it is bounded, hence there exists a $K>0$ such that each $\left|s_{n}\right| \leq K$. Now

$$
\left(T_{1} s\right)_{m}=\sum_{n=0}^{\infty} t_{m n} s_{n} \leq \sum_{n=0}^{\infty}\left|t_{m n} s_{n}\right| \leq \sum_{n=0}^{\infty}\left|t_{m n}\right|\left|s_{n}\right| \leq K \sum_{n=0}^{\infty}\left|t_{m n}\right|<\infty
$$

where the last inequality follows by (i). Hence each $\left(T_{1} s\right)_{m}$ is well-defined and we can write

$$
\begin{equation*}
\left(T_{1} s\right)_{m}=\sum_{n=0}^{\infty} t_{m n}\left(s_{n}-L\right)+L \sum_{n=0}^{\infty} t_{m n} \tag{2.1}
\end{equation*}
$$

By (i) and the boundedness and convergence of $\left(s_{n}\right)$, we can introduce some $I>0$ such that each

$$
\begin{equation*}
\sum_{n=0}^{\infty}\left|t_{m n}\right|<I \text { and }\left|s_{n}-L\right|<I \tag{2.2}
\end{equation*}
$$

By the convergence of $\left(s_{n}\right)$, we see furthermore that for each $\varepsilon>0$ there exists a $N \in \mathbb{N}$ such that $n>N$ implies

$$
\begin{equation*}
\left|s_{n}-L\right|<\frac{\varepsilon}{2 I} \tag{2.3}
\end{equation*}
$$

For the first term in the right hand side of (2.1) we see that

$$
\begin{aligned}
\left|\sum_{n=0}^{\infty} t_{m n}\left(s_{n}-L\right)\right| & =\left|\sum_{n=0}^{N} t_{m n}\left(s_{n}-L\right)+\sum_{n=N+1}^{\infty} t_{m n}\left(s_{n}-L\right)\right| \\
& \leq\left|\sum_{n=0}^{N} t_{m n}\left(s_{n}-L\right)\right|+\left|\sum_{n=N+1}^{\infty} t_{m n}\left(s_{n}-L\right)\right| \\
& \leq \sum_{n=0}^{N}\left|t_{m n}\right|\left|\left(s_{n}-L\right)\right|+\sum_{n=N+1}^{\infty}\left|t_{m n}\right|\left|\left(s_{n}-L\right)\right|
\end{aligned}
$$

Using (2.3) and (2.2) we see that

$$
\sum_{n=N+1}^{\infty}\left|t_{m n}\right|\left|\left(s_{n}-L\right)\right|<\frac{\varepsilon}{2 I} \sum_{n=N+1}^{\infty}\left|t_{m n}\right| \leq \frac{\varepsilon}{2 I} \sum_{n=0}^{\infty}\left|t_{m n}\right|<\frac{\varepsilon}{2} .
$$

Moreover, by (ii), we see that there exists a $N^{\prime} \in \mathbb{N}$ such that for each $n=0, \ldots, N$ and $m>N^{\prime}$ it holds that $\left|t_{m n}\right|<\frac{\varepsilon}{2 I(N+1)}$ which in turn implies (using (2.2))

$$
\sum_{n=0}^{N}\left|t_{m n}\right|\left|\left(s_{n}-L\right)\right|<I \sum_{n=0}^{N}\left|t_{m n}\right|<I(N+1) \frac{\varepsilon}{2 I(N+1)}=\frac{\varepsilon}{2}
$$

for $m>N^{\prime}$. Consequently, for every $\varepsilon$ we now have that $m>N^{\prime}$ implies

$$
\left|\sum_{n=0}^{\infty} t_{m n}\left(s_{n}-L\right)\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,
$$

hence

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \sum_{n=0}^{\infty} t_{m n}\left(s_{n}-L\right)=0 \tag{2.4}
\end{equation*}
$$

Now by (2.4), our assumption (iii) and applying the limiting operator $T_{2}$ to (2.1) we get

$$
T(s)=\lim _{m \rightarrow \infty}\left(T_{1} s\right)_{m}=L
$$

This proves the regularity of $T$. For a proof of the necessity, we refer to theorem 1.1 in [2].

We can also impose further conditions on $T$ to let it be a totally regular method. In this sense one might say that $T$ can be seen as a proper extension of ordinary summation which will give us an answer to subquestion 1 .

Corollary 1. The linear summation method $T$ with operator $T_{1}=\left(t_{m n}\right)$ is totally regular if it is regular and it holds that all $t_{m n} \geq 0$.

Proof. Let us give the series $a$ with sequence of partial sums $\left(s_{n}\right)$ such that $\lim _{n \rightarrow \infty} s_{n}=$ $\infty$. We can now say that for each $L>0$ there exists some $N \in \mathbb{N}$ such that $n>N$ implies $s_{n}>L$. Now

$$
\left(T_{1} s\right)_{m}=\sum_{n=0}^{\infty} t_{m n} s_{n}=\sum_{n=0}^{N} t_{m n} s_{n}+\sum_{n=N+1}^{\infty} t_{m n} s_{n}>\sum_{n=0}^{N} t_{m n} s_{n}+L \sum_{n=N+1}^{\infty} t_{m n} .
$$

The first term in the most right hand part is a finite quantity and the second term can be made sufficiently large by considering the fact that all respective $t_{m n}$ are nonnegative where at least one $t_{m n}>0$. This concludes that

$$
T(s)=\lim _{m \rightarrow \infty}\left(T_{1} s\right)_{m}=\infty
$$

The case in which $\lim _{n \rightarrow \infty} s_{n}=-\infty$ is similar.

In the next chapter we wil dive deeper into a particular type of summation methods which one in an unified way could call Cesàro summation. After giving the most elementary variant, we will gradually generalize it and show that this Cesàro type of methods possess nice properties (such as linearity and regularity) which makes it a good alternative to ordinary summation. There are in fact many types of methods which possess this nice properties, notable mentions are Abel, Borel and Euler summation. Due to the fact that we simply cannot explain the whole mathematical field of summability theory in one thesis, we tend to only focus on the theory of Cesàro summation. Furthermore, we can say that Cesàro summation is an interesting choice due to its historical relevance and applicability.

## 3 Cesàro summation

## 3.1 ( $C, 1$ )-method

In the late 19th-century, Cesàro proposed a summation method which is now generally accepted to be the first rigourous defined summation method in history; it could even be seen as the starting point of summability theory. We will see that this method provides a good alternative to ordinary summation. Furthermore, (generalizations of) the method has a lot of applications in other subfields of mathematics such as Fourier analysis and, as we will see, surprising connections with open questions within the field of analytic number theory.

Definition $6\left((C, 1)\right.$-method). Let us give some series $a=\sum_{k=0}^{\infty} a_{k}$ with an associated sequence of partial sums $s=\left(s_{n}\right)$. Now let $\sigma=\left(\sigma_{m}\right)$ represent the sequence of arithmetic means of $\left(s_{n}\right)$, i.e.

$$
\sigma_{m}=\frac{s_{0}+s_{1}+\cdots+s_{m}}{m+1} .
$$

If $\lim _{m \rightarrow \infty} \sigma_{m}$ exists and is equal to $L \in \mathbb{R}$, we say that the Cesàro sum $(C, 1)$ of $a$ is defined by $(C, 1)(s)=L$.

Strictly spoken, the ( $C, 1$ )-method is a special case of a so-called Nörlund mean method, we will however not discuss this in further detail. It is easy to see that $(C, 1)$ is a linear summation method, the proof of its stability will be postponed to the next section. By its linearity, we may regard $(C, 1)$ as a composed linear transformation: The first part can be seen as an infinite matrix (that maps the 'vector's to the 'vector' $\sigma$ ) and the second part as the limiting procedure $\left(\lim _{m \rightarrow \infty} \sigma_{m}\right)$. Especially the infinite matrix will be of importance in our further analysis and therefore we will from now on call it the Cesàro operator $C$. Note that the elements in its matrix representation are given by

$$
\begin{equation*}
c_{m n}=\frac{1}{m+1}(0 \leq n \leq m), \quad c_{m n}=0 \quad(n>m) . \tag{3.1}
\end{equation*}
$$

We also note that $\sigma_{m}=(C s)_{m}$ and it thus essentially boils down to a matrix and vector multiplication. By using the matrix representation of the Cesàro operator we can now show that the $(C, 1)$-method is (totally) regular.
Proposition 1. The ( $C, 1$ )-method is totally regular.
Proof. Consider the matrix representation of the Cesàro operator as given in (3.1). In order to show regularity, we have to check whether the conditions of the Silverman-Toeplitz theorem are satisfied:
(i) Consider, for some $m \in \mathbb{N}$, the series $\sum_{n=0}^{\infty}\left|c_{m n}\right|$. We see that they are all equal to $\sum_{n=0}^{m} \frac{1}{m+1}+\sum_{n=m+1}^{\infty} 0=1<\infty$ and hence $\sup \left\{\sum_{n=0}^{\infty}\left|c_{m n}\right|: m \in \mathbb{N}\right\}<\infty$.
(ii) Consider, for some $n \in \mathbb{N}, \lim _{m \rightarrow \infty} c_{m n}$. We see that this expression is either equal to $\lim _{m \rightarrow \infty} \frac{1}{m+1}$ or $\lim _{m \rightarrow \infty} 0$ and both are equal to 0 .
(iii) Consider the sequence of series $\left(\sum_{n=0}^{\infty} c_{m n}\right)_{m}$ for $m \in \mathbb{N}$. We immediately see that each term in the sequence is equal to $\sum_{n=0}^{m} \frac{1}{m+1}+\sum_{n=m+1}^{\infty} 0=1$ hence it is a sequence of 1 's which obviously converges to 1 .

Now that we have proven regularity, we immediately see that the ( $C, 1$ )-method is totally regular since all $c_{m n} \geq 0$.

We can also give an example of how the ( $C, 1$ )-method 'extends' the notion of summation to a series which is not convergent in its ordinary sense. This, in fact, gives an answer to subquestion 2 ; namely that we have found a method which can assign finite values to divergent series in a quite justified way.

Example 1. Consider the series $g=\sum_{k=0}^{\infty}(-1)^{k}$ as written in (1.2), this series is also known as Grandi's series. One sees that the accompanying sequence of partial sums is $\left(s_{n}\right)_{n=0}^{\infty}=(1,0,1,0, \ldots)$. Since this sequence does not converge to a finite limit, $g$ is a divergent series. Now let $\sigma=\left(\sigma_{m}\right)_{m=0}^{\infty}$ be the sequence of arithmic means as given in definition 4, i.e. $\sigma=\left(\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{2}{4}, \frac{3}{5}, \frac{3}{6}, \frac{4}{7}, \frac{4}{8}, \ldots\right)$. It follows that $\lim _{m \rightarrow \infty} \sigma_{m}=\frac{1}{2}$ and hence the Cesàro sum $(\mathrm{C}, 1)$ of $g$ is defined and equal to $\frac{1}{2}$.

## $3.2(C, \alpha)$-method

Later, Cesàro generalized his $(C, 1)$-method to a method we will call the $(C, \alpha)$-method, we initially assume that $\alpha \in \mathbb{N}$.

Definition $7\left((C, \alpha)\right.$-method). Let us give some series $a=\sum_{k=0}^{\infty} a_{k}$ with an associated sequence of partial sums $s=\left(s_{n}\right)$. Now define $s_{n}^{(0)}=s_{n}$ and in general $s_{n}^{(\alpha)}=\sum_{k=0}^{n} s_{k}^{(\alpha-1)}$ for $\alpha \in \mathbb{N}^{+}$. Aditionally, let $e_{n}^{(\alpha)}$ represent $s_{n}^{(\alpha)}$ in the special case where $a_{0}=1$ and $a_{k}=0$ for $k=1,2, \ldots$ If $\lim _{n \rightarrow \infty} \frac{s_{n}^{(\alpha)}}{e_{n}^{(\alpha)}}$ exists and is equal to $L \in \mathbb{R}$, we say that the Cesàro sum $(C, \alpha)$ of $a$ is defined by $(C, \alpha)(s)=L$.

In order to make more sense of this definition, we state (without derivation) two useful equations. These equations reveal the underlying relationships between $s_{n}^{(\alpha)}$ and $a_{k}$ and give us a nice 'tool' for applying the $(C, \alpha)$-method. The derivation of these equations is based on a so-called binomial expansion and is written in $\S 5.4$ of [3] (the relevant equations are (5.4.5) and (5.4.6)).

$$
\begin{align*}
& s_{n}^{(\alpha)}=\sum_{k=0}^{n}\binom{n-k+\alpha}{\alpha} a_{k},  \tag{3.2}\\
& e_{n}^{(\alpha)}=\binom{n+\alpha}{\alpha} \sim \frac{n^{\alpha}}{(\alpha)!} \tag{3.3}
\end{align*}
$$

where $\sim$ means is asymptotic to ${ }^{1}$.
Due to these facts and the definition of the $(C, \alpha)$-method, one thus also says that the series $a=\sum_{k=0}^{\infty} a_{k}$ is ( $C, \alpha$ )-summable to $L$ if and only if

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{1}{n^{\alpha}} \sum_{k=0}^{n} \frac{(n-k+\alpha)!}{(n-k)!} a_{k}=L \tag{3.4}
\end{equation*}
$$

One observes that this generalization reduces to ordinary summation when $\alpha=0$ and complies with the $(C, 1)$-method when $\alpha=1$. Since the $(C, 1)$-method can be seen as an 'extension' of ordinary summation, one can hypothesize that the method becomes more powerful each time we increase $\alpha$; as we will see later, this indeed is partially true. Although one might expect that, for example, the ( $C, 1000000$ )-method is a very strong method, it is still not able to sum all divergent series; see the next example.

