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Wallpaper Groups

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Abstract

In this text, we repeat the definition of a wallpaper group, and the notion of equivalence of such groups. We present a proof of the classical fact that there are precisely *seventeen equivalence classes* of wallpaper groups. This is done by discussing an earlier proof presented by Schwarzenberger, and adding more details to his argument. This thesis contains a fully explicit proof using the underlying idea.

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Notation	Description
$\mathrm{Isom}(\mathbb{R}^2)$	The group of all isometries of $\mathbb{R}^2$ under composition.
$\mathrm{O}(2)$	The group of all linear isometries of $\mathbb{R}^2$ under composition.
$\circ$	The function composition.
$\langle\cdot,\cdot\rangle$	The standard Euclidean inner product.
$\\|\cdot\\|$	The standard Euclidean norm.
id	The identity map.
τ_v	The translation over the vector v.
ρ_θ	The rotation by the angle θ about the origin.
μ_l	The reflection across the line l.
$[a]$	The integral part of a real number a.
$\angle(l,m)$	The slope angle of m minus the slope angle of l.
$v \in l$	The vector v is parallel to the line l through the origin.
$v \in l_\perp$	The vector v is perpendicular to the line l through the origin.
$\mathrm{sftvec}(\mu_l)$	The set of all shift vectors of μ_l.
q	The order of the rotation subgroup of the point group.

1 Introduction

We classify plane isometries into four types – translations, rotations, reflections, and glide reflections. We categorize non-identity discrete groups of plane isometries into three types – rosette groups (finite dihedral groups and their cyclic subgroups), frieze groups, and wallpaper groups. A rosette group contains no translation; A frieze group contains translations on one single direction; and a wallpaper group contains translations on more than one directions.

Fedorov [1891], Fricke and Klein [1897], Niggli [1924], and Pólya [1924] gave early classifications of the 17 types of wallpaper groups [Martin, 1982]. Their approaches are geometric and *ad hoc* [Hiller, 1986]. Schwarzenberger [1974] gave an algebraic proof, in which the theories are correct but necessary non-trivial details are missing. Martin [1982] proved the classification by discussing the possibilities of generators around a motif, the approach is also geometric. Geometric approach is helpful for construction and visual discrimination of wallpaper patterns [see e.g. Schattschneider, 1978], but often forces the thought of the audience onto a specific picture instead of the group of transformations in the plane. Armstrong [1988] presented the 17 types of wallpaper groups and proved that the different types are indeed not isomorphic. However, he did not give a definition of equivalence. Artin [2011] explained two groups completely, which correspond to the subsection $q = 2$ in Section 4.3 in this text. From this non-exhaustive research of literature, we found mostly geometric approaches of the proof and incomplete algebraic approaches.

This thesis is based on Schwarzenberger's ideas. In this thesis we recall the equivalence relation between wallpaper groups, and prove that under this definition there are precisely seventeen equivalence classes of wallpaper groups.

Firstly, we recall the related concepts about plane isometries; secondly, we prove the so-called crystallographic restriction; thirdly, we separate the situation into three cases – no reflection, one reflection, more than one reflection. Under each case, we prove that there are certain numbers of equivalent classes of wallpaper groups, and the numbers sum up to seventeen.

2 Plane Isometries

This is a chapter of preliminaries. We recall a few definitions, propositions and remarks to be used later. One can use this chapter to ease into the notation. In the end of this chapter, we prove the classification of plane isometries to make sure that all the definitions in the beginning are indeed exhaustive.

2.1 Definitions and Group Structure of Plane Isometries

Definition 2.1 (Plane isometry). If $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a transformation of the plane satisfying $\|\sigma(x)-\sigma(y)\| = \|x-y\|$ for every pair of points $x, y \in \mathbb{R}^2$, we call σ an *isometry* of the plane.

Definition 2.2 (Plane isometry group). Equipped with composition as the binary operation, the set $\operatorname{Isom}\mathbb{R}^2$ of the isometries of the plane forms a group (see Proposition 2.18). We call this group $(\operatorname{Isom}(\mathbb{R}^2), \circ, \mathrm{id})$ the *isometry group* of the plane.

Definition 2.3 (Translation of the plane). If a map $\tau_v : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $x \mapsto x+v$ for some $v \in \mathbb{R}^2$, we call τ_v the *translation* about the vector v.

Remark 2.4. A translation is a plane isometry, and has an inverse, which is also a translation. For any translations τ_v and τ_w of the plane, we have $\tau_v^{-1} = \tau_{-v}$ and $\tau_v \circ \tau_w = \tau_{v+w} = \tau_w \circ \tau_v$.

Definition 2.5 (Rotation of the plane). If a map $\rho_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by $x \mapsto \left[\begin{smallmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{smallmatrix}\right] x$ for some angle θ, we call ρ_θ the *rotation by angle θ about the origin*. A *rotation ρ about any point A by angle θ* is defined as the composition $\rho = \tau_A \circ \rho_\theta \circ \tau_{-A}$.

Remark 2.6. A rotation is a plane isometry, and has an inverse, which is also a rotation. For any rotation ρ_θ about the origin, we have its inverse $\rho_\theta^{-1} = \rho_{-\theta}$. We also have $\rho_\alpha \circ \rho_\beta = \rho_{\alpha+\beta}$ for any two rotations ρ_α and ρ_β about the origin. An arbtrary rotation $\rho = \tau_A \circ \rho_\theta \circ \tau_{-A}$ is a composition of isometries, therefore, also an isometry, it has an inverse $\tau_A \circ \rho_{-\theta} \circ \tau_{-A}$, which is, by Definition 2.5, a rotation.

Definition 2.7 (Reflection of the plane). If a map $\mu\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by $x \mapsto \left[\begin{smallmatrix}1 & 0 \\ 0 & -1\end{smallmatrix}\right] x$, we call μ the *reflection across the e_1-axis*. A *reflection across any line l through the origin* is defined as the composition $\mu_l = \rho_{\angle(l)} \circ \mu$, where $\angle(l)$ is the slope angle of l. A *reflection across a line m parallel to l through an arbitrary point A* is defined as the composition $\mu_m = \tau_A \circ \mu_l \circ \tau_{-A}$.

Remark 2.8. A reflection μ across the e_1-axis is a plane isometry, the inverse is μ itself. An arbitrary reflection $\mu_l = \tau_A \circ \rho_\theta \circ \mu \circ \tau_{-A}$ is a composition of isometries, therefore, also an isometry. The inverse of a reflection is the reflection itself. In particular, for lines l and m we have $\mu_l \circ \mu_l = \mathrm{id}$, and $\mu_l \circ \mu_m = \rho_\theta$ where $\theta = 2\angle(m,l)$. The notation $\angle(m,l)$ means the angle from m to l.

Proposition 2.9. *Let μ_l be a reflection across the line l through the origin. Let $t \in \mathbb{R}^2$. Then $t - \mu_l(t) \in l_\perp$ and $t + \mu_l(t) \in l$.*

Proof. We have $\mu_l(t - \mu_l(t)) = \mu_l(t) - \mathrm{id}(t) = -(t - \mu_l(t))$. Therefore, $t - \mu_l(t) \in l_\perp$. We have $\mu_l(t + \mu_l(t)) = \mu_l(t) + \mathrm{id}(t) = t + \mu_l(t)$. Therefore, $t + \mu_l(t) \in l$. ∎

Remark 2.10. Let l be a line through the origin. Then $\mu_l(t) = t$ if and only if $t \in l$; and $\mu_l(t) = -t$ if and only if $t \in l_\perp$.

Definition 2.11 (Glide reflection of the plane). If a map $\gamma_{l,v} : \mathbb{R}^2 \to \mathbb{R}^2$ is a composition $\gamma_{v,l} = \tau_v \circ \mu_l$ where τ_v is a non-identity translation, and μ_l is a reflection across a line l through the origin, and the vector v is parallel with the reflection axis l, we call $\gamma_{l,v}$ a *glide reflection* across the line l through the origin over the vector v. A glide reflection across a line m through an arbitrary point A parallel to l about the vector v is defined as the composition $\mu_m = \tau_A \circ \tau_v \circ \mu_l \circ \tau_{-A}$.

Remark 2.12. A glide reflection is a plane isometry, and has an inverse, which is also a glide reflection. Let γ be a glide reflection. Then by definition, $\gamma = \tau_v \circ \mu_l$, where $v \in l$. It is a isometry because τ_v and μ_l are isometries by Remark 2.4 and 2.8. Its inverse is $\mu_l \circ \tau_{-v} = \tau_A \circ \rho_\theta \circ \mu \circ \tau_{-A} \circ \tau_{-v}$, which is also a composition of isometries, therefore, an isometry.

Remark 2.13. Inherited from the matrix definition, a rotation about the origin is a linear map, a reflection across a line through the origin is a linear map.

Proposition 2.14. *If φ is a linear isometry, then $\varphi \circ \tau_v = \tau_{\varphi(v)} \circ \varphi$.*

Proof. For any $x \in \mathbb{R}^2$, we have $(\varphi \circ \tau_v)(x) = \varphi(\tau_v(x)) = \varphi(v + x) = \varphi(v) + \varphi(x) = (\tau_{\varphi(v)} \circ \varphi)(x)$. ∎

Proposition 2.15. *For a non-identity rotation ρ_θ about the origin and any translation τ_v, the composition $\tau_v \circ \rho_\theta$ is a rotation.*

Proof. We look for a fixed point of $\tau_v \circ \rho_\theta$ using the equation $(\tau_v \circ \rho_\theta)(x) = x$. Since ρ_θ is defined as $x \mapsto Ax$ for some $A \in \mathrm{SO}(2)$, we can write the equation as $v + Ax = x$. Then $x = [I - A]^{-1}v$. This inversion make sense because the condition that $\rho_\theta \neq \mathrm{id}$ implies that $Az \neq z$ for all $z \neq 0$. Hence, $\ker(A - I) = \{0\}$. Therefore, $A - I$ is invertible. Then for an arbitrary $y \in \mathbb{R}^2$, we have $(\tau_{-x} \circ \tau_v \circ \rho_\theta \circ \tau_x)(y) = -[I-A]^{-1}v + v + A(y + [I-A]^{-1}v) = Ay = \rho_\theta(y)$. Therefore, $\tau_v \circ \rho_\theta = \tau_x \circ \rho_\theta \circ \tau_{-x}$. By Definition 2.5, $\tau_v \circ \rho_\theta$ is a rotation. ∎

Remark 2.16. A composition $\rho_\theta \circ \tau_v$ is a rotation. By Proposition 2.14, $\rho_\theta \circ \tau_v = \tau_{\rho_\theta(v)} \circ \rho_\theta$, which is a rotation as above.

Remark 2.17. A composition $\mu_l \circ \tau_v$ with $v \in l$ is a glide reflection. By Proposition 2.14, $\mu_l \circ \tau_v = \tau_{\mu_l(v)} \circ \mu_l = \tau_v \circ \mu_l$, which is a glide reflection.

Equipped with composition as the binary operation, the isometries of the plane form a group which is denoted $(\mathrm{Isom}\,\mathbb{R}^2, \circ, \mathrm{id})$. We recall the following standard properties of this group.

Proposition 2.18 (Group structure of plane isometries). *The isometries of the plane form a group under composition.*

Proof. Closure: Composition of isometries are still isometries. Because for all isometries f and g, for all $x, y \in \mathbb{R}^2$, we have $\|(f \circ g)(x) - (f \circ g)(y)\| = \|f(g(x)) - f(g(y))\| = \|g(x) - g(y)\| = \|x - y\|$.

Associativity: Isometries of the plane are associative under composition. Because for arbitrary isometries f, g and h, for every $x \in \mathbb{R}^2$, we have $(f \circ (g \circ h))(x) = f(g(h(x))) = (f \circ g)(h(x)) = ((f \circ g) \circ h)(x)$.

Unit element: The identity map serves as a unit element. Because for arbitrary isometry f, for every $x \in \mathbb{R}^2$, we have $(\mathrm{id} \circ f)(x) = \mathrm{id}(f(x)) = f(x) = f(\mathrm{id}(x)) = (f \circ \mathrm{id})(x)$.

Inverse: Every plane isometry has an inverse which is also a plane isometry. Let f be an isometry of the plane. Then by Proposition 2.23, we can write uniquely that $f = \tau_v \circ \varphi$, where τ is a translation and φ is a linear isometry. Then $\varphi^{-1} \circ \tau_{-v}$ is the inverse of f and an isometry. ∎

Proposition 2.19. *The composition $\tau_v \circ \mu_l$ with $v \notin l$ is either a glide reflection or a reflection.*

Proof. Let μ_l be a reflection across the line l through the origin. Let τ_v be a translation over the vector $v \notin l$. Then we can write $v = r + s$, where $r \in l$ and $s \in l_\perp$. Then $\tau_v \circ \mu_l = \tau_r \circ \tau_s \circ \mu_l = \tau_r \circ \tau_{s/2} \circ \tau_{s/2} \circ \mu_l = \tau_r \circ \tau_{s/2} \circ \mu_l \circ \tau_{\mu_l(s/2)}$. We know that $\mu(s/2) = -s/2$. Therefore, $\tau_v \circ \mu_l = \tau_r \circ \tau_{s/2} \circ \mu_l \circ \tau_{-s/2} = \tau_{s/2} \circ \tau_r \circ \mu_l \circ \tau_{-s/2}$. If $r \neq 0$, then $\tau_v \circ \mu_l$ is a glide reflection across a line through $s/2$ parallel to l, by definition. If $r = 0$, then $\tau_v \circ \mu_l$ is a reflection across a line through $s/2$ parallel to l, by definition. ∎

2.2 Classification of Plane Isometries

Lemma 2.20. *A plane isometry fixing the origin preserves the standard Euclidean norm.*

Proof. Let $\sigma \in \mathrm{Isom}(\mathbb{R}^2)$. Let $a \in \mathbb{R}^2$. Then we have $\|\sigma(a)\| = \|\sigma(a) - \sigma(0)\| = \|a - 0\| = \|a\|$. ∎

Lemma 2.21. *If $\sigma \in \mathrm{Isom}(\mathbb{R}^2)$ safisfy $\sigma(0) = 0$ then $\langle \sigma(a), \sigma(b)\rangle = \langle a, b\rangle$ for all $a, b \in \mathbb{R}^2$. In other words, a plane isometry fixing the origin preserves the standard Euclidean inner product.*

Proof. Let $\sigma \in \mathrm{Isom}(\mathbb{R}^2)$ satisfy $\sigma(0) = 0$. Let $a, b \in \mathbb{R}^2$. Then we have $\|\sigma(a)\|^2 - 2\langle \sigma(a), \sigma(b)\rangle + \|\sigma(b)\|^2 = \|\sigma(a) - \sigma(b)\|^2 = \|a - b\|^2 = \|a\|^2 - 2\langle a, b\rangle + \|b\|^2$. By Lemma 2.20 we have $\|\sigma(a)\| = \|a\|$ and $\|\sigma(b)\| = \|b\|$. It follows that $\langle \sigma(a), \sigma(b)\rangle = \langle a, b\rangle$. ∎

Proposition 2.22. *A plane isometry is $\mathbb{R}$-linear if and only if it fixes the origin.*

Proof. Let $\sigma \in \mathrm{Isom}(\mathbb{R}^2)$ be linear over $\mathbb{R}$. For any $x, y \in \mathbb{R}^2$, any $c \in \mathbb{R}^2$, we have $\sigma(x+y) = \sigma(x) + \sigma(y)$ and $\sigma(cx) = c\sigma(x)$. Therefore, $\sigma(0) = \sigma(x + (-x)) = \sigma(x) - \sigma(x) = 0$.

Now we prove the converse. Let $\sigma \in \mathrm{Isom}(\mathbb{R}^2)$ satisfy $\sigma(0) = 0$. Let $x, y \in \mathbb{R}^2$, and $c \in \mathbb{R}$. Then we have

$$\begin{aligned}
&\|\sigma(x+y) - \sigma(x) - \sigma(y)\|^2 \\
=&\|\sigma(x+y)\|^2 + \|\sigma(x)\|^2 + \|\sigma(y)\|^2 - 2\langle \sigma(x+y), \sigma(x)\rangle - 2\langle \sigma(x+y), \sigma(y)\rangle + 2\langle \sigma(x), \sigma(y)\rangle \\
=&\|x+y\|^2 + \|x\|^2 + \|y\|^2 - 2\langle x+y, x\rangle - 2\langle x+y, y\rangle + 2\langle x, y\rangle && \text{(by Lemma 2.20, 2.21)} \\
=&\|x+y-x-y\|^2 \\
=&0.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\|\sigma(cx) - c\sigma(x)\|^2 \\
=&\|\sigma(cx)\|^2 + \|c\sigma(x)\|^2 - 2\langle \sigma(cx), c\sigma(x)\rangle \\
=&\|\sigma(cx)\|^2 + |c|^2\,\|\sigma(x)\|^2 - 2c\langle \sigma(cx), \sigma(x)\rangle \\
=&\|cx\|^2 + |c|^2\,\|x\|^2 - 2c\langle cx, x\rangle && \text{(by Lemma 2.20, 2.21)} \\
=&|c|^2\,\|x\|^2 + |c|^2\,\|x\|^2 - 2c^2\langle x, x\rangle \\
=&|c|^2(\|x\|^2 + \|x\|^2 - 2\langle x, x\rangle) \\
=&|c|^2\,\|x-x\|^2 \\
=&0.
\end{aligned}$$

Therefore, $\sigma(x+y)=\sigma(x)+\sigma(y)$ and $\sigma(cx)=c\sigma(x)$. In other words, σ is linear.

In the plane, the linear isometries are precisely those isometries fixing the origin. ∎

Proposition 2.23 (The unique expression of a plane isometry). *Every plane isometry can be written uniquely as a composition of a translation and a linear isometry.*

Proof. Existence: Let $\sigma \in \mathrm{Isom}(\mathbb{R}^2)$. We can write $\sigma = \tau_{\sigma(0)} \circ (\tau_{-\sigma(0)} \circ \sigma)$. The expression $\tau_{-\sigma(0)} \circ \sigma$ is a composition of plane isometries, therefore, also an isometry of the plane. Moreover, we have $(\tau_{-\sigma(0)} \circ \sigma)(0) = \tau_{-\sigma(0)}(\sigma(0)) = 0$. By Proposition 2.22, $\tau_{-\sigma(0)} \circ \sigma$ is an linear isometry.

Uniqueness: Let σ be any plane isometry. From above, we can express σ as $\sigma = \tau_{\sigma(0)} \circ (\tau_{-\sigma(0)} \circ \sigma)$. Assume we can also express σ as $\sigma = \tau_v \circ \varphi$ for some translation τ_v and linear isometry φ. We want to prove that $v = \sigma(0)$ and $\varphi = \tau_{-\sigma(0)} \circ \sigma$. The two expressions for the same isometry σ must be equal, in other words, $\tau_v \circ \varphi = \sigma = \tau_{\sigma(0)} \circ (\tau_{-\sigma(0)} \circ \sigma)$. Hence, we have $v = \tau_v(0) = \tau_v(\varphi(0)) = (\tau_v \circ \varphi)(0) = (\tau_{\sigma(0)} \circ (\tau_{-\sigma(0)} \circ \sigma))(0) = \tau_{\sigma(0)}\big((\tau_{-\sigma(0)} \circ \sigma)(0)\big) = \tau_{\sigma(0)}(0) = \sigma(0)$. and $\varphi = \tau_{-v} \circ \tau_v \circ \varphi = \tau_{-v} \circ \sigma = \tau_{-\sigma(0)} \circ \sigma$. Thus the two expression coincide. ∎

Remark 2.24. We can write any element σ in $\mathrm{Isom}(\mathbb{R}^2)$ as $\sigma = \tau_v \circ \varphi$ uniquely with a translation τ_v and a linear isometry φ. From now on we simply do so without mentioning again.

Proposition 2.25 (Classification of linear isometries). *A linear isometry of the plane is either a reflection or a rotation.*

Proof. Let φ be a linear isometry of the plane. By linearity, φ can be represented by a 2×2 matrix A. Since φ is an isometry, for an arbitrary $x \in \mathbb{R}^2$ we have $x^T x = \|x\|^2 = \|\varphi(x)\|^2 = \|Ax\|^2 = [Ax]^T[Ax] = x^T A^T A x$. Hence, $A^T A = I$. It follows that $(\det(A))^2 = 1$. Therefore, $\det(A) = \pm 1$. Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Then

$$\begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A^T A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

By far we have four equalities $a^2+c^2 = 1$, $b^2+d^2 = 1$, $ab+cd = 0$, and $ad-bc = \det(A) = \pm 1$. The first two equalities $a^2+c^2 = 1$ and $b^2+d^2 = 1$ give us $a = \cos\theta_1$, $c = \sin\theta_1$, $b = \sin\theta_2$, and $d = \cos\theta_2$ for some angles $\theta_1, \theta_2 \in [0, 2\pi)$. (Note that switching to $b = \cos\theta_2$ and $d = \sin\theta_2$, or similarly $a = \sin\theta_1$ and $c = \sin\theta_1$, we would obtain the same final conclusion.) After substitution, the third equality $ab+cd = 0$ gives $0 = \cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2 = \sin(\theta_1+\theta_2)$. Therefore, the only posibilities are $\theta_1+\theta_2 \in \{0, \pi, 2\pi, 3\pi\}$. In other words, $\theta_2 = -\theta_1$ or $\theta_2 = \pi-\theta_1$ or $\theta_2 = 2\pi-\theta_1$ or $\theta_2 = 3\pi-\theta_1$. The latter two cases merges with the former two respectively up to the evaluation of sin and cos. Hence, we have either $\theta_2 = -\theta_1$ or $\theta_2 = \pi-\theta_1$. Now we only need one angle to express the matrix A.

For the former case $\theta_2 = -\theta_1$, let $\theta_1 = \theta$, and $\theta_2 = -\theta$. Then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin(-\theta) \\ \sin\theta & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

By Definition the matrix A represents a rotation about the origin by angle θ.

For the latter case $\theta_2 = \pi-\theta_1$, let $\theta_1 = \theta$, and $\theta_2 = \pi-\theta$. Then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin(\pi-\theta) \\ \sin\theta & \cos(\pi-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

By Definition the matrix A represents a reflection across a line l through the origin with the slope angle θ.

In both cases above, the fourth equality $ad-bc = \det(A) = \pm 1$ is satisfied.

Therefore, every linear isometry is either a reflection across a line through the origin or a rotation about the origin. ∎

Proposition 2.26 (classification of plane isometries). *An isometry of the plane is either an identity, a translation, a rotation, a reflection, or a glide reflection.*

Proof. Let σ be a plane isometry, then by Proposition 2.23 we can write uniquely $\sigma = \tau_v \circ \varphi$ where τ_v is a translation and φ is a linear isometry. We have the following possible cases by toggling τ and φ.

1. If $\tau_v = \mathrm{id}$ and $\varphi = \mathrm{id}$, then σ is the identity transformation.
2. If $\tau_v \neq \mathrm{id}$ and $\varphi = \mathrm{id}$, then σ is a translation.
3. If $\tau_v = \mathrm{id}$ and $\varphi \neq \mathrm{id}$, then σ is an linear isometry, and
   a) if φ is a reflection across a line l through the origin, then so is σ;
   b) if φ is a rotation about the origin, then so is σ.
4. If $\tau_v \neq \mathrm{id}$ and $\varphi \neq \mathrm{id}$, then $v \neq 0$, and
   a) if φ is a reflection across a line l through the origin, and
      i. if $v \in l$, then, by definition, $\sigma = \tau_v \circ \varphi$ is a glide reflection with the axis l;
      ii. if $v \notin l$, then $\sigma = \tau_v \circ \varphi$ is either a glide reflection or a reflection by Proposition 2.19.
   b) if φ is a rotation about the origin, then σ is a rotation by Proposition 2.15.

Therefore, exhausted all the possibilities, we conclude that every isometry of the plane is either an identity, or a translation, or a rotation, or a reflection, or a glide reflection. ∎

Notation 2.27 (shared name $\mathrm{O}(2)$*).* Note that $\mathrm{O}(2)$ is the group of orthogonal matrices under matrix multiplication instead of the group of linear isometries under function composition. Because these two groups are isomorphic through the mapping $A \longmapsto [x \mapsto Ax]$, we choose to let them share the name $\mathrm{O}(2)$.

