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Tsirelson's space

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1. Introduction

Banach spaces arose in the beginning of the 20th century out of the study of function spaces. Classical examples are the spaces of all continuous, differentiable or integrable functions. Those classical Banach spaces shared the property that they all have subspaces isomorphic to c_0 or l_p for $1 \leq p < \infty$.

If this held true in the general case, so that every Banach space had subspaces isomorphic to either c_0 or ℓ_p, $1 \leq p < \infty$, then at least those subspaces had rather simple and intuitive properties. Some findings further gave hope in the direction of isomorphs of ℓ_p or c_0 existing in every Banach space, e.g. as given by [Casazza and Shura, 1989]

- Any Banach space contains a subspace isomorphic to c_0 if and only if X contains a sequence $\{x_n\}_{n=1}^{\infty}$ such that $\sum_n x_n$ does not converge with $\sum_n |x^*(x_n)| < \infty$ for all x^* in the dual X^* of X, and
- Every bounded sequence in a Banach space has a subsequence which is either weakly Cauchy or equivalent to the unit vector basis of ℓ_1.

In 1974, the Russian-Israeli mathematician Boris Tsirelson constructed a space as a counterexample to those "classical" Banach spaces showing that one can build a Banach space, which is reflexive *and* finitely universal. By having chosen those specific properties, and proving that there indeed exists such a Banach space, Tsirelson managed to force the non-existence of c_0 and ℓ_1 isomorphics on the one hand and of subspaces isomorphic to ℓ_p for $1 < p < \infty$ on the other.

Mentions of Tsirelson spaces show up in different fields of mathematics using Banach space, as Tsirelson gave an example to a new form of Banach spaces. Even though often the Tsirelson space constructed by [Figiel and Johnson, 1974] is the space of interest instead of its dual, the original space by Tsirelson, the space is referred to in geometry on Banach spaces for example with regard to distortion, but also in the applied field of computer aided mathematics to compute the norm and in the study of polynomial functions on Banach spaces.

In this paper, Tsirelson's original construction will be retraced and proved in greater detail and an alternative counterexample based on his space will be introduced.

We do so by first examining isomorphs of ℓ_p for $1 < p < \infty$. As we will show, ℓ_p are uniformly convex, giving us a way of ensuring that there are no isomorphs of ℓ_p for $1 < p < \infty$, i.e. find a Banach space of which all infinite-dimensional subspaces are not uniformly convex. For that, we present the definition of finite universality, which implies not being uniformly convex.
Then, we will introduce four properties from which, at the end of this paper, we will conclude reflexivity, among other implications. Also, here we prove the existence of a subset of c_0 with the afore mentioned properties which are share with its closed convex hull as well.
Having now hinted the use of reflexivity, we will see that reflexivity does imply that there are no isomorphs of neither c_0 nor ℓ_1 contained in a Banach space. So, we bring reflexivity and finite universality together to get a Banach space without isomorphs of c_0 or ℓ_p for $1 \leq p < \infty$.
Finally, we prove the existence of a Banach space of Tsirelson type, that is confirming that

> *There exists a reflexive Banach space in which each infinite-dimensional subspace is finitely universal.*

2. Finitely universal spaces

In this section, we show that the property of finite universality implies that there is no uniform convexity. This is key to the construction of the Tsirelson space as we will later provide the link between uniform convexity and isomorphism to ℓ_p for $1 < p < \infty$. Hence, if we manage to find a Banach space of which every infinite-dimensional subspace is finitely universal, then we can conclude that said Banach space does not have a subspace isomorphic to ℓ_p for $1 < p < \infty$. That leaves only the matter of c_0 and ℓ_1 which is discussed in the other sections of this paper.

Definition 2.1 (Finite universality)
Let X be a Banach space. Then, X is called finite universal, if there exists a constant $C \geq 1$ such that for each finitely dimensional normed space E there exist a subspace $F \subset X$ of the same dimension as E and an invertible operator $T : E \to F$ such that $\|T\| \cdot \|T^{-1}\| \leq C$.

Remark. *We can restrict ourselves to spaces E of the form ℓ_∞^N, the N-dimensional space with maximum norm, without loss of generality, since every finite-dimensional space E can be ϵ-isometrically embedded in ℓ_∞^N by choosing linear functionals $f_1, \ldots, f_N \in E^*$ with $\max_{1\leq i\leq N} |f_i(x)| \leq \|x\| \leq (1+\epsilon) \max_{1\leq i\leq N} |f_i(x)|$ since we can define the appropriate operator by $Ux = (f_1(x), \ldots, f_N(x))$ then.*

Definition 2.2 (Uniformly convex)
A vector space X with norm $\|\cdot\|$ is called uniformly convex if for every $0 < \epsilon \leq 1$ there exists some $0 < \delta < 1$ such that for vectors $x, y \in X$ with $\|x\| = 1 = \|y\|$

$$\left\| \frac{x-y}{2} \right\| \geq \epsilon \Rightarrow \left\| \frac{x+y}{2} \right\| \leq \delta.$$

Remark. *Note, that ℓ_1 is not for $p = 1$. To show the latter, suppose ℓ_1 is uniformly convex. Then for $x, y \in \ell_1$ with $x \neq y$ and $\|x\| = 1 = \|y\|$, we know that $\|\frac{x+y}{2}\| < 1$. Now, choose $x = (1, 0, 0, \ldots), y = (0, 1, 0, \ldots) \in \ell_1$, which are clearly $\|x\| = 1 = \|y\|$. But we have $\|\frac{x+y}{2}\| = 1$. Contradiction. ℓ_1 is not uniformly convex.*

Proposition 2.3. *If a space is uniformly convex, then it is not finitely universal.*

Proof. We prove this by contradiction. Suppose that X is a uniformly convex space that is indeed finitely universal with a constant $C \geq 1$.
As X is uniformly convex, there exists $\delta > 0$ for $1/C$ such that for $\|x\| = \|y\| = 1$ and $\|(x-y)/2\| \geq 1/C$, it holds that $\|(x+y)/2\| \leq \delta$. Also, following that X is finitely universal, $\|T\| \cdot \|T^{-1}\| \leq C$. Using induction, we will show by contradiction that the opposite is true, leading to an overall contradiction for the premise that X is uniformly convex and finitely universal. For that, we show that in fact $\|T\| \cdot \|T^{-1}\| > \min(C, \delta^{2-N})$.
For $n = 1$, the base step is clear as

$$\|T\| \cdot \|T^{-1}\| \geq \|T\| \cdot \frac{1}{\|T\|} = 1 > \min(C, \delta^{2-1}) = \delta.$$

Choose U to be the restriction of T in ℓ_∞^N to ℓ_∞^{N-1} with ℓ_∞^{N-1} being canonically injected into ℓ_∞^N. Suppose also, that $\|U\| \cdot \|U^{-1}\| > \min(C, \delta^{2-(N-1)}$ holds and that the same does not hold for $\|T\| \cdot \|T^{-1}\|$, so $\|T\| \cdot \|T^{-1}\| \leq \min(C, \delta^{2-N})$. Then, for each $z \in \ell_\infty^{N-1}$ with $\|z\| \leq 1$, there are

$x, y \in \ell_\infty^N$ with $z = (x+y)/2$ for $\|x\|, \|y\| \le 1$ and $\|(x-y)/2\| \ge 1$. But by uniform convexity, this means that $\|U\| \le \delta\|T\|$ and thus $\|U\| \cdot \|U^{-1}\| \le \min(C, \delta^{2-N+1})$.

Claim. *Given U and T as above in Proposition 2.3, we have $\|U\| \le \delta\|T\|$.*

Proof of claim. Note, that $\|U\| = \sup_{\|u\|\le 1} \|U(u)\|$. As ℓ_∞^{N-1} is canonically embedded into ℓ_∞^N we can write every $\|u\| \le 1$ in ℓ_∞^{N-1} as $u = (x+y)/2$ for $x, y \in \ell_p^N$ with $\|x\|, \|y\| \le 1$ and $\|(x-y)/2\| \ge 1 \ge 1/C$ since $C > 1$. Since a subspace of a uniformly convex space in uniformly convex, we get that $\|(x+y)/2\| = \|u\| \le \delta \le 1$. Then,

$$\|U\| = \sup_{\|u\|_{N-1}\le 1} |U(u)| \le \sup_{\|u\|_N \le \delta} |T(u)| = \sup_{\|u\|_N\le 1} |T(\delta u)| = \delta\|T\|$$

with the index N of $\|\cdot\|_N$ indicating the dimension of the finite-dimensional normed space.

This is a contradiction for this step in the induction.

So, by this mathematical induction, we have shown that $\|T\| \cdot \|T^{-1}\| > \min(C, \delta^{2-N})$. But as the space X is finitely universal, so there has to be some invertible operator T with $\|T\| \cdot \|T^{-1}\| \le C$. We can choose N sufficiently large, such that $\min(C, \delta^{2-N}) = C$, to get another contradiction, this time to finite universality. ∎

The following inequality by Clarkson ([Clarkson, 1936]) will be used gain an inequality to estimate the bounds needed for uniform convexity of ℓ_p with $1 < p < \infty$. We make use of the fact that $\frac{1}{p} + \frac{1}{q} = 1$, so it suffices to proof the Clarkson inequality for $1 < p < \infty$. For the case of $1 < p \le 2$, we can, when proving uniform convexity, apply the inequality on $q = p/(p-1) \ge 2$ and then in the last step rewrite the result in terms of p.

Lemma 2.4 (Clarkson inequality)**.** *The following inequality holds for ℓ_p with $p \ge 2$ with x, y arbitrary elements and $\frac{1}{p} + \frac{1}{q} = 1$*

$$\|x+y\|^p + \|x-y\|^p \le 2^{p-1}(\|x\|^p + \|y\|^p).$$

Proof. This proof is the combination of the following two claims:

Claim. *For non-negative $a, b, p, q \in \mathbb{R}$ with $1 = \frac{1}{p} + \frac{1}{q}$ and $p \ge 2$, we have*

$$2\left(a^q + b^q\right)^{p-1} \le 2^{p-1}\left(a^p + b^p\right).$$

Proof of claim. Without loss of generality, we can assume that $a \le b$ with non-zero b (otherwise we get the trivial case $a = b = 0$). Note, that $p = q(p-1)$. Then dividing by $b^p = b^{q(p-1)}$

$$\begin{aligned}
&2\left(a^q + b^q\right)^{p-1} \le 2^{p-1}\left(a^p + b^p\right) \\
\Leftrightarrow\ &\left(c^q + 1\right)^{p-1} \le 2^{p-1}\left(c^p + 1\right) && \left(\text{with } c := \tfrac{a}{b}\right) \\
\Leftrightarrow\ &1 \le 2^{p-2}\frac{c^p + 1}{\left(c^q + 1\right)^{p-1}} \\
\Leftrightarrow\ &1 \le 2^{(p-2)/p}\frac{\left(c^p + 1\right)^{1/p}}{\left(c^q + 1\right)^{1/q}} =: f(c). && (\text{by taking to the power of } 1/p)
\end{aligned}$$

First, we see that $f(1) = 1$ as with $q = p/(p-1)$

$$f(1) = 2^{(p-2)/p} \cdot \frac{2^{1/p}}{2^{1/q}} = 2^{1/p+(p-2)/p-(p-1)/p} = 2^0 = 1.$$

The derivative $f'(c)$ is with $0 < c \leq 1$ (since $a \leq b$)

$$f'(c) = 2^{(p-2)/p} \cdot c^{p-1}(c^p+1)^{1/p-1}(c^q+1)^{-1/q} - c^{q-1}(c^p+1)^{1/p}(c^q+1)^{-1/q-1} < 0$$

is decreasing for $0 < c \to 1$, hence $f(c) \geq 1$ for all $0 < c < 1$. As we have only used equivalences in our argument, the claim is proved.