[^1]Example 2. Consider the series $a=\sum_{k=0}^{\infty} 1$, as written in (1.3). In order to determine the $(C, 1000000)$-sum of it, we have to check whether $\lim _{n \rightarrow \infty} s_{n}^{(1000000)} / e_{n}^{(1000000)}$ exists and is equal to a finite value. This task can be made more efficient by checking, according to (3.4), whether

$$
\lim _{n \rightarrow \infty} \frac{1}{n^{1000} 000} \sum_{k=0}^{n} \frac{(n-k+1000000)!}{(n-k)!}=\lim _{n \rightarrow \infty} \sum_{k=0}^{n} \frac{(k+1000000)!}{n^{1000} 000}(k)!
$$

exists and approaches a finite value. Since the last term of the series in the RHS of the equation is equal to

$$
\frac{(n+1000000)(n+999999) \cdots(n+1)}{n^{1000000}}
$$

which for any $n$ is larger than 1 and thus is not equal to 0 whenever $n \rightarrow \infty$, we see that the above series is not convergent. Hence the series $a$ is not ( $C, 1000000$ )-summable.

We can even further extend the definition of $(C, \alpha)$-method to the case where $\alpha$ can be any real number greater than -1 . Due to some technicalities, it is proven infeasible that we include negative integers $\alpha$ or more generally that $\alpha \leq-1$ (see also $\S 5.5$ of [3]). Also, the to be stated definition will in principle make sense when we regard $\alpha \in \mathbb{C}$ (with $\Re(\alpha)>-1)$ but according to the literature and the results we shall prove we will restrict $\alpha$ to be a real number larger than -1 . First we note that the combinations and factorials appearing equations (3.2) and (3.3) can be rewritten and hence extended in terms of the gamma function:

$$
\begin{align*}
& s_{n}^{(\alpha)}=\sum_{k=0}^{n} \frac{\Gamma(n-k+\alpha+1)}{\Gamma(\alpha+1) \Gamma(n-k+1)} a_{k},  \tag{3.5}\\
& e_{n}^{(\alpha)}=\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1) \Gamma(n+1)} \sim \frac{n^{\alpha}}{\Gamma(\alpha+1)} . \tag{3.6}
\end{align*}
$$

Definition $8\left((C, \alpha)\right.$-method, revisited). Let us give some series $a=\sum_{k=0}^{\infty} a_{k}$ with an associated sequence of partial sums $s=\left(s_{n}\right)$ and let $\alpha>-1$. If $\lim _{n \rightarrow \infty} \frac{s_{n}^{(\alpha)}}{e_{n}^{(\alpha)}}$ exists and is equal to $L \in \mathbb{R}$, or rather using (3.5) and (3.6) if

$$
\lim _{n \rightarrow \infty} \frac{1}{n^{\alpha}} \sum_{k=0}^{n} \frac{\Gamma(n-k+\alpha+1)}{\Gamma(n-k+1)} a_{k}=L
$$

we say that the Cesàro sum $(C, \alpha)$ of $a$ is defined by $(C, \alpha)(s)=L$.
It is easy to see that the $(C, \alpha)$-method is linear. Its stability will be shown below.
Proposition 2. For each $\alpha>-1$ we have that the ( $C, \alpha$ )-method is stable.
Proof. Let us give two sequences $s, s^{\prime}$ and the associated series $a, a^{\prime}$ where $s_{n}^{\prime}=s_{n+1}-s_{0}$ and $a_{k}^{\prime}=a_{k+1}$. It suffices to prove that $a$ is $(C, \alpha)$-summable $(s \in \mathcal{S})$ implies $s^{\prime} \in \mathcal{S}$ and $(C, \alpha)\left(s^{\prime}\right)=(C, \alpha)(s)-s_{0}$; the proof that $s^{\prime} \in \mathcal{S}$ implies $s \in \mathcal{S}$ and $(C, \alpha)(s)=$ $(C, \alpha)\left(s^{\prime}\right)+s_{0}$ is essentially the same.

We note that $\lim _{n \rightarrow \infty}{s^{\prime}}_{n}^{(\alpha)}=\lim _{n \rightarrow \infty} s^{\prime^{\prime(\alpha)}}{ }_{n-1}$ with

$$
\begin{aligned}
s_{n-1}^{\prime(\alpha)} & =\sum_{k=0}^{n-1} \frac{\Gamma((n-1)-k+\alpha+1)}{\Gamma(\alpha+1) \Gamma((n-1)-k+1)} a_{k}^{\prime} \\
& =\sum_{k=0}^{n-1} \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1) \Gamma(k+1)} a_{n-1-k}^{\prime} \\
& =\sum_{k=0}^{n-1} \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1) \Gamma(k+1)} a_{n-k} \\
& =\sum_{k=0}^{n} \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1) \Gamma(k+1)} a_{n-k}-\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1) \Gamma(n+1)} a_{0} .
\end{aligned}
$$

We can thus say that

$$
\begin{aligned}
(C, \alpha)\left(s^{\prime}\right) & =\lim _{n \rightarrow \infty} \frac{{\frac{s_{n}^{\prime}}{n}}_{(\alpha)}^{e_{n}^{(\alpha)}}}{} \\
& =\lim _{n \rightarrow \infty} \frac{\sum_{k=0}^{n} \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1) \Gamma(k+1)} a_{n-k}-\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1) \Gamma(n+1)} a_{0}}{\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1) \Gamma(n+1)}} \\
& =\lim _{n \rightarrow \infty} \frac{s_{n}^{(\alpha)}-e_{n}^{(\alpha)} s_{0}}{e_{n}^{(\alpha)}} \\
& =(C, \alpha)(s)-s_{0}
\end{aligned}
$$

and thus $s^{\prime} \in \mathcal{S}$.
In order to show that the $(C, \alpha)$-method, like the $(C, 1)$-method, can in general be regarded as a proper extension to the notion of ordinary summation (i.e. the $(C, 0)$ method!) we state two interesting results. This furthermore leads to the conclusion that the introduction of the ( $C, \alpha$ )-method, when taking $\alpha$ arbitrarily large enough, opens indeed to some new possibilities in the assigning of finite values to divergent series. First we state a useful equation:

$$
\begin{equation*}
s_{n}^{(\alpha)}=\sum_{k=0}^{n} \frac{\Gamma(n-k+\alpha-\beta)}{\Gamma(\alpha-\beta) \Gamma(n-k+1)} s_{k}^{(\beta)}, \quad(\alpha>\beta>-1) . \tag{3.7}
\end{equation*}
$$

Although we not will derive this equation here, we remark it is written in a slightly different form in equation (5.4.8) of [3] and can be derived along the same way as (3.5).
Proposition 3. For $\alpha \geq 0$ we have that the ( $C, \alpha$ )-method is totally regular.
Proof. If $\alpha=0$ we have that the $(C, \alpha)$-method reduces to ordinary sumation hence the statement is trivial; therefore we consider $\alpha>0$. To show that the $(C, \alpha)$-method is regular, it suffices to verify the three conditions of the Silverman-Toeplitz theorem. In order to verify this, we must know the matrix representation of the concerning operator $C_{\alpha}:=\left(c_{m n}\right)$ (note that $C_{1}=C$ ). Keeping in mind the fact that $\alpha>0$ and $s_{n}^{(0)}=s_{n}$ we see that (3.7) reduces to

$$
\begin{equation*}
s_{m}^{(\alpha)}=\sum_{n=0}^{m} \frac{\Gamma(m-n+\alpha)}{\Gamma(\alpha) \Gamma(m-n+1)} s_{n} . \tag{3.8}
\end{equation*}
$$

Hence

$$
\frac{s_{m}^{(\alpha)}}{e_{m}^{(\alpha)}}=\frac{\Gamma(\alpha+1) \Gamma(m+1)}{\Gamma(m+\alpha+1)} \sum_{n=0}^{m} \frac{\Gamma(m-n+\alpha)}{\Gamma(\alpha) \Gamma(m-n+1)} s_{n}
$$

and thus the matrix representation is given by

$$
\begin{equation*}
c_{m n}=\frac{\Gamma(\alpha+1) \Gamma(m+1) \Gamma(m-n+\alpha)}{\Gamma(\alpha) \Gamma(m+\alpha+1) \Gamma(m-n+1)} \quad(0 \leq n \leq m), \quad c_{m n}=0 \quad(n>m) \tag{3.9}
\end{equation*}
$$

It is easy to see that, when $\alpha>0$, each $\left(c_{m n}\right) \geq 0$; in order to prove the proposition we therefore, according to corollary 1 , only have to show that the method is regular. We can indeed verify the three conditions:
(i) Consider, for some $m \in \mathbb{N}$, the series $\sum_{n=0}^{\infty}\left|c_{m n}\right|$. We see that each

$$
\sum_{n=0}^{\infty}\left|c_{m n}\right|=\frac{\Gamma(\alpha+1) \Gamma(m+1)}{\Gamma(m+\alpha+1)} \sum_{n=0}^{m} \frac{\Gamma(m-n+\alpha)}{\Gamma(\alpha) \Gamma(m-n+1)}
$$

Now we remark from (3.6) that

$$
\frac{\Gamma(\alpha+1) \Gamma(m+1)}{\Gamma(m+\alpha+1)}=\frac{1}{e_{m}^{(\alpha)}}
$$

and from (3.8) and (3.6) that

$$
e_{m}^{(\alpha)}=\sum_{n=0}^{m} \frac{\Gamma(m-n+\alpha)}{\Gamma(\alpha) \Gamma(m-n+1)} e_{n}^{(0)}=\sum_{n=0}^{m} \frac{\Gamma(m-n+\alpha)}{\Gamma(\alpha) \Gamma(m-n+1)} .
$$

Combining these two expressions we see that each $\sum_{n=0}^{\infty}\left|c_{m n}\right|=1<\infty$ and hence $\sup \left\{\sum_{n=0}^{\infty}\left|c_{m n}\right|: m \in \mathbb{N}\right\}<\infty$.
(ii) Consider, for some $n \in \mathbb{N}, \lim _{m \rightarrow \infty} c_{m n}$. We see that this expression is either equal to $\lim _{m \rightarrow \infty} 0$ or

$$
\lim _{m \rightarrow \infty} \frac{\Gamma(\alpha+1) \Gamma(m+1) \Gamma(m-n+\alpha)}{\Gamma(\alpha) \Gamma(m+\alpha+1) \Gamma(m-n+1)}
$$

From (3.6), we note that

$$
\frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)} \sim \frac{1}{m^{\alpha}}
$$

and

$$
\frac{\Gamma(m-n+\alpha)}{\Gamma(m-n+1)} \sim m^{\alpha-1}
$$

which concludes that

$$
\lim _{m \rightarrow \infty} \frac{\Gamma(\alpha+1) \Gamma(m+1) \Gamma(m-n+\alpha)}{\Gamma(\alpha) \Gamma(m+\alpha+1) \Gamma(m-n+1)}=\lim _{m \rightarrow \infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha) m}=0
$$

The expression must in any way thus to be equal to zero.
(iii) In (i) we saw that each $\sum_{n=0}^{\infty} c_{m n}=1$ which defines $\left(\sum_{n=0}^{\infty} c_{m n}\right)_{m}$ to be a sequence of 1 's which obviously converges to 1 .

Proposition 4. For $\alpha>\beta>-1$ we have that the ( $C, \alpha$ )-method is compatible to the ( $C, \beta$ )-method.

Proof. Consider some series $a$ with an associated sequence of partial sums $s$ such that $(C, \beta)(s)=\lim _{n \rightarrow \infty} \frac{s_{n}^{(\beta)}}{e_{n}^{(\beta)}}=L$. Note from (3.6) and (3.7) that

$$
\begin{aligned}
\frac{s_{m}^{(\alpha)}}{e_{m}^{(\alpha)}} & =\sum_{n=0}^{m} \frac{\Gamma(\alpha+1) \Gamma(m+1) \Gamma(m-n+\alpha-\beta)}{\Gamma(m+\alpha+1) \Gamma(\alpha-\beta) \Gamma(m-n+1)} s_{n}^{(\beta)} \\
& =\sum_{n=0}^{m} \frac{\Gamma(n+\beta+1) \Gamma(\alpha+1) \Gamma(m+1) \Gamma(m-n+\alpha-\beta)}{\Gamma(\beta+1) \Gamma(n+1) \Gamma(m+\alpha+1) \Gamma(\alpha-\beta) \Gamma(m-n+1)} \frac{s_{n}^{(\beta)}}{e_{n}^{(\beta)}} .
\end{aligned}
$$

Since by our assumption that $\frac{s_{n}^{(\beta)}}{e_{n}^{(\beta)}}$ is a convergent sequence to $L$ and the fraction inside the summation term reduces to elements in the matrix representation of the $C_{\alpha-\beta}$ operator as described in (3.9) (which is in the previous proposition shown to be regular if $\alpha-\beta>0$ ), we see that $(C, \alpha)(s)=\lim _{n \rightarrow \infty} \frac{s_{n}^{(\alpha)}}{e_{n}^{(\alpha)}}=\lim _{n \rightarrow \infty} \frac{s_{n}^{(\beta)}}{e_{n}^{(\beta)}}=L$.