3 Wallpaper Groups

The goal of this chapter is to recall the definition of wallpaper groups, equivalence relation of such groups, and prove that there are 17 equivalence classes.

3.1 Wallpaper Groups and Related Concepts

We start from the following map, which removes the translation component of a plane isometry. It firstly is proven to be a homomorphism. Then its kernel and image are subgroups of corresponding atmosphere groups. These subgroups are the ingredients for defining wallpaper groups.

Proposition 3.1. *Define* $\pi\colon \mathrm{Isom}(\mathbb{R}^2) \to \mathrm{O}(2)$ *by* $\tau_v \circ \varphi \mapsto \varphi$. *Then* π *is a group homomorphism.*

Proof. Let $\tau_v \circ \varphi, \tau_w \circ \psi \in \mathrm{Isom}(\mathbb{R}^2)$. Then $\pi(\tau_v \circ \varphi \circ \tau_w \circ \psi) = \pi(\tau_v \circ \tau_{\varphi(w)} \circ \varphi \circ \psi) = \pi(\tau_{v+\varphi(w)} \circ \varphi \circ \psi) = \varphi \circ \psi = \pi(\tau_v \circ \varphi) \circ \pi(\tau_w \circ \psi)$. ∎

Proposition 3.2. *Define* $\pi\colon \mathrm{Isom}(\mathbb{R}^2) \to \mathrm{O}(2)$ *by* $\tau_v \circ \varphi \mapsto \varphi$. *Let* G *be a group of plane isometries. Let* T *be the set of all translations in* G. *Let* $\pi|_G\colon G \to \pi(G)$ *be the restriction of* π *to* G. *Then* $T = \ker(\pi|_G)$*, the set* T *is a normal subgroup of* G*, and the set* $\pi(G)$ *is a subgroup of* $\mathrm{O}(2)$. *Moreover,* $G/T \cong \pi(G)$.

Proof. We know that $\pi\colon \mathrm{Isom}(\mathbb{R}^2) \to \mathrm{O}(2)$ is a homomorphism from Proposition 3.1. Since G is a subgroup of $\mathrm{Isom}(\mathbb{R}^2)$, the restriction $\pi|_G\colon G \to \pi(G)$ is also a homomorphism. Hence, $\pi(G)$ is a subgroup of $\mathrm{O}(2)$. By the definition of kernel, we have $\ker(\pi|_G) = \{\tau_v \circ \varphi \in G \mid \varphi = \mathrm{id}\} = \{\tau_v \mid \tau_v \in G\} = T$. By the homomorphism theorem [see e.g. Top and Müller, 2018, thm. VII.2.11], the set T is a normal subgroup of G and $G/T \cong \pi(G)$. ∎

Definition 3.3 (The point group of a plane isometry group). Define $\pi\colon \mathrm{Isom}(\mathbb{R}^2) \to \mathrm{O}(2)$ by $\tau_v \circ \varphi \mapsto \varphi$. Let G be group of plane isometries. We call the subgroup $(\pi(G), \circ, \mathrm{id})$ of $\mathrm{O}(2)$ the *point group* of G. In particular, $\pi(G) = \{\varphi \in \mathrm{O}(2) \mid \exists v \in \mathbb{R}^2 \colon \tau_v \circ \varphi \in G\}$. In other words, $\varphi \in \pi(G)$ if and only if there exists $v \in \mathbb{R}^2$ such that $\tau_v \circ \varphi \in G$.

Remark 3.4. Recall that the determinant mapping $\det\colon (\mathrm{O}(2), \circ, \mathrm{id}) \to (\{\pm 1\}, \cdot, 1)$ is a homomorphism. Since the point group $\pi(G)$ of G is a subgroup of $\mathrm{O}(2)$, the restriction $\det|_{\pi(G)}\colon \pi(G) \to \{\pm 1\}$ is also an homomorphism. In particular, $\ker(\det|_{\pi(G)})$ is the set of all matrices in $\pi(G)$ with determinant 1, in other words, all rotations in $\pi(G)$. Therefore, the rotations in $\pi(G)$ form a (normal) subgroup.

Definition 3.5 (The translation subgroup of a plane isometry group). Let G be a group of plane isometries. Its subgroup $(T, \circ, \mathrm{id})$ of all translations is called the *translation subgroup* of G.

Remark 3.6. Let G be a group of plane isometries. Let T be its translation subgroup. Then the set $L = \{v \in \mathbb{R}^2 \mid \tau_v \in G\}$ forms a group under addition. The groups $(T, \circ, \mathrm{id})$ and $(L, +, 0)$ are isomorphic under the mapping $\iota\colon \tau_v \mapsto v$, since $\iota(\tau_v \circ \tau_w) = \iota(\tau_{v+w}) = v + w = \iota(\tau_v) + \iota(\tau_w)$, and ι has an obvious inverse $\iota^{-1}\colon v \mapsto \tau_v$.

Definition 3.7 (The lattice group of a plane isometry group). Let G be a group of plane isometries. The set $L = \{v \in \mathbb{R}^2 \mid \tau_v \in G\}$ forms a group under addition. We call this group $(L, +, 0)$ the *lattice group* of G.

Recall the following notion of wallpaper groups [see e.g. Schwarzenberger, 1974, def. p.127]. The two axioms in the definition make sure the group is discrete.

Definition 3.8 (Wallpaper group). Define $\pi\colon \operatorname{Isom}(\mathbb{R}^2) \to \mathrm{O}(2)$ by $\tau_v \circ \varphi \mapsto \varphi$. A group $(W, \circ, \mathrm{id})$ of plane isometries is called a *wallpaper group* if

1. its point group $(\pi(W), \circ, \mathrm{id})$ is finite, and
2. its lattice group $(L, +, 0)$ is generated by two $\mathbb{R}$-linearly independent vectors in $\mathbb{R}^2$.

Remark 3.9. Note that $\mathbb{Z}^2 \cong L \cong T$.

The second condition in the Definition of wallpaper group, which is a restriction of lattice group of a plane isometry group, can be interpreted in the following way. There exists $\mathbb{R}$-linearly independent vectors v and w generating the lattice group of W. In other words, $L = \{nv + mw \in \mathbb{R}^2 \mid n, m \in \mathbb{Z}\}$.

Equivalently, there exists vectors $v, w \in \mathbb{R}^2$ independent over $\mathbb{R}$ such that the translation subgroup is generated by two translations which are respectively over two $\mathbb{R}$-linearly independent vectors. In other words, $T = \{\tau_{nv+mw} \in \operatorname{Isom}\mathbb{R}^2 \mid n, m \in \mathbb{Z}\}$.

The following notion of shift vectors is needed in later proofs involving reflections [see also Schwarzenberger, 1974, sec. 1, def. (iv)].

Definition 3.10 (The shift vector of a linear isometry). Let W be a wallpaper group. Let $\varphi \in \pi(W)$ satisfy $\operatorname{ord}(\varphi) = q$. A vector $a \in \mathbb{R}^2$ is a *shift vector of* φ in W if there exists $v \in \mathbb{R}^2$ such that $\tau_v \circ \varphi \in W$ and $a = v + \varphi(v) + \varphi^2(v) + \cdots + \varphi^{q-1}(v)$.

Remark 3.11. Note that $\tau_a = \tau_{v+\varphi(v)+\varphi^2(v)+\cdots+\varphi^{\operatorname{ord}(\varphi)-1}(v)} = (\tau_v \circ \varphi)^{\operatorname{ord}(\varphi)}$ is a translation in W. Hence, $a \in L$. We also have $\varphi(a) = \varphi(v) + \varphi^2(v) + \cdots + \varphi^{q-1}(v) + \mathrm{id}(v) = a$. It follows that if φ is a non-identity rotation then $a = 0$ and therefore $(\tau_v \circ \varphi)^q = \tau_0 = \mathrm{id}$; and if φ is a non-identity reflection then a is in the reflection axis.

Notation 3.12. We denote the set of all the shift vectors of a linear isometry φ as $\operatorname{sftvec}(\varphi)$.

Proposition 3.13. *Let W be a wallpaper group. Let $\mu_l \in \pi(W)$. Then every shift vector of μ_l is in l. Any shift vectors of μ_l differ by a vector of the form $t + \mu_l(t)$ for some $t \in L$.*

Proof. All shift vectors of μ_l are in l. Let $a \in \mathbb{R}^2$ be a shift vector of μ_l. Then there exists $v \in \mathbb{R}^2$ such that $\tau_v \circ \mu_l \in W$ and $a = v + \mu_l(v)$. We know that μ_l fixes only the line l. and $\mu_l(a) = a$. Hence, $a \in l$.

Let $a, a' \in \mathbb{R}^2$ be two distinct shift vectors of μ_l. Then there exists $v, v' \in \mathbb{R}^2$ such that $\tau_v \circ \mu_l, \tau_{v'} \circ \mu_l \in W$, $a = v + \mu_l(v)$ and $a' = v + \mu_l(v')$. Denote $t = v - v'$. Then $a - a' = t + \mu(t)$. But $\tau_t = \tau_v \circ \tau_{-v'} = \tau_v \circ \mu_l \circ \tau_{\mu_l^{-1}(-v)} \circ \mu_l^{-1} = \tau_v \circ \mu_l \circ (\tau_{v'} \circ \mu_l^{-1}) \in W$. Hence, $t \in L$. In other words, a and a' differ by a vector of the form $t + \mu_l(t)$ for some $t \in L$. ∎

3.2 Equivalence of Wallpaper Groups

Recall the following notion of wallpaper group equivalence [see e.g. Schwarzenberger, 1974, sec. 3].

Definition 3.14 (Equivalence of wallpaper groups). Two wallpaper groups W and W' are *equivalent*, denoted as $W \sim W'$, if there is a isomorphism $\eta\colon W \to W'$, such that the restriction $\eta|_T : T \to T'$ is also an isomorphism, where T and T' are corresponding translation subgroups of W and W'.

In other words, all wallpaper groups have isomorphic translation subgroup, and if the wallpaper groups are equivalent then also their point groups are isomorphic. However, it is possible that two wallpaper groups have both isomorphic point groups and isomorphic translation groups but they are not equivalent.

For example, consider two wallpaper groups W and W', where $\pi(W) = \langle \mu_l \rangle$ and $\pi(W') = \langle \rho_\pi \rangle$. The point groups $\pi(W)$ and $\pi(W')$ are isomorphic but these two wallpaper groups should not be considered equivalent, and they are indeed not equivalent under our definition of equivalence.

Proposition 3.15. *Let W and W' be wallpaper groups such that $\pi(W) = \langle \mu_l \rangle$ and $\varphi(W') = \langle \rho_\pi \rangle$. Then W and W' are not equivalent.*

Proof. Since $\mu_l \in \pi(W)$, we know that there exists $u \in \mathbb{R}^2$ such that $\tau_u \circ \mu_l \in W$. If $u \neq 0$ and $u \notin l_\perp$, we let $v = u$. If $u = 0$, then take $v \in L \setminus l_\perp$. If $u \in l_\perp$, then take $v = u + w$ where $w \in L$ such that $v \notin l_\perp$. Then we found $v \in \mathbb{R}^2 \setminus \{0\}$ such that $\tau_v \circ \mu_l \in W$ and $v \notin l_\perp$. Assume W and W' are equivalent. Then we have an isomorphism $\eta\colon W \to W'$ such that η and $\eta|_T$ are isomorphisms. Then we shall have $\tau_v \circ \mu_l \mapsto \tau_{v'} \circ \rho_\pi$ for some $v' \in \mathbb{R}^2$. Because η is an isomorphism, we have $(\tau_v \circ \mu_l)^2 \mapsto (\tau_{v'} \circ \rho_\pi)^2$. However, $(\tau_v \circ \mu_l)^2 = \tau_{v+\mu_l(v)} \neq \mathrm{id}$ and $(\tau_{v'} \circ \rho_\pi)^2 = \mathrm{id}$. We know that isomorphisms map the identity to the identity. Hence, we have a contradiction. ∎

3.3 Possibilities of Point Groups

The finiteness of the point group restrict the possibilities of the point groups of a wallpaper group. In this section we recall this so-called the crystallographic restriction (Theorem 3.21), the proof of which contains a key ingredient – Proposition 3.19 stating that the lattice group of a wallpaper group admits a non-zero vector of minimal length.

Lemma 3.16. *Let G be a non-trivial finite group of plane rotations about the origin. Every non-identity element of G can be written as the form $(\rho_{2\pi/b})^a$ for some positive coprime integers a and b.*

Proof. Let G be a non-trivial finite group of plane rotations. Let ρ_θ be a non-identity element in G. Because of the finiteness of G, there exists a integer $i > 0$ such that $\rho_{i\theta} = (\rho_\theta)^i = \mathrm{id}$. Then $i\theta = j2\pi$ for some integer $j > 0$. We have $\theta = j/i \cdot 2\pi$. Let $a = j/\gcd(i,j)$ and $b = i/\gcd(i,j)$. Then $\rho_\theta = (\rho_{2\pi/b})^a$. ∎

Lemma 3.17. *Let G be a non-trivial finite group of plane rotations about the origin. If there exists an element $(\rho_{2\pi/b})^a \in G$ where a and b are positive coprime integers, then $\rho_{2\pi/b} \in G$.*

Proof. Let $(\rho_{2\pi/b})^a \in G$ where a and b are positive coprime integers. By Bézout's Identity [see Top and Müller, 2018, thm. I.1.12], there exist integers c and d satisfying $ac + bd = 1$. Then we have $cr = ca/b = (1 - bd)/b = 1/b - d$. It follows that $c(2\pi r) = 2\pi/b - 2\pi d$. Therefore, $(\rho_{2\pi r})^c = \rho_{2\pi/b - 2\pi d} = \rho_{2\pi/b}$. By the group structure of G, we have $\rho_{2\pi/b} = (\rho_{2\pi r})^c \in G$. ∎

The following lemma is an analogue of Lemma V.2.6 in Top and Müller [2018].

Proposition 3.18. *Let G be a non-trivial finite group of rotations in the plane about the origin. Then $G = \langle \rho_{2\pi/\mathrm{ord}(G)} \rangle$.*

Proof. Let G be a non-trivial finite group of rotations in the plane about the origin. Because of the finiteness of G, we can find a rotation of minimal angle ρ_α in G. By Lemma 3.16, $\rho_\alpha = (\rho_{2\pi/n})^m$ for some coprime positive integers m and n. By Lemma 3.17, $\rho_{2\pi/n} \in G$. Because of minimality, $m = 1$. Hence, the rotation of minimal angle is of the form $\rho_{2\pi/n}$ for a positive integer n.

Now we prove that $G = \langle \rho_{2\pi/n} \rangle$. By Lemma 3.16, we can write any element of G in the form $(\rho_{2\pi/b})^a$ where a and b are coprime positive integers. Let $(\rho_{2\pi/b})^a$ be such a representation of an arbitrary element of G. By Lemma 3.17, $\rho_{2\pi/b}$ is also an element of G. By the group structure of G, $\rho_{2\pi/b} \circ \rho_{2\pi/n} = \rho_{2\pi/b + 2\pi/n}$ is also an element of G. We know that $2\pi/b + 2\pi/n = 2\pi \cdot p/\mathrm{lcm}(b,n)$ for some positive integer p, where p and $\mathrm{lcm}(b,n)$ are coprime. Hence, $\rho_{2\pi/b+2\pi/n} = (\rho_{2\pi/\mathrm{lcm}(b,n)})^p$. By Lemma 3.17, $\rho_{2\pi/\mathrm{lcm}(b,n)}$ is also an element of G. Because of the minimality of $2\pi/n$ we have $2\pi/\mathrm{lcm}(b,n) \geq 2\pi/n$. In other words, $\mathrm{lcm}(b,n) \leq n$. But we know that $\mathrm{lcm}(b,n) \geq n$. Hence, $\mathrm{lcm}(b,n) = n$. It follows that $b \mid n$. In other words, $n = kb$ for some integer k. Hence, $(\rho_{2\pi/b})^a = (\rho_{2\pi/n})^{ak}$. An arbitrary element of G can be written as a power of $\rho_{2\pi/n}$. Therefore, G is generated by $\rho_{2\pi/n}$, in other words, cyclic.

Moreover. $n = \mathrm{ord}(\rho_{2\pi/n}) = \mathrm{ord}(\langle \rho_{2\pi/n} \rangle) = \mathrm{ord}(G)$. It follows that $G = \langle \rho_{2\pi/\mathrm{ord}(G)} \rangle$. ∎

Proposition 3.19. *Let L be a lattice group of a wallpaper group. Then for every $d > 0$, there are only finitely many vectors in L with their length smaller than d. Moreover, L admits a non-zero vector of minimal length.*

Proof. Let $(L, +, 0)$ be the lattice group of a wallpaper group. Then by Definition there exists $\mathbb{R}$-linearly independent vectors v and w in $\mathbb{R}^2$ such that $L = \mathbb{Z}v + \mathbb{Z}w$. Take a nonzero vector $mv + nw$ from L. Its length is $\|mv + nw\|$.

$$\begin{aligned}\|mv + nw\|^2 =& \langle mv + nw, mv + nw \rangle \\ =& m^2 \langle v, v \rangle + 2mn \langle v, w \rangle + n^2 \langle w, w \rangle \\ =& m^2 \|v\|^2 + 2mn \langle v, w \rangle + n^2 \|w\|^2.\end{aligned}$$

Let $a = \|v^2\| > 0$ and $b = \langle v, w \rangle$ and $c = \|w\|^2 > 0$. Then

$$\begin{aligned}\|mv + nw\|^2 =& am^2 + 2bmn + cn^2 \\ =& a \cdot (m^2 + 2b/a \cdot mn + c/a \cdot n^2) \\ =& a \cdot ((m + b/a \cdot n)^2 + (-b^2/a^2 + c/a) \cdot n^2) \\ =& (1/a) \cdot (am + bn)^2 + (1/a^2)(ac - b^2) \cdot n^2.\end{aligned}$$

By Cauchy Schwartz' inequality, we have $\|v\|\|w\| \geq |\langle v, w \rangle|$. In our case, the equality does not happen since v and w are $\mathbb{R}$-linearly independent. Thus, we have $\|v\|^2 \|w\|^2 > |\langle v, w \rangle|^2$. Equivalently, $ac > b^2$. Denote $x = 1/a > 0$ and $y = (1/a^2)(ac - b^2) > 0$ then

$$\|mv + nw\|^2 = (ma + nb)^2 x + n^2 y.$$

Let $d > 0$. Let $(ma+nb)^2x+n^2y < d$. Then we have $n^2y < d$. Therefore, $|n| < \sqrt{d/y}$. We can only have finitely many options for the integer n. Let n be fixed and satisfy $n^2y < d$. Then $(ma+nb)^2x < d-n^2y$. We have $|ma+nb| < \sqrt{(d-n^2y)/x}$. Denote $e = \sqrt{(d-n^2y)/x}$, By triangular inequality, we have $\big||ma|-|nb|\big| \le |ma+nb| < e$. Hence, $|ma| \in (|nb|-e, |nb|+e)$. It follows that $|m| \in \big((|nb|-e)/a, (|nb|+e)/a\big)$. We have finitely many options of the integer m. Therefore, for any $d > 0$, we have finitely many options of $(m,n) \in \mathbb{Z}$ such that $\|nv+nw\| < d$.

In other words, if $L = \langle v, w\rangle$ is the lattice group of a wallpaper group, then for any $d > 0$ the set $\{(m,n) \in \mathbb{Z} \mid mv+nw \in L,\ 0 < |mv+nw| < d\}$ is finite.

Fix $d > \min\{\|v\|, \|w\|\}$. Then the set $\{(m,n) \in \mathbb{Z} \mid mv+nw \in L,\ 0 < |mv+nw| < d\}$ is finite. It is also non-empty because it contains at least one of v and w. Therefore, we can pick a pair (m,n) of integers such that $\|nv+nw\|$ is nonzero and minimal. In other words, we can pick a non-zero vector of minimal length in L. ∎

Proposition 3.20. *Let W be a wallpaper group with point group $\pi(W)$ and lattice group L. If $t \in L$ and $\varphi \in \pi(W)$, then $\varphi(t) \in L$.*

Proof. Let W be a wallpaper group. Let L be the lattice group of W, and $\pi(W)$ be the point group of W. Let $t \in L$. Then $\tau_t \in W$. Let $\varphi \in \pi(W)$. Then there exists $v \in \mathbb{R}^2$ such that $\tau_v \circ \varphi \in W$. By the group structure of W, we have $\tau_v \circ \varphi \circ \tau_t \in W$. By Proposition 2.14, $\varphi \circ \tau_t = \tau_{\varphi(t)} \circ \varphi$. Hence, $\tau_v \circ \tau_{\varphi(t)} \circ \varphi \in W$. In other words, $\tau_{\varphi(t)} \circ \tau_v \circ \varphi \in W$. It follows that $\tau_{\varphi(t)} = (\tau_{\varphi(t)} \circ \tau_v \circ \varphi) \circ (\tau_v \circ \varphi)^{-1} \in W$. In other words, $\varphi(t) \in L$. ∎

Theorem 3.21 (Crystallographic restriction). *The rotations in the point group of a wallpaper group form a finite cyclic group of order* 1*,* 2*,* 3*,* 4 *or* 6*.*

Proof. Let W be a wallpaper group. Let $\pi(W)$ be the point group of W. Then $\pi(W)$ is finite.

Let H_0 be the set of rotations in the point group H of W. By Remark 3.4, H_0 is a subgroup of $\pi(W)$. H_0 is finite since H is finite. Let $q = \operatorname{ord}(H_0)$.

If $q = 1$, then $H_0 = \{\mathrm{id}\}$. This is possible because the group generated by translations over two $\mathbb{R}$-linearly independent vectors in $\mathbb{R}^2$ can form a wallpaper group.

Suppose $q > 1$. By Proposition 3.18, H_0 is cyclic group of order q generated by $\rho_{2\pi/q}$. We only need to prove q can only be picked from 1, 2, 3, 4 or 6. Let L be the lattice group of W. By Lemma 3.19, L admits a vector of minimal length. Let t be such a vector. By Proposition 3.20, $\rho_{2\pi/q}(t) \in L$. Hence, $\rho_{2\pi/q}(t) - t \in L$. Because of the minimality of t, we must have

$$\|t\| \le \|\rho_{2\pi/q}(t) - t\| = \left|\det\left(\begin{bmatrix} -1+\cos(2\pi/q) & -\sin(2\pi/q) \\ \sin(2\pi/q) & -1+\cos(2\pi/q) \end{bmatrix}\right)\right| \|t\| = |2-2\cos(2\pi/q)|\|t\|.$$

Hence, $1 \le |2-2\cos(2\pi/q)| = 2-2\cos(2\pi/q)$. It follows that $\cos(2\pi/q) \le 1/2$. This means $2\pi/q \in [\pi/3, 5\pi/3]$. Hence, $q \in [6/5, 6] \cap \mathbb{Z}_{>1} = \{2,3,4,5,6\}$.

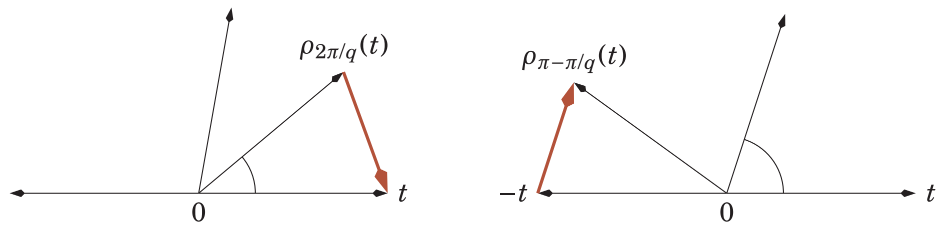
Let q be odd and larger than 1. Then $(q-1)/2$ is an integer, and $(q-1)\pi/q = \pi - \pi/q$ is an integral multiple of $2\pi/q$. Hence, $\rho_{\pi-\pi/q} \in H_0$. By Proposition 3.20, $\rho_{\pi-\pi/q}(t) \in L$. It follows that $\rho_{\pi-\pi/q}(t) + t \in L$. Because of the minimality of t we have

$$\|t\| \le \|t + \rho_{\pi-\pi/q}(t)\| = \left|\det\left(\begin{bmatrix} 1+\cos(\pi-\pi/q) & -\sin(\pi-\pi/q) \\ \sin(\pi-\pi/q) & 1+\cos(\pi-\pi/q) \end{bmatrix}\right)\right| \|t\| = |2+2\cos(\pi-\pi/q)|\|t\|.$$

Hence, $1 \le |2+2\cos(\pi-\pi/q)| = |2-2\cos(\pi/q)| = 2-2\cos(\pi/q)$. It follows that $\cos(\pi/q) \le 1/2$. This means $\pi/q \in [\pi/3, 5\pi/3]$. Hence, $q \in [3/5, 3] \cap \mathbb{Z}_{>1,\mathrm{odd}} = \{3\}$.