What is left is to show that $2\left(\|x\|^q + \|y\|^q\right)^{p-1}$ is an upper bound for $\|x+y\|^p + \|x-y\|^p$.

Claim. *For $x, y \in \mathbb{K} = \mathbb{C}$ and $p \geq 2$, we get*

$$|x+y|^p + |x-y|^p \leq 2(|x|^q + |y|^q)^{p-1}.$$

Proof of claim. We can again assume $|x| \geq |y|$ and x non-zero without loss of generality. Also, we define $c := y/x$ (as we assume x to be non-zero) therefore $0 \leq |c| \leq 1$ (note $|c| = |re^{i\varphi}| = |r|$ for some real r and angle φ). Since $q = p/(p-1)$, we divide the inequality in the claim by $|x|^p$ so we get

$$|1+c|^p + |1-c|^p \leq 2\left(|x|^{q-(p/(p-1))} + \frac{|y|^q}{|x|^{p/(p-1)}}\right)^{p-1} = 2(1+|c|^q)^{p-1}.$$

Following Clarkson's proof, we can transform the inequality with $c = (1-z)/(1+z)$ for $0 < z < 1$ (the cases $z = 0$ and $z = 1$ are obvious) and expand the result

$$T := \frac{1}{2}\left((1+z)^q + (1-z)^q\right) - (1+z^p)^{p-1} \geq 0$$

using a Taylor series around zero. As the odd derivatives are vanish at zero for $(1+z)^q + (1-z)^q$, we get the terms

$$\begin{aligned}\frac{1}{2}((1+z)^q + (1-z)^q) =& 1 + \frac{q(q-1)}{2!}z^2 + \frac{q(q-1)(2-q)(3-q)}{4!}z^4 + \cdots \\ &+ \frac{q(q-1)(2-q)\cdots(2k-1-q)}{(2k)!}z^{2k} + \cdots\end{aligned}$$

and

$$\begin{aligned}(1+z^p)^{q-1} =& 1 + (q-1)z^p - \frac{(q-1)(2-q)}{2!}z^{2q} + \cdots \\ &+ \frac{(q-1)(2-q)\ldots(2k-1-q)}{(2k-1)!}z^{2k-1}p \\ &- \frac{(q-1)(2-q)\ldots(2k-q)}{(2k)!}z^{2kp} + \cdots.\end{aligned}$$

T becomes then

$$\begin{aligned}T = \sum_{i=1}^{\infty} \Bigg[&\frac{q(q-1)(2-q)\cdots(2k-1-q)}{(2k)!} z^{2k} \\ &- \frac{(q-1)(2-q)\cdots(2k-1-q)}{(2k-1)!} z^{(2k-1)p} \\ &+ \frac{(q-1)(2-q)\cdots(2k-q)}{(2k)!} z^{2kp} \Bigg] \\ = \sum_{i=1}^{\infty} &\frac{(2-q)(3-q)\cdots(2k-q)}{(2k-1)!} z^{2k} \left[\frac{1-z^{(2k-q)/(q-1)}}{(2k-q)/(q-1)} - \frac{1-z^{2k/(q-1)}}{2k/(q-1)} \right].\end{aligned}$$

The last part is non-negative as $(1-z^t)/t$ for $0 < z < 1$ and positive t is decreasing, therefore T as a whole is non-negative.

We can now build elements of ℓ_p for $p \geq 2$ by taking elements $x_1, x_2, \dots \in \mathbb{K}$ and $y_1, y_2, \dots \in \mathbb{K}$ to form sets $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots) \in \ell_p$. As we have $p \geq 2$, we can use the fact that $p \geq q$ as $\frac{1}{p} + \frac{1}{q} = 1$ to apply the Minkowski inequality which states for $1 \leq p' < \infty$ and $a, b \in \ell_{p'}$ with $p' = p/q$ it holds that

$$\left(\sum_{i=1}^{\infty} |a_i + b_i|^{p'} \right)^{1/p'} \leq \left(\sum_{i=1}^{\infty} |a_i|^{p'} \right)^{1/p'} + \left(\sum_{i=1}^{\infty} |b_i|^{p'} \right)^{1/p'}$$

by choosing $a_i = |x_i + y_i|^q$ and $b_i = |x_i - y_i|^q$. We get

$$\left[\sum_{i=1}^{\infty} |x_i + y_i|^q \right]^{p/q} + \left[\sum_{i=1}^{\infty} |x_i - y_i|^q \right]^{p/q} \leq \left[\sum_{i=1}^{\infty} (|x_i + y_i|^p + |x_i - y_i|^p)^{q/p} \right]^{p/q}.$$

By the previous claim, for the right-hand side it holds that

$$\begin{aligned}\left[\sum_{i=1}^{\infty} (|x_i + y_i|^p + |x_i - y_i|^p)^{q/p} \right]^{p/q} &\leq \left[\sum_{i=1}^{\infty} (2(|x_i|^q + |y_i|^q)^{p/q})^{q/p} \right]^{p/q} \\ &= \left[\sum_{i=1}^{\infty} 2^{p/q}(|x_i|^q + |y_i|^q) \right]^{p/q} \\ &= 2^{p-1} \left(\sum_{i=1}^{\infty} |x_i|^q + |y_i|^q \right)^{p/q} \\ &\leq 2^{p-1}(\|x\|^p + \|y\|^p).\end{aligned}$$

■

Proposition 2.5. *The space ℓ_p is uniformly convex for $1 < p < \infty$.*

Proof. Let $p \geq 2$ and $\|x\|, \|y\| = 1$. Then by using Clarkson's inequality from Lemma 2.4 we get

$$\|x + y\|^p + \|x - y\|^p \leq 2^{p-1}(1 + 1) = 2^p.$$

Now, if $\|\frac{x-y}{2}\| \geq \epsilon$ for $0 < \epsilon \leq 1$, we get

$$\begin{aligned} &\|x+y\|^p + \|x-y\|^p \leq 2^p \qquad (\star) \\ \Leftrightarrow &\left\|\frac{x+y}{2}\right\|^p + \epsilon^p \leq \left\|\frac{x+y}{2}\right\|^p + \left\|\frac{x-y}{2}\right\|^p \leq 1 \\ \Leftrightarrow &\left\|\frac{x+y}{2}\right\|^p \leq 1-\epsilon^p \\ \Leftrightarrow &\left\|\frac{x+y}{2}\right\| \leq (1-\epsilon^p)^{1/p}. \end{aligned}$$

Hence, we can choose for every $0 < \epsilon \leq 1$ a $0 < \delta < 1$ by

$$\delta := (1-\epsilon^p)^{1/p},$$

fitting the definition of uniform convexity.
Note, that if $1 < p \leq 2$, then $2 \leq q < \infty$ and that the equivalent steps following the Clarkson inequality for $2 \leq p < \infty$ do not require $p \geq 2$. It is just the Clarkson inequality itself that we have only proven it for $2 \leq p < \infty$. So let $1 < p \leq 2$. Then, we have $2 \leq q < \infty$ as $q = p/(p-1) \geq 2$. Thus, by Lemma 2.4 in terms of q, we get with $\|x\|, \|y\| = 1$,

$$\|x+y\|^q + \|x-y\|^q \leq 2^{q-1} = 2^q.$$

We proceed similarly to $(\star)$ with $\|\frac{x-y}{2}\| \geq \epsilon$ for $0 < \epsilon \leq 1$, this time choosing q instead of p in $(\star)$. Then we get

$$\left\|\frac{x+y}{2}\right\| \leq (1-\epsilon^q)^{1/q} = (1-\epsilon^{p/(p-1)})^{(p-1)/p} =: \delta > 0,$$

implying that for $1 < p \leq 2$, ℓ_p is also uniform convex.
Hence, ℓ_p is uniform convex for $1 < p < \infty$. ∎

3. Subsets of ℓ_∞

Remark. *In a set of elements of ℓ_∞, we use indices to denote the position within the set, e.g. $\{x_1, x_2, x_3, \dots, x_N\}$. To refer to the n-th elements of a single $x_i \in \ell_\infty$, we write said x_i as function on the set of natural numbers. Then, the n-th component of the i-th element of $\{x_1, \dots, x_N\} \in \ell_\infty^N$ would be referred to as $x_i(n)$.*

Definition 3.1 (Block disjoint)
A set $\{x_1, \dots, x_N\}$ of elements of ℓ_∞ is called block disjoint if there exists a set $\{a_i, b_i\}_{i=1,\dots,N}$ with $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_n \leq b_n$ such that for all $n, j = 1, \dots, N$, either $j < a_n$ or $j > b_n$ both imply $x_n(j) = 0$.

Example. A simple example of a block disjoint set would be the ordered set of unit vectors $\{e_i\}_{i=1}^n$ with $e_i(j) = 1$ for $j = i$ and otherwise equal to zero. The set has to be ordered since being block disjoint means that for one, at most one elements is non-zero at each position, and also, that elements of the set with smaller indices $j < i$ will always be zero at positions after the i-th elements first non-zero position. This is clearly the case for the unit vectors e_i. We could also add a finite amount of zero-elements to the set, not changing the fact that the set is block disjoint.

Definition 3.2 (Operator of multiplication by characteristic function P_n)
Sticking to the notation of Tsirelson's original paper, we will indicate the operator indicating pointwise multiplication by the characteristic function $\mathbb{1}_{[n+1,\infty)}$, which maps the first n positions to zero, by P_n. For some n and $x \in \ell_\infty$, we then get $P_n(x) = (0, \dots, 0, x(n+1), x(n+2), \dots)$.

Corollary 3.3. *Let a set $\{x_1, \dots, x_N\}$ of elements of the space ℓ_∞ be block disjoint. There exist some $i = 1, \dots, N$ and n such that $x_i = P_n(x_1 + \dots + x_N)$.*

Proof. Choose i to be the index of the last non-zero x_j. Then choose n to be the corresponding a_i. Clearly, then $x_i = P_n(x_1 + \dots + x_N)$. ■

In the following, we give four properties that some subsets $S \subset \ell_\infty$ are supposed to have. Later, we will show that the properties imply reflexivity and also finite universality.

P(1) Each basis vector e_i of ℓ_∞, with $e_i(i) = 1$ and zero elsewhere, belongs to S and furthermore, S is contained in the unit ball.

P(2) If for $x \in \ell_\infty$ its norm is pointwise smaller or equal than the norm of any element $s \in S$, than $x \in S$, i.e. $\forall\, y \in S\ \forall\, x \in \ell_\infty$ we have $|x| \leq |y| \Rightarrow x \in S$.

P(3) Let $\{x_1, \dots, x_N\}$, a set of N elements in S, be block disjoint, then $\frac{1}{2}P_N(x_1 + \dots + x_N) \in S$.

P(4) For every element $s \in S$ there exists some n such that $2P_n(x) \in S$.

Claim. *Let S have P(4). For all $s \in S$ and all $q \in \mathbb{N}_{>0}$ there exists n such that $2^q P_n(x) \in S$.*

Proof of claim. The claim is proved by repeated application of P(4).
So, take $s \in S$ and n_1 such that $2P_{n_1}(x) \in S$. As $2P_{n_1}(x)$ is in S, we can reapply property (4) with some n_2, getting $2P_{n_2}(2P_{n_1}(x)) \in S$. By definition of P_n, $2P_n(x)$ and $P_n(2x)$ are equivalent. We get

$$S \ni 2P_{n_2}(2P_{n_1}(x)) = 2(2P_{n_2}(P_{n_1}(x)) = 2^2 P_{\max\{n_2,n_1\}}(x).$$

Hence, by applying P(4) q times, we get $2^q P_n(x) \in S$ for some n.