In some cases we even have that the $(C, \alpha)$-method is stronger than the $(C, \beta)$-method $(\alpha>\beta)$; i.e. there exists a series which is $(C, \alpha)$-summable but not $(C, \beta)$-summable. The following gives an example for the case $\alpha=2$ and $\beta=1$.

Example 3. Consider the series

$$
a=1+\sum_{k=1}^{\infty} 2 k(-1)^{k}
$$

with sequence of partial sums $s=(1,-1,3,-3,5,-5, \cdots)$, we clearly see that $a$ is a divergent series.
Now we try to apply the $(C, 1)$-method on it and hence we multiply the Cesàro operator $C$ (actually its matrix representation as given in (3.1)) with $s$ :

$$
\left(\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & \cdots \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \cdots \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)\left(\begin{array}{c}
1 \\
-1 \\
3 \\
-3 \\
5 \\
\vdots
\end{array}\right)=\left(\begin{array}{c}
1 \\
0 \\
1 \\
0 \\
1 \\
\vdots
\end{array}\right) .
$$

As we saw in example 1 , the obtained sequence (vector) is divergent hence the $(C, 1)$ method can not assign a finite value to $a$.
When we try to apply the $(C, 2)$-method on $a$, we can check whether there can be assigned a finite value by checking whether the limit(s) as stated in the according definitions will converge. Since this is a bit of a tedious computation, it is more efficient to multiply the square of the Cesàro operator, $C^{2}$, with it and check whether the obtained sequence converges; as will be made clear in theorem 2 of the next section, multiplying the Cesàro to the $\alpha$-power and taking the limit of the obtained sequence gives (in general) equal results as applying the $(C, \alpha)$-method through the definition. Since we have already calculated $C s=\sigma$, we note that $C^{2} s=C \sigma=u$ where $u$ is in fact the same sequence as $\sigma$ of example 1 and hence converges to $\frac{1}{2}$. This concludes that the Cesàro sum $(C, 2)$ of $a$ is well-defined and is equal to $\frac{1}{2}$.

Although the $(C, \alpha)$-method is stronger than the ( $C, 1$ )-method (for at least $\alpha \geq 2$ ), it can undoubtly still not sum all possible divergent series; we in some way also concluded this in example 2. Besides the series (1.3) from example 2, we can also show that the series (1.4) is, for each $\alpha>-1$, not summable by the $(C, \alpha)$-method. Furthermore, as we will see in chapter 4 , the harmonic series (written in (1.1)) will shown to be unsummable by the ( $C, \alpha$ )-method.

Example 4. Consider the series $a=\sum_{k=0}^{\infty}(k+1)$ which is also written in (1.4) in a slightly different form. Now we try to apply the $(C, \alpha)$-method on it $(\alpha>-1)$; by definition we have that there must be assigned a finite value to

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{1}{n^{\alpha}} \sum_{k=0}^{n} \frac{\Gamma(n-k+\alpha+1)}{\Gamma(n-k+1)}(k+1)=\lim _{n \rightarrow \infty} \frac{1}{n^{\alpha}} \sum_{k=0}^{n} \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}(n-k+1) \tag{3.10}
\end{equation*}
$$

which equals

$$
\lim _{n \rightarrow \infty}\left(\frac{1}{n^{\alpha-1}} \sum_{k=0}^{n} \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}-\frac{1}{n^{\alpha}} \sum_{k=0}^{n} \frac{\Gamma(k+\alpha+1)}{\Gamma(k)}+\frac{1}{n^{\alpha}} \sum_{k=0}^{n} \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}\right)
$$

Noting the similar work we did in proposition 3 to sum fractions of gamma functions and looking to the "asymptotic expansion" (i.e. $\frac{\Gamma(n+a)}{\Gamma(n+b)} \sim n^{a-b}$ ) as written in equation (3.6), we see that:

$$
\begin{gathered}
\lim _{n \rightarrow \infty} \frac{1}{n^{\alpha-1}} \sum_{k=0}^{n} \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}=\lim _{n \rightarrow \infty} \frac{1}{n^{\alpha-1}} \frac{(n+1) \Gamma(n+\alpha+2)}{(\alpha+1) \Gamma(n+2)}=\lim _{n \rightarrow \infty} \frac{n^{2}+n}{\alpha+1} \\
\lim _{n \rightarrow \infty} \frac{-1}{n^{\alpha}} \sum_{k=0}^{n} \frac{\Gamma(k+\alpha+1)}{\Gamma(k)}=\lim _{n \rightarrow \infty} \frac{-1}{n^{\alpha}} \frac{n \Gamma(n+\alpha+2)}{(\alpha+2) \Gamma(n+1)}=\lim _{n \rightarrow \infty} \frac{-n^{2}}{\alpha+2}
\end{gathered}
$$

and

$$
\lim _{n \rightarrow \infty} \frac{1}{n^{\alpha}} \sum_{k=0}^{n} \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}=\lim _{n \rightarrow \infty} \frac{1}{n^{\alpha}} \frac{(n+1) \Gamma(n+\alpha+2)}{(\alpha+1) \Gamma(n+2)}=\lim _{n \rightarrow \infty} \frac{n+1}{\alpha+1}
$$

Hence (3.10) reduces to

$$
\begin{aligned}
\lim _{n \rightarrow \infty}\left(\frac{n^{2}+n}{\alpha+1}-\frac{n^{2}}{\alpha+2}+\frac{n+1}{\alpha+1}\right) & =\lim _{n \rightarrow \infty}\left(\frac{n^{2}+2 n+1}{\alpha+1}-\frac{n^{2}}{\alpha+2}\right) \\
& =\lim _{n \rightarrow \infty} \frac{\left(n^{2}+2 n+1\right)(\alpha+2)-n^{2}(\alpha+1)}{(\alpha+1)(\alpha+2)} \\
& =\lim _{n \rightarrow \infty} \frac{\left(n^{2}+2 n+1\right)(\alpha+2)-n^{2}(\alpha+1)}{(\alpha+1)(\alpha+2)} \\
& =\lim _{n \rightarrow \infty} \frac{n^{2}+(\alpha+2) 2 n+(\alpha+2)}{(\alpha+1)(\alpha+2)} \\
& =\infty
\end{aligned}
$$

by our assumption of $\alpha>-1$. This leads to the conclusion that (3.10) tends to infinity and the $(C, \alpha)$-method is therefore incapable to assign a finite value to the concerned series.

This concludes that we have to find even more general methods which can 'sum' this series.

### 3.3 Further generalization

### 3.3.1 Hölder summation

In 1882, Otto Hölder (1859-1937) proposed a method similar to the ( $C, \alpha$ )-method (where $\alpha \in \mathbb{N}$ ). This method, now called Hölder summation (abbevriated as the ( $H, \alpha$ )-method), is almost equally defined as the $(C, 1)$-method but in this case the Cèsaro operator is matrix-multiplied $\alpha$ times; i.e. we define the Hölder operator by $H_{\alpha}:=C^{\alpha}$. After proving the following lemma and theorem, we will immediately see that properties such as linearity, stability, (total) regularity and compatibility will carry over to the ( $H, \alpha$ )-method.

Lemma 1. Let us give the sequences $s=\left(s_{n}\right)$ and $\sigma=\left(\sigma_{n}\right)$ with $\sigma_{n}=\frac{s_{0}+s_{1}+\cdots+s_{n}}{n+1}$. Then, for $\alpha \in \mathbb{N}$, we have that $(C, \alpha)(s)=L$ if and only if $(C, \alpha-1)(\sigma)=L$.

Proof. According to definition 7, equation (3.6) and the nature of the operator $C_{\alpha}$ we can write

$$
\begin{equation*}
s_{k}^{(\alpha)}=\binom{k+\alpha}{\alpha}\left(C_{\alpha} s\right)_{k} \text { and } \sigma_{k}^{(\alpha)}=\binom{k+\alpha}{\alpha}\left(C_{\alpha} \sigma\right)_{k} \tag{3.11}
\end{equation*}
$$

and in particular $s_{k}^{(1)}=(k+1) \sigma_{k}^{(0)}=(k+1) \sigma_{k}$. We know by the method of 'summation by parts' (the discrete analogue of integration by parts and sometimes also called the Abel transformation) that

$$
\sum_{k=0}^{n}(k+p) u_{k}=(n+p) u_{n}^{(1)}-\sum_{k=0}^{n-1} u_{k}^{(1)}=(n+p+1) u_{n}^{(1)}-u_{n}^{(2)}
$$

for any $p, u_{k} \in \mathbb{R}$ and $k, n \in \mathbb{N}$. Therefore we see that

$$
s_{n}^{(2)}=\sum_{k=0}^{n} s_{k}^{(1)}=\sum_{k=0}^{n}(k+1) \sigma_{k}=(n+2) \sigma_{n}^{(1)}-\sigma_{n}^{(2)}
$$

and successively

$$
\begin{equation*}
s_{n}^{(3)}=(n+3) \sigma_{n}^{(2)}-2 \sigma_{n}^{(3)}, \cdots, s_{n}^{(\alpha)}=(n+\alpha) \sigma_{n}^{(\alpha-1)}-(\alpha-1) \sigma_{n}^{(\alpha)} \tag{3.12}
\end{equation*}
$$

From (3.11) we now see that

$$
\begin{aligned}
& \binom{n+\alpha}{\alpha}\left(C_{\alpha} s\right)_{n}=(n+\alpha)\binom{n+\alpha-1}{\alpha-1}\left(C_{\alpha-1} \sigma\right)_{n}-(\alpha-1)\binom{n+\alpha}{\alpha}\left(C_{\alpha} \sigma\right)_{n} \\
\Longrightarrow & \frac{(n+\alpha)!}{\alpha!n!}\left(C_{\alpha} s\right)_{n}=(n+\alpha) \frac{(n+\alpha-1)!}{(\alpha-1)!n!}\left(C_{\alpha-1} \sigma\right)_{n}-(\alpha-1) \frac{(n+\alpha)!}{\alpha!n!}\left(C_{\alpha} \sigma\right)_{n} \\
\Longrightarrow & (n+\alpha)!\left(C_{\alpha} s\right)_{n}=(n+\alpha) \alpha(n+\alpha-1)!\left(C_{\alpha-1} \sigma\right)_{n}-(\alpha-1)(n+\alpha)!\left(C_{\alpha} \sigma\right)_{n}
\end{aligned}
$$

which eventually implies

$$
\begin{equation*}
\left(C_{\alpha} s\right)_{n}=\alpha\left(C_{\alpha-1} \sigma\right)_{n}-(\alpha-1)\left(C_{\alpha} \sigma\right)_{n} \tag{3.13}
\end{equation*}
$$

Since the ( $C, \alpha$ )-method is stronger (or becomes at least compatible) whenever $\alpha$ increases, we know that $\lim _{n \rightarrow \infty}\left(C_{\alpha-1} \sigma\right)_{n}=L$ implies $\lim _{n \rightarrow \infty}\left(C_{\alpha} \sigma\right)_{n}=L$ and therefore by (3.13) also $\lim _{n \rightarrow \infty}\left(C_{\alpha} s\right)_{n}=L$.

Secondly, suppose that $\lim _{n \rightarrow \infty}\left(C_{\alpha} s\right)_{n}=L$. Since $\sigma_{n}^{(\alpha-1)}=\sigma_{n}^{(\alpha)}-\sigma_{n-1}^{(\alpha)}$, equation (3.12) reduces to

$$
s_{n}^{(\alpha)}=(n+\alpha)\left(\sigma_{n}^{(\alpha)}-\sigma_{n-1}^{(\alpha)}\right)-(\alpha-1) \sigma_{n}^{(\alpha)}=(n+1) \sigma_{n}^{(\alpha)}-(n+\alpha) \sigma_{n-1}^{(\alpha)} .
$$

By doing the same steps as we did to go from equations (3.11) and (3.12) to (3.13) we now get

$$
\left(C_{\alpha} s\right)_{n}=(n+1)\left(C_{\alpha} \sigma\right)_{n}-n\left(C_{\alpha} \sigma\right)_{n-1} .
$$

From this, it follows that

$$
\begin{align*}
\left(C_{\alpha} \sigma\right)_{n} & =\frac{n\left(C_{\alpha} \sigma\right)_{n-1}+\left(C_{\alpha} s\right)_{n}}{n+1} \\
& =\frac{\left(C_{\alpha} s\right)_{0}+\left(C_{\alpha} s\right)_{1}+\cdots+\left(C_{\alpha} s\right)_{n}}{n+1} \tag{3.14}
\end{align*}
$$

by the relation between $\sigma$ and $s$ and the linearity of the $C_{\alpha}$-operator. Hence $\lim _{n \rightarrow \infty}\left(C_{\alpha} s\right)_{n}=$ $L$ implies $\lim _{n \rightarrow \infty}\left(C_{\alpha} \sigma\right)_{n}=L$ and by (3.13) we now see that $\lim _{n \rightarrow \infty}\left(C_{\alpha-1} \sigma\right)_{n}=L$.

