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# The Geometry of Attractors in Inflationary Cosmology

*Bachelor's Thesis in Physics and Mathematics*

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**Abstract**

Recent developments in observational cosmology have led to the notion of "cosmological attractors", trajectories of inflationary cosmologies that seem to attract the dynamics from a large range of initial conditions. In this work, we aim to give a rigorous description of this behaviour, by reviewing the mathematics of non-dissipative and dissipative Hamiltonian systems. Then, we give an introduction to cosmology and inflation, to familiarize the reader with concepts needed in the remainder of the thesis, in which we discuss attractor behaviour in the Hamiltonian setting of inflationary cosmology. We discover that for flat universes, a remarkable simplification in the dynamics arises, that allows us to express them in terms of the field variable and its time derivative alone. Finally, we use the formalism of Hamiltonian systems to derive the conditions for the existence of a conserved measure on this "effective" phase space.

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**CONTENTS**

*"Sadly, cosmological attractors have nothing at all to do with the hypothetical notion of attractive cosmologists."*

- Sean Carroll

## I. INTRODUCTION

Cosmology is arguably one of the most exciting fields of research in modern physics. Next to many developments on the theoretical side, there have been huge successes in observational cosmology as well. Large surveys of Type Ia supernovae and the cosmic microwave background have fundamentally changed the way we think about the universe [1–4]. For example, we now know that its geometry is almost completely flat, and that its expansion is accelerating. We know that ordinary matter makes up only approximately 5% of its energy content, with the other 95% veiled in mystery. With the successes of modern cosmology, also new challenges came to light. The observed homogeneity of the CMB led to the horizon problem, an example of a cosmological fine-tuning problem. In standard Big Bang cosmology, the last scattering surface consisted of many causally disconnected regions. The homogeneity of the CMB could therefore either be coined as a coincidence, or as a sign that the existing theory was incomplete [5]. In the same manner, the striking flatness of the universe could not be explained by Big Bang cosmology alone.

To make the classical theory consistent, flatness and homogeneity have to be implemented as assumptions. It goes without saying that an extended theory that could explain these initial conditions on its own is prefered. Inflation is the leading paradigm that provides such an extension. It says that the universe underwent a period of accelerated expansion, only $10^{-36}$ seconds after the Big Bang. As we will see in this thesis, such an era offers a solution to the horizon and flatness problem. Furthermore, it gives an explanation for the formation of large scale structure, by transforming microscopic quantum fluctuations into the macroscopic density perturbations that we can observe in the CMB today [6].

Since its formulation in the 1980s, a large number of different models of inflation have been proposed. Many of them build upon the existence of a scalar field, the inflaton, to drive inflation. For such models, the degree of freedom is the inflaton potential and the way it is coupled to gravity. The observational data of the WMAP and Planck experiments in 2012 and 2013 pointed out that several broad classes of models were favored, all making similar predictions for a specific set of parameters [7, 8]. These observational predictions appear to be invariant under large modifications of the inflaton potential, leading to the notion of a cosmological attractor.

In mathematics, an attractor is a region of phase space toward which trajectories converge [9, 10]. For such systems, phase space volume shrinks, corresponding to dissipation. In this thesis, we will review the geometry of non-dissipative systems, described by symplectic mechanics. Then, we consider contact mechanics, a framework in which dissipation can be implemented naturally, without affecting the Hamiltonian structure of the theory. After this discussion, we review cosmology and the motivations for inflation, as well as its proposed physical origin. Finally, we follow the work of from Carroll and Remmen [11], discussing the mathematical structure of cosmological attractors.

## II. SYMPLECTIC AND CONTACT MECHANICS

### II.1. Symplectic Mechanics

In this section, we will give an introduction to the framework of symplectic Hamiltonian mechanics for non-dissipative systems. It will be the starting point for generalizing to dissipative systems, which will be described by the formalism of contact mechanics.

#### *II.1.1. Symplectic Manifolds*

Hamiltonian mechanics describes the dynamics of a Hamiltonian system, which is a dynamical system on an $n$-dimensional configuration manifold $Q$. The cotangent bundle of this manifold, $T^*Q$, is of dimension $2n$ and is usually referred to as the phase space of the system (from now on denoted $M$). In local coordinates, a point $q \in T^*Q$ is written as $(p_a, q^a)$, where $q^a \in Q$, $p_a \in T^*_qQ$ and $a$ takes values $1, \ldots, n$. Physically, $q^a$ usually corresponds to a position and $p_a$, being a function on the tangent space, corresponds to momentum. The Hamiltonian function $H$ is a function on the phase space, i.e., $H = H(p_a, q^a)$. Before we use it to derive the dynamics of the system, we introduce the canonical 1-form $\theta = p_a \, \mathrm{d}q^a$. It has the property that its exterior derivative,

$$\Omega = \mathrm{d}\theta = \mathrm{d}p_a \wedge \mathrm{d}q^a \tag{1}$$

is closed and non-degenerate. The latter means that $\Omega_q$ is a non-degenerate bilinear form on the vector space $T_qM \times T_qM$ for all $q \in M$, i.e., $\Omega_q(X_q, Y_q) = 0$ for all $Y_q$ implies that $X_q = 0$. This is easily shown for $\Omega$: Choose vector fields

$$Y_i = b_i \frac{\partial}{\partial p_i}, \quad Z_i = c^i \frac{\partial}{\partial q^i} \tag{2}$$

and impose that $\Omega(X, Y_i)$ and $\Omega(X, Z_i)$ vanish for all $i = 1, \ldots, n$ and at all points $q$. It follows that $X$ is the zero vector field.

A smooth manifold $M$ together with a closed, non-degenerate 2-form $\Omega$ is called a symplectic manifold, in which case $\Omega$ is called the symplectic structure or symplectic form. In fact, it is a result from symplectic geometry [12], called Darboux's Theorem, that allowed us to locally write $\Omega$ in coordinates used above.

#### *II.1.2. Dynamics of Symplectic Hamiltonian Systems*

An important property of a symplectic structure is that it establishes (pointwise) isomorphisms between the tangent and cotangent spaces of the symplectic manifold [13]. Consequently, we can associate a vector field $X_f$ to each smooth function $f$. In particular, the Hamiltonian function $H$ has a corresponding vector field $X_H$, which is called the Hamiltonian vector field. It satisfies [14]

$$-\,\mathrm{d}H = \iota_{X_H}\Omega \,, \tag{3}$$

where the operation $\iota_X$ is the contraction of a differential form with the vector field $X$. This equation allows us to determine $X_H$ in terms of the canonical coordinates $(p_a, q^a)$. Notice that $X_H$ has the form

$$X_H = b_a \frac{\partial}{\partial p_a} + c^a \frac{\partial}{\partial q^a} \tag{4}$$

and therefore

$$\iota_{X_H}\Omega = b_a \, \mathrm{d}q^a - c^a \, \mathrm{d}p_a \; . \tag{5}$$

Furthermore, the left-hand side of (3) can be written as

$$-\,\mathrm{d}H = -\frac{\partial H}{\partial p_a} \, \mathrm{d}p_a - \frac{\partial H}{\partial q^a} \, \mathrm{d}q^a \; . \tag{6}$$

By comparing (5) and (6) it follows that

$$X_H = -\frac{\partial H}{\partial q^a} \frac{\partial}{\partial p_a} + \frac{\partial H}{\partial p_a} \frac{\partial}{\partial q^a} \; . \tag{7}$$

The equations of motion for this system are the integral curves of the Hamiltonian vector field. This implies that they are solutions to the first order differential equations induced by $X_H$:

$$\dot{q}^a = \frac{\partial H}{\partial p_a}, \quad \dot{p}_a = -\frac{\partial H}{\partial q^a} \; . \tag{8}$$

These are the Hamilton equations.

To conclude this section, we introduce the Poisson bracket $\{\cdot,\cdot\}$. This is a bilinear operation on the space of smooth functions on $M$, that allows us to determine the time evolution of any function that does not depend explicitly on time. For two functions $f, g \in C^\infty(M)$, it is defined as

$$\{f, g\} = \Omega(X_f, X_g) \; . \tag{9}$$

Notice that the right-hand side can be written in several equivalent ways:

$$\Omega(X_f, X_g) = \iota_{X_f}\Omega(X_g) = -\,\mathrm{d}g(X_f) = -X_f g = -\pounds_{X_f} g \; , \tag{10}$$

where $\pounds_X$ denotes the Lie derivative along the flow of $X$. Using the Hamilton equations, we can write the time derivative of a function $f \in C^\infty(M)$ as

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial q^a} \, \dot{q}^a + \frac{\partial f}{\partial p_a} \, \dot{p}_a = \frac{\partial f}{\partial q^a} \frac{\partial H}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial H}{\partial q^a} \\ &= X_H f = \Omega(X_H, X_f) \\ &= \{f, H\} \; , \end{aligned} \tag{11}$$

which shows that the time derivative is given by the Poisson bracket of $f$ with the Hamiltonian.

### *II.1.3. Canonical Transformations and Phase Space Volume*

The objective in this section is to derive a quantity that is conserved under canonical transformations. These are transformations of phase space, that preserve the form of the Hamilton equations. More precisely, for a transformation

$$p_a \to P_a(p_a, q^a), \quad q^a \to Q^a(p_a, q^a) \tag{12}$$

there exists a function $K(P_a, Q^a)$ that satisfies the Hamilton equations in terms of the new coordinates [15], i.e.

$$\dot{Q}^a = \frac{\partial K}{\partial P_a}, \quad \dot{P}_a = -\frac{\partial K}{\partial Q^a} \ . \tag{13}$$

These equations imply that $Q^a$ and $P_a$ are integral curves of a vector field $X_K$ that has the same form as (7) and satisfies (3). Therefore, the transformation preserves the symplectic form $\Omega$. In more general context, a differentiable map $f$ between symplectic manifolds whose pullback satisfies

$$f^*\Omega = \Omega \tag{14}$$

is called a symplectomorphism [13, 16]. A useful family of symplectomorphisms is the flow that arises from the Hamiltonian vector field $X_H$:

$$\pounds_{X_H}\Omega = \iota_{X_H}\, \mathrm{d}\Omega + \mathrm{d}\iota_{X_H}\Omega = \mathrm{d}(-\,\mathrm{d}H) = 0 \ . \tag{15}$$

This allows us to prove that the volume form $\Omega^n$ is conserved:

$$\begin{aligned} \pounds_{X_H}\Omega^n &= \pounds_{X_H}(\Omega) \wedge \Omega \wedge \cdots \wedge \Omega \\ &\quad + \Omega \wedge \pounds_{X_H}(\Omega) \wedge \Omega \wedge \cdots \wedge \Omega + \ldots \\ &\quad + \Omega \wedge \cdots \wedge \Omega \wedge \pounds_{X_H}(\Omega) \\ &= 0 \ . \end{aligned} \tag{16}$$

This result is one of many formulations of Liouville's theorem, which states that the symplectic phase space volume is preserved under Hamiltonian flow [16].

The reason that symplectic Hamiltonian systems are called non-dissipative, is exactly because of this property. Consequently, the framework lends itself well for the description of conservative systems, but breaks down when trying to describe processes that have dissipative terms in their equations of motion [17]. We will see in the next section that, contrary to symplectic systems, the standard volume form of contact Hamiltonian systems is not preserved, which is why they are called dissipative. They allow for a Hamiltonian description of both conservative as non-conservative systems and are therefore considered to be a natural extension of symplectic systems.

### II.2. Contact Mechanics

#### *II.2.1. Contact Manifolds*

We now assume our phase space to be a $(2n+1)$-dimensional manifold $\mathcal{M}$, with a differential 1-form $\eta$ that gives rise to the standard volume form

$$\eta \wedge (\mathrm{d}\eta)^n \neq 0 \ . \tag{17}$$

The form $\eta$ is refered to as a contact form, and the pair $(\mathcal{M}, \eta)$ is called a contact manifold. The motivation for such a construction is that for each point $p$ on the manifold, the equation $\eta_{\mathrm{p}} = 0$ describes a hyperplane in $T_pM$. It is said that $\eta$ describes locally a field of hyperplanes, called the contact structure. Equation (17) is called the non-integrability condition, as it implies that the hyperplane field is non-integrable [16]. The interpretation of this property is beyond the scope of this text.

#### *II.2.2. Dynamics of Contact Hamiltonian Systems*

In the same way that the Hamiltonian vector field $X_H$ was defined for a Hamiltonian function $H$ through equation (3) (or equivalently, through the correspondence between smooth functions and vector fields), we can associate a Hamiltonian vector field to every contact Hamiltonian function $\mathcal{H}$ on the contact manifold [18]. The contact Hamiltonian function satisfies the relations [14]

$$\pounds_{X_{\mathcal{H}}}\eta = f_{\mathcal{H}}\eta \ , \qquad -\mathcal{H} = \iota_{X_{\mathcal{H}}}\eta \ , \tag{18}$$

for some smooth function $f_{\mathcal{H}}$. We can use these relations to write a similar equation for (3) in the symplectic case. We first expand the Lie derivative using the Cartan formula:

$$\pounds_{X_{\mathcal{H}}}\eta = \iota_{X_{\mathcal{H}}}\,\mathrm{d}\eta + \mathrm{d}\Big(\iota_{X_{\mathcal{H}}}\eta\Big) \ . \tag{19}$$

But $\mathrm{d}(\iota_{X_{\mathcal{H}}}\eta) = -\,\mathrm{d}\mathcal{H}$, so using (18) we can write

$$\mathrm{d}\mathcal{H} = \iota_{X_{\mathcal{H}}}\,\mathrm{d}\eta - \pounds_{X_{\mathcal{H}}}\eta \ . \tag{20}$$

(20) is similar to (3), the difference being the term $\pounds_{X_{\mathcal{H}}}\eta = f_{\mathcal{H}}\eta$, which reflects the fact that contrary to the symplectic case, the 1-form $\eta$ is not conserved under the flow of $X_{\mathcal{H}}$. Instead, the flow induces a contact transformation, which is the contact counterpart to the symplectomorphism defined in section II.1.3 and will be defined in section II.2.3.

