## UNIVERSITY OF GRONINGEN

**BACHELOR'S THESIS MATHEMATICS** 

# **Controllability of RLC electrical circuits with ideal components**

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## Abstract

#### Controllability of RLC electrical circuits with ideal components

by Maico Engelaar

In this paper, we investigate the *controllability of RLC electrical circuits with ideal components*. For this, we make use of *Differential Algebraic Equations* (DAE) and model the circuits by the use of the so-called *branch-oriented model*. When considering the controllability of the circuits, we will make use of the *Kalman Controllability Decomposition* (KCD) and in particular *the augmented Wong sequences* which are tremendously help-full in constructing KCDs that decouple the system into a *completely controllable* part, a *behavioral controllable* part and an *uncontrollable* part. This paper will be concluded with some observations regarding the controllability of electrical circuits.

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# **List of Abbreviations**

- AL All Loop
- ANA Augmented Nodal Analysis
- ANI All Node Incidence
- DAE Differential Algebraic Equations
- KCD Kalman Controllability Decomposition
- KCL Kirchhoff Current Law
- KVL Kirchhoff Voltage Law
- MLA Modified Loop Analysis
- MNA Modified Nodal Analysis
- ODE Ordinary Differential Equations
- QWF Quasi Weierstrass Form

# **List of Symbols**

- Q.E.D.
- Strict subset
- $\subseteq$ non-strict subset
- $\mathbb{N}$ Natural Numbers
- Natural Numbers including 0  $\mathbb{N}_0$
- **Real Numbers**  $\mathbb{R}$
- **Complex Numbers**  $\mathbb{C}$
- R Resistance
- L Inductance
- С Capacitance
- i current
- v voltage

## Chapter 1

# Introduction

Nowadays, electrical circuits are everywhere. Examples of electrical appliances containing electrical circuits are flashlights, microwaves, computers, vacuum cleaners and many others. In our day-to-day life, people might not realize, but without understanding these complex circuits, our current way of living would not be possible. It would not be a stretch to even state that people nowadays are completely dependent on these circuits. Therefore, understanding electrical circuits is and has been over the last two centuries, an important research topic for many scientists.

In this paper, we are going to study the controllability of electrical circuits. In particular, we will consider RLC electrical circuits with ideal components. Besides this main objective, we have the following questions that are of interest to us:

- What are RLC electrical circuits?
- How can we model these RLC electrical circuits?
- What is controllability with regard to our model?
- What is the Kalman Controllability Decomposition (KCD) and how can we use the KCD to investigate the controllability of our system?

In this paper, we are going to answer all of these questions. Of course, we will also include the results regarding the main objective. But first, let us consider the following overview regarding the topics that will be discussed in this paper.

## 1.1 Overview

A particular kind of electrical circuits are those consisting of only resistors(R), inductors(L), capacitors(C), current sources and voltage sources. These circuits are called RLC electrical circuits. These electrical circuits will be the main focus of this paper. Furthermore, for simplicity, it is also assumed that the resistor, inductor and capacitor components are all "ideal". In chapter 2 we will discuss RLC electrical circuits in more detail, including, Kirchhoff's laws, the components themselves and the many relations between voltages and currents with regards to each of the components.

When looking at systems in general, there is a need for mathematical models. Mathematical models are used to describe the system into a form which can be mathematically analyzed. Since electrical circuits are also systems, there is a need to find an appropriate mathematical model. A few possible models includes the branch-oriented model, the nodal analysis models and the loop analysis models. All of these models have in common, they make use of Differential Algebraic Equations (DAE). In chapter 3 we will use graphs and graph theory to model the many relations, between the currents and the voltages, into a DAE system of the form  $E\dot{x} = Ax + Bu$  according to the branch-oriented model. At the end of the chapter, we will consider some examples of DAE systems constructed from electrical circuits and also study briefly the nodal analysis models and the loop analysis models.

When considering ODE systems of the form  $\dot{x} = Ax + Bu$  the definition of controllability is rather straightforward: "every state is reachable from every state." However, in case of DAE systems of the form  $E\dot{x} = Ax + Bu$  there are two kinds of controllabilities that are of interest, namely complete controllability and behavioral controllability. In chapter 4 we will discuss the different kinds of controllabilities with regards to both ODE and DAE systems. Thereafter we will discuss the Kalman Controllability Decomposition and in particular the KCD that decomposes the DAE system with regard to complete-and behavioral controllability. Furthermore, we will consider KCDs of both general and regular DAE systems. We will end this chapter with some examples regarding KCDs of DAE systems both general and regular.

In chapter 5 we will discuss some examples of electrical circuits. In particular, we will use the previous established methods to investigate the controllability of these examples. First some basic examples will be considered. Thereafter, we will consider some examples more related to the structure of electrical circuits and how the structure influences the controllability. The chapter will be concluded with some hypotheses regarding the structure and the controllability of electrical circuits.

We will finish this paper in chapter 6 with a conclusion/summary. After chapter 6 there will be an appendix containing some data and a bibliography.

## **Chapter 2**

# **RLC** electrical circuits

Electrical circuits exists in many shapes and forms. In this paper only circuits consisting of resistors(R), inductors( $L^1$ ), capacitors(C) current sources and voltage sources will be considered. These circuits are called RLC electrical circuits. These kind of circuits are interesting since they have multiple applications, among which applications in oscillator circuits.

The resistor, inductor and capacitor components each describe a relation between the current and the voltage through that particular component. It will be assumed that all three of these components are "ideal". This will simplify the relation between the current and voltage in each component (see chapter 2.2). The sources are our input, which means that we decide what the current/voltage through a particular current/voltage source shall be.

Using these relations and the inputs of the sources together with Kirchhoff's laws, it becomes possible to model RLC circuits into DAE systems. In this chapter we will mostly focus on the components, the relations between the currents and voltages with regards to these components and Kirchhoff's laws. At the end of the chapter, we will also discuss a special kind of RLC circuit. In chapter 3 the construction of DAE systems will be discussed.

**Remark 2.1.** From now on when mentioning electrical circuits or just circuits, this always refers to RLC electrical circuits with ideal components.

### 2.1 Kirchhoff's laws

In 1845, German Physicist, Gustav Robert Kirchhoff devised two different laws with respect to electrical circuits [2, Ch. 2.3 and 2.4]. These laws are called Kirchhoff's current law (KCL) and Kirchhoff's voltage law (KVL). KCL states that the sum of the currents entering a node should equal the sum of the currents exiting the node. Here a node represents the point where two or more components are connected. KVL states that in a loop the sum of the voltages with the same orientation should equal the sum of the voltages with the opposite orientation.

<sup>&</sup>lt;sup>1</sup>Because I is already used for the current, instead L is used for the inductance in honor of Emil Lenz, known for Lenz's law [1]



**Remark 2.2.** It will be assumed that during the entirety of this paper, the orientation of current and voltage are always the same. This is just a mathematically convention for simplicity.

**Example 2.3.** Take a look at figure 2.1. The KCL would state that at node 2  $i_R = i_C + i$ . The KVL implies that in the loop from node 1 to node 1, going through the capacitor,  $v_R + v_C + v_L + v = 0$ .

**Remark 2.4.** It will also be assumed that both current i and voltage v are functions of time even though this will not always be specifically stated.

For a more in-dept explanation on how to find the Kirchhoff's laws, see [3, Ch. 2.3].

### 2.2 RLC Components

The three main components in a RLC circuit, besides the sources, are the resistor, the inductor and the capacitor. As stated before, each one of them will be assumed to be "ideal" for simplicity. For a more in-dept discussion on the relation between the current and the voltage, within each component, using Maxwell's equations, see [3, Ch. 2.5]. The symbols used for a resistor, inductor and capacitor are given in figure 2.2.



FIGURE 2.2: Left is resistor. Middle is inductor. Right is capacitor

#### 2.2.1 Resistor

Resistors are used to regulate both the current and the voltage in the circuit. In particular, it limits the flow of electrical current. A good comparison to how this works, is with water tubes. If one narrows a certain part of the tube, the flow (rate) of the water will decrease. A resistor does the same, but then with electrical current.

The relation between the current and the voltage, within a resistor, was first described by Ohm in 1827 [4] and for an ideal resistor is described by Ohm's law:

$$v_R = Ri_R \tag{2.1}$$

Here R is the positive resistance of the resistor measured in  $\Omega$  (Ohm).

#### 2.2.2 Inductor

An inductor has the property that when electricity runs through it, it will store this electrical energy in the form of a magnetic field. Within an ideal inductor, the following relation holds true between the current and the voltage:

$$L\frac{d}{dt}i_L = v_L \tag{2.2}$$

Here L is the positive inductance of the inductor measured in *H*(enry).

#### 2.2.3 Capacitor

A capacitor has the property that when electricity runs through it, it will store this electrical energy in the form of an electrical field. Within an ideal capacitor, the following relation holds true between the current and the voltage:

$$C\frac{d}{dt}v_C = i_C \tag{2.3}$$

Here C is the positive capacitance of the capacitor measured in *F*(arad).

### 2.3 Sources

There are two kinds of sources that are of interest to us, namely current sources and voltage sources. The symbols for current and voltage sources are given in figure 2.3.



FIGURE 2.3: Left is current. Right is voltage

### 2.3.1 Current source

A current source generates a current, independent of the voltage. Because one can choose what current it will generate, this is one of two ways for us to influence the system/circuit.

### 2.3.2 Voltage source

A voltage source generates a voltage, independent of the current. Because one can choose what voltage it will generate, this is one of two ways for us to influence the system/circuit.

## 2.4 Special RLC circuits

Take a look at figure 2.4. What is represented here are two voltage sources being connected to each other. In this specific example there are a total of 4 unknowns, namely  $v_1$ ,  $v_2$ ,  $i_1$  and  $i_2$ . Furthermore, both  $v_1$  and  $v_2$  are inputs, so both of them are known. However, two different problems occur in this example. First of all, by KVL it is given that  $v_1 = v_2$ , so the inputs are not without restrictions. Secondly, both  $i_1$  and  $i_2$  cannot be determined in this specific example. The implications of this example are that not all circuits can be uniquely solved and that not all circuits have a solution for any given input. Later on when discussing specific examples of DAE systems, using the DAE system constructed from this particular example, it will be shown that indeed this circuit cannot be uniquely solved (see chapter 3.4).