Theorem 2. The $(C, \alpha)$-method (with domain $\mathcal{S}$ ) and the $(H, \alpha)$-method (with domain $\mathcal{S}^{\prime}$ ) are equivalent; this means that $\mathcal{S}=\mathcal{S}^{\prime}$ and, for each $\alpha \in \mathbb{N}$, that $(C, \alpha)(s)=(H, \alpha)(s)=$ $L \in \mathbb{R}$.

Proof. Take some $s \in \mathcal{S}$ such that $(C, \alpha)(s)=L$. By lemma 1 we see that this is equivalent to saying that $(C, \alpha-1)\left(H_{1} s\right)=L$ where $H_{1}=C$ is the Cesàro operator/matrix. This, in turn, is equivalent to $(\mathrm{C}, \alpha-2)\left(\mathrm{H}_{2} s\right)=L$ and gradually this turns out to be equivalent to saying that $(C, 0)\left(H_{\alpha} s\right)=L$ hence $(H, \alpha)(s)=\lim _{m \rightarrow \infty}\left(H_{\alpha} s\right)_{m}=L$. Furthermore, every $(C, \alpha)$-summable series is thus $(H, \alpha)$-summable and vice-versa, hence $\mathcal{S}=\mathcal{S}^{\prime}$.

Although the $(C, \alpha)$ - and $(H, \alpha)$-methods are equivalent, they do not behave in the same way when the 'sums' ought to be infinite (i.e. at series/sequences on which the methods are not defined). We refer to theorem 54 in [3] for a proof on this matter. More precisely, we have that $(C, \alpha)(s)= \pm \infty$ implies $(H, \alpha)(s)= \pm \infty$ but the converse is false when $\alpha>1$. It is true that this statement can conjectured to be valid when one observes the according operators $C_{\alpha}$ (as described in (3.9)) and $H_{\alpha}=C^{\alpha}$ (the matrix as described in (3.1) multiplied $\alpha$ times) which are not necessarily equal.

### 3.3.2 $\left(C_{k}, g\right)$-method

By considering the equivalence between the $(C, \alpha)$ - and $(H, \alpha)$-methods and the fact that they are both linear, stable, and totally regular, we can now introduce an even more general summation method. By making use of polynomials $p(C)$ in terms of the operator $C$, we can obtain a method which we call from now on Generalized Cesáro summation (abbevriated as the ( $C, g$ )-method); see also [8] and [9] (in the second reference, it is 'more naturally' called "Remainder Cesáro summation"). It is true indeed that the naming is not very convenient and "Generalized Hölder summation" seems in technical perspective to be a better choice of the name; to be consistent with the literature we however keep calling it Generalized Cesáro summation.

For practical and conventional reasons, we initially consider the soon to be defined summability kernel $C_{k}$ and the corresponding $\left(C_{k}, g\right)$-method (in a sense the 'uncountable'
variant of the $(C, g)$-method). The summability kernel $C_{k}$ is used in the determination of the Cesàro (or Hölder) limit of divergent functions $s:[0, \infty) \rightarrow \mathbb{C}$ (and thus also the by $s$ defined divergent improper integrals $\left.s(t)=\int_{0}^{t} a(\tilde{t}) d \tilde{t}\right)$. Since the sequence-to-sequence operator $C$ has the matrix representation (3.1), it becomes naturally that we define the function-to-function operator $C_{k}$ as

$$
\left(C_{k}(x)\right)[s]:=\frac{1}{x} \int_{0}^{x} s(t) d t
$$

and say that $s$ has Cesáro limit $L$ if $\lim _{x \rightarrow \infty}\left(C_{k}^{\alpha}(x)\right)[s]=L$ for some positive integer $\alpha$. If we now let $s$ be in the function space $\mathcal{F}$ in which

$$
\mathcal{F}:=\left\{s: \int_{0}^{x}\left|s(t) \ln (t)^{m}\right| d t<\infty \text { for all } x \geq 0 \text { and } m \in \mathbb{N}\right\}
$$

we are ensured that $C_{k}$ sends $\mathcal{F}$ to itself and hence $C_{k}^{\alpha}$ becomes in turn a well-defined operator from $\mathcal{F}$ to $\mathcal{F}$; see page 333 in [8]. Concludingly, we remark that each $C_{k}^{\alpha}$ is a linear, stable and totally regular operator ${ }^{2}$ and can be used to sum divergent series in the case we introduce the partial sum function $s(t):=\sum_{k \leq t} a_{k}$.

Definition $9\left(\left(C_{k}, g\right)\right.$-method). Let us give some series $a$ with an associated partial sum function $s \in \mathcal{F}$. If there exists some polynomial of the operator $C_{k}, p\left(C_{k}\right)$, such that $p(1)=1$ and

$$
\lim _{x \rightarrow \infty}\left(p\left(C_{k}\right)(x)\right)[s]=L
$$

we say that the generalized Cesàro sum $\left(C_{k}, g\right)$ of $a$ is defined by $\left(C_{k}, g\right)(s)=L$.
Remark. The condition $p(1)=1$ ensures that the method is regular (although, as we will see later on, it is not totally regular!). This can be verified by the fact that the coefficients of the polynomial terms must sum to 1 and by the regularity of all the $C_{k}^{\alpha}$ operators it will turn out that $p\left(C_{k}\right)$ will give the exact same values as the individual $C_{k}^{\alpha}$ do in the case of convergent series. The imposed condition on the polynomial furthermore implies that $L$ is uniquely determined hence $\left(C_{k}, g\right)$ summmation can said to be well-defined; namely, if we have for two concerning polynomials $p_{1}$, $p_{2}$ that $\lim _{x \rightarrow \infty}\left(p_{1}\left(C_{k}\right)(x)\right)[s]=L_{1}$ and $\lim _{x \rightarrow \infty}\left(p_{2}\left(C_{k}\right)(x)\right)[s]=L_{2}$ we see that

$$
\begin{aligned}
L_{2} & =\lim _{x \rightarrow \infty}\left(p_{1}\left(C_{k}\right)(x)\right)\left[\left(p_{2}\left(C_{k}\right)(x)\right)[s]\right] \\
& =\lim _{x \rightarrow \infty}\left(p_{1}\left(C_{k}\right) p_{2}\left(C_{k}\right)(x)\right)[s] \\
& =\lim _{x \rightarrow \infty}\left(p_{2}\left(C_{k}\right) p_{1}\left(C_{k}\right)(x)\right)[s] \\
& =\lim _{x \rightarrow \infty}\left(p_{2}\left(C_{k}\right)(x)\right)\left[\left(p_{1}\left(C_{k}\right)(x)\right)[s]\right]=L_{1}
\end{aligned}
$$

since each of the transformations of $s$ is convergent in the usual sense. Furthermore we note, without giving the proof ${ }^{3}$, that the $\left(C_{k}, g\right)$-method is linear.

[^2]Since $C_{k}$ is a linear operator, we can consider its eigenvalues and eigenfunctions; the reason why this is interesting will soon become clear. If $\lambda \in \mathbb{C}$ is an eigenvalue of $C_{k}$ with eigenfunction $f \in \mathcal{F}$, we see that $\left(\left(C_{k}-\lambda\right)(x)\right)[f]=0$ for all $x \geq 0$. Although the operator $\left(C_{k}-\lambda\right)$ will not give us a regular summation method, the 'scaled' operator $\frac{1}{1-\lambda}\left(C_{k}-\lambda\right)$ with $\lambda \neq 1$ does indeed give us a regular method (since $\left.\frac{1}{1-\lambda}(1-\lambda)=1\right)$. We can thus say that for any eigenfunction $f$ with eigenvalue $\lambda \neq 1$ it holds that $\left(C_{k}, g\right)(f)=0$ by taking $p\left(C_{k}\right)=\frac{1}{1-\lambda}\left(C_{k}-\lambda\right)$ as the concerned polynomial. Note that the only $f$ 's which satisfy $\frac{1}{x} \int_{0}^{x} f(t) d t=f(t)$ are the constants and hence these are the eigenfunctions associated with $\lambda=1$; their $\left(C_{k}, g\right)$-sums are simply their values. We can also, for each eigenvalue $\lambda \neq 1$, consider the notion of a generalized eigenfunction; this is a function $\tilde{f} \in \mathcal{F}$ such that $\left(\left(C_{k}-\lambda\right)^{m}(x)\right)[\tilde{f}]=0$ for all $x \geq 0$ and some integer $m \geq 1$. The set spanned by all generalized eigenfunctions for a given $\lambda$, forms the generalized eigenspace for that $\lambda$. Since the polynomials $\frac{1}{(1-\lambda)^{m}}\left(C_{k}-\lambda\right)^{m}$ are regular, we see that for each generalized eigenfunction $\tilde{f}$ it holds that $\left(C_{k}, g\right)(\tilde{f})=0$. If we can now write a function $s \in \mathcal{F}$ in the form (ignoring the trivial and useless case that $s=R$ )

$$
\begin{equation*}
s(t)=R(t)+\sum_{i=1}^{n} c_{i} f_{i}(t) \tag{3.15}
\end{equation*}
$$

in which $c_{i} \in \mathbb{C}, f_{i}$ is some (generalized) eigenfunction and $R$ is a 'remainder' function satisfying $\lim _{x \rightarrow \infty}\left(C_{k}^{\alpha}(x)\right)[R]=L$, it holds by the linearity of the $\left(C_{k}, g\right)$-method that $\left(C_{k}, g\right)(s)=L$. To make more sense of this, we can explicitely identify some (generalized) eigenfunctions.

Theorem 3. Let us give the operator $C_{k}$ with an eigenvalue of the form $\frac{1}{\rho+1}$ ( $\rho \in \mathbb{C} \backslash\{0\}$ and $\Re(\rho)>-1)$. The corresponding eigenfunction is given by
(i) $f(t)=t^{\rho}$.

Moreover, the generalized eigenspace is spanned by the generalized eigenfunctions
(ii) $\tilde{f}(t)=t^{\rho} \ln (t)^{m}, m \in \mathbb{N}$.

Proof. (i) It is easy to see that $f(t)=t^{\rho}$ is an eigenfunction since

$$
\left(C_{k}(x)\right)[f]=\frac{1}{x} \int_{0}^{x} t^{\rho} d t=\frac{1}{x} \frac{1}{\rho+1} x^{\rho+1}=\frac{1}{\rho+1} x^{\rho}=\frac{1}{\rho+1} f(x)
$$

To see that $f$ must explicitely be given in the stated form, we have to check whether the dimension of the eigenspace is 1 . Consider some function $g:[0, \infty) \rightarrow \mathbb{C}$ such that $\left(C_{k}(x)\right)[g]=\frac{1}{\rho+1} g$ which means that $\int_{0}^{x} g(t) d t=\frac{1}{\rho+1} x g(x)$. Differentiating both sides with respect to $x$ gives us $g(x)=\frac{1}{\rho+1}\left(g(x)+x g^{\prime}(x)\right)$ which implies that

$$
x g^{\prime}(x)-\rho g(x)=0
$$

Since this is a homogeneous first-order linear ODE, its solution space must be onedimensional hence the dimension of the eigenspace is 1 and indeed $g=f$.
(ii) First we check whether each $\tilde{f}(t)=t^{\rho} \ln (t)^{m}(m \in \mathbb{N})$ satisfies $\left(\left(C_{k}-\frac{1}{\rho+1}\right)^{m+1}(x)\right)[\tilde{f}]=$ 0 and hence is a generalized eigenfunction; this can be verified by an induction argument. For $m=0$ this is indeed true, as we have seen in (i). Now assume it holds for $m=m^{\prime}$, hence

$$
\begin{equation*}
\left(\left(C_{k}-\frac{1}{\rho+1}\right)^{m^{\prime}+1}(x)\right)\left[\tilde{t}^{\rho} \ln (\tilde{t})^{m^{\prime}}\right]=0 \tag{3.16}
\end{equation*}
$$

where the variable $\tilde{t}$ is written here to make notation more convenient later on. Now we will check for $m=m^{\prime}+1$. We note that

$$
\begin{align*}
& \left(\left(C_{k}-\frac{1}{\rho+1}\right)^{m^{\prime}+2}(x)\right)\left[t^{\rho} \ln (t)^{m^{\prime}+1}\right] \\
= & \left(\left(C_{k}-\frac{1}{\rho+1}\right)^{m^{\prime}+1}(x)\right)\left[\left(\left(C_{k}-\frac{1}{\rho+1}\right)(\tilde{t})\right)\left[t^{\rho} \ln (t)^{m^{\prime}+1}\right]\right] \tag{3.17}
\end{align*}
$$

whereas

$$
\begin{equation*}
\left(\left(C_{k}-\frac{1}{\rho+1}\right)(\tilde{t})\right)\left[t^{\rho} \ln (t)^{m^{\prime}+1}\right]=\frac{1}{\tilde{t}} \int_{0}^{\tilde{t}} t^{\rho} \ln (t)^{m^{\prime}+1} d t-\frac{1}{\rho+1}\left(\tilde{t}^{\rho} \ln (\tilde{t})^{m^{\prime}+1}\right) \tag{3.18}
\end{equation*}
$$