Note a couple of things concerning the foregone properties:

Remark.

R(1) Clearly, the properties P(1) and P(2) also apply to the closed and the convex hull of S.

R(2) For $S \subset \ell_\infty$ such that S has the properties P(1), P(2) and/or P(3), the closure of S has the same properties.

R(3) Property P(4) is in general not preserved when taking the closure.

R(4) Properties P(1) and P(4) imply that $S \subset c_0$.

In the following we need to prove Banach-Alaoglu. To do so, we first need to prove Tychonoff's theorem.

Theorem 3.4 (Tychonoff)**.** *Let X_i be compact topological spaces for each $i \in I \subset \mathbb{N}$. Then, the Cartesian product $\prod_{i \in I} X_i$ is compact in the product topology.*

Proof. Let X_i be compact space for each $i \in I$ and $X := \prod_{i \in I} X_i$. Suppose that O is a family of open subsets of X. We assume that there is no finite subfamily of O that covers X, meaning that X would not be compact which then leads to a contradiction. To do to, we try to find a point $x \in X$ which does not belong to any finite subfamily of U. We introduce some notation which will only be used within this proof:

Define $X_J := \prod_{i \in J} X_i$ for $J \subset I$ and $X_I = X$. The *canonical projection* is given by $\pi_{J',J} : X_{J'} \to X_J$ for $J \subset J' \subset I$ and $\pi_J : X \to X_J$ for $J' = I$. Elements of the union P of all X_J are called *partial points*, which means that every partial point has a unique domain $J \subset I$ such that it is contained in X_J. The natural extension ordering is denoted by $\preceq$ on P, i.e. for partial points $p \in X_J$ and $q \in X_{J'}$ we get $p \preceq q$ for $J \subset J'$. We say that a partial point p with domain J is *bad*, if there is no neighbourhood V of p such that $\pi_J^{-1} : X_J \to X$ is covered by a finite subfamily of O. The set of all bad partial points will be denoted by B. The concatenation of two partial points p, q with disjoint domains J and J' is given by $p \vee q \in \prod_{i \in J \cup J'} X_i$.

What we try to prove now is that there is a partial point p with domain $J = I$. But first we realise that

Claim. *If p_0 is a bad partial point then any partial point p with $p \preceq p_0$ is a bad partial point as well, i.e. B is downward closed for $\preceq$.*

Proof of claim. Fix $p_0 \in B$ to be a bad partial point with domain J_0 and let $p \in P$ with domain $J \subset J_0$. Let V be any neighbourhood of $p \in X_J$, then by the definition of the canonical projection we have a $V_0 := \pi_{J_0,J}^{-1}(V)$ which is a neighbourhood of $p_0 \in X_{J_0}$. But $\pi_{J_0}(V_0)$ cannot be covered by a finite subfamily of O. Then p is bad follows by rewriting $V_0 = \pi_{J_0,J}^{-1}(V) = \pi_{J_0}\left(\pi_J^{-1}(V)\right)$ as $\pi_J^{-1}(V) = \pi_{J_0}^{-1}(V_0)$.

In the following, we will formulate two lemmas. The first one revolves around finding another bad partial point such that it is "greater" with regard to $\preceq$ for initial domain $J \neq I$. That means that as long as we do not have $J = I$, we can find a greater, "more maximal" bad partial point with regard to $\preceq$. The second lemma states that that the is indeed a $\preceq$-maximal bad partial point. But if $J \neq I$, we would find a $\preceq$-greater bad partial point. Hence, we get that J is equal to I. However, we just found a bad partial point for domain I and by definition of the "bad" property, we cannot cover any of its neighbourhoods V by a finite subfamily of O. Therefore, for the neighbourhood being the point p itself, it is also not possible to cover it by a finite subfamily of O, in particular by any elements of O. But $p \in X_I = X$ and it is not contained in O, so O does not cover X. By this contradiction, X will be compact.

Lemma 3.5. *Let p be a bad partial point with domain $J \subsetneq I$. For any $i_0 \in I \backslash J$, there is a point $a \in X_{\{i_0\}}$ such that $p \vee a \in B$.*

Proof. Suppose that this was not the case, so $p \vee a \notin B$ for all $a \in X_{\{i_0\}}$. So, we can find an open neighbourhood V_a of $p \vee a$ in $X_{J \cup \{i_0\}}$ for any a such that $\pi_{J \cup \{i_0\}}^{-1}(V_a)$ can be covered by a finite subfamily of O. We can write the neighbourhood V_a in terms of open neighbourhoods Np, a and N_a for p in X_J and a in $X_{\{i_0\}}$, respectively, as $V_a = N_{p,a} \times N'_a$.

Now, $X_{\{i_0\}}$ is compact that means that we can cover is by a finite amount of open set, i.e. there are $a_1, \ldots, a_n \in X_{\{i_0\}}$ such that $X_{\{i_0\}} = N'_{a_1} \cup \cdots \cup N'_{a_n}$. In turn, $N_p := N_{p,a_1} \cap \ldots N_{p,a_n}$ is a neighbourhood of p in X_J. We can then write

$$\pi_J^{-1}(N_p) = \pi_{J\cup\{i_0\}}^{-1}(N_p \times N'_{a_1}) \cup \cdots \cup \pi_{J\cup\{i_0\}}^{-1}(N_p \times N'_{a_n}) \subset \pi_{J\cup\{i_0\}}^{-1}(V_{a_1}) \cup \ldots \pi_{J\cup\{i_0\}}^{-1}(V_{a_n}).$$

But then we can cover $\pi_J^{-1}(N_p)$ by a finite subfamily of O meaning that p is not bad. Contradiction.
There is a point $a \in X_{\{i_0\}}$ for every i_0 and p such that $p \vee a$ is bad. ∎

Lemma 3.6. *Under $\preceq$, there is a maximal bad partial point.*

Proof. Following Zorn's lemma, for $(B, \preceq)$ to contain a maximal element, it suffices to show that any ordered subset C of $(B, \preceq)$ is bounded in B. Note, that C is an ordered set in $(P, \preceq)$. Now, combining all points $q \in C$ gives us a partial point $p \in P$ as P is the union of all spaces $X_J, J \subset I$. So, C bounded in P by p.
To show that indeed $p \in B$, let J be the domain of p and V an arbitrary neighbourhood of p in X_J. V has is of the form $\pi_{J,F}^{-1}(W)$ with a finite subset $F \subset J$ and W an open set in X_F with $p_{|F} := \pi_{J,F}(p) \in W$. Since F is finite, we can find $q_0 \in C$ with $p_{|F} \preceq q_0$. This q_0 is a bad partial point as C is an ordered subset in $(B, \preceq)$. But then, by the preceding Lemma 3.5, $p_{|F}$ is a bad partial point as well. Then, $\pi_F^{-1}(W)$ can also not be covered by a finite subfamily of O and additionally, similar to above, $\pi_F^{-1}(W) = \pi_J^{-1}(V)$ indicates that p is a bad partial point as well, i.e. the bound p of any ordered set C in $(B, \preceq)$ is in B. ∎

As by the explanation given before the lemmas, this concludes the proof.

∎

With Tychonoff's theorem proved, we can approach Banach-Alaoglu's theorem since we are now able to make use of the compactness of the Cartesian product of compact spaces.
First, we take the Cartesian product over closed unit balls of different radii. As those are compact, so is their Cartesian product. Then, we note that the closed unit ball of X^* is contained in said Cartesian product and in the end show that the product topology and the weak topology coincide on said ball.

We denote the weak topology on X by $\sigma(X, X^*)$ and the weak* topology on X^* by $\sigma(X^*, X)$.

Theorem 3.7 (Banach-Alaoglu)**.** *Let X be a normed space and X^* be its topological dual-space. Then* $\text{ball}(X^*) := \{x^* \in X^* : \|x^*\| \leq 1\}$*, which denotes the closed unit ball in X^*, is $\sigma(X^*, X)$-compact.*

Proof. We define

$$D_x = \{z \in \mathbb{C} : |z| \leq \|x\|\}$$

for all $x \in X$ to be the closed unit ball of radius $\|x\|$ in $\mathbb{C}$. As D_x is compact in $\mathbb{C}$ for all x, Tychonoff's theorem implies that $D := \prod_{x\in X} D_x$ is compact in the product topology. Note, that we can denote elements of D as sequences $x^* = \{x^*(x)\}_{x\in X}$ since x^* maps X into $\bigcup_{x\in X} D_x = \mathbb{C}$ with $|x^*(x)| \leq \|x\|$ for all $x \in X$. So x^* is a functional and if it is linear, then $\|x^*\| = \sup_{\|x\|=1} |x^*(x)| \leq \sup_{\|x\|=1} \|x\| = 1$ and thus $x^* \in \text{ball}(X^*)$. This even means that $\text{ball}(X^*)$ contains exactly all linear elements of D. Take a sequence $\{x^*(i)\}_{i\in I}$ in $\text{ball}(X^*)$ with $x^*(i) \to x^* \in D$. Note, that the canonical projections π_x for $x \in X$ in the product topology are continuous, so we get that x^* is linear as

$$\pi_{ax+by}(x^*(i)) = a\pi_x(x^*(i)) + b\pi_x(x^*(i)) \to a\pi_x(x^*) + b\pi_y(x^*).$$

But ball(X^*) contains all linear functionals so ball(X^*) $\subset D$ is closed. As a closed subset of a compact space is compact, so is ball(X^*) in the product topology.

What is left to show is that the product topology of D restricted to ball(X^*) coincides with the weak* topology.
Suppose, $\{x^*(i)\}_{i\in I} \subset$ ball(X^*) with $x^*(i) \to x^*$ in the product topology on D. Then also $x^* \in$ ball(X^*) and $x_i^* \to x^*$ in the weak* topology. So, every closed subset of the closed unit ball of X^* with respect to the weak* topology is also closed with respect to the product topology.
Now, suppose that $\{x_i^*\}_{i\in I} \subset$ ball(X^*) with $x_i^* \to x^*$ with respect to the weak* topology. For any fixed $x \in X$, the canonical projection is continuous on ball(X^*) w.r.t. the weak* topology. But the product topology is the smallest topology in which π_x is continuous of every $x \in X$ thus weaker than the weak* topology on ball(X^*).
Hence, we get that both topologies coincide on the closed unit ball of X^*. As ball(X^*) is compact in the product topology, so it is w.r.t. the weak* topology. ■

Proposition 3.8 (Existence)**.** *There exists a weakly compact set $K \subset c_0$ which has the properties P(1)-P(4).*

Proof. First, we construct S such that it has the properties P(1)-P(3). Then, we show that its closure, which by R(2) has said properties as well, additionally has P(4). We stick to the topology of pointwise convergence.
Define S_1 to be the set of all basis vectors e_i scaled by some α within the unit ball, namely $S_1 := \{\alpha e_i : i \in \mathbb{N}, |\alpha| \leq 1\}$. Thus, S_1 has P(1) and P(2). For P(3), define the set $T_1 := \{\frac{1}{2}P_N(x_1, \dots, x_N) : x_i \in S_1 \ \forall 1 \leq i \leq N, N \in \mathbb{N}\}$ as the set of $\frac{1}{2}P_N$ over all block disjoint sets of elements of S_1 of arbitrary length N. But we need to include arbitrary set of elements of T_1 (or a combination of elements of S_1 and T_1) with regard to P(3) as well. Thus, we define $S_2 := S_1 \cup T_1$ and T_2 analogously to the previous definition of T_1, and so forth.
Hence, we constructed the smallest set fulfilling the properties P(1), P(2) and P(3), denote by $S := S_1 \cup S_2 \cup \dots$.
Now, let K be the closure of said $S \subset \ell_\infty$ in the topology of pointwise convergence; it has P(1)-P(3).