FIGURE 2.4

## Chapter 3

## Graphs, DAEs and Circuit models

There are many ways to model RLC electrical circuits. There is the so-called branch-oriented model, the loop analysis models, the nodal analysis models and many others. Even though there are many ways to model electrical circuit, most of these models have one thing in common, namely they make use of Differential Algebraic Equations (DAE). In this chapter, it will be assumed that we are using the branch-oriented model [3, 5, 6]. This model expresses the many circuit equations in terms of the currents and the voltages. More information regarding other models and why we use the branch-oriented model, will be discussed in chapter 3.5.

If a RLC circuit consists of n components, each of these components has a current and a voltage going through it. This implies that there are 2n unknowns. The component relations (see chapter 2.2) together with the inputs of the sources, gives rise to n linearly independent equations. Together with Kirchhoff's laws, 2n equations can be found, making it possible to solve for the 2n unknowns. These 2n equations, together with the assumption of using the branch-oriented model, can then be written into a DAE system of the following form:

$$E\dot{x} = Ax + Bu \tag{3.1}$$

where *E*, *A* are matrices in  $\mathbb{R}^{2n \times 2n}$  and *B* is a matrix in  $\mathbb{R}^{2n \times (k+p)}$  with k the number of voltage sources and p the number of current sources. *x* is a column vector in  $\mathbb{R}^{2n}$  containing the 2n currents and voltages. *u* is a column vector in  $\mathbb{R}^{k+p}$  containing the k + p inputs of the sources.

While it is easy to find the first *n* equations, when considering both the inputs of the sources and the component relations, the same cannot be said about the other *n* equations one gets from Kirchhoff's laws.

When looking at Kirchhoff's laws, they give more equations then is needed. Also some of these equations are linearly dependent (see chapter 3.2). Consequently, there is a need to find n linearly independent equations from Kirchhoff's laws. This can, however, be done by using graphs and graph theory. Using graphs and graph theory, Kirchhoff's laws can be rewritten into a form which can be directly applied when constructing the DAE system (3.1).

In this chapter we will first discuss the required background on graphs. After that, this newly obtained knowledge will be applied to find n linearly independent equations from the Kirchhoff's laws. Next, a "blueprint" will be given on how to construct the DAE system (3.1). Thereafter, there will be some examples of DAE systems regarding various circuits. We will end this chapter with a brief discussion regarding different kinds of circuit models.

### 3.1 Introduction to Graphs

In 1736, Swiss mathematician, Leonhard Euler wrote one of his most famous works, his paper about the Seven Bridges of Königsberg [7]. This paper laid the foundations of graph theory among others. In 1847, Gustav Robert Kirchhoff wrote a paper about a certain relation between the number of wires, the number of junctions, and the number of loops in a circuit [8]. In his paper he made use of a certain kind of graph called Trees (see definition 3.9). However, it was in 1857 that, British mathematician, Arthur Cayley coined the term Tree [9]. Nevertheless, the paper written by Kirchhoff shows that using graphs, to study electrical circuits, has been a practice for almost two centuries now.

Before looking at graph theory, some elemental properties of graphs need to be established, among which the definition of a graph and what it means to be a connected graph. [3, 5]

**Definition 3.1.** A *directed graph* (or just *graph*) is a triple  $G = (V, E, \phi)$ . Here V is the set of nodes (also called vertices), E is the set of branches (also called edges) and  $\phi$  is the incidence map:  $\phi : E \to V \times V$ ;  $\phi(e) = (\phi_1(e), \phi_2(e))$  where  $\phi_1(e) = v_1$  is the initial node and  $\phi_2(e) = v_2$  is the terminal node.

**Remark 3.2.** In this paper when talking about graphs, this always refers to directed graphs. There also exists graphs which are not directed, however these are of no interest to us.

**Definition 3.3.** Let  $V' \subseteq V, E' \subseteq E|_{V'} = \{e \in E \mid \phi(e) \in V' \times V'\}$  and  $\phi|_{E'}$  be the restriction of  $\phi$  on E' then  $K = (V', E', \phi|_{E'})$  is a *subgraph*. If V' = V, then K is called a *spanning* subgraph. If  $E \neq E'$ , then K is called a *proper* subgraph.

**Definition 3.4.** Let  $e \in E$ , let  $\phi(-e) = (\phi_2(e), \phi_1(e))$  and let  $E_0 = \{e, -e | e \in E\}$ . Let  $w = \{w_1, ..., w_r\} \in E_0^r$ , where  $v_i = \phi_2(w_i) = \phi_1(w_{i+1})$  for i = 1, ..., r - 1, then this tuple is called a *path* from  $v_0$  to  $v_r$ . Furthermore if  $\forall i \in \{0, ..., r\}$  the  $v_i$  are distinct them w is called an *elementary path*.

**Definition 3.5.** A *loop* is an elementary path with  $v_0 = v_r$ . A *self-loop* is a loop with only 1 branch.

**Definition 3.6.** Two nodes v and v' are called *connected* if there exists a path from v to v'.

**Definition 3.7.** A graph is called *connected* if any two nodes in the graph are connected.

**Definition 3.8.** A graph *G* is called finite if both *V* and *E* are finite.

**Definition 3.9.** A *Tree* is a minimally connected (spanning sub)graph, that is, it is connected without having any connected proper spanning subgraphs. If a Tree consists of n nodes, it will also consist of exactly n - 1 branches.

Now that these definitions on graphs have been established, lets take a look at the following example.

**Example 3.10.** In figure 3.1 there is both an example of a connected graph and an example of a Tree. For simplification the directions of the branches are not drawn in this example.



FIGURE 3.1: Left is a graph. Right is a Tree

When talking about graphs of electrical circuits, the idea is that nodes refer to the points were two or more components are connected and branches refer to the components themselves. Because not every graphs is a representation of an electrical circuit, there will be some restrictions. The following remarks state and explain some of these restrictions [3, Ch. 2.4.2].

**Remark 3.11.** When talking about electrical circuits, its obvious that the corresponding graphs will contain at least two nodes and two branches. Therefore, no cases in which there are less then two branches and/or two nodes will be considered.

**Remark 3.12.** When talking about electrical circuits, its obvious that the corresponding graphs will be finite and connected. Furthermore, it can be assumed that the graph does not contain any self-loops. Even if electrical circuits would contain self-loops, these can be deleted without influencing the circuits themself, because both current and voltage, over a self-loop, vanishes.

## 3.2 Kirchhoff's laws and Graphs

Now that the most important definitions on graphs have been established, next is to try finding *n* linearly independent equations from Kirchhoff's laws [3, 5, 6]. For this, the following definition is needed:

**Definition 3.13.** Let  $G = (V, E, \phi)$  be a finite graph with *n* branches  $E = \{e_1, ..., e_n\}$  and *m* nodes  $V = \{1, ..., m\}$ . Assume the graph does not contain

any self-loops. The *All-node incidence* (ANI) matrix of *G* is defined by  $A_0 = \{a_{ik}\} \in \mathbb{R}^{m \times n}$ , where

$$a_{jk} = \begin{cases} 1 & \text{if branch k leaves node j} \\ -1 & \text{if branch k enters node j} \\ 0 & \text{otherwise} \end{cases}$$

Let  $L = \{l_1, ..., l_b\}$  be the set of loops of *G*. Then the *All-loop* (AL) matrix  $B_0 = \{b_{jk}\} \in \mathbb{R}^{b \times n}$  is defined by:

$$b_{jk} = \begin{cases} 1 & \text{if branch k is in loop j and has the same orientation} \\ -1 & \text{if branch k is in loop j and has the opposite orientation} \\ 0 & \text{otherwise} \end{cases}$$

**Remark 3.14.** When talking about matrices  $A_0$  and  $B_0$  in relation to electrical circuits, it is always assumed that G is the graph corresponding to the electrical circuits.

Using definition 3.13, Kirchhoff's laws can be rewritten into the following forms:

$$A_0 i(t) = 0 \tag{3.2}$$

$$B_0 v(t) = 0 \tag{3.3}$$

where  $i(t) = (i_1(t), ..., i_n(t))^T$  are the currents through component/branches 1 till *n* and  $v(t) = (v_1(t), ..., v_n(t))^T$  are the voltage drops/rises between the initial nodes and terminal nodes of the branches 1 till *n*. Here it is assumed that branch *i* corresponds to column *i* in both the matrices  $A_0$  and  $B_0$ . The equalities come from the Kirchhoff's Current Law (KCL) and the Kirchhoff's Voltage Law (KVL), respectively.

**Remark 3.15.** Important to see is that each row of matrix  $A_0$  corresponds to one of the equations from the KCL. The same is true for matrix  $B_0$ . Each row of  $B_0$  corresponds to one of the equation from the KVL.

At the beginning of this chapter, it was stated that not all the equations from Kirchhoff's laws are linearly independent. To show this, see matrix  $A_0$ . If one takes the transpose of  $A_0$  and looks at the rows, one will see that they contain exactly two nonzero entries, namely 1 and -1. This implies that

$$A_0^T (1, ..., 1)^T = 0 (3.4)$$

which implies that the columns of  $A_0^T$  are linearly dependent, which results in that the rows of  $A_0$  are linearly dependent. So indeed the KCL does not give linearly independent equations. The same is true for the KVL since also matrix  $B_0$  might have linearly dependent rows by construction. As an example, think about the case that two loops might overlap in one branch. It can then be shown that there exists a third loop of non-overlapping edges that is linearly dependent to the first two loops.

Now that it is known that both the matrices  $A_0$  and  $B_0$  have rank lower then the number of rows they contain, next is to find their specific ranks and a constructive method to delete the unnecessary rows in the matrices  $A_0$  and  $B_0$ .

When first considering the matrix  $A_0$  the following theorem will be of great use:

**Theorem 3.16.** Let  $G = (V, E, \phi)$  be a finite connected graph with *n* branches  $E = \{e_1, ..., e_n\}$  and m > 1 nodes  $V = \{1, ..., m\}$  and no self-loops. Let matrix  $A_0 \in \mathbb{R}^{m \times n}$  be the ANI matrix of G. Then  $Rank(A_0) = m - 1$ .

*Proof.* Let  $A_0$  be the ANI matrix of graph G and assume that  $A_0^T x = 0$  for some  $x \in \mathbb{R}^m$ . Assuming it is known that x does not contain any zero entries, it can be concluded that all entries of x are the same. This can easily be seen by first assuming that x has at least one entry that is different from the other entries, say entry k and its value is l. If one then subtract the vector given in (3.4), l-times from the previous mentioned vector, one gets a vector that is in the kernel of  $A_0^T$ , is not the trivial vector and has a zero entry at entry k, hence contradicting the assumption that all entries are non-zero.