The function $f_{X_{\mathcal{H}}}$ can be expressed in terms of the Reeb vector field $\xi$. This vector field on $\mathcal{M}$ is an element of $\ker \mathrm{d}\eta$ and satisfies $\iota_\xi\eta = 1$. Using (18) and (19) we can write

$$\begin{aligned} f_{\mathcal{H}} = f_{\mathcal{H}}\,\iota_\xi\eta = \iota_\xi\left(\iota_{X_{\mathcal{H}}}\,\mathrm{d}\eta + \mathrm{d}\Big(\iota_{X_{\mathcal{H}}}\eta\Big)\right) \\ = \iota_\xi\,\mathrm{d}\Big(\iota_{X_{\mathcal{H}}}\eta\Big) = -\iota_\xi\,\mathrm{d}\mathcal{H} \\ = -\xi(\mathcal{H}) \ . \end{aligned} \tag{21}$$

In section II.1.1 we used the Darboux theorem to assert the existence of local coordinates that allowed us to write the symplectic form as in (1). There is also a Darboux theorem for contact manifolds [16]. The local coordinates will be denoted $(p_a, q^a, S)$, such that $\eta$ and $\xi$ take the form [14, 18]

$$\eta = \mathrm{d}S - p_a\,\mathrm{d}q^a\,, \qquad \xi = \frac{\partial}{\partial S}\,. \tag{22}$$

We are almost ready to derive the contact analogues of the Hamilton equations. This amounts to expressing $X_{\mathcal{H}}$ in terms of the local coordinates. Notice that we can combine (18), (21) and (22) to rewrite (20) as

$$\mathrm{d}\mathcal{H} = \iota_{X_{\mathcal{H}}}\,\mathrm{d}\eta + \frac{\partial \mathcal{H}}{\partial S}\eta\,. \tag{23}$$

Now, we use the same method that was used to derive (7). Writing $X_{\mathcal{H}}$ in a general form, and taking exterior derivatives of $\mathcal{H}$ and $\eta$ gives

$$X_{\mathcal{H}} = b_a\frac{\partial}{\partial p_a} + c^a\frac{\partial}{\partial q^a} + s\frac{\partial}{\partial S}\,, \tag{24}$$

$$\mathrm{d}\mathcal{H} = \frac{\partial \mathcal{H}}{\partial p_a}\,\mathrm{d}p_a + \frac{\partial \mathcal{H}}{\partial q^a}\,\mathrm{d}q^a + \frac{\partial \mathcal{H}}{\partial S}\,\mathrm{d}S\,, \tag{25}$$

$$\mathrm{d}\eta = -\,\mathrm{d}p_a \wedge \mathrm{d}q^a\,. \tag{26}$$

It follows from (26) and (24) that

$$\iota_{X_{\mathcal{H}}}\,\mathrm{d}\eta = -\left(\iota_{X_{\mathcal{H}}}\,\mathrm{d}p_a\right)\mathrm{d}q^a + \left(\iota_{X_{\mathcal{H}}}\,\mathrm{d}q^a\right)\mathrm{d}p_a = -b_a\,\mathrm{d}q^a + c^a\,\mathrm{d}p_a\,. \tag{27}$$

Equations (24) to (27) can be used to compare the left- and right-hand side of (23), to find that

$$X_{\mathcal{H}} = \left(p_a\frac{\partial \mathcal{H}}{\partial p_a} - \mathcal{H}\right)\frac{\partial}{\partial S} - \left(p_a\frac{\partial \mathcal{H}}{\partial S} + \frac{\partial \mathcal{H}}{\partial q^a}\right)\frac{\partial}{\partial p_a} + \frac{\partial \mathcal{H}}{\partial p_a}\frac{\partial}{\partial q^a}\,. \tag{28}$$

Recall that the Hamilton equations are the ordinary differential equations for the integral curves of the Hamiltonian vector field. Evidently, the (contact) Hamilton equations are

$$\dot{q}^a = \frac{\partial \mathcal{H}}{\partial p_a}\,, \tag{29}$$

$$\dot{p}_a = -\frac{\partial \mathcal{H}}{\partial q^a} - p_a\frac{\partial \mathcal{H}}{\partial S}\,, \tag{30}$$

$$\dot{S} = p_a\frac{\partial \mathcal{H}}{\partial p_a} - \mathcal{H}\,. \tag{31}$$

These equations show great resemblence to the symplectic Hamilton equations. There are two main differences:

(i) In (30), there is a dissipative term when $\partial\mathcal{H}/\partial S$ is nonzero, which characterizes the nonconservative nature of contact Hamiltonian systems.

(ii) There is a third nontrivial equation (31), which coincides with Hamilton's principle function. [14]

The contact Hamilton equations can now be used to calculate the time dependence of any smooth function $f$ on the contact manifold. It is given by

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial S}\dot{S} + \frac{\partial f}{\partial p_a}\dot{p}_a + \frac{\partial f}{\partial q^a}\dot{q}^a \\ &= \frac{\partial f}{\partial S}\left[p_a\frac{\partial \mathcal{H}}{\partial p_a} - \mathcal{H}\right] + \frac{\partial f}{\partial p_a}\left[-\frac{\partial \mathcal{H}}{\partial q^a} - p_a\frac{\partial \mathcal{H}}{\partial S}\right] + \frac{\partial f}{\partial q^a}\frac{\partial \mathcal{H}}{\partial p_a} \qquad (32) \\ &= X_{\mathcal{H}} f \ . \qquad (33)\end{aligned}$$

Notice that for the contact Hamiltonian itself, (32) reduces to

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = -\mathcal{H}\frac{\partial \mathcal{H}}{\partial S} \ , \qquad (34)$$

which implies that the contact Hamiltonian is not a constant of motion when it depends on $S$. This agrees with observation (i) that was made above.

Equation (34) can be used to obtain another result, that will be used later. For a smooth function $g = g(\mathcal{H})$, the time evolution is given by

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\mathrm{d}g}{\mathrm{d}\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = -\mathcal{H}\frac{\mathrm{d}g}{\mathrm{d}\mathcal{H}}\frac{\partial \mathcal{H}}{\partial S} \ . \qquad (35)$$

In the next section, we will introduce the contact transformation, the contact counterpart of the canonical transformation (or symplectomorphism). We will see that the standard volume form is not conserved under the contact flow, but that there exists another quanitity that does have this property [17].

### *II.2.3. Contact Transformations and Dissipation*

In section II.1.3, we studied the canonical transformation, which was defined as a special type of phase space transformation that preserved the symplectic form $\Omega$. For contact systems, we define a contact transformation to be a phase space transformation that transforms the contact form according to

$$\eta' = f\eta \ , \qquad (36)$$

where $f$ is a nowhere-vanishing function [19]. It follows that

$$\mathrm{d}\eta' = \mathrm{d}f \wedge \eta + f\,\mathrm{d}\eta \ . \qquad (37)$$

The transformed volume form therefore is

$$\eta' \wedge (\mathrm{d}\eta')^n = f\eta \wedge (\mathrm{d}f \wedge \eta + f\,\mathrm{d}\eta)^n = f^{n+1}\eta \wedge (\mathrm{d}\eta)^n \ , \qquad (38)$$

where in the last step we have used the fact that $\eta$ is of odd degree and therefore $\eta \wedge \eta$ vanishes. When $f = 1$, the contact volume form is preserved and it can be shown that transformations for which this holds correspond to canonical transformations [14]. For general contact transformations however, $f$ will be different from 1. A particular case is (again) the contact Hamiltonian vector flow, whose effect on $\eta$ we have seen in section II.2.2 in the form of

$$\pounds_{X_\mathcal{H}}\eta = -\frac{\partial \mathcal{H}}{\partial S}\eta\ . \tag{39}$$

This result can now be used to prove that the volume form is indeed not preserved under the Hamiltonian flow:

$$\pounds_{X_\mathcal{H}}\left[\eta \wedge (\mathrm{d}\eta)^n\right] = \pounds_{X_\mathcal{H}}\eta \wedge (\mathrm{d}\eta)^n + \eta \wedge \left(\pounds_{X_\mathcal{H}}(\mathrm{d}\eta)^n\right)\ . \tag{40}$$

Notice that

$$\begin{aligned}\pounds_{X_\mathcal{H}}(\mathrm{d}\eta)^n &= \pounds_{X_\mathcal{H}}(\mathrm{d}\eta) \wedge \mathrm{d}\eta \wedge \cdots \wedge \mathrm{d}\eta \\ &\quad + \mathrm{d}\eta \wedge \pounds_{X_\mathcal{H}}(\mathrm{d}\eta) \wedge \mathrm{d}\eta \wedge \cdots \wedge \mathrm{d}\eta + \dots \\ &\quad + \mathrm{d}\eta \wedge \cdots \wedge \mathrm{d}\eta \wedge \pounds_{X_\mathcal{H}}(\mathrm{d}\eta) \\ &= -n\frac{\partial \mathcal{H}}{\partial S}(\mathrm{d}\eta)^n\ ,\end{aligned} \tag{41}$$

where in the last step we have used the commutation property of the Lie derivative and exterior derivative. Equations (40) and (41) can now be combined:

$$\pounds_{X_\mathcal{H}}\left[\eta \wedge (\mathrm{d}\eta)^n\right] = -(n+1)\frac{\partial \mathcal{H}}{\partial S}\,\eta \wedge (\mathrm{d}\eta)^n\ , \tag{42}$$

which proves that Liouville's theorem indeed does not hold for contact Hamiltonian flow.

To conclude this section, we will outline the derivation of an invariant volume form, an analogue to Liouville's theorem for contact systems [17]. Consider a general volume form $\rho\,\eta \wedge (\mathrm{d}\eta)^n$ on $\mathcal{M}$, for some distribution function $\rho$. We impose that this form is conserved under the Hamiltonian flow:

$$\pounds_{X_\mathcal{H}}\left[\rho\,\eta \wedge (\mathrm{d}\eta)^n\right] = 0\ . \tag{43}$$

Using the properties of the Lie derivative, we can write this as

$$\begin{aligned}\pounds_{X_\mathcal{H}}\left[\rho\,\eta \wedge (\mathrm{d}\eta)^n\right] &= \pounds_{X_\mathcal{H}}(\rho\,\eta) \wedge (\mathrm{d}\eta)^n + (\rho\eta) \wedge \pounds_{X_\mathcal{H}}(\mathrm{d}\eta)^n \\ &= \left[\left(\pounds_{X_\mathcal{H}}\rho\right)\eta + \rho\pounds_{X_\mathcal{H}}\eta\right] \wedge (\mathrm{d}\eta)^n + (\rho\eta) \wedge \pounds_{X_\mathcal{H}}(\mathrm{d}\eta)^n \\ &= \left[X_\mathcal{H}\rho - \rho\frac{\partial \mathcal{H}}{\partial S} - n\rho\frac{\partial \mathcal{H}}{\partial S}\right]\eta \wedge (\mathrm{d}\eta)^n = 0\ .\end{aligned} \tag{44}$$

Clearly, the expression within the square brackets must be zero. Now, we assume that $\rho = \rho(\mathcal{H})$, such that $X_\mathcal{H}\rho$ is given by (35). But then (44) implies

$$-\mathcal{H}\frac{\mathrm{d}\rho}{\mathrm{d}\mathcal{H}}\frac{\partial \mathcal{H}}{\partial S} - (n+1)\rho\frac{\partial \mathcal{H}}{\partial S} = 0 \quad \Longrightarrow \quad \mathcal{H}\frac{\mathrm{d}\rho}{\mathrm{d}\mathcal{H}} + (n+1)\rho = 0\ . \tag{45}$$

This is an ordinary differential equation that is easily solved by seperation of variables:

$$\rho(\mathcal{H}) = |\mathcal{H}|^{-(n+1)}\ , \tag{46}$$

where we have chosen only a positive solution as the distribution function is intrinsically nonnegative. We have arrived at the main result of this section: the volume form

$$\mathrm{d}\mu = |\mathcal{H}|^{-(n+1)}\eta \wedge (\mathrm{d}\eta)^n \tag{47}$$

is invariant under the contact Hamiltonian flow.

*II.2.4. An Example*

We will now showcase the advantage of using contact mechanics in the description of dissipative dynamics, as opposed to an effective symplectic approach. Consider a 1-dimensional mechanical system, describing a particle moving in a potential $V$, subject to a friction force that is linear in velocity. An effective Hamiltonian description of this system was introduced by Caldirola and Kanai in the 1940s [20, 21]. It is given by the time-dependent Hamiltonian

$$\tilde{H} = e^{-\alpha t} \frac{\tilde{p}^2}{2m} + e^{\alpha t} V(\tilde{q}) \,, \tag{48}$$

where $\tilde{p}$ and $\tilde{q}$ are the canonical coordinates of the system, but do not coincide with the physical coordinates. These are obtained by

$$\tilde{p} = e^{\alpha t} p \,, \quad \tilde{q} = e^{\alpha t} q \,. \tag{49}$$

This Hamiltonian gives the correct equation of motion, since

$$\dot{\tilde{q}} = e^{\alpha t} (\gamma q + \dot{q}) = \frac{\partial \tilde{H}}{\partial \tilde{p}} = e^{\alpha t} \frac{\tilde{p}}{m}, \quad \Longrightarrow \quad \dot{q} + \gamma q - \frac{1}{m} \tilde{p} = 0 \,. \tag{50}$$

Differentiating once more gives

$$\ddot{q} + \alpha \dot{q} - \frac{1}{m} \dot{\tilde{p}} = \ddot{q} + \alpha \dot{q} + \frac{1}{m} \frac{\partial \tilde{H}}{\partial \tilde{q}} = \ddot{q} + \alpha \dot{q} + \frac{e^{\alpha t}}{m} \frac{\partial V}{\partial \tilde{q}} = 0 \,. \tag{51}$$

Notice that $\partial V / \partial \tilde{q} = e^{-\alpha t} \partial V / \partial q$. Hence

$$\ddot{q} + \alpha \dot{q} + \frac{1}{m} \frac{\partial V}{\partial q} = 0 \,. \tag{52}$$

Although the dynamics can be derived without difficulties, the approach has a disadvantage, lying in (49). This relation is clearly non-canonical, leading to problems when the theory is quantized [22].