In case that every $x \in K$ is finite, it is clear that K has P(4), since given x finite with length N, we choose $n > N$. Then we get that $P_n(x) = 0$ (per definition of P_n), implying that $2P_n(x) = 0 \in K$ for this n with $0 \in K$ by P(2). Therefore, we need to take a closer look at the case $x \in K$ infinite.
So, let $x \in K$ by infinite. We select a sequence $x^{(j)}$ of elements in S which converges pointwise to x in the given topology. For sufficiently large j, $x^{(j)}$ will reasonably close to the infinite x, thus given more than one non-zero mapping. But this means that $x^{(j)} \notin S_1$, as S_1 only contains scaled standard basis vectors.
This means that $x^{(j)} \in T_n$ for some $n \in \mathbb{N}$; that is $x^{(j)} = \frac{1}{2}P_{N_j}(x_1^{(j)} + \dots + x_{N_j}^{(j)})$ for some block disjoint set $\{x_1^{(j)}, \dots, x_{N_j}^{(j)}\}$ of S.

Let $k_{\min} := \{k : x(k) \neq 0\}$ be the smallest position in which x maps non-zero. Note, that $k_{\min} > N_j$ as the point is mapped non-zero. Since $k_{\min}$ is fixed for $x \in K$ and is independent of j, we can choose $N = N_j$. This leaves us with $x^{(j)} = \frac{1}{2}P_N(x_1^{(j)} + \dots + x_N^{(j)})$ for the set $\{x_1^{(j)}, \dots, x_N^{(j)}\}$ of S. Since $x^{(j)} \to x$ pointwise, we can assume that each $x_i^{(j)}$ converges to some

$x_i \in K$ for all $i = 1, \dots, N$. Then we get that

$$x^{(j)} \longrightarrow x$$
$$\frac{1}{2}P_N(x_1^{(j)} + \dots + x_N^{(j)}) \longrightarrow \frac{1}{2}P_N(x_1 + \dots + x_N)$$

and a corresponding convergence to the set of K-elements

$$\{x_1^{(j)}, \dots, x_N^{(j)}\} \longrightarrow \{x_1, \dots, x_n\}$$

Following the definition of block disjointedness, we are working on the set of natural numbers for the corresponding set $\{a_i, b_i\}_{i=1}^{\infty}$. Clearly, the limit of a sequence $b_i' := \{b_{i_n}\}_{n=1}^{\infty}$ will be strictly smaller than the limit of $a_i' := \{a_{(i+1)_n}\}_{n=1}^{\infty}$ for $b_{i_n} < a_{(i+1)_n}$ and all i and n. We end up with a sequence $\{a_i', b_i'\}_{i=1}^{\infty}$ with $a_1' \leq b_1' < a_2' \leq b_2' < \dots$. Therefore, the limit of a sequence of block disjoint sets is block disjoint. By Corollary 3.3, there exist then i and n' such that $x_i = P_{n'}(x_1 + \dots + x_N)$. Then, we get for $n := \max\{n', N\}$

$$\begin{aligned}
2P_n(x) &= 2P_{\max\{n',N\}}(x) \\
&= 2P_{n'}\left(\frac{1}{2}P_N(x_1 + \dots, x_N)\right) \\
&= P_{n'}P_N(x_1 + \dots + x_N) \\
&= P_N P_{n'}(x_1 + \dots + x_N) \\
&= P_N(x_i) \in K
\end{aligned}$$

Note, that characteristic functions are abelian, i.e. $P_N P_{n'}(x) = P_{n'}P_N(x)$. Also, $P_{\max\{n',N\}}(x) = P_{n'}P_N(x)$ directly following the definition of the characteristic function.

Altogether, this means that for any arbitrary $x \in K$ we have found an n such that $2P_n(x) \in K$. Thus, K has has all the listed properties. Since K has the properties P(1) to P(4), it is a subset of c_0. In particular, K is contained in the unit ball of c_0 by P(1).
As K is pointwise compact by Banach-Alaoglu and contained in c_0, it is weakly compact in c_0. ■

Theorem 3.9 (Hahn-Banach separation theorem)**.** *Let X be a $\mathbb{K}$ topological with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $A, B \subset X$ non-empty, disjoint, convex set. Furthermore, A let be open. Then there exists a linear continuous map $\varphi : X \to \mathbb{K}$ and a real $\alpha > 0$ such that*

$$\varphi(a) < \alpha \leq \varphi(b), \quad \forall a \in A, b \in B.$$

Proof. We first show the real case $\mathbb{K} = \mathbb{R}$. Define

$$C = A - B + x_0$$

by fixing some $a_0 \in A, b_0 \in B$ and defining $x_0 := b_0 - a_0 \neq 0$ since A and B are disjoint. As A and B are convex, so is C and $0 \in C$ as $a_0 - b_0 + b_0 - a_0 \in C$. Note, that C is open as well as it can be written as union over all $b \in B$ of open sets $A - b + x_0$. Since $x_0 \notin C$, we know from the course on Functional Analysis [de Snoo and Sterk, 2019] that then by Hahn-Banach there exists a linear continuous map $\varphi \in X^*$ such that $\varphi(x_0) = 1$ and $\varphi(c) < 1$ for all $c \in C$. By the linearity of φ, we get that $\varphi(b_0) - \varphi(a_0) = 1$ and for all $a \in A, b \in B$

$$\begin{aligned}
\varphi(C) \ni &\varphi(a) - \varphi(b) + \varphi(x_0) < 1 \\
\Leftrightarrow &\varphi(a) < \varphi(b) - (\varphi(b_0) - \varphi(a_0)) + 1 \\
\Leftrightarrow &\varphi(a) < \varphi(b).
\end{aligned}$$

Let $\alpha = \inf_{b\in B} \varphi(b)$, then we get $\varphi(a) \leq \alpha \leq \varphi(b)$ for all $a \in A, b \in B$. To get a strict inequality between $\varphi(a)$ and α, suppose that there is some $a_1 \in A$ such that $\varphi(a_1) = \alpha$. Then, for some $\epsilon > 0$ we have $a_1 + \epsilon x_0 \in A$. But then

$$\alpha \geq \varphi(a_1 + \epsilon x_0) = \varphi(a_1) + \epsilon\varphi(x_0) = \varphi(a_1) + \epsilon = \alpha + \epsilon.$$

Contradiction. There is a strict inequality between $\varphi(a)$ and α, resulting in

$$\varphi(a) < \alpha \leq \varphi(b), \quad \forall a \in A, b \in B.$$

In case $\mathbb{K} = \mathbb{C}$, we can proceed the same way as above for the real case and then define a function $\varphi' = \varphi(x) - i\varphi(ix)$. ■

Definition 3.10 (Local convexity)
We say that a topological vectorspace X is <u>*locally convex*</u>*, if for every element $x \in X$ and every neighbourhood N, there exists an open convex set O such that $x \in O \subset N$.*

Theorem 3.11 (Hahn-Banach separation theorem for local convexity)**.** *Let X be a locally convex $\mathbb{C}$-vectorspace, $A \subset X$ locally convex and compact and $B \subset X$ convex and closed with A, B disjoint. Then there exists a continuous linear map $\varphi : X \to \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$ such that*

$$\operatorname{Re}\varphi(a) \leq \alpha < \beta \leq \operatorname{Re}\varphi(b)$$

for all $a \in A, b \in B$.

Proof. Let $C = B - A$, then C is closed and $0 \in C$ as A and B are disjoint. As C is closed, this means that $X \backslash C$ is a neighbourhood of 0. So, we have some open convex D such that $0 \in D \subset X \backslash C$ as X is locally convex which is disjoint to C. Noting that $C \cap D = \emptyset$, we can apply the Hahn-Banach separation theorem to get

$$\operatorname{Re}\varphi(c) \leq \gamma \leq \operatorname{Re}\varphi(d), \quad \forall c \in C, d \in D.$$

But $0 \in C$, so $\gamma > 0$ and

$$\operatorname{Re}\varphi(d) - \operatorname{Re}\varphi(c) \geq \gamma > 0, \quad \forall c \in C, d \in D.$$

The proof is completed by choosing $\alpha = \sup_{c\in C} \operatorname{Re}\varphi(c)$ and $\beta = \inf_{d\in D} \operatorname{Re}\varphi(d)$ since $\alpha + \gamma \leq \beta$. ■

Proposition 3.12 (Kakutani's Theorem)**.** *The closed unit ball in a normed space X is compact in the weak topology if and only if X is reflexive.*

Proof. Suppose the closed unit ball in X is compact in the weak topology $\sigma(X, X^*)$. Then $\text{ball}(X)$ is weakly*-closed in X^{**} as the weak* topology on X^{**} restricted to X is the weak topology of X, i.e. $\sigma(X^{**}, X^*)\mid_X = \sigma(X, X^*)$.

Claim. *Let X be a normed space, then the* $\text{ball}(X)$ *is* $\sigma(X^{**}, X^*)$*-dense in* $\text{ball}(X^{**})$.

Proof of claim. Let B be the closure of the ball of X under $\sigma(X^{**}, X^*)$ in X^{**}, this means that $B \subset \text{ball}(X^{**})$. Suppose that $\text{ball}(X)$ is not dense in X^{**}. Then there is some $x_0^{**} \in \text{ball}(X^{**}) \backslash B$. Following the Hahn-Banach separation theorem for local convexity, there then is some scalar $\alpha \in \mathbb{R}$, $\epsilon' > 0$ and $x^* \in X^*$ such that for all $x \in \text{ball}(X)$ we have

$$\begin{aligned} &\operatorname{Re}\varphi'(x) < \alpha < \alpha + \epsilon' < \operatorname{Re}\varphi'(x^{**}) \\ \Leftrightarrow &\operatorname{Re}\varphi(x) < 1 < 1 + \epsilon < \operatorname{Re}\varphi(x^{**}). \qquad (\text{with } \varphi := \alpha^{-1}\varphi', \epsilon := \alpha^{-1}\epsilon') \end{aligned}$$

Note, that for all $x \in \text{ball}(X)$, we have that $e^{i\theta}x \in \text{ball}(X)$ as well, implying that $|\varphi(x)| \leq 1$ for $\|x\| \leq 1$. This in turn means that $\varphi \in \text{ball}(X^*)$. We get

$$1 + \epsilon < \operatorname{Re}\varphi(x^{**}) \leq |\varphi(x^{**})| \leq \|x^{**}\| \leq 1.$$

Contradiction. $\text{ball}(X)$ is $\sigma(X^{**}, X^*)$-dense in $\text{ball}(X^{**})$.

Therefore, $\text{ball}(X)$ is $\sigma(X^{**}, X^*)$-dense in $\text{ball}(X^{**})$. This means that $\text{ball}(X) = \text{ball}(X^{**})$, which in turn implies that X is reflexive.
Suppose now that X is reflexive. Then, by Banach-Alaoglu, $\text{ball}(X^{**})$ is $\sigma(X^{**}, X^*)$-compact. As X is reflexive, we get that X is $\sigma(X, X^*)$ compact. ■

Definition 3.13 (Absolutely convex)
A set X is called absolutely convex if for all scalars α, β such that $|\alpha| + |\beta| \leq 1$ and for all elements $x, y \in X$, it holds that $\alpha x + \beta y \in X$.
Equivalently, a set X is absolute convex if and only if it is convex, for $0 \leq \alpha \leq 1$ and all $x, y \in X$ we get $\alpha x + (1-\alpha)y \in X$, and balanced, for all $|\alpha| \leq 1$ and $x \in X$ there is $\alpha x \in X$.

