university of groningen

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Snap-back repellers in one- and two-dimensional maps

Abstract
In this thesis we will study snap-back repellers in one- and two dimensional maps. A repelling fixed point is a snap-back repeller if a point unequal to the fixed point in a repelling neighborhood of the fixed points returns to the fixed point after a finite number of iterations. The main reason to study snap-back repellers is to show chaos. Marotto has proven that a dynamical system is chaotic if a fixed point is a snap-back repeller. We will discuss some examples of one-dimensional maps and we will go to two-dimensional maps using a rotation. We also did numerical computations to find snap-back repellers in two-dimensional maps, for example the logistic map with a rotation, to show how snap-back repellers can be found.

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Snap back to reality

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1 Introduction

In the study of dynamical systems, mathematicians are trying to get a grip on chaos. To come up with a proof of chaos in certain dynamical systems can be tricky and sometimes involves a lot of computing or numerical computing power. Also there is a variety of definitions of chaos given by mathematicians. The most well known and most used definition is the one by R.L. Devaney[1]. He came up with three requirements which a system has to satisfy in order to call the system chaotic:

Definition 1. *Let $J \subseteq \mathbb{R}^n$ be a set. $F : J \to J$ is said to be chaotic in the sense of Devaney on J if:*

1. *F has sensitive dependence on initial conditions; that is, there exists $\delta > 0$ such that, for any $x \in J$ and any neigborhood N of x there exists $y \in N$ and $n \geq 0$ such that*

$$||F^n(x) - F^n(y)|| > \delta,$$

2. *F is topologically transitive; that is, for any pair of open sets $U, V \subset J$ there exists $k > 0$ such that $F^k(U) \cap V \neq \emptyset$,*
3. *periodic points are dense in J.*

Devaney's definition of chaos is used often, but it is hard to mathematically prove chaos for some dynamical systems in the sense of Devaney. An often used method is for example calculating the Lyapunov exponent of a system, which is an easy tool to prove chaos, but it needs sometimes a lot of computing time. We won't go further into detail of other methods for proving Devaney chaos. In this paper we will look at the definition of chaos by Marotto instead. Marotto gave the following definition:

Definition 2. *[9] Let $F : \mathbb{R}^n \to \mathbb{R}^n$. $X_{n+1} = F(X_n)$ is chaotic in the sense of Marotto if there exists:*

1. *a positive integer N such that for each integer $p \geq N$, F has a point of period p;*
2. *a "scrambled set" of F, i.e. an uncountable set S containing no periodic points of F such that:*
   (a) *$F[S] \subset S$,*
   (b) *for every $X, Y \in S$ with $X \neq Y$*

$$\limsup_{k\to\infty} ||F^k(X) - F^k(Y)|| > 0,$$

   (c) *for every $X \in S$ and any periodic point Y of F*

$$\limsup_{k\to\infty} ||F^k(X) - F^k(Y)|| > 0;$$

3. *an uncountable subset S_0 of S such that for every $X, Y \in S_0$:*

$$\liminf_{k\to\infty} ||F^k(X) - F^k(Y)|| = 0.$$

This definition still seems hard to verify, but Marotto stated a theorem, which will help us with determining whether a system is chaotic. Before we will state the theorem we need to give the definition of a snap-back repeller:

Definition 3. *[9] Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map. A fixed point p of F is called a snap-back repeller if the following two conditions are satisfied:*

1. *the fixed point p is expanding, which means that there exists an $r > 0$ such that the eigenvalues of $DF(x)$ exceed 1 in absolute value for all $x \in B_r(p)$,*
2. *there exists a point $x_0 \in B_r(p)$ with $x_0 \neq p$ and $m \in \mathbb{N}$ such that $x_m = p$ and $det(DF(x_k)) \neq 0$ for all $1 \leq k \leq m$ where $x_k = F^k(x_0)$.*

Notice that the map needs to be non-invertible, in order to have a fixed point that is a snap-back repeller. This because there should be a point $x_{m-1} \neq p$, where p is the fixed point, and $f(x_{m-1}) = x_m = p$. Now we can state the theorem given by Marotto [9]:

Theorem 1. *If $F : \mathbb{R}^n \to \mathbb{R}^n$ possesses a snap-back repeller then $X_{n+1} = F(X_n)$ is chaotic in the sense of Marotto.*

For the proof of theorem 1 we refer to [9]. The definitions of chaos given by Marotto and Devaney seem very different and since the Devaney definition is the most used, we need to know how these definitions are similar. In article [12] a theorem is given on whether a snap-back repeller implies chaos in the sense of Devaney, but before stating the theorem we need to provide the following definition:

Definition 4. *Let $U, V \subset \mathbb{R}^n$. A map $H : U \to V$ is a topological conjugacy from $F : U \to U$ to $G : V \to V$ if:*

1. *H is a homeomorphism from U to V (in other words H is a bijection and both H and H^{-1} are continuous),*
2. *$H \circ F = G \circ H$, or equivalently $G = H \circ F \circ H^{-1}$, or $F = H^{-1} \circ G \circ H$.*

Now we can state the theorem given in [12]:

Theorem 2. *If f satisfies all the assumptions of definition 3, then, for each neigborhood U of z, there exist a positive integer $k > m$ and a Cantor set $\Lambda \subset U$ such that $f^k : \Lambda \to \Lambda$ is topologically conjugate to the one-sided symbolic dynamical system $\sigma : \Sigma_2^+ \to \Sigma_2^+$. Consequently, f^k is chaotic on Λ in the sense of Devaney.*

The proof of this theorem is given in [11]. This last theorem shows that if we can find a snap-back repeller we have chaos in the sense of Marotto by Marotto's theorem, but it also implies chaos in the sense of Devaney for non-invertible maps. So snap-back repellers are a very usefull tool in finding chaos in non-invertible maps. In this thesis we will answer the question:

How can a snap-back repeller be found, given a one- or two-dimensional map?

We will show how this can be done analytically for simple maps and numerically for more difficult maps. Although Marotto's theorem was found already in 1978, the study on snap-back repellers is still very relevant and there is a lot of recent research on this topic, for example [2, 5, 7, 8] in which the dynamics of different continuous maps is studied and snap-back repellers are used to show chaos, and the relations with other definitions of chaos are proven. In [3, 4] the theory of snap-back repellers is even extended to non-continuous maps.

2 Preliminaries

In this section we introduce the basic mathematical context and notation which is used in this thesis. The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. The n'th iterate of a map $F : \mathbb{R}^n \to \mathbb{R}^n$ of a point x is denoted by $x_n = F^n(x)$. A fixed point of a map F is a point x which maps back to itself under iteration of the map, i.e. $F(x) = x$. A periodic point of period n is a point x which maps back to itself under n iterations of the map, i.e. $F^n(x) = x$ and $F^m(x) \neq x$ for all $0 < m < n$. A sequence or trajectory of a map is the a sequence of points $x_0, x_1 = F(x_0), x_2 = F^2(x_0), \ldots, x_n = F^n(x_0)$ and is obtained by iterating the map F on a point x_0. The inverse or pre-image of a map F is given as follows. Let $F : X \to Y$. The pre-image of a set $B \subseteq Y$ under F is the subset of X defined by

$$F^{-1}(B) = \{x \in X | F(x) \in B\}.$$

A non-invertible k-to-1 map can be devided into k invertible parts. We will call the inverses of this k invertible parts the partial inverses of the map. The A differentiable function is a function of which the derivative exists in each point of its domain. The Euclidean norm is the norm that gives length of a vector $x = (x_1, x_2, \ldots, x_n)$ by the formula

$$||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

and gives the ordinary distance from the origin to the point X in the n-dimensional Euclidean space.

Given two normed vector spaces V and W, a linear map $A : V \to W$ is continuous if and only if there exist a real number c such that

$$||Av|| \leq c||v|| \qquad \text{for all } v \in V.$$

The induced operator norm is given by

$$||A||_{op} = \inf\{c \geq 0 : ||Av|| \leq c||v|| \text{ for all } v \in V\},$$

where inf denotes the infimum of the set. In this thesis we will use small letters to denote one-dimensional functions, whereas capital letters denote multi-dimensional functions.

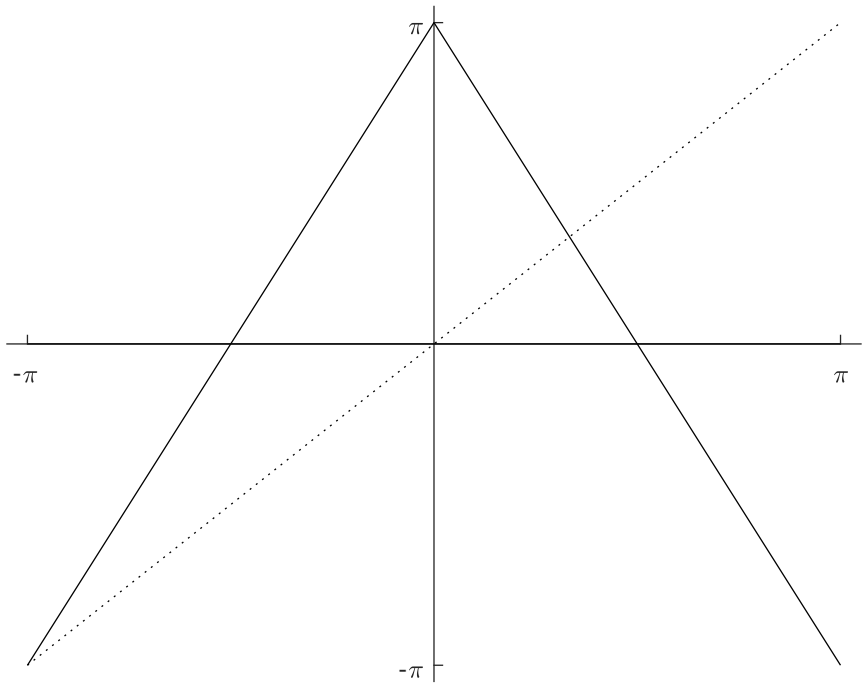


Figure 1 – The tent map from $-\pi$ to π.