Now consider the contact Hamiltonian

$$\mathcal{H} = \frac{p^2}{2m} + V(q) + \alpha S \,. \tag{53}$$

It depends only on the coordinates of the (contact) phase space, without explicit time dependence. From (29)-(31) we obtain

$$\begin{aligned} \dot{q} &= \frac{p}{m} \,, \\ \dot{p} &= -\frac{\partial V}{\partial q} - \alpha p \,, \\ \dot{S} &= \frac{p^2}{2m} - V(q) - \alpha S \,. \end{aligned} \tag{54}$$

Differentiating the first equation and inserting the second one gives

$$\ddot{q} = \frac{\dot{p}}{m} = \frac{1}{m} \left( -\frac{\partial V}{\partial q} - \alpha p \right) = -\frac{1}{m} \frac{\partial V}{\partial q} - \alpha \dot{q} \,. \tag{55}$$

Clearly, we obtain the same equation of motion as before. In this case howerver, the contact coordinates coincide with the physical ones, and there is no need to resort to some ad hoc non-canonical transformation. In that sense, the formalism of contact mechanics provides a consistent way of describing Hamiltonian systems with dissipative dynamics.

In the last section of this thesis, we will use the theory of Hamiltonian systems to describe the dynamical systems of inflationary cosmologies. They have an inherently symplectic structure, that allows us to discuss conserved measures and Liouville's theorem in the cosmological setting. First however, we will review elementary cosmology and inflation.

## III. COSMOLOGY AND INFLATION

Before we can understand and study inflation, we have to familiarize ourselves with the framework of cosmology. This is essential, as inflation was formulated by Alan Guth as a solution to two ”fine-tuning problems” that arise in classical cosmology [23]; the standard Hot Big Bang scenario requires a set of highly fine-tuned initial conditions, in order to evolve to the state of the universe that we observe today. In particular, the motivation behind the first formulation of inflation was the flatness and horizon problem, which we will study in detail. Then we will formulate the conditions which have to be satisfied to solve these problems, and argue that the presence of a single scalar field is natural a candidate. In the discussions below, we follow the arguments of D. Baumann [5, 6] and M. Postma [24].

### III.1. Cosmology

Modern cosmology revolves around the cosmological principle, which is the idea that our universe is homogeneous and isotropic on large scales. Homogeneity means that it is invariant under translations, implying that there is no prefered position in the universe. Isotropy refers to rotational invariance, such that there is no prefered direction as well. For a very long time, this principle was a mere assumption, that was initially formulated on philosophical grounds. It was only in recent history that measurements of the cosmic microwave background (CMB) and large scale structures showed that at scales of $\sim 100\text{Mpc}$ and larger, the universe is statistically homogeneous and isotropic [1, 25]. The cosmological principle will be important in the sections that are to come, playing a prominent role in the description of the dynamics of our universe.

#### *III.1.1. The FRW Metric*

In relativity, you are usually interested in quantities that are invariant under spacetime isometries. It is therefore useful to define the metric tensor $g_{\mu\nu}$, a concept from Riemannian geometry that generalizes the first fundamental form on tangent planes of Euclidean surfaces to the tangent spaces of more general Riemannian manifolds. Metric tensors can be used to write the invariant (space-time) line element $\mathrm{d}s$ as

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}X^\mu\,\mathrm{d}X^\nu \ , \tag{56}$$

where $X^\mu = (t, x^i)$ is the coordinate 4-vector. Here, we use indices from the Greek and Latin alphabets to distinquish between spatial and more general space-time indices. Imposing spatial homogeneity and isotropy foliates spacetime into spatial hypersurfaces $\Sigma_t$, parametrized by time. It can be shown that these homogeneous and isotropic hypersurfaces are maximally symmetric spaces, which have the property of constant scalar curvature [26]. Therefore, the cosmological principle tells us that there are three options for the geometry of the universe [5]:

- flat geometry, for which the spatial line element is

  $$\mathrm{d}l^2 = \mathrm{d}x^2 \ ,$$

  where $x$ is the spatial 3-vector.

- positively curved geometry, such that $\Sigma$ is a 3-sphere of radius $a$ with

$$\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}u^2, \quad x^2 + u^2 = a^2 \ .$$

- negatively curved geometry. In this case $\Sigma$ is a hyperbolic space, with spatial line element

$$\mathrm{d}l^2 = \mathrm{d}x^2 - \mathrm{d}u^2, \quad x^2 - u^2 = -a^2 \ .$$

By rescaling $x$ and $u$ with the constant $a$, we can account for the spherical and hyperbolic cases by writing

$$\mathrm{d}l^2 = a^2 \left(\mathrm{d}x^2 \pm \mathrm{d}u^2\right), \quad x^2 \pm u^2 = \pm 1 \ . \tag{57}$$

Taking exterior derivatives of the second equation gives

$$2x\,\mathrm{d}x \pm 2u\,\mathrm{d}u = 0 \quad \Longrightarrow \quad \mathrm{d}u^2 = \frac{(x\cdot\mathrm{d}x)^2}{u^2} \quad \Longrightarrow \quad \mathrm{d}u^2 = \frac{(x\cdot\mathrm{d}x)^2}{1\mp x^2} \ . \tag{58}$$

To include the case of a flat geometry, we define

$$k \equiv \begin{cases} 0 & \text{flat} \\ +1 & \text{positively curved} \\ -1 & \text{negatively curved} \end{cases} \tag{59}$$

By combining (57) and (58) we obtain

$$\mathrm{d}l^2 = a^2 \left(\mathrm{d}x^2 + k\frac{(x\cdot\mathrm{d}x)^2}{1-kx^2}\right) = a^2\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \quad \text{where} \quad \gamma_{ij} = \delta_{ij} + k\frac{x_i x_j}{1-kx_k x^k} \ . \tag{60}$$

It will prove useful to write the metric in spherical coordinates. Therefore let $x = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$. It follows that

$$\mathrm{d}x^2 = \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \quad x\cdot\mathrm{d}x = r\,\mathrm{d}r \ . \tag{61}$$

Then (60) takes the form

$$\mathrm{d}l^2 = a^2\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2\right] \ , \tag{62}$$

where $\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$. The full spacetime interval is then written as

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}l^2 = -\,\mathrm{d}t^2 + a(t)^2\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2\right] \ . \tag{63}$$

Another useful alternative is obtained when the radial and time coordinates are redefined, according to

$$\begin{aligned} \mathrm{d}\chi &\equiv \mathrm{d}r/\sqrt{1-kr^2} \ , \\ \mathrm{d}\tau &\equiv \frac{\mathrm{d}t}{a(t)} \ . \end{aligned} \tag{64}$$

(63) then takes the form

$$\mathrm{d}s^2 = a(t)^2\left[-\,\mathrm{d}\tau^2 + \mathrm{d}\chi^2 + S_k^2(\chi)\mathrm{d}\Omega^2\right] \ , \tag{65}$$

where

$$S_k(\chi) \equiv \begin{cases} \sinh\chi & k=-1 \\ \chi & k=0 \\ \sin\chi & k=+1 \end{cases} \tag{66}$$

This formulation of the metric emphasizes the difference between physical coordinates and so-called comoving coordinates. Two points in spacetime have the same comoving coordinates when the evolution from one point to the other is only due to the Hubble flow (that is, the expansion of the universe). You can always obtain physical coordinates by multiplying with the scale factor.

The metric, independent of the form in which it is written, is often refered to as the Friedmann-Robertson-Walker (FRW) metric. Notice that the homogeneity and isotropy of each spatial hypersurface $\Sigma$ is not affected by letting $a$ depend on time. This parameter is then called the scale factor, and measures the expansion of the universe. Determining its time dependence for universes with different contents is often one of the main topics when studying cosmology.

Note that, when implementing the spatial metric into the full one, we were essentially free to choose a nontrivial time-time component of the metric, as this does not break our assumption of spatial homogeneity and isotropy. This is in fact what we will do in section IV when we consider inflation in the Hamiltonian setting.

#### *III.1.2. The Friedmann Equations*

As mentioned earlier, an important aspect of cosmology is studying the expansion of the universe through the scale factor $a(t)$. Its evolution is obtained by solving the equations of general relativity; the Einstein field equations. We will not do these calculations in full detail in this section, but give an overview of the procedure instead. For the interested reader; [5, 26] contain the complete derivations.

In general, the Einstein field equations are a set of ten differential equations, given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} , \tag{67}$$

where $R_{\mu\nu}$ is the Ricci tensor, $R$ is its trace, called the Ricci scalar, and $T_{\mu\nu}$ is the stress-energy tensor. These equations essentially couple the energy content of the universe to its geometry; $T_{\mu\nu}$ describes the energy-momentum flux and density, while $R_{\mu\nu}$ and $R$ contain information about the curvature of the universe. We will now consider these components individually and see how the cosmological principle reduces their complexity.

For arbitrary configurations, the stress-energy tensor can take complicated forms, including off-diagonal elements to account for fluxes in orthogonal directions. The assumption of homogeneity and isotropy forces the stress-energy tensor to have a highly simplified form: the one of a perfect fluid. A perfect fluid is characterised only by its time-dependent energy density $\rho(t)$ and isotropic pressure $P(t)$ [27]. The corresponding stress-energy tensor is simply

$$T^{\mu}_{\ \nu} = \mathrm{diag}\left(-\rho, P, P, P\right) . \tag{68}$$

Now we turn our attention to the Ricci curvature tensor $R_{\mu\nu}$. It is given in terms of the Christoffel symbols:

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} . \tag{69}$$

In differential geometry, Christoffel symbols arise when derivatives of tangent space basis vectors are decomposed in the original ones and can therefore be completely specified by the metric tensor, according to

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda} \left( \partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta} \right) . \tag{70}$$

As we have seen earlier, the cosmological principle reduces the metric tensor $g_{\mu\nu}$ to the compact expression given in (63). This greatly simplifies the Christoffel symbols and consequently, the Ricci tensor. The only nonzero components are

$$R_{00} = -3\frac{\ddot{a}}{a}, \quad R_{ij} = \left[ \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} \right] g_{ij} , \tag{71}$$

such that the Ricci scalar is given by

$$R = g^{\mu\nu} R_{\mu\nu} = 6 \left[ \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \right] . \tag{72}$$

We now obtain two independent equations from (67); one for $\mu\nu = 00$ and one for $\mu\nu = ij$:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} , \tag{73}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) . \tag{74}$$

These are the Friedmann equations. Given a flat, positively curved or negatively curved universe with total energy density $\rho(t)$ causing a pressure $P(t)$, the scale factor as a function of time is completely determined. In practice, these computations are difficult, since not all types of energies contribute equally to the pressure that drives the expansion of the universe. Rather, the effect of each species is determined by its equation of state:

$$P = w\rho . \tag{75}$$

The contribution of all species according to their equation of state parameter and energy density then have to be added to obtain a total pressure.

As you would expect, the energy density of most species decreases when the universe expands. There is a third equation (though not independent), called the continuity equation, that is often used to measure this relationship. It is essentially a statement of conservation of energy, and is obtained by requiring

$$\nabla_\mu T^{\mu}_{\ \nu} = 0 , \tag{76}$$

where $\nabla_\mu$ is the covariant derivative operator. It gives us

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{77}$$

. Using (75) and solving the differential equation for $\rho$, we obtain

$$\rho = C a^{-3(1+w)} . \tag{78}$$

The fall-off of the energy density therefore depends on the equation of state parameter $w$. All currently known matter sources satisfy $w > -1/3$, which is called the strong energy condition, or SEC for short. For example, non-relativistic matter has $w = 0$, such that $\rho \propto a^{-3}$, which is to be expected since the volume increases as $V \propto a^3$. For radiation, including relativistic species, $w = 1/3$. This gives $\rho \propto a^{-4}$, where the extra factor of $a^{-1}$ is accounted for by the redshifting wavelength of the radiation.

### III.2. Inflation

Our discussion so far has been entirely theoretical, describing universes of arbitrary content and saying nothing about our own. In the past few centuries, various types of experiments have been conducted to determine cosmological parameters like energy densities and curvature. In particular, spacecraft missions like WMAP and Planck that measured CMB anisotropies [1, 3] and large surveys of Type Ia supernovae redshifts have constrained the Big Bang model to a great extent [2, 4]. It was discovered that the expansion of the universe is accelerating, suggesting the existence of a component with a negative equation of state parameter. Furthermore, there is a great amount of observational evidence for the existence of dark matter, an unknown type of matter that interacts only through gravity (and possibly the weak force) [28, 29]. These results come together in the highly succesful ΛCDM model, sometimes called the standard model of cosmology.