Krein and Smulian proved that the closed convex hull of a weakly compact set in a Banach space is weakly compact. While it is important to for proving Proposition 3.15, it is only used in this instance and given the proof for Krein-Smulian is non-trivial and relies on other findings such as Eberlein-Smulian, it would be beyond the scope of this paper to provide a complete proof. We refer to the original paper by [Krein and Smulian, 1940]. Of course, there are other approaches to prove the theorem. One of them is making use of measures, for which we will outline the steps in the following.

Theorem 3.14 (Krein-Smulian). *The closed convex hull of a weakly compact set K in a Banach space X is weakly compact.*

Proof.

- Given that Eberlein-Smulian's theorem on the equivalence of different forms of compactness is known (again, we refer to the original proof [Eberlein, 1947]), we can assume that X is separable.
- Then, we have to show that the identity on K, $f : (K, \sigma(X, X^*)) \to (X, \|\cdot\|)$ with $k \mapsto k$, has the following property with respect to every measure of the dual of the space of all continuous functions on K with supremum norm, $C(K)^*$, has the property

$$\int \|f\| d\mu < \infty,$$

i.e. it is Bochner-integrable.

- Then, we need to prove that the operator

$$T : C(K)^* \to X, \quad \mu \mapsto \int_K f d\mu$$

is weakly continuous.

- Finally, we use T to map the unit ball of $C(K)^*$ onto a weakly compact set containing the closed convex hull of K.

■

Proposition 3.15. *Take a weakly compact set $K \subset c_0$ such that it has all the given properties. Then V, its closed convex hull, is an absolutely convex weakly compact subset of c_0. Additionally, V also has the properties P(1)-P(4).*

Proof. By Krein-Smulian, we get given weakly compact K, its closed convex hull V is weakly compact as well. Note, that $x \in K$ implies that $-x \in K$ as well by absolute homogeneity of norms. Then for all $|\alpha| \leq 1$, we have that $\alpha x \in K \Rightarrow -\alpha x \in K$. So, the convex hull is balanced, meaning that V is absolutely convex.
By R(1), P(1) and P(2) already hold for a convex hull V of a K which fulfils P(1) and P(2).
To prove P(3), we limit ourselves to the convex hull M of K. We can do this since P(3) extends to the closure (see rR(2)). For P(4), we have to use the absolutely convex weakly compact V.
Take a set $\{x_1, \dots, x_N\}$ of elements of M and suppose it is block disjoint. Since M is the convex hull of K, it is the set of all finite convex combinations, implying that we can write each x_i as

$$x_i = \alpha_i^{(1)} x_i^{(1)} + \dots + \alpha_i^{(n)} x_i^{(n)}$$

with $x_i^{(j)} \in K$ and scalars $\alpha_i^{(j)} \geq 0$ for $j = 1, \dots, n$ and $\sum\limits_{k=1}^{n} \alpha_k = 1$. Using property P(2) with $K \subset M$, we get that $x_i(k) = 0$ implies $x_i^{(j)}(k) = 0$. Then a set of a summand from each of the convex combination of all x_i, so $\{x_1^{(j_1)}, \dots, x_N^{(j_N)}\}$ with $j_1, \dots, j_N \in [1, N]$ is block disjoint. This in turn means that $\frac{1}{2} P_N(x_1^{(j_1)} + \dots + x_N^{(j_N)}) \in K$ as K has all the properties.
Note that since $\{x_1, \dots, x_N\}$ is block disjoint, for each $k \in \mathbb{N}$ there is at most one x_i such that $x_i(k) \neq 0$. We can write $x_1 + \dots + x_N$ as convex combination of elements of the form $x_1^{(j_1)} + \dots + x_N^{(j_N)}$. Then, we get for those scalars $\sum_{k=1}^{n'} \beta_k = 1$ that

$$\begin{aligned}
\frac{1}{2} P_N(x_1 + \dots + x_N) &= \frac{1}{2} P_N(\beta_1(x_1^{(j_1,1)} + \dots + x_N^{(j_N,1)}) + \dots + \beta_{n'}(x_1^{(j_1,n')} + \dots + x_N^{(j_N,n')})) \\
&= \frac{1}{2} P_N(\beta_1(x_1^{(j_1,1)} + \dots + x_N^{(j_N,1)})) + \dots + \frac{1}{2} P_N(\beta_{n'}(x_1^{(j_1,n')} + \dots + x_N^{(j_N,n')}))
\end{aligned}$$

Since M is the convex hull, which in particular means that it is the set of all convex combinations of elements of K, and we have written $\frac{1}{2} P_N(x_1 + \dots + x_N)$ as convex combination of elements of K with scalars β_k, we get $\frac{1}{2} P_N(x_1 + \dots + x_N) \in M$. Hence, M, and with that also its closure V, has property (3).

We make use of some concepts of Measure Theory to show P(4) for V. For that, define the set $D_n := \{x \in K : 4P_n(x) \in K\}$. Note, that for increasing n, we "ignore" more and more

initial positions of the mapping x, thus increasing n allows for more elements to included in the respective D_n. Also, if some x is contained in D_n for some n, it will be contained in all of the following $D_{n'}$ for $n' \geq n$. In particular, the countable union of all those D_n is exactly K. Now, fix $x_0 \in V$ and Riesz's representation theorem states that for a linear functional $\varphi : c_0 \to \mathbb{K}$ all elements f in c_0 there exists a unique probability measure μ on K such that $\varphi(f) = \int_K f d\mu$. Let φ map $\varphi(f) = f(x_0)$. For this μ, we get $\mu(D_n) \underset{n}{\to} 1$ since $D_1 \subset D_2 \subset \dots$ and $K = \cup_{i=1}^{\infty} D_i$ imply that $1 = \mu(K) = \lim_{n\to\infty} \mu(D_n)$. Then, there is some n_0 for which $\mu(D_{n_0}) \geq \frac{3}{4}$.
Suppose that $2P_{n_0}(x_0) \notin V$. As V is the closed convex hull of K and μ is a probability measure on K, there then exists a functional f with $f(x_0) > 1$. As we have a non-negative measurable function, we can approach it by simple functions of sets D_{n_0} and $K \backslash D_{n_0}$ as given in [de Snoo and Winkler, 2019]. We get

$$1 < \int_K f d\mu = \int_{D_{n_0}} f d\mu + \int_{K \backslash D_{n_0}} f d\mu \leq \frac{1}{2}\mu(D_{n_0}) + 2\mu(K \backslash D_{n_0}) \leq \frac{3}{8} + \frac{1}{2} < 1.$$

Contradiction. Indeed, $2P_{n_0}(x_0)$ is in V. ■

4. Bases of Banach spaces

Every Banach space can be seen as a vector space. A basis in the common vector space sense spans the space and is also linearly independent. But while a finite set of linearly independent vectors spans a finite vector space in the way that it is used in linear algebra, we run into problems with infinite Banach spaces. Indeed, it can be shown that such a finite basis for an infinite Banach space is uncountable.
Thus, we will introduce the term of a Schauder basis to have a proper description of the spaces' bases. Furthermore, we will show that an unconditional Schauder basis has, given certain properties, implications for isomorphisms of c_0 and ℓ_1. Said properties will then be combined in the following section to the property of reflexivity.

Definition 4.1 (Schauder basis)
A sequence $\{e_i\}_{i=1}^{\infty}$ is called Schauder basis of a Banach space X, if for every $x \in X$ there exists a unique scalar sequence $\{\alpha_i\}_{i=1}^{\infty}$ such that x can be written as convergent series $\sum_{i=1}^{\infty} \alpha_i e_i$, i.e.

$$\lim_{n\to\infty} \left\| x - \sum_{i=1}^{n} \alpha_i e_i \right\| = 0.$$

In the following, let Q_n denote a linear, bounded projection for a normed vectorspace X and a Schauder basis $\{e_i\}_{i\in\mathbb{N}}$ with

$$Q_n : X \to \operatorname{span}\{e_i : i \in \mathbb{N}_{\leq n}\}, \quad \sum_{i=1}^{\infty} a_i e_i \mapsto \sum_{i=1}^{n} a_i e_i.$$

Note, that if $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis for X, then $\|Q_n x - x\| \to 0$ for all $x \in X$ and $n \to \infty$.

Definition 4.2 (Unconditional Schauder basis)
A Schauder basis $\{e_i\}_{i=1}^{\infty}$ is called unconditional (Schauder) basis if for every $x \in X$ there is a unique sequence of scalars $\{\alpha_i\}_{i=1}^{\infty}$ such that the series $\sum_{i=1}^{\infty} \alpha_i e_i$ converges to x for all

rearrangements of indices; that is for all bijective maps $\pi : \mathbb{N} \to \mathbb{N}$, we have

$$\lim_{n\to\infty} \sum_{j=1}^{n} \alpha_{\pi(j)} e_{\pi(j)} = x.$$

Definition 4.3 (Normed block-system)
A sequence of the form $\{\sum_{j=n_r}^{n_{r+1}-1} \lambda_j e_j\}_{i=1}^{\infty}$ is called normed block-system with respect to the basis $\{e_j\}_j^{\infty}$ if the norm for each of its terms is equal to one for some increasing integer sequence $\{n_r\}_{r=1}^{\infty}$ with $n_1 = 1$.

Proposition 4.4. *Suppose that V is an absolutely convex weakly compact subset of c_0 with P(1)-P(4). Let X be its linear hull with a norm $\|\cdot\|_X$ chosen in a way that V is the unit ball. Then X is a reflexive Banach space. The sequence of unit vectors $\{e_j\}_1^{\infty}$ forms, for which $e_i(j) = 1$ for $i = j$ and zero otherwise, an unconditional basis in X, and the conjugate system of functions $\{e_j^*\}_1^{\infty}$ is an unconditional basis in the dual X^*. If $\{x_i\}_1^{\infty}$ is a normed block-system with respect to the basis $\{e_j\}_1^{\infty}$, then for arbitrary N and $\lambda_1, \dots, \lambda_N$*

$$\|P_N(\lambda_1 x_1 + \cdots + \lambda_N x_N)\|_X \leq 2 \max_{1\leq i\leq N} |\lambda_i|.$$