3 One-dimensional maps

In this chapter we will discuss a few one-dimensional examples with fixed points that are snap-back repellers.

3.1 Tent map

We will start with an example of a one-dimensional map; the tent map (figure 1). The tent map $f : [-\pi, \pi] \to [-\pi, \pi]$ is a 2 to 1, differentiable map. Note that the map is not invertible and we observe that a snap-back repeller can only occur in non-invertible maps.

The tent map is defined as:

$$f(x) = \begin{cases} 2x + \pi & -\pi \leq x \leq 0 \\ -2x + \pi & 0 < x \leq \pi \end{cases} \tag{1}$$

A fixed point p can be found at $p = \frac{\pi}{3}$. We note that the tent map f maps two points onto p, namely p itself and $-p$. Suppose we take $x_0 = \frac{5\pi}{12}$ and $r = \frac{\pi}{8}$, where x_0 and r are

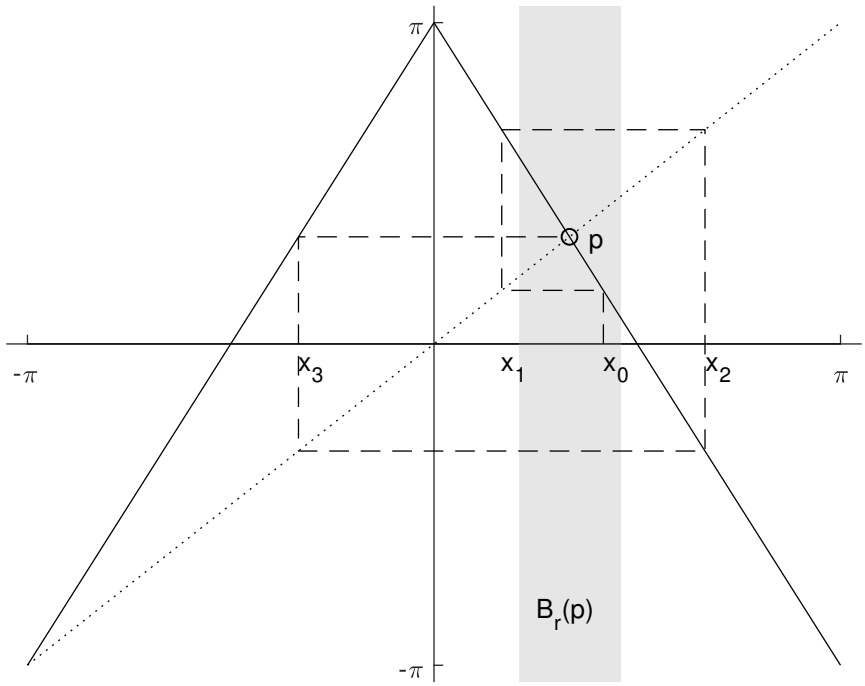


Figure 2 – The tent map from $-\pi$ to π with the trajecory from $x_0 = \frac{5\pi}{12}$ to $x_4 = p$ and $B_r(p) = B_{\frac{\pi}{8}}(\frac{\pi}{3})$ indicated as the gray area.

defined as in the definition of a snap-back repeller. Then, $B_r(p)$ is the interval $(\frac{5\pi}{24}, \frac{11\pi}{24})$. We notice that the derivative of f equals 2 in absolute value for all x, so condition 1 in the definition is verified.

For the verification of the second condition we will iterate the map from x_0 untill we reach the fixed point (figure 2);

$$
\begin{aligned}
x_1 &= -2x_0 + \pi = -2\frac{5\pi}{12} + \pi = \frac{\pi}{6} \\
x_2 &= -2x_1 + \pi = -2\frac{\pi}{6} + \pi = \frac{2\pi}{3} \\
x_3 &= -2x_2 + \pi = -2\frac{2\pi}{3} + \pi = \frac{-\pi}{3} \\
x_4 &= 2x_3 + \pi = 2\frac{-\pi}{3} + \pi = \frac{\pi}{3} = p.
\end{aligned}
$$

Hence, the fixed point in the tent map is a snap-back repeller. Notice that r is not unique. For example chosing $r = \frac{\pi}{4}$ and $x_0 = \frac{\pi}{6}$ would also have been sufficient to show there is a

snap-back repeller in the tent map.

In order to show the relationship between a snap-back repeller and chaos we will verify the first condition of Marotto's theorem; *There exists a positive integer N such that for each integer $p \geq N$, F has a point of period p*, by considering the tent map. We know already that the fixed point of the tent map is a fixed point.

We can find periodic points for the tent map by solving the equation $f^n(q) = q$ where q is a periodic point with period n. We can write the n-th iterate of the tent map as follows:

$$f^n(x) = \begin{cases} 2^n x + \pi(2^n - 1 - 4i) & x \in [\pi(\frac{2i}{2^{n-1}} - 1), \pi(\frac{(2i+1)}{2^{n-1}} - 1)] \\ -2^n x + \pi(3 + 4i - 2^n) & x \in [\pi(\frac{(2i+1)}{2^{n-1}} - 1), \pi(\frac{(2i+2)}{2^{n-1}} - 1)] \end{cases} \tag{2}$$

for $i = 0, 1, ..., 2^{n-1} - 1$.

This will give us two n-periodic points at

$$q_i^+ = \frac{4i + 1 - 2^n}{2^n - 1}\pi$$

and

$$q_i^- = \frac{3 + 4i - 2^n}{1 + 2^n}\pi$$

for every interval of size $\frac{2\pi}{2^{n-1}}$ and $i = 0, 1, ..., 2^{n-1} - 1$. Since $q_i^+ = q_i^- \Leftrightarrow n = 0$ and furthermore

$$\sum_{k=1}^{n-1} 2^k = 2^n - 2,$$

we can conclude there are at least two n-periodic points in the interval $[-\pi, \pi]$ for all $n \in \mathbb{N}$. So taking $N = 1$ will give the desired result.

To verify the second and third point we need to find a scrambled set as defined in Marotto's theorem. Although Marotto has proven his theorem, it is a hard task to do so. Therefore, we will continue with other examples and won't go into detail on this part of the theorem.

3.2 Tripling map

Another example would be the tripling map (figure 3), which is a 3 to 1 differentiable map $f : [0, 1] \to [0, 1]$ and it can be written as:

$$f(x) = 3x \mod 1 \tag{3}$$

Note that the derivative is equal to 3 for all x and there is a fixed point $p = \frac{1}{2}$. To show that the fixed point is a snap-back repeller we choose $r = \frac{1}{12}$ and $x_0 = \frac{1}{2} \pm \frac{1}{27}$ (see figure

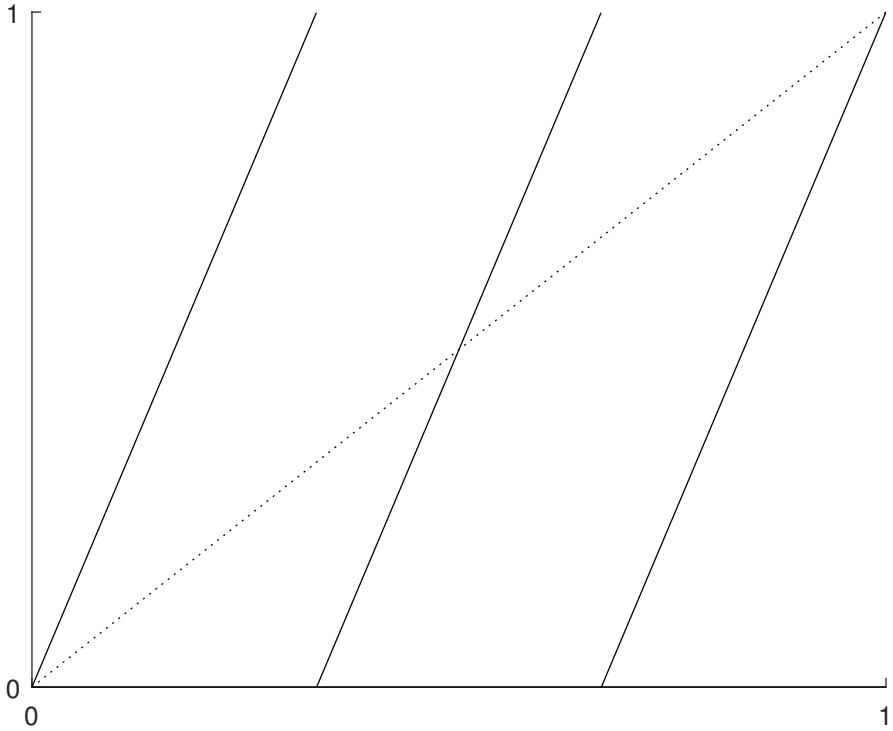


Figure 3 – The tripling map from 0 to 1

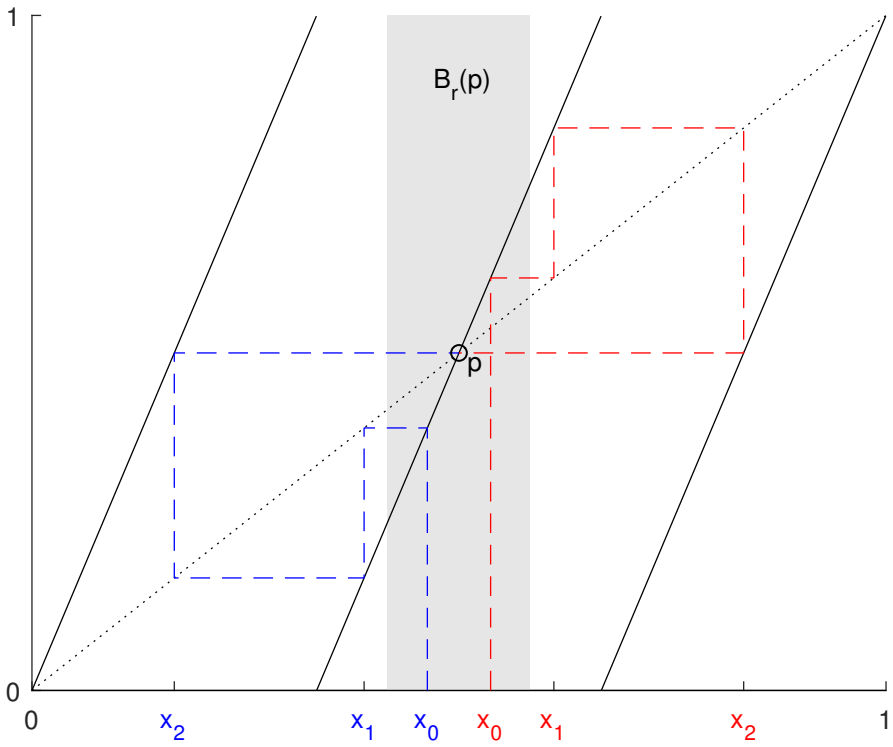


Figure 4 – The tripling map from $-\pi$ to π with the trajecory from $x_0 = \frac{25}{54}$ to $x_3 = p$ in blue and $x_0 = \frac{29}{54}$ to $x_3 = p$ in red. $B_r(p) = B_{\frac{1}{12}}(\frac{1}{2})$ is indicated as the gray area.