Despite the great successes, there were still unresolved problems that could not be explained by standard Big Bang cosmology alone. In particular, the striking flatness of the observable universe and the homogeneity of the CMB temperature were still considered open problems. The horizon and flatness problems, as they are called, led theorists to explore alternatives and extensions to the classical Big Bang model. They came up with inflation, the idea that there was a period of accelerated expansion in the very early universe. We will now consider the horizon and flatness problems seperately and see how inflation offers a solution.

#### *III.2.1. The Horizon Problem*

Measurements of the CMB show that it is extremely uniform. Its temperature has fluctuations of order $\mathcal{O}(10^{-5})$ K [1]. On first sight, this does not seem to pose much of a problem. After all, we were assuming a homogeneous and isotropic universe when developing cosmological theory. The problem arises, when you take a closer look at the causal structure of the universe. To make the discussion more precise, we will define the particle horizon and the event horizon, distance measures that set the boundary for the existence of causal connections between points in spacetime. This boundary is determined by the maximum distance that light can travel in a specific amount of time. Consider the trajectory of a photon. We are free to set up our coordinate system in such a way that the angular seperation $\mathrm{d}\Omega$ vanishes. Then (65) reduces to

$$\mathrm{d}s^2 = a^2(t)\left[\mathrm{d}\tau^2 - \mathrm{d}\chi^2\right] . \tag{79}$$

As photon trajectories are null geodesics ($\mathrm{d}s = 0$), (65) gives

$$\Delta\chi(\tau) = \pm\Delta\tau , \tag{80}$$

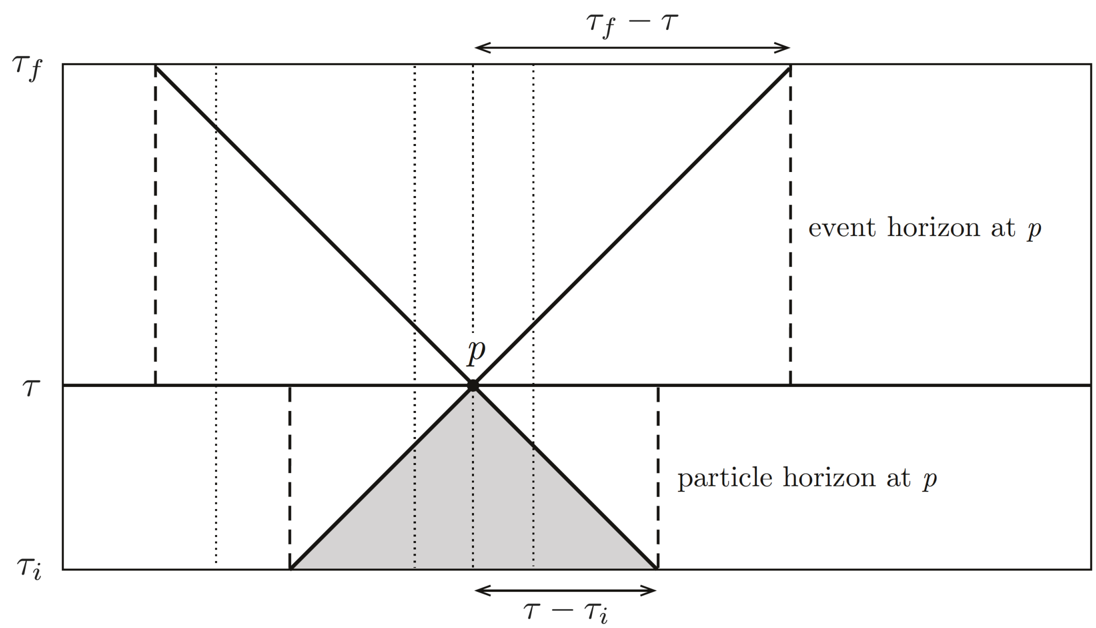


**FIG. 1:** Comoving spacetime diagram. The dotted lines indicate observers at rest with respect to the Hubble flow. Image taken from D. Baumann [5].

where the plus-minus sign accounts for incoming and outgoing photons. It follows from (80) that the maximum comoving distance that a photon can travel between some time $\tau_1$ and $\tau_2$ is just $\tau_2 - \tau_1$.

The particle horizon $\chi_{\text{ph}}(\tau)$ is defined as the maximum comoving distance from which an observer at (conformal) time $\tau$ can detect a photon that was emited at time $\tau_i$, corresponding to the Big Bang singularity. It is therefore given by

$$\chi_{\text{ph}}(\tau) = \tau - \tau_i = \int_{t_i}^{t} \frac{\mathrm{d}t}{a(t)} \,. \tag{81}$$

Analogously, we can define the event horizon, the maximum comoving distance from which an observer at some final time $t_f$ can detect photons that were emitted later than time $t$. We have

$$\chi_{\text{eh}}(\tau) = \tau_f - \tau = \int_{t}^{t_f} \frac{\mathrm{d}t}{a(t)} \,. \tag{82}$$

Notice that the conformal time $\tau_f$ is not necessarily infinite here, even when physical time $t_f$ is infinite. This depends on the functional form of the scale factor $a(t)$. In figure 1 the notions of particle and event horizons are visualized in a comoving spacetime diagram. The particle horizon corresponds to the boundary of the past lightcone, whereas the event horizon is the boundary of the future lightcone. In other words, the particle horizon tells us from what distance we can detect photons today and the event horizon gives us the maximum distance that photons, emitted today, can travel to reach us in the future. The particle horizon can be written in terms of the comoving Hubble radius $(aH)^{-1}$. This is the comoving counterpart of the Hubble radius, which is the distance beyond which all objects that are at rest with respect to each other in a comoving frame, recede at a rate faster than the speed of light [30]. This quantity is of special interest in discussions on inflation, as will become clear later. Notice that $\mathrm{d}t = \mathrm{d}a/\dot{a}$, so

$$\chi_{\text{ph}}(\tau) = \int_{t_i}^{t} \frac{\mathrm{d}t}{a} = \int_{a_i}^{a} \frac{\mathrm{d}a}{a\dot{a}} \, . \tag{83}$$

Using $\mathrm{d}\ln a = \mathrm{d}a/a$ and $1/\dot{a} = (aH)^{-1}$, we can write

$$\chi_{\text{ph}}(\tau) = \int_{\ln a_i}^{\ln a} \frac{\mathrm{d}\ln a}{\dot{a}} = \int_{\ln a_i}^{\ln a} (aH)^{-1} \, \mathrm{d}\ln a \, . \tag{84}$$

Now suppose that the early universe was dominated by a perfect fluid with equation of state parameter $w$. Furthermore, assume that the universe was spatially flat (we will see later why this is a reasonable assumption). From (73) and (78) we obtain

$$\dot{a} \propto a^{-\frac{1}{2}(1+3w)} \, . \tag{85}$$

Setting $a_0 = a(t_0) = 1$ where $t_0$ corresponds to the present moment, we can write (85) as

$$(aH)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3w)} \, . \tag{86}$$

Notice that the strong energy condition, established in section III.1.2, forces the comoving hubble radius to be an increasing function of time; when $w > -1/3$, the exponent of $a$ is positive and $a(t)$ increases monotonically. We now plug (86) into the particle horizon integral:

$$\begin{aligned} \chi_{\text{ph}}(a) = H_0^{-1} \int_{\ln a_i}^{\ln a} a^{\frac{1}{2}(1+3w)} \, \mathrm{d}\ln a &= H_0^{-1} \int_{\ln a_i}^{\ln a} \exp\left[\frac{1}{2}(1+3w)\ln a\right] \mathrm{d}\ln a \\ &= \frac{2H_0^{-1}}{(1+3w)} \left[a^{\frac{1}{2}(1+3w)} - a_i^{\frac{1}{2}(1+3w)}\right] \\ &\equiv \tau - \tau_i \, . \end{aligned} \tag{87}$$

Notice that the second term, corresponding to the conformal time at the Big Bang singularity, vanishes under the strong energy condition. Hence, we can write

$$\chi_{\text{ph}}(t) = \frac{2H_0^{-1}}{(1+3w)} a(t)^{\frac{1}{2}(1+3w)} = \frac{2}{(1+3w)} (aH)^{-1} \, . \tag{88}$$

It follows that the particle horizon is proportional to the comoving Hubble radius. The fact that the Hubble radius (and therefore the particle horizon) is an increasing function of time in conventional cosmology, is the essence of the horizon problem, as we will now explain. Consider two CMB photons, observed now at $t = t_0$. As we can only detect photons from within our Hubble horizon (the spherical region set by the Hubble radius), the comoving distance between their points of emission $\lambda(\tau_0)$ should satisfy

$$\lambda(\tau_0) < 2\chi_{\text{ph}}(\tau_0) \, . \tag{89}$$

In particular, let $\lambda(\tau_0) = \chi_{\text{ph}}(\tau_0)$. Note that $\lambda$ does not change in time, since it is a comoving quantity. However, $\chi_{\text{ph}} \propto (aH)^{-1}$. Therefore, at the time of last scattering $t_{\text{ls}}$ (when the photons decoupled from matter and the CMB was created), we had

$$\lambda(\tau_{\text{ls}}) > \chi_{\text{ph}}(\tau_{\text{ls}}) \, . \tag{90}$$

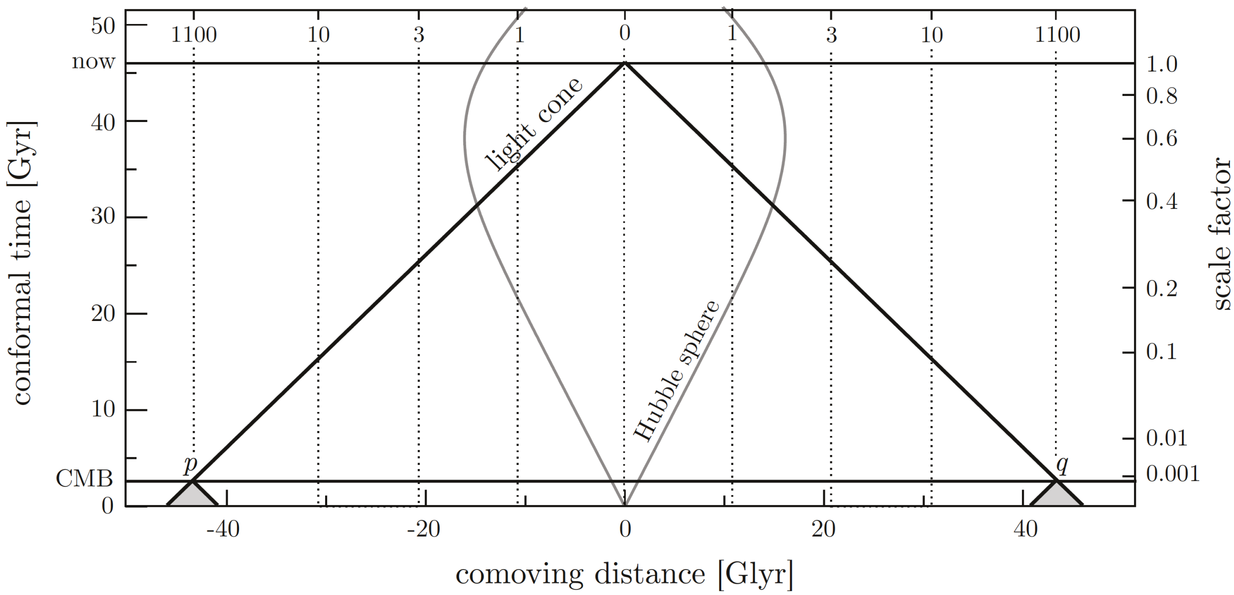


**FIG. 2:** Comoving spacetime diagram for two photons reaching us at $t = t_0$. Their past lightcones do not intersect, suggesting that the regions of the last scattering surface they emmanated from were not in causal contact. Image taken from D. Baumann [5].

In other words, the points of emission were not in causal contact at the time that the CMB was created. In fact, the sky can be divided into 40.000 patches, about 0.02 radians apart, that were not causally connected at the time of last scattering [30]. This is in clear conflict with measurements; the CMB is near-homogeneous, suggesting an early universe in which these patches could reach thermal equilibrium. The situation is depicted in figure 2. As only a finite amount of time had passed between the Big Bang singularity and last scattering, the past lightcones of the two photons do not intersect and therefore they are not causally connected.

### *III.2.2. The Flatness Problem*

The results of the Planck mission show that the spatial geometry of the universe today is very close to flat [31]. This in itself can already be considered an unlikely coincidence, but as we will see, spatial flatness is an unstable equilibrium, from which the universe diverges. Before we proceed, we will rewrite the Friedmann equation in a way that is convenient for this discussion. Recall that for a flat universe ($\kappa = 0$), the Friedmann equation reads

$$H^2(t) = \frac{8\pi G}{3}\rho(t) \ . \tag{91}$$

The energy density corresponding to a flat universe is often refered to as the critical density $\rho_c(t)$, i.e.

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G} \ . \tag{92}$$

Using the critical density, we can define the dimensionless density parameter $\Omega(t)$, given by

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_c(t)} \ . \tag{93}$$

Dividing the full Friedmann equation (73) by $\rho_c$ and using (92) yields

$$1 - \Omega = -\frac{k}{(aH)^2} \ . \tag{94}$$

Notice that $1-\Omega$ is a measure of spatial curvature; for flat universes we have $\rho = \rho_c$ and consequently $\Omega = 1$, such that the left-hand side of (94) vanishes. The right-hand side is written in terms of the comoving Hubble radius $(aH)^{-1}$, a convenience that will be used later on. It follows from (94) that

$$|1 - \Omega| \propto (aH)^{-2} \ . \tag{95}$$

As established in section III.2.1, the Hubble radius is a strictly increasing function of time in classical cosmology. This implies that, as time passes, any spatial curvature that was there in the very early universe is amplified. The extent to which this amplification occurs depends on the scale factor. At present, we have $|1-\Omega| \approx 0.001$ [32]. By using data from the ΛCDM model and (95), it can be shown that during the planck era,

$$\Big|1 - \Omega(t_{\rm pl})\Big| \leq \mathcal{O}(10^{-61}) \ . \tag{96}$$

In other words, the universe must have been astonishingly flat near the origin. This is the flatness problem; a priori, there is no reason for the universe to have such a highly fine-tuned initial condition.