Using the fact that the entries of *x* are all the same, it can be concluded that the kernel of  $A_0^T$  has dimension 1. By the rank-nullity theorem of matrix  $A_0^T$  the result is that the rank of  $A_0^T$  equals m - 1 and therefore the rank of  $A_0$  equals m - 1.

Now to prove that x has indeed no zero entries see the following: Assume that  $x \in \mathbb{R}^m$  contains at least one zero entry. Rearrange now the rows of matrix  $A_0$  in such a way that the first k entries of x are nonzero, whereas the last m - k entries are zero, that is,  $x = [x_1^T 0^T]^T$ , where  $x_1$  is an element of  $\mathbb{R}^k$ . By further reordering the columns of  $A_0$ , it can be assumed that  $A_0$  will be of the form:

$$A_0 = \begin{pmatrix} A_{11} & \mathbf{0} \\ A_{21} & A_{22} \end{pmatrix} \tag{3.5}$$

where each column of  $A_{11}$  is not the zero vector. This gives  $A_{11}^T x_1 = 0$ . Now take an arbitrary column  $a_{21,i}$  from  $A_{21}$ . Since each column of  $A_0$  has exactly two nonzero entries, namely 1 and -1, either column  $a_{21,i}$  has no, one or two nonzero entries. If assuming that  $a_{21,i}$  has two nonzero entries, this would contradict the statement that  $A_{11}$  has no zero columns. If assuming that  $a_{21,i}$ has one nonzero entry, and the other nonzero entry is at the *j*th position in column  $a_{11,i}$ , the relation  $x_1^T A_{11} = 0$  would imply that the *j*th entry in  $x_1$ would be equal to zero. Since this gives a contradiction, it can be concluded that the matrix  $A_{21}$  is the zero matrix. However by construction of the matrix  $A_0$  this would imply that the graph *G* is not connected. Therefore our assumption that *x* contains zero entries is wrong.

Since a graph of an electrical circuit is finite, connected and has no self-loops (see remark 3.12), by theorem 3.16, it holds that the rank of  $A_0$  equals m - 1

when considering electrical circuits. Using this together with equation (3.4), it can be seen that by arbitrarily deleting one row, the newly constructed matrix  $\tilde{A} \in \mathbb{R}^{(m-1)\times n}$  has m-1 linearly independent rows. This matrix  $\tilde{A}$  will be called the *Incidence* matrix of *G*. The equation

$$\tilde{A}i(t) = 0 \tag{3.6}$$

where i(t) is the same as in equation (3.2), contains all linearly independent equations that can be found by the KCL.

Let us next consider the matrix  $B_0$ . However, lets first take a look at the following 2 remarks.

**Remark 3.17.** If  $K = (V, E', \phi|_{E'})$  is a spanning subgraph of  $G = (V, E, \phi)$ , one may, by suitable reordering of the columns, perform a partition of the loop matrix according to the branches of K and  $G - K = (V, E - E', \phi|_{E-E'})$ , that is,  $B_0 = [B_K B_{G-K}]$ .

**Remark 3.18.** If a subgraph *T* is a Tree, then each branch *e* in G - T defines a loop via  $(e, w_1, ..., w_l)$ , where  $(w_1, ..., w_l)$  is the elementary path in *T* from the terminal node to the initial node of *e*. This, together with the fact that a Tree consisting of *m* nodes has exactly m - 1 branches, gives that the following form can be obtained by reordering the rows of  $B_T$  and  $B_{G-T}$ :

$$B_T = \begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix}, \quad B_{G-T} = \begin{pmatrix} I_{n-m+1} \\ B_{22} \end{pmatrix}$$
(3.7)

Using these two remarks, the following theorem can be proven.

**Theorem 3.19.** Let  $G = (V, E, \phi)$  be a finite connected graph with n branches  $E = \{e_1, ..., e_n\}$  and m > 1 nodes  $V = \{1, ..., m\}$  and no self-loops. Let matrix  $B_0 \in \mathbb{R}^{b \times n}$  be the AL matrix of G, where b is the number of loops in G. Then  $Rank(B_0) = n - m + 1$ .

*Proof.* First, let us prove that  $rank(B_0) \le n - m + 1$ . To show this, it is enough to prove that  $im(B_0^T) \subseteq ker(A_0)$ , where  $A_0$  is the ANI matrix of *G*. Showing this is enough because the dimension of the kernel of  $A_0$  is equal to n - m + 1 by the rank-nullity theorem.

Let *l* be a loop and let the vector  $b_l = [b_{l1}, ..., b_{ln}] \in \mathbb{R}^{1 \times n} \setminus \{0\}$  with

 $b_{jk} = \begin{cases} 1 & \text{if branch k belongs to l and has the same orientation} \\ -1 & \text{if branch k belongs to l and has the opposite orientation} \\ 0 & \text{otherwise} \end{cases}$ 

Let  $a_1, ..., a_n$  be the column vectors of  $A_0$ . Then, by the construction of  $b_l$ , each row of the matrix  $[b_{l1}a_1...b_{ln}a_n]$  contains exactly two nonzero entries, namely 1 and -1. This implies that  $A_0b_l^T = b_lA_0^T = 0$ . Since  $b_l$  is a row of the AL matrix  $B_0$ , it can be concluded that indeed im $(B_0^T) \subseteq \text{ker}(A_0)$  and therefore rank $(B_0) \leq n - m + 1$ .

The next thing that needs to be proven is that  $\operatorname{rank}(B_0) \ge n - m + 1$ . To proof this, first consider a Tree *T* in *G*. Using remark 3.18, the AL matrix  $B_0$  can be rewritten into the form given by (3.7). Because the matrix  $B_{G-T}$  has full column rank and n - m + 1 columns, it holds that  $\operatorname{rank}(B_0) \ge \operatorname{rank}(B_{G-T}) = n - m + 1$ . This proves that indeed the rank of  $B_0$  is equal to n - m + 1.

As stated before, a graph of an electrical circuit is finite, connected and has no self-loops, so by theorem 3.19, the rank of  $B_0$  is n - m + 1 when considering electrical circuits. So if one takes n - m + 1 linearly independent rows from matrix  $B_0$  and construct matrix  $\tilde{B} \in \mathbb{R}^{(n-m+1)\times n}$  out of those, one will have the so called *loop* matrix of *G*. One way of constructing the loop matrix  $\tilde{B}$  is by doing the following.

Let *G* be the graph of an electrical circuit and delete branches in such a way that you are left with a Tree *T*. The loop matrix can then be determined by  $\tilde{B} = [B_{11} I_{n-m+1}]$ , where the *j*th row of  $B_{11}$  contains the information on the path in *T* between the initial and terminal nodes of the (m - 1 + j)th branch of *G*. By doing this one obviously gets a matrix  $\tilde{B} \in \mathbb{R}^{(n-m+1)\times n}$  which has n - m + 1 linearly independent rows. The equation

$$\tilde{B}v(t) = 0 \tag{3.8}$$

where v(t) is the same as in equation (3.3), contains all linearly independent equations that can be found by the KVL.

Now that both, a method to find the incidence matrix  $\tilde{A}$  and a method to find the the loop matrix  $\tilde{B}$  have been established, all that is left is to check whether the equations (3.6) and (3.8) are linearly independent. This is trivial because in equation (3.6) there are only relations between currents, while in equation (3.8) there are only relations between voltages. So indeed the equations (3.6) and (3.8) are linearly independent and together give rise to *n* linearly independent equations that are given by Kirchhoff's laws.

Using equations (3.6) and (3.8) together with the inputs of the sources and the component relations, we will, in chapter 3.3, construct a "blueprint" for the DAE system (3.1).

### 3.3 The construction of the DAE system

As stated before, the following form of the DAE system will be used to model the electrical circuits:

$$E\dot{x} = Ax + Bu \tag{3.1}$$

Before it is possible to construct the matrices *E*, *A* and *B*, first the vectors *x* and *u* need to be established. While it is known that *x* consist of all the currents and voltages, when considering the branch-oriented model, the particular ordering has not yet been decided. The same holds true for the vector *u*.

Order	х	u
1	resistors (currents)	Input voltage sources
2	capacitors (currents)	Input current sources
3	inductors (currents)	_
4	voltage sources (currents)	
5	current sources (currents)	
6	resistors (voltages)	
7	capacitor (voltages)	
8	inductors (voltages)	
9	voltage sources (voltages)	
10	current sources (voltages)	

The ordering that will be used in this paper, for *x* and *u*, is given in table 3.2.

TABLE 3.2: The order of the component	ts
---------------------------------------	----

It is assumed that the component groups, i.e. resistors, capacitors, inductors, voltage sources and current sources, have a separate ordering among themselves. Furthermore, it is assumed that this separate ordering is used consistently throughout the construction. For example, the *ith* voltage source's voltage in *x* and the *ith* voltage source's input in *u*, correspond to the same voltage source.

Now that x and u are formally established, next the incidence matrix (3.6) and the loop matrix (3.8) need to be considered. For both matrices it holds that each column corresponds to one of the components in the circuit. Hence, it is important that the ordering of columns corresponds to the ordering of the vector x. If needed the columns can be rearranged in such a way that this condition is satisfied.

Now that all of this has been established, the definition, which expresses how to construct the matrices *E*, *A* and *B*, can be formulated as following:

**Definition 3.20** (The construction of DAE systems). Let *G* be a graph of an electrical circuit with *n* branches and *m* nodes. Let  $\tilde{A} \in \mathbb{R}^{(m-1)\times n}$  and  $\tilde{B} \in \mathbb{R}^{(n-m+1)\times n}$  be the incidence- and loop matrix, respectively. Let *r*, *c*, *l*, *k*, *p* be the number of resistors, capacitors, inductors, voltage sources and current sources, respectively. Let  $R \in \mathbb{R}^{r\times r}$  be a diagonal matrix where entry  $R_{ii}$  contains the *ith* resistor's resistance. Let  $C \in \mathbb{R}^{c\times c}$  be a diagonal matrix where entry  $C_{ii}$  contains the *ith* capacitor's capacitance. Let  $L \in \mathbb{R}^{l\times l}$  be a diagonal matrix where entry  $L_{ii}$  contains the *ith* inductor's inductors. Then



	#	r	С	1	k	р	r	С	1	k	p	
	<i>m</i> -1			Ã		0						
	<i>n-m</i> +1			0			Ĩ					
A =	r	- <i>R</i>			0		I 0					
<u> </u>	С	0	Ι				(	)				
	1	0								I = 0		
	k				0					Ι	0	
	р		0			Ι			0			
		Г		#		k	1)					
				#			- <i>p</i>					
		B =	n+	r+c	:+l	(	)					
			1	k+p		-	Ι					

**Remark 3.21.** *Note that* r + c + l + k + p = n*.* 

Lets take a closer look at the constructed matrices and the DAE system (3.1). Clearly, it can be concluded that each of the rows in the DAE system now corresponds to either Kirchhoff's laws, the component relations or the input of the sources.