Evaluating the integral using integration by parts now gives us

$$
\int_{0}^{\tilde{t}} t^{\rho} \ln (t)^{m^{\prime}+1} d t=\frac{1}{\rho+1}\left(\tilde{t}^{\rho+1} \ln (\tilde{t})^{m^{\prime}+1}\right)-\frac{m^{\prime}+1}{\rho+1} \int_{0}^{\tilde{t}} t^{\rho+1} \frac{\ln (t)^{m^{\prime}}}{t} d t
$$

hence we see that (3.18) reduces to

$$
\left(\left(C_{k}-\frac{1}{\rho+1}\right)(\tilde{t})\right)\left[t^{\rho} \ln (t)^{m^{\prime}+1}\right]=-\frac{1}{\tilde{t}} \frac{m^{\prime}+1}{\rho+1} \int_{0}^{\tilde{t}} t^{\rho} \ln (t)^{m^{\prime}} d t
$$

and thus (3.17) gives us

$$
\begin{align*}
& \left(\left(C_{k}-\frac{1}{\rho+1}\right)^{m^{\prime}+2}(x)\right)\left[t^{\rho} \ln (t)^{m^{\prime}+1}\right] \\
= & \left(\left(C_{k}-\frac{1}{\rho+1}\right)^{m^{\prime}+1}(x)\right)\left[-\frac{1}{\tilde{t}} \frac{m^{\prime}+1}{\rho+1} \int_{0}^{\tilde{t}} t^{\rho} \ln (t)^{m^{\prime}} d t\right]  \tag{3.19}\\
= & \left(\left(C_{k}-\frac{1}{\rho+1}\right)^{m^{\prime}+1}(x)\right)\left[-\frac{1}{\tilde{t}} \frac{m^{\prime}+1}{\rho+1}\left(\frac{1}{\rho+1} \tilde{t}^{\rho+1} \ln (\tilde{t})^{m^{\prime}}-\int_{0}^{\tilde{t}} \frac{1}{\rho+1} t^{\rho+1} m^{\prime} \frac{\ln (t)^{m^{\prime}-1}}{t} d t\right)\right] \\
= & \left(\left(C_{k}-\frac{1}{\rho+1}\right)^{m^{\prime}+1}(x)\right)\left[-\frac{m^{\prime}+1}{(\rho+1)^{2}} \tilde{t}^{\rho} \ln (\tilde{t})^{m^{\prime}}+\frac{1}{\tilde{t}} \frac{\left(m^{\prime}+1\right) m^{\prime}}{(\rho+1)^{2}} \int_{0}^{\tilde{t}} t^{\rho} \ln (t)^{m^{\prime}-1} d t\right]
\end{align*}
$$

by using integration by parts. In conclusion, we see by using (3.16) that

$$
\left(\left(C_{k}-\frac{1}{\rho+1}\right)^{m^{\prime}+1}(x)\right)\left[-\frac{m^{\prime}+1}{(\rho+1)^{2}} \tilde{t}^{\rho} \ln (\tilde{t})^{m^{\prime}}\right]=0
$$

and in addition using the same methodology to derive (3.19) we see that

$$
\begin{aligned}
& \left(\left(C_{k}-\frac{1}{\rho+1}\right)^{m^{\prime}+1}(x)\right)\left[\frac{1}{\tilde{t}} \frac{\left(m^{\prime}+1\right) m^{\prime}}{(\rho+1)^{2}} \int_{0}^{\tilde{t}} t^{\rho} \ln (t)^{m^{\prime}-1} d t\right] \\
= & \left(\left(C_{k}-\frac{1}{\rho+1}\right)^{m^{\prime}+2}(x)\right)\left[-\frac{m^{\prime}+1}{\rho+1} t^{\rho} \ln (t)^{m^{\prime}}\right] \\
= & \left(\left(C_{k}-\frac{1}{\rho+1}\right)(x)\right)\left[\left(\left(C_{k}-\frac{1}{\rho+1}\right)^{m^{\prime}+1}(\tilde{t})\right)\left[-\frac{m^{\prime}+1}{\rho+1} t^{\rho} \ln (t)^{m^{\prime}}\right]\right] \\
= & \left(\left(C_{k}-\frac{1}{\rho+1}\right)(x)\right)[0]=0 .
\end{aligned}
$$

Hence

$$
\left(\left(C_{k}-\frac{1}{\rho+1}\right)^{m^{\prime}+2}(x)\right)\left[t^{\rho} \ln (t)^{m^{\prime}+1}\right]=0
$$

and the statement holds.
To see that this are all generalized eigenfunctions and therefore span the generalized eigenspace, we have to check whether the solution space of $\left(\left(C_{k}-\frac{1}{\rho+1}\right)^{d}(x)\right)[\tilde{g}]=0$ $\left(d \in \mathbb{N}^{+}\right)$is $d$-dimensional and $\tilde{g}(t)$ can be written as a linear combination of $d$ independent terms $t^{\rho} \ln (t)^{m}(m=0, \ldots, d-1)$. We do this by an induction argument as well. For $d=1$ this is indeed true, as we have seen in (i). Assume it is true that for $d=d^{\prime} \in \mathbb{N}^{+}$the solution space of $\left(\left(C_{k}-\frac{1}{\rho+1}\right)^{d^{\prime}}(x)\right)[\tilde{g}]=0$ is $d^{\prime}$-dimensional and that $\tilde{g}(t)$ can be written as a linear combination of $d^{\prime}$ independent terms $t^{\rho} \ln (t)^{m}$ $\left(m=0, \ldots, d^{\prime}-1\right)$. Now we will check for $d^{\prime}+1$ and note that

$$
\begin{equation*}
\left(\left(C_{k}-\frac{1}{\rho+1}\right)^{d^{\prime}+1}(x)\right)[\tilde{g}]=\left(\left(C_{k}-\frac{1}{\rho+1}\right)^{d^{\prime}}(x)\right)\left[\left(\left(C_{k}-\frac{1}{\rho+1}\right)(\tilde{t})\right)[\tilde{g}]\right] \tag{3.20}
\end{equation*}
$$

where we again take $\tilde{t}$ in order to retain convenient notation. From this we see, by our induction hypothesis, that $\left(\left(C_{k}-\frac{1}{\rho+1}\right)(\tilde{t})\right)[\tilde{g}]$ must be equal to an arbitrary linear combination of $d^{\prime}$ independent terms $\tilde{t}^{\rho} \ln (\tilde{t})^{m}\left(m=0, \ldots, d^{\prime}-1\right)$ which we will briefly call $I(\tilde{t})$. Reordering terms and differentiating both sides with respect to $\tilde{t}$ gives rise to the differential equation

$$
\tilde{g}^{\prime}(\tilde{t})-\frac{\rho}{\tilde{t}} \tilde{g}(\tilde{t})=\tilde{I}(\tilde{t})
$$

with $\tilde{I}(\tilde{t})$ being derived from $I(\tilde{t})$ and is an addition of $d^{\prime}$ linear independent logarithmical terms with exponent up to $d^{\prime}-1$. We see that this is an inhomogeneous first-order linear ODE hence its solution space must be of one dimension higher than the vector space spanned by the terms in $\tilde{I}(t)$ (hence is equal to $d^{\prime}+1$ ); this proves the statement.

By using the previous theorem and the observation that for $\Re(\rho) \leq-1$ it holds that $\lim _{t \rightarrow \infty} t^{\rho} \ln (t)^{m}=0$, we see that (3.15) translates to

$$
\begin{equation*}
s(t)=R(t)+\sum_{i=1}^{n} c_{i} t^{\rho_{i}} \ln (t)^{m_{i}} \tag{3.21}
\end{equation*}
$$

with $\rho_{i} \in \mathbb{C} \backslash\{0\}$ and $m_{i} \in \mathbb{N}$. It is also interesting to explore the relationship between the collection of constants $\rho_{i}, m_{i}$ and the (simplest) concerning polynomial $p\left(C_{k}\right)$ such that $\left(C_{k}, g\right)(s)=\lim _{x \rightarrow \infty}\left(p\left(C_{k}\right)(x)\right)[s]=L$. This polynomial has to be chosen in order to ensure that the (generalized) eigenfunctions $c_{i} t^{\rho_{i}} \ln (t)^{m_{i}}$ are assigned 0 and the remainder function $R(t)$ is assigned $L$. We do this as follows:

1. Take the list $\rho_{1}, \ldots, \rho_{n}$. For each distinct $\rho$ in this list consider those $\rho_{i}$ with $\rho_{i}=\rho$ and let $m$ be the largest of the corresponding values of $m_{i}$.
2. Consider, for each distinct $\rho$ and the corresponding chosen $m$, terms of the form $\left(\frac{\rho+1}{\rho}\right)^{m+1}\left(C_{k}-\frac{1}{\rho+1}\right)^{m+1}$.
3. Construct the polynomial $p\left(C_{k}\right)$ by the multiplication of all the terms from step 2 in addition with the factor $C_{k}^{\alpha}$ (with $\alpha$ being the smallest integer such that $\left.\lim _{x \rightarrow \infty}\left(C_{k}^{\alpha}(x)\right)[R]=L\right)$. Since $p(1)=1$, we are ensured that the summation method is regular.

After this introduction of a more generalized Cesàro summation scheme, we can now return to our original motivation and try to assign a finite value to the series as described in (1.4).

Example 5. Consider the series $a=\sum_{k=1}^{\infty} k$ with partial sums up till $n_{1}$ equal to $\frac{n_{1}\left(n_{1}+1\right)}{2}$. In order to obtain the partial sum function $s\left(t_{1}\right)$ we note that the continuous variable $t_{1} \geq 1$ can be written as $t_{1}=n_{1}+\gamma_{1}$ with $n_{1}=\left\lfloor t_{1}\right\rfloor$ and $\gamma_{1} \in[0,1)$ as a 'saw-tooth' function of $t_{1}$. In this example, we write subscripts $1,2,3$ in order to retain convenient notation (i.e. $t_{2}=n_{2}+\gamma_{2}$ and $t_{3}=n_{3}+\gamma_{3}$ ); this is similar with the reason why we wrote $\tilde{t}$ earlier on. We now see that $s\left(t_{1}\right)$ can be explicitly written as

$$
s\left(t_{1}\right)=\frac{n_{1}^{2}+n_{1}}{2}=\frac{\left(n_{1}+\gamma_{1}\right)^{2}}{2}+\left(\frac{1}{2}-\gamma_{1}\right) n_{1}-\frac{\gamma_{1}^{2}}{2}=\frac{t_{1}^{2}}{2}+R\left(t_{1}\right)
$$

with $R\left(t_{1}\right)=R\left(n_{1}+\gamma_{1}\right)=\left(\frac{1}{2}-\gamma_{1}\right) n_{1}-\frac{1}{2} \gamma_{1}^{2}$. Since $t_{1}^{2}$ is an eigenvalue of the operator $C_{k}$, we see by (3.21) that $\left(C_{k}, g\right)(s)=\lim _{x \rightarrow \infty}\left(C_{k}^{\alpha}(x)\right)[R]$ for some $\alpha \in \mathbb{N}$. We note that

$$
\begin{aligned}
\left(C_{k}\left(t_{2}\right)\right)[R] & =\frac{1}{t_{2}} \int_{0}^{t_{2}}\left(\left(\frac{1}{2}-\gamma_{1}\right) n_{1}-\frac{1}{2} \gamma_{1}^{2}\right) d t_{1} \\
& =\frac{1}{n_{2}+\gamma_{2}}\left(\int_{0}^{n_{2}}\left(\left(\frac{1}{2}-\gamma_{1}\right) n_{1}-\frac{1}{2} \gamma_{1}^{2}\right) d t_{1}\right. \\
& \left.+\int_{n_{2}}^{n_{2}+\gamma_{2}}\left(\left(\frac{1}{2}-\gamma_{1}\right) n_{1}-\frac{1}{2} \gamma_{1}^{2}\right) d t_{1}\right) \\
& =\frac{1}{n_{2}+\gamma_{2}}\left(\int_{0}^{n_{2}}\left(\left(\frac{1}{2}-\left(t_{1}-\left\lfloor t_{1}\right\rfloor\right)\right)\left\lfloor t_{1}\right\rfloor-\frac{1}{2}\left(t_{1}-\left\lfloor t_{1}\right\rfloor\right)^{2}\right) d t_{1}\right. \\
& \left.+\int_{n_{2}}^{n_{2}+\gamma_{2}}\left(\left(\frac{1}{2}-\left(t_{1}-\left\lfloor t_{1}\right\rfloor\right)\right)\left\lfloor t_{1}\right\rfloor-\frac{1}{2}\left(t_{1}-\left\lfloor t_{1}\right\rfloor\right)^{2}\right) d t_{1}\right)
\end{aligned}
$$