### *III.2.3. The Inflationary Solution*

Now that we have seen two problems of classical Big Bang cosmology, we will argue that they can be solved with a single solution. Notice that both the horizon and flatness problem were formulated using the comoving Hubble radius $(aH)^{-1}$, a quantity that increases monotonically in the conventional theory. Indeed, the growing Hubble sphere ensured that early time particle horizons were small and that spatial curvature was driven away from flatness. The idea of inflation is therefore simple; a phase in which the Hubble sphere shrinks for long enough to solve the horizon and flatness problem. Notice that (86) implies that inflation must violate the strong energy condition, requiring the existence of a fluid with $w < -1/3$. Furthermore, a decreasing Hubble radius also corresponds to accelerated expansion of the universe:

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{a})^{-1} = -\frac{\ddot{a}}{(\dot{a})^2} \ , \tag{97}$$

from which we conclude that $\ddot{a} > 0$.

With the shrinking Hubble spere in mind, we take another look at (87). The condition $w < -1/3$ now implies that the term representing $\tau_i$ does not vanish, but rather approaches $-\infty$ when the scale factor is taken to 0. Put differently, the Big Bang singularity is pushed back to $\tau_i = -\infty$, implying that conformal time was negative during inflation. This negative

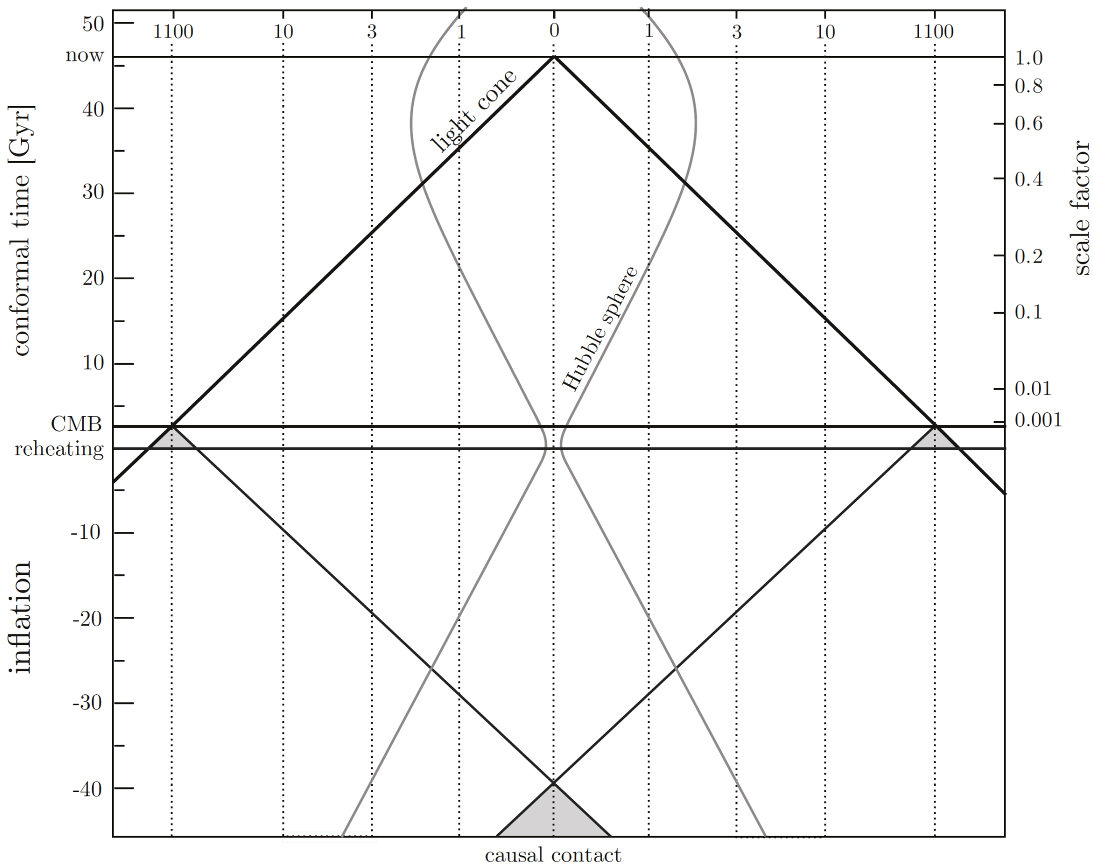


**FIG. 3:** Comoving spacetime diagram, with the Big Bang singularity now at $\tau = -\infty$. Image taken from D. Baumann [5].

conformal time allows us to trace back the past lightcones of patches of the last scattering surface until they overlap, as shown in figure 3. This solves the horizon problem; inflation asserts the existence of a period in which causal regions were large enough to establish thermal equilibrium.

The flatness problem is solved quite straightforwardly. Recall that $|1 - \Omega|$, a measure of spatial curvature of the universe, satisfies

$$|1 - \Omega| \propto (aH)^{-2} \ . \tag{98}$$

A period of decreasing Hubble radius would clearly suppress any spatial curvature present at the Big Bang singularity. If inflation is allowed to last long enough, the curvature at $\tau = 0$ would be consistent with the values given in III.2.2. This solves the flatness problem.

### III.3. The Physics of Inflation

We have seen that inflation, in the form of a sufficiently long period of decreasing Hubble radius, solves the horizon and flatness problems. In this section, we develop some useful concepts to help us simplify the conditions for inflation and propose a physical source that could have led to the shrinking Hubble sphere.

### III.3.1. *Useful Concepts*

We introduce two additional parameters to describe the characteristics of specific inflationary scenarios. Firstly, let $\varepsilon \equiv -\dot{H}/H^2$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1-\varepsilon)\ . \tag{99}$$

Requiring a shrinking Hubble sphere is therefore equivalent to requiring $\varepsilon < 1$. In literature on inflation, authors often quantify the amount of expansion during the inflationary phase in terms of the number of $e$-folds $N$. If $t_i$ is the time at which inflation starts, then the number of $e$-folds of expansion up to time $t$ is given by

$$N(t) = \int_t^{t_f} H(t)\,\mathrm{d}t\ . \tag{100}$$

It follows that $\mathrm{d}N = H\,\mathrm{d}t$, such that

$$\varepsilon = -\frac{\mathrm{d}\ln H}{\mathrm{d}N}\ . \tag{101}$$

As mentioned earlier, inflation must have lasted long enough to solve the horizon and flatness problems. The parameter $\varepsilon$ should therefore not only be small, but also maintain a small value for a sufficient amount of time, or equivalently, for a sufficient number of $e$-folds. We define the second parameter $\eta$, a measure of the change in $\varepsilon$, as

$$\eta \equiv \frac{\mathrm{d}\ln\varepsilon}{\mathrm{d}N} = \frac{\dot{\varepsilon}}{H\varepsilon}\ , \tag{102}$$

with the corresponding requirement $|\eta| < 1$.

### III.3.2. *The Inflaton*

So far we have discussed inflation as an unspecified mechanism that sets the values of certain parameters. We will consider a scalar field as the physical origin for this mechanism and see how the constraints of inflation affects its characteristics. This scalar field $\phi(t,x)$ is called the inflaton. Notice that the requirement of isotropy and homogeneity suppresses the dependence on spatial coordinates of this field at large scales. Therefore we will often denote the inflaton simply as $\phi(t)$. It will be convenient for later sections to describe the dynamics of $\phi$ in the framework of classical field theory. To that end, consider the action $S_\phi$ associated to $\phi$, which is just the standard action for a dynamical scalar field with potential $V(\phi)$:

$$S_\phi = \int \mathrm{d}^4x\sqrt{-g}\,\mathcal{L}_\phi = \int \mathrm{d}^4\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]\ , \tag{103}$$

where $g = \det g_{\mu\nu}$ and $\mathcal{L}_\phi$ is the Lagrangian density of the inflaton. The stress-energy tensor for $\phi$ is calculated from the identity

$$T_{\mu\nu} = -2\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}\ . \tag{104}$$

We have

$$g_{\mu\nu}\mathcal{L}_\phi = -g_{\mu\nu}\left(\frac{1}{2}\partial^\alpha\phi\partial_\alpha\phi + V(\phi)\right), \qquad \frac{\partial\mathcal{L}_\phi}{\partial g^{\mu\nu}} = -\frac{1}{2}\partial_\mu\phi\partial_\nu\phi \ , \tag{105}$$

and therefore

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\alpha\phi\partial_\alpha\phi - V(\phi)\right) \ . \tag{106}$$

From (68), we know that $-\rho_\phi = T^0_{\ 0}$ and $P_\phi\delta^i_{\ j} = T^i_{\ j}$. Using $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -\dot{\phi}^2$ and the induced independence of $\phi$ on spatial coordinates, we obtain

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \ . \tag{107}$$

Recall that in general, the density and pressure are related by the equation of state parameter $w$. It follows that for the inflaton,

$$w_\phi = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \ . \tag{108}$$

Violation of the strong energy condition, i.e. $w_\phi < -1/3$, now manifests itself in the configuration of the field; the potential energy $V(\phi)$ must dominate over the kinetic energy $\dot{\phi}^2/2$.

In the remainder of this section, we will derive $\varepsilon$ and $\eta$, the slow-roll parameters for the inflaton. To do so, we will find the Hubble parameter using the Friedmann equation, in the approximation of a flat universe:

$$H^2 = \frac{8\pi G}{3}\rho_\phi = \frac{1}{3M_{\rm Pl}^2}\rho_\phi = \frac{1}{3M_{\rm Pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \ , \tag{109}$$

where we have used $M_{\rm Pl} = (8\pi G)^{-1/2}$. The last equation can be differentiated to obtain $\dot{H}$. Recalling that $\varepsilon = -\dot{H}/H^2$, we get

$$\varepsilon = \frac{\dot{\phi}^2}{2M_{\rm Pl}^2 H^2} \ . \tag{110}$$

For inflation to occur, we require $\varepsilon < 1$. It follows that the kinetic energy $\dot{\phi}^2/2$ should be small compared to $M_{\rm Pl}^2 H^2 = \rho_\phi/3$. Again, this implies that the kinetic energy of $\phi$ should be small compared to its potential energy. This is called slow-roll inflation.

For inflation to persist, $\varepsilon$ needs to be small for a sustained period of time. This corresponds to $|\eta| < 1$. From (102) we obtain

$$\eta = \frac{\dot{\varepsilon}}{H\varepsilon} = 2\left(\frac{\ddot{\phi}}{H\dot{\phi}} - \frac{\dot{H}}{H^2}\right) \ . \tag{111}$$

Let $\delta \equiv -\ddot{\phi}/H\dot{\phi}$. We can rewrite $\eta$ in terms of $\delta$ and $\varepsilon$, according to

$$\eta = 2(\varepsilon - \delta) \ . \tag{112}$$

Notice that if both $\varepsilon$ and $\delta$ are much smaller than unity, than so is $|\eta|$. In the next section, the condition $\{\varepsilon, \delta\} \ll 1$ will be used to to develop the slow-roll approximation, which offers a great simplification of the dynamics, as well as a way to assess theories of inflation on their ability to drive the accelerated expansion.

First, consider the condition $\varepsilon \ll 1$. Then $\dot{\phi}^2/2 \ll \rho_\phi$ and therefore $\dot{\phi}^2/2 \ll V(\phi)$. This can be used to simplify the Friedmann equation (109):

$$H^2 \approx \frac{V(\phi)}{3M_{\mathrm{Pl}}^2} \ . \tag{113}$$

Another useful simplification comes from the equation of motion for $\phi$, which can be obtained from the two Friedmann equations, or by deriving it directly fromt the action $S_\phi$. The latter method amounts to finding the Euler-Lagrange equation for $\phi$ from (103). This yields the Klein-Gordon equation:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \ . \tag{114}$$

Notice that when $\delta \ll 1$, then $\ddot{\phi}$ is insignificant compared to $3H\dot{\phi}$. The Klein-Gordon equation then reduces to

$$3H\dot{\phi} \approx -V'(\phi), \qquad 3\dot{H}\dot{\phi} + 3H\ddot{\phi} \approx -V''\dot{\phi} \ . \tag{115}$$

Expressions (113) and (115) can now be combined in (110) to obtain an approximation for $\varepsilon$:

$$\varepsilon = \frac{\frac{1}{2}\dot{\phi}^2}{M_{\mathrm{pl}}^2 H^2} \approx \frac{M_{\mathrm{pl}}^2}{2}\left(\frac{V'}{V}\right)^2 \ . \tag{116}$$

We can make a similar approximation for $\eta$. Notice that when $\varepsilon$ and $\delta$ are very small, then $2(\varepsilon - \delta) \approx \varepsilon - \delta \approx \varepsilon + \delta$. Using (115),

$$\begin{aligned} \varepsilon + \delta = -\frac{\ddot{\phi}}{H\dot{\phi}} - \frac{\dot{H}}{H^2} &\approx -\frac{\ddot{\phi}H}{\dot{\phi}}\left(\frac{3M_{\mathrm{Pl}}^2}{V}\right) - \dot{H}\left(\frac{3M_{\mathrm{Pl}}^2}{V}\right) \\ &= \frac{-M_{\mathrm{Pl}}^2}{V}\left[\frac{3\ddot{\phi}H}{\dot{\phi}} + 3\dot{H}\right] \\ &\approx M_{\mathrm{Pl}}^2 \frac{V''}{V} \ . \end{aligned} \tag{117}$$

These approximations provide a straightforward method to test the capability of a model with potential $V(\phi)$ to initiate slow-roll inflation; the slow-roll parameters

$$\epsilon_{\mathrm{v}} \equiv \frac{M_{\mathrm{pl}}^2}{2}\left(\frac{V'}{V}\right)^2 \quad , \quad |\eta_{\mathrm{v}}| \equiv M_{\mathrm{pl}}^2 \frac{|V''|}{V} \ . \tag{118}$$

should satisfy $\{\varepsilon_v, |\eta_v|\} \ll 1$. An example of a typical potential that leads to slow-roll inflation is shown in figure 4.