### **3.4 Examples of DAE systems**

Now that it is known how to construct the matrices *E*, *A* and *B* in equation (3.1), lets take a look at the following 2 examples.

**Example 3.22.** Let us again consider example 2.3 and figure 2.1. In this figure there is exactly one of each component. Before one can find the matrices *E*, *A* and *B*, first the vector x(t), the incidence matrix and the loop matrix need to be constructed. Using what is written in chapters 3.2 and 3.3 about the construction of x(t),  $\tilde{A}$  and  $\tilde{B}$ , one gets the following:

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 \end{pmatrix} \quad x(t) = \begin{pmatrix} I_R(t) \\ I_C(t) \\ I_L(t) \\ I_V(t) \\ I(t) \\ V_R(t) \\ V_C(t) \\ V_L(t) \\ V(t) \\ V_I(t) \end{pmatrix}$$
(3.9)

Here  $\tilde{A}$  was constructed by ignoring node 4 and  $\tilde{B}$  by considering the Tree constructed by removing the branches corresponding to both sources. Using

(3.9) together with definition 3.20, the matrices *E*, *A* and *B* are as follows:

	(0	0	0	0	0	0	0	0	0	0)		(1	0	0	-1	0	0	0	0	0	0)		( 0	0 \							
	0	0	0	0	0	0	0	0	0	0			-1	1	0	0	1	0	0	0	0	0		0	0						
	0	0	0	0	0	0	0	0	0	0						0	-1	1	0	-1	0	0	0	0	0		0	0			
	0	0	0	0	0	0	0	0	0	0		0	0	0	0	0	1	1	1	1	0		0	0							
г	0	0	0	0	0	0	0	0	0	0		0	0	0	0	0	0	-1	0	0	1	D	0	0							
E =	0	0	0	0	0	0	0	0	0	0	, A =	-R	0	0	0	0	1	0	0	0	0	, в=	0	0							
	0	0	0	0	0	0	С	0	0	0		0	1	0	0	0	0	0	0	0	0		0	0							
	0	0	L	0	0	0	0	0	0	0						1			0	0	0	0	0	0	0	1	0	0		0	0
	0	0	0	0	0	0	0	0	0	0		0	0	0	0	0	0	0	0	1	0		-1	0							
	0	0	0	0	0	0	0	0	0	0/		0	0	0	0	1	0	0	0	0	0/		0	-1/							

where *R*, *C* and *L* are the resistance, the capacitance and the inductance, respectively.

**Example 3.23.** Let us again take a look at figure 2.4. This figure has already been discussed in chapter 2.4. Let us now, however, consider the DAE system corresponding to this electrical circuit. Without going into too much details, the matrices *E*, *A* and *B* are as follows:

$$E = 0^{4 \times 4}, \quad A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$
(3.10)

As can be seen from (3.10), the matrix E is the zero matrix and the matrix A has zero determinant, so it can indeed be concluded that the DAE system, described by (3.10), does not have an unique solution.

### 3.5 Circuit Models

As stated in the beginning of this chapter, there are many ways of modelling electrical circuits. In this chapter, only the branch-oriented model was used. Besides the branch-oriented model, one could also have used the nodal analysis models or the loop analysis models. [3, 5, 6]

While each model has their own advantages, the main reason why the branchoriented model is used, during the entirety of this paper, is because when one uses the branch-oriented model, the vector x, in equation (3.1), is written in terms of the currents and the voltages only. When discussing the controllability of RLC electrical circuits, having the system modelled using only the currents and the voltages, it becomes much easier to understand which part of the system is controllable and which part is not. When using other models the vector x might include more abstract variables, which do not directly tell us to which part of the system they refer to. So, with regards to this, only the branch-oriented model was used.

The main difference between the branch-oriented model and the nodal analysis models/the loop analysis models is that the nodal analysis models/the loop analysis models make use of node potentials respectively loop currents. The node potentials and loop currents are defined as follows:

$$v(t) = \tilde{A}^T \phi(t) \tag{3.11}$$

$$i(t) = \tilde{B}^T \iota(t) \tag{3.12}$$

where matrices  $\tilde{A}$  and  $\tilde{B}$  are the same matrices as in equations (3.6) and (3.8).  $\phi \in \mathbb{R}^{m-1}$  is the vector containing the node potentials and  $\iota \in \mathbb{R}^{n-m+1}$  is the vector containing the loop currents. v(t) and i(t) are the same vectors as in equations (3.2) and (3.3).

Loop analysis begins with the description of Kirchhoff's voltage laws in the form of equation (3.8) and then proceeds to replace, as far as possible, the voltages of current-controlled components in terms of currents and eventually loop currents [6, Ch. 2.6]. For nodal analysis models the same happens only instead of using equation (3.8), one uses equation (3.6) (Kirchhoff's current law), and instead rewrites the currents of voltage-controlled components in terms of voltages and eventually nodal potentials. In the case of the nodal analysis models, these kind of models are called Augmented Nodal Analysis (ANA) models.

There are many types of nodal- and loop analysis models, but two interesting ones are the Modified Nodal Analysis (MNA) model and the Modified Loop Analysis (MLA) model. Both of them use the idea of reducing the number of variables, while still attaining a form which is easy to set up in an automatic way that makes them well-suited for computational purposes. [6, Ch. 2.3]

**Remark 3.24.** If instead of the branch-oriented model one would use either the nodal analysis models or loop analysis models, one can still use equation (3.1). The only differences would be the dimensions of the matrices and the elements of vectors x and *u*.

For more information on nodal analysis models and loop analysis models and in particular the DAE systems of these kind of models, see [3, 5, 6].

For more information on DAEs and in particularly time-invariant DAEs, see [5]. [5] gives both an analytic study on time-invariant and time-variant DAEs and might be of interest to those who want to do a more analytic study on DAEs.

Now that all of these models have been briefly discussed, next we can take a look at the controllability of DAE systems.

## Chapter 4

# Controllability and Kalman Controllability Decomposition

There are many ways to check the controllability of a system. One wellknown method is the so-called Kalman Controllability Decomposition (KCD). This method uses the idea of splitting the system into a controllable and an uncontrollable part. While indeed this method will be used, when considering systems of the form

$$E\dot{x} = Ax + Bu \tag{4.1}$$

where *E*, *A* are matrices in  $\mathbb{R}^{l \times n}$  and *B* is a matrix in  $\mathbb{R}^{l \times m}$  (short-hand notation:  $(E, A, B) \in \sum_{m}^{l \times n}$ ), it first needs to be extended on.

When talking about systems of the form (4.1), there are two kinds of controllability that are of interest, namely complete controllability and behavioral controllability (see definition 4.5). This is why the standard KCD is not applicable in this case. Instead, the KCD has to be modified in such a way that it also separates the system with respect to complete and behavioral controllability.

Furthermore, when considering both general and regular systems (see remark 4.1) of the form (4.1), the method on how to modify the KCD will differ. While in both cases (augmented) Wong sequences will be used to construct the KCDs, regular systems can be decomposed into a even more detailed KCD.

In this chapter we will first discuss what controllability means with respect to systems of Ordinary Differential Equations (ODE) and systems of the form (4.1). After that, we will take a look at the Kalman Controllability Decomposition with respect to ODE systems and systems of the form (4.1). In particular, when considering systems of the form (4.1), we will consider the general case and the regular case. We will end this chapter with some examples regarding KCDs of systems of the form (4.1).

**Remark 4.1.** A DAE (4.1) is called regular if l = n and  $det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$ . *If not regular, the system is called singular.* [10, Ch. 1]

**Remark 4.2.** When considering systems of electrical circuits, all one has to do, is to take  $l = n = 2\tilde{n}$  and m = k + p where  $\tilde{n}$  is the number of components and k, p the number of voltage/current sources, respectively.

## 4.1 Controllability

Before discussing the controllability of systems of the form (4.1), lets first take a look at the controllability of linear ODE time-invariant control systems of the form:

$$\dot{x} = Ax + Bu \tag{4.2}$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$ .

**Remark 4.3.** From now on ODE-systems will refer to systems of the form (4.2) and DAE-systems will refer to systems of the form (4.1), including the dimensions, unless stated otherwise.

### 4.1.1 Controllability of ODEs

For ODE-systems, one way of describing the controllability is as follows [11, Ch. 3.2]:

**Definition 4.4.** An ODE-system is said to be *controllable*, if  $\forall x_0, x_1 \in \mathbb{R}^n$  one can find a solution (x(t), u(t)) and  $\exists t_1 > 0$  such that  $x(0) = x_0$  and  $x(t_1) = x_1$  i.e. every state is reachable from every state.

When talking about the controllability of ODE-systems, a well known theorem states that such systems are controllable if, and only if, Rank([ $B AB ... A^{n-1}B$ ]) = n. This is one of the many ways to check whether an ODE-system is controllable or not. [11]

Another way of analyzing the controllability of an ODE-system is by using Kalman Controllability Decomposition. We will discuss more about this in chapter 4.2. For more information on ODE-systems and the controllability of ODE-systems, see [11].

### 4.1.2 Controllability of DAEs

For DAE-systems a distinction can be made between two kinds of controllabilities, namely complete controllability and behavioral controllability. Both of these forms of controllability are defined in the following definition: [10, Definition 2.1]

**Definition 4.5.** Let  $(E, A, B) \in \sum_{m}^{l \times n}$  and let  $\mathfrak{B}_{(E,A,B)} = \{(x, u) \in \mathscr{W}_{loc}^{1}(\mathbb{R} \to \mathbb{R}^{n}) \times \mathscr{L}_{loc}^{1}(\mathbb{R} \to \mathbb{R}^{m}) | (x, u) \text{ satisfies (4.1) for almost all } t \in \mathbb{R} \}$ , where  $\mathscr{L}_{loc}^{1}$  and  $\mathscr{W}_{loc}^{1}$  denote the space of locally (lebesgue) integrable or weakly differentiable functions with locally integrable derivatives, respectively, i.e. x and u need to be sufficiently differentiable and integrable, respectively.