4). This implies

$$\begin{aligned} x_1 &= 3x_0 \mod 1 = \frac{1}{2} \pm \frac{1}{9} \\ x_2 &= 3x_1 \mod 1 = \frac{1}{2} \pm \frac{1}{3} \\ x_3 &= 3x_2 \mod 1 = \frac{1}{2} = p \end{aligned}$$

Hence, the fixed point $p = \frac{1}{2}$ of the tripling map is a snap-back repeller.

4 Rotations of one-dimensional maps

In this section we will discuss a specific class of two-dimensional maps and snap-back repellers in these maps. Similar maps have been studied in [5]. They studied three non-inverible planar maps, which twist and fold the plane. They have shown the dynamics of these maps and discussed snap-back repellers in these maps as well. We first define the map

$$G : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} f(x) \\ y \end{pmatrix},$$

where $f : \mathbb{R} \to \mathbb{R}$ is a differentiable non-invertible map.

Next, we consider a rigid rotation around the origin:

$$R_\phi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \end{pmatrix}.$$

We will investigate the composition

$$F_\phi = R_\phi \circ G : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} f(x)\cos\phi - y\sin\phi \\ f(x)\sin\phi + y\cos\phi \end{pmatrix}. \tag{4}$$

4.1 Rotation of $\phi = \frac{\pi}{2}$

First we will choose the rotation angle $\phi = \frac{\pi}{2}$.

Propostion 1. *If $f : \mathbb{R} \to \mathbb{R}$ is a symmetric differentiable map, i.e. $f(-x) = f(x)$ and the fixed point p is a snap-back repeller. Then the fixed point of $F_{\frac{\pi}{2}} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y \\ f(x) \end{pmatrix}$ is a snap-back repeller.*

Proof. For convenience we will write $F_{\frac{\pi}{2}}$ as F during this proof. First, we note that since f is symmetric, F has the fixed point $\bar{p} = \begin{pmatrix} -p \\ p \end{pmatrix}$.

We know that the fixed point p of $f(x)$ is a snap-back repeller. Therefore, there exists an $r > 0$ such that $|f'(x)| > 1$ for all $x \in B_r(p)$. Note that $DF(X) = \begin{pmatrix} 0 & -1 \\ f'(x) & 0 \end{pmatrix}$, which has $\sqrt{-f'(x)}$ as eigenvalue of order 2. The eigenvalue is larger than 1 in absolute value as long as $f'(x)$ is larger than 1 in absolute value. This is the case since $f'(-x) = -f'(x)$. And hence the fixed point $\bar{p}$ is repelling.

For the second part of the proof we know, from p being a snap-back repeller, that there exists a point $\bar{x}_0 \in B_r(p)$ with $\bar{x}_0 \neq p$ and $m \in \mathbb{N}$ such that $\bar{x}_m = p$ and $f'(\bar{x}_k) \neq 0$, for all $1 \leq k \leq m$ where $\bar{x}_k = f^k(\bar{x}_0)$.

First we notice that, since f is symmetric, we can write

$$F^{2k}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -f^k(x) \\ f^k(y) \end{pmatrix} \tag{5}$$

and

$$F^{2k-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -f^{k-1}(y) \\ f^k(x) \end{pmatrix}. \tag{6}$$

Taking $X_0 = \begin{pmatrix} -\bar{x}_0 \\ \bar{x}_0 \end{pmatrix} \in B_r(\bar{p})$ gives us from equation (5) that

$$F^{2m}(X_0) = \begin{pmatrix} -f^k(\bar{x}_0) \\ f^k(\bar{x}_0) \end{pmatrix} = \begin{pmatrix} -\bar{x}_m \\ \bar{x}_m \end{pmatrix} = \begin{pmatrix} -p \\ p \end{pmatrix}.$$

The last thing we need to verify is that $\det DF^k(X_0) \neq 0$ for all $1 \leq k \leq 2m$ and indeed:

$$\det DF^{2k}(X_0) = -(f'(x_k))^2 \neq 0$$

and

$$\det DF^{2k-1}(X_0) = f'(x_{k-1})f'(x_k) \neq 0$$

for all $1 \leq k \leq m$. And hence the fixed point $\bar{p}$ of F is a snapback repeller. □

A logical next step would be to prove the case that $f : \mathbb{R} \to \mathbb{R}$ is anti-symmetric, i.e. $f(-x) = -f(x)$. First we observe that any continuous antisymmetric function has a fixed point at $p = 0$. We can state the following proposition:

Propostion 2. *If $f : \mathbb{R} \to \mathbb{R}$ is a anti-symmetric differentiable map and the fixed point $p = 0$ is a snap-back repeller. Then, the fixed point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of $F_{\frac{\pi}{2}} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y \\ f(x) \end{pmatrix}$, is a snap-back repeller.*

Proof. Since f is anti-symmetric we can state the following results for f and F:

1. $f'(-x) = f'(x)$,
2. $F^{2k-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (-1)^k f^{k-1}(y) \\ (-1)^{k-1} f^k(x) \end{pmatrix}$,
3. $F^{2k}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (-1)^k f^k(x) \\ (-1)^k f^k(y) \end{pmatrix}$.

The rest of the proof is similar to the proof of proposition 1. □

We notice that in this case the point $\begin{pmatrix} \pm p \\ \pm p \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a fixed point of F^4 where p is the fixed point of the anti-symmetric $f(x)$. So there exist a subsequence $\begin{pmatrix} x_0 \\ x_0 \end{pmatrix}, \begin{pmatrix} x_2 \\ x_2 \end{pmatrix}, \ldots, \begin{pmatrix} p \\ p \end{pmatrix}$ which provides that the fixed point $\begin{pmatrix} \pm p \\ \pm p \end{pmatrix}$ of F^4 is a snap-back repeller. Notice that

$$\left| DF^4 \begin{pmatrix} x_k \\ x_k \end{pmatrix} \right| = 2f'(f(x_k))f'(x_k) \neq 0$$

since $f'(x_k) \neq 0$ for all $1 \leq k \leq m$, where $x_m = p$.

Propostion 3. *If $f : \mathbb{R} \to \mathbb{R}$ is a anti-symmetric differentiable map and the fixed point $p \neq 0$ is a snap-back repeller. Then, the fixed point $\begin{pmatrix} \pm p \\ \pm p \end{pmatrix}$ of*

$$F^4 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f^2(x) \\ f^2(y) \end{pmatrix}$$

is a snapback repeller provided that $|f'(f(x))f'(x)| > 1$ and $|f'(f(y))f'(y)| > 1$ for all $(x, y) \in B_r(p)$. F is chosen as before.

Proof. The proof can be completed by the arguments above. □

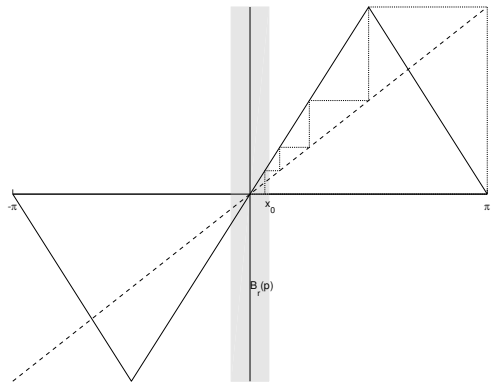
It seems logical to use the fact that every function can be written as the sum of a symmetric and an anti-symmetric function. Although we have proven proposition 1 and 2, it is not necessarily true that we can prove a similar statement for any function $f(x)$. This is because the sequence $X_0, F(X_0), \ldots, F^m(X_0)$ was easy to find when the sequence in the one-dimensional case was known. For the case that $f(x)$ is not symmetric nor anti-symmetric, we can start with $X_0 = \begin{pmatrix} x_0 \\ x_0 \end{pmatrix}$ and then $F(X_0) = \begin{pmatrix} -x_0 \\ x_1 \end{pmatrix}$. However, the next iterate gives us:

$$F^2(X_0) = \begin{pmatrix} -x_1 \\ x_1 - h(x_0) \end{pmatrix}$$

where $h(x) = \frac{1}{2}(f(x) - f(-x))$ is an antisymmetric function and $h(x_0) \neq 0$. So this gives a very different sequence. Note: a sequence $X_0, \ldots, X_m$ may exist so that the fixed point of F is a snap-back repeller, but we can not prove this by the assumption that f has a snap-back repeller.