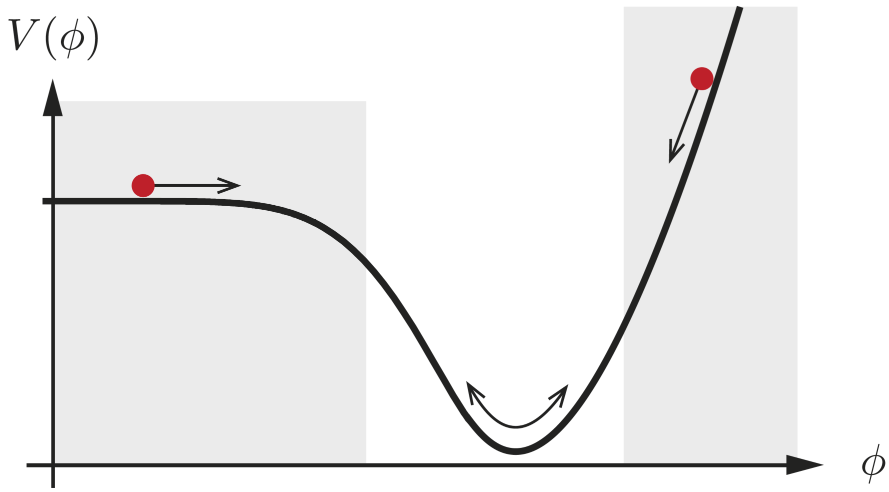


**FIG. 4:** A typical slow-roll potential. The grey area corresponds to $\varepsilon < 1$. Image by D. Baumann [5].

## IV. ATTRACTORS IN INFLATIONARY COSMOLOGY

Now we will study the evolution of inflationary cosmologies. These are dynamical systems whose equations of motion are the solutions to Einstein's equations for FRW universes in the presence of a the inflaton $\phi$. The system allows for a Hamiltonian description, which we will soon develop. Then, we will take a closer look at the notion of an (inflationary) attractor and compare it to its definition in mathematical literature. It will become clear that there are fundamental differences and that no true attractor behaviour can exist in scalar field cosmologies. Finally, we will follow the steps of Carroll and Remmen [11] to develop an effective phase space in the $\phi$-$\dot{\phi}$ coordinates for universes without spatial curvature.

### IV.1. Hamiltonian Formulation of Inflationary Cosmology

Our starting point will be the generic action for a scalar field, minimally coupled to gravity, i.e., without terms that couple $\phi$ directly to $R$.

$$\begin{aligned} S &= S_{\text{EH}} + S_\phi \\ &= \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\text{Pl}}}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] , \end{aligned} \tag{119}$$

where $S_{EH}$ is the Einstein-Hilbert action and $S_\phi$ is the action for the inflaton with potential $V(\phi)$, as given in III.3.2. Recall that the FRW metric $g_{\mu\nu}$ can be expressed as the infinitesimal line element

$$\mathrm{d}s^2 = -N^2(t)\,\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\,\mathrm{d}\Omega^2\right) , \tag{120}$$

with $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$. The factor $N(t)$ is called the lapse function. It arises in the Hamiltonian formulation of general relativity as an additional degree of freedom when spacetime is foliated into spatial hypersurfaces, parametrized by time [33]. This will not be important for the rest of this discussion.

We will now use (119) and (120) to express the Lagrangian density in the coordinates $a$, $\phi$ and their derivatives. The diagonal form of the metric allows us to write

$$\sqrt{-g} = \sqrt{-\det g_{\mu\nu}} = \sqrt{\frac{N^2 a^6 r^4 \sin^2\theta}{1-\kappa r^2}} = \frac{N a^3 r^2 \sin\theta}{\sqrt{1-\kappa r^2}} \; . \tag{121}$$

Furthermore, The requirement of isotropy and homogeneity forces $\phi$ to depend only on time. Hence

$$-\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \frac{\dot{\phi}}{2N^2} \; . \tag{122}$$

The Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ is computed from the metric according to (69) and (70). We find

$$R = \frac{6}{N^2}\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) + 6\frac{\kappa}{a^2} \; . \tag{123}$$

Combining the results gives

$$\mathcal{L} = \frac{r^2\sin\theta}{\sqrt{1-\kappa r^2}}\left[3M_{\mathrm{Pl}}^2\left(Na\kappa \; + \frac{\ddot{a}a^2}{N} + \frac{\dot{a}^2 a}{N}\right) + a^3\left(\frac{\dot{\phi}}{2N} - NV(\phi)\right)\right] \; . \tag{124}$$

Only the factor on the left depends on the spatial coordinates, as $a$ and $\phi$ depend only on time. The spatial dependence can be integrated out to yield a multiplicative constant, which can be discarded without affecting the dynamics (i.e. the equations of motions remain the same). Another simplification is obtained by subtracting a total derivative from the Lagrangian, under which the dynamics are invariant as well. We have

$$\ddot{a}a^2 + 2\dot{a}^2 a = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{a}a^2\right) \; . \tag{125}$$

Setting $M_{\mathrm{Pl}} = 1$ for notational convenience further reduces the expression to

$$\begin{aligned} L(t) &= \int \mathrm{d}^3x\,\mathcal{L} \\ &= 3\left(Na\kappa - \frac{a\dot{a}}{N}\right) + a^3\left[\frac{\dot{\phi}}{2N} - NV(\phi)\right] \; . \end{aligned} \tag{126}$$

From this form of the Lagrangian, we can compute the conjugate momenta according to

$$p_i = \partial L/\partial \dot{q}^i \; . \tag{127}$$

It follows that

$$p_N = 0, \quad p_a = \frac{6a\dot{a}}{N}, \quad p_\phi = \frac{a^3\dot{\phi}}{N} \; . \tag{128}$$

We can now substitute the conjugate momenta in the Lagrangian, and perform a Legendre transformation to obtain the Hamiltonian:

$$\begin{aligned}\mathcal{H} &= p_\phi \dot{\phi} + p_a \dot{a} - L \\ &= N\left[-\frac{p_a^2}{12a} + \frac{p_\phi^2}{2a^3} + a^3 V(\phi) - 3a\kappa\right] \\ &= -3a^3 N\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} - \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)\right] .\end{aligned} \tag{129}$$

From (129) it is clear that N is not dynamical; the Hamilton equation for $N$ gives $\dot{N} = 0$. The equation for $p_N$ is

$$\dot{p}_N = -\frac{\partial \mathcal{H}}{\partial N} = 3a^3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} - \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)\right] = 0 . \tag{130}$$

However, this can only hold when

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) - \frac{\kappa}{a^2} , \tag{131}$$

where $H(t) = \dot{a}/a$ is the Hubble parameter. This is the Friedmann equation for a universe whose only consituent is a scalar field $\phi$. In the setting of Hamiltonian systems, it is thought of as a constraint equation; given the full four-dimensional phase space $\Gamma = (\phi, p_\phi, a, p_a)$ and an (arbitrary) choice for $\kappa$, we can use (131) to eliminate one phase space coordinate to obtain the three-dimensional constraint manifold $C$.

### IV.2. Inflationary Attractors

In literature on inflation, states to which the system evolves from a large set of initial conditions are often called attractor solutions. This makes sense intuitively; these trajectories seem to attract the dynamics from a neighbourhood of the space that they are a part of. Often, such statements are made for the space spanned by $\phi$ and $\dot{\phi}$. As we will see later on, this is a natural choice as well. There is, however, an apparent contradiction between this description of attractors and the theory that was developed in section IV.1. Indeed, Hamiltonian systems do not allow for attractors, as they are subject to Liouville's theorem (see also section II.1.3).

The solution to this issue is found when taking a closer look at the mathematical definition of an attractor. In dynamical systems theory, an attractor is a subset of the phase space of a system, for which there exists a region such that every trajectory that intersects the region eventually converges to the attractor. Of course, this conceptual description is based on more rigorous definitions [9, 10], but it is all we really need here. The important thing is that attractors are defined for the complete phase space which, by definition, is spanned by coordinates that are canonically conjugate; they satisfy (127). This is clearly not true for $\phi$-$\dot{\phi}$ space, as $\phi$ satisfies (128). Attractor-like behaviour in these variables is coordinate dependend; it disappears when choosing a different set of coordinates. This can be seen

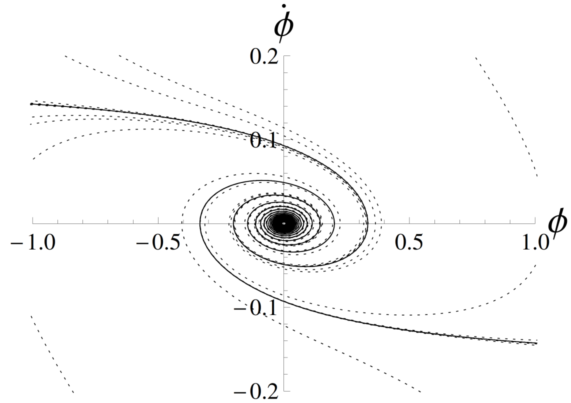


**FIG. 5:** Apparant attractor behaviour around $\phi = 0$ in the $\phi$-$\dot{\phi}$ coordinates. Image take from Remmen and Carroll [11].

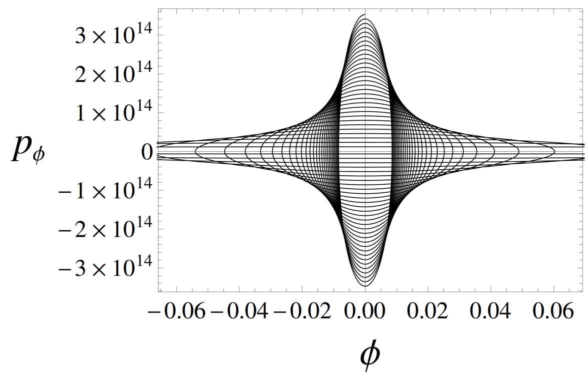


**FIG. 6:** In $\phi$-$p_\phi$ coordinates, there is no such attractor behaviour. In fact, trajectories diverge at $\phi = 0$. Image take from Remmen and Carroll [11].

from figure 5 and 6, in which trajectories for a system with $V(\phi) = \frac{1}{2}m^2\phi^2$ are projected onto $\phi$-$\dot{\phi}$ and $\phi$-$p_\phi$ space. Moreover, $\phi$ and $\dot{\phi}$ do not span the complete phase space, which was determined to be four-dimensional. Liouville's theorem dictates that when trajectories converge in some subset of variables of $\Gamma$, they should diverge in the complement in order to preserve phase space volume.

Strictly speaking, use of the word "attractor" in $\phi$-$\dot{\phi}$ coordinates is not justified in the context of inflation. Nevertheless, there is way in which this choice of variables is more than just an arbitrary projection onto a subspace of $\Gamma$. We will see that it constitutes an "effective" phase space for flat universes ($\kappa = 0$), in the sense that the dynamics of the system can be expressed in $\phi$ and $\dot{\phi}$ alone. More precisely, this space has the important property that te evolution of a phase space point is completely fixed once the initial data is specificied.

### IV.3. Vector Field Invariance and Effective Phase Spaces

It is well known that phase space trajectories of autonomous systems do not cross [34]. In this section, we will show that for flat universes, $\phi$-$\dot{\phi}$ space possesses this property. First we introduce the notion of vector field invariance.

#### *IV.3.1. Vector Field Invariance*

Let $M$ and $N$ be smooth manifolds and let $X$ be a smooth vector field on $M$. Furthermore, let $\mathcal{F}(M)$ and $\mathcal{F}(N)$ be the spaces of smooth functions on $M$ and $N$, respectively. Then the map $\psi : M \to N$ is called vector field invariant with respect to X if for any $f \in \mathcal{F}(N)$ and all $q \in N$, we have

$$X_p(\psi^* f) = X_{p'}(\psi^* f) \tag{132}$$

for all $p, p' \in \psi^{-1}(q)$ [11]. Notice that this unambiguously defines a vector field $\tilde{X}$ on N; the pushforward of X at $q \in N$ is given by

$$\tilde{X}_q(f) = (\psi_* X)_{\psi(p)}(f) = X_p(\psi^* f)\ . \tag{133}$$

If $X$ defines the same vector for all $p, p' \in \psi^{-1}(q)$, then the above equality is well defined. Now consider two distinct integral curves of $X$ on $M$. Intersection of their images under the vector field invariant map $\psi : M \to N$ would contradict the uniqueness of the induced vector field $\tilde{X}$ on $N$. Therefore, no such intersections can exist. We know from section II.1.2 that trajectories in phase space are just integral curves of the Hamiltonian vector field. Consequently, a vector field invariant map $\psi$ from the phase space $M$ of a Hamiltonian system to some (possibly smaller) space $N$, defines an "effective" phase space, in which trajectories do not intersect.