Now we have that q can be 1; if $q \ne 1$, then $q \in \{2,3,4,5,6\}$; if q is odd and larger than 1, then $q = 3$. In conclusion, $q \in \{1,2,3,4,6\}$. ∎

Remark 3.22. An illustration of the proof is as follows. If $q \in 2\mathbb{Z}_{>0} \setminus \{1,2,3,4,6\}$, then we have a contradiction as Figure 3.1a. If $q \in (1+2\mathbb{Z}_{>0}) \setminus \{1,2,3,4,6\}$, then we have a contradiction as Figure 3.1b.



(a) $\|t\| > \|\rho_{2\pi/q}(t) - t\|$ (b) $\|t\| > \|t + \rho_{\pi-\pi/q}(t)\|$

Figure 3.1: Two contradictions from $q \in \mathbb{Z} \setminus \{1,2,3,4,6\}$

4 Classification of Wallpaper Groups

The goal of this chapter is to reproduce a proof that there exist precisely 17 equivalence classes of wallpaper groups. We separate cases and prove that the wallpaper groups are equivalent under each case. Then we discuss how the wallpaper groups under different cases are indeed not equivalent. We eventually exhaust the possibilities, and reach our conclusion.

4.1 Point Group Containing No Reflection

In this section we prove that there are precisely five equivalence classes of wallpaper groups with no reflection in their point groups using Definition 3.14. In order to do so, we need to create an isomorphism mapping one group to another. The idea is to create an isomorphism between the translation subgroups first. Then expand this isomorphism to the whole wallpaper group. We know that the translation subgroup of a wallpaper group is a normal subset. It is natural to expect this expansion to map cosets to cosets.

Lemma 4.1 (Writing a wallpaper as the union of translation subgroup cosets). *Let W be a wallpaper group with $\pi(W) = \langle \rho_{2\pi/q} \rangle$. Let T be the translation subgroup of W. Then there exists $v \in \mathbb{R}^2$ such that $\tau_v \circ \rho_{2\pi/q} \in W$. Meanwhile, $W = \bigcup_{i=0}^{q-1} T \circ (\tau_v \circ \rho_{2\pi/q})^i$ and this union is disjoint. Moreover, every element in W can be expressed uniquely in the form $\tau_w \circ (\tau_u \circ \rho)^i$ for some $w \in L$ and $i \in \{0, \dots, q-1\}$.*

Proof. Let W, T and $\pi(W)$ satisfy the premises. Denote $\rho = \rho_{2\pi/q}$.

Since $\rho \in \pi(W)$, by Definition 3.3, there exists $v \in \mathbb{R}^2$ such that $\tau_v \circ \rho \in W$. Then for every $i \in \mathbb{Z}$ we have $(\tau_v \circ \rho)^i \in W$. By Proposition 3.2, T is a normal subgroup of W, meanwhile, $W/T \cong \pi(W)$. Hence, the representatives of the cosets of T can be picked in the fiber on each element of $\pi(W)$. We see that π maps $\{(\tau_v \circ \rho)^i\}_{i=0}^{q-1}$ to all the elements of $\pi(W)$. Therefore, for each $i \in \{0, 1, \dots, q-1\}$, we can take $(\tau_v \circ \rho)^i$ as a representative of $T \circ (\tau_v \circ \rho)^i$. Note that when we pick a different i, the image of $(\tau_v \circ \rho)^i$ in $\pi(W)$ is different. Hence, $(\tau_v \circ \rho)^i|_{i=0}^{q-1}$ are indeed elements in different fibers. Therefore, $\{T \circ (\tau_v \circ \rho)^i\}_{i=0}^{q-1}$ is a partition of W, and we can write W as a disjoint union $W = \bigcup_{i=0}^{q-1} T \circ (\tau_v \circ \rho)^i$.

By Proposition 2.23, we can express any element of W uniquely as $\tau_u \circ \rho^i$ for some $u \in \mathbb{R}^2$ and $i \in \{0, \dots, q-1\}$. Observe that $\tau_u \circ \rho^i = \tau_{u-v-\rho(v)-\cdots-\rho^{i-1}(v)} \circ (\tau_v \circ \rho)^i$. Denote $w = u - v - \rho(v) - \cdots - \rho^{i-1}(v)$. Then $\tau_w \circ (\tau_v \circ \rho)^i = \tau_u \circ \rho^i$ is unique.

We see $\tau_w = \tau_u \circ \rho^i \circ (\tau_v \circ \rho)^{-i} \in W$. Therefore, $\tau_w \in T$. ■

Remark 4.2. If W is a wallpaper group with $\pi(W) = \langle \rho_{2\pi/q} \rangle$, for a choice of $v \in \mathbb{R}^2$ such that $\tau_v \circ \rho_{2\pi/q} \in W$, without mentioning again, we write an element of W in the form $\tau_w \circ (\tau_v \circ \rho)^i$, where $\tau_w \in T$.

Remark 4.3 (It is possible to create a bijection). Let W and W' be wallpaper groups with $\pi(W) = \langle \rho_{2\pi/q} \rangle$. Denote $\rho = \rho_{2\pi/q}$. Let L and L' be the lattice groups of W and W' respectively. Let $\lambda : L \to L'$ be an isomorphism.

By Lemma 4.1, the cosets $\{T \circ \tau_v \circ \rho\}_{i=1}^{q-1}$ form a partition of W, and the cosets$\{T' \circ \tau_{v'} \circ \rho\}_{i=1}^{q-1}$ form a partition of W'. We can, therefore, define $\eta : W \to W'$ by $\tau_w \circ (\tau_v \circ \rho_\pi)^i \mapsto \tau_{\lambda(w)} \circ (\tau_{v'} \circ \rho_\pi)^i$ for $i \in \{1, \dots, q-1\}$. Meanwhile, we can also define its inverse $\eta^{-1} : W' \to W$ by $\tau_{w'} \circ (\tau_{v'} \circ \rho_\pi)^i \mapsto \tau_{\lambda^{-1}(w)} \circ (\tau_v \circ \rho_\pi)^i$ for $i \in \{1, \dots, q-1\}$.

Remark 4.4 (A "natural" isomorphism between translation subgroups). Let W and W' be two wallpaper groups. Let $L = \langle v, w\rangle$ and $L' = \langle v', w'\rangle$ be the lattice groups of W and W' respectively. If we define $\lambda\colon L \to L'$ by $mv + nw \mapsto mv' + nw'$ for all $m, n \in \mathbb{Z}$, then λ is an isomorphism. Moreover, if we define $\gamma\colon T \to T'$ by $\tau_t \mapsto \tau_{\lambda(t)}$, then γ is an isomorphism.

In the case that the point groups are trivial, we can simply map the translation subgroups (the whole groups) to each other with this "natural" isomorphism.

Theorem 4.5 (0 reflection, $q = 1$). *Let W and W' be wallpaper groups with $\pi(W) = \pi(W') = \{\mathrm{id}\}$. Let $L = \langle v, w\rangle$ and $L' = \langle v', w'\rangle$ be the lattice groups of W and W' respectively. Let T and T' be the translation subgroups respectively. Define $\lambda\colon L \to L$ by $mv+nw \mapsto mv'+nw'$ for all $m, n \in \mathbb{Z}$. Then we can define $\eta\colon W \to W'$ by $\tau_w \mapsto \tau_{\lambda(w)}$. Moreover, $\eta\colon W \to W'$ and $\eta|_T\colon T \mapsto T'$ are isomorphisms.*

Proof. Since $\pi(W) = \pi(W') = \{\mathrm{id}\}$, we have $W = T$ and $W' = T'$. By Remark 4.4, $\eta = \eta|_T$ is an isomorphism. ∎

Corollary 4.6 (0 reflection, $q = 1$, equivalence). *Let W and W' be wallpaper groups with $\pi(W) = \pi(W') = \{\mathrm{id}\}$. Then $W \sim W'$.*

Theorem 4.7 (0 reflection, $q \in \{2,3,4,6\}$, isomorphism*). *Let $q \in \{2,3,4,6\}$. Let W and W' be wallpaper groups with point group $\pi(W) = \pi(W') = \langle \rho_{2\pi/q}\rangle$, lattice groups L and L', translation subgroups T and T', respectively. Denote $\rho = \rho_{2\pi/q}$. Let $v, v' \in \mathbb{R}^2$ satisfy $\tau_v \circ \rho \in W$ and $\tau_{v'} \circ \rho \in W'$. Let $\lambda\colon L \to L'$ be an isomorphism. Define $\eta\colon W \to W'$ by $\tau_w \circ (\tau_v \circ \rho)^i \mapsto \tau_{\lambda(w)} \circ (\tau_{v'} \circ \rho)^i$ for all $i \in \{0, \dots, q-1\}$. If $\rho \circ \lambda = \lambda \circ \rho$, then $\eta\colon W \mapsto W'$ and $\eta|_T\colon T \mapsto T'$ are isomorphisms.*

Proof. By Remark 4.3, we know that η has an inverse. Now we only need to prove that η is an homomorphism. Let $v \in \mathbb{R}^2$ such that $\tau_v \circ \rho \in W$. Let $\tau_{w_1} \circ (\tau_v \circ \rho)^i \in W$ and $\tau_{w_2} \circ (\tau_v \circ \rho)^j \in W$. By repetitively applying Proposition 2.14 We have $(\tau_{w_1} \circ (\tau_u \circ \rho)^i) \circ (\tau_{w_2} \circ (\tau_u \circ \rho)^j) = \tau_{w_1 + \rho^i(w_2)} \circ (\tau_u \circ \rho)^{i+j}$. Since when $i + j \geq q$ we can bring the power down using the fact $(\tau_v \circ \rho)^q = \mathrm{id}$ in Remark 3.11, we know that η maps $\tau_{w_1+\rho^i(w_2)} \circ (\tau_u \circ \rho)^{i+j}$ to $\tau_{\lambda(w_1+\rho^i(w_2))} \circ (\tau_{u'} \circ \rho)^{i+j}$.

Suppose $\rho \circ \lambda = \lambda \circ \rho$. By Proposition 4.11, we have $\rho^i \circ \lambda = \lambda \circ \rho^i$ for every $i \in \{0, \dots, q-1\}$. Then $\lambda(w_1 + \rho^i(w_2)) = \lambda(w_1) + (\lambda \circ \rho^i)(w_2) = \lambda(w_1) + (\rho^i \circ \lambda)(w_2) = \lambda(w_1) + \rho^i(\lambda(w_2))$. Therefore, $\tau_{\lambda(w_1+\rho^i(w_2))} \circ (\tau_{u'} \circ \rho)^{i+j} = \tau_{\lambda(w_1)+\rho^i(\lambda(w_2))} \circ (\tau_{u'} \circ \rho)^{i+j}$.

Again by repetitively applying Proposition 2.14, we obtain $\tau_{\lambda(w_1)+\rho^i(\lambda(w_2))} \circ (\tau_{u'} \circ \rho)^{i+j} = (\tau_{\lambda(w_1)} \circ (\tau_{u'} \circ \rho)^i) \circ (\tau_{\lambda(w_2)} \circ (\tau_{u'} \circ \rho)^j)$. We know that $(\tau_{\lambda(w_1)} \circ (\tau_{u'} \circ \rho)^i) \circ (\tau_{\lambda(w_2)} \circ (\tau_{u'} \circ \rho)^j) = \eta((\tau_{w_1} \circ (\tau_u \circ \rho)^i)) \circ \eta((\tau_{w_2} \circ (\tau_u \circ \rho)^j))$. In other words, $\eta((\tau_{w_1} \circ (\tau_u \circ \rho)^i) = \eta((\tau_{w_1} \circ (\tau_u \circ \rho)^i)) \circ \eta((\tau_{w_2} \circ (\tau_u \circ \rho)^j))$. Hence, η is a homomorphism.

Having an inverse, η is an isomorphism.

By construction, the restriction $\eta|_T\colon T \to T'$ is an isomorphism. In fact, $\eta|_T = \eta|_{i=0}$ is simply the map $\tau_t \mapsto \tau_{\lambda(v)}$. ∎

Theorem 4.8 (0 reflection, $q = 2$). *Let W and W' be wallpaper groups with $\pi(W) = \pi(W') = \langle \rho_\pi\rangle$. Let $L = \langle v, w\rangle$ and $L' = \langle v', w'\rangle$ be the lattice groups of W and W' respectively. Let T and T' be the translation subgroups respectively. Define $\lambda\colon L \to L'$ by $mv + nw \mapsto mv' + nw'$ for all $m, n \in \mathbb{Z}$. Let $u, u' \in \mathbb{R}^2$ satisfy $\tau_u \circ \rho_\pi \in W$ and $\tau_{u'} \circ \rho_\pi \in W'$. Then we can define $\eta\colon W \to W'$ by $\tau_w \circ (\tau_u \circ \rho_\pi)^i \mapsto \tau_{\lambda(w)} \circ (\tau_{u'} \circ \rho_\pi)^i$ for $i = 1, 2$. Moreover, $\eta\colon W \to W'$ and $\eta|_T\colon T \mapsto T'$ are isomorphisms.*

Proof. By Remark 4.4, λ is an isomorphism. For all $m,n \in \mathbb{Z}$, we have $\rho_\pi(\lambda(mv+nw)) = \rho_\pi(mv'+nw') = -mv'-nw' = \lambda(-mv-nw) = \lambda(\rho_\pi(mv+nw))$. Therefore, $\rho_\pi \circ \lambda = \lambda \circ \rho_\pi$. By Theorem 4.7, η and $\eta|_T$ are isomorphisms. ∎

Corollary 4.9 (0 reflection, $q=2$, equivalence). *If W and W' be wallpaper groups with $\pi(W) = \pi(W') = \langle \rho_\pi \rangle$, then $W \sim W'$.*

Lemma 4.10. *Let W be a wallpaper group with $\pi(W) = \langle \rho_{2\pi/q} \rangle$, where $q \in \{3,4,6\}$. Denote $\rho = \rho_{2\pi/q}$. Let L be the lattice groups of W. Pick $t \in L$ of minimal length. Then t and $\rho(t)$ form a basis of L. In other words, $L = \mathbb{Z}t + \mathbb{Z}\rho(t)$.*

Proof. Fix $q \in \{3,4,6\}$. Denote $\rho = \rho_{2\pi/q}$. Let W be a wallpaper group with $\pi(W) = \langle \rho \rangle$. Let L be the lattice groups of W. Let $t \in L$ be of minimal length. By Proposition 3.20, $\rho(t) \in L$. Therefore, $\{mt + n\rho(t) \mid m,n \in \mathbb{Z}\} \subseteq L$.

Assume for contradiction that $L \neq \mathbb{Z}t + \mathbb{Z}\rho(t)$. Then we can pick $v \in L \setminus (\mathbb{Z}t + \mathbb{Z}\rho(t))$. Such a vector must be of the form $v = at + b\rho(t)$ where $a,b \in \mathbb{R}$ are not both integers.

Assume $b \in \mathbb{Z}$. Then $(a-[a])t = v - [a]t - b\rho(t) \in L$, but $(a-[a])t$ is shorter than t, which contradicts the minimality of t. Assume $a \in \mathbb{Z}$. Then $(b-[b])\rho(t) = v - at - [b]\rho(t) \in L$, but $(b-[b])\rho(t)$ is shorter than t, which contradicts the minimality of t.

Assume $a,b \in \mathbb{R}\setminus\mathbb{Z}$. Then $|a-[a]| \le 1/2$ or $|a-[a]-1| \le 1/2$, and $|b-[b]| \le 1/2$ or $|b-[b]-1| \le 1/2$. Pick $c = \min\{|a-[a]|, |a-[a]-1|\} \le 1/2$, and $d = \min\{|b-[b]|, |b-[b]-1|\} \le 1/2$. Then $ct + d\rho(t) \in L$. However, $\|ct + d\rho(t)\| \le c\|t\| + d\|\rho(t)\| = (c+d)\|t\|$. We know that the equality does not happen because $0 < 2\pi/q < \pi$. Hence, we have $\|ct + d\rho(t)\| < (c+d)\|t\| \le \|t\|$. We have found a vector $ct + d\rho(t)$ shorter than t, which contradicts the minimality of t.

Therefore, $L = \mathbb{Z}t + \mathbb{Z}\rho(t)$. ∎

Proposition 4.11. *Let $t, t' \in \mathbb{R}^2 \setminus \{0\}$. Let $q \in \mathbb{Z}_{>2}$. Denote $\rho = \rho_{2\pi/q}$. Define $\lambda \colon \mathbb{R}^2 \to \mathbb{R}^2$ by $at + b\rho(t) \mapsto at' + b\rho(t')$ for all $a,b \in \mathbb{R}$. Then for every $i \in \{1,2,\ldots,q-1\}$ we have $\rho^i \circ \lambda = \lambda \circ \rho^i$.*

Proof. Suppose $\rho(t) = At$ where $A = \begin{bmatrix} \cos(2\pi/q) & -\sin(2\pi/q) \\ \sin(2\pi/q) & \cos(2\pi/q) \end{bmatrix}$. Then we can find its eigenvalues $z = e^{j2\pi/q}$ and $\bar{z} = e^{-j2\pi/q}$, where $j = \sqrt{-1}$. Hence, $A = X^{-1}DX$ for a non-singular matrix X and $D = \begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix}$.

Pick any $i \in \{0,\ldots,q-1\}$. Let $\rho^i t = at + b\rho(t)$ for some $a,b \in \mathbb{R}$. Then $A^i t = aIt + bAt$. Then $[A^i - bA - aI]t = 0$. Since $t \neq 0$, the matrix $A^i - bA - aI$ is singular. In other words, $\det(A^i - bA - aI) = 0$. But $X^{-1}(A^i - bA - aI)X = D^i - bA - aI = \begin{bmatrix} z^i - bz - a & 0 \\ 0 & \bar{z}^i - b\bar{z} - a \end{bmatrix}$. Hence, we have $\det(D) = (z^i - bz - a)(\bar{z}^i - b\bar{z} - a) = |z^i - bz - a|^2 = |\bar{z}^i - b\bar{z} - a|^2 = 0$. It follows that $z^i - bz - a = 0$ and $\bar{z}^i - b\bar{z} - a = 0$. Therefore, $A^i - bA - aI = X(D^i - bA - aI)X^{-1} = 0$. In other words, $\rho^i = a\,\mathrm{id} + b\rho$.

It follows that $\rho^i t' = at' + b\rho(t')$. We know $\lambda(\rho^i t) = \lambda(at + b\rho(t)) = at' + b\rho(t')$ by the premises. Hence, $\lambda(\rho^i(t)) = \rho^i t'$. Moreover, $\rho(\lambda(at + b\rho(t))) = \rho(at' + b\rho(t')) = a\rho(t') + b\rho^2(t') = a\lambda(\rho(t)) + b\lambda(\rho^2(t)) = \lambda(\rho(at + b\rho(t)))$ for all $a,b \in \mathbb{R}$. Hence, $\rho^i \circ \lambda = \lambda \circ \rho^i$. ∎

Theorem 4.12 (0 reflection, $q \in \{3,4,6\}$). *Let W and W' be wallpaper groups with $\pi(W) = \pi(W') = \langle \rho_{2\pi/q} \rangle$, lattice groups L and L', translation sugroups T and T', respectively, where $q \in \{3,4,6\}$. Denote $\rho = \rho_{2\pi/q}$. Let $t \in T$ and $t' \in T'$ be of minimal length. Then we can define $\lambda \colon L \to L'$ by $mt + n\rho(t) \mapsto mt' + n\rho(t')$ for all $m,n \in \mathbb{Z}$. Let $v, v' \in \mathbb{R}^2$ satisfy $\tau_v \circ \rho_\pi \in W$ and $\tau_{v'} \circ \rho_\pi \in W'$. Define $\eta \colon W \to W'$ by $\tau_w \circ (\tau_v \circ \rho)^i \mapsto \tau_{\lambda(w)} \circ (\tau_{v'} \circ \rho)^i$ for $i \in \{0,\ldots,q-1\}$. Then $\eta \colon W \to W'$ and $\eta|_T \colon T \mapsto T'$ are isomorphisms.*

Proof. By Lemma 4.10, we can define such a map λ. By Remark 4.4, λ is an isomorphism. By Proposition 4.11, $\rho^i \circ \lambda = \lambda \circ \rho^i$ for every $i \in \{0,\ldots,q-1\}$. By Theorem 4.7, η and $\eta|_T$ are isomorphisms. ∎

Corollary 4.13 (0 reflection, $q \in \{3,4,6\}$, equivalence). *If W and W' be wallpaper groups with $\pi(W) = \pi(W') = \langle \rho_{2\pi/q} \rangle$, where $q \in \{3,4,6\}$, then $W \sim W'$.*

Remark 4.14. Let W and W' be wallpaper groups with $\pi(W) = \langle \rho_{2\pi/q} \rangle$ and $\pi(W') = \langle \rho_{2\pi/q'} \rangle$, where $q, q' \in \{3,4,6\}$. If $q \neq q'$ then $W \nsim W'$. Because equivalent wallpaper groups admit isomorphic point groups, and these two point groups are not isomorphic when $q \neq q'$.

Theorem 4.15. *We have five equivalence classes of wallpaper groups with their point group containing only rotations.*

Proof. This follows directly from Corollary 4.6, Corollary 4.9, Corollary 4.13, and Remark 4.14. ∎

4.2 Point Group Containing Only One Reflection

In this section we follow a similar recipe.

Lemma 4.16 (Writing a wallpaper group as a union of cosets). *Let W be a wallpaper group with point group $\pi(W) = \langle \mu_l \rangle$ and translation subgroup T. Pick $v \in \mathbb{R}^2$ such that $\tau_v \circ \rho_{\mu_l} \in W$. Then $W = T \cup T \circ (\tau_v \circ \mu_l)$ and this union is disjoint. Moreover, every element in W can be expressed uniquely in the form $\tau_w \circ (\tau_u \circ \mu_l)^i$ for some $w \in L$ and $i \in \{0,1\}$.*

Proof. This is an analogue of Lemma 4.1. Denote $\mu = \mu_l$ By Proposition 3.2, T is a normal subgroup of W, meanwhile, $W/T \cong \pi(W)$. Hence, the representatives of the cosets of T can be picked in the fiber on each element of $\pi(W)$. We see that $\pi(\mathrm{id}) \cup \pi(\tau_v \circ \mu) = \{\mathrm{id}, \mu\} = \pi(W)$. Therefore, id and $(\tau_v \circ \rho)$ are representatives of the cosets T and $T \circ (\tau_v \circ \mu_l)$ respectively. Therefore, $\{T, T \circ (\tau_v \circ \mu_l)\}$ is a partition of W, and we can write W as a disjoint union $W = T \cup T \circ (\tau_v \circ \mu)$.