We will discuss some examples of anti-symmetric functions and the snap-back repellers that can be found. First of all we consider a variation of the tent map (figure 5):

$$f(x) = \begin{cases} -2x - 2\pi & -\pi \leq x \leq -\frac{\pi}{2} \\ 2x & -\frac{\pi}{2} < x \leq \frac{\pi}{2} \\ -2x + 2\pi & \frac{\pi}{2} < x \leq \pi \end{cases} \tag{7}$$



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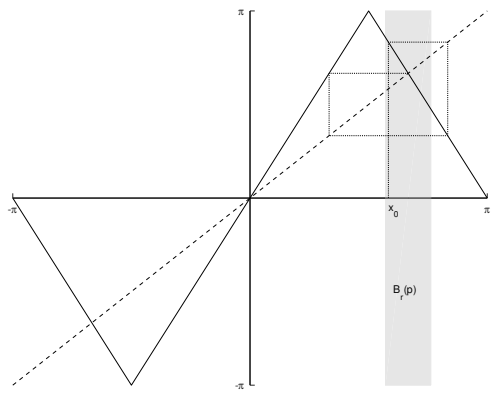


(b)

Figure 5 – An anti-symmetric variation of the tent map. Fixed points can be found at $x = 0$ and $x = \pm\frac{2\pi}{3}$. In (a) the sequence is given and in (b) the sequence is given to show $x = \frac{2\pi}{3}$ is a snap-back repeller.

Since this map is anti-symmetric it has a fixed point at $x = 0$, the other fixed points can be found at $x = \pm\frac{2\pi}{3}$. Note that in the sequence to $p = 0$ there is a point x_i which is not differentiable. But from article [2] we know that the fixed point is still a snap-back repeller. Now, if we rotate this map by $\pi/2$ as we did before we obtain the two-dimensional map $F_{\frac{\pi}{2}}$. The point $(0, 0)$ is indeed a fixed point of $F_{\frac{\pi}{2}}$ and by proposition 2 also a snap-back repeller. On the other hand, the points $(\pm\frac{2\pi}{3}, \pm\frac{2\pi}{3})$ are not fixed points of $F_{\frac{\pi}{2}}$, but these points are periodic points of period 4. We need to verify the fact that the eigenvalues of $|DF^4|$ are larger than 1. The eigenvalues of $|DF^4|$ are given by:

$$\lambda_1 = f'(f(x))f'(x) \text{ and } \lambda_2 = f'(f(y))f'(y)$$

and since $|f'(x)| = 2$ for all $x \in [-\pi, \pi]$ the eigenvalues are larger than 1. Therefore, the points $(\pm\frac{2\pi}{3}, \pm\frac{2\pi}{3})$ are snap-back repellers.

Another example for an anti-symmetric function would be $f(x) = \pi \sin x$. Obviously it has a fixed point at $x = 0$, but this fixed point is not a snap-back repeller because $x_{m-2} = \pm\pi/2$. At this point the derivative of f is equal to zero. The other fixed points are not easy to find by hand, but by using the Newton-Raphson method, we can still find the other fixed points (figure 6). As can be seen in this figure, the fixed point is a snap-back repeller. $B_r(p)$ is chosen in such a way that the derivative of $f^2(x)$ is also larger than 1 in absolute value. So by proposition 3, the fixed point is also snap-back repeller for the two-dimensional function F^4 rotated by an angle $\pi/2$.

The sequence is found using an Octave-script. Given any radius around the fixed point, the script iterates until a point x_0 is found in the interval. One needs to separate the one-to-one parts of the map. Then identify in which part the fixed point lies. After this, it iterates back one of the other parts where the point $x_{m-1} \neq p$ lies where $f(x_{m-1}) = p$ and then iterate back to the part with the fixed point until the interval is reached using the local inverse of the function. The script is given in appendix A.1.

4.2 Other rotation angles

In this section we are going investigate other rotation angles ϕ. As we came up with some propositions in the previous section for a rotation angle $\phi = \frac{\pi}{2}$, we will come with some results which hold for other rotation angles. For any choice of ϕ the derivative of F_ϕ is

$$DF(x) = \begin{pmatrix} f'(x)\cos\phi & -\sin\phi \\ f'(x)\sin\phi & \cos\phi \end{pmatrix}$$

and hence the determinant equals $f'(x)$ for all angles ϕ.

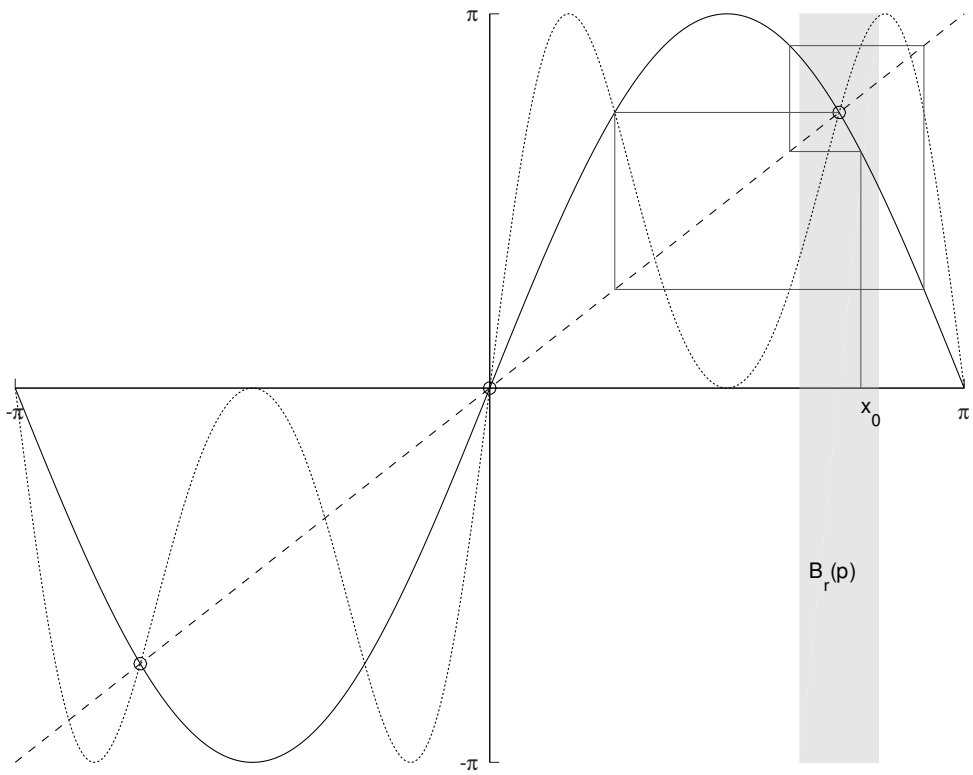


Figure 6 – The map $f(x) = \pi \sin x$ on the interval $[-\pi, \pi]$. The fixed points are found using the Newton-Raphson method. Also the graph of $f^2(x)$ is shown. The sequence is given to show that the fixed point is a snap-back repeller.

The eigenvalues of $DF(x)$ are equal to

$$\lambda_+ = \frac{1}{2}((f'(x)+1)\cos\phi + \sqrt{(f'(x)+1)^2\cos^2\phi - 4f'(x)})$$
$$\lambda_- = \frac{1}{2}((f'(x)+1)\cos\phi - \sqrt{(f'(x)+1)^2\cos^2\phi - 4f'(x)}).$$

These eigenvalues may be smaller than one in absolute value for some cases of f. Therefore we will continue with specific examples of one-dimensional functions.

For example, we consider the tent map:

$$F_\phi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} (-2|x|+\pi)\cos\phi - y\sin\phi \\ (-2|x|+\pi)\sin\phi + y\cos\phi \end{pmatrix}.$$

Two fixed points can be found. For $x < 0$ we can find a fixed point at $\begin{pmatrix} -\pi/3 \\ \frac{\pi}{3}\frac{\sin\phi}{1-\cos\phi} \end{pmatrix}$ and for $x > 0$ we can find a fixed point at $\begin{pmatrix} \pi \\ \frac{\pi\sin\phi}{\cos\phi-1} \end{pmatrix}$. In both cases ϕ must be unequal to $k\pi$ for $k \in \mathbb{Z}$. Notice that the determinant of DF is nowhere zero.

For $x < 0$ the eigenvalues of DF are equal to:

$$\lambda_+ = \frac{1}{2}(3\cos\phi + \sqrt{9\cos^2\phi - 8})$$
$$\lambda_- = \frac{1}{2}(3\cos\phi - \sqrt{9\cos^2\phi - 8})$$

and for $x > 0$ the eigenvalues of DF are equal to:

$$\lambda_+ = \frac{1}{2}(-\cos\phi + \sqrt{\cos^2\phi + 8})$$
$$\lambda_- = \frac{1}{2}(-\cos\phi - \sqrt{\cos^2\phi + 8}).$$

The absolute value of the eigenvalues are given in figure 7. We notice that for any rotation (except for $\phi = k\pi$ for $k \in \mathbb{Z}$) the eigenvalues are larger than one in absolute value. We choose $r = \frac{2\pi}{3}$ and $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \frac{\pi}{3}\begin{pmatrix} \cos\phi - 1 \\ \sin\phi\frac{2\cos\phi-1}{1-\cos\phi} \end{pmatrix}$. The fixed point $p = \begin{pmatrix} -\pi/3 \\ \frac{\pi}{3}\frac{\sin\phi}{1-\cos\phi} \end{pmatrix}$ is a snap-back repeller, namely:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{\pi}{3}\begin{pmatrix} 1 \\ \frac{\sin\phi}{1-\cos\phi} \end{pmatrix}$$
$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{\pi}{3}\begin{pmatrix} -1 \\ \frac{\sin\phi}{1-\cos\phi} \end{pmatrix}$$
$$= \begin{pmatrix} x_p \\ y_p \end{pmatrix}$$

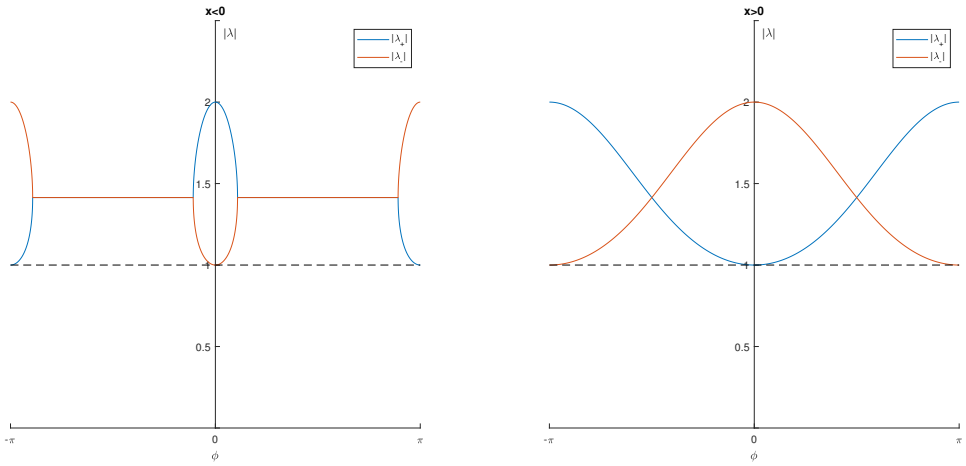


Figure 7 – The absolute value of the eigenvalues of DF_ϕ where f is the tent map, for any rotation ϕ.