The first integral can be 'split up' into $n_{2}$ subsequent integrals and we observe that the according $\left\lfloor t_{1}\right\rfloor$ 's in each subsequent integral can be regarded as a constant. Moreover, we
see that $t_{1}-\left\lfloor t_{1}\right\rfloor$ is periodic with period 1 and hence for $j \in \mathbb{N}$ and $\tau \in \mathbb{R}$ we get that

$$
\int_{j}^{j+\tau}\left(t_{1}-\left\lfloor t_{1}\right\rfloor\right) d t_{1}=\int_{0}^{\tau} t_{1} d t_{1}
$$

By this observations we see that $\left(C_{k}\left(t_{2}\right)\right)[R]$ is equal to

$$
\begin{aligned}
& \frac{1}{n_{2}+\gamma_{2}}\left(\sum_{j=0}^{n_{2}-1}\left(\frac{1}{2}-\int_{0}^{1} t_{1} d t_{1}\right) j-\frac{1}{2} n_{2} \int_{0}^{1} t_{1}^{2} d t_{1}\right. \\
& \left.+n_{2} \int_{0}^{\gamma_{2}}\left(\frac{1}{2}-t_{1}\right) d t_{1}-\frac{1}{2} \int_{0}^{\gamma_{2}} t_{1}^{2} d t_{1}\right) \\
& =\frac{1}{n_{2}+\gamma_{2}}\left(-\frac{n_{2}}{6}+\frac{n_{2} \gamma_{2}}{2}-\frac{n_{2} \gamma_{2}^{2}}{2}-\frac{\gamma_{2}^{3}}{6}\right) \\
& =-\frac{1}{6}+\frac{\gamma_{2}}{2}-\frac{\gamma_{2}^{2}}{2}+O\left(\frac{1}{n_{2}}\right)
\end{aligned}
$$

Applying again the $C_{k}$ operator on the above expression we get that

$$
\begin{aligned}
\left(C_{k}^{2}\left(t_{3}\right)\right)[R] & =\left(C_{k}\left(t_{3}\right)\right)\left[-\frac{1}{6}+\frac{\gamma_{2}}{2}-\frac{\gamma_{2}^{2}}{2}+O\left(\frac{1}{n_{2}}\right)\right] \\
& =-\frac{1}{6}+\frac{1}{t_{3}} \int_{0}^{n_{3}+\gamma_{3}}\left(\frac{\gamma_{2}}{2}-\frac{\gamma_{2}^{2}}{2}\right) d t_{2} \\
& =-\frac{1}{6}+\frac{1}{n_{3}+\gamma_{3}}\left(\frac{1}{2} n_{3} \int_{0}^{1}\left(t_{2}-t_{2}^{2}\right) d t_{2}+\frac{1}{2} \int_{0}^{\gamma_{3}}\left(t_{2}-t_{2}^{2}\right) d t_{2}\right) \\
& =-\frac{1}{6}+\frac{1}{n_{3}+\gamma_{3}}\left(\frac{1}{2} n_{3}\left(\frac{1}{2}-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{\gamma_{3}^{2}}{2}-\frac{\gamma_{3}^{3}}{3}\right)\right) \\
& =-\frac{1}{6}+\frac{1}{1+\frac{\gamma_{3}}{n_{3}}}\left(\frac{1}{4}-\frac{1}{6}+\frac{1}{2}\left(\frac{\gamma_{3}^{2}}{2 n_{3}}-\frac{\gamma_{3}^{3}}{3 n_{3}}\right)\right)
\end{aligned}
$$

Taking $\alpha=2$ and $x=t_{3}$ we now see that the generalized Cesàro sum of the concerning series is $\left(C_{k}, g\right)(s)=\lim _{t_{3} \rightarrow \infty}\left(C_{k}^{2}\left(t_{3}\right)\right)[R]=\lim _{n_{3} \rightarrow \infty}\left(C_{k}^{2}\left(t_{3}\right)\right)[R]=-\frac{1}{12}$; this makes the $\left(C_{k}, g\right)$-method stronger than the $(C, \alpha)$-method.

In the next chapter we will see that there is a deeper reason why the $\left(C_{k}, g\right)$-sum of (1.4) equals $-\frac{1}{12}$; this furthermore might lead to very surprising applications of summability theory within other fields of mathematics. Although the value of $-\frac{1}{12}$ definitely equals not the 'true' value of (1.4) (which ought to be $\infty$ ), we will see that there might be hidden relationships between this assigned value and the limiting value obtained from ordinary summation. This observations may provide insights into the nature of our summation methods and the conditions on which to apply them in other fields of mathematics and science/engineering in general.

## 4 Epilogue

### 4.1 Connections with the Riemann zeta function

### 4.1.1 Zeta function regularization

In the previous chapter, we saw that the series (1.2) can be reasonable summed by the ( $C, \alpha$ )-method. We were, however, not able to assign a finite value to the series (1.4) by this method; also it seemed that (1.3) is unsummable using this method. Later on we saw that we in fact, using the $\left(C_{k}, g\right)$-method, can assign a finite value to the series (1.4); namely, the (rather unexpected) value of $-\frac{1}{12}$. In order for a method to assign a finite value to this kind of series, it has to have weakened properties in comparison to ordinary summation. Obviously, such a method would not be totally regular. In order to retain consistency with the existing 'mathematical building', we have the following observation. Note that this is in principle related to the negation of theorem 2 in [6].

Proposition 5. Any summation method $T$ which assigns a finite value $L$ to $a=\sum_{k=1}^{\infty} k$ (with corresponding sequence of partial sums $\left.s=\left(s_{n}\right)=\left(\frac{n(n+1)}{2}\right), n \in \mathbb{N}^{+}\right)$is either not linear or not stable.

Proof. Assume the opposite, that is that $T$ is linear and stable. Since $s$, by assumption, is in the domain $\mathcal{S}$ of $T$ and considering the fact that $T$ is stable we see that we can 'shift' $a$ to the right and 'plug in' the term 0, i.e. $s^{\prime}=\left(s_{n}^{\prime}\right)=\left(s_{n-1}+s_{1}^{\prime}\right) \in \mathcal{S}$ (with $s_{1}^{\prime}=0$ ) and $T\left(s^{\prime}\right)=T(s)+s_{1}^{\prime}=T(s)$. By the linearity of $T$ we have that $T\left(s-s^{\prime}\right)=T(s)-T\left(s^{\prime}\right)$ and so $T(\tilde{s})=0$ with $\tilde{s}:=s-s^{\prime}=\left(s_{n}-s_{n-1}\right)=\left(\frac{n(n+1)}{2}-\frac{(n-1) n}{2}\right)=(n)$.
Now we again shift the corresponding series of $\tilde{s}, \tilde{a}$, to the right and plug in the term 0 . We get that $\tilde{s}^{\prime}=\left(\tilde{s}_{n}^{\prime}\right)=\left(\tilde{s}_{n-1}+\tilde{s}_{1}^{\prime}\right) \in \mathcal{S}\left(\right.$ with $\left.\tilde{s}_{1}^{\prime}=0\right)$ and $T\left(\tilde{s}^{\prime}\right)=T(\tilde{s})+\tilde{s}_{1}^{\prime}=T(\tilde{s})$. Now by the linearity of $T$ we get that both

$$
0=T(\tilde{s})-T\left(\tilde{s}^{\prime}\right)=T\left(\tilde{s}-\tilde{s}^{\prime}\right)=T((n-(n-1)))=T((1))
$$

and

$$
0=T\left(\tilde{s}^{\prime}\right)-T(\tilde{s})=T\left(\tilde{s}^{\prime}-\tilde{s}\right)=T(((n-1)-n))=T((-1)) .
$$

Now, concludingly, it is by the stability of $T$ easy to see that

$$
-1+T((0))=T((-1))=0=T((1))=1+T((0))
$$

hence $-1=1$. This leads to a contradiction since it is inconsistent with the existing building of mathematics and therefore $T$ must either be nonlinear or unstable.

Since in subsection 3.3.2 we said that the $\left(C_{k}, g\right)$-method is linear, it follows from the above proposition that this method must be unstable. We can thus conclude that this method, by its non-total regularity and unstability, does not possess all properties which are at least required in order to have a very nice alternative to ordinary summation.

We can now pose the question whether there exists other summation methods which assigns a finite value, preferably $-\frac{1}{12}$, to the series (1.4). The answer to this question is in
fact positive and one of these methods is called zeta function regularization which is based on the Riemann zeta function. Zeta function regularization can in general be regarded as a somewhat different method to the methods we discussed earlier; namely, there is no use of a linear operator and there is even no 'limiting procedure'. Although zeta function regularization in general not bears the desirable properties from ordinary summation (such as linearity and total regularity) we can however say that zeta function regularization in conjuction with so-called 'renormalization' offers a great tool within theoretical physics; see also the next section.

To see how zeta function regularization assigns a value to $a=\sum_{k=1}^{\infty} k$, note that this series is equal to $\sum_{k=1}^{\infty} \frac{1}{k^{-1}}$. We then consider the Riemann zeta function

$$
\begin{equation*}
\zeta(s)=\sum_{k=1}^{\infty} \frac{1}{k^{s}} \quad s \in \mathbb{C}, \Re(s)>1 \tag{4.1}
\end{equation*}
$$

and, although the defining series diverges when $\Re(s) \leq 1$, we 'in some way' could say that $a$ is equal to $\zeta(-1)$. Bernhard Riemann (1826-1866) derived a functional equation for the zeta function in which it was showed that the function could be analytically continued to the whole complex plane with the exception of a simple pole with residue 1 at $s=1$. We namely have that (see also [7])

$$
\begin{equation*}
\zeta(s)=2^{s} \pi^{s-1} \sin \left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \tag{4.2}
\end{equation*}
$$

If we now let $s=-1$, we see that indeed

$$
\begin{aligned}
\zeta(-1) & =2^{-1} \pi^{-2} \sin \left(-\frac{\pi}{2}\right) \Gamma(2) \sum_{k=1}^{\infty} \frac{1}{k^{2}} \\
& =2^{-1} \pi^{-2}(-1)(1) \frac{\pi^{2}}{6} \\
& =-\frac{1}{12}
\end{aligned}
$$

Besides this series, we can now in fact show that (1.3) is also summable but the series (1.1) will however still remain unsummable by using zeta function regularization.

Example 6. Consider the series $\sum_{k=1}^{\infty} 1$ and we see that if we let $s=0$ in (4.1), the series becomes 'equal' to $\zeta(0)$ (remind that this formally is not permitted!). By substituting $s=0$ in the functional equation (4.2) we now see that we have to deal with the simple pole at $s=1$. We can however make use of the Taylor and Laurent expansions for the sine and Riemann zeta functions respectively, by using the fact that the Riemann zeta function is analytic hence continuous at $s=0$ we get that

$$
\begin{aligned}
\zeta(0) & =2^{0} \pi^{-1} \Gamma(1) \lim _{s \rightarrow 0} \sin \left(\frac{\pi s}{2}\right) \zeta(1-s) \\
& =\pi^{-1} \lim _{s \rightarrow 0}\left(\frac{\pi s}{2}-\frac{\pi^{3} s^{3}}{48}+\cdots\right)\left(-\frac{1}{s}+\gamma-\cdots\right) \\
& =-\frac{1}{2}
\end{aligned}
$$

where $\gamma=0.57721 \ldots$ is the Euler-Mascheroni constant. Furthermore, we see that the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is not 'summable' since it corresponds to the pole at $s=1$. There do
however exist alternative methods (also strongly incomparable with ordinary summation) which can assign a finite value to this series. One of these methods, called Ramanujan summation, assigns the Euler-Mascheroni constant $\gamma$ to it, as was slightly expected from the Laurent series of the Riemann zeta function.

We saw that both the $\left(C_{k}, g\right)$-method and zeta function regularization assigned the value $-\frac{1}{12}$ to the series $\sum_{k=1}^{\infty} \frac{1}{k^{-1}}$. This leads us to pose the question whether the $\left(C_{k}, g\right)$ method does assign exactly the same values on the series $\sum_{k=1}^{\infty} \frac{1}{k^{s}}(s \in \mathbb{C})$ as zeta function regularization does, i.e. that

$$
\left(C_{k}, g\right)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{s}}\right)\right)=\zeta(s)
$$

for all $s \in \mathbb{C} \backslash\{1\}$ and has a simple pole with residue 1 at $s=1$ (and therefore (1.3) is summable and (1.1) is unsummable by the ( $C_{k}, g$ )-method)? According to $\S 3$ in [8], this indeed follows to be true; here the link is made between the ( $C_{k}, g$ )-sum of the defining series $\sum_{k=1}^{n} \frac{1}{k^{s}}$ and the Riemann zeta function by considering the respective Euler-Maclaurin expansion and the (generalized) eigenvalues of the $C_{k}$-operator. Furthermore, when one considers the in $\S 3.3 .2$ introduced discrete version of generalized Cesàro summation, $(C, g)$ (a polynomial of the operator $C$ instead of $C_{k}$ ), it is in $\S 4$ of [8] proven that

$$
(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{s}}\right)\right)=\left\{\begin{array}{lc}
1, & s \in \mathbb{Z}_{\leq 0} \\
\zeta(s), & \text { otherwise }
\end{array} .\right.
$$

The reason that the $(C, g)$-sum of the defining series does not equal the $\left(C_{k}, g\right)$-sum of this series (and in fact $\zeta(s)$ ) for all $s \in \mathbb{C}$ can be explained by the fact that the eigenvectors of $C$ are of a different nature than the eigenvectors of $C_{k}$.