The system is called

(i) completely controllable if, and only if,

$$\forall x_0, x_f \in \mathbb{R}^n \ \exists t_f > 0 \ \exists (x, u) \in \mathfrak{B}_{(E, A, B)} : x(0) = x_0 \land x(t_f) = x_f$$

i.e. it is possible to control the state  $x(\cdot)$  from any given initial value  $x_0$  to any final value  $x_f$ .

(ii) behaviorally controllable if, and only if,

$$\begin{aligned} \forall (x_1, u_1), (x_2, u_2) \in \mathfrak{B}_{(E,A,B)} \ \exists t_f > 0 \ \exists (x, u) \in \mathfrak{B}_{(E,A,B)} :\\ (x(t), u(t)) = \begin{cases} (x_1(t), u_1(t)), & \text{if } t < 0\\ (x_2(t), u_2(t)), & \text{if } t > t_f \end{cases} \end{aligned}$$

i.e. it is possible to connect any two feasible trajectories via a third feasible trajectory.

Both of these forms of controllability are equivalent for ODE-systems (take matrix *E* to be square non-singular), however this does not hold in general for DAE-systems. For general DAE-systems, complete controllability is stronger then behavioral controllability [10, Ch. 2]. This because, being able to find a solution for any initial value, is not always possible when considering DAE-systems. Most of the time algebraic constraints make this impossible. This is in contrast to ODE-systems, where one can always find a solution for any given initial value.

To better understand these concepts of controllability, let us take a look at the following example:

**Example 4.6.** Consider the following 3 systems:

1. 
$$\dot{x}_1(t) = u_1(t)$$

2. 
$$\dot{x}_2(t) = u_2(t)$$
 and  $0 = x_2(t)$ 

3. 
$$\dot{x}_3(t) = x_3(t)$$

System 1 is completely and behaviorally controllable. System 2 is behaviorally controllable, but not completely controllable. System 3 is neither completely nor behaviorally controllable.

Now that the definition of controllability of systems of the form (4.1) has been established, let us next take a look at the Kalman Controllability Decomposition.

## 4.2 Kalman Controllability Decomposition

It has been almost 60 years since Kalman derived his famous decomposition of linear ODE control systems [12]. This decomposition has later been generalized to regular DAEs by Verghese et al. [13]. Besides that, a Kalman decomposition of general discrete-time DAE systems has been provided by Banaszuk et al. [14] in a very nice way using the augmented Wong sequences [15, Ch. 6 and 7]. However, before discussing the KCD with respect to DAE-systems, let us first take a look at the KCD with respect to ODE-systems.

#### 4.2.1 KCD of ODE-systems

The Kalman Controllability Decomposition for ODE-systems states that by doing some suitable coordination transformation on x, say x = Tz, the system of the form (4.2) can be rewritten into the following form:

$$\dot{z} = \begin{pmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{pmatrix} z + \begin{pmatrix} B_1 \\ \mathbf{0} \end{pmatrix} u$$
(4.3)

where the system constructed by  $(A_{11}, B_1)$  is controllable and the system constructed by  $(A_{22}, 0)$  is uncontrollable. This implies that the system is decomposed into a controllable part and an uncontrollable part. [10, 11, 15]

#### 4.2.2 KCD of DAE-systems

Just like ODE-systems, also for DAE-systems the Kalman Controllability Decomposition is available (even in the singular case), see [15, Theorem 7.1] (which is based on a result for the discrete time case in [14]). The main difference is that one now has to take into account matrix *E*. Applying suitable transformation matrices *S* and *T* to the DAE-system, one gets the following KCD: [10, 15]:

$$(SET, SAT, SB) = \left( \begin{pmatrix} E_{11} & E_{12} \\ \mathbf{0} & E_{22} \end{pmatrix}, \begin{pmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{pmatrix}, \begin{pmatrix} B_1 \\ \mathbf{0} \end{pmatrix} \right)$$
(4.4)

where  $(E_{11}, A_{11}, B_1)$  is controllable and  $(E_{22}, A_{22}, 0)$  is uncontrollable.

There is, however, a problem with (4.4). In the case of the trivial DAE 0 = x, while this system is behaviorally controllability, the KCD will only consist of the uncontrollable part. This is a rather displeasing situation and is due to the fact that for certain DAE-systems, particular states can be inconsistent. For these inconsistent states, it does not make sense to label those as either controllable or uncontrollable. To solve this, one can use the following more detailed form of the KCD. [10, Ch. 1]

$$(\tilde{S}E\tilde{T}, \tilde{S}A\tilde{T}, \tilde{S}B) = \begin{pmatrix} \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ \mathbf{0} & E_{22} & E_{23} \\ \mathbf{0} & \mathbf{0} & E_{33} \end{pmatrix}, \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ \mathbf{0} & A_{22} & A_{23} \\ \mathbf{0} & \mathbf{0} & A_{33} \end{pmatrix}, \begin{pmatrix} B_1 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \end{pmatrix}$$
(4.5)

In this form, the system  $(E_{11}, A_{11}, B_1) \in \sum_{m}^{l_1 \times n_1}$  is completely controllable. Furthermore,  $E_{22}$  is invertible and the system  $(E_{33}, A_{33}, 0) \in \sum_{m}^{l_3 \times n_3}$  is such that it only has the trivial solution. Hence, the system is now decomposited into a (completely) controllable part, a classical uncontrollable part (given by the ODE) and an inconsistent part (which is behaviorally controllable but contains no completely controllable part). Furthermore, if restricted to only regular DAE-systems, the completely controllable part (given by a controllable ODE) and an instantaneously controllable part (given by a controllable ODE) and an instantaneously controllable part (given by a controllable pure DAE) [10, Ch. 1]. Before delving into the particulars regarding the KCD of DAE-systems, let us first give a formal definition of what it means for a DAE-system to be in KCD. [10, Definition 3.1]

**Definition 4.7** (Kalman Controllability Decomposition). A system of the form (4.1) is said to be in *Kalman Controllability Decomposition* if, and only if,

$$(E, A, B) = \left( \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ \mathbf{0} & E_{22} & E_{23} \\ \mathbf{0} & \mathbf{0} & E_{33} \end{pmatrix}, \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ \mathbf{0} & A_{22} & A_{23} \\ \mathbf{0} & \mathbf{0} & A_{33} \end{pmatrix}, \begin{pmatrix} B_1 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \right)$$
(4.6)

where

- (i)  $(E_{11}, A_{11}, B_1) \in \sum_{m}^{l_1 \times n_1}$  with  $l_1 = rank[E_{11}, B_1] \le n_1 + m$  is completely controllable,
- (ii)  $(E_{22}, A_{22}, 0) \in \sum_{m}^{l_2 \times n_2}$  with  $l_2 = n_2$  and  $E_{22}$  is invertible, i.e.  $(E_{22}, A_{22}, 0)$  is uncontrollable
- (iii)  $(E_{33}, A_{33}, 0) \in \sum_{m}^{l_3 \times n_3}$  with  $l_3 \ge n_3$  satisfies  $\operatorname{Rank}_{\mathbb{C}}(\lambda E_{33} A_{33}) = n_3$  for all  $\lambda \in \mathbb{C}$  i.e.  $(E_{33}, A_{33}, 0)$  only has the zero solution and, consequently, is behaviorally controllable

Let us now look into the particulars on finding the KCD when considering both general and regular DAE-systems.

#### 4.2.2.1 KCD general case

To be able to find the KCD of a general DAE-system, first the augmented Wong sequences need to be introduced. These sequences are an important tool in finding the KCD of general DAE-systems. [10, Ch. 2]

**Definition 4.8.** *The augmented Wong sequences* are defined, with regards to (4.1), as follows:

$$\begin{aligned} \mathscr{V}^{0}_{(E,A,B)} &:= \mathbb{R}^{n}, \quad \mathscr{V}^{i+1}_{(E,A,B)} &:= A^{-1}(E\mathscr{V}^{i}_{(E,A,B)} + \operatorname{im}B) \subseteq \mathbb{R}^{n} \\ \mathscr{V}^{*}_{(E,A,B)} &:= \bigcap_{i \in \mathbb{N}_{0}} \mathscr{V}^{i}_{(E,A,B)'} \\ \mathscr{W}^{0}_{(E,A,B)} &:= \{0\}, \quad \mathscr{W}^{i+1}_{(E,A,B)} &:= E^{-1}(A\mathscr{W}^{i}_{(E,A,B)} + \operatorname{im}B) \subseteq \mathbb{R}^{n} \\ \mathscr{W}^{*}_{(E,A,B)} &:= \bigcup_{i \in \mathbb{N}_{0}} \mathscr{W}^{i}_{(E,A,B)}. \end{aligned}$$

**Remark 4.9.** Recall that, for some matrix  $M \in \mathbb{R}^{l \times n}$ ,  $M\mathscr{S} = \{Mx \in \mathbb{R}^{l} | x \in \mathscr{S}\}$ denotes the image of  $\mathscr{S} \subseteq \mathbb{R}^{n}$  under M and  $M^{-1}\mathscr{S} = \{x \in \mathbb{R}^{n} | Mx \in \mathscr{S}\}$ denotes the preimage of  $\mathscr{S} \subseteq \mathbb{R}^{l}$  under M.

The sequences  $(\mathscr{V}_{(E,A,B)}^{i})_{i\in\mathbb{N}}$  and  $(\mathscr{W}_{(E,A,B)}^{i})_{i\in\mathbb{N}}$  are called augmented Wong sequences, since they are based on the Wong sequences (B = 0) which have their origin in Wong [16].