To verify that $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in B_{\frac{2\pi}{3}} \begin{pmatrix} x_p \\ y_p \end{pmatrix}$ we need to calculate the distance between the two points:

$$\begin{aligned} d\left(\begin{pmatrix} x_p \\ y_p \end{pmatrix}, \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right) &= \sqrt{(\frac{\pi}{3}(\cos\phi - 1) + \frac{\pi}{3})^2 + (\frac{\pi}{3}\sin\phi\frac{2\cos\phi - 1}{1-\cos\phi} - \frac{\pi}{3}\frac{\sin\phi}{1-\cos\phi})^2} \\ &= \sqrt{(\frac{\pi}{3}\cos\phi)^2 + (\frac{2\pi}{3}\sin\phi)^2} \\ &= \frac{\pi}{3}\sqrt{1 + 3\sin\phi^2} \leq \frac{2\pi}{3} \end{aligned}$$

And hence the fixed point $\begin{pmatrix} x_p \\ y_p \end{pmatrix}$ of F_ϕ is a snap-back repeller.

Propostion 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be the tent map given by $f(x) = -2|x| + \pi$. Then the fixed point p of*

$$F_\phi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} f(x)\cos\phi - y\sin\phi \\ f(x)\sin\phi + y\cos\phi \end{pmatrix}$$

is a snap-back repeller.

Proof. The proof can be completed by using the arguments above. □

4.3 The logistic map with a rigid rotation

Another example would be the logistic map. We consider the logistic map as

$$L : x \mapsto -2x^2 + 1 \tag{8}$$

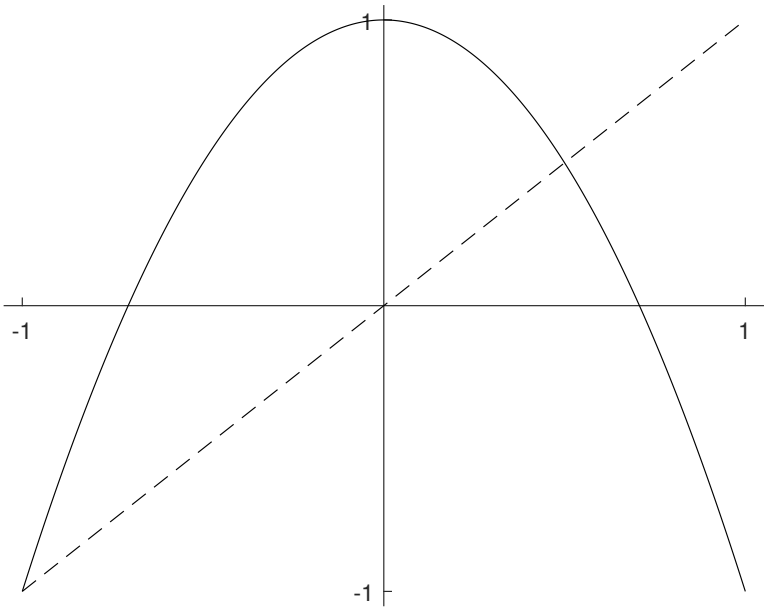


Figure 8 – The logistic map on the interval $[-1, 1]$.

on the interval $[-1, 1]$. The logistic map can be found in figure 8. Notice that this map is conjugate to a more common used map of the logistic family $g(x) = 4x(1 - x)$. For $\psi(x) = 2x - 1$ we have that $L \circ \psi = \psi \circ g$. The logistic map is conjugate to the tent map, but the derivative is depending on x and therefore the eigenvalues are not everywhere larger than 1 in absolute value.

We will consider the logistic map under the rigid rotation R_ϕ. For a rotation of $\frac{\pi}{2}$ the sequence is easy to find to show that there exist a snap-back repeller. A similar numerical method can be extended to two dimensions. This is shown in figure 9. For other angles of ϕ finding the sequence is a little harder and needs some more numerical computation. First of all we will find the basin of no escape for the logistic map, to estimate whether a point in the map escapes rapidly from the interval under iteration of the map. The basin of no escape for the logistic map under a rotation of $\frac{62\pi}{180}$ can be found together with the sequence in figure 10. Originally we have taken the interval $[-1, 1] \times [-1, 1]$, but since we are rotating, this interval may change. Of course this is no problem, since the logistic map can be extended to a larger interval. We have chosen the ball around the fixed point sufficiently small, so that the eigenvalues are larger than one in absolute value. Also the only points where the derivative equals zero are the points where $x = 0$. No point in the sequence lies on the y-axis and therefore the fixed point is a snap-back repeller.

Of course similar procedures can be done for any angle ϕ. On the other hand, if we are

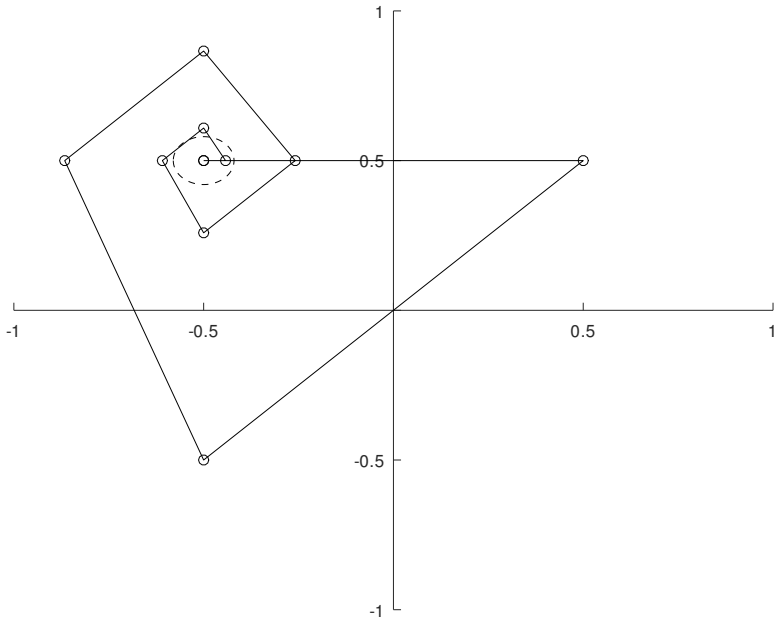


Figure 9 – The sequence of the logistic map under a rotation $\frac{\pi}{2}$, the radius of the ball around the fixed point is chosen sufficiently small to make sure that the eigenvalues are larger than one in absolute value.

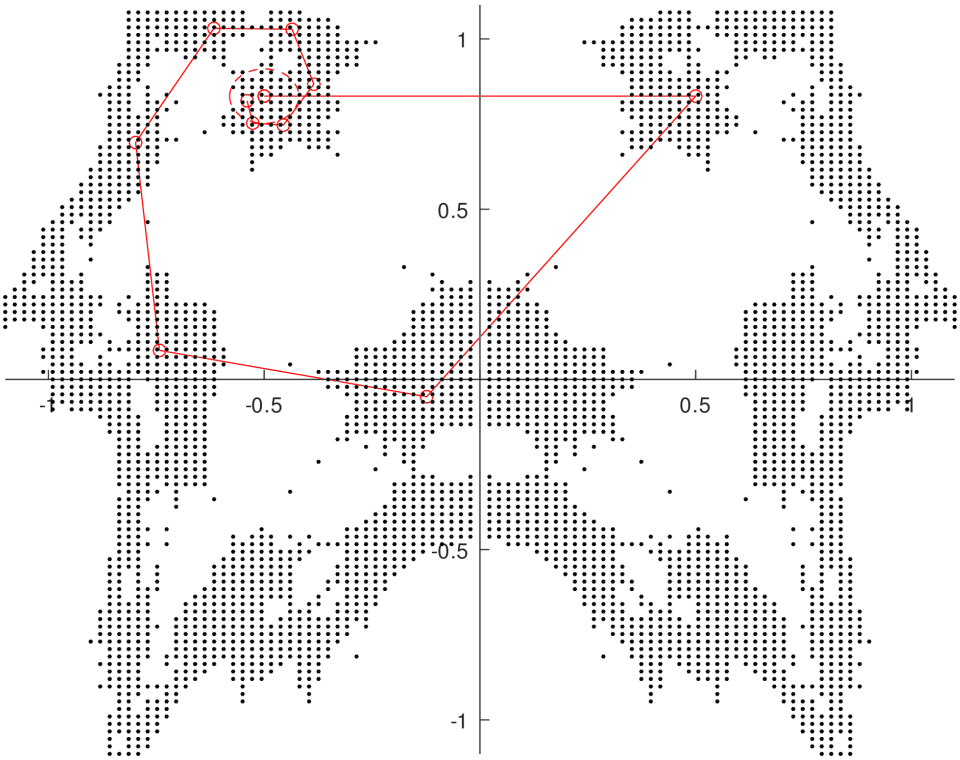


Figure 10 – The basin of no escape for the logistic map as given in equation 9 under a rotation $\phi = \frac{62\pi}{180}$ with the sequence to show that the fixed point is a snap-back repeller