By Proposition 2.23, we can express any element of W uniquely as $\tau_u \circ \mu_l^i$ for some $u \in \mathbb{R}^2$ and $i \in \{0,1\}$. Observe that if $i = 0$, then $\tau_u \circ \mu^i = \tau_u \in T$; if $i = 1$ then $\tau_u \circ \mu = \tau_{u-v} \circ \tau_v \circ \mu$. Denote $w = u - v$. Then $\tau_w = \tau_u \circ \mu \circ (\tau_v \circ \mu)^{-1} \in W$. Hence, $\tau_w \in T$. ∎

Remark 4.17 (It is possible to create a bijection). Let W and W' be wallpaper groups with $\pi(W) = \langle \mu_l \rangle$ and $\pi(W') = \langle \mu_{l'} \rangle$. Let L and L' be the lattice groups of W and W' respectively. Let $\lambda \colon L \to L'$ be an isomorphism. Pick $v, v' \in \mathbb{R}^2$ such that $\tau_v \circ \mu_l \in W$ and $\tau_{v'} \circ \mu_{l'} \in W$. By Lemma 4.16, the cosets T and $T \circ (\tau_v \circ \mu_l)$ form a partition of W, and the cosets T' and $T' \circ (\tau_{v'} \circ \mu_{l'})$ form a partition of W'. We can, therefore, define $\eta \colon W \to W'$ by $\tau_w \circ (\tau_u \circ \mu_l)^i \mapsto \tau_{\lambda(w)} \circ (\tau_{u'} \circ \mu_{l'})^i$ for $i \in \{0,1\}$. Meanwhile, we can also define its inverse $\eta^{-1} \colon W' \to W$ by $\tau_{w'} \circ (\tau_{u'} \circ \mu_{l'})^i \mapsto \tau_{\lambda^{-1}(w')} \circ (\tau_u \circ \mu_l)^i$ for $i \in \{0,1\}$.

Lemma 4.18. *Let W be a wallpaper group with its point group $\pi(W)$ containing a reflection μ_l. Let L be the lattice group of W. Then there exists $r \in L \cap l$ and $s \in L \cap l_\perp$ of minimal length.*

Proof. Since $\mu_l \in \pi(W)$, there exists $u \in \mathbb{R}^2$ such that $\tau_u \circ \mu_l \in W$. We can always pick a $w \in L$ to make $v = u + w$ non-zero and not perpendicular to l and not parallel to l. Then $\tau_v \circ \mu_l \in W$. Then $\tau_{v+\mu_l(v)} = (\tau_v \circ \mu_l)^2 \in W$.

Denote $a = v + \mu_l(v)$. Note that $\mu_l(a) = a$. Hence, $a \in L \cap l$. Because $v \neq 0$ and $v \notin l_\perp$, we have $a \neq 0$. Therefore, $L \cap l \setminus \{0\} \neq \emptyset$. Let $d > \|a\|$. Then $a \in \{v \in L \cap l \mid \|v\| < d\}$. Then $\{v \in L \cap l \mid \|v\| < d\} \neq \emptyset$. By Lemma 3.19, $\{v \in L \mid 0 < \|v\| < d\}$ is finite. Hence, its subset $\{v \in L \cap l \mid 0 < \|v\| < d\}$ is also finite. Therefore, there exists $r \in L \cap l$ of minimal length.

Denote $b = v - \mu_l(v)$. Then $\mu_l(b) = \mu_l(v) - \mathrm{id}(v) = -b$. Hence, $b \in L \cap l_\perp$. Because $v \neq 0$ and $v \notin l$, we have $b \neq 0$. Therefore, $L \cap l_\perp \setminus \{0\} \neq \emptyset$. Let $e > \|b\|$. Then $b \in \{v \in L \cap l_\perp \mid 0 < \|v\| < e\}$. Then $\{v \in L \cap l_\perp \mid 0 < \|v\| < e\} \neq \emptyset$. By Lemma 3.19, the set $\{v \in L \mid 0 < \|v\| < d\}$ is finite. Hence, its subset $\{v \in L \cap l_\perp \mid 0 < \|v\| < d\}$ is also finite. Therefore, there exists $s \in L \cap l_\perp$ of minimal length. ∎

Lemma 4.19. *Let W be a wallpaper group with $\pi(W) = \langle \mu_l \rangle = \{\mathrm{id}, \mu_l\}$, where l is a line through the origin. Let $r \in L \cap l$ and $s \in L \cap l_\perp$ be of minimal length. Suppose $t \in L$. Then either $t \in L \setminus (\mathbb{Z}r + \mathbb{Z}s)$ or $t \in (r+s)/2 + \mathbb{Z}r + \mathbb{Z}s$. In other words, $L \subseteq (\mathbb{Z}r + \mathbb{Z}s) \cup ((r+s)/2 + \mathbb{Z}r + \mathbb{Z}s)$.*

Proof. If $t \in L \setminus (\mathbb{Z}r + \mathbb{Z}s)$ then the proof is finished.

Assume $t = ar + bs \in L \setminus (\mathbb{Z}r + \mathbb{Z}s)$, where $a, b \in \mathbb{R}$. We know that $\mu_l(t) = ar - bs$. Then $2ar = t + \mu_l(t) \in L \cap l$ and $2bs = t - \mu_l(t) \in L \cap l_\perp$.

The numbers $2a$ and $2b$ are both integers. Otherwise, $(2a - [2a])r \in L \cap l$ is non-zero and shorter than r, or $(2b - [2b])s \in L \cap l_\perp$ is non-zero and shorter than s, which contradict the minimality of r or s.

Also neither a nor b can be an integer. If $a, b \in \mathbb{Z}$, then $t \in \langle r, s \rangle$, which contradicts the premise. If $a \in \mathbb{Z}$ and $b \notin \mathbb{Z}$, then $(b - [b])s \in L \cap l_\perp$ is nonzero and shorter than s, which contradicts the minimality of s. If $a \notin \mathbb{Z}$ and $b \in \mathbb{Z}$, then $(a - [a])r \in L \cap l$ is nonzero and shorter than s, which contradicts the minimality of r.

Therefore, we have $2a, 2b \in \mathbb{Z}$ but $a, b \notin \mathbb{Z}$. In other words, it holds that $a, b \in 1/2 + \mathbb{Z}$. Therefore, $t \in (r+s)/2 + \mathbb{Z}r + \mathbb{Z}s$. ∎

Lemma 4.20. *Let W be a wallpaper group with $\pi(W) = \langle \mu_l \rangle$, where l is a line through the origin. Let $r \in L \cap l$ and $s \in L \cap l_\perp$ be of minimal length. Then either $L = \mathbb{Z}r + \mathbb{Z}s$ or $L = \mathbb{Z}(r+s)/2 + \mathbb{Z}r$.*

Proof. If $L = \mathbb{Z}r + \mathbb{Z}s$, the proof is finished. Assume $L \neq \mathbb{Z}r + \mathbb{Z}s$. Then there exists $t \in L \setminus (\mathbb{Z}r + \mathbb{Z}s)$. By Lemma 4.19, we have $t \in (r+s)/2 + \mathbb{Z}r + \mathbb{Z}s$. Let $t = (r+s)/2 = mr + ns$ for some $m, n \in \mathbb{Z}$. Because of the group structure of L, we have $t + \mathbb{Z}r + \mathbb{Z}s \subseteq L$. We know $t + \mathbb{Z}r + \mathbb{Z}s = (r+s)/2 + \mathbb{Z}r + \mathbb{Z}s$. Hence, $(r+s)/2 + \mathbb{Z}r + \mathbb{Z}s \subseteq L$. In other words, $(\mathbb{Z}r + \mathbb{Z}s) \cup ((r+s)/2 + \mathbb{Z}r + \mathbb{Z}s) \subseteq L$. By Lemma 4.19, $L \subseteq (\mathbb{Z}r + \mathbb{Z}s) \cup ((r+s)/2 + \mathbb{Z}r + \mathbb{Z}s)$. This means $L = (\mathbb{Z}r + \mathbb{Z}s) \cup ((r+s)/2 + \mathbb{Z}r + \mathbb{Z}s)$. We see that $(\mathbb{Z}r + \mathbb{Z}s) \cup ((r+s)/2 + \mathbb{Z}r + \mathbb{Z}s) = \mathbb{Z}(r+s)/2 + \mathbb{Z}r$. ∎

Proposition 4.21. *Let W be a wallpaper group with point group $\pi(W)$. Suppose $\mu_l \in \pi(W)$. Let $r \in L \cap l$ be of minimal length. Then every shift vector of μ_l is in the set $\mathbb{Z}r$.*

Proof. By Proposition 3.13, every shift vector of μ_l is in l. Let $a = kr$ be a shift vector of μ_l. Then k must be an integer. Otherwise, $(k - [k])r \in L \cap l$ is nonzero and shorter than r, contradicts the minimality of r. Therefore, all shift vectors of μ_l are in $\mathbb{Z}r$. ∎

Proposition 4.22. *Let W be a wallpaper group with $\pi(W)$. Suppose $\mu_l \in \pi(W)$. Let $r \in L \cap l$ be of minimal length. Suppose there exists $t \in L$ such that $t + \mu_l(t) = r$. Then $\mathbb{Z}r$ is the set of all the shift vectors of μ_l.*

Proof. By Proposition 4.21, all shift vectors of μ_l are contained in $\mathbb{Z}r$. Since $\mu_l \in \pi(W)$, there exists $v \in \mathbb{R}^2$ such that $\tau_v \circ \mu_l \in W$. Then $v + \mu_l(v)$ is a shift vector of μ_l. Then $v + \mu_l(v) = (m+1)r$ for some $m \in \mathbb{Z}$. Then $r = v + \mu_l(v) - mr = v - mt + \mu_l(v - mt)$. We know $\tau_{v-mt} \circ \mu_l \in W$. Therefore, r is a shift vector of μ_l. Then we can write $r = w + \mu_l(w)$ where $w \in \mathbb{R}^2$ satisfies $\tau_w \circ \mu_l \in W$.

Let $n \in \mathbb{Z}$. Since $r = t + \mu_l(t)$, we have $nr = r + (n-1)r = w + \mu_l(w) + (n-1)(t + \mu_l(t)) = w + (n-1)t + \mu_l(w + (n-1)t)$. We know $\tau_{w+(n-1)t} \circ \mu_l \in W$, since $t \in L$. Hence, nr is a shift vector of μ_l. Thus, every element of $\mathbb{Z}r$ is a shift vector of μ_l. Therefore, $\mathbb{Z}r$ is the set of all the shift vectors of μ_l. ∎

Proposition 4.23. *Let W be a wallpaper group with point group $\pi(W)$ and lattice group L. Suppose $\mu_l \in \pi(W)$. Let $r \in L \cap l$ be of minimal length. Suppose that there exists no $t \in L$ such that $t + \mu_l(t) = r$. Then either $2\mathbb{Z}r$ or $r + 2\mathbb{Z}r$ is the set of all the shift vectors of μ_l.*

Proof. By Proposition 4.21, every shift vector of μ_l is in $\mathbb{Z}r$. Let $cr \in L$ be a shift vector of μ_l, where $c \in \mathbb{Z}$. Then there exists $v \in \mathbb{R}^2$ such that $\tau_v \circ \mu_l \in W$ and $cr = v + \mu_l(v)$. Let $n \in \mathbb{Z}$. Then $cr + 2nr = v + nr + \mu_l(v + nr)$. We know that $\tau_{v+nr} \circ \mu_l = \tau_{nr} \circ \tau_v \circ \mu_l \in W$. Hence, $cr + 2nr$ is a shift vector of μ_l. Therefore, every element of $cr + 2\mathbb{Z}r$ is a shift vector of μ_l. By Proposition 3.13, there are no other shift vectors in L other than the ones in $cr + 2\mathbb{Z}r$. Therefore, every shift vector of μ_l is in $cr + 2\mathbb{Z}r$. Since $c \in \mathbb{Z}$, the set $cr + 2\mathbb{Z}r$ is either $2\mathbb{Z}r$ or $r + 2\mathbb{Z}r$. ∎

Lemma 4.24. *Let W be a wallpaper group with $\pi(W) = \langle \mu_l \rangle$, where l is a line through the origin. Let $r \in L \cap l$ and $s \in L \cap l_\perp$ be of minimal length. If $L = \mathbb{Z}r + \mathbb{Z}s$, then either $2\mathbb{Z}r$ or $r + 2\mathbb{Z}r$ is the set of all the shift vectors of μ_l.*

Proof. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$. By Proposition 4.23, either $2\mathbb{Z}r$ or $r + 2\mathbb{Z}r$ is the set of all the shift vectors of μ_l. ∎

Lemma 4.25. *Let W be a wallpaper group with $\pi(W) = \langle \mu_l \rangle = \{\mathrm{id}, \mu_l\}$, where l is a line through the origin. Let $r \in L \cap l$ and $s \in L \cap l_\perp$ be of minimal length. If $L = \mathbb{Z}(r+s)/2 + \mathbb{Z}r$, then $\mathbb{Z}r$ is the set of all the shift vectors of μ_l.*

Proof. Pick $w = (r+s)/2 \in L$. Then $w + \mu_l(w) = r$. By Proposition 4.22, $\mathbb{Z}r$ is the set of all the shift vectors of μ_l. ∎

Lemma 4.26 (1 reflection, cases of $\mathrm{sftvec}(\mu_l)$**).** *Let W be a wallpaper group with $\pi(W) = \langle \mu_l \rangle = \{\mathrm{id}, \mu_l\}$, where l is a line through the origin. Let $r \in L \cap l$ and $s \in L \cap l_\perp$ be of minimal length. Let $L = \mathbb{Z}r + \mathbb{Z}s$. Then one of the sets $\mathbb{Z}r$, $2\mathbb{Z}r$ and $r + 2\mathbb{Z}r$ is the set of all the shift vectors of μ_l.*

Proof. By Lemma 4.20, either $L = \mathbb{Z}r + \mathbb{Z}s$ or $L = \mathbb{Z}(r+s)/2 + \mathbb{Z}r$. Assume $L = \mathbb{Z}(r+s)/2 + \mathbb{Z}r$. By Lemma 4.25, $\mathbb{Z}r$ is the set of shift vectors of μ_l. Assume $L = \mathbb{Z}r + \mathbb{Z}s$. By Lemma 4.24, either $2\mathbb{Z}r$ or $r + 2\mathbb{Z}r$ is the set of shift vectors of μ_l. ∎

Proposition 4.27. *Let W and W' be a wallpaper group with point groups $\pi(W) = \langle \mu_l \rangle$ and $\pi(W) = \langle \mu_{l'} \rangle$ respectively, and lattice groups L and L' respectively. Let $r \in L \cap l$, $s \in L \cap l_\perp$, $r' \in L \cap l'$, and $s' \in L \cap l'_\perp$ be of minimal length. Define $\lambda : L \to L'$ such that $\lambda(r) = r'$ and $\lambda(s) = s'$. Then $\lambda \circ \mu_l = \mu_{l'} \circ \lambda$.*

Proof. Take any $u \in L$. We can write $u = ar + bs$ for $a, b \in \mathbb{R}$. Then $\lambda(\mu_l(u)) = \lambda(ar - bs) = ar' - bs'$, and $\mu_{l'}(\lambda(u)) = \mu_{l'}(ar' + bs') = ar' - bs'$. Hence, $\lambda(\mu_l(u)) = \mu_{l'}(\lambda(u))$. Therefore, $\lambda \circ \mu_l = \mu_{l'} \circ \lambda$. ∎

Theorem 4.28 (1 reflection, $q = 1$, isomorphism*). *Let W and W' be two wallpaper groups, with point groups $\pi(W) = \langle \mu_l \rangle$ and $\pi(W') = \langle \mu_{l'} \rangle$, lattice groups L and L', translation subgroups T and T' respectively. Let $r \in L \cap l$, $s \in L \cap l_\perp$, $r' \in L \cap l'$ and $s' \in L \cap l'_\perp$ be of minimal length. Suppose $\lambda : L \to L'$ is an isomorphism such that $\lambda(r) = r'$ and $\lambda(s) = s'$. Let $v, v' \in \mathbb{R}^2$ satisfy $\tau_v \circ \mu_l \in W$ and $\tau_{v'} \circ \mu_{l'} \in W'$ and $\lambda(v + \mu_l(v)) = v' + \mu_l(v')$. Define $\eta : W \to W'$ by $\tau_w \circ (\tau_v \circ \mu_l)^i \mapsto \tau_{\lambda(w)} \circ (\tau_{v'} \circ \mu_{l'})^i$ for $i \in \{0, 1\}$. Then $\eta : W \to W'$ and $\eta|_T : T \to T'$ are isomorphisms.*

Proof. Let $u \in \mathbb{R}^2$. Write $u = ar + bs$ for $a, b \in \mathbb{R}$. Then $\lambda(\mu_l(u)) = \lambda(ar - bs) = ar' - bs'$, and $\mu_{l'}(\lambda(u)) = \mu_{l'}(ar' + bs') = ar' - bs'$. Therefore, $\lambda \circ \mu_l = \mu_{l'} \circ \lambda$.

Let $\tau_{w_1} \circ (\tau_v \circ \mu_l)^i$ and $\tau_{w_2} \circ (\tau_v \circ \mu_l)^j$ be elements of W. By repetitively applying Proposition 2.14, $(\tau_{w_1} \circ (\tau_v \circ \mu_l)^i) \circ (\tau_{w_2} \circ (\tau_v \circ \mu_l)^j) = \tau_{w_1 + \mu_l^i(w_2)} \circ (\tau_v \circ \mu_l)^{i+j}$.

Since $\lambda(v + \mu_l(v)) = v' + \mu_{l'}(v')$, we have that η maps $(\tau_v \circ \mu_l)^2 = \tau_{v + \mu_l(v)}$ to $(\tau_{v'} \circ \mu_{l'})^2 = \tau_{v' + \mu_{l'}(v')}$. Therefore, for all $i, j \in \{0, 1\}$, including the cases that $i + j > 1$, η maps $\tau_{w_1 + \mu_l^i(w_2)} \circ (\tau_v \circ \mu_l)^{i+j}$ to $\tau_{\lambda(w_1 + \mu_l^i(w_2))} \circ (\tau_{v'} \circ \mu_{l'})^{i+j}$.

By linearity of λ, we have $\tau_{\lambda(w_1 + \mu_l^i(w_2))} \circ (\tau_{v'} \circ \mu_{l'})^{i+j} = \tau_{\lambda(w_1) + \lambda(\mu_l^i(w_2))} \circ (\tau_{v'} \circ \mu_{l'})^{i+j}$. Since $\lambda \circ \mu_l = \mu_{l'} \circ \lambda$, we have $\tau_{\lambda(w_1) + \lambda(\mu_l^i(w_2))} \circ (\tau_{v'} \circ \mu_{l'})^{i+j} = \tau_{\lambda(w_1) + \mu_{l'}^i(\lambda(w_2))} \circ (\tau_{v'} \circ \mu_{l'})^{i+j}$. By repetitively applying Proposition 2.14, $\tau_{\lambda(w_1) + \mu_{l'}^i(\lambda(w_2))} \circ (\tau_{v'} \circ \mu_{l'})^{i+j} = \eta(\tau_{w_1} \circ (\tau_v \circ \mu_l)^i)) \circ \eta(\tau_{w_2} \circ (\tau_v \circ \mu_l)^j))$.

Therefore, η is an homomorphism. By Remark 4.17, the inverse of η exists. Therefore, η is an isomorphism. We know that $\eta|_T$ is an isomorphism by construction. ∎

Remark 4.29 (why different ones are different). Suppose we have an isomorphism between the groups, which is also an isomorphism between the translation subgroups. Then there must be some consequences of this map. If we violate them then they are not equivalent.

Remark 4.30. Let W be a wallpaper group with $\pi(W) = \langle \mu_l \rangle = \{\mathrm{id}, \mu_l\}$, where l is a line through the origin. Let $r \in L \cap l$ and $s \in L \cap l_\perp$ be of minimal length.

Let $L = \mathbb{Z}(r + s)/2 + \mathbb{Z}r$. Let $\mathbb{Z}r$ be the set of all the shift vectors of μ_l. Then 0 is a shift vector of μ_l. Then there exists $v \in \mathbb{R}^2$ such that Then $0 = v + \mu_l(v)$.

Let $L = \mathbb{Z}r + \mathbb{Z}s$. Let $2\mathbb{Z}r$ be the set of all the shift vectors of μ_l. Then 0 is a shift vector of μ_l. Then there exists $v \in \mathbb{R}^2$ such that $0 = v + \mu_l(v)$.

Let $L = \mathbb{Z}r + \mathbb{Z}s$. Let $r + 2\mathbb{Z}r$ be the set of all the shift vectors of μ_l. Then r is a shift vector of μ_l Then there exists $v \in \mathbb{R}^2$ such that $r = v + \mu_l(v)$.

Theorem 4.31 (1 reflection, $q = 1$, $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$). *Let W and W' be two wallpaper groups, with point groups $\pi(W) = \langle \mu_l \rangle$ and $\pi(W') = \langle \mu_{l'} \rangle$, lattice group L and L', translation subgroups T and T' respectively. Let $r \in L \cap l$, $s \in L \cap l_\perp$, $r' \in L \cap l'$ and $s' \in L \cap l'_\perp$ be of minimal length. Suppose $L = \mathbb{Z}(r + s)/2 + \mathbb{Z}r$ and $L' = \mathbb{Z}(r' + s')/2 + \mathbb{Z}r'$. Suppose $\mathbb{Z}r$ and $\mathbb{Z}r'$ are the sets of all the shift vectors of μ_l and $\mu_{l'}$ respectively. Let $v, v' \in \mathbb{R}^2$ satisfy $0 = v + \mu_l(v)$ and $0 = v' + \mu_{l'}(v')$. Define $\lambda : L \to L'$ by $m(r + s)/2 + nr \mapsto m(r' + s')/2 + nr'$ for all $m, n \in \mathbb{Z}$. Define $\eta : W \to W'$ by $\tau_w \circ (\tau_v \circ \mu_l)^i \mapsto \tau_{\lambda(w)} \circ (\tau_{v'} \circ \mu_{l'})^i$ for $i \in \{0, 1\}$. Then η and $\eta|_T$ are isomorphisms.*

Proof. From the premises, $0 = v + \mu_l(v)$ and $0 = v' + \mu_{l'}(v')$. Therefore, $\lambda(v + \mu_l(v)) = 0 = v' + \mu_l(v')$. By Remark 4.4, λ is an isomorphism. By Theorem 4.28, η and $\eta|_T$ are isomorphisms. ∎

Corollary 4.32 (1 reflection, $q = 1$, $\text{sftvec}(\mu_l) = \mathbb{Z}r$, equivalence). *Let W and W' be two wallpaper groups, with point groups $\pi(W) = \langle \mu_l \rangle$ and $\pi(W') = \langle \mu_{l'} \rangle$, lattice group L and L' respectively. Let $r \in L \cap l$, $s \in L \cap l_\perp$, $r' \in L \cap l'$ and $s' \in L \cap l'_\perp$ be of minimal length. Suppose $L = \mathbb{Z}(r+s)/2 + \mathbb{Z}r$ and $L' = \mathbb{Z}(r'+s')/2 + \mathbb{Z}r'$. Suppose $\mathbb{Z}r$ and $\mathbb{Z}r'$ be the sets of all the shift vectors of μ_l and $\mu_{l'}$ respectively. Then $W \sim W'$.*

Theorem 4.33 (1 reflection, $q = 1$, $\text{sftvec}(\mu_l) = 2\mathbb{Z}r$). *Let W and W' be two wallpaper groups, with point groups $\pi(W) = \langle \mu_l \rangle$ and $\pi(W') = \langle \mu_{l'} \rangle$, lattice group L and L', translation subgroups T and T' respectively. Let $r \in L \cap l$, $s \in L \cap l_\perp$, $r' \in L \cap l'$ and $s' \in L \cap l'_\perp$ be of minimal length. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Suppose $2\mathbb{Z}r$ and $2\mathbb{Z}r'$ are the sets of all the shift vectors of μ_l and $\mu_{l'}$ respectively. Let $v, v' \in \mathbb{R}^2$ satisfy $0 = v + \mu_l(v)$ and $0 = v' + \mu_{l'}(v')$. Define $\lambda \colon L \to L'$ by $mr + ns \mapsto mr' + ns'$ for all $m, n \in \mathbb{Z}$. Define $\eta \colon W \to W'$ by $\tau_w \circ (\tau_v \circ \mu_l)^i \mapsto \tau_{\lambda(w)} \circ (\tau_{v'} \circ \mu_{l'})^i$ for $i \in \{0,1\}$. Then η and $\eta|_T$ are isomorphisms.*

Proof. From the premises, $0 = v + \mu_l(v)$ and $0 = v' + \mu_{l'}(v')$. Therefore, $\lambda(v + \mu_l(v)) = 0 = v' + \mu_l(v')$. By Remark 4.4, λ is an isomorphism. By Theorem 4.28, η and $\eta|_T$ are isomorphisms. ∎