Using these augmented Wong sequences, the following theorem with regards to finding the KCD of general DAE-systems can be stated. [10, Theorem 3.3]

**Theorem 4.10** (Kalman Controllability Decomposition). Consider the DAE system of the form (4.1) and the limits  $\mathscr{V}^*_{(E,A,B)}$  and  $\mathscr{W}^*_{(E,A,B)}$  of the augmented Wong sequences. Choose any full rank matrices  $R_1 \in \mathbb{R}^{n \times n_1}$ ,  $P_1 \in \mathbb{R}^{n \times n_2}$ ,  $Q_1 \in \mathbb{R}^{n \times n_3}$ ,  $R_2 \in \mathbb{R}^{l \times l_1}$ ,  $P_2 \in \mathbb{R}^{l \times l_2}$ ,  $Q_2 \in \mathbb{R}^{l \times l_3}$  such that

$$\begin{split} &\text{im} R_{1} = \mathscr{V}_{(E,A,B)}^{*} \cap \mathscr{W}_{(E,A,B)}^{*}, \\ &\text{im} R_{2} = (E\mathscr{V}_{(E,A,B)}^{*} + \text{im}B) \cap (A\mathscr{W}_{(E,A,B)}^{*} + \text{im}B), \\ &\text{im} R_{1} \oplus \text{im} P_{1} = \mathscr{V}_{(E,A,B)}^{*}, \quad \text{im} R_{2} \oplus \text{im} P_{2} = E\mathscr{V}_{(E,A,B)}^{*} + \text{im}B, \\ &\text{im} [R_{1}, P_{1}] \oplus \text{im} Q_{1} = \mathbb{R}^{n}, \quad \text{im} [R_{2}, P_{2}] \oplus \text{im} Q_{2} = \mathbb{R}^{l}. \\ &\text{Then } T := [R_{1}, P_{1}, Q_{1}] \in \mathbf{GL}_{n}, S := [R_{2}, P_{2}, Q_{2}]^{-1} \in \mathbf{GL}_{l} \text{ and } (SET, SAT, SB) \\ &\text{is in } KCD (4.6). \end{split}$$

**Remark 4.11. GL**<sub>k</sub> denotes the space of invertible real-valued  $k \times k$  matrices.

Using this theorem, one is now able to find the KCD of any general DAEsystem. This of course doesn't imply that finding the KCD will be a simple task. As can be seen from the theorem, multiple steps, which do not have to be unique, need to be taken. In particular, the choose of matrices  $R_i$ ,  $P_i$ ,  $Q_i$  do not need to be unique, resulting in multiple possible KCDs. More on how to find these matrices will be discussed in chapter 4.3 when considering some examples of KCDs.

Both the proof and additional information, regarding theorem 4.10, can be found in [10, Ch. 3]. We will end this section with the following theorem regarding the uniqueness of general KCDs [10, Theorem 3.5]. However, before considering this theorem, take a look at the following remark.

**Remark 4.12.** Two DAE-systems (E, A, B) and  $(\tilde{E}, \tilde{A}, \tilde{B})$  are called equivalent if, and only if,  $\exists S \in \mathbf{GL}_l, T \in \mathbf{GL}_n : (SET, SAT, SB) = (\tilde{E}, \tilde{A}, \tilde{B})$ . A short-hand notation of this is  $(E, A, B) \cong (\tilde{E}, \tilde{A}, \tilde{B})$ . In case one wants to highlight the involved transformation matrices, one uses  $\cong$  instead of  $\cong$ .

**Theorem 4.13** (Uniqueness of KCD). *Consider the DAE system of the form* (4.1) *and let*  $S_1, S_2 \in \mathbf{GL}_l, T_1, T_2 \in \mathbf{GL}_n$  *be such that for* i = 1, 2

$$(E, A, B) \stackrel{S_{i}, T_{i}}{\cong} (E_{i}, A_{i}, B_{i}) = \begin{pmatrix} \begin{pmatrix} E_{11,i} & E_{12,i} & E_{13,i} \\ \mathbf{0} & E_{22,i} & E_{23,i} \\ \mathbf{0} & \mathbf{0} & E_{33,i} \end{pmatrix}, \begin{pmatrix} A_{11,i} & A_{12,i} & A_{13,i} \\ \mathbf{0} & A_{22,i} & A_{23,i} \\ \mathbf{0} & \mathbf{0} & A_{33,i} \end{pmatrix}, \begin{pmatrix} B_{1,i} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \end{pmatrix}$$

where  $(E_i, A_i, B_i)$  is in KCD (4.6) with corresponding block sizes given by  $l_{1,i}$ ,  $n_{1,i}$ ,  $l_{2,i}$ ,  $n_{2,i}$ ,  $l_{3,i}$ ,  $n_{3,i}$ .

Then  $l_{1,1} = l_{1,2}$ ,  $l_{2,1} = l_{2,2}$ ,  $l_{3,1} = l_{3,2}$ ,  $n_{1,1} = n_{1,2}$ ,  $n_{2,1} = n_{2,2}$ ,  $n_{3,1} = n_{3,2}$ and, moreover, for some  $S_{11} \in \mathbf{GL}_{l_{1,1}}$ ,  $S_{22} \in \mathbf{GL}_{l_{2,1}}$ ,  $S_{33} \in \mathbf{GL}_{l_{3,1}}$ ,  $T_{11} \in \mathbf{GL}_{n_{1,1}}$ ,  $T_{22} \in \mathbf{GL}_{n_{2,1}}$ ,  $T_{33} \in \mathbf{GL}_{n_{3,1}}$  and  $S_{12}$ ,  $S_{13}$ ,  $S_{23}$ ,  $T_{12}$ ,  $T_{13}$ ,  $T_{23}$  of appropriate sizes we have that

$$S_2 S_1^{-1} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ \mathbf{0} & S_{22} & S_{23} \\ \mathbf{0} & \mathbf{0} & S_{33} \end{pmatrix}, \quad T_1^{-1} T_2 = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ \mathbf{0} & T_{22} & T_{23} \\ \mathbf{0} & \mathbf{0} & T_{33} \end{pmatrix}$$

In particular,

 $(E_{11,1}, A_{11,1}, B_{1,1}) \cong (E_{11,2}, A_{11,2}, B_{1,2}),$   $(E_{22,1}, A_{22,1}, 0) \cong (E_{22,2}, A_{22,2}, 0),$  $(E_{33,1}, A_{33,1}, 0) \cong (E_{33,2}, A_{33,2}, 0).$ 

#### 4.2.2.2 KCD Regular case

Regularity implies that equation (4.1) has a solution for any (sufficiently smooth) input u and each such solution is uniquely determined by the initial value x(0). Therefore, it is often assumed that DAE systems of the form (4.1) are regular for the analysis and numerical simulations. [10, Ch. 4]

When considering DAE-systems which are known to be regular, one is able to find KCDs which are more detailed then when using theorem 4.10. Furthermore, there is no need to use the augmented Wong sequences. In this case the original Wong sequences (B = 0) are enough to find the KCD. While there is a relation between the original and the augmented Wong sequences, in case of regularity, this paper will not go into details about this. For more details about this relation, see [10, Theorem 4.4].

Using the original Wong sequences one can obtain the quasi-Weierstrass form (QWF): [10, Proposition 4.1]

**Proposition 4.14** (Quasi-Weierstrass Form). *The DAE system*  $(E, A, B) \in \sum_{m}^{n \times n}$  *is regular if, and only if,* 

$$(E, A, B) \stackrel{S,T}{\cong} \left( \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & N \end{pmatrix}, \begin{pmatrix} J & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right),$$
(4.7)

where  $N \in \mathbb{R}^{n_2 \times n_2}$ ,  $0 \le n_2 \le n$  is nilpotent and  $J \in \mathbb{R}^{n_1 \times n_1}$ ,  $B_1 \in \mathbb{R}^{n_1 \times m}$ ,  $B_2 \in \mathbb{R}^{n_2 \times m}$ ,  $n_1 := n - n_2$ . Furthermore, the transformation matrices  $T = [T_1, T_2] \in \mathbf{GL}_n$  and  $S \in \mathbf{GL}_n$  achieve the QWF (4.7) if, and only if,

$$\operatorname{im} T_{1} = \mathscr{V}_{(E,A)}^{*}, \quad \operatorname{im} T_{2} = \mathscr{W}_{(E,A)}^{*}, \quad S = [ET_{1}, AT_{2}]^{-1},$$
  
where  $\mathscr{V}_{(E,A)}^{*} = \mathscr{V}_{(E,A,0)}^{*}$  and  $\mathscr{W}_{(E,A)}^{*} = \mathscr{W}_{(E,A,0)}^{*}.$ 

By proposition 4.14, the original Wong sequences yield a decoupling of the DAE into an ODE and into a so-called pure DAE.

The Wong sequences are coordinate free in the sense that the specific choice of matrices  $T_1$  and  $T_2$  is not relevant. Once the QWF is obtained for a specific choice of the coordinate transformation matrix T, it is not difficult to obtain a KCD for each block separately: [10, Ch. 4 and Proposition 4.2]

**Proposition 4.15.** *Consider the regular DAE system*  $(E, A, B) \in \sum_{m=1}^{n \times n}$ . *Then* 

$$(E, A, B) \stackrel{S,T}{\cong} \left( \begin{pmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & N_{11} & N_{12} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & N_{22} \end{pmatrix}, \begin{pmatrix} J_{11} & J_{12} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I \end{pmatrix}, \begin{pmatrix} B_{11} \\ \mathbf{0} \\ B_{21} \\ \mathbf{0} \end{pmatrix} \right), \quad (4.8)$$

where  $\begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & N_{11} \end{pmatrix}$ ,  $\begin{pmatrix} J_{11} & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}$ ,  $\begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix} \end{pmatrix}$  is completely controllable and  $N_{11}$  and  $N_{22}$  are nilpotent.

Clearly the QWF-KCD (4.8) obtained via the QWF (4.7) matches the general KCD (4.6) after rearrangement of the corresponding blocks. In particular

$$(E_{11}, A_{11}, B_1) = \left( \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & N_{11} \end{pmatrix}, \begin{pmatrix} J_{11} & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}, \begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix} \right),$$
  
$$(E_{22}, A_{22}, 0) = (I, J_{22}, 0), \quad (E_{33}, A_{33}, 0) = (N_{22}, I, 0).$$

However, the form (4.8) is not really satisfactory as its derivation needs two separate coordinate transformations: first, one needs to transform the DAE into QWF and then the separate ODE and pure DAE need to be transformed again to get the KCD. In particular, the latter transformation depends on the former (because *J* and *N* depend on *T*) and is therefore not coordinate free. [10, Ch. 4]

Instead, using the following definition, one is able to define a way of finding the KCD without being restricted by the choice of coordinate transformation. [10, Definition 4.3]

**Definition 4.16** (Consistency, Differential and Impulse Projector). With the notation of Proposition 4.14 define the consistency projector

$$\Pi_{(E,A)} := T \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} T^{-1},$$

the differential projector

$$\Pi_{(E,A)}^{\operatorname{diff}} := T \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} S,$$

and the impulse projector

$$\Pi_{(E,A)}^{\operatorname{imp}} := T \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} S,$$

where the block matrix sizes correspond to the block sizes in the QWF (4.7). Furthermore, let

$$A^{\text{diff}} := \Pi^{\text{diff}} A, \quad B^{\text{diff}} := \Pi^{\text{diff}} B,$$

 $E^{\rm imp} := \Pi^{\rm imp} E, \quad B^{\rm imp} := \Pi^{\rm imp} B.$ 

It is easy to see that all projectors (and consequently  $A^{\text{diff}}, B^{\text{diff}}, E^{\text{imp}}, B^{\text{imp}}$ ) do not depend on the specific choice of the transformation matrices *T* and *S* (and only on the spaces  $\mathscr{V}^*_{(E,A)}$  and  $\mathscr{W}^*_{(E,A)}$ ).