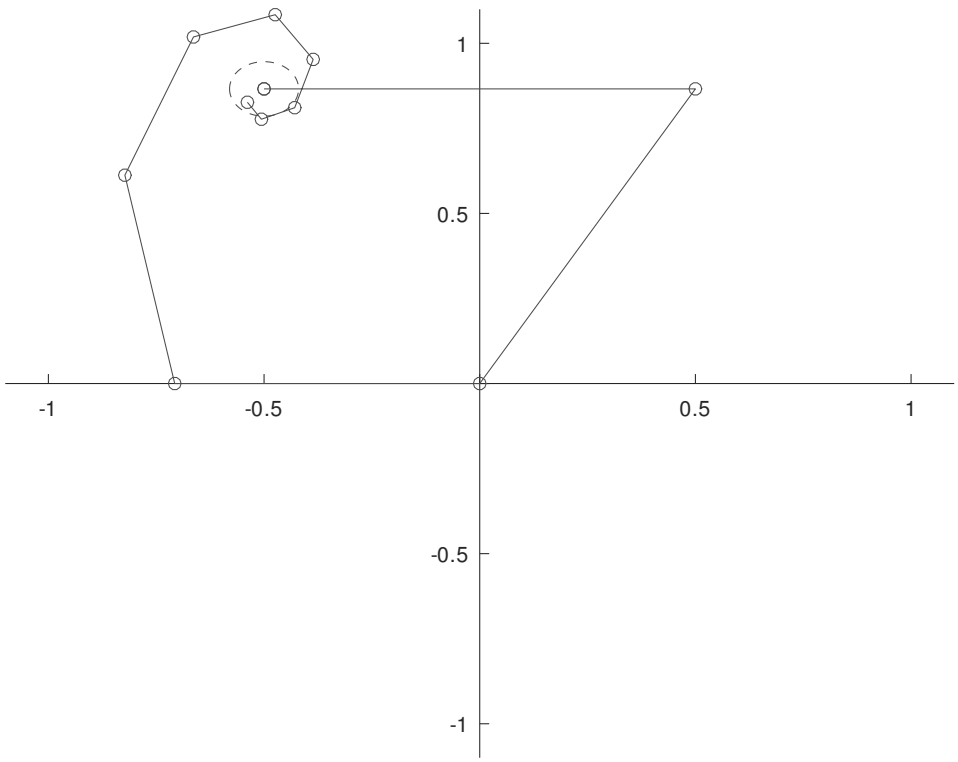


Figure 11 – The sequence in the logistic map for a rotation of $\phi = \frac{\pi}{3}$.

sufficiently close to a known case, the fixed point will still be a snap-back repeller. This is stated in the following theorem and proven in [7].

Theorem 3. *[5, 7]. Denote by $||\cdot||$ and $||\cdot||_{op}$ the Euclidean norm and the induced operator norm, respectively. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map with a snap-back repeller. If $G : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 map such that $||F - G|| + ||DF - DG||_{op}$ is sufficiently small, then G has a snap-back repeller.*

This says, that any function close enough to a known function has a snap-back repeller. Although this may have some issues, since we don't know what sufficiently small is in this case. In the previous case, we have found a snap-back repeller for the rotation $\phi = \frac{62\pi}{180}$. If we use the rotation $\phi = \frac{\pi}{3}$ we obtain a function, close to the previous one, but in the point $x_{m-2} = (0, 0)$ is in the sequence. Hence, for the logistic map, the derivative $DF(0, 0) = 0$ and therefore, the fixed point is not a snap-back repeller (see figure 11).

The Octave script used in this case uses a Newton method to find one fixed point. One needs to find at least one fixed point and a guess close to the fixed point is needed. To make sure there are any fixed points can be done by counting the number of intersections

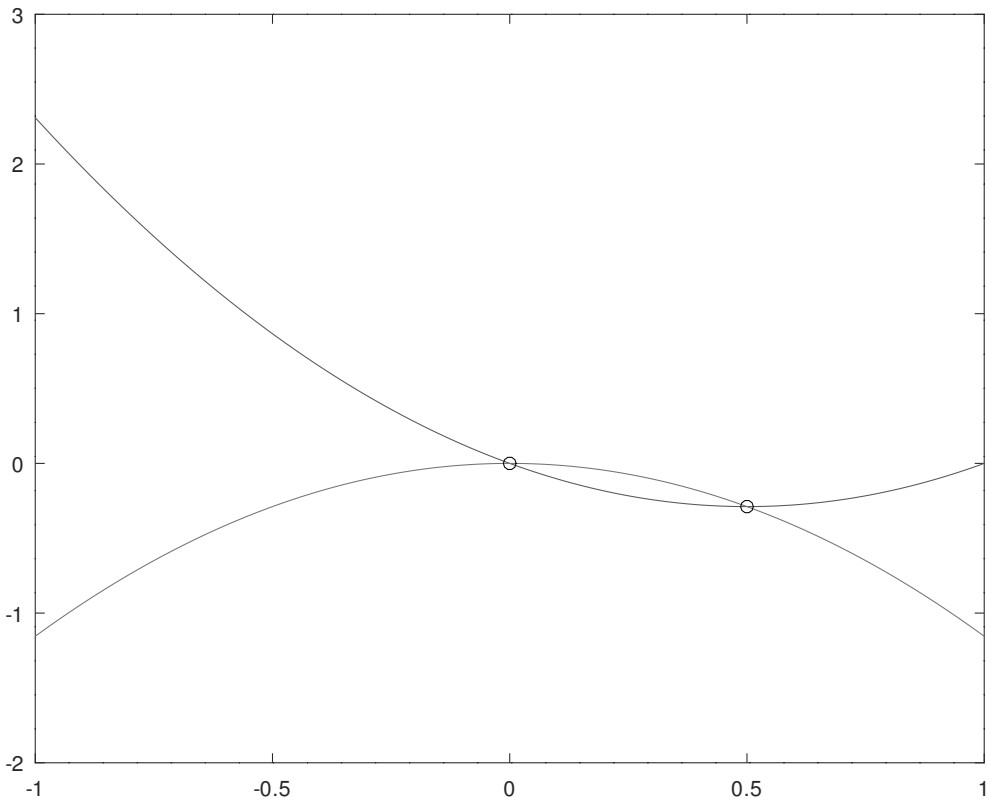


Figure 12 – The graph of the functions g_1 and g_2 to show that the logistic map, with a rotation of $\phi = \frac{62\pi}{180}$, has two fixed points.

(see figure 12) of the maps

$$g_1 = f_1(x, y) - x = 0$$

and

$$g_2 = f_2(x, y) - y = 0.$$

In this case we were able to write g_1 and g_2 into explicit expressions such that $y = j(x)$. This might not be the case for other examples. For any given r a neighborhood is defined then, but make we need to make sure that, for this r, the eigenvalues of the derivative are larger than 1 in absolute value. After this, the script finds all the sequences possible, untill one point in the last iterate is in the neighborhood around the fixed point. Then the script identifies which sequence this is and plots the desired sequence.

The computing time for simple functions like the logistic map is low, because the inverse has only two parts and therefore for every iterate the number of sequences possible multiplies by 2. The user needs to fill in the parameters as the rotation angle and the radius of the neighborhood. Also the inverse of the one-to-one parts of the map need to be calculated.

Note: This may be a lot harder for different maps than the logistic or the tent map and the computing time will increase, because the number of sequences is multiplied by at least 2 for every iterate.

4.4 A family of logistic maps

At this point we only looked at one case of the logistic map. The logistic map is better know as a family of maps with a paramater a defined on the interval $[0, 1]$:

$$g(x) = ax(1 - x) \tag{9}$$

We will consider a similar family of maps with a parameter a:

$$f(x) = \frac{1}{4}(a - 2) - ax^2. \tag{10}$$

This map was also studied in [5]. Equation (10) is conjugate to equation (9). With $\psi(x) = x - \frac{1}{2}$ we have that $f \circ \psi = \psi \circ g$. Notice that this map, in contrary to the map given in equation (9), is symmetric in the y-axis.

We are considering the map as given in equation (10). The fixed points of the one-dimensional equation are $p_a = \frac{a-2}{2a}$ and $p_0 = -\frac{1}{2}$. The first one, is the most interesting one, because it depends on a. If we want the fixed point to be a snap-back repeller, the fixed point needs to be expanding. The eigenvalue of the derivative equals $-2ax$ which is only larger than 1 in absolute value if $x > \frac{1}{2a}$ or $x < -\frac{1}{2a}$. A neighborhood around the fixed point needs to satisfy this condition, so the if we consider only positive values of a, a needs to be larger than 3 and the radius $r \leq \frac{a-3}{2a}$. Then, the first condition for the fixed point p_a being a snap-back repeller is satisfied. In figure 13 can be seen for which parametervalues a and ϕ the eigenvalues are strictly larger than one in absolute value at the fixed point p. Of course this needs to be verified in a region, but since the eigenvalues are stricly larger than 1, there exist an $r > 0$ such that this is satisfied. Hence, if $a > 3$ there exist a neighborhood which satisfies the conditions. The sequence to prove that the fixed point is a snap-back repeller is harder. However, using a similar script as in the previous section can help out for specific rotation angles and values for a.

As mentioned before, a periodic point can be a snap-back repeller as well. If we, for example, iterate the logistic map as defined in equation (10) 4 times under a rotation of $\phi = \frac{\pi}{2}$ we obtain:

$$F^4(x, y) = \begin{pmatrix} -f^2(x) \\ f^2(y) \end{pmatrix}.$$

Notice that a fixed point of F^4 is a periodic point of period 4 of F. We have included the Lyapunov diagram of the map F from [5] in figure 14 to investigate the value of the parameters a and ϕ. As can be seen we are looking for values in the red or black area. Otherwise it won't be possible to find a snap-back repeller, since there will be no chaos.