The rest of this section will be dedicated to showing that the map $\mathcal{P} : C \to K$ is vector field invariant with respect to the Hamiltonian vector field $X_{\mathcal{H}}$. $C$ is the the constraint manifold induced by (131) and $K$ is a two-dimensional manifold that has the property that its inverse image under $\mathcal{P}$ contains all points in $C$ that have the same (specific combination of) $\phi$ and $\dot{\phi}$ values. Notice that this implies that $K$ is isomorphic to $\phi$-$\dot{\phi}$; there is a one-to-one correspondence between points $q \in K$ and $(\phi_q, \dot{\phi}_q)$ in $\phi$-$\dot{\phi}$ space. Recall that the Hamiltonian vector field to a corresponding Hamiltonian function $\mathcal{H}$ is given by

$$\begin{aligned} X_{\mathcal{H}} &= -\frac{\partial \mathcal{H}}{\partial q^a}\frac{\partial}{\partial p_a} + \frac{\partial \mathcal{H}}{\partial p_a}\frac{\partial}{\partial q^a} \\ &= X_{\mathcal{H}}^{(p_a)}\frac{\partial}{\partial p_a} + X_{\mathcal{H}}^{(q^a)}\frac{\partial}{\partial q^a} \ . \end{aligned} \tag{134}$$

From (129), we obtain

$$\begin{aligned} X_{\mathcal{H}}^{(\phi)} &= \frac{p_\phi}{3} \ , \\ X_{\mathcal{H}}^{(p_\phi)} &= -a^3 V'(\phi) \ , \\ X_{\mathcal{H}}^{(a)} &= -\frac{p_a}{6a} \ , \\ X_{\mathcal{H}}^{(p_a)} &= -\frac{p_a^2}{12a^2} + \frac{3p_\phi^2}{2a^4} - 3a^2 V(\phi) + 3\kappa \ . \end{aligned} \tag{135}$$

We now make a change of variables in $\Gamma$. We choose the four independent coordinates $(\phi, \dot{\phi}, a, H)$. Using (128), we rewrite (135) as

$$\begin{aligned} X_{\mathcal{H}}^{(\phi)} &= \dot{\phi} \ , \\ X_{\mathcal{H}}^{(\dot{\phi})} &= \frac{1}{a^3} X_{\mathcal{H}}^{(p_\phi)} = -V'(\phi) \ , \\ X_{\mathcal{H}}^{(a)} &= \dot{a} \ , \\ X_{\mathcal{H}}^{(H)} &= -\frac{1}{6a^2} X_{\mathcal{H}}^{(p_a)} = \frac{1}{2}H^2 - \frac{1}{4}\dot{\phi}^2 + \frac{1}{2}V(\phi) - \frac{\kappa}{2a^2} \ . \end{aligned} \tag{136}$$

As can be seen from the $H$-component of the vector field, a great simplification of the dynamics presents itself when $\kappa = 0$. The otherwise coupled equations of motion now decouple, such that $a$ and $\dot{a}$ do not affect the motion in the $H$, $\phi$ and $\dot{\phi}$ directions. The constraint equation (131) allows us to eliminate $H$ for $\phi$ and $\dot{\phi}$. Thus, the $H$, $\phi$ and $\dot{\phi}$ components of $X_{\mathcal{H}}$ can be expressed solely in $\phi$ and $\dot{\phi}$. For the projection map $\mathcal{P}$ given by

$$\mathcal{P}(a, \phi, \dot{\phi}, H) = (\phi, \dot{\phi}) \ , \tag{137}$$

the requirement of vector field invariance with respect to $X_{\mathcal{H}}$ reduces exactly to this condition; the components of vectors $X_{\mathcal{H}}(p)$ with $p \in \mathcal{P}^{-1}(q)$ only depend on the point $q = (\phi_q, \dot{\phi}_q)$. Thus, for a flat universe, the projection $\mathcal{P}$ is vector field invariant and we can conclude that $\phi$-$\dot{\phi}$ is an "effective" phase space, in the sense that was discussed earlier. It is important to note that this is not a trivial property. Maps that project $\Gamma$ onto a different two-dimensional subspace are not vector field invariant and the arguments above do not hold for $\kappa \neq 0$. It is therefore justified to consider $\phi$ and $\dot{\phi}$ as "special" coordinates, and study the properties of this parametrization in more detail.

#### *IV.3.2. The Geometrical Approach*

To conclude this section, we will prove the vector field invariance of the projection $\mathcal{P}$ using more geometrical arguments, for the quadratic potential $V(\phi) = \frac{1}{2}m^2\phi^2$. Notice that (131) defines a distinct constraint equation for each value of $\kappa$. We can therefore define an equivalence relation on $\Gamma$, such that two points are in the same equivalence class when they satisfy the same constraint equation. This relation defines a foliation on $\Gamma$; each leaf of the foliation corresponds to a three-dimensional constraint manifold $C$. Choosing a specific value $a = a_*$ yields a two-dimensional surface $C_{a_*}$, which is contained in the three-dimensional manifold $\Gamma_{a_*}$, the set of points in $\Gamma$ with $a = a_*$. Notice that in general, $C$ does not have a fibre bundle structure; the fibre space $C_a$ is different for each choice of $a$, because (131) is $a$-dependent. Rather, $C$ is obtained by taking the fibration of the manifolds $C_a$ with $\mathbb{R}_{>0}$, the positive real numbers given by $a$. For flat universes, the Hamiltonian constraint becomes independent of $a$. This is the one case for which each $C$ becomes factorizable and can thus be written as the fibre bundle $C_{a_*} \times \mathbb{R}_{>0}$, where the choice of $a_*$ can be made arbitrarily since each fibre defines the same space. In other words, varying $a$ does not affect the manifold on which the dynamics are defined. This can be seen from figure 7. For the cases $\kappa = 0.5$ and $\kappa = -0.5$ the constraint surface clearly changes its appearence when varying $a$. For $\kappa = 0$ however, it is the same for each $a$.

In the previous paragaph, the decoupling of the dynamics for $\kappa = 0$ was shown to be equivalent to vector field invariance of the projection $\mathcal{P}$ onto $\phi$-$\dot{\phi}$ space with respect to $X_{\mathcal{H}}$. This too can be seen from figure 7. $X_{\mathcal{H}}$ is a vector field on each surface $C_a$, describing the transformation of the surface under its flow. The field can pushed forward pointwise from the tangent spaces of $C_a$ to the tangent spaces of $\phi$-$\dot{\phi}$ space. This process can be visualized by projecting the tangent vectors of the $C_a$'s down onto $\phi$-$\dot{\phi}$ space. Vector field invariance of $\mathcal{P}$ for flat universes now follows: the projected vector fields are the same for all values of $a$, so a unique vector field $\tilde{X}_{\mathcal{H}}$ is defined on $\phi$-$\dot{\phi}$ space.

### IV.4. Effective Phase Space Measures

In section IV.3 we have seen that for flat universes, $\phi$ and $\dot{\phi}$ are special variables, in the sense that the space that they span allows for a complete description of the dynamics. We have made this precise by showing that the map $\mathcal{P}$ that projects the full phase space $\Gamma$ onto $\phi$-$\dot{\phi}$ space is vector field invariant with respect to the Hamiltonian vector field. This implies that trajectories do not intersect, such that every point is part of a unique trajectory, whose evolution is completely fixed. This property justifies the use of the term "effective" phase space. However, in chapter II.1 we have seen that phase spaces are essentially cotan-

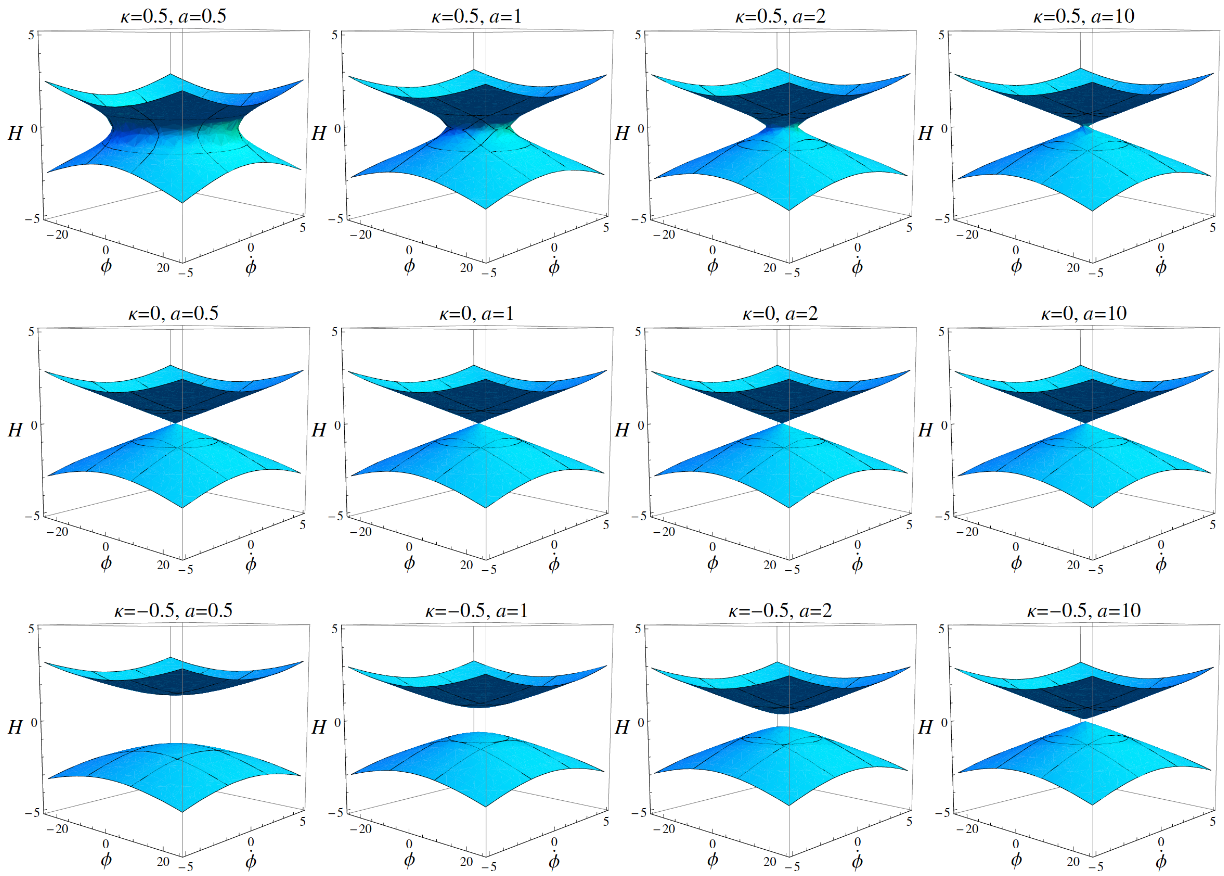


**FIG. 7:** Plots of two-dimensional constraint surfaces $C_a$ corresponding to $V(\phi) = (1/2)m^2\phi^2$, embedded in $\Gamma_a$ for different values of $a$ and $\kappa$.

gent bundles of configuration manifolds, which constitute an important class of symplectic manifolds. $\phi$-$\dot{\phi}$ space is not symplectic: $\phi$ and $\dot{\phi}$ are not canonically conjugate. Hence, we cannot simply assert the existence of a conserved volume form, or measure, by invoking Liouville's theorem. In this section, we will develop conditions for a conserved measure to exist. More precisely, we will show that a conserved measure exists if and only if there exists a Lagrangian from which the dynamics in $\phi$-$\dot{\phi}$ space can be derived.

### *IV.4.1. Conserved Measures*

Let the vector field on $\phi$-$\dot{\phi}$ space that is induced by $X_{\mathcal{H}}$ under $\mathcal{P}$ be denoted by $h$. Also, let $x = \phi$ and $y = \dot{\phi}$. In general, a volume form on a manifold is a differential form of top degree. Therefore, a measure on $\phi$-$\dot{\phi}$ space should be a 2-form $\tau$. It can always be written as

$$\tau = f(x, y)\, \mathrm{d}x \wedge \mathrm{d}y \;, \tag{138}$$

where $f(x, y)$ is a function on $\phi$-$\dot{\phi}$ space. We have already seen in section II.1.3 that conservation of $\tau$ along the flow of $h$ corresponds to the requirement

$$\pounds_h \tau = 0 \;. \tag{139}$$

We can expand the left-hand side of (139) using the Cartan formula (19):

$$\pounds_h \tau = \iota_h \, \mathrm{d}\tau + \mathrm{d}(\iota_h \tau) \ . \tag{140}$$

The first term on the right is an exterior derivative of a volume form, and hence vanishes identically. Notice that

$$\begin{aligned} \iota_h(\tau) &= \iota_h(f \, \mathrm{d}x \wedge \mathrm{d}y) \\ &= f \iota_h(\mathrm{d}x) \, \mathrm{d}y - f \iota_h(\mathrm{d}y) \, \mathrm{d}x \\ &= f h_x \, \mathrm{d}y - f h_y \, \mathrm{d}x \ , \end{aligned} \tag{141}$$

where $h_x$ and $h_y$ are the $x$ and $y$ components of $h$. Consequently, (140) reduces to

$$\begin{aligned} \pounds_h \tau &= \mathrm{d}(f h_x \, \mathrm{d}y - f h_y \, \mathrm{d}x) \\ &= h_x \, \mathrm{d}f \wedge \mathrm{d}y + f \, \mathrm{d}v_x \wedge \mathrm{d}y - v_y \, \mathrm{d}f \wedge \mathrm{d}x - f \, \mathrm{d}v_y \wedge \mathrm{d}x \\ &= \left( v_x \frac{\partial f}{\partial x} + f \frac{\partial v_x}{\partial x} + v_y \frac{\partial f}{\partial y} + f \frac{\partial v_y}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y = 0 \ . \end{aligned} \tag{142}$$

Notice that the terms within the brackets can be written as

$$v_x \frac{\partial f}{\partial x} + f \frac{\partial v_x}{\partial x} + v_y \frac{\partial f}{\partial y} + f \frac{\partial v_y}{\partial y} = f(\nabla \cdot h) + h \cdot (\nabla f) = \nabla \cdot (fh) \ . \tag{143}$$

Therefore, the condition for conservation of the measure can be written in vector form:

$$\nabla \cdot (fh) = 0 \ . \tag{144}$$

This looks very similar to the (continuity) Euler equation for incompressible fluids [35]:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho h) = 0 \ . \tag{145}$$

where in our case $\partial \rho / \partial t = 0$. It tells us that the change in density over time must equal the in- or outflux of "fluid", which implies conservation of mass. Therefore, we can think of $f$ as the density of system points in effective phase space, knowing that it is conserved under the flow of the Hamiltonian vector field.