Corollary 4.34 (1 reflection, $q = 1$, $\text{sftvec}(\mu_l) = 2\mathbb{Z}r$, equivalence). *Let W and W' be two wallpaper groups, with point groups $\pi(W) = \langle \mu_l \rangle$ and $\pi(W') = \langle \mu_{l'} \rangle$, lattice group L and L' respectively. Let $r \in L \cap l$, $s \in L \cap l_\perp$, $r' \in L \cap l'$ and $s' \in L \cap l'_\perp$ be of minimal length. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Suppose $2\mathbb{Z}r$ and $2\mathbb{Z}r'$ are the sets of all the shift vectors of μ_l and $\mu_{l'}$ respectively. Then $W \sim W'$.*

Theorem 4.35 (1 reflection, $q = 1$, $\text{sftvec}(\mu_l) = r + 2\mathbb{Z}r$). *Let W and W' be two wallpaper groups, with point groups $\pi(W) = \langle \mu_l \rangle$ and $\pi(W') = \langle \mu_{l'} \rangle$, lattice group L and L', translation subgroups T and T' respectively. Let $r \in L \cap l$, $s \in L \cap l_\perp$, $r' \in L \cap l'$ and $s' \in L \cap l'_\perp$ be of minimal length. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Suppose $r + 2\mathbb{Z}r$ and $r' + 2\mathbb{Z}r'$ are the sets of all the shift vectors of μ_l and $\mu_{l'}$ respectively. Let $v, v' \in \mathbb{R}^2$ satisfy $r = v + \mu_l(v)$ and $r = v' + \mu_{l'}(v')$. Define $\lambda \colon L \to L'$ by $mr + ns \mapsto mr' + ns'$ for all $m, n \in \mathbb{Z}$. Define $\eta \colon W \to W'$ by $\tau_w \circ (\tau_v \circ \mu_l)^i \mapsto \tau_{\lambda(w)} \circ (\tau_{v'} \circ \mu_{l'})^i$ for $i \in \{0,1\}$. Then η and $\eta|_T$ are isomorphisms.*

Proof. From the premises, $r = v + \mu_l(v)$ and $r = v' + \mu_{l'}(v')$. Therefore, $\lambda(v + \mu_l(v)) = \lambda(r) = r' = v' + \mu_l(v')$. By Remark 4.4, λ is an isomorphism. By Theorem 4.28, η and $\eta|_T$ are isomorphisms. ∎

Corollary 4.36 (1 reflection, $q = 1$, $\text{sftvec}(\mu_l) = r + 2\mathbb{Z}r$, equivalence). *Let W and W' be two wallpaper groups, with point groups $\pi(W) = \langle \mu_l \rangle$ and $\pi(W') = \langle \mu_{l'} \rangle$, lattice group L and L' respectively. Let $r \in L \cap l$, $s \in L \cap l_\perp$, $r' \in L \cap l'$ and $s' \in L \cap l'_\perp$ be of minimal length. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Suppose $r + 2\mathbb{Z}r$ and $r' + 2\mathbb{Z}r'$ are the sets of all the shift vectors of μ_l and $\mu_{l'}$ respectively. Then $W \sim W'$.*

Proposition 4.37. *Let W and W' be two wallpaper groups, with point groups $\pi(W) = \langle \mu_l \rangle$ and $\pi(W') = \langle \mu_{l'} \rangle$, lattice group L and L' respectively. Let $r \in L \cap l$, $s \in L \cap l_\perp$, $r' \in L \cap l'$ and $s' \in L \cap l'_\perp$ be of minimal length. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Suppose $2\mathbb{Z}r$ and $r' + 2\mathbb{Z}r'$ are the sets of all the shift vectors of μ_l and $\mu_{l'}$ respectively. Then $W \nsim W'$.*

Proof. Assume there is such an isomorphism. Let $v \in \mathbb{R}^2$ such that $\tau_v \circ \mu_l \in W$ and $(\tau_v \circ \mu_l)^2 = \text{id}$. Then $\tau_v \circ \mu_l \mapsto \tau_{v'} \circ \mu_{l'}$ for some $v' \in \mathbb{R}^2$. Then $(\tau_v \circ \mu_l)^2 \mapsto (\tau_{v'} \circ \mu_{l'})^2$. We know that $(\tau_{v'} \circ \mu_{l'})^2 \in \tau_{r'+2\mathbb{Z}r'}$. Hence, $(\tau_{v'} \circ \mu_{l'})^2 \neq \text{id}$. However, this is impossible, since an isomorphism maps identity to identity. ∎

Proposition 4.38. *Let W and W' be two wallpaper groups, with point groups $\pi(W) = \langle \mu_l \rangle$ and $\pi(W') = \langle \mu_{l'} \rangle$, lattice group L and L' respectively. Let $r \in L \cap l$, $s \in L \cap l_\perp$, $r' \in L \cap l'$ and $s' \in L \cap l'_\perp$ be of minimal length. Suppose $L = \mathbb{Z}(r+s)/2 + \mathbb{Z}r$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Suppose $\mathbb{Z}r$ and $r' + 2\mathbb{Z}r'$ are the sets of all the shift vectors of μ_l and $\mu_{l'}$ respectively. Then $W \not\simeq W'$.*

Proof. The proof is identical as Proposition 4.37.

Proposition 4.39. *Let W and W' be two wallpaper groups, with point groups $\pi(W) = \langle \mu_l \rangle$ and $\pi(W') = \langle \mu_{l'} \rangle$, lattice group L and L' respectively. Let $r \in L \cap l$, $s \in L \cap l_\perp$, $r' \in L \cap l'$ and $s' \in L \cap l'_\perp$ be of minimal length. Suppose $L = \mathbb{Z}(r+s)/2 + \mathbb{Z}r$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Suppose $\mathbb{Z}r$ and $2\mathbb{Z}r'$ are the sets of all the shift vectors of μ_l and $\mu_{l'}$ respectively. Then $W \not\simeq W'$.*

Proof. Assume there is such an isomorphism. Let $v \in \mathbb{R}^2$ such that $\tau_v \circ \mu_l \in W$ and $(\tau_v \circ \mu_l)^2 = \tau_r$. Then $\tau_v \circ \mu_l \mapsto \tau_{v'} \circ \mu_{l'}$ for some $v' \in \mathbb{R}^2$. Then $(\tau_v \circ \mu_l)^2 \mapsto (\tau_{v'} \circ \mu_{l'})^2$. We know that $(\tau_{v'} \circ \mu_{l'})^2 \in \tau_{2\mathbb{Z}r'}$. But we know that there is no generator of L' in $2\mathbb{Z}r'$ and r is one of the generators of L. We have a contradiction. ∎

Theorem 4.40. *We have three equivalence classes of wallpaper groups with their point group containing one single reflection.*

Proof. This follows directly from Corollary 4.32, Corollary 4.34, Corollary 4.36, Proposition 4.37, Proposition 4.38, and Proposition 4.39. ∎

4.3 Point Group Containing More Than One Reflection

Lemma 4.41. *Let $H = \langle \mu_k, \mu_m \rangle$ be finite, where k and m are distinct lines through the origin. Then all rotations in H form a cyclic subgroup H_0. Moreover, there exists a line l through the origin such that $\angle(l, m) = \pi/\mathrm{ord}(H_0)$ and $H = \langle \mu_l, \mu_m \rangle$.*

Proof. Let $H = \langle \mu_k, \mu_m \rangle$ be finite, where k and m are distinct lines through the origin. Then $\mu_k \circ \mu_m = \rho_{2\angle(m,k)}$. Then H admits at least one subgroup $\langle \rho_{2\angle(m,k)} \rangle$ of rotations. Let H_0 be the set of all the rotations in H. Then H_0 is finite because H is finite. Let $q = \mathrm{ord}(H_0)$. By Proposition 3.18, $H_0 = \langle \rho_{2\pi/q} \rangle$.

Let $l = \rho_{\pi/q}(m)$. Then $\angle(m, l) = \pi/q$. Then $\mu_l \circ \mu_m = \rho_{2\pi/q}$. Then $\mu_l \in H$.

Now we want to prove $H = \langle \mu_l, \mu_m \rangle$. Assume that $H \neq \langle \mu_l, \mu_m \rangle$. Then we can pick $\mu_n \in H \setminus \langle \mu_l, \mu_m \rangle$, where n is a line through the origin. Then $n \neq (\rho_{\pi/q})^i(m)$ for any $i \in \mathbb{Z}$. Otherwise, $\angle(n, m) = i\pi/q$, and $\mu_n \circ \mu_m = \rho_{2i\pi/q} = (\mu_l \circ \mu_m)^i$, then $\mu_n \in \langle \mu_l, \mu_m \rangle$, contradicting the assumption. Now we have $n \neq (\rho_{\pi/q})^i(m)$ for any $i \in \mathbb{Z}$. Then $\angle(n, m) \neq i\pi/q$ for any $i \in \mathbb{Z}$. Then $\mu_m \circ \mu_n \neq \rho_{2i\pi/q}$ for any $i \in \mathbb{Z}$. This is not possible since H_0 contains all the rotations in H, and is cyclic. This is a new rotation in H_0 that is not a power of the generator, which is impossible. Therefore, $H = \langle \mu_l, \mu_m \rangle$. ∎

Remark 4.42. By the preceding lemma, any finite group of linear isometries generated by two reflections can be expressed as $\langle \mu_l, \mu_m \rangle$ where $\angle(m, l) = \pi/q$ and q is the order of the rotation subgroup. We simply do so without mentioning this fact again.

Lemma 4.43. *Let W be a wallpaper group with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$ and translation subgroup T, where $\angle(m,l) = \pi/q$ and $q \in \{2,3,4,6\}$. Pick $u, v \in \mathbb{R}^2$ such that $\tau_u \circ \rho_{\mu_l}, \tau_v \circ \rho_{\mu_l} \in W$. Then $W = \bigcup_{i\in\{0,1\}, j\in\{0,\dots,q-1\}} T \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j$ and this union is disjoint. Moreover, every element in W can be expressed uniquely in the form $\tau_u \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j$ for some $u \in L$, $i \in \{0,1\}$ and $j \in \{0,\dots,q-1\}$.*

Proof. By Proposition 3.2, T is a normal subgroup of W, meanwhile, $W/T \cong \pi(W)$. Hence, the representatives of the cosets of T can be picked in the fiber on each element of $\pi(W)$. We see that $\{\pi((\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j)\}_{i,j} = \{\mu_l^i \circ \rho_{2\pi/q}^j\}_{i,j} = \pi(W)$. Therefore, $(\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j \in W$ are the representatives of the cosets $T \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j$. Therefore, $\{T \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j\}_{i,j}$ is a partition of W, and we can write W as a disjoint union $W = \bigcup_{i\in\{0,1\}, j\in\{0,\dots,q-1\}} T \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j$.

By Proposition 2.23, we can express any element of W uniquely as $\tau_t \circ \mu_l^i \circ \rho^j$ for some $t \in \mathbb{R}^2$, $i \in \{0,1\}$ and $j \in \{0,\dots,q-1\}$. By repetitively applying Proposition 2.14, $\tau_t \circ \mu_l^{i+j} \circ \mu_m^j = \tau_u \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j$, where $u = t - \mu_l(v) - \dots - \mu_l^{i+j-1}(v) - \mu_l^{i+j}(w) - \mu_l^{i+j}(\mu_m(w)) - \dots - \mu_l^{i+j}(\mu_m^{j-1}(w))$, which is also unique. Then $\tau_u = \tau_t \circ \mu_l^{i+j} \circ \mu_m^j \circ ((\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j)^{-1} \in W$. Hence, $\tau_u \in T$. The expression $\tau_t \circ \mu_l^i \circ \rho^j = \tau_u \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j$ is a unique expression in $T \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j$. ■

Remark 4.44. Let W and W' be wallpaper groups with $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$. Let L and L' be the lattice groups of W and W' respectively. Let $\lambda: L \to L'$ be an isomorphism. Let $v, w, v', w' \in \mathbb{R}^2$ satisfy $\tau_v \circ \mu_l, \tau_w \circ \mu_m \in W$ and $\tau_{v'} \circ \mu_{l'}, \tau_{w'} \circ \mu_{m'} \in W'$.

We can define $\eta: W \to W'$ by $\tau_u \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j \mapsto \tau_{\lambda(u)} \circ (\tau_{v'} \circ \mu_{l'})^i \circ (\tau_{w'} \circ \mu_{m'} \circ \tau_{v'} \circ \mu_{l'})^j$. Meanwhile, we can also define its inverse $\eta^{-1}: W' \to W$ by $\tau_{u'} \circ (\tau_{v'} \circ \mu_{l'})^i \circ (\tau_{w'} \circ \mu_{m'} \circ \tau_{v'} \circ \mu_{l'})^j \mapsto \tau_{\lambda^{-1}(u')} \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j$.

Lemma 4.45. *Let W be a wallpaper group with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$, where $\angle(l,m) = \pi/q$ and $q \in \{3,4,6\}$. Let L and L' be their lattice groups respectively, Let $r \in L \cap l$ and $s \in L \cap m$ be of minimal length. Then there exist $k, n \in \mathbb{Z}_+$ such that $r + \mu_m(r) = ks$ and $s + \mu_l(s) = nr$. Moreover, if $q = 3$, then either $(k,n) = (1,1)$ and equivalently $\|s\| = \|r\|$; if $q = 4$, then either $(k,n) = (2,1)$ and equivalently $\|r\| = \sqrt{2}\|s\|$, or $(k,n) = (1,2)$ and equivalently $\|s\| = \sqrt{2}\|r\|$; if $q = 6$, then either $(k,n) = (3,1)$ and equivalently $\|r\| = \sqrt{3}\|s\|$, or $(k,n) = (1,3)$ and equivalently $\|s\| = \sqrt{3}\|r\|$.*

Proof. Since $r, s \in L$, we know $r + \mu_m(r) \in L \cap m$ and $s + \mu_l(s) = nr \in L \cap l$. Then we can write $k, n \in \mathbb{R}$ such that $r + \mu_m(r) = ks$ and $s + \mu_l(s) = nr$. Assume k and n are not both integers. Then $(k - [k])s \in L \cap m$ or $(n - [n])r \in L \cap l$, but they are shorter than s and r respectively, which contradicts the minimality of r and s. Hence, k and l are integers. They must be positive since $\angle(r,s) \in (0, 2\pi)$. Therefore, there exist $k, n \in \mathbb{Z}_+$ such that $r + \mu_m(r) = ks$ and $s + \mu_l(s) = nr$.

We know $\|r + \mu_m(r)\| = 2\cos(\pi/q)\|r\|$ and $\|s + \mu_l(s)\| = 2\cos(\pi/q)\|s\|$.

Suppose $q = 3$. Then $|k|\|s\| = \|ks\| = \|r + \mu_m(r)\| = \|r\|$ and $|n|\|r\| = \|nr\| = \|s + \mu_l(s)\| = \|s\|$. It follows that $|kn| = 1$. Then $(k,n) = (1,1)$ and equivalently $\|r\| = \|s\|$.

Suppose $q = 4$. Then $|k|\|s\| = \|ks\| = \|r + \mu_m(r)\| = \sqrt{2}\|r\|$ and $|n|\|r\| = \|nr\| = \|s + \mu_l(s)\| = \sqrt{2}\|s\|$. It follows that $|kn| = 2$. Then either $(k,n) = (2,1)$ and equivalently $\|r\| = \sqrt{2}\|s\|$, or $(k,n) = (1,2)$ and equivalently $\|s\| = \sqrt{2}\|r\|$.

Suppose $q = 6$. Then $|k|\|s\| = \|ks\| = \|r + \mu_m(r)\| = \sqrt{3}\|r\|$ and $|n|\|r\| = \|nr\| = \|s + \mu_l(s)\| = \sqrt{3}\|s\|$. It follows that $|kn| = 3$. Then either $(k,n) = (3,1)$ and equivalently $\|r\| = \sqrt{3}\|s\|$, or $(k,n) = (1,3)$ and equivalently $\|s\| = \sqrt{3}\|r\|$. ■

Proposition 4.46. *Let W and W' be wallpaper groups with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$ respectively, where $\angle(l,m) = \pi/2$. Let L and L' be the lattice groups respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L' \cap l'$ and $s' \in L' \cap m'$ be non-zero and of minimal length. Suppose $\lambda\colon L \to L'$ gives $\lambda(r) = r'$ and $\lambda(s) = s'$. Then $\lambda \circ \mu_l = \mu_{l'} \circ \lambda$ and $\lambda \circ \mu_m = \mu_{m'} \circ \lambda$.*

Proof. Let $u \in L$. Since we don't know if r and s forms a basis of L, we write $u = ar + bs$ for $a, b \in \mathbb{R}$. Then $\lambda(\mu_l(u)) = \lambda(ar - bs) = ar' - bs'$, and $\mu_{l'}(\lambda(u)) = \mu_{l'}(ar' + bs') = ar' - bs'$. Moreover, $\lambda(\mu_m(u)) = \lambda(-ar + bs) = -ar' + bs'$, and $\mu_{l'}(\lambda(u)) = \mu_{l'}(ar' + bs') = -ar' + bs'$. Therefore, $\lambda \circ \mu_m = \mu_{m'} \circ \lambda$. ∎

Proposition 4.47. *Let W and W' be a wallpaper group with point groups $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$ resepectively, where $\angle(l,m) = \langle(l'm') = \pi/q$ and $q \in \{3,4,6\}$. Let L and L' be their lattice groups respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L \cap l'$, and $s' \in L \cap m'$ be of minimal length. Suppose $\|r\| \geq \|s\|$ and $\|r'\| \geq \|s'\|$. Define a linear map $\lambda\colon L \to L'$ such that $\lambda(r) = r'$ and $\lambda(s) = s'$. Then $\lambda \circ \mu_l = \mu_{l'} \circ \lambda$ and $\lambda \circ \mu_m = \mu_{m'} \circ \lambda$.*

Proof. By Lemma 4.45, there exist $k, n \in \mathbb{Z}_+$ such that $r + \mu_m(r) = ks$ and $s + \mu_l(s) = nr$, there exist $k', l' \in \mathbb{Z}_+$ such that $r' + \mu_{m'}(r') = k's'$ and $s + \mu_{l'}(s') = l'r'$. Note that by the premises $\|r\| \geq \|s\|$ and $\|r'\| \geq \|s'\|$, we have $\|r\|/\|s\| = \|r'\|/\|s'\|$. Suppose $q = 3$. By Lemma 4.45, $(k,n) = (k',n') = (1,1)$. Suppose $q = 4$. By Lemma 4.45, either $(k,n) = (k',n') = (2,1)$ or $(k,n) = (k',n') = (1,2)$. Suppose $q = 6$. By Lemma 4.45, either $(k,n) = (k',n') = (3,1)$ or $(k,n) = (k',n') = (1,3)$. Therefore, in all cases, $(k,l) = (k',l')$. Then we have $\mu_m(r) = ks - r$, $\mu_l(s) = nr - s$, $\mu_m(r') = ks' - r'$, and $\mu_l(s') = nr' - s'$.

Let $u \in L$. Then we can write $u = ar + bs$ for $a, b \in \mathbb{R}$. Then $\lambda(\mu_l(u)) = \lambda(\mu_l(ar + bs)) = \lambda(ar + b\mu_l(s)) = \lambda(ar + b(nr - s)) = \lambda((a + bn)r - s) = (a + bn)r' - s'$. Moreover, $\mu_{l'}(\lambda(u)) = \mu_{l'}(\lambda(ar + bs)) = \mu_{l'}(ar' + bs') = ar' + b\mu_{l'}(s') = ar' + b(nr' - s') = (a + bn)r' - s'$. Hence, $\lambda(\mu_l(u)) = \mu_{l'}(\lambda(u))$, similarly, $\lambda(\mu_m(u)) = \mu_{m'}(\lambda(u))$. Therefore, $\lambda \circ \mu_l = \mu_{l'} \circ \lambda$ and $\lambda \circ \mu_m = \mu_{m'} \circ \lambda$. ∎

Theorem 4.48 (>1 reflection, $q \in \{2,3,4,6\}$, isomorphism*). *Let W and W' be two wallpaper groups, with $\pi(W) = \langle \mu_l, \mu_m \rangle$, and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$, where $\angle(l,m) = \angle(l',m') = \pi/q$ and $q \in \{2,3,4,6\}$. Let L and L' be the lattice groups of W and W' respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L' \cap l'$ and $s' \in L' \cap m'$ be non-zero and of minimal length. Let $\lambda\colon L \to L'$ be an isomorphism such that $\lambda(r) = r'$ and $\lambda(s) = s'$. Let $v, w, v', w' \in \mathbb{R}^2$ satisfy $\tau_v \circ \mu_l \in W$, $\tau_w \circ \mu_m \in W$, $\tau_{v'} \circ \mu_{l'} \in W'$, $\tau_{w'} \circ \mu_{m'} \in W'$. Suppose $\lambda(v + \mu_l(v)) = v' + \mu_{l'}(v')$ and $\lambda(w + \mu_m(w)) = w' + \mu_{m'}(w')$. Suppose $\lambda \circ \mu_l = \mu_{l'} \circ \lambda$ and $\lambda \circ \mu_m = \mu_{m'} \circ \lambda$. Define $\eta\colon W \to W'$ by $\tau_u \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j \mapsto \tau_{\lambda(u)} \circ (\tau_{v'} \circ \mu_{l'})^i \circ (\tau_{w'} \circ \mu_{m'} \circ \tau_{v'} \circ \mu_{l'})^j$ for every $i \in \{0,1\}$ and $j \in \{0,\ldots,q-1\}$. Then η and $\eta|_T$ are isomorphisms.*

Proof. Firstly, we prove that η is a homomorphism. We want to prove that $\eta(\sigma_1 \circ \sigma_2) = \eta(\sigma_1) \circ \eta(\sigma_2)$ for all $\sigma_1, \sigma_2 \in W$.

Let $\sigma_1 = \tau_{u_1} \circ (\tau_v \circ \mu_l)^{i_1} \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^{j_1} \in W$ and $\sigma_2 = \tau_{u_2} \circ (\tau_v \circ \mu_l)^{i_2} \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^{j_2} \in W$. For a shorter expression, we omit the composition symbol and parenthesis, when possible. We write $\sigma_1 = \tau_{u_1}(\tau_v \circ \mu_l)^{i_1}(\tau_w \mu_m \tau_v \mu_l)^{j_1}$ and $\sigma_2 = \tau_{u_2}(\tau_v \mu_l)^{i_2}(\tau_w \mu_m \tau_v \mu_l)^{j_2}$. Then by repeatedly applying Proposition 2.14, $\sigma_1 \circ \sigma_2 = \tau_{u_1} \tau_{\mu_l^i \mu_m^j \mu_l^j (u_2)} (\tau_v \mu_l)^{i_1} (\tau_w \mu_m \tau_v \mu_l)^{j_1} (\tau_v \mu_l)^{i_2} (\tau_w \mu_m \tau_v \mu_l)^{j_2}$.