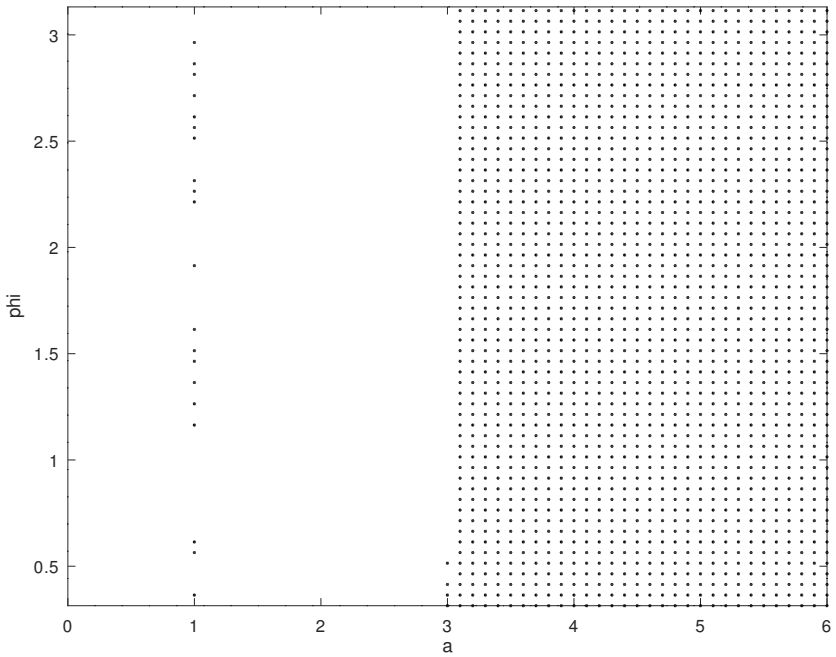
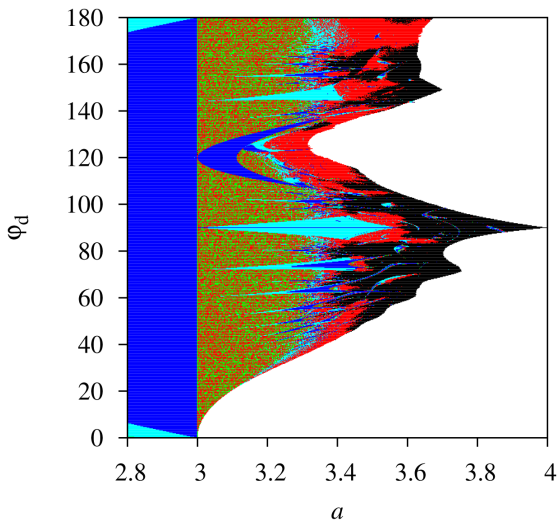


Figure 13 – In this figure the points are indicating where the eigenvalues λ of the derivative $DF(p)$ in the fixed point are strictly larger than 1 for values $a \in [0, 6]$ and rotation angle $\phi \in [\frac{\pi}{10}, \pi - 0.01]$

number of iterates	runtime 2-to-1 map	runtime 16-to-1 map
4	0.314055 seconds	0.41635 seconds
6	0.367323 seconds	0.925561 seconds
8	0.403225 seconds	50.3118 seconds

Table 1 – In this table the time to run the octave-scripts are given to show the increase in runtime of the 16-to-1 logistic map compared to the 2-to-1 logistic map



Color	Lyapunov Exponents	Attractor Type
Cyan	$0 > \lambda_1 > \lambda_2$	Periodic point of node type
Blue	$0 > \lambda_1 = \lambda_2$	Periodic point of focus type
Green	$0 = \lambda_1 > \lambda_2$	Invariant circle
Red	$\lambda_1 > 0 \geq \lambda_2$	Chaotic attractor
Black	$\lambda_1 \geq \lambda_2 > 0$	Chaotic attractor
White		No attractor detected

Figure 14 – The Lyapunov diagram from [5] for the map F on the left, with the corresponding colortable on the right to indicate the dynamics.

We already chose $\phi = \frac{\pi}{2}$ and we choose $a = 4$. For this values we can indeed find a snap-back repeller numerically (see figure 15). Notice that $f^2(x)$ is 4-to-1 instead of 2-to-1 and therefore we have 4 pre-images. This also holds for the y part of the map, so F^4 is a 16-to-1 map. Therefore, the number of pre-images is 16^i where i is the number of iterates. This is a lot more than the 2^i from the previous examples. So already a few more iterations require a lot more computing time (see table 1). Also, decreasing the radius of the neighborhood around the periodic point may lead to extra iterations and therefore to more computing time. Notice that the sequence that is found is not unique, so similar sequences can be found. The only thing that needs to be checked is the derivative at each point in the sequence.

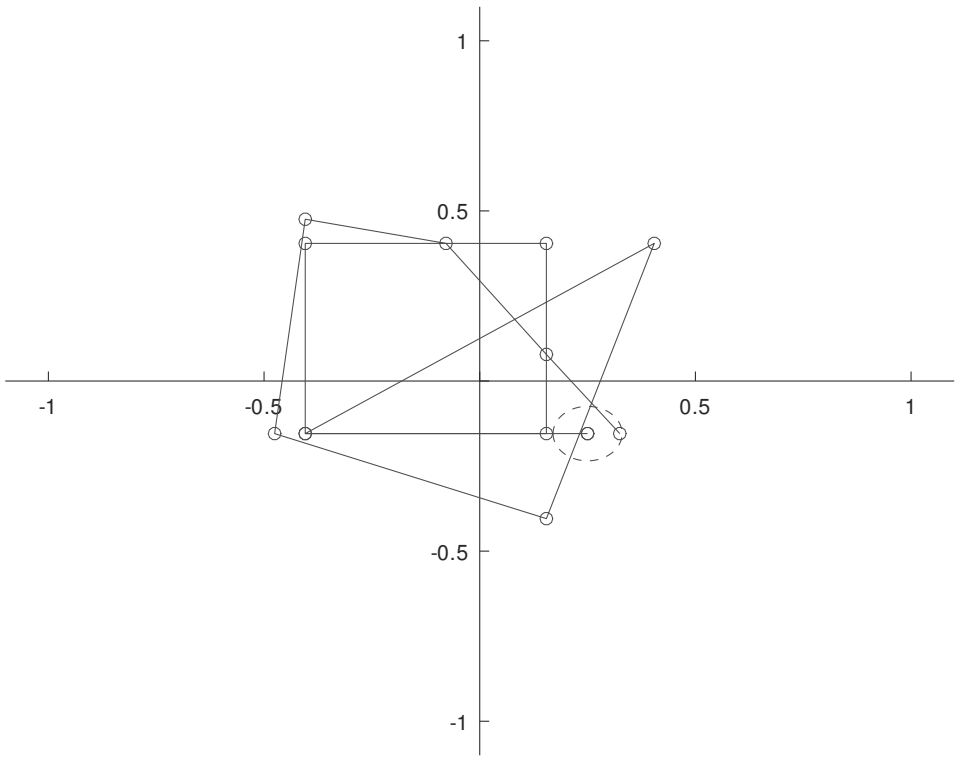


Figure 15 – The sequence in the 2-dimensional map F^4 with a rotation angle $\phi = \frac{\pi}{2}$ and $a = 4$. All iterates of F are given. The fixed point of F^4 is a snap-back repeller. The radius of the ball around the periodic point of F is $r = 0.08$. The lines between the points are indicated for clarity. The octave algorithm is shown in appendix A.3.

5 Conclusion

In this thesis we wanted to come with a ”recipe” to show that snap-back repellers exist in order to prove chaos. We came with a few examples to show how this can be done by analytically. First we will look if there are any fixed points or periodic points and calculate them. Then we calculate the maximum radius of the neighborhood around the fixed point (or periodic point) by calculating the eigenvalues of the derivative. After that we calculate the pre-images of the fixed point and see if one point lies in the neighborhood of the fixed point. Of course, calculating the pre-images takes a few seconds or even a few minutes because for a k-to-1 map, the number of pre-images is k. So after i iterates the number of points is k^i. Then, it is useful to come with a numerical algorithm to show a fixed point is a snap-back repeller. We have done this for some one-dimensional examples first, by calculating all the pre-images. The recipe is the same as for the analytical computation. Also, we have done similar procedures to some specific 2-dimensional examples and we came with some propositions for the relation between symmetric and anti-symmetric 1-dimensional maps and 2-dimensional rotated maps.

In conclusion one needs to follow these steps to find a snap-back repeller:

1. Find fixed or periodic points:
   (a) Identify the fixed or periodic points.
   (b) Calculate the fixed or periodic points by solving $F^k(X) = X$. For example with the Newton-Raphson method.
2. Calculate maximum radius:
   (a) Calculate the eigenvalues of the derivative of F^k.
   (b) Calculate for which values of X these eigenvalues are larger than 1 in absolute value.
3. Calculate pre-images:
   (a) Calculate all pre-images
   (b) Discard complex pre-images, pre-images that are equal to the fixed or periodic point and pre-images that have a derivative equal to 0.
   (c) Check if one of these points is in the neighborhood of the fixed or periodic point.
   (d) If not, repeat step 3.

We have shown a procedure to show that a fixed or periodic point is a snap-back repeller, but there are some issues that can be encountered while following this procedure. First of all, we only investigated 1 and 2-dimensional examples. Although it might be interesting to look at higher dimensional example, finding a fixed point is harder, and there are more pre-images than in the lower dimensional cases. Therefore, it requires more computing

time and the algorithm is more difficult. Also, in 2 dimensions, we only looked at rotations of 1-dimensional maps, but there are of course a lot of 2-dimensional examples which are also interesting to study. We only looked at 2-dimensional examples that are related to 1-dimensional examples, so that we can predict the dynamics easier from the knowledge of the one-dimensional case. Also, a similar study to other examples is interesting, but this involves less easy computations in finding the pre-images. When computing the pre-images some sequences contain complex values. One wants to exclude these sequences, but then more iterations are needed to show the fixed or periodic point is a snap-back repeller. Numerical methods are an easy way to make a lot of computations, but one needs to be careful with looking for snap-back repellers, since the number of pre-images increases with at least a factor 2 in every iterate and therefore the computing increases exponentially. Also, due to the amount pre-images one needs to be very exact in the way these points are stored. As a last note we need to stress that finding the pre-images is easy in simple examples, but finding the pre-images can be a lot of work in k-to-1 maps with high values of k.