Since the Riemann zeta function, and especially its distribution of zeroes, is an important object of study within pure mathematics (analytic number theory) it comes to an unexpected surprise that we can now evaluate values of the zeta function using our generalized Cesàro method. In the next subsection we will learn more about the importance of knowing the zeroes of the zeta function and we will see how generalized Cesàro summation might provide a helping hand in this.

### 4.1.2 Riemann hypothesis

In 1859, Riemann published the article "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" ("On the Number of Primes Less Than a Given Magnitude"); see [7]. In this article, Riemann considers the relationship between the prime counting function and the zeroes of the zeta function and mentions that the zeroes of the zeta function must have an as 'tight' possible structure in order for the prime numbers to be 'nicely' distributed. The zeroes of the zeta function can in fact be divided into two types: trivial and nontrivial zeroes. The trivial zeroes are easily obtained from the functional equation (4.2) and are in fact all negative even integers. Riemann hypothesizes that the nontrivial zeroes all lie on the so-called trivial line (all complex numbers with real part $\frac{1}{2}$ ), this is now called the Riemann hypothesis and remains after 160 years still unproven despite the efforts of the best mathematicians in the world. Since a large part of the mathematical community believes the hypothesis is true, a lot of future results are based upon its validity.

It is currently known that all zeros $s=\sigma+i t$ must lie within the critical strip $(0<\sigma<1)$ and that there are in fact infinitely many zeroes on the critical line. Also it is known that, for $s=\sigma+i t$ to be a zero, we have that $s=\sigma-i t, s=(1-\sigma)+i t$ and $s=(1-\sigma)-i t$ are zeroes as well; this is due to symmetrical properties of the functional equation (4.2).

Now it is interesting to observe that $(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{s}}\right)\right)$ is equal to 1 if $s$ is a negative even integer, hence the set of nontrivial zeroes $s$ correspond to all $s$ for which $(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{s}}\right)\right)$ is equal to 0 . Since we know that both the $(C, g)$ - and the $\left(C_{k}, g\right)$ methods assign the same values on the concerned series when we regard $s$ to be in the critical strip, it does not matter which particular method we are using; for practical reasons we will however use the $(C, g)$-method. In example 5 we used an expression for the partial sums of the concerned series; for general $s$, the partial sums $\sum_{k=1}^{n} \frac{1}{k^{s}}$ (also called the generalized harmonic numbers) do however not possess such a closed-form expression (or they are at least not simply to be found). In [8] it is furthermore made clear that for some other sequence $r=\left(r_{n}\right)$ such that $\sum_{k=1}^{n} \frac{1}{k^{s}} \sim r_{n}$ we have that $(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{s}}\right)\right)=(C, g)\left(\left(r_{n}\right)\right)$ but it is not easy to find such an $r$ without using the Euler-Maclaurin expansion of the defining series and therefore obtaining the zeta function itself (see also theorem 2 of [8]).

What the $(C, g)$-method can however do, is to give an equivalent formulation of the Riemann hypothesis in terms of (real analytic) summability theory. Suppose $s=\sigma+i t$ is a zero of $\zeta(s)$, hence

$$
(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{\sigma+i t}}\right)\right)=(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{s}}\right)\right)=\zeta(s)=0
$$

but by symmetry of the zeroes also

$$
\begin{aligned}
(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{\sigma+i t}}\right)\right) & =(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{\sigma-i t}}\right)\right)=0 \\
\text { and }(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{(1-\sigma)+i t}}\right)\right) & =(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{(1-\sigma)-i t}}\right)\right)=0 .
\end{aligned}
$$

By the linearity of the $(C, g)$-method we now see that

$$
\begin{aligned}
&(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{\sigma}}\left(\frac{1}{k^{i t}}+\frac{1}{k^{-i t}}\right)\right)\right)=(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{\sigma}}\left(\frac{1}{k^{i t}}-\frac{1}{k^{-i t}}\right)\right)\right) \\
&=(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{\sigma}}\left(k^{-i t}+k^{i t}\right)\right)\right)=(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{\sigma}}\left(k^{-i t}-k^{i t}\right)\right)\right)=0,
\end{aligned}
$$

this holds as well for $(1-\sigma)$ instead of $\sigma$ and $-t$ instead of $t$. By considering the fact that the cosine and sine functions can be written in terms of complex exponentials and using
again the linearity of the method we now have that

$$
\begin{aligned}
0 & =(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{\sigma}}\left(k^{i t}+k^{-i t}\right)\right)=(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{\sigma}}\left(k^{i t}-k^{-i t}\right)\right)\right)\right. \\
& =(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{\sigma}}\left(e^{\ln (k) t i}+e^{-\ln (k) t i}\right)\right)\right)=(C, g)\left(\left(\sum_{k=1}^{n} \frac{1}{k^{\sigma}}\left(e^{\ln (k) t i}-e^{-\ln (k) t i}\right)\right)\right) \\
& =(C, g)\left(\left(\sum_{k=1}^{n} \frac{2 \cos (t \ln (k))}{k^{\sigma}}\right)\right)=(C, g)\left(\left(\sum_{k=1}^{n} \frac{2 i \sin (t \ln (k))}{k^{\sigma}}\right)\right) \\
& =(C, g)\left(\left(\sum_{k=1}^{n} \frac{\cos (t \ln (k))}{k^{\sigma}}\right)\right)=(C, g)\left(\left(\sum_{k=1}^{n} \frac{\sin (t \ln (k))}{k^{\sigma}}\right)\right),
\end{aligned}
$$

this is also true when we consider $-t$ and $(1-\sigma)$. Note that all variables and function-values in the partial sums $\sum_{k=1}^{n} \frac{\cos (t \ln (k))}{k^{\sigma}}$ and $\sum_{k=1}^{n} \frac{\sin (t \ln (k))}{k^{\sigma}}$ are real-valued. If we can now find closed-form expressions of the partial sums (which is not known to be easier/harder than deriving closed-form expressions for the generalized harmonic numbers) and apply the $(C, g)$-method on the according sequences we hopefully arrive at the necessary conditions that $\sigma=\frac{1}{2}$ (hence the Riemann hypothesis would be proved).

In the first three figures of Appendix A, we show some plots of the partial sum sequences $\left(\sum_{k=1}^{n} \frac{\cos (t \ln (k))}{k^{\sigma}}\right)$ for several values of $\sigma$ and $t$. It seems that there are $\ln (n)$ and $\cos (t \ln (n))$ terms involved in its growth; also the growth is larger when both $\sigma$ and $t$ are smaller and there seems to be a higher vertical displacement when $t$ is larger. Furthermore we may hypothesize that all partial sum sequences are divergent and are not $(C, \alpha)$-summable because of its unbounded nature. If it is true that the partial sum sequences are of a simple form containing $\ln (n)$ and $\cos (t \ln (n))$ terms, we can apply the $(C, g)$-method in a clever way: By considering their respective Taylor series and the fact that the eigensequences of the $C$-operator are of the form $n^{a} \ln (n)^{b}(a, b \in \mathbb{C}$ with $a \notin \mathbb{N}$, see also $\S 4.1$ of [8]), we are potentially be able to arrive at sequences on which we can apply ordinary summation or the $(C, \alpha)$-method and thus eventually retrieve the necessary value(s) of $\sigma$.

In conclusion, it is interesting to consider the defining partial sums of $-\frac{\zeta(s)^{\prime}}{\zeta(s)}$, namely

$$
\begin{equation*}
\sum_{k=1}^{n} \frac{\Lambda(k)}{k^{s}} \quad s \in \mathbb{C}, \Re(s)>1 \tag{4.3}
\end{equation*}
$$

with $\Lambda(k)$ being the von Mangoldt function given by

$$
\Lambda(k)= \begin{cases}\ln (p), & k=p^{r} \text { with } p \text { prime and } r \in \mathbb{N} \\ 0, & \text { otherwise }\end{cases}
$$

It turns out that the nontrivial zeroes of $\zeta(s)$ correspond to the simple poles of $-\frac{\zeta(s)^{\prime}}{\zeta(s)}$ in the critical strip $0<\Re(s)<1$. Without giving the full explanation, in $\S 6.2$ of [8] it is stated that poles of order $\tilde{m}$ at $s$ of a function that is equal to the $(C, g)$-limit of a certain defining partial sum sequence (restricting only to certain classes of Dirichlet series) correspond to the $s$ values for which the partial sums or the $m$-th derivatives of them contain pure logarithmic terms of power $\tilde{m}+m$. Since the von Mangoldt function has quite irregular behaviour, it is uncertain whether the $(C, g)$-method is able to assign
a limit to the defining series (4.3) outside its classical domain of convergence; furthermore it is not known whether the series (4.3) is contained in one of the restricted classes of Dirichlet series. If we assume for the moment this is true, we see that the nontrivial zeroes of $\zeta(s)$ should precisely occur at the $s$-values for which the closed-form expression of the partial sums

$$
(-1)^{m} \sum_{k=1}^{n} \frac{\ln (k)^{m} \Lambda(k)}{k^{s}}
$$

for some $m \in \mathbb{Z}_{\geq 0}$ contain a pure logarithmic term.
Note that this is all highly speculative and there is no guarantee given that the staten assumptions are satisfied, moreover the real difficulty is again to obtain closed-form expressions of the according partial sum sequences. What is interesting, though, is that we see surprising connections between summability theory and other fields of mathematics and hence the investigation of divergent series might not be meaningless at all. Finally it is possible that (an extension of) generalized Cesàro summation might disprove the Riemann Hypothesis; see [9], the author leaves some open questions concerning the rigourness of the article however and moreover the article is not peer-reviewed at the moment of writing this thesis.

### 4.2 Assessing the veracity and applicability

We have seen that we can 'sum' a series such as (1.2) using linear, stable and totally regular methods; a method which is linear and stable complies in great extent with ordinary summation and can thus according to [6] be seen as a quite 'natural' extension of it. Moreover it turns out that, due to their linearity and stability, methods that possess this properties often will assign the same value to the same series. In order to sum a series such as (1.3) and (1.4) we however have to use a summation method which is either nonlinear or nonstable. Since these methods are to certain extent incomparable with ordinary summation, we expect that there is little applicability in assigning values to divergent series using these methods. It turns out to be, surprising enough, that this is generally not the case.

In theoretical physics, (zeta function) regularization is often used in conjuction with so-called renormalization. To give an example: One could come across upon two divergent series (or integrals) which represent energy-levels at unobservable high scales, he/she however only has to determine their difference which ought to be an observable quantity. One can then introduce a so-called cutoff term to make the series/integrals finite, subtract the finite series/integrals and hope to arrive at an (observable) quantity which is in turn independent of the cutoff term. In particular, one could come across upon an infinite determinant/trace of a certain matrix/operator but to know this determinant/trace is in fact not the 'end-goal' (it can indeed merely be seen as an 'immediate' value representing a quantity at unobservable scales and not the 'final' quantity which eventually has to be observed). The determinant/trace can then be 'zeta regularized' and act like the series (4.1) at some specific $s$ and by using (4.2) one could consequently assign the corresponding function-value to this infinite determinant/trace. Since the eventually to be observed quantity must again be independent of the determinant/trace itself, there do not occur mathematical/physical inconsistencies and thus zeta function regularization is in fact a
very powerful method to apply; see also $\S 2.3, \S 4$ and $\S 5$ of [5]. The assigned value of $-\frac{1}{12}$ to (1.4) is for example used within the calculation of the critical dimension 26 of bosonic string theory and one can assign the value of $\zeta(-3)=\frac{1}{120}$ to $\sum_{k=1}^{\infty} k^{3}$ in order to derive the (experimentally verifiable) Casimir effect in 3-dimensional space.

Of course there is zero truth in the statement that e.g. $1-1+1-\cdots$ will tend to $\frac{1}{2}$ and $1+2+3+\cdots$ will tend to $-\frac{1}{12}$. The reason that precisely these values are assigned to the according series is however not completely arbitrary. For this to see, it can make sense to shift our reasoning from the context of real numbers to the context of hyperreal numbers. Within the hyperreal number system, the (smallest) infinity that is larger than any finite quantity is being handled as a well-defined object (we call this object $\omega$ ); note that $\varepsilon:=\frac{1}{\omega}$ represents the infinitesemal object that is smaller than any positive number but larger than 0 . When we substitute $\omega$ for $\infty$ within the calculations with divergent series, these infinite series may be regarded as finite series. According to [4] it now makes sense that linear and stable summation methods assign the value of $\frac{1}{2}$ to (1.2) (by using the fact that ordinary summation is also linear and stable together with 'probabilistic' arguments) and that (1.4) remains unsummable when using a linear and stable method.