Using definition 4.16, the KCD can directly be obtained in terms of the original system's matrices (and in the original coordinate system) using the following corollary: [10, Corollary 4.6]

**Corollary 4.17** (Regular KCD). *Choose full column rank matrices* P<sub>1</sub>, P<sub>2</sub>, R, Q as *follows:* 

im  $P_1 = \operatorname{im}\langle A^{\operatorname{diff}}, B^{\operatorname{diff}} \rangle$ , im  $\langle A^{\operatorname{diff}}, B^{\operatorname{diff}} \rangle \bigoplus \operatorname{im} R = V^*_{(E,A)}$ 

im  $P_2 = \operatorname{im}\langle E^{\operatorname{imp}}, B^{\operatorname{imp}}\rangle$ , im  $\langle E^{\operatorname{imp}}, B^{\operatorname{imp}}\rangle \oplus \operatorname{im} Q = W^*_{(E,A)}$ 

where  $\langle A, B \rangle = [B, AB, A^2B, ..., A^nB]$ . Then  $T = [[P_1, P_2], R, Q] \in \mathbf{GL}_n$  and  $S = [[EP_1, AP_2], ER, AQ]^{-1} \in \mathbf{GL}_n$  transform the DAE system  $(E, A, B) \in \sum_{m=1}^{n \times n}$  into KCD (4.6) with some additional zero blocks:

$$(E, A, B) \stackrel{S,T}{\cong} \tag{4.9}$$

$$\left(\begin{pmatrix} \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & N_{11} \end{pmatrix} & \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} & \begin{pmatrix} \mathbf{0} \\ N_{12} \end{pmatrix} \\ \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & N_{22} \end{pmatrix}, \begin{pmatrix} \begin{pmatrix} J_{11} & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} & \begin{pmatrix} J_{12} \\ \mathbf{0} \end{pmatrix} & \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \\ \mathbf{0} & J_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{pmatrix}, \begin{pmatrix} \begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix} \\ \mathbf{0} \end{pmatrix} \end{pmatrix}, \right)$$

where  $\begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & N_{11} \end{pmatrix}$ ,  $\begin{pmatrix} J_{11} & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}$ ,  $\begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix} \end{pmatrix}$  is completely controllable and  $N_{11}$  and  $N_{22}$  are nilpotent.

Now that we know how to construct the KCD in case of both general and regular DAE-systems, lets look at some examples of both the general and the regular cases.

## 4.3 Examples of Kalman Controllability Decomposition

In this section we will look at some examples of DAE-systems. In particular, we will consider both the general and regular case of one DAE system.

**Example 4.18.** Let  $(E, A, B) \in \sum_{2}^{3 \times 3}$  be defined as follows:

$$(E, A, B) = \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$
(4.10)

Clearly this system is regular as det(sE - A) = (1 - s)(s - 1). Let us, however, first consider the general case by use of theorem 4.10. First the limits of the augmented Wong sequences need to be calculated (see definition 4.8). Using either pen and paper or matlab, the following limits are obtained:

$$V_{(E,A,B)}^{*} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad W_{(E,A,B)}^{*} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Using these limits the matrices  $R_1$  and  $R_2$  are given by:

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Using the limits and the matrices  $R_1$  and  $R_2$ , the matrices  $P_1$  and  $P_2$  can be found as follows. First calculate the ranks of  $P_1$  and  $P_2$ . These are given by  $n_2 = \text{Rank}(V_{(E,A,B)}^*) - \text{Rank}(R_1)$  and  $l_2 = \text{Rank}(EV_{(E,A,B)}^* + \text{im}B) - \text{Rank}(R_2)$ , respectively. Next construct a random n by  $n_2$  matrix out of the columns of  $V_{(E,A,B)}^*$ . Because this newly constructed matrix is randomly constructed, there is a good change that the image of this newly constructed matrix and the image of the matrix  $R_1$  are disjoint<sup>1</sup>. If not, just construct a new random matrix out of the columns of  $V_{(E,A,B)}^*$ . After checking that the images are indeed disjoint, it can be concluded that this newly constructed matrix can be chosen as  $P_1$ . For  $P_2$  the exact same is done only this time construct a random l by  $l_2$  matrix out of the columns of  $EV_{(E,A,B)}^*$  + imB. Also, instead of  $R_1$  use  $R_2$ . Using this method, the following matrices for  $P_1$  and  $P_2$  are obtained:

$$P_1 = \begin{pmatrix} 0\\1\\0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0\\1\\0 \end{pmatrix}$$

Using the matrices  $R_1$ ,  $R_2$ ,  $P_1$  and  $P_2$ , the matrices  $Q_1$  and  $Q_2$  can be found as follows. First calculate the rank of  $Q_1$  and  $Q_2$ . These are given by  $n_3 = n - \text{Rank}([R_1, P_1])$  and  $l_3 = l - \text{Rank}([R_2, P_2])$ , respectively. Next construct a random n by  $n_3$  matrix and check whether the image of this newly constructed matrix and the image of the matrix  $[R_1, P_1]$  are disjoint. If true, it can be concluded that this newly created matrix can be chosen as  $Q_1$ . If false, just construct a new random matrix. For  $Q_2$  the exact same is done only this time construct a random l by  $l_3$  matrix. Also, instead of  $[R_1, P_1]$  use  $[R_2, P_2]$ . Using this method, the following matrices for  $Q_1$  and  $Q_2$  are obtained:

<sup>&</sup>lt;sup>1</sup>Disjoint means that they only share the zero vector

$$Q_1 = \left( \right)$$
,  $Q_2 = \left( \right)$ 

Here the matrices  $Q_1$  and  $Q_2$  are elements of  $\mathbb{R}^{3\times 0}$  because  $n_3 = l_3 = 0$ . Using all of these matrices the following transformation matrices *T* and *S* are obtained:

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Using the matrices *S* and *T* the following KCD is obtained:

$$(E, A, B) \stackrel{S,T}{\cong} \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \right)$$
(4.11)

So, it can be concluded that the system

$$(E_{11}, A_{11}, B_1) = \left( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

is completely controllable, that the system

 $(E_{22}, A_{22}, 0) = (1, 1, 0)$ 

is uncontrollable and that there is no inconsistent part in this example.

**Example 4.19.** Consider again the same system (4.10) as in the previous example, only now consider the regular case by use of corollary 4.17. While not entirely the same, the steps that need to be taken are rather similar to those in the previous example. Therefore, while the intermediate steps and matrices will be given, the details themselves will be omitted.

First, the limits of the original Wong sequences need to be calculated. These are given by:

$$V_{(E,A)}^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$
,  $W_{(E,A)}^* = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ 

Next the transformation matrices  $\tilde{S}$  and  $\tilde{T}$  of the QWF (4.7) need to be found. Using the Wong limits they are as follows:

$$ilde{T} = egin{pmatrix} 0 & 0 & 1 \ 1 & 0 & 0 \ 0 & 1 & 0 \end{pmatrix}, \quad ilde{S} = egin{pmatrix} 0 & 1 & 0 \ 0 & 0 & 1 \ 1 & 0 & 0 \end{pmatrix}$$

After finding the matrices  $\tilde{S}$  and  $\tilde{T}$ , they need to be used to find the projectors and in particular the differential and the impulse projector (see definition 4.16). These two projectors are as follows:

$$\Pi^{\text{diff}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Pi^{\text{imp}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Using these projectors, the following matrices for  $A^{\text{diff}}, B^{\text{diff}}, B^{\text{imp}}, E^{\text{imp}}$  can be found:

$$A^{\text{diff}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B^{\text{diff}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$B^{\text{imp}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad E^{\text{imp}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Finally, using these four matrices, the transformation matrices *S* and *T* regarding corollary 4.17 can be found. But first let us find the matrices  $P_1$ ,  $P_2$ , *R* and *Q*:

$$P_1 = \begin{pmatrix} 0\\0\\1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1\\0\\0 \end{pmatrix}, \quad R = \begin{pmatrix} 0\\1\\0 \end{pmatrix}, \quad Q = \begin{pmatrix} \end{pmatrix}$$

Use these four matrices the following transformation matrices *S* and *T* are obtained:

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Using the matrices *S* and *T* the following regular KCD is obtained:

$$(E, A, B) \stackrel{S,T}{\cong} \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \right)$$
(4.12)

So it can be concluded that the system

$$(E_{11}, A_{11}, B_1) = \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

is completely controllable, where  $N_{11} = [0]$  and  $J_{11} = [1]$ , that the system

$$(E_{22}, A_{22}, 0) = (1, 1, 0)$$

is uncontrollable, where  $J_{22} = [1]$  and that there is no inconsistent part in this example.

In the next chapter we will take a look at KCDs of electrical circuits and see what we can conclude regarding the controllability of electrical circuits.

## **Chapter 5**

# **Controllability of RLC Electrical Circuits**

Now that the needed background information regarding RLC electrical circuits, modelling RLC electrical circuits and Kalman Controllability Decomposition has been established, they can now be applied to investigate the controllability of RLC electrical circuits.

In this chapter, we will consider multiple examples of electrical circuits. Regarding these examples, we will restrict ourselves to only KCDs of the form (4.6). Of course, one could also use the form (4.9). However, this only works if the DAE-system is regular, something which isn't always the case. Of course, when considering practical purposes, one can just assume that only regular system occur, as singular systems correspond to electrical circuits that have no particular physical meaning. Regardless, equation (4.6) already gives us enough information to understand which part of the system is completely controllable, behaviourally controllable and uncontrollable.