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A Octave scripts

A.1 The map $f(x) = \pi \sin x$

```
x=-pi:2*pi/1000:pi;
y=pi*sin(x);

%find fixed point
p=2*pi/3;
err=1;
while err>10^-12
  p0=p;
  p=p-(pi*sin(p)-p)/(pi*cos(p)-p);
  err=abs(p0-p);
endwhile

%find maximum possible radius
r=1;
t=3*pi/4;
while r>10^-12
  t0=t;
  t=t+(pi^2*cos(pi*sin(t))*cos(t)-1)/(pi^3*sin(pi*sin(t))*cos(
     t)*cos(t)+pi^2*cos(pi*sin(t))*sin(t));
  r=abs(t0-t);
endwhile

%find trajectory to show the fixed point is a snap-back
   repeller
%start at fixed point
s(1)=p;

%inverse of the one-to-one part of f without fixed point
s(3)=asin(s(1)/pi);

%inverse of the one-to-one part of f with fixed point
m=1;
while abs(s(2*m+1)-p)>abs(t-p)
  m=m+1;
  s(2*m+1)=pi-asin(s(2*m-1)/pi);
endwhile
for i=1:m+1
  s(2*i)=s(2*i-1);
```

```
endfor


rectangle("Position",[p-(t-p), -pi, 2*(t-p), 2*pi],"Edgecolor
   ",0.9*[1,1,1],"Facecolor",0.9*[1,1,1])
hold on
plot(x,y,'k',x,x,'k--',[p,0,-p],[p,0,-p],'ko',x,zeros(length(x
   ),1),'k',zeros(length(x),1),x,'k')
plot([s(2:((m+1)*2))],[s(1:(m*2)),0])
axis([-pi pi -pi pi])
text (p-0.2,-pi/2, "B_r(p)");
text(s(2*m+1),-0.2,"x_0")
xticks([-pi pi])
xticklabels({'-\pi','\pi'})
yticks([-pi pi])
yticklabels({'-\pi','\pi'})
set (gca, "xaxislocation", "origin");
set (gca, "yaxislocation", "origin");
box off
hold off
```

A.2 The logistic map under rotation

```
%SET INTERVAL
x=-1:0.001:1;
y=-1:0.001:1;
%
%SET ROTATION ANGLE
phi=60*pi/180;
%
%FIND FIXED POINT USING NEWTON RAPHSON METHOD
err=1;
p=[-0.5;0.5];
while err>10^-12
  p0=p;
  z=1-2*p(1)*p(1);
  z_prime=-4*p(1);
  F=[z*cos(phi)-p(2)*sin(phi);z*sin(phi)+p(2)*cos(phi)];
  G=F-p;
  J=[z_prime*cos(phi)-1,-sin(phi);z_prime*sin(phi),cos(phi)
     -1];
```

```
  p=p-J\G;
  err=sqrt((p0(1)-p(1))^2+(p0(2)-p(2))^2);
endwhile
%
%DEFINE INTERVAL AROUND FIXED POINT
Rad=0.08;
th=-pi:0.001:pi;
xt=Rad*cos(th)+p(1);
yt=Rad*sin(th)+p(2);
%
%FIND SHORTEST SEQUENCE
r=p;
c=1;
i=1;
while c>Rad
  i=i+1;
  for j=1:2^(i-2)
    r(4*j-3,i)=sqrt((r(1+2*(j-1),i-1)*cos(phi)+r(2+2*(j-1),i
       -1)*sin(phi)-1)/-2);
    r(4*j-2,i)=r(2+2*(j-1),i-1)*cos(phi)-r(1+2*(j-1),i-1)*sin(
       phi);
    r(4*j-1,i)=-sqrt((r(1+2*(j-1),i-1)*cos(phi)+r(2+2*(j-1),i
       -1)*sin(phi)-1)/-2);
    r(4*j,i)=r(2+2*(j-1),i-1)*cos(phi)-r(1+2*(j-1),i-1)*sin(
       phi);
  endfor
  for k=1:length(r)/2
  dist(k)=sqrt((r(1,1)-r(2*k-1,end))^2+(r(2,1)-r(2*k,end))^2);
  c=min(dist(dist>0));
endfor
l=1;
while c~=dist(l)
  l=l+1;
endwhile
endwhile
i=0;
seq=[1;1];
while l>0
seq(:,i+1)=[r(2*l-1,end-i);r(2*l,end-i)];
l=floor(l/2);
i=i+1;
endwhile
%
```

```
%PLOT RESULTS
plot(p(1),p(2),'ro',[seq(1,:),p(1)],[seq(2,:),p(2)],'ro-',xt,
   yt,'r--')
axis([-1.1 1.1 -1.1 1.1])
set (gca, "xaxislocation", "origin");
set (gca, "yaxislocation", "origin");
box off
```

A.3 The fourth iterate of the logistic map under rotation

```
tic
%SET INTERVAL
x=-1:0.001:1;
y=-1:0.001:1;
%
%SET ROTATION ANGLE PHI & PARAMETER A
phi=pi/2;
a=4;
%
%FIND FIXED POINT USING NEWTON RAPHSON METHOD
err=1;
p=[0.2;-0.2];
while err>10^-12
  p0=p;
  z=0.25*(a-2)-a*p(1)*p(1);
  F1=[-p(2);z];
  z=0.25*(a-2)-a*F1(1)*F1(1);
  F2=[F1(2);z];
  z=0.25*(a-2)-a*F2(1)*F2(1);
  F3=[F2(2);z];
  z=0.25*(a-2)-a*F3(1)*F3(1);
  F4=[F3(2);z];
  G=F4-p;
  z_prime1=4*a*a*p(1)*(0.25*(a-2)-a*p(1)*p(1));
  z_prime2=4*a*a*p(2)*(0.25*(a-2)-a*p(2)*p(2));
  J=[z_prime1-1,0;0,z_prime2-1];
  p=p-J\G;
  err=sqrt((p0(1)-p(1))^2+(p0(2)-p(2))^2);
endwhile
%
%DEFINE INTERVAL AROUND FIXED POINT
```

```
Rad=0.15;
th=-pi:0.001:pi;
xt=Rad*cos(th)+p(1);
yt=Rad*sin(th)+p(2);
%
%FIND SHORTEST SEQUENCE, USING THE FOUR PARTIAL INVERSES OF
   THE MAP
r=p;
c=1;
i=1;
while c>Rad
  i=i+1;
  for j=1:4^(i-2)
    r((j-1)*8+1,i)=sqrt(1/a*(0.25*(a-2)+sqrt(1/a*(0.25*(a-2)+r
       (j*2-1,i-1)))));
    r((j-1)*8+2,i)=sqrt(1/a*(0.25*(a-2)-sqrt(1/a*(0.25*(a-2)+r
       (j*2-1,i-1)))));
    r((j-1)*8+3,i)=-sqrt(1/a*(0.25*(a-2)+sqrt(1/a*(0.25*(a-2)+
       r(j*2-1,i-1)))));
    r((j-1)*8+4,i)=-sqrt(1/a*(0.25*(a-2)-sqrt(1/a*(0.25*(a-2)+
       r(j*2-1,i-1)))));
    r((j-1)*8+5,i)=sqrt(1/a*(0.25*(a-2)+sqrt(1/a*(0.25*(a-2)-r
       (j*2,i-1)))));
    r((j-1)*8+6,i)=sqrt(1/a*(0.25*(a-2)-sqrt(1/a*(0.25*(a-2)-r
       (j*2,i-1)))));
    r((j-1)*8+7,i)=-sqrt(1/a*(0.25*(a-2)+sqrt(1/a*(0.25*(a-2)-
       r(j*2,i-1)))));
    r((j-1)*8+8,i)=-sqrt(1/a*(0.25*(a-2)-sqrt(1/a*(0.25*(a-2)-
       r(j*2,i-1)))));
  endfor
  u=1;
  for b=1:length(r)/8
    for k=1:4
    dist(u)=sqrt((r(1,1)-r((b-1)*8+k,end))^2+(r(2,1)-r((b-1)
       *8+5,end))^2);
    u=u+1;
    dist(u)=sqrt((r(1,1)-r((b-1)*8+k,end))^2+(r(2,1)-r((b-1)
       *8+6,end))^2);
    u=u+1;
    dist(u)=sqrt((r(1,1)-r((b-1)*8+k,end))^2+(r(2,1)-r((b-1)
       *8+7,end))^2);
    u=u+1;
```

```
    dist(u)=sqrt((r(1,1)-r((b-1)*8+k,end))^2+(r(2,1)-r((b-1)
       *8+8,end))^2);
    u=u+1;
    endfor
  endfor
c=min(dist(dist>0));
l=1;
while c~=dist(l)
  l=l+1;
endwhile
endwhile
b=1+(l-mod(l-1,16)-1)/16;
k=1+(mod(l-1,16)+1-mod(mod(l-1,16),4)-1)/4;
t=mod(mod(l-1,16),4)+5;
seq=[r((b-1)*8+k,end);r((b-1)*8+t,end)];
[m,n]=size(r);
for i=2:4*n
  seq(1,i)=-seq(2,i-1);
  seq(2,i)=0.25*(a-2)-a*seq(1,i-1)*seq(1,i-1);
endfor
%
%PLOT RESULTS
plot(p(1),p(2),'ro',[seq(1,:),p(1)],[seq(2,:),p(2)],'ro-',xt,
   yt,'r--')
axis([-1.1 1.1 -1.1 1.1])
set (gca, "xaxislocation", "origin");
set (gca, "yaxislocation", "origin");
box off
toc
```