Proof. Clearly, X is a Banach space, as V is closed in c_0 which in turn is a complete metric space, and X is its linear span. Note, that the Banach space X is reflexive if and only if its closed unit ball is weakly compact in the weak topology of X (see Kakutani's Theorem), and in this case said unit ball is named V. So take any $f \in X^*$. We can represent this f restricted to the unit ball V by elements $f_n \in c_0^*$, assuming that there is a basis $\{e_i^*\}_{i=1}^{\infty}$ for X^* (the existence of which we prove below), as follows

$$f_n = f(e_1)e_1^* + f(e_2)e_2^* + \cdots + f(e_n)e_n^* = f - P_n^* f,$$

So, as $f_n \to f$ with $n \to \infty$, every $f\ |_V$ is the uniform limit of some sequence $\{f_n\ |_V\}_{n=1}^{\infty}$, implying that f is continuous on the weak topology of c_0. Hence, we can identify both the topologies on V and V is not only compact in c_0 but also in the weak topology of X. X is a reflexive Banach space.
To prove that the sequence of unit vectors $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis in X, we have to show that $\lim_{n\to\infty} \|x - \sum_{i=1}^{\infty} \alpha_i e_i\| = \lim_{n\to\infty} \|P_n x\| = 0$ for each $x \in X$ with respective unique scalar sequence $\{\alpha_i\}_{i=1}^{\infty}$. Since V has P(4) and X is the linear span of V, we get that for all $x \in X$ and all $q \in \mathbb{N}$ there exists n such that $2^q P_n(x) \in X \subset c_0$. As we are in c_0, this has to bounded and by choosing $q \to \infty$, we get that $\|P_n(x)\|_X \to 0$. As two distinct unit vectors always map on distinct non-zero points, one non-zero point each, the arrangement in the sum under the limit does not matter. $\{e_i\}_{i=1}^{\infty}$ is an unconditional Schauder basis.
Let $\{x_i\}_{i=1}^{\infty}$ be a normed block-system with $\|x_i\|_X = 1$ for all i. This particularly means that $x_i \in V$ the x_i do not share any e_j for $1 \leq i, j \leq N$ for arbitrary N form the unconditional basis. This is the very idea of block disjointedness. Then by P(3) we have $\frac{1}{2}P_N(x_1 + \cdots + x_N) \in V$ which implies

$$\begin{aligned}
&\left\|\frac{1}{2}P_N(x_1 + \cdots + x_N)\right\|_X \leq 1 \\
\Leftrightarrow &\|P_N(x_1 + \cdots + x_N)\|_X \leq 2 \\
\Leftrightarrow &\|P_N(\lambda_1 x_1 + \cdots + \lambda_N x_N)\|_X \leq \max_{1\leq i\leq N} |\lambda_i| \cdot \|P_N(x_1 + \cdots + x_N)\|_X \leq 2 \max_{1\leq i\leq N} |\lambda_i|
\end{aligned}$$

for arbitrary N and scalars $\lambda_1, \dots, \lambda_N$.
To prove that $\{e_i^*\}_{i=1}^\infty$ is an unconditional Schauder basis in X^*, we need again to show that $\|P_n^* f\| \underset{n}{\to} 0$ for all $f \in X^*$. Suppose that this was not the case, i.e. $\|P_n^* f\| > \epsilon > 0$ for all n. We construct a normed block-system $\{x_i\}_{i=1}^\infty$ to the basis $\{e_i\}_{i=1}^\infty$ to show a contradiction by using P(3) the way we did above. For that, let $x_1 \in X$ be finite (remember, only the last non-zero x_i in a block disjoint set can be infinite) with $\|x_1\| = 1$ and $f(x_1) > \epsilon$. Then, let $x_2 \in X$, also with $\|x_2\| = 1$ and $f(x_2) > \epsilon$, be block disjoint to x_1, meaning it maps natural numbers larger than those of x_1 to non-zero points. By repetition, we get a block disjoint normed block-system $\{x_i\}_{i=1}^\infty$ with $f(x_i) > \epsilon$ for all i. Choose an arbitrary N. By property (3) we can say that $\frac{1}{2} P_N(x_{N+1} + \cdots + x_{2N}) \in X$. Also, $P_N(x_{N+1} + \cdots + x_{2N}) = x_{N+1} + \cdots + x_{2N}$ as $\|x_i\| = 1$ means that the x_i are non-zero, so for each i we have that x_{i+1} is mapping non-zero at least one position to the right of of those of x_i. Now according to the inequality above, we have

$$\|x_{N+1} + \cdots + x_{2N}\|_X = \|P_N(x_{N+1} + \cdots + x_{2N})\|_X \leq 2.$$

And yet, there is $f(x_{N+1} + \cdots + x_{2N}) = f(x_{N+1}) + \cdots + f(x_{2N}) > N\epsilon$. Note, that the contraposition for a continuous function $f' \in X^*$ at zero is

$$\exists \epsilon' > 0 \ \forall \delta' > 0 \ \exists x \in X : \|x\|_X \leq \delta' \wedge |f'(x)| \geq \epsilon'.$$

Thus, with $x = x_{N+1} + \cdots + x_{2N}$ and $\delta' = 2, \epsilon' = N\epsilon$, we have shown that f is not continuous. Contradiction.
Since $\lim_{n\to\infty} \|P_n^* f\| = 0$ does hold for all $f \in X^*$, $\{e_i^*\}_{i=1}^\infty$ is an unconditional basis in X^*. ■

Definition 4.5 (Shrinking basis)
A Schauder basis $\{e_i\}_{i=1}^\infty$ of a Banach space X is shrinking if for every bounded linear functional on X and with x element of the span of a subset of basis vectors, the limit

$$\lim_{n\to\infty} \left(\sup_{x \in \text{span}\{e_i : i \geq n\}} \{|f(x)| : \|x\| = 1\} \right) = 0$$

tends to zero.

Lemma 4.6. *A Schauder basis $\{e_i\}_{i=1}^\infty$ of a Banach space X is shrinking, if and only if the biorthogonal functionals $\{e_i^*\}_{i=1}^\infty$ form a Schauder basis of X^*.*

Proof. Let $\{e_i^*\}_{i=1}^\infty$ be a basis of X^*, this means that $\|Q_n^* x^* - x^*\| \to 0$ for all $x^* \in X^*$. Note, that $(Q_n^* e^*) \mid_{\{e_i\}_{i=n+1}^\infty} = 0$, so $\lim_{n\to\infty} \|e^* \mid_{\{e_i\}_{i=n+1}^\infty}\| = 0$. Thus $\{e_i\}_{i=1}^\infty$ shrinking.
Let $\{e_i\}_{i=1}^\infty$ be shrinking, i.e. $\lim_{n\to\infty} \|e^* \mid_{\{e_i\}_{i=n}^\infty} = 0$, and let $x \in X$ such that $\|x\| = 1$. We get

$$(e^* - Q_n^* e^*)(x) = e^*((I - Q_n)x) \leq \|e^* \mid_{\{e_i\}_{i=n+1}^\infty}\| (\text{bc}(e_i) + 1).$$

Then $\lim_{n\to\infty} \|e^* \mid_{\{e_i\}_{i=n}^\infty}\| = 0$ implies $\|Q_n^* e^* - e^*\| \to 0$. ■

Definition 4.7 (Boundedly-complete basis)
A Schauder basis $\{e_i\}_{i=1}^\infty$ is called boundedly-complete if $\sup_n \|\sum_{i=1}^n \alpha_i e_i\| < \infty$ for a sequence of scalars $\{\alpha_i\}_{i=1}^\infty$ implies that $\sum_{i=1}^\infty \alpha_i e_i$ converges.

Theorem 4.8. *If a Banach space B with basis $\{e_i\}_{i=1}^\infty$ is reflexive then $\{e_i\}_{i=1}^\infty$ is both shrinking and boundedly-complete.*

Proof.

- Suppose $\{e_i\}_{i=1}^\infty$ was not shrinking. Then there exists a linear functional f and some fixed $\epsilon > 0$ for a bijective mapping $\pi : \mathbb{N} \to \mathbb{N}\backslash\{0\}, i \mapsto \pi(i)$ and a sequence $\{x_i\}_{i=1}^\infty = \{\sum_{i=1}^\infty \alpha_{\pi(i)} e_{\pi(i)}\}$ with $\|x_i\| \leq 1$ such that $f(x_i) > \epsilon$ for all i and $\lim_{i\to\infty} \pi(i) = \infty$. But as the x_i are bounded, the sequence must have a weakly converging subsequence. But construction of $\{x_i\}_{i=1}^\infty$, this limit has to be zero, but this is a contradiction to $f(x_i) > \epsilon$ for all i.
- To prove that $\{e_i\}_{i=1}^\infty$ is a boundedly-complete basis, let $\{f_i\}_{i=1}^\infty$ be a sequence of linear functionals defined by $f_i(e_j) = \delta_j^i$. Note, that this sequence is a basis on B^* since we can see every f_i has 'extracting' the i-th coordinate scalar from some $x \in B$. Also, let $\sup_n \|\sum_{i=1}^n \alpha_i e_i\| < \infty$ and $b_n = \sum_{i=1}^n \alpha_i e_i \in B$. Therefore, b_n is bounded and thus, some subsequence $\{b_i\}_{i\in I}$ must have a weak limit b (since the unit ball of a reflexive Banach space is weakly compact according to Banach-Alaoglu). This b can be written in terms of the basis $\{e_i\}_{i=1}^\infty$ as $b = \sum_{i=1}^\infty \beta_i e_i$. Then $f_i(b) = \beta_i$ and $f_i(b_j) = \alpha_i$ for $j \geq i$. But choice of f_i, we get that $f_i(b) = f_i(b_j)$ for $j \geq i$ for all i. This means that $\sum_{i=1}^\infty \alpha_i e_i$ is convergent.

∎

Definition 4.9 (Basic sequence)
A sequence $\{e_i\}_{i=1}^\infty$ in a Banach space X is called <u>*basic sequence*</u>*, if the sequence is a Schauder basis of* $\overline{\text{span}}\{e_i : i \in \mathbb{N}\}$.
Furthermore, a basic sequence $\{e_i\}_{i=1}^\infty$ of Banach space X is called <u>*equivalent*</u> *to a basic sequence $\{e_i'\}_{i=1}^\infty$ of Banach space Y, if for every scalar sequence $\{a_i\}_{i=1}^\infty$ with convergent series $\sum_{i=1}^\infty a_i e_i$ the series $\sum_{i=1}^\infty a_i e_i'$ also converges.*

Lemma 4.10. *Let $\{Q_i\}_{i\in\mathbb{N}}$ be a sequence of said projections in X with*

(i) $\dim \operatorname{ran} Q_n = n$

(ii) $\lim_n Q_n(x) = x$ *for all* $x \in X$,

then we get a Schauder basis by choosing arbitrary non-zero $e_i \in \operatorname{ran} Q_i \cap \ker Q_{i-1}$ for $i \in \mathbb{N}$ and with $e_i \neq e_j$ whenever $i \neq j$.

Proof. Note, that since we are working with projections, that $X = \operatorname{ran} Q_n + \ker Q_n$ and by (i), there is $\operatorname{ran} Q_{n-1} \subsetneq \operatorname{ran} Q_n$ with one dimension difference between them. Since now $\operatorname{ran} Q_n \cap \ker Q_{n-1}$ is of dimension one (just keeping the n-th term of the sum), we represent this as the projection $Q_n - Q_{n-1}$. Using this notation, we can write $Q_n(x)$ as sum

$$Q_n(x) = \sum_{i=1}^n (Q_n - Q_{n-1})(x).$$

Now, with property (ii) and the e_i as given in the statement of this lemma, we can write the projections $(Q_n - Q_{n-1})(x)$ using a linear functional α_i as $\alpha_i(x)e_i$:

$$\sum_{i=1}^\infty \alpha_i(x) e_i = \lim_n Q_n(x) = x.$$

We get uniqueness of representation when we look at some $x = \sum_{i=1}^{\infty} \beta_i e_i \in X$

$$\alpha_n(x)e_n = (Q_n - Q_{n-1}) \left(\sum_{i=1}^{\infty} \beta_i e_i \right) = \beta_n e_n.$$

Note, that for all $i \leq n$, we get $Q_n(e_i) = e_i$, otherwise we get $Q_n(e_i) = 0$. ■

Lemma 4.11. *A sequence $\{e_i\}_{i=1}^{\infty}$ in a Banach space X is a basic sequence if and only if there is a constant $C > 0$ for all scalar sequences $\{a_i\}_{i=1}^{\infty}$, for all n, m with $n \leq m$ such that*

$$\left\| \sum_{i=1}^{n} a_i e_i \right\| \leq C \left\| \sum_{i=1}^{m} a_i e_i \right\| \tag{1}$$

and for all $i \in \mathbb{N}$ we have non-zero e_i.

Proof.

- $\Rightarrow$ Suppose $\{e_i\}_{i=1}^{\infty}$ is a basic sequence, then

$$\left\| \sum_{i=1}^{n} a_i e_i \right\| = \left\| Q_n \sum_{i=1}^{m} a_i e_i \right\| \leq \sup_n \|Q_n\| \cdot \left\| \sum_{i=1}^{m} a_i e_i \right\|,$$

  so this direction immediately follows.