Referring to [1] and figure A. 4 however, we see that a 'perturbed' variant of (1.4) does tend to $-\frac{1}{12}$ (in its ordinary sense!); namely

$$
\begin{equation*}
\sum_{k=1}^{\omega} k e^{-\tilde{\varepsilon} k} \cos (\tilde{\varepsilon} k)=-\frac{1}{12} \tag{4.4}
\end{equation*}
$$

where the infinitesemal quantity $\tilde{\varepsilon}$ can accordingly be expressed in terms of $\varepsilon$. If we indeed can solve $\tilde{\varepsilon}$ in terms of $\omega$ (or $\varepsilon$ ), we would arrive at sufficient conditions for which a perturbed variant of (1.4) would be equal to $-\frac{1}{12}$. Unfortunately, despite numerous efforts, we are not able to give a closed-form expression of $\tilde{\varepsilon}$. It is interesting though to see for which values of $\tilde{\varepsilon}$, given $\omega$, the perturbed sum converges to $-\frac{1}{12}$ (see figure A. 5 and the corresponding $\mathbf{R}$-code in appendix B from which it looks like that $20 \varepsilon<\tilde{\varepsilon}<0.5$ ). Furthermore it is interesting that, since $\tilde{\varepsilon}$ can in theory be regarded as an infinitesemal quantity, the terms in $\sum_{k=1}^{\omega} k e^{-\tilde{\varepsilon} k} \cos (\tilde{\varepsilon} k)$ are in the same halo as the terms of $\sum_{k=1}^{\omega} k$ (this means that, in theory, the terms in both series have an arbitrarily small difference since taking the limit to 0 with respect to $\tilde{\varepsilon}$ inside the series yield equal terms). We can thus say that, although both terms represent different hyperreal numbers, the terms are approximately equal in the real number system. Because within physics and other sciences there is often dealt with perturbed series, we might have another good explanation why it 'works' that e.g. the value $-\frac{1}{12}$ can be assigned to (1.4). Of course one has to really know what he/she is doing before he/she assigns a finite value to a divergent series but when one can replace the series (1.4) with (4.4) and knows an appropriate value of $\tilde{\varepsilon}$, the assignment of $-\frac{1}{12}$ is completely valid; for this reason it is also interesting to find out which other terms in the halo of $k$ yield the according finite value of $-\frac{1}{12}$.

It is also interesting to consider the smoothed version of (1.4), namely

$$
\lim _{N \rightarrow \infty} \sum_{k=0}^{\infty} k \eta\left(\frac{k}{N}\right),
$$

where $\eta$ is a so-called cutoff function such that $\eta(0)=1$ and is required to be smooth, bounded and compactly supported. It is fairly easy to deduce the asymptotic properties
of this sum, according to [10] it namely turns out that

$$
\sum_{k=0}^{\infty} k \eta\left(\frac{k}{N}\right)=-\frac{1}{12}+C N^{2}+O\left(\frac{1}{N}\right)
$$

with $C$ being a constant dependent on the choice of $\eta$. One can subsequently interpret this asymptotic expression as a parabola with $y$-intercept $-\frac{1}{12}$ and note that the partial sum function of the smoothed series has this y-intercept as well. More generally, the smoothed sums $\lim _{N \rightarrow \infty} \sum_{k=0}^{\infty} \frac{1}{k^{s}} \eta\left(\frac{k}{N}\right)$ have the asymptotic expansion

$$
\begin{equation*}
\sum_{k=0}^{\infty} \frac{1}{k^{s}} \eta\left(\frac{k}{N}\right)=\zeta(s)+C N^{1-s}+O\left(N^{-B}\right) \sim \zeta(s)+C N^{1-s} \tag{4.5}
\end{equation*}
$$

with $B>0$ and $C$ a constant depending on $\eta$ and $s$. From these observations we see that the values $\zeta(s)$ assigned to the series $\sum_{k=0}^{\infty} \frac{1}{k^{s}}$ (by zeta function regularization as well by $\left(C_{k}, g\right)$-summation) are nothing more than just the constant values of their (smoothed or Euler-Maclaurin) asymptotic expansion and are in the hyperreal setting equal to the infinitely large object obtained by ordinary summation 'modulo' matching infinitely large object(s) (i.e. the eigenvectors of the $C_{k}$ operator in the case of $\left(C_{k}, g\right)$-summation); they can furthermore also be perceived as the y-intercept of the partial sum function corresponding to the smoothed series. Moreover, in [10], we see that the asymptotic expression of the smoothed variant of Grandi's series (1.2) is equal to $\frac{1}{2}+O\left(\frac{1}{N}\right)$ and hence (as expected) agrees with the, through Cesàro methods, assigned value of $\frac{1}{2}$.

Since we now see that the assigned values of a divergent series, by totally regular as well as not totally regular methods, are not completely picked out of thin air, we have a decent answer to subquestion 3 of this thesis: They arise naturally (by linearity/stability as well as analytic continuation) from the values of related convergent series, are equal to the constant term in its (smoothed) asymptotic expansion and can in fact be perceived as the $y$-intercept of the partial sum function obtained by smoothing the corresponding series.

### 4.3 Concluding remarks

We have seen that there is a good reason for applying summation methods to assign a finite value to divergent series of oscillatory nature (in which case we assign the 'average value'). Besides this, particularly in the case of series which tend to infinity, we saw that there is a good reason for assigning the constant term arising in its asymptotic expansion; this especially proves useful within various applications in which there is a need to cancel the arising infinities. In a unified sense, we thus say that summation methods 'extend' the notion of convergence in a simply logical (i.e. linear/stable or analytic) manner although one does always has to ensure that using a particular summation method is appropriate for the given context.

Besides applications to other sciences, we have seen that summability theory can also prove useful within other fields of (pure) mathematics; this is as well by the creation of new mathematical techniques as the obtained insights into deep relationships between seemingly different objects of study. Such a relationship, for example, is that it turns out that the harmonic series (1.1) can be shown (generalized) Cesàro unsummable by
observing the fact that $\zeta(s)$ has a simple pole at $s=1$. The reason that the harmonic series turns out to be difficulty summable can be explained by the fact that it diverges 'too slow', this is also observable when we let $s=1$ in (4.5) from which we see that this series just 'falls in between' ordinary convergent series and divergent series which can be assigned the constant value from its smoothed asymptotic expansion.

The answer to our main question of this thesis is thus that assigning finite values to divergent series gives rise to a whole new mathematical field called summability theory which not only provides us a nice 'tool' in mathematical applications but also gives insight into the nature of divergent series, a better guidance of handling infinite(simal) objects and reveals deep relationships between seemingly different mathematical topics. It is therefore ironically that we can assign an unexpected veracity to the famous quote of Abel that "The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever."; summability theory might in potential, according to [9], namely give us a disproof to the Riemann hypothesis hence the prime numbers would not be as perfectly distributed as one would wish. Nevertheless we have seen that closed-form expressions of the sequences of partial sums

$$
\begin{equation*}
\sum_{k=1}^{n} \frac{\cos (t \ln (k))}{k^{\sigma}}, \sum_{k=1}^{n} \frac{\sin (t \ln (k))}{k^{\sigma}} \tag{4.6}
\end{equation*}
$$

are potentially be able to give a proof of the Riemann hypothesis in which it also possibly reveals deep connections between the respective theories of divergent series and prime numbers. Due to these obervations that, in the end, all turns out to be connected we can thus say that Dirac's statement "God is a very skilled mathematician" bears a lot of significance. To however obtain the, seemingly impossible, closed-form expressions of (4.6) from which we might prove the Riemann hypothesis, could require stuff even God does not know its existence of.

## A Numerical observations: Plots



Figure A.1: Plots of $\left(\sum_{k=1}^{n} \frac{\cos (t \ln (k))}{k^{\sigma}}\right)$ for several values of $\sigma$ with $t=0$.


Figure A.2: Plots of $\left(\sum_{k=1}^{n} \frac{\cos (t \ln (k))}{k^{\sigma}}\right)$ for several values of $\sigma$ with $t \approx 14.1$ (the middle one corresponds to a nontrivial zero of $\zeta(s)$ ).


Figure A.3: Plots of $\left(\sum_{k=1}^{n} \frac{\cos (t \ln (k))}{k^{\sigma}}\right)$ for several values of $\sigma$ with $t=25$.

The convergence of the 'perturbed' Riemann zeta function evaluated at $\mathbf{s}=\mathbf{- 1}$


Figure A.4: Plot of the convergence of $\sum_{k=1}^{\omega} k e^{-\tilde{\varepsilon} k} \cos (\tilde{\varepsilon} n k)$ to $-\frac{1}{12}$ when taking sufficient large $\omega=100000$ and small $\tilde{\varepsilon}=0.001$.

The value of the 'perturbed' Riemann zeta function at $\mathbf{s = - 1}$ for different values of epsilon


The value of the 'perturbed' Riemann zeta function at $\mathbf{s}=\mathbf{- 1}$ for different values of epsilon


Figure A.5: The convergence values of $\sum_{k=1}^{\omega} k e^{-\tilde{\varepsilon} k} \cos (\tilde{\varepsilon} k)$ as a function of $\tilde{\varepsilon}$ with $\omega=$ 10000 and $\omega=100000$ respectively.

## B Numerical observations: Code

$\mathbf{R}$-code to reproduce the plots of the partial sum sequences $\left(\sum_{k=1}^{n} \frac{\cos (t \ln (k))}{k^{\sigma}}\right)$ :

```
old.par = par (mfrow =c (3, 1))
N = 10000
sigmavector = c (0.2,0.5,0.99)
tvector = c(0,14.134725142,25)
t = tvector[3] # Can also be set to 1 or 2
for(sigma in sigmavector){
partialsum = vector()
partialsum [1] = cos (t*log(1))
for(i in 2:N){
    partialsum [i] = partialsum [i - 1] +cos(t*log(i))/(i^(sigma))}
plot(partialsum, type = "l', xlab = "n", ylab = "Partial sum value",main =
    paste("sigma =",sigma,", t =",t))}
par(old.par)
```

$\mathbf{R}$-code to evaluate the convergence of $\sum_{k=1}^{\omega} k e^{-\tilde{\varepsilon} k} \cos (\tilde{\varepsilon} k)$ for sufficient large $\omega$ and small ह:

```
omega = 40000
eps = 0.001
a_i = seq(from=1,to=omega, length.out = omega)
perturb = exp(-eps*a_i)*\operatorname{cos}(eps*\mp@subsup{a}{_}{}i)
a_i_perturb = a_i * perturb
partialsum = vector(length = omega)
partialsum [1] = a_i_perturb [1]
for(i in seq(from = 2, to =omega)){
    partialsum[i] = partialsum[i-1] + a_i_perturb[i]}
plot(partialsum, type = "l", main = "The convergence of the 'perturbed'
    Riemann zeta function evaluated at s=-1", xlab = "Number of terms in the
    series", ylab = "Partial sum value")
partialsum [omega]
```

R-code to evaluate, for several magnitudes of $\omega$, the convergence values of $\sum_{k=1}^{\omega} k e^{-\tilde{\varepsilon} k} \cos (\tilde{\varepsilon} k)$ as a function of $\tilde{\varepsilon}$ :

```
omega = 100000 # Can also be set to 10000 or anything else
length=1000
epsvector = seq(from = 0.0002, to = 10, length.out = length) # If omega
    =10000 set the left-bound to 0.002
perturbsum = vector(length = length)
a_i}=\operatorname{seq}(\mathrm{ from = 1,to=omega, length.out = omega)
for(i in seq(from = 1, to = length)){
perturb = exp(-epsvector[i]*a_i)*\operatorname{cos}(epsvector[i]*a_i)
a_i_perturb = a_i * perturb
perturbsum[i] = sum(a_i_perturb)}
plot(epsvector, perturbsum, type = "l", main = "The value of the 'perturbed'
    Riemann zeta function at s=-1 for different values of epsilon", xlab = "
    Epsilon", ylab = "Convergence value")
```


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[^0]:    ${ }^{1}$ Note that there exists a one-to-one correspondence between the elements of $\mathcal{S}$ and $\mathcal{A}$, we namely have that $a_{0}=s_{0}$ and $a_{k}=s_{k}-s_{k-1}, k \in \mathbb{N}^{+}$.

[^1]:    ${ }^{1}$ i.e. $f(n) \sim g(n)$ if and only if $\lim _{n \rightarrow \infty} \frac{f(n)}{g(n)}=1$.

[^2]:    ${ }^{2}$ For the linearity and stability we refer to $\S 5.15$ in [3] and for the total regularity one can verify the conditions stated in theorems 6 and 11 of [3].
    ${ }^{3}$ The proof relies on commutativity properties of polynomials and can be found in lemma 3 of [8].