Suppose $j_1 = 0$ or $i_2 = 0$. Then $\sigma_1 \circ \sigma_2 = \tau_{u_1} \tau_{\mu_l^{i_1} \mu_m^{j_1} \mu_l^{j_1}(u_2)} (\tau_v \mu_l)^{i_1+i_2} (\tau_w \mu_m \tau_v \mu_l)^{j_1+j_2}$. Then $\eta(\sigma_1 \circ \sigma_2) = \tau_{\lambda u_1} \tau_{\lambda \mu_l^{i_1} \mu_m^{j_1} \mu_l^{j_1}(u_2)} (\tau_{v'} \mu_{l'})^{i_1+i_2} (\tau_{w'} \mu_{m'} \tau_{v'} \mu_{l'})^{j_1+j_2}$. We know that this is well-defined also when $i_1 + i_2 \geq 2$ or $j_1 + j_2 \geq q$. Assume $i_1 + i_2 \geq 2$. Then we can express the part $(\tau_v \mu_l)^{i_1+i_2}$ in $\sigma_1 \circ \sigma_2$ as

$\tau_{v+\mu_l v}(\tau_v\mu_l)^{i_1+i_2-q}$. The corresponding part in $\eta(\sigma_1\circ\sigma_2)$ is $\tau_{\lambda(v+\mu_l v)}(\tau_{v'}\mu_{l'})^{i_1+i_2-q}$. In the premises, we have $\lambda(v+\mu_l v) = v'+\mu_l v'$. Hence, $\tau_{\lambda(v+\mu_l v)}(\tau_{v'}\mu_{l'})^{i_1+i_2-q} = \tau_{v'+\mu_l v'}(\tau_{v'}\mu_{l'})^{i_1+i_2-q} = (\tau_{v'}\mu_{l'})^{i_1+i_2}$. Assume $j_1+j_2 \geq q$. Since $\tau_w\mu_m\tau_v\mu_l$ is a rotation about the origin, its shift vector is 0. In other words, $(\tau_w\mu_m\tau_v\mu_l)^q = \text{id}$. Then we can express the part $(\tau_w\mu_m\tau_v\mu_l)^{j_1+j_2}$ as $(\tau_w\mu_m\tau_v\mu_l)^{i_1+i_2-q}$. Then in $\eta(\sigma_1\circ\sigma_2)$, the corresponding part is $(\tau_{w'}\mu_{m'}\tau_{v'}\mu_{l'})^{i_1+i_2-q} = (\tau_{w'}\mu_{m'}\tau_{v'}\mu_{l'})^{i_1+i_2}$. Now we consider the part $\tau_{\lambda u_1}\tau_{\lambda\mu_l^{i_1}\mu_m^{j_1}\mu_l^{j_1}(u_2)}$ in $\eta(\sigma_1\circ\sigma_2)$. We know $\lambda\circ\mu_l = \mu_{l'}\circ\lambda$. Hence, $\lambda\mu_l^{i_1}\mu_m^{j_1}\mu_l^{j_1}(u_2) = \mu_{l'}^{i_1}\mu_{m'}^{j_1}\mu_{l'}^{j_1}\lambda(u_2)$. Then $\tau_{\lambda u_1}\tau_{\lambda\mu_l^{i_1}\mu_m^{j_1}\mu_l^{j_1}(u_2)} = \tau_{\lambda u_1}\tau_{\mu_{l'}^{i_1}\mu_{m'}^{j_1}\mu_{l'}^{j_1}\lambda(u_2)}$. By assembling, we have $\eta(\sigma_1\circ\sigma_2) = \tau_{\lambda u_1}\tau_{\mu_{l'}^{i_1}\mu_{m'}^{j_1}\mu_{l'}^{j_1}\lambda(u_2)}(\tau_{v'}\mu_{l'})^{i_1+i_2}(\tau_{w'}\mu_{m'}\tau_{v'}\mu_{l'})^{j_1+j_2}$. Then by Repetitively applying Proposition 2.14, we have $\eta(\sigma_1\circ\sigma_2) = \tau_{\lambda u_1}(\tau_{v'}\mu_{l'})^{i_1}(\tau_{w'}\mu_{m'}\tau_{v'}\mu_{l'})^{j_1}\circ\tau_{\lambda u_2}(\tau_{v'}\mu_{l'})^{i_2}(\tau_{w'}\mu_{m'}\tau_{v'}\mu_{l'})^{j_2} = \eta(\sigma_1)\circ\eta(\sigma_2)$.

Suppose $j_1 \neq 0$ and $i_2 = 1$. Then $\sigma_1\circ\sigma_2 = \tau_{u_1}\tau_{\mu_l^{i_1}\mu_m^{j_1}\mu_l^{j_1}(u_2)}(\tau_v\mu_l)^{i_1}(\tau_w\mu_m\tau_v\mu_l)^{j_1}(\tau_v\mu_l)^1(\tau_w\mu_m\tau_v\mu_l)^{j_2}$. We see that $(\tau_w\mu_m\tau_v\mu_l)^{j_1}(\tau_v\mu_l)^1$ gives us $(\tau_w\mu_m\tau_v\mu_l)^{j_1-1}(\tau_w\mu_m)\tau_{v+\mu_l v}$. We move this $\tau_{v+\mu_l v}$ to the front of $(\tau_v\mu_l)^{i_1}$ by repetitively applying Proposition 2.14, then it becomes $\tau_{\mu_l^{i_1}\mu_m^{j_1}\mu_l^{j_1-1}(v+\mu_l v)}$. Hence, we have $\sigma_1\circ\sigma_2 = \tau_{u_1}\tau_{\mu_l^{i_1}\mu_m^{j_1}\mu_l^{j_1}(u_2)}\tau_{\mu_l^{i_1}\mu_m^{j_1}\mu_l^{j_1-1}(v+\mu_l v)}(\tau_v\mu_l)^{i_1}(\tau_w\mu_m\tau_v\mu_l)^{j_1-1}(\tau_w\mu_m)^1(\tau_w\mu_m\tau_v\mu_l)^{j_2}$. We can repeat a similar procedure with the left over $(\tau_w\mu_m)^1$, and then with $(\tau_v\mu_l)^{i_1}$, and keep going until we acheive the following. $\sigma_1\circ\sigma_2 = \tau\circ(\tau_v\mu_l)^x(\tau_w\mu_m\tau_v\mu_l)^y$ where $\tau = \tau_{u_1}\tau_{\mu_l^{i_1}\mu_m^{j_1}\mu_l^{j_1}(u_2)}\circ\tau_{\sum_k\varphi_k(v+\mu_l v)}\tau_{\sum_n\varphi_n(w+\mu_m w)}$ and $x,y\in\mathbb{Z}_{\geq 0}$ and each φ_k or φ_n represent an element in $\pi(W)$, which is a complicated composition of μ_l and μ_m. We know that $\lambda\circ\mu_l = \mu_{l'}\circ\lambda$ and $\lambda\circ\mu_m = \mu_{m'}\circ\lambda$.We use φ'_n to represent the expression corresponding to φ_n with μ_l replaced by $\mu_{l'}$ and μ_m replaced by $\mu_{m'}$, as is φ'_k. Then $\lambda\circ\varphi_k = \varphi'_k\circ\lambda$ and $\lambda\circ\varphi_n = \varphi'_n\circ\lambda$ for every possible k and n. Then we have $\eta(\sigma_1\circ\sigma_2) = \tau'\circ(\tau_{v'}\mu_{l'})^x(\tau_{w'}\mu_{m'}\tau_{v'}\mu_{l'})^y$, where $\tau' = \tau_{\lambda(u_1)}\tau_{\lambda\mu_l^{i_1}\mu_m^{j_1}\mu_l^{j_1}(u_2)}\circ\tau_{\lambda\sum_k\varphi_k(v+\mu_l v)}\tau_{\lambda\sum_n\varphi_n(w+\mu_m w)} = \tau_{\lambda(u_1)}\tau_{\mu_l^{i_1}\mu_m^{j_1}\mu_l^{j_1}\lambda(u_2)}\circ\tau_{\sum_k\varphi'_k\lambda(v+\mu_l v)}\tau_{\sum_n\varphi'_n\lambda(w+\mu_m w)}$. By the premises, we have $\lambda(v+\mu_l v) = v'+\mu_{l'}v'$ and $\lambda(w+\mu_m w) = w'+\mu_{m'}w'$. Therefore, $\tau' = \tau_{\lambda(u_1)}\tau_{\mu_{l'}^{i_1}\mu_{m'}^{j_1}\mu_{l'}^{j_1}\lambda(u_2)}\circ\tau_{\sum_k\varphi'_k(v'+\mu_{l'}v')}\tau_{\sum_n\varphi'_n(w'+\mu_{m'}w')}$. By repetitively applying Proposition 2.14 and $v'+\mu_{l'}v' = (\tau_{v'}\mu_{l'})^2$ and $w'+\mu_{m'}w' = (\tau_{w'}\mu_{m'})^2$, we reverse the procedure in W' according to what we did in W for $\sigma_1\circ\sigma_2$. Then we have $\eta(\sigma_1\circ\sigma_2) = \tau_{\lambda(u_1)}(\tau_{v'}\mu_{l'})^{i_1}(\tau_{w'}\mu_{m'}\tau_{v'}\mu_{l'})^{j_1}(\tau_{v'}\mu_{l'})^1(\tau_{w'}\mu_{m'}\tau_{v'}\mu_{l'})^{j_2}$. In other words, $\eta(\sigma_1\circ\sigma_2) = \eta(\sigma_1)\circ\eta(\sigma_2)$.

Hence, η is a homomorphism. By Remark 4.44, η has an inverse η^{-1}. Therefore, η is an isomorphism. $\eta|_T$ is simply $T\to T'$ given by $\tau_t\mapsto\tau_{\lambda(t)}$, which is an isomorphism by construction. ∎

$q = 2$

Lemma 4.49. *Let W be a wallpaper group with $\pi(W) = \langle\mu_l,\mu_m\rangle$, where $\angle(l,m) = \pi/2$. Let L be its lattice group. Let $r\in L\cap l$, $s\in L\cap m$ be of minimal length. Then either $L = \mathbb{Z}r+\mathbb{Z}s$ or $L = \mathbb{Z}(r+s)/2+\mathbb{Z}r$.*

Proof. This proof is a quick version of the proofs of Lemma 4.19 and Lemma 4.20. If $L = \mathbb{Z}r+\mathbb{Z}s$, the the proof is finished. Assume $L \neq \mathbb{Z}r+\mathbb{Z}s$. Let $t\in L\setminus(\mathbb{Z}r+\mathbb{Z}s)$. Write $t = ar+bs$, where $a,b\in\mathbb{R}$. We know $\mu_l(t) = ar-bs$ and $\mu_m(t) = -ar+bs$. Then $2ar = t+\mu_l(t)\in L$ and $2bs = t+\mu_m(t)\in L$. Thus, $2a,2b\in\mathbb{Z}$. We know that $a,b\notin\mathbb{Z}$. Otherwise, either $L = \mathbb{Z}r+\mathbb{Z}s$ contradicts the premise or we violate the minimality of r or s. Therefore, $a,b\in 1/2+\mathbb{Z}$, which means $t\in(r+s)/2+\mathbb{Z}r+\mathbb{Z}s$. Hence, $L\setminus(\mathbb{Z}r+\mathbb{Z}s)\subseteq((r+s)/2+\mathbb{Z}r+\mathbb{Z}s)$. In other words, $L\subseteq(\mathbb{Z}r+\mathbb{Z}s)\cup((r+s)/2+\mathbb{Z}r+\mathbb{Z}s)$. We know that there exists this $t\in(r+s)/2+\mathbb{Z}r+\mathbb{Z}s$. Then $t+\mathbb{Z}r+\mathbb{Z}s\subseteq L$. But $t+\mathbb{Z}r+\mathbb{Z}s = (r+s)/s+\mathbb{Z}r+\mathbb{Z}s$. Hence, $(r+s)/2+\mathbb{Z}r+\mathbb{Z}s\subseteq L$. In other words, $(\mathbb{Z}r+\mathbb{Z}s)\cup((r+s)/2+\mathbb{Z}r+\mathbb{Z}s)\subseteq L$. Therefore, $L = (\mathbb{Z}r+\mathbb{Z}s)\cup((r+s)/2+\mathbb{Z}r+\mathbb{Z}s) = \mathbb{Z}(r+s)/2+\mathbb{Z}r$. ∎

Lemma 4.50. *Let W be a wallpaper group with $\pi(W)$. Suppose $\mu_l, \mu_m \in \pi(W)$. Let $r \in L \cap l$ and $s \in L \cap m$ be of minimal length. Suppose $L = \mathbb{Z}(r+s)/2 + \mathbb{Z}r$. Then $\mathbb{Z}r$ is the set of all the shift vectors of μ_l, and $s\mathbb{Z}$ is the set of all the shift vectors of μ_m.*

Proof. Denote $w = (r+s)/2 \in L$. Then $r = w + \mu_l(w)$ and $s = w + \mu_l(w)$. By applying Proposition 4.22 to μ_l, we have that $\mathbb{Z}r$ is the set of all the shift vectors of μ_l. By applying Proposition 4.22 to μ_m, we have that $s\mathbb{Z}$ is the set of all the shift vectors of μ_m. ∎

Lemma 4.51. *Let W be a wallpaper group with $\pi(W) = \langle \mu_l, \mu_m \rangle$, where $\angle(l,m) = \pi/2$. Let $r \in L \cap l$ and $s \in L \cap m$ be of minimal length. If $L = \mathbb{Z}r + \mathbb{Z}s$, then either $2\mathbb{Z}r$ or $r + 2\mathbb{Z}r$ is the set of all the shift vectors of μ_l, and either $2\mathbb{Z}s$ or $s + 2\mathbb{Z}s$ is the set of all the shift vectors of μ_m.*

Proof. By applying Proposition 4.23 to μ_l, either $2\mathbb{Z}r$ or $r + 2\mathbb{Z}r$ is the set of all the shift vectors of μ_l. By applying Proposition 4.23 to μ_m, either $2\mathbb{Z}s$ or $s + 2\mathbb{Z}s$ is the set of all the shift vectors of μ_m. ∎

Lemma 4.52 (>1 reflection, $p = 2$, cases of $\mathrm{sftvec}(\mu_l)$ **and** $\mathrm{sftvec}(\mu_m)$**).** *] Let W be a wallpaper group with $\pi(W) = \langle \mu_l, \mu_m \rangle$, where $\angle(l,m) = \pi/2$. Let $r \in L \cap l$ and $s \in L \cap m$ be of minimal length. Then one of the following situation happens.*

1. *$L = \mathbb{Z}(r+s)/2 + \mathbb{Z}r$, the sets of shift vectors of μ_l and μ_m are $\mathbb{Z}r$ and $\mathbb{Z}s$ respectively;*
2. *$L = \mathbb{Z}r + \mathbb{Z}s$, the sets of shift vectors of μ_l and μ_m are $2\mathbb{Z}r$ and $2\mathbb{Z}s$ respectively;*
3. *$L = \mathbb{Z}r + \mathbb{Z}s$, the sets of shift vectors of μ_l and μ_m are $r + 2\mathbb{Z}r$ and $s + 2\mathbb{Z}s$ respectively;*
4. *$L = \mathbb{Z}r + \mathbb{Z}s$, the sets of shift vectors of μ_l and μ_m are $r + 2\mathbb{Z}r$ and $2\mathbb{Z}s$ respectively, or the sets of shift vectors of μ_l and μ_m are $2\mathbb{Z}r$ and $s + 2\mathbb{Z}s$ respectively.*

Proof. This is a direct result of Lemma 4.49, Lemma 4.50 and Lemma 4.51. ∎

Theorem 4.53 (>1 reflection, $q = 2$, $L = \mathbb{Z}(r+s)/2 + \mathbb{Z}r$, $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$**,** $\mathrm{sftvec}(\mu_m) = \mathbb{Z}s$**).** *Let W and W' be wallpaper groups with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$ respectively, where $\angle(l,m) = \pi/2$. Let L and L' be the lattice groups respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L' \cap l'$ and $s' \in L' \cap m'$ be non-zero and of minimal length. Suppose $L = \mathbb{Z}(r+s)/2 + \mathbb{Z}r$ and $L' = \mathbb{Z}(r'+s')/2 + \mathbb{Z}r'$. Suppose* $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$ *and* $\mathrm{sftvec}(\mu_m) = \mathbb{Z}s$*. Let $v, w, v', w' \in \mathbb{R}$ satisfy $v + \mu_l(v) = 0$, $w + \mu_m(w) = 0$, $v' + \mu_{l'}(v') = 0$ and $w' + \mu_{m'}(w') = 0$. Define $\lambda \colon L \to L'$ by $m(r+s)/2 + nr \mapsto m(r'+s')/2 + nr'$ for all $m, n \in \mathbb{Z}$. Define $\eta \colon W \to W'$ by $\tau_u \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j \mapsto \tau_{\lambda(u)} \circ (\tau_{v'} \circ \mu_{l'})^i \circ (\tau_{w'} \circ \mu_{m'} \circ \tau_{v'} \circ \mu_{l'})^j$ for all $i \in \{0,1\}$ and $j \in \{0,1\}$. Then η and $\eta|_T$ are isomorphisms.*

Proof. By Remark 4.4, λ is an isomorphism. Therefore, $\eta|_T$ is an isomorphism. Note $\lambda(r) = r'$ and $\lambda(s) = s'$. By the premises, $\lambda(v + \mu_l(v)) = 0 = v' + \mu_{l'}(v)$ and $\lambda(w + \mu_m(w)) = 0 = w' + \mu_{m'}(w)$. Let $u \in L$. Write $u = mr + ns$ for $m, n \in \mathbb{Z}$. Then $\lambda(\mu_l(u)) = \lambda(mr - ns) = mr' - ns'$, and $\mu_{l'}(\lambda(u)) = \mu_{l'}(mr' + ns') = mr' - ns'$. Therefore, $\lambda \circ \mu_l = \mu_{l'} \circ \lambda$. Similarly, $\lambda \circ \mu_m = \mu_{m'} \circ \lambda$. By Theorem 4.48, η is an isomorphism. ∎

Corollary 4.54 (>1 reflection, $q = 2$, $L = \mathbb{Z}(r+s)/2 + \mathbb{Z}r$, $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$**,** $\mathrm{sftvec}(\mu_m) = \mathbb{Z}s$**, equiv).** *Let W and W' be wallpaper groups with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$ respectively, where $\angle(l,m) = \pi/2$. Let L and L' be the lattice groups respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L' \cap l'$ and $s' \in L' \cap m'$ be non-zero and of minimal length. Suppose $L = \mathbb{Z}(r+s)/2 + \mathbb{Z}r$ and $L' = \mathbb{Z}(r'+s')/2 + \mathbb{Z}r'$. Suppose* $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$ *and* $\mathrm{sftvec}(\mu_m) = \mathbb{Z}s$*. Then $W \sim W'$.*

Theorem 4.55 (>1 reflection, $q = 2$**,** $L = \mathbb{Z}r + \mathbb{Z}s$**,** $\operatorname{sftvec}(\mu_l) = 2\mathbb{Z}r$**,** $\operatorname{sftvec}(\mu_m) = 2\mathbb{Z}s$**).** *Let W and W' be wallpaper groups with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$ respectively, where $\angle(l,m) = \pi/2$. Let L and L' be the lattice groups respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L' \cap l'$ and $s' \in L' \cap m'$ be non-zero and of minimal length. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Suppose* $\operatorname{sftvec}(\mu_l) = 2\mathbb{Z}r$ *and* $\operatorname{sftvec}(\mu_m) = 2\mathbb{Z}s$. *Let $v, w, v', w' \in \mathbb{R}$ satisfy $v + \mu_l(v) = 0$, $w + \mu_m(w) = 0$, $v' + \mu_{l'}(v') = 0$ and $w' + \mu_{m'}(w') = 0$. Define $\lambda \colon L \to L'$ by $mr + ns \mapsto mr' + ns'$ for all $m, n \in \mathbb{Z}$. Define $\eta \colon W \to W'$ by $\tau_u \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j \mapsto \tau_{\lambda}(u) \circ (\tau_{v'} \circ \mu_{l'})^i \circ (\tau_{w'} \circ \mu_{m'} \circ \tau_{v'} \circ \mu_{l'})^j$ for all $i \in \{0,1\}$ and $j \in \{0,1\}$. Then η and $\eta|_T$ are isomorphisms.*

Proof. By Remark 4.4, λ is an isomorphism. Therefore, $\eta|_T$ is an isomorphism. Note $\lambda(r) = r'$ and $\lambda(s) = s'$. By the premises, $\lambda(v + \mu_l(v)) = 0 = v' + \mu_{l'}(v)$ and $\lambda(w + \mu_m(w)) = 0 = w' + \mu_{m'}(w)$. By Proposition 4.46, $\lambda \circ \mu_l = \mu_{l'} \circ \lambda$ and $\lambda \circ \mu_m = \mu_{m'} \circ \lambda$. By Theorem 4.48, η is an isomorphism. ∎

Corollary 4.56 (>1 reflection, $q = 2$**,** $L = \mathbb{Z}r + \mathbb{Z}s$**,** $\operatorname{sftvec}(\mu_l) = 2\mathbb{Z}r$**,** $\operatorname{sftvec}(\mu_m) = 2\mathbb{Z}s$**, equiv).** *Let W and W' be wallpaper groups with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$ respectively, where $\angle(l,m) = \pi/2$. Let L and L' be the lattice groups respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L' \cap l'$ and $s' \in L' \cap m'$ be non-zero and of minimal length. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Suppose* $\operatorname{sftvec}(\mu_l) = 2\mathbb{Z}r$ *and* $\operatorname{sftvec}(\mu_m) = 2\mathbb{Z}s$. *Then $W \sim W'$.*

Theorem 4.57 (>1 reflection, $q = 2$**,** $L = \mathbb{Z}r + \mathbb{Z}s$**,** $\operatorname{sftvec}(\mu_l) = r + 2\mathbb{Z}r$**,** $\operatorname{sftvec}(\mu_m) = s + 2\mathbb{Z}s$**).** *Let W and W' be wallpaper groups with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$ respectively, where $\angle(l,m) = \pi/2$. Let L and L' be the lattice groups respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L' \cap l'$ and $s' \in L' \cap m'$ be non-zero and of minimal length. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Suppose* $\operatorname{sftvec}(\mu_l) = r + 2\mathbb{Z}r$ *and* $\operatorname{sftvec}(\mu_m) = s + 2\mathbb{Z}s$. *Let $v, w, v', w' \in \mathbb{R}$ satisfy $v + \mu_l(v) = r$, $w + \mu_m(w) = s$, $v' + \mu_{l'}(v') = r'$ and $w' + \mu_{m'}(w') = s'$. Define $\lambda \colon L \to L'$ by $mr + ns \mapsto mr' + ns'$ for all $m, n \in \mathbb{Z}$. Define $\eta \colon W \to W'$ by $\tau_u \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j \mapsto \tau_{\lambda}(u) \circ (\tau_{v'} \circ \mu_{l'})^i \circ (\tau_{w'} \circ \mu_{m'} \circ \tau_{v'} \circ \mu_{l'})^j$ for all $i \in \{0,1\}$ and $j \in \{0,1\}$. Then η and $\eta|_T$ are isomorphisms.*

Proof. By Remark 4.4, λ is an isomorphism. Therefore, $\eta|_T$ is an isomorphism. Note $\lambda(r) = r'$ and $\lambda(s) = s'$. By the premises, $\lambda(v + \mu_l(v)) = \lambda(r) = r' = v' + \mu_{l'}(v)$ and $\lambda(w + \mu_m(w)) = \lambda(s) = s' = w' + \mu_{m'}(w)$. By Proposition 4.46, $\lambda \circ \mu_l = \mu_{l'} \circ \lambda$ and $\lambda \circ \mu_m = \mu_{m'} \circ \lambda$. By Theorem 4.48, η is an isomorphism. ∎

Corollary 4.58 (>1 reflection, $q = 2$**,** $L = \mathbb{Z}r + \mathbb{Z}s$**,** $\operatorname{sftvec}(\mu_l) = r + 2\mathbb{Z}r$**,** $\operatorname{sftvec}(\mu_m) = s + 2\mathbb{Z}s$**, equiv).** *Let W and W' be wallpaper groups with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$ respectively, where $\angle(l,m) = \pi/2$. Let L and L' be the lattice groups respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L' \cap l'$ and $s' \in L' \cap m'$ be non-zero and of minimal length. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Suppose* $\operatorname{sftvec}(\mu_l) = r + 2\mathbb{Z}r$ *and* $\operatorname{sftvec}(\mu_m) = s + 2\mathbb{Z}s$. *Then $W \sim W'$.*