In chapter 5.1, we will look at four basic electrical circuits. After finding a KCD for each of these four basic examples, we are going to consider the complete controllable part, the behavioral controllable part and the uncontrollable part of the systems. In chapter 5.2, we will consider some simple examples regarding controllability and structure. Using these examples, we will show how the structure of an electrical circuit can influence the dimensions of the complete controllable part, the behavioral controllable part and the uncontrollable part. We will conclude this chapter with some hypotheses regarding controllability and the structure of electrical circuits.

## 5.1 Basic RLC electrical circuits

Let us take a look at the following four circuits described in figures 5.1 and 5.2. Without going into too much details and by use of theorem 4.10 and Matlab, the general KCDs and transformation matrices of these four circuits are described in appendix A.1.

Looking at the KCDs given in appendix A.1, it can be concluded that for all of these circuits, there is a 3-dimensional subspace of  $\mathbb{R}^8$  that describes the part



(A) Serie and Voltage

(B) Serie and Current

FIGURE 5.1: The basic serie electrical circuits



FIGURE 5.2: The basic parallel electrical circuits

of the system that is completely controllable and a 5-dimensional subspace of  $\mathbb{R}^8$  that describes the part of the system that is behaviorally controllable. No part of the systems is uncontrollable. This implies that, at least to a certain extent, it is possible to control the entire system and nothing within the systems is completely out of our hands.

Why the dimension of the behavioral controllable part is higher than the dimension of the complete controllable part, this is probably because a large part of the system consist of algebraic constraints, coming from Kirchhoff's laws, the input of the sources and Ohm's law. More algebraic constraints restricting the system, results in less initial values having a solution. Because the number of initial values with solutions decreases, the dimension of the complete controllable part will also decrease (see chapter 4.1.2). This in turn leads to an increase in dimensions of the other parts.

As the three relations previously mentioned, make up more than 50% of the equations in the examples, it isn't strange that the dimension of the behavioral controllable part is higher than the dimension of the complete controllable part, when also taking into account that the uncontrollable part has dimension zero.

This also brings up the idea that, if one increases the number of algebraic constrains, by replacing inductors or capacitors with resistors for example, the dimension of the complete controllable part will decrease or stay the same, while the combined dimensions of the behavioral controllable part and the uncontrollable part will, respectively, increase or stay the same. Furthermore, because at least 50% of the equations are algebraic constraints, one could bring up the idea that the dimension of the complete controllable part can never exceed the combined dimensions of the behavioral controllable part and the uncontrollable part. Whether these ideas also hold true is still unknown. More about this in chapter 5.3.

## 5.2 Controllability and structure

Now that the four basic examples have been discussed, let us look at some other simple examples more focused on the relation between controllability and the structure of circuits. For these examples no matrices are given. Instead, the result will be given in terms of the dimensions of the different kinds of controllabilities. But first, let us describe the examples using figures 5.3, 5.4, 5.5, 5.6, 5.7 and 5.8.



(A) Voltage source

 $i \bigoplus_{i \in \mathcal{L}_{2}} C_{2}$ 

(B) Current source

FIGURE 5.3: Serie and Capacitors



FIGURE 5.4: Parallel and Capacitors



FIGURE 5.5: Serie and Inductors



FIGURE 5.6: Parallel and Inductors





(A) Voltage source

(B) Current source





FIGURE 5.8: Parallel and Resistors

The results regarding the dimensions of the different kinds of controllability are given in table 5.9. As can be seen from the table, circuits 5.3a, 5.3b, 5.6a and 5.6b contain an uncontrollable part, which is in contrast to the other circuits mentioned in this section. That these circuits contain an uncontrollable part, implies that structure can indeed influence the controllability of an electrical circuit.

Circuit	Dim. complete	Dim. uncontrollable	Dim. behavioral
5.3a	2	1	3
5.3b	2	1	3
5.4a	2	0	4
5.4b	2	0	4
5.5a	2	0	4
5.5b	2	0	4
5.6a	2	1	3
5.6b	2	1	3
5.7a	1	0	5
5.7b	1	0	5
5.8a	1	0	5
5.8b	1	0	5

#### TABLE 5.9: The results

When considering the circuits described in figures 5.7 and 5.8, the difference in dimensions between the complete controllable part and the sum of the behavioral controllable part and the uncontrollable part is greater, then compared to the other circuits mentioned in this section. This again confirms the idea, that the number of algebraic constraints influences the ratio of the dimensions of the complete controllable part and the sum of the behavioral controllable part and the uncontrollable part. This can be seen from the fact that the examples described in figures 5.7 and 5.8 consist mainly of resistors, which implies that they contain more algebraic constraint compared to the other examples mentioned in this section.

Using these results together with the results in chapter 5.1, in the next section we will construct some hypotheses regarding electrical circuits and controllability.

### 5.3 Hypotheses

Considering the examples discussed in chapters 5.1 and 5.2, we can state the following hypotheses.

**Hypothesis 5.1.** Any electrical circuit that contains either two or more capacitors in series and/or two or more inductors in parallel will have an uncontrollable part.

**Hypothesis 5.2.** In an electrical circuit the dimension of the complete controllable part can never exceed the combined dimensions of the behavioral controllable part and the uncontrollable part.

**Hypothesis 5.3.** If in an electrical circuit one increases the algebraic constraints, by replacing inductors and/or capacitors with resistors and/or sources, the dimension of the complete controllable part will either decrease or stay the same, while the combined dimensions of the behavioral controllable part and the uncontrollable part will, respectively, increase or stay the same.

Regarding hypothesis 5.1, there is no real prove for this hypothesis except for the fact that no counter example has been found after considering multiple examples. Furthermore, intuitively it makes sense for this hypothesis to be true, when considering the examples in chapter 5.2.

Regarding hypothesis 5.2, just from considering the examples that have been analyzed in this paper, one could even state that the dimension of the complete controllable part can never exceed the dimension of the behavioral controllable part. However, for this no explanation has been found as of yet. Meanwhile, for hypothesis 5.2 at least an explanation is given in chapter 5.1. This is why hypothesis 5.2 also includes the uncontrollable part.

Regarding hypothesis 5.3, using this as a basis, one can go even further and state the following hypothesis:

**Hypothesis 5.4.** *If, by changing the electrical circuit, the ratio of the number of algebraic constrains and the number of other equations shifts in the direction of the other equations, it can be concluded that the ratio of dimensions of the complete controllable part and the other parts will either stay the same or shifts in the direction of the other parts.* 

And with these hypotheses, we will end this chapter. Whether the hypotheses turn out to be true, still has to be investigated. During the writing of this paper, multiple examples have been studied and analyzed, but none have disputed any of these hypotheses. Therefore, these hypotheses would be good subjects for later studies.

## Chapter 6

# Conclusion

We have discussed RLC electrical circuits and in particular discussed the different kinds of relations between currents and voltages. We talked about how to model these electrical circuits by using both the branch-oriented model and DAE systems of the form  $E\dot{x} = Ax + Bu$ . We discussed controllability for both ODE systems of the form  $\dot{x} = Ax + Bu$  and for DAE systems, where we had to distinguish between complete and behavioral controllability. We explained how to construct the Kalman Controllability Decomposition for both ODE and DAE systems, where we distinguished between general DAE systems and regular DAE systems. And finally we finished this paper with some observations and hypotheses regarding electrical circuits and controllability.

## Appendix A

# System equations and KCDs

## A.1 KCDs of basic electrical circuits

The general KCDs and transformation matrices of the electrical circuits described in chapter 5.1 in figures 5.1 and 5.2 are given by the following four combinations of matrices:

where  $l_1 = 3$ ,  $l_2 = 0$ ,  $l_3 = 5$ ,  $n_1 = 3$ ,  $n_2 = 0$  and  $n_3 = 5$ .

where  $l_1 = 3$ ,  $l_2 = 0$ ,  $l_3 = 5$ ,  $n_1 = 3$ ,  $n_2 = 0$  and  $n_3 = 5$ .

$$T_{3} = \begin{pmatrix}
 0 & 0 & -1/R & -1 & 0 & -1 & -1 & 1 \\
 -1 & 1 & 1/R & 1 & -1 & 0 & -1 & -1 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 0 & 1 & 0 & -1 & 0 & 1 & -1 & 0 \\
 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 \\
 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 \\
 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
 0 & -1/2 & -1/2 & -1/2 & 10 & 0 \\
 0 & -1/2 & -1/2 & -1 & 0 & 0 & 0 & 1 \\
 -1/2 & -1/2 & -1 & 0 & 0 & 0 & 1 \\
 -1/2 & -1/2 & -1/2 & 0 & 0 & 0 \\
 -1/2 & 3/2 & 1/2 & -1/2 & 0 & 0 & 0 \\
 -1 & 0 & 3/2 & 1/2 & -1/2 & 0 & 0 & 0 \\
 -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 -1 & -1/2 & -1/2 & -1/2 & 1/2 & 0 & 0 & 0 \\
 -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 -1 & -1/2 & -1/2 & -1/2 & 0 & 0 & 0 \\
 -1 & -1/2 & -1/2 & -1/2 & -1/2 & -1/2 \\
 0 & 0 & -1 & -3/2 & 0 & -3/2 & 3-R/2 & 1+R/2 \\
 0 & 0 & 0 & -7/2 & -R/2 & -1/2 & -1/2 & -1/2 \\
 0 & 0 & 0 & -3/2 & 0 & -3/2 & 3/2 & -1/2 \\
 0 & 0 & 0 & -5/2 & -1 & 3/2 & 9/2 & 1/2 \\
 0 & 0 & 0 & -3/2 & -1 & -1/2 & 3/2 & 3/2
 \end{pmatrix}$$

$$F_{3} = \begin{pmatrix}
 0 & 0 & -C & 0 & -C & C & 0 & C \\
 L & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\$$

where  $l_1 = 3$ ,  $l_2 = 0$ ,  $l_3 = 5$ ,  $n_1 = 3$ ,  $n_2 = 0$  and  $n_3 = 5$ .

where  $l_1 = 3$ ,  $l_2 = 0$ ,  $l_3 = 5$ ,  $n_1 = 3$ ,  $n_2 = 0$  and  $n_3 = 5$ .

Here matrices (A.1) correspond to figure 5.1a, matrices (A.2) correspond to figure 5.1b, matrices (A.3) correspond to figure 5.2a and matrices (A.4) correspond to figure 5.2b.

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