We will now find an explicit expression for the vector field $h$. The Hamilton equation for $p_\phi$ can be derived from (129) and (128):

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \ . \tag{146}$$

Combining this with the constraint equation for $\kappa = 0$ (131) yields

$$\ddot{\phi} = -\sqrt{3}\dot{\phi}\sqrt{\frac{1}{2}\dot{\phi}^2 + V(\phi)} - V'(\phi) \ . \tag{147}$$

Recall that $h$ has the form

$$h = h_x \frac{\partial}{\partial x} + h_y \frac{\partial}{\partial y} = (h_x, h_y) \ . \tag{148}$$

The dynamics in $\phi$-$\dot{\phi}$ space are given by the flow along $h$, so $h_x = \dot{x} = y$ and $h_y = \dot{y} = \ddot{x}$. We thus have

$$h = \left( y, -\sqrt{3}y\sqrt{V(x) + \frac{1}{2}y^2} - V'(x) \right) . \tag{149}$$

In principle, we could now continue and try to solve (144) for $f$ given the expression for $h$ in (149). It turns out that solving the Euler equations analytically is generally difficult or impossible, so instead we will seek a Lagrangian description of the dynamics in $\phi$-$\dot{\phi}$ space. This has the advantage that the existence of a Lagrangian immediately leads to a conserved measure; we could calculate the momentum $\pi_\phi$ conjugate to $\phi$ and conclude that $\mathrm{d}\pi_\phi \wedge \mathrm{d}\phi$ is conserved, since this is a standard symplectic form.

### *IV.4.2. Lagrangian Formulation*

In the Lagrangian formulation of mechanics, one usually derives the equations of motion of a system from Lagrangian function. However, it is generally not true that the dynamics of a system allow for a Lagrangian description. Conditions for the existence of a Lagrangian from which the equations of motions can be derived are given in a theorem by Jesse Douglas [36]. It states that, given an equation of motion

$$\ddot{x} = F(x, \dot{x}) = F(x, y) , \tag{150}$$

a Lagrangian exists if and only if there exists a function $g$ that satisfies the Helmholtz condition, given by

$$\frac{\mathrm{d}g}{\mathrm{d}t} + \frac{\partial F}{\partial \dot{x}} g = 0. \tag{151}$$

By expanding the derivatives using the chain rule and (150) we get

$$\frac{\partial g}{\partial t} + \frac{\partial}{\partial x}(\dot{x}g) + \frac{\partial}{\partial y}(Fg) = 0 . \tag{152}$$

We can now apply this theorem to the equation of motion that was obtained in (147), written in the form given in (150):

$$F(x, y) = -\sqrt{3}y\sqrt{\frac{1}{2}y^2 + V(x)} - V'(x) \tag{153}$$

and hence $h$ can be written as $h = (h_x, h_y) = (y, F)$. In these coordinates, the Helmholtz condition is

$$\frac{\partial g}{\partial t} + \frac{\partial}{\partial x}(h_x g) + \frac{\partial}{\partial y}(h_y g) = \frac{\partial g}{\partial t} + \nabla \cdot (gh) = 0 . \tag{154}$$

When $g$ is time-independent, this equation is nothing more than the Euler equation that already came up when we derived a condition for the measure $\tau = f\,\mathrm{d}x \wedge \mathrm{d}y$ to be conserved along the flow of $h$. We can therefore conclude that a time-independent conserved measure on $\phi$-$\dot{\phi}$ space exists if and only if the dynamics can be derived from a Lagrangian.

Now we take a closer look at this Lagrangian formulation. We will again follow the steps of Grant and Remmen [37], whose analysis is based on the book by Santilli [38]. Suppose that the Helmholtz condition is satisfied for some $g(\phi, \dot{\phi}, t)$. The equations of motion can then be expressed as

$$A(t, \phi, \dot{\phi})\ddot{\phi} + B(t, \phi, \dot{\phi}) = 0 \ , \tag{155}$$

such that

$$\frac{\partial B}{\partial \dot{\phi}} = \left(\frac{\partial}{\partial t} + \dot{\phi}\frac{\partial}{\partial \phi}\right) A \ . \tag{156}$$

We will show that for our specific case, the equations hold for $A = g$ and $B = -gF$ with $F$ as in (153). We have

$$\frac{\partial B}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}(-gF) = -\left(F\frac{\partial g}{\partial \dot{\phi}} + g\frac{\partial F}{\partial \dot{\phi}}\right) \ . \tag{157}$$

From (154) we can write

$$\frac{\partial g}{\partial t} + \dot{\phi}\frac{\partial g}{\partial \phi} + F\frac{\partial g}{\partial \dot{\phi}} + g\frac{\partial F}{\partial \dot{\phi}} = 0 \quad \Longrightarrow \quad F\frac{\partial g}{\partial \dot{\phi}} + g\frac{\partial F}{\partial \dot{\phi}} = -\left(\frac{\partial g}{\partial t} + \dot{\phi}\frac{\partial g}{\partial \phi}\right) \ , \tag{158}$$

such that

$$\frac{\partial B}{\partial \dot{\phi}} = \frac{\partial g}{\partial t} + \dot{\phi}\frac{\partial g}{\partial \phi} = \left(\frac{\partial}{\partial t} + \dot{\phi}\frac{\partial}{\partial \phi}\right) A \ . \tag{159}$$

Also, it is immediately clear from (153) that (155) is the correct equation of motion for this choice of $A$ and $B$. Writing (153) in this form allows the Lagrangian for the dynamics in $\phi$-$\dot{\phi}$ space to be expressed as

$$\mathcal{L}_{\phi\text{-}\dot{\phi}}(t, \phi, \dot{\phi}) = G(t, \phi, \dot{\phi}) + C(t, \phi) \ , \tag{160}$$

where

$$\begin{aligned} G(t, \phi, \dot{\phi}) &= \dot{\phi}\int_0^1 \mathrm{d}\tau' \left[\dot{\phi}\int_0^1 \mathrm{d}\tau A(t, \phi, \tau\dot{\phi})\right]\left(t, \phi, \tau'\phi\right) \ , \\ C(t, \phi) &= \phi\int_0^1 \mathrm{d}\tau W(t, \tau\phi) \ , \\ W(t, \phi) &= -B - \frac{\partial G}{\partial \phi} + \frac{\partial^2 G}{\partial \dot{\phi}\partial t} + \frac{\partial G}{\partial \phi \partial \dot{\phi}}\dot{\phi} \ . \end{aligned} \tag{161}$$

Therefore, the conjugate momentum $\pi_\phi$ and its differential in $\phi$-$\dot{\phi}$ space are given by

$$\pi_\phi = \frac{\partial \mathcal{L}_{\phi\text{-}\dot{\phi}}}{\partial \dot{\phi}} = \frac{\partial G}{\partial \dot{\phi}}, \qquad \mathrm{d}\pi_\phi = \frac{\partial^2 G}{\partial \phi \partial \dot{\phi}}\,\mathrm{d}\phi + \frac{\partial^2 G}{\partial \dot{\phi}^2}\,\mathrm{d}\dot{\phi} \ , \tag{162}$$

so that the measure $\mathrm{d}\pi_\phi \wedge \mathrm{d}\phi$ is

$$\mathrm{d}\pi_\phi \wedge \mathrm{d}\phi = \frac{\partial^2 G}{\partial \dot{\phi}^2}\,\mathrm{d}\dot{\phi} \wedge \mathrm{d}\phi \ . \tag{163}$$

Using (161) and $A = g$, it follows from the fundamental theorem of calculus that $(\partial^2/\partial\dot{\phi}^2)G = f$. Therefore

$$\mathrm{d}\pi_\phi \wedge \mathrm{d}\phi = g\,\mathrm{d}\dot{\phi} \wedge \mathrm{d}\phi \ . \tag{164}$$

In other words, if there exists a Lagrangian from which the dynamics can be derived, the measure that is obtained from it is equal to the one that is found by constructing it explicitly.

Notice that we have not assumed anything about the potential $V(\phi)$. The results above therefore hold in general.

This concludes the general discussion of conserved measures on the effective phase space generated by $\phi$ and $\dot{\phi}$. From here, it is possible to investigate individual models (corresponding to specific potentials) to try to prove the existence of a conserved measure. As discussed above, this can be done by finding either a closed form solution to (143) or by deriving it from a Lagrangian, from which it is known that it correctly describes the dynamics in $\phi$-$\dot{\phi}$ space. Carroll and Remmen [11] have proven the existence of the measure for quadratic potentials. They circumvent the problem of finding an explicit form of the measure by deriving its early and late time limits and show that the measure diverges along the apparant attractor solutions, shown in figure 5.

## V. CONCLUDING REMARKS

In this work, we set out to accomplish three objectives. First, we aimed to review the formalisms of symplectic and contact Hamiltonian systems. We have seen that the first lends itself well for describing non-dissipative dynamics, with the Hamiltonian being a constant of motion. Furthermore, we have derived the famous Liouville's theorem for symplectic manifolds. Then, we described contact systems, the odd-dimensional counterpart of symplectic systems. We have shown that its dynamics can include dissapitive terms, whenever the contact Hamiltonian depends on the extension variable $S$. This dissipation also led to the conclusion that the standard volume form was not preserved under the contact Hamiltonian flow, showing that the Liouville theorem does not hold. However, we did derive an analogue to this theorem, finding a unique conserved measure by explicitly constructing it.

In the second part of the thesis, we gave an extensive overview of theoretical comsology and inflation, deriving the metric for a homogeneous and isotropic spacetime and sketching the procedure obtaining the Friedmann equations from the Einstein field equations. The horizon and flatness problems, the classical motivations for inflation, were presented as well. Both these problems were attributed to the increasing Hubble radius, such that inflation could be introduced as an era in which this quantity decreased. We proposed a scalar field, the inflaton, as the physical origin of inflation and derived the slow roll paramaters as tools to constrain its potential energy function.

In the last section, we considered the Hamiltonian system obtained by (minimally) coupling the inflaton to gravity. We argued that for such systems, apparant attractor behaviour that is often reported in literature, is illusory. It arises as a consequence of expressing the dynamics in the variables $\phi$ and $\dot{\phi}$ only, knowing that this coordinate system is non-canonical and more importantly, only a subset of the full system. However, it was shown that $\phi$-$\dot{\phi}$ space is special for flat universes, in the sense that it completely captures the dynamics of the full system, making it an effective phase space. We have made this precise using the notion of vector field invariance. Lastly, we aimed to derive the conditions for a conserved measure to exist on $\phi$-$\dot{\phi}$ space, by requiring it to vanish under the projection of the Hamiltonain flow. We showed that it exists if and only if there exists a Lagrangian from which all dynamics can be derived. Furthermore, the canonical measure that is obtained from such a Lagrangian coincides with the one found by explicitly constructing it.

## VI. FUTURE DIRECTIONS

In the first section, we introduced contact mechanics as a framework that could account for dissipative terms. Such a term arises naturally in inflationary cosmologies through the Klein-Gordon equation, given by

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \ . \tag{165}$$

We have seen that for flat universes, the dynamics may be expressed in $\phi$ and $\dot{\phi}$ alone. This led to the effective system, described by the equation

$$\ddot{\phi} + \sqrt{3}\dot{\phi}\sqrt{\frac{1}{2}\dot{\phi}^2 + V(\phi)} + V'(\phi) = 0 \ . \tag{166}$$

This equation resembles (55), the equation of motion of a damped mechanical system that was derived from both symplectic and contact principles in II.2.4. This suggests that a description using contact mechanics may be possible, unlocking the full theoretical framework we have developed. There are some subtleties however, since the coefficient of $\dot{\phi}$ in (165) depends on the variables $\phi$ and $\dot{\phi}$ itself. This implies that the standard contact description given in II.2 is not sufficient. Recently however, promising efforts have been made to deal with these nonlinearlities of dissipative field theories in the contact setting [39].

Another suggestion that we would like to make, is the further study of the apparent attractor behaviour in $\phi$-$\dot{\phi}$ space. It was shown that the flow in this space conserves the measure $\mathrm{d}\pi_\phi \wedge \mathrm{d}\phi$, if it exists. This indicates that the $\phi$-$\dot{\phi}$ space is indeed still hamiltonian, and that the attractor behaviour is just an artifact of the choice of coordinates. It would be interesting to rewrite the system in terms of the $\pi_\phi$-$\phi$ coordinates, and investigate the presence of hyperbolic structures that could justify the appearance of the apparent attractor behaviour.


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