- $\Leftarrow$ Clearly, the e_i are linearly independent, otherwise we could find n and scalars $\{a_i\}_{i=1}^{n}$ such that $0 \neq \sum_{i=1}^{n-1} a_i e_i > \sum_{i=1}^{n} a_i e_i = 0$, so the equation in the lemma would not hold for any constant $C > 0$. Hence, $\text{span}\{e_i \; i \in \mathbb{N}\}$ has dimension n and we can take projections

$$Q_n : \text{span}\{e_i : i \in \mathbb{N}\} \to \text{span}\{e_i : i \in \mathbb{N}\}, \quad \sum_{i=1}^{m} a_i e_i \mapsto \sum_{i=1}^{\min(n,m)} a_i e_i.$$

  By Lemma 4.10, $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis of $\text{span}\{e_i : i \in \mathbb{N}\}$ and also for the closure of this span.

■

Remark. *In particular, the smallest constant C is defined as the basic constant* $\text{bc}(e_i)$, *i.e.* $\text{bc}(e_i) := \sup_n \|Q_n\|$.

Lemma 4.12. *Let $\{e_i\}_{i \in \mathbb{N}}$ be a basic sequence of a Banach space X and $\{b_i\}_{i \in \mathbb{N}}$ a sequence of another Banach space Y. Then the following statements are equivalent:*

(1) $\{b_i\}_{i \in \mathbb{N}}$ is a basic sequence equivalent to $\{e_i\}_{i \in \mathbb{N}}$.

(2) There exists a linear bijection $\varphi : \overline{\text{span}}\{e_i : i \in \mathbb{N}\} \to \overline{\text{span}}\{b_i : i \in \mathbb{N}\}$, such that $\varphi(e_i) = b_i$ for each i.

(3) There are constants $C_1, C_2 > 0$, such that for all scalars $\{a_i\}_{i=1}^{n}$

$$\frac{1}{C_1} \left\| \sum_{j=1}^{n} a_i e_i \right\|_X \leq \left\| \sum_{i=1}^{n} a_i b_i \right\|_Y \leq C_2 \left\| \sum_{i=1}^{n} a_i e_i \right\|_X$$

with $\| \cdot \|_X, \| \cdot \|_Y$ indicating the norms on X and Y, respectively.

Proof.

(1) ⇒ (2) We define $\varphi : \overline{\operatorname{span}}\{e_i : i \in \mathbb{N}\} \to \overline{\operatorname{span}}\{b_i : i \in \mathbb{N}\}$ with $\varphi(\sum_{i=1}^{\infty} e_i'(x)e_i) = \sum_{i=1}^{\infty} b_i'(x)b_i$ for $x = \sum_{i=1}^{\infty} e_i'(x)e_i$. By the fact that both are equivalent basic sequences, this map is linear, well-defined and bijective. So, for $x_n \underset{n}{\longrightarrow} x$ and $\varphi(x_n) \underset{n}{\longrightarrow} y$, we get

$$e_i'(x) = \lim_{n\to\infty} e_i'(x_n) = \lim_{n\to\infty} b_i'(\varphi(x_n)) = b_i'(y).$$

(2) ⇒ (3) Simply choose $C_1 = \|\varphi^{-1}\|$ and $C_2 = \|\varphi\|$.

(3) ⇒ (1) For all $m \leq n$ and scalars $\{a_i\}_{i=1}^n$, we have

$$\left\|\sum_{i=1}^{n} a_i b_i\right\|_Y \leq C_2 \left\|\sum_{i=1}^{n} a_i e_i\right\|_X \leq C_2 \operatorname{bc}(e_i) \left\|\sum_{i=1}^{m} a_i e_i\right\|_X \leq C_1 C_2 \operatorname{bc}(e_i) \left\|\sum_{i=1}^{m} a_i b_i\right\|_Y .$$

By Lemma 4.11, $\{b_i\}_{i=1}^{\infty}$ is a basic sequence.
Let $\sum_{i=1}^{\infty} a_i e_i$ be a converging series, meaning that $\|\sum_{i=m+1}^{n} a_i e_i\|_X < \frac{\epsilon}{C_2}$ for $n, (m+1) \geq N \in \mathbb{N}$ following the Cauchy criterion. Then using the inequality above

$$\left\|\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{m} a_i b_i\right\|_Y = \left\|\sum_{i=m+1}^{n} a_i b_i\right\|_Y \leq C_2 \left\|\sum_{i=m+1}^{n} a_i e_i\right\|_X < \epsilon$$

indicates the convergence of the Cauchy sequence $\{\sum_{i=1}^{n} a_i b_i\}_{n=1}^{\infty}$. We proceed similarly for the other way of the equivalence by choosing a convergent Cauchy sequence in $\|\cdot\|_Y$.

■

Definition 4.13 (Unconditional basic constant)
Similarly to the basic constant $\operatorname{bc}(e_i)$, *we introduce the* <u>*unconditional basic constant*</u> $\operatorname{ubc}(e_i) := \sup_{\{A\subset\mathbb{N}: A \text{ finite}\}} \|Q_A\|$ *with projection* $Q_A : X \to \operatorname{span}\{e_i : i \in A\}$, $\sum_{i\in\mathbb{N}} a_i e_i \mapsto \sum_{i\in A} a_i e_i$.

4.1. Reflexivity and isomorphisms of c_0 and ℓ_1

As much as not being uniformly convex, and with that finite universality, is important for the examination of isomorphs of ℓ_p for $1 < p < \infty$, in this segment we show the analogous applies to reflexivity with regard to c_0 and ℓ_1.
We have shown that reflexivity is equivalent to having a boundedly-complete and shrinking Schauder basis. In the following we will prove the those properties imply the non-existence of isomorphs of c_0 and ℓ_1, respectively.

Lemma 4.14. *If a Banach space* X *has an unconditional Schauder basis* $\{e_i\}_{i=1}^{\infty}$ *which is not boundedly-complete, then* X *contains an isomorph of* c_0.

Proof. As $\{e_i\}_{i=1}^{\infty}$ is not boundedly-complete, there is a scalar sequence $\{a_i\}_{i=1}^{\infty}$ with $\sup_{n\in\mathbb{N}} \|\sum_{i=1}^{\infty} a_i e_i\| < \infty$ and yet $\sum_{i=1}^{\infty} a_i e_i$ does not converge. This means that there is $\epsilon > 0$ such that there are $q > p \geq n$ for all $n \in \mathbb{N}$ such that $\|\sum_{i=o}^{q} a_i e_i\| > \epsilon$. We define $z_j := \sum_{i=p_j}^{q_j} a_i e_i$

for sequences $\{p_j\}_{j=1}^\infty, \{q_j\}_{i=1}^\infty$ with $p_j < q_j < p_{j+1}$. This implies that $\|z_j\| > \epsilon$ for all j. We define

$$\mu_j = \begin{cases} \lambda_i, & i \text{ such that } j \in [p_i, q_i], \\ 0, & \text{otherwise,} \end{cases}$$

such that for all $m \in \mathbb{N}$, we can state with $\lambda_i = 0$ for $i > m$

$$\left\| \sum_{i=1}^m \lambda_i u_i \right\| = \left\| \sum_{j=1}^{q_m} \mu_j a_j e_j \right\| \leq K' \cdot \sup_{i=1,\ldots,m} |\lambda_i| \left\| \sum_{j=1}^{q_m} a_j e_j \right\| \leq K \cdot \sup_{i=1,\ldots,m} |\lambda_i|$$

for some constants K, K'. But now we can see that a lower bound for $\| \sum_{i=1}^m \lambda_i z_i \|$ by Lemma 4.14, giving us

$$\frac{\epsilon}{\operatorname{ubc}(e_i)} \sup_{i=1,\ldots,m} |\lambda_i| \leq \left\| \sum_{i=1}^m \lambda_i z_i \right\| \leq K \sup_{i=1,\ldots,m} |\lambda|.$$

By Lemma 4.12, $\{z_i\}_{i=1}^\infty$ is equivalent to the canonical Schauder basis of c_0 and there is a bijection between the $\overline{\operatorname{span}}\{z_i : i \in \mathbb{N}\} \subset X$ and c_0. ■

Corollary 4.15. *If a Banach space X with unconditional Schauder basis $\{e_i\}_{i=1}^\infty$ does not contain an isomorph of c_0, then $\{e_i\}_{i=1}^\infty$ is boundedly-complete.*

Lemma 4.16. *Let a Banach space X have an unconditional Schauder basis $\{e_i\}_{i=1}^\infty$. Said Schauder basis is not shrinking, if and only if X contains an isomorph of ℓ_1.*

Proof. As $\{e_i\}_{i=1}^\infty$ is shrinking, there exists $f \in X^*$ and $f \notin \overline{\operatorname{span}}\{e_i^* : i \in \mathbb{N}\}$ with $\|f\| = 1$ by Lemma 4.6. Now, $\{e_i\}_{i=1}^\infty$ is a Schauder basis, therefore

$$f(x) = f\left(\sum_{i=1}^\infty e_i^*(x) e_i\right) = \sum_{i=1}^\infty e_i^*(x) f(e_i) = \sum_{i=1}^\infty \iota(e_i)(f) e_i^*(x)$$

with $\iota : X \to X^{**}$ the canonical embedding of X into X^{**}. But as $f \notin \overline{\operatorname{span}}\{e_i^* : i \in \mathbb{N}\}$, $\sum_{i=1}^\infty \iota(e_i)(f) e_i^*(x)$ cannot converge towards f in the norm of X^*, i.e. there is an $\epsilon > 0$ such that

$$\limsup_{n \in \mathbb{N}} \sup_{\|x\|=1} \left| f\left(\sum_{i=n+1}^\infty e_i^*(x) e_i\right) \right| > 2\epsilon.$$

So, there is an infinite subset $S \subset \mathbb{N}$ with elements s and $x_s \in X$, $\|x_s\| = 1$ such that $|f(\sum_{i=s}^\infty e_i^*(x_s) e_i)| > \epsilon$. We can then say that there is an bijection $k : \mathbb{N} \to S$ such that for every $n \in \mathbb{N}$ we have $|f(\sum_{i=s(n)}^\infty e_i^*(x_{s(n)}) e_i)| > \epsilon$. We know that f is continuous as $f \in X^*$ and also, that $\sum_{i=s(n)}^\infty e_i^*(x_{s(n)}) e_i$ converge since $\{e_i\}_{i=1}^\infty$ is a Schauder basis. Therefore, we can find some $q_n \in \mathbb{N}$ with $|f(\sum_{i=s(n)}^{q_n - 1} e_i^*(x_{s(n)}) e_i)| > \epsilon$.
We can to proceed similarly to the proof of Lemma 4.14, so we want to define a sequence $\{z_n\}_{n=1}^\infty$ such that we can apply Lemma 4.12 again, seeing isomorphism between the closed span and ℓ_1. To do so, we choose z_n in a way that $f(z_n) > \epsilon$ and introduce a sequence $\{p_n\}_{n=1}^\infty$ with $p_1 = 1$ and $p_{n+1} = q_{p_n}$. This way, we have distinct indices for each z_i the same way we did for Lemma 4.14. We get

$$z_n := \frac{|f(\sum_{i=s(p_n)}^{p_{n+1}-1} e_i^*(x_{s(p_n)}) e_i)|}{f(\sum_{i=s(p_n)}^{p_{n+1}-1} e_i^*(x_{s(p_n)}) e_i)} \sum_{i=s(p_n)}^{p_{n+1}-1} e_i^*(x_{s(p_n)}) e_i.$$