Theorem 4.59 (>1 reflection, $q = 2$**,** $L = \mathbb{Z}r + \mathbb{Z}s$**,** $\operatorname{sftvec}(\mu_l) = r + 2\mathbb{Z}r$**,** $\operatorname{sftvec}(\mu_m) = 2\mathbb{Z}s$**).** *Let W and W' be wallpaper groups with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$ respectively, where $\angle(l,m) = \pi/2$. Let L and L' be the lattice groups respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L' \cap l'$ and $s' \in L' \cap m'$ be non-zero and of minimal length. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Suppose* $\operatorname{sftvec}(\mu_l) = r + 2\mathbb{Z}r$ *and* $\operatorname{sftvec}(\mu_m) = 2\mathbb{Z}s$. *Let $v, w, v', w' \in \mathbb{R}$ satisfy $v + \mu_l(v) = r$, $w + \mu_m(w) = 0$, $v' + \mu_{l'}(v') = r'$ and $w' + \mu_{m'}(w') = 0$. Define $\lambda \colon L \to L'$ by $mr + ns \mapsto mr' + ns'$ for all $m, n \in \mathbb{Z}$. Define $\eta \colon W \to W'$ by $\tau_u \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j \mapsto \tau_{\lambda}(u) \circ (\tau_{v'} \circ \mu_{l'})^i \circ (\tau_{w'} \circ \mu_{m'} \circ \tau_{v'} \circ \mu_{l'})^j$ for all $i \in \{0,1\}$ and $j \in \{0,1\}$. Then η and $\eta|_T$ are isomorphisms.*

Proof. By Remark 4.4, λ is an isomorphism. Therefore, $\eta|_T$ is an isomorphism. Note $\lambda(r) = r'$ and $\lambda(s) = s'$. By the premises, $\lambda(v + \mu_l(v)) = \lambda(r) = r' = v' + \mu_{l'}(v)$ and $\lambda(w + \mu_m(w)) = 0 = w' + \mu_{m'}(w)$. By Proposition 4.46, $\lambda \circ \mu_l = \mu_{l'} \circ \lambda$ and $\lambda \circ \mu_m = \mu_{m'} \circ \lambda$. By Theorem 4.48, η is an isomorphism. ∎

Corollary 4.60 (>1 reflection, $q = 2$**,** $L = \mathbb{Z}r + \mathbb{Z}s$**,** $\mathrm{sftvec}(\mu_l) = r + 2\mathbb{Z}r$**,** $\mathrm{sftvec}(\mu_m) = 2\mathbb{Z}s$**).** *Let W and W' be wallpaper groups with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$ respectively, where $\angle(l,m) = \pi/2$. Let L and L' be the lattice groups respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L' \cap l'$ and $s' \in L' \cap m'$ be non-zero and of minimal length. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Suppose* $\mathrm{sftvec}(\mu_l) = r + 2\mathbb{Z}r$ *and* $\mathrm{sftvec}(\mu_m) = 2\mathbb{Z}s$*. Then $W \sim W'$.*

Remark 4.61 (>1 reflection, $q = 2$, $L = \mathbb{Z}r + \mathbb{Z}s$, $\mathrm{sftvec}(\mu_l) = 2\mathbb{Z}r$*,* $\mathrm{sftvec}(\mu_m) = s + 2\mathbb{Z}s$*).* Let W and W' be wallpaper groups with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$ respectively, where $\angle(l,m) = \pi/2$. Let L and L' be the lattice groups respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L' \cap l'$ and $s' \in L' \cap m'$ be non-zero and of minimal length. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Let $v, w, v', w' \in \mathbb{R}$ satisfy $v + \mu_l(v) = 0$, $w + \mu_m(w) = s$, $v' + \mu_{l'}(v') = 0$ and $w' + \mu_{m'}(w') = s'$. Define $\lambda\colon L \to L'$ by $mr + ns \mapsto mr' + ns'$ for all $m, n \in \mathbb{Z}$. Define $\eta\colon W \to W'$ by $\tau_u \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j \mapsto \tau_{\lambda(u)} \circ (\tau_{v'} \circ \mu_{l'})^i \circ (\tau_{w'} \circ \mu_{m'} \circ \tau_{v'} \circ \mu_{l'})^j$ for all $i \in \{0,1\}$ and $j \in \{0,1\}$. Then η and $\eta|_T$ are isomorphisms.

Note that this is simply Theorem 4.59 with r and s flipped, and therefore, this can not be counted as a different case.

Remark 4.62. The different cases are indeed not equivalent because of similar reasoning as Proposition 4.37, Proposition 4.38, and Proposition 4.39.

Theorem 4.63. *There are four equivalence classes with point groups containing more than one reflection and rotation subgroups of order 2.*

Proof. This follows from Corollary 4.54, Corollary 4.56, Corollary 4.58, Corollary 4.60, Remark 4.61 and Remark 4.62. ∎

$q = 3$

Lemma 4.64. *Let W be a wallpaper group with $\pi(W) = \langle \mu_l, \mu_m \rangle$, where $\angle(l,m) = \pi/3$. Let $r \in L \cap l$, $s \in L \cap m$ be of minimal length and $\angle(r,s) = \pi/3$. Then either $L = \mathbb{Z}r + \mathbb{Z}s$ or $L = \mathbb{Z}(r+s)/3 + \mathbb{Z}r$.*

Proof. If $L = \mathbb{Z}r + \mathbb{Z}s$, the proof is finished. Assume $L \neq \mathbb{Z}r + \mathbb{Z}s$.

Take $t \in L \setminus (\mathbb{Z}r + \mathbb{Z}s)$. Write $t = ar + bs$, where $a, b \in \mathbb{R}$. We know that $r + \mu_m(r) \in L \cap m$ and $s + \mu_l(s) \in L \cap l$. Hence, $r + \mu_m(r) = js$ and $s + \mu_l(s) = kr$ for some $j, k \in \mathbb{Z}_+$. We know $\|r + \mu_m(r)\| = r$ and $\|s + \mu_l(s)\| = s$. Hence, $|jk| = 1$. Hence, either $j = 1$ and $k = 1$. In other words, $\|r\| = \|s\|$. Then $\mu_m(r) = s - r$ and $\mu_l(s) = r - s$. Then $\mu_m(t) = a(s-r) + bs$ and $\mu_l(t) = ar + b(r-s)$. Then $t + \mu_m(t) = (a+2b)s \in L$ and $t + \mu_l(t) = (2a+b)r \in L$. Then $a + 2b \in \mathbb{Z}$ and $2a + b \in \mathbb{Z}$. Then $3a, 3b \in \mathbb{Z}$, $a - b \in \mathbb{Z}$ and $3a + 3b \in \mathbb{Z}$. Since $a, b \notin \mathbb{Z}$, either $a, b \in 1/3 + \mathbb{Z}$ or $a, b \in 2/3 + \mathbb{Z}$. Hence, $t \in ((r+s)/3 + \mathbb{Z}r + \mathbb{Z}s) \cup (2(r+s)/3 + \mathbb{Z}r + \mathbb{Z}s)$. Hence, $L \setminus (\mathbb{Z}r + \mathbb{Z}s) \subseteq ((r+s)/3 + \mathbb{Z}r + \mathbb{Z}s) \cup (2(r+s)/3 + \mathbb{Z}r + \mathbb{Z}s$. Then $L \subseteq (\mathbb{Z}r + \mathbb{Z}s) \cup ((r+s)/3 + \mathbb{Z}r + \mathbb{Z}s) \cup (2(r+s)/3 + \mathbb{Z}r + \mathbb{Z}s)$.

We know there exist this $t \in ((r+s)/3 + \mathbb{Z}r + \mathbb{Z}s) \cup (2(r+s)/3 + \mathbb{Z}r + \mathbb{Z}s)$, such that $t \in L$. Then either $t \in (r+s)/3 + \mathbb{Z}r + \mathbb{Z}s$ or $t \in 2(r+s)/3 + \mathbb{Z}r + \mathbb{Z}s$. Assume $t \in (r+s)/3 + \mathbb{Z}r + \mathbb{Z}s$. Then $t + \mathbb{Z}r + \mathbb{Z}s \subseteq L$, which means $(r+s)/3 + \mathbb{Z}r + \mathbb{Z}s \subseteq L$. Meanwhile, $-t + \mathbb{Z}r + \mathbb{Z}s \subseteq L$, which means $2(r+s)/3 + \mathbb{Z}r + \mathbb{Z}s \subseteq L$. Hence, $L \supseteq (\mathbb{Z}r + \mathbb{Z}s) \cup ((r+s)/3 + \mathbb{Z}r + \mathbb{Z}s) \cup (2(r+s)/3 + \mathbb{Z}r + \mathbb{Z}s)$. Assume $t \in 2(r+s)/3 + \mathbb{Z}r + \mathbb{Z}s$.

Then $t+\mathbb{Z}r+\mathbb{Z}s \subseteq L$, which means $2(r+s)/3+\mathbb{Z}r+\mathbb{Z}s \subseteq L$. Meanwhile, $-t+\mathbb{Z}r+\mathbb{Z}s \subseteq L$, which means $(r+s)/3+\mathbb{Z}r+\mathbb{Z}s \subseteq L$. Hence, $L \supseteq (\mathbb{Z}r+\mathbb{Z}s)\cup((r+s)/3+\mathbb{Z}r+\mathbb{Z}s)\cup(2(r+s)/3+\mathbb{Z}r+\mathbb{Z}s)$.

In conclusion, $L=(\mathbb{Z}r+\mathbb{Z}s)\cup((r+s)/3+\mathbb{Z}r+\mathbb{Z}s)\cup(2(r+s)/3+\mathbb{Z}r+\mathbb{Z}s)$. We see that $(\mathbb{Z}r+\mathbb{Z}s)\cup((r+s)/3+\mathbb{Z}r+\mathbb{Z}s)\cup(2(r+s)/3+\mathbb{Z}r+\mathbb{Z}s)=\mathbb{Z}(r+s)/3+\mathbb{Z}r$. ∎

Lemma 4.65. *Let W be a wallpaper group with $\pi(W)=\langle\mu_l,\mu_m\rangle$, where $\angle(l,m)=\pi/3$. Let $r\in L\cap l$, $s\in L\cap m$ be of minimal length and $\angle(r,s)=\pi/3$. Then the set of all shift vectors of μ_l is $\mathbb{Z}r$ and the set of all shift vectors of μ_m is $\mathbb{Z}s$.*

Proof. For both cases $L=\mathbb{Z}r+\mathbb{Z}s$ or $L=\mathbb{Z}r+\mathbb{Z}s$. We know $s+\mu_l(s)=r$ and $r+\mu_m(r)=s$. By applying Proposition 4.22 to μ_l, the set of all shift vectors of μ_l is $\mathbb{Z}r$. By applying Proposition 4.22 to μ_m, the set of all shift vectors of μ_m is $\mathbb{Z}s$. ∎

Lemma 4.66. *Let W be a wallpaper group with $\pi(W)=\langle\mu_l,\mu_m\rangle$, where $\angle(l,m)=\pi/3$. Let $r\in L\cap l$, $s\in L\cap m$ be of minimal length and $\angle(r,s)=\pi/3$. Then*

Lemma 4.67 (>1 reflection, $q=3$, cases of $\mathrm{sftvec}(\mu_l)$ and $\mathrm{sftvec}(\mu_m)$). *Let W be a wallpaper group with $\pi(W)=\langle\mu_l,\mu_m\rangle$, where $\angle(l,m)=\pi/3$. Let $r\in L\cap l$, $s\in L\cap m$ be of minimal length and $\angle(r,s)=\pi/3$. Then one of the following cases happens.*

1. *$L=\mathbb{Z}r+\mathbb{Z}s$, the set of all shift vectors of μ_l is $\mathbb{Z}r$ and the set of all shift vectors of μ_m is $\mathbb{Z}s$;*
2. *$L=\mathbb{Z}(r+s)/3+\mathbb{Z}r$, the set of all shift vectors of μ_l is $\mathbb{Z}r$ and the set of all shift vectors of μ_m is $\mathbb{Z}s$.*

Proof. This is a direct consequence of Lemma 4.64 and Lemma 4.65. ∎

Theorem 4.68 (>1 reflection, $q=3$, $L=\mathbb{Z}r+\mathbb{Z}s$, $\mathrm{sftvec}(\mu_l)=\mathbb{Z}r$, $\mathrm{sftvec}(\mu_m)=\mathbb{Z}s$). *Let W and W' be wallpaper groups with point group $\pi(W)=\langle\mu_l,\mu_m\rangle$ and $\pi(W')=\langle\mu_{l'},\mu_{m'}\rangle$ respectively, where $\angle(l,m)=\pi/3$. Let L and L' be the lattice groups respectively. Let $r\in L\cap l$, $s\in L\cap m$, $r'\in L'\cap l'$ and $s'\in L'\cap m'$ be non-zero and of minimal length. Suppose $\|r\|\geq\|s\|$ and $\|r'\|\geq\|s'\|$. Suppose $L=\mathbb{Z}r+\mathbb{Z}s$ and $L'=\mathbb{Z}r'+\mathbb{Z}s'$. Let $v,w,v',w'\in\mathbb{R}$ satisfy $v+\mu_l(v)=0$, $w+\mu_m(w)=0$, $v'+\mu_{l'}(v')=0$ and $w'+\mu_{m'}(w')=0$. Define $\lambda\colon L\to L'$ by $mr+ns\mapsto mr'+ns'$ for all $m,n\in\mathbb{Z}$. Define $\eta\colon W\to W'$ by $\tau_u\circ(\tau_v\circ\mu_l)^i\circ(\tau_w\circ\mu_m\circ\tau_v\circ\mu_l)^j\mapsto\tau_{\lambda}(u)\circ(\tau_{v'}\circ\mu_{l'})^i\circ(\tau_{w'}\circ\mu_{m'}\circ\tau_{v'}\circ\mu_{l'})^j$ for all $i\in\{0,1\}$ and $j\in\{0,1,2\}$. Then η and $\eta|_T$ are isomorphisms.*

Proof. By Remark 4.4, λ is an isomorphism. Therefore, $\eta|_T$ is an isomorphism. Note $\lambda(r)=r'$ and $\lambda(s)=s'$. By the premises, $\lambda(v+\mu_l(v))=0=v'+\mu_{l'}(v)$ and $\lambda(w+\mu_m(w))=0=w'+\mu_{m'}(w)$. By Proposition 4.47, $\lambda\circ\mu_l=\mu_{l'}\circ\lambda$ and $\lambda\circ\mu_m=\mu_{m'}\circ\lambda$. By Theorem 4.48, η is an isomorphism. ∎

Corollary 4.69 (>1 reflection, $q=3$, $L=\mathbb{Z}r+\mathbb{Z}s$, $\mathrm{sftvec}(\mu_l)=\mathbb{Z}r$, $\mathrm{sftvec}(\mu_m)=\mathbb{Z}s$, equiv). *Let W and W' be wallpaper groups with point group $\pi(W)=\langle\mu_l,\mu_m\rangle$ and $\pi(W')=\langle\mu_{l'},\mu_{m'}\rangle$ respectively, where $\angle(l,m)=\pi/3$. Let L and L' be the lattice groups respectively. Let $r\in L\cap l$, $s\in L\cap m$, $r'\in L'\cap l'$ and $s'\in L'\cap m'$ be non-zero and of minimal length. Suppose $\|r\|\geq\|s\|$ and $\|r'\|\geq\|s'\|$. Then $W\sim W'$*

Theorem 4.70 (>1 reflection, $q=3$, $L=\mathbb{Z}(r+s)/3+\mathbb{Z}r$, $\mathrm{sftvec}(\mu_l)=\mathbb{Z}r$, $\mathrm{sftvec}(\mu_m)=\mathbb{Z}s$). *Let W and W' be wallpaper groups with point group $\pi(W)=\langle\mu_l,\mu_m\rangle$ and $\pi(W')=\langle\mu_{l'},\mu_{m'}\rangle$ respectively, where $\angle(l,m)=\pi/3$. Let L and L' be the lattice groups respectively. Let $r\in L\cap l$, $s\in L\cap m$, $r'\in L'\cap l'$ and $s'\in L'\cap m'$ be non-zero and of minimal length. Suppose $\|r\|\geq\|s\|$ and $\|r'\|\geq\|s'\|$. Suppose*

$L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Let $v, w, v', w' \in \mathbb{R}$ satisfy $v + \mu_l(v) = 0$, $w + \mu_m(w) = 0$, $v' + \mu_{l'}(v') = 0$ and $w' + \mu_{m'}(w') = 0$. Define $\lambda\colon L \to L'$ by $m(r+s)/3 + nr \mapsto m(r'+s')/3 + nr'$ for all $m, n \in \mathbb{Z}$. Define $\eta\colon W \to W'$ by $\tau_u \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j \mapsto \tau_{\lambda}(u) \circ (\tau_{v'} \circ \mu_{l'})^i \circ (\tau_{w'} \circ \mu_{m'} \circ \tau_{v'} \circ \mu_{l'})^j$ for all $i \in \{0,1\}$ and $j \in \{0,1,2\}$. Then η and $\eta|_T$ are isomorphisms.

Proof. By Remark 4.4, λ is an isomorphism. Therefore, $\eta|_T$ is an isomorphism. Note $\lambda(r) = r'$ and $\lambda(s) = s'$. By the premises, $\lambda(v + \mu_l(v)) = 0 = v' + \mu_{l'}(v)$ and $\lambda(w + \mu_m(w)) = 0 = w' + \mu_{m'}(w)$. By Proposition 4.47, $\lambda \circ \mu_l = \mu_{l'} \circ \lambda$ and $\lambda \circ \mu_m = \mu_{m'} \circ \lambda$. By Theorem 4.48, η is an isomorphism. ∎

Corollary 4.71 (>1 reflection, $q = 3$, $L = \mathbb{Z}(r+s)/3 + \mathbb{Z}r$, sftvec$(\mu_l) = \mathbb{Z}r$, sftvec$(\mu_m) = \mathbb{Z}s$, equiv). *Let W and W' be wallpaper groups with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$ respectively, where $\angle(l, m) = \pi/3$. Let L and L' be the lattice groups respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L' \cap l'$ and $s' \in L' \cap m'$ be non-zero and of minimal length. Suppose $\|r\| \geq \|s\|$ and $\|r'\| \geq \|s'\|$. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Then $W \sim W'$.*

Proposition 4.72. *Let W and W' be wallpaper groups with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$ respectively, where $\angle(l, m) = \pi/3$. Let L and L' be the lattice groups respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L' \cap l'$ and $s' \in L' \cap m'$ be non-zero and of minimal length. Suppose $\|r\| \geq \|s\|$ and $\|r'\| \geq \|s'\|$. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L = \mathbb{Z}(r'+s')/3 + \mathbb{Z}r'$. Then $W \nsim W'$.*

Proof. Assume $W \sim W'$. Then there exists an isomorphism η with an isomorphic restriction on translation subgroups. Let $v, w \in \mathbb{R}^2$ satisfy $\tau_v \circ \mu_l \in W$, $\tau_w \circ \mu_m \in W$, $(\tau_v \circ \mu_l)^2 = \tau_r$, and $(\tau_w \circ \mu_m)^2 = \tau_s$. Since isomorphism maps generators to generators, we must have at least one of τ_r and τ_s mapped to $\tau_{(r'+s')/3}$. Without loss of generality, assume $\tau_r \mapsto \tau_{(r'+s')/3}$. We know $(r'+s')/3 \notin l \cup m$. Hence, $(r'+s')/3$ is not a shift vector. Hence, there is no $v' \in \mathbb{R}^2$ such that $\tau_{v'} \circ \mu_{l'} \in W$ and $(\tau_{v'} \circ \mu_{l'})^2 = \tau_{(r'+s')/3}$, and there is no $w' \in \mathbb{R}^2$ such that $\tau_{w'} \circ \mu_{m'} \in W$ and $(\tau_{w'} \circ \mu_{m'})^2 = \tau_{(r'+s')/3}$. This is a contradiction, since the isomorphism maps $(\tau_v \circ \mu_l)^2$ to such an element. ∎

Theorem 4.73. *There are two equivalence classes of wallpaper groups with point groups containing more than one reflections and the rotation subgroups of the point groups are of order 3.*

Proof. This follows from Corollary 4.69, Corollary 4.71, and Proposition 4.72. ∎

$q = 4$

Lemma 4.74. *Let W be a wallpaper group with $\pi(W) = \langle \mu_l, \mu_m \rangle$, where $\angle(l, m) = \pi/4$. Let $r \in L \cap l$, $s \in L \cap m$ be of minimal length and $\angle(r, s) = \pi/4$. Then $L = \mathbb{Z}r + \mathbb{Z}s$.*

Proof. If $L = \mathbb{Z}r + \mathbb{Z}s$, the proof is finished. Assume $L \neq \mathbb{Z}r + \mathbb{Z}s$. Take $t \in L \setminus (\mathbb{Z}r + \mathbb{Z}s)$. Write $t = ar + bs$, where $a, b \in \mathbb{R}$. We know that $r + \mu_m(r) \in L \cap m$ and $s + \mu_l(s) \in L \cap l$. Hence, $r + \mu_m(r) = js$ and $s + \mu_l(s) = kr$ for some $j, k \in \mathbb{Z}_+$. We know $\|r + \mu_m(r)\| = \sqrt{2}r$ and $\|s + \mu_l(s)\| = \sqrt{2}s$. Hence, $|jk| = 2$. Hence, either $j = 2$ and $k = 1$, or $j = 1$ and $k = 2$.

Suppose $j = 2$ and $k = 1$. Then $\mu_m(r) = 2s - r$ and $\mu_l(s) = r - s$. Then $\mu_m(t) = a(2s - r) + bs$ and $\mu_l(t) = ar + b(r - s)$. Then $t + \mu_m(t) = (2a + 2b)s \in L$ and $t + \mu_l(t) = (2a + b)r \in L$. Then $2a + 2b \in \mathbb{Z}$ and $2a + b \in \mathbb{Z}$. Then $a, b \in \mathbb{Z}$, contradicts that $t \in L \setminus (\mathbb{Z}r + \mathbb{Z}s)$.

Suppose $j = 1$ and $k = 2$. Then $\mu_m(r) = s - r$ and $\mu_l(s) = 2r - s$. Then $\mu_m(t) = a(s - r) + bs$ and $\mu_l(t) = ar + b(2r - s)$. Then $t + \mu_m(t) = (a + 2b)s \in L$ and $t + \mu_l(t) = (2a + 2b)r \in L$. Then $a + 2b \in \mathbb{Z}$ and $2a + 2b \in \mathbb{Z}$. Then $a, b \in \mathbb{Z}$, contradicts that $t \in L \setminus (\mathbb{Z}r + \mathbb{Z}s)$. ∎

Lemma 4.75 (>1 reflection, $p = 4$, cases of $\mathrm{sftvec}(\mu_l)$ and $\mathrm{sftvec}(\mu_m)$). *Let W be a wallpaper group with $\pi(W) = \langle \mu_l, \mu_m \rangle$, where $\angle(l,m) = \pi/4$. Let $r \in L \cap l$, $s \in L \cap m$ be of minimal length and $\angle(r,s) = \pi/4$. Suppose $\|r\| \geq \|s\|$. Then one of the following situation happens.*

1. *The set of shift vectors of μ_l is $\mathbb{Z}r$, and the set of shift vectors of μ_m is $2\mathbb{Z}s$.*
2. *The set of shift vectors of μ_l is $\mathbb{Z}r$, and the set of shift vectors of μ_m is $s + 2\mathbb{Z}s$.*

Proof. We know that $r + \mu_m(r) \in L \cap m$ and $s + \mu_l(s) \in L \cap l$. Hence, $r + \mu_m(r) = js$ and $s + \mu_l(s) = kr$ for some $j, k \in \mathbb{Z}_+$. We know $\|r + \mu_m(r)\| = \sqrt{2}r$ and $\|s + \mu_l(s)\| = \sqrt{2}s$. Hence, $|jk| = 2$. Hence, either $j = 2$ and $k = 1$, or $j = 1$ and $k = 2$. Since $\|r\| \geq \|s\|$, we have $k = 1$ and $j = 2$. Then $\mu_m(r) = 2s - r$ and $s + \mu_l(s) = r$.

Because $s + \mu_l(s) = r$, by Proposition 4.22, the set of all shift vectors of μ_l is $\mathbb{Z}r$.

Assume there exists $t \in L$ such that $s = t + \mu_m(t)$. Write $t = ar + bs$. Then $\mu_m(t) = a\mu_m(r) + bs = a(2s - r) + bs$. Then $s = t + \mu_m(t) = (2a + 2b)s$. This is impossible since by Lemma 4.74, $a, b \in \mathbb{Z}$. Therefore, there exists no $t \in L$ such that $s = t + \mu_m(t)$. By Proposition 4.23, the set of shift vectors of μ_m is either $2\mathbb{Z}s$ or $s + 2\mathbb{Z}s$. ∎

Theorem 4.76 (>1 reflection, $q = 4$, $L = \mathbb{Z}r + \mathbb{Z}s$, $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$, $\mathrm{sftvec}(\mu_m) = 2\mathbb{Z}s$). *Let W and W' be wallpaper groups with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$ respectively, where $\angle(l,m) = \pi/4$. Let L and L' be the lattice groups respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L' \cap l'$ and $s' \in L' \cap m'$ be non-zero and of minimal length. Suppose $\|r\| \geq \|s\|$ and $\|r'\| \geq \|s'\|$. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Suppose $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$ and $\mathrm{sftvec}(\mu_m) = 2\mathbb{Z}s$. Let $v, w, v', w' \in \mathbb{R}$ satisfy $v + \mu_l(v) = 0$, $w + \mu_m(w) = 0$, $v' + \mu_{l'}(v') = 0$ and $w' + \mu_{m'}(w') = 0$. Define $\lambda \colon L \to L'$ by $mr + ns \mapsto mr' + ns'$ for all $m, n \in \mathbb{Z}$. Define $\eta \colon W \to W'$ by $\tau_u \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j \mapsto \tau_{\lambda(u)} \circ (\tau_{v'} \circ \mu_{l'})^i \circ (\tau_{w'} \circ \mu_{m'} \circ \tau_{v'} \circ \mu_{l'})^j$ for all $i \in \{0,1\}$ and $j \in \{0,1,2,3\}$. Then η and $\eta|_T$ are isomorphisms.*

Proof. By Remark 4.4, λ is an isomorphism. Therefore, $\eta|_T$ is an isomorphism. Note $\lambda(r) = r'$ and $\lambda(s) = s'$. By the premises, $\lambda(v + \mu_l(v)) = 0 = v' + \mu_{l'}(v)$ and $\lambda(w + \mu_m(w)) = 0 = w' + \mu_{m'}(w)$. By Proposition 4.47, $\lambda \circ \mu_l = \mu_{l'} \circ \lambda$ and $\lambda \circ \mu_m = \mu_{m'} \circ \lambda$. By Theorem 4.48, η is an isomorphism. ∎

Corollary 4.77 (>1 reflection, $q = 4$, $L = \mathbb{Z}r + \mathbb{Z}s$, $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$, $\mathrm{sftvec}(\mu_m) = 2\mathbb{Z}s$, equiv). *Let W and W' be wallpaper groups with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$ respectively, where $\angle(l,m) = \pi/4$. Let L and L' be the lattice groups respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L' \cap l'$ and $s' \in L' \cap m'$ be non-zero and of minimal length. Suppose $\|r\| \geq \|s\|$ and $\|r'\| \geq \|s'\|$. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Suppose $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$ and $\mathrm{sftvec}(\mu_m) = 2\mathbb{Z}s$. Then $W \sim W'$.*

Theorem 4.78 (>1 reflection, $q = 4$, $L = \mathbb{Z}r + \mathbb{Z}s$, $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$, $\mathrm{sftvec}(\mu_m) = s + 2\mathbb{Z}s$). *Let W and W' be wallpaper groups with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$ respectively, where $\angle(l,m) = \pi/4$. Let L and L' be the lattice groups respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L' \cap l'$ and $s' \in L' \cap m'$ be non-zero and of minimal length. Suppose $\|r\| \geq \|s\|$ and $\|r'\| \geq \|s'\|$. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Suppose $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$ and $\mathrm{sftvec}(\mu_m) = s + 2\mathbb{Z}s$. Let $v, w, v', w' \in \mathbb{R}$ satisfy $v + \mu_l(v) = 0$, $w + \mu_m(w) = s$, $v' + \mu_{l'}(v') = 0$ and $w' + \mu_{m'}(w') = s'$. Define $\lambda \colon L \to L'$ by $mr + ns \mapsto mr' + ns'$ for all $m, n \in \mathbb{Z}$. Define $\eta \colon W \to W'$ by $\tau_u \circ (\tau_v \circ \mu_l)^i \circ (\tau_w \circ \mu_m \circ \tau_v \circ \mu_l)^j \mapsto \tau_{\lambda(u)} \circ (\tau_{v'} \circ \mu_{l'})^i \circ (\tau_{w'} \circ \mu_{m'} \circ \tau_{v'} \circ \mu_{l'})^j$ for all $i \in \{0,1\}$ and $j \in \{0,1,2,3\}$. Then η and $\eta|_T$ are isomorphisms.*

Proof. By Remark 4.4, λ is an isomorphism. Therefore, $\eta|_T$ is an isomorphism. Note $\lambda(r) = r'$ and $\lambda(s) = s'$. By the premises, $\lambda(v + \mu_l(v)) = 0 = v' + \mu_{l'}(v)$ and $\lambda(w + \mu_m(w)) = \lambda(s) = s' = w' + \mu_{m'}(w)$. By Proposition 4.47, $\lambda \circ \mu_l = \mu_{l'} \circ \lambda$ and $\lambda \circ \mu_m = \mu_{m'} \circ \lambda$. By Theorem 4.48, η is an isomorphism. ∎

Corollary 4.79 (>1 reflection, $q = 4$, $L = \mathbb{Z}r + \mathbb{Z}s$, $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$, $\mathrm{sftvec}(\mu_m) = s + 2\mathbb{Z}s$, equiv). *Let W and W' be wallpaper groups with point group $\pi(W) = \langle \mu_l, \mu_m \rangle$ and $\pi(W') = \langle \mu_{l'}, \mu_{m'} \rangle$ respectively, where $\angle(l,m) = \pi/4$. Let L and L' be the lattice groups respectively. Let $r \in L \cap l$, $s \in L \cap m$, $r' \in L' \cap l'$ and $s' \in L' \cap m'$ be non-zero and of minimal length. Suppose $\|r\| \geq \|s\|$ and $\|r'\| \geq \|s'\|$. Suppose $L = \mathbb{Z}r + \mathbb{Z}s$ and $L' = \mathbb{Z}r' + \mathbb{Z}s'$. Suppose $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$ and $\mathrm{sftvec}(\mu_m) = s + 2\mathbb{Z}s$. Then $W \sim W'$.*

Remark 4.80. The different cases are indeed not equivalent because of similar reasoning as Proposition 4.37.

Theorem 4.81. *There are two equivalence classes of wallpaper groups with point groups containing more than one reflections and the rotation subgroup of the point groups are of order 4.*

Proof. This follows from Corollary 4.77, Corollary 4.79, and Remark 4.80. ∎

$q = 6$

Lemma 4.82. *Let W be a wallpaper group with $\pi(W) = \langle \mu_l, \mu_m \rangle$, where $\angle(l,m) = \pi/6$. Let $r \in L \cap l$, $s \in L \cap m$ be of minimal length and $\angle(r,s) = \pi/6$. Then $L = \mathbb{Z}r + \mathbb{Z}s$.*

Proof. If $L = \mathbb{Z}r + \mathbb{Z}s$, the proof is finished.

Assume $L \neq \mathbb{Z}r + \mathbb{Z}s$. Take $t \in L \setminus (\mathbb{Z}r + \mathbb{Z}s)$. Write $t = ar + bs$, where $a, b \in \mathbb{R}$. We know that $r + \mu_m(r) \in L \cap m$ and $s + \mu_l(s) \in L \cap l$. Hence, $r + \mu_m(r) = js$ and $s + \mu_l(s) = kr$ for some $j, k \in \mathbb{Z}_+$. We know $\|r + \mu_m(r)\| = \sqrt{3}r$ and $\|s + \mu_l(s)\| = \sqrt{3}s$. Hence, $|jk| = 3$. Hence, either $j = 3$ and $k = 1$, or $j = 1$ and $k = 3$.

Assume $j = 3$ and $k = 1$. Then $\mu_m(r) = 3s - r$ and $\mu_l(s) = r - s$. Then $\mu_m(t) = a(3s - r) + bs$ and $\mu_l(t) = ar + b(r - s)$. Then $t + \mu_m(t) = (3a + 2b)s \in L$ and $t + \mu_l(t) = (2a + b)r \in L$. Then $3a + 2b \in \mathbb{Z}$ and $2a + b \in \mathbb{Z}$. Then $a, b \in \mathbb{Z}$, contradicts that $t \in L \setminus (\mathbb{Z}r + \mathbb{Z}s)$.

Assume $j = 1$ and $k = 3$. Then $\mu_m(r) = s - r$ and $\mu_l(s) = 3r - s$. Then $\mu_m(t) = a(s - r) + bs$ and $\mu_l(t) = ar + b(3r - s)$. Then $t + \mu_m(t) = (a + 2b)s \in L$ and $t + \mu_l(t) = (2a + 3b)r \in L$. Then $a + 2b \in \mathbb{Z}$ and $2a + 3b \in \mathbb{Z}$. Then $a, b \in \mathbb{Z}$, contradicts that $t \in L \setminus (\mathbb{Z}r + \mathbb{Z}s)$.

Therefore, $L = \mathbb{Z}r + \mathbb{Z}s$. ∎

Lemma 4.83 (>1 reflection, $p = 6$, cases of $\mathrm{sftvec}(\mu_l)$ and $\mathrm{sftvec}(\mu_m)$). *Let W be a wallpaper group with $\pi(W) = \langle \mu_l, \mu_m \rangle$, where $\angle(l,m) = \pi/6$. Let $r \in L \cap l$, $s \in L \cap m$ be of minimal length and $\angle(r,s) = \pi/6$. Then the sets of shift vectors of μ_l is $\mathbb{Z}r$, and the sets of shift vectors of μ_m is $\mathbb{Z}s$.*

Proof. We know that $r + \mu_m(r) \in L \cap m$ and $s + \mu_l(s) \in L \cap l$. Hence, $r + \mu_m(r) = js$ and $s + \mu_l(s) = kr$ for some $j, k \in \mathbb{Z}_+$. We know $\|r + \mu_m(r)\| = \sqrt{3}r$ and $\|s + \mu_l(s)\| = \sqrt{3}s$. Hence, $|jk| = 3$. Hence, either $j = 3$ and $k = 1$, or $j = 1$ and $k = 3$.

Assume $j = 3$ and $k = 1$. Then $s + \mu_l(s) = r$ and $\mu_l(s) + \mu_m(\mu_l(s)) = s$. By Proposition 4.22, the sets of shift vectors of μ_l is $\mathbb{Z}r$, and the sets of shift vectors of μ_m is $\mathbb{Z}s$.

Assume $j = 1$ and $k = 3$. Then $r + \mu_m(r) = s$ and $\mu_m(r) + \mu_l(\mu_m(r)) = r$. By Proposition 4.22, the sets of shift vectors of μ_l is $\mathbb{Z}r$, and the sets of shift vectors of μ_m is $\mathbb{Z}s$. ∎

Theorem 4.84 (>1 reflection, $q=6$**,** $L=\mathbb{Z}r+\mathbb{Z}s$**,** $\text{sftvec}(\mu_l)=\mathbb{Z}r$**,** $\text{sftvec}(\mu_m)=\mathbb{Z}s$**).** *Let W and W' be wallpaper groups with point group $\pi(W)=\langle\mu_l,\mu_m\rangle$ and $\pi(W')=\langle\mu_{l'},\mu_{m'}\rangle$ respectively, where $\angle(l,m)=\pi/6$. Let L and L' be the lattice groups respectively. Let $r\in L\cap l$, $s\in L\cap m$, $r'\in L'\cap l'$ and $s'\in L'\cap m'$ be non-zero and of minimal length. Suppose $\|r\|\geq\|s\|$ and $\|r'\|\geq\|s'\|$. Suppose $L=\mathbb{Z}r+\mathbb{Z}s$ and $L'=\mathbb{Z}r'+\mathbb{Z}s'$. Let $v,w,v',w'\in\mathbb{R}$ satisfy $v+\mu_l(v)=0$, $w+\mu_m(w)=0$, $v'+\mu_{l'}(v')=0$ and $w'+\mu_{m'}(w')=0$. Define $\lambda\colon L\to L'$ by $mr+ns\mapsto mr'+ns'$ for all $m,n\in\mathbb{Z}$. Define $\eta\colon W\to W'$ by $\tau_u\circ(\tau_v\circ\mu_l)^i\circ(\tau_w\circ\mu_m\circ\tau_v\circ\mu_l)^j\mapsto\tau_{\lambda}(u)\circ(\tau_{v'}\circ\mu_{l'})^i\circ(\tau_{w'}\circ\mu_{m'}\circ\tau_{v'}\circ\mu_{l'})^j$ for all $i\in\{0,1\}$ and $j\in\{0,1,2,3,4,5\}$. Then η and $\eta|_T$ are isomorphisms.*

Proof. By Remark 4.4, λ is an isomorphism. Therefore, $\eta|_T$ is an isomorphism. Note $\lambda(r)=r'$ and $\lambda(s)=s'$. By the premises, $\lambda(v+\mu_l(v))=0=v'+\mu_{l'}(v)$ and $\lambda(w+\mu_m(w))=0=w'+\mu_{m'}(w)$. By Proposition 4.47, $\lambda\circ\mu_l=\mu_{l'}\circ\lambda$ and $\lambda\circ\mu_m=\mu_{m'}\circ\lambda$. By Theorem 4.48, η is an isomorphism. ∎

Corollary 4.85 (>1 reflection, $q=6$**,** $L=\mathbb{Z}r+\mathbb{Z}s$**,** $\text{sftvec}(\mu_l)=\mathbb{Z}r$**,** $\text{sftvec}(\mu_m)=\mathbb{Z}s$**, equiv).** *Let W and W' be wallpaper groups with point group $\pi(W)=\langle\mu_l,\mu_m\rangle$ and $\pi(W')=\langle\mu_{l'},\mu_{m'}\rangle$ respectively, where $\angle(l,m)=\pi/6$. Then $W\sim W'$*

Theorem 4.86. *There exists only one equivalence class of wallpaper groups with point groups containing more than two reflections and rotation subgroups of order 6.*

Proof. This follows from Corollary 4.85. ∎

Theorem 4.87 (17 wallpaper groups). *There exists seventeen equivalence classes of wallpaper groups.*

Proof. This follows from Theorem 4.15, Theorem 4.40, Theorem 4.63, Theorem 4.73, Theorem 4.81, and Theorem 4.86. ∎

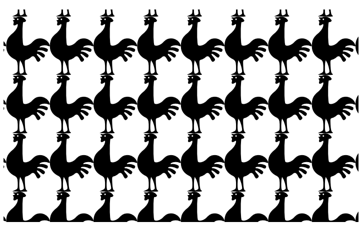
5 Examples of Wallpaper Patterns

In this appendix, we present a corresponding wallpaper pattern to each wallpaper group equivalence class. The table before each group of illustrations shows the correspondence between the theorems and the wallpaper patterns. A good explanation of the notation to each wallpaper group equivalence class can be found in Schattschneider [1978].

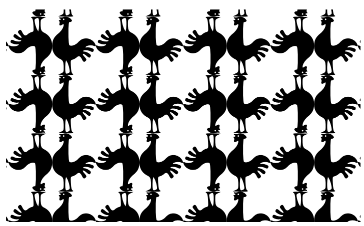
No reflection The following illustrations are wallpaper patterns corresponding to wallpaper groups with no reflection.

Table 5.1: No reflection

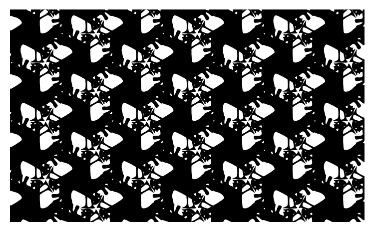
Notation	Description
p1	0 reflection, $q = 1$.
p2	0 reflection, $q = 2$.
p3	0 reflection, $q = 3$.
p4	0 reflection, $q = 4$.
p6	0 reflection, $q = 6$.



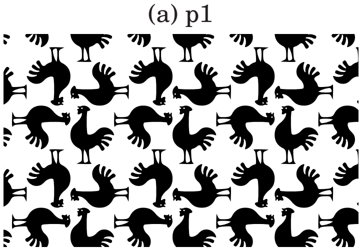
(a) p1



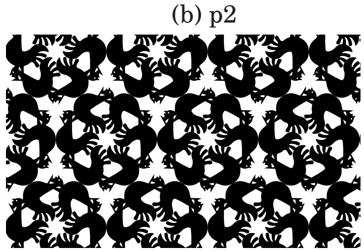
(b) p2



(c) p3



(d) p4



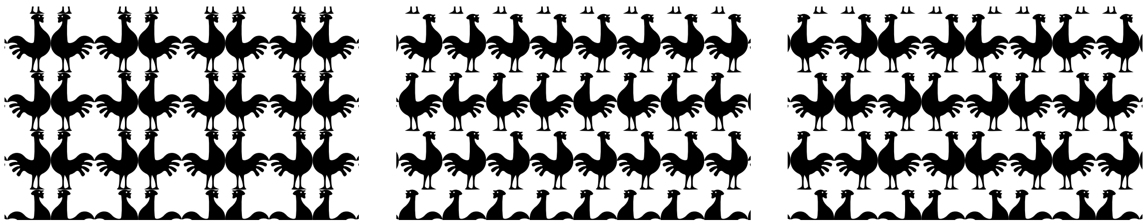
(e) p6

Figure 5.1: No reflection

One reflection The following illustrations are wallpaper patterns corresponding to wallpaper groups with only one reflection.

Table 5.2: One reflection

Notation	Description
pm	0 reflection, $q = 1$, $\mathrm{sftvec}(\mu_l) = 2\mathbb{Z}r$.
pg	0 reflection, $q = 1$, $\mathrm{sftvec}(\mu_l) = r + \mathbb{Z}r$.
cm	0 reflection, $q = 1$, $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$.



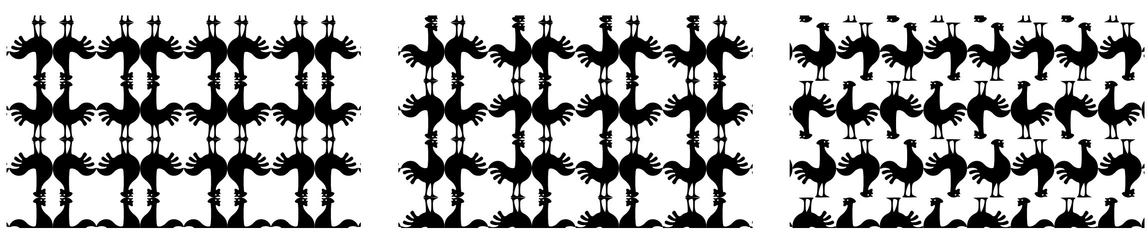
(a) pm (b) pg (c) cm

Figure 5.2: One reflection

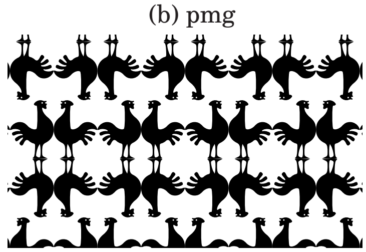
More than one reflection The following illustrations are wallpaper patterns corresponding to wallpaper groups with more than one reflection.

Table 5.3: More than one reflections, $q = 2$

Notation	Description
pmm	>1 reflection, $q = 2$, $L = \mathbb{Z}r + \mathbb{Z}s$, $\mathrm{sftvec}(\mu_l) = 2\mathbb{Z}r$, $\mathrm{sftvec}(\mu_m) = 2\mathbb{Z}s$.
pmg	>1 reflection, $q = 2$, $L = \mathbb{Z}r + \mathbb{Z}s$, $\mathrm{sftvec}(\mu_l) = r + \mathbb{Z}r$, $\mathrm{sftvec}(\mu_m) = 2\mathbb{Z}s$.
pgg	>1 reflection, $q = 2$, $L = \mathbb{Z}r + \mathbb{Z}s$, $\mathrm{sftvec}(\mu_l) = r + 2\mathbb{Z}r$, $\mathrm{sftvec}(\mu_m) = s + 2\mathbb{Z}s$.
cmm	>1 reflection, $q = 2$, $L = \mathbb{Z}(r+s)/2 + \mathbb{Z}r$, $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$, $\mathrm{sftvec}(\mu_m) = \mathbb{Z}s$.
p31m	>1 reflection, $q = 3$, $L = \mathbb{Z}r + \mathbb{Z}s$, $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$, $\mathrm{sftvec}(\mu_m) = \mathbb{Z}s$.
p3m1	>1 reflection, $q = 3$, $L = \mathbb{Z}(r+s)/3 + \mathbb{Z}r$, $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$, $\mathrm{sftvec}(\mu_m) = \mathbb{Z}s$.
p4m	>1 reflection, $q = 4$, $L = \mathbb{Z}r + \mathbb{Z}s$, $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$, $\mathrm{sftvec}(\mu_m) = 2\mathbb{Z}s$.
p4g	>1 reflection, $q = 4$, $L = \mathbb{Z}r + \mathbb{Z}s$, $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$, $\mathrm{sftvec}(\mu_m) = s + 2\mathbb{Z}s$.
p6m	>1 reflection, $q = 6$, $L = \mathbb{Z}r + \mathbb{Z}s$, $\mathrm{sftvec}(\mu_l) = \mathbb{Z}r$, $\mathrm{sftvec}(\mu_m) = \mathbb{Z}s$.

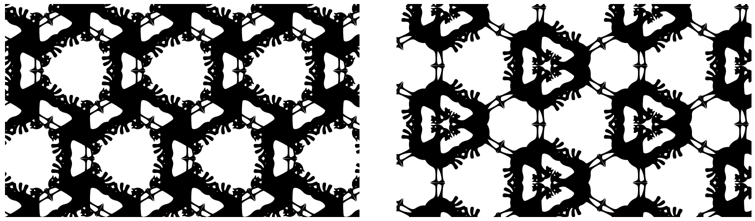


(a) pmm (b) pmg (c) pgg



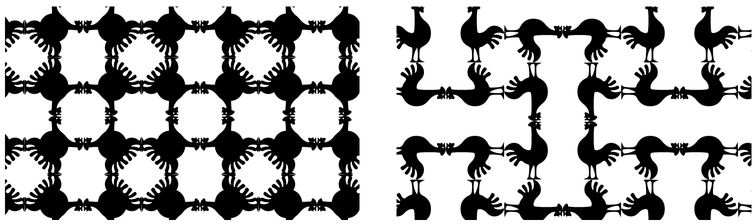
(d) cmm

Figure 5.3: $q = 2$



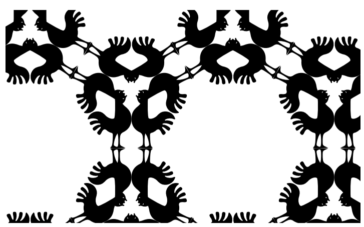
(a) p31m (b) p3m1

Figure 5.4: $q = 3$



(a) p4m (b) p4g

Figure 5.5: $q = 4$



(a) p6m

Figure 5.6: $q = 6$

6 Discussion and Conclusion

In this manuscript we have reproduced Schwarzenberger's proof in detail. We first recalled the notions of plane symmetries and their group structure. We proved the unique expression of a plane isometry, the classification of linear isometries, and classification of plane isometries. We recalled the development of the definition of wallpaper groups. we recalled the definition of shift vectors and related results for later use. We recalled the definition of equivalence of wallpaper groups and analyzed its consequence. We reproduced the proof of the so-called crystallographic restriction. We then separated the possible cases. In particular the point group of a wallpaper groups can contain no reflection, one reflection, or more than one reflection. We proved that when the point group contains no reflection, there are five possibilities, under each possibility the wallpaper groups are equivalent, and the wallpaper groups from different equivalence classes are indeed not equivalent. We then did the same for the case that the point group contains no reflection, and the case that the point group contains more than one reflection. In the end, we conclude that there are indeed seventeen wallpaper group equivalence classes.

This proof should be able to stimulate a recognition procedure. Seeing a wallpaper pattern, we first look for reflections / glide reflections to see how many reflections are in the point group, then we look for rotation orders, then we distinguish the lattice basis, then we look for non-trivial shift vectors (the case that the shift vectors are odd multiples of the shortest translation vector on the reflection axis).

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