Clearly, if we take $f(z_n)$, the rightmost sum cancels the denominator, leaving only the absolute value so $f(z_n) = |f(\sum_{i=s(p_n)}^{p_{n+1}-1} e_i^*(x_{s(p_n)})e_i)| > \epsilon$ and $\|z_n\| \leq \mathrm{ubc}(e_i)$. Using the triangle inequality, we can get the upper bound $\|\sum_{i=1}^m a_i z_i\| \leq \sum_{i=1}^m |a_i|\|z_i\| \leq \mathrm{ubc}(e_i)\sum_{i=1}^m |a_i|$.
Define the sets

$$M_+ := \{1 \leq i \leq m : \mathrm{Re}\, a_i \geq 0\}, \quad M_- := \{1 \leq i \leq m : \mathrm{Re}\, a_i < 0\},$$

assume for now that $\sum_{i\in M_+} \mathrm{Re}\, a_i \geq -\sum_{i\in M_-} \mathrm{Re}\, a_i$ and fix the scalar sequence $\{a_i\}_{i\in\mathbb{N}}$ and m. We get the lower bound

$$\begin{aligned}\left\|\sum_{i=1}^m a_i z_i\right\| &\geq \frac{1}{\mathrm{ubc}(e_i)}\left\|\sum_{i\in M_+} a_i z_i\right\| \geq \frac{1}{\mathrm{ubc}(e_i)}\sum_{i\in M_+} \mathrm{Re}\, a_i f(z_i)\\ &\geq \frac{\epsilon}{2\,\mathrm{ubc}(e_i)}\left(\sum_{i\in M_+} \mathrm{Re}\, a_i - \sum_{i\in M_-} \mathrm{Re}\, a_i\right)\\ &= \frac{\epsilon}{2\,\mathrm{ubc}(e_i)}\sum_{i=1}^m |\mathrm{Re}\, a_i|.\end{aligned}$$

If $\sum_{i\in M_+} \mathrm{Re}\, a_i < -\sum_{i\in M_-} \mathrm{Re}\, a_i$ was the case, we could simply choose the scalar sequence $-\{a_i\}_{i=1}^\infty$ to come to the same result. Also, in the same fashion it can be shown that the same inequality holds for $\mathrm{Im}\, a_i$ instead of $\mathrm{Re}\, a_i$. Combining them, we get

$$\left\|\sum_{i=1}^m a_i z_i\right\| \geq \frac{\epsilon}{4\,\mathrm{ubc}(e_i)}\sum_{i=1}^m |a_i|$$

for every $m \in \mathbb{N}$. Finishing the implication in this direction with Lemma 4.12, we have shown that a not shrinking unconditional Schauder basis implies an isomorph of ℓ_1.

Now let X contain an isomorph $Y \subset X$ of ℓ_1. We use [de Snoo and Sterk, 2019] to prove that X is not shrinking. First, we know that $Y^* \cong \ell_\infty$. Also, by Hahn-Banach, we can extend every bounded functional on Y to X with equal norm restricted to Y. This means that X^* contains an isomorph of ℓ_∞. But this in turn implies that X^* is not separable, then it is not shrinking as otherwise it would have a countable basis by Lemma 4.6. ■

5. Proof of Tsirelson's theorem

Before we finally prove Tsirelson's theorem, we sum up the previous results of Lemma 4.14 and Lemma 4.16 on properties of reflexivity in a single powerful statement. As Proposition 4.4 shows the existence of this specific reflexive Banach space X, we are only one step away from proving Tsirelson's claim.

Proposition 5.1. *A Banach space B with unconditional basis $\{x_i\}_{i=1}^\infty$ is reflexive, if and only if B does not contain any subspaces isomorphic to c_0 or ℓ_1.*

Proof. If B does not contain isomorphs of c_0 and ℓ_1, then, by Corollary 4.15 and Lemma 4.16, B is shrinking and boundedly-complete, thus reflexive following Theorem 4.8.
If B is reflexive, then again by Lemma 4.16, B does not contain an isomorph of ℓ_1. Suppose B contained an isomorph of c_0, then as $(c_0)^* \cong \ell_1$ and $(\ell_1)^* \cong \ell_\infty$, B^{**} has an isomorphic copy of ℓ_∞. But this means that B^{**} is not separable and B being reflexive implies that B^{**} is separable. Contradiction. B does not contain an isomorph of c_0. ■

The following last lemma before this paper's main theorem concerns the use of different bases to a Banach space. If we have one basis and there is another sequence such that the sum of the distance between elements of both sequences is finite, thus tending to zero, then the second sequence can be chosen as basis for the Banach space, too. This way one can relatively easy show that, since there is an invertible operator taking one sequence to the other, there is an isomorphism between the two.

We use this approach to isomorphically map the basis of a finitely universal space to the basis of another space. Hence, the latter has to be finitely universal as well.

Lemma 5.2. *Let $\{x_i\}_{i=1}^{\infty}$ be a norm in a Banach space X and sequence $\{y_i\}_{i=1}^{\infty}$ satisfies*

$$\sum_{i=1}^{\infty} \|x_i - y_i\| < \infty.$$

Then $\{y_i\}_{i=1}^{\infty}$ is a basis to the Banach space X.

Proof. The satisfied condition means that for all $\epsilon > 0$ there is some N such that we have $\sum_{i=N}^{\infty} \|x_i - y_i\| < \epsilon$. We can express each $x \in X$ as $x = \sum_{i=1}^{\infty} a_i x_i$ and identify the scalars a_i with linear functionals $f_i(x)$ giving $x = \sum_{i=1}^{\infty} f_i(x)x_i$. Then we get get for $\epsilon = \frac{1}{C}$ with $C = \sup_{n\in\mathbb{N}} \|f_n\|$

$$\sum_{i=N}^{\infty} \|x_i - y_i\| < \frac{1}{C}.$$

We can define some $\tilde{y}_i := y_i$ if $i \geq N$ and otherwise $\tilde{y}_i := x_i$ to rewrite the summation in terms of an index going from 1 to infinity

$$\sum_{i=N}^{\infty} \|x_i - y_i\| = \sum_{i=1}^{\infty} \|x_i - \tilde{y}_i\| < \frac{1}{C}.$$

It is sufficient to show that $\{\tilde{y}_i\}_{i=1}^{\infty}$ is a basis on X as, since we are working in infinite spaces, $\{y_i\}_{i=1}^{\infty}$ is a basis as well then.
We introduce operators S following [Krein et al., 1940] and U by

$$Sx := \sum_{i=1}^{\infty} f_i(x)(x_i - \tilde{y}_i) \quad \text{and} \quad Ux := x - Sx = \sum_{i=1}^{\infty} f_i(x)\tilde{y}_i.$$

Note, that U is invertible since it has the inverse $U^{-1} = (I - S)^{-1} = I + S + S^2 + \dots$ as

$$(I - S) \cdot (I + S + S^2 + \dots) = I + (S - S) + (S^2 - S^2) + (S^3 - S^3) + \dots = I$$

and that $Ux_i = \tilde{y}_i$. With $y = U^{-1}x$, we can express the sum representation of x in terms of y, i.e. $y = \sum_{i=1}^{\infty} f_i(y)x_i$. This means that

$$\begin{aligned} y &= \sum_{i=1}^{\infty} f_i(y)x_i \\ \Leftrightarrow U^{-1}x &\sum_{i=1}^{\infty} f_i\left(U^{-1}x\right)x_i \\ \Leftrightarrow x &= \sum_{i=1}^{\infty} f_i\left(U^{-1}x\right)\tilde{y}_i. \end{aligned}$$

Denote $b_i := f_i\left(U^{-1}x\right)$ for all $i \in \mathbb{N}$. Since the linear functional implies a unique representation for x, so does b_i. Thus, $\{\tilde{y}_i\}_{i=1}^{\infty}$ is a basis in X. ■

Combining our work, we can finally approach the theorem we have been aiming at.
Note, that as the mentioned Banach space is reflexive and has an unconditional basis, no isomorphism of either c_0 or ℓ_1 is contained, and as each infinite-dimensional subspace is finitely universal, said subspaces are not uniformly convex. Since ℓ_p is uniformly convex for $1 < p < \infty$, the Banach space does not contain ℓ_p for $1 < p < \infty$ either.

Theorem 5.3. *There exists a reflexive Banach space in which each infinite-dimensional subspace is finitely universal.*

Proof. We have shown that there exists a reflexive Banach space X which is the linear span of an absolutely convex weakly compact set $V \subset c_0$ with a norm such that V is the unit ball. In the following, let $Y \subset X$ be an arbitrary infinite-dimensional subspace.
If $\{x_i\}_{i=1}^{\infty}$ is a normed block-system with respect to the basis $\{e_i\}_{i=1}^{\infty}$ of X, then by Proposition 4.4 we get for all arbitrary N and $\lambda_1, \dots, \lambda_N$ that

$$\max_{1 \le i \le N} |\lambda_i| \le \|\lambda_1 x_{N+1} + \cdots + \lambda_N x_{2N}\| \le 2 \max_{1 \le i \le N} |\lambda_i|.$$

But this means that the subspace $X_S \subset X$, generated by $\{x_i\}_{i=1}^{\infty}$, is finitely universal. Hence, all we have to show that there is an isomorphism between X_S and some subspace $Y_S \subset Y$.
To do so, let Y_S be generated by the sequence $\{y_i\}_{i=1}^{\infty}$ and $\{x_i\}_{i=1}^{\infty}$ be a normed block-system such that for all $i \in \mathbb{N}$, we get that $\|y_i - x_i\| \le 2^{-i}$. Then by Lemma 5.2, $\{x_i\}_{i=1}^{\infty}$ there exists an invertible operator U such that by $Uy_i = x_i$ X_S is mapped onto Y_S.
Therefore, as Y was chosen arbitrary, we can find a finitely universal subspace in each infinite-dimensional subspace $Y \subset X$. Note, that by the definition of finite universality, it suffices that each finite-dimensional normed space has an invertible operator mapping on some subspace of $Y_{S_S} \subset Y_S$ as Y_{S_S} in turn is subspace a of Y, expanding the finite universality from Y_S onto Y.
Each infinite-dimensional subspace of the given reflexive Banach space X is finitely universal. ■

6. A (more) analytic construction

As we have now established, there is indeed a Banach space which does not contain isomorphs of c_0 and ℓ_p for $1 \le p < \infty$. Yet, the original construction by Tsirelson is in itself only what is ought to be, an example of a certain archetype of Banach space. Within a year, T. Figiel and W.B. Johnson followed up on Tsirelson's construct providing another space of the type of Tsirelson. Their space has the nice property of giving an analytical, even computable, description of the norm on the space. While presenting proof to the validity of the properties is not within the scope of this paper, the construction is shortly stated and an outline of the approach to the proof is given.
Firstly though, we need to fix some notation. We make use of the notation given in [Casazza and Shura, 1989] to avoid introducing new notations for this brief mention:
Let $\mathbb{R}^{(\mathbb{N})}$ denote the space of real scalar sequences which are eventually zero, meaning they have finite support. $\{t_n\}_{n=1}^{\infty}$ is the unit vector basis of $\mathbb{R}^{(\mathbb{N})}$ with zero at every position except in n for all n. Let E, F be non-empty, finite subspaces of $\mathbb{N}$ and let $E \le F$ indicate that for all $e \in E, f \in F$, we have that $e \le f$. Analogously, $E < F$ is defined. Lastly, given any $x = \sum_{n\in\mathbb{N}} a_n t_n \in \mathbb{R}^{(\mathbb{N})}$ and $1 \le e$ for all $e \in E$, we define $Ex = \sum_{n\in E} a_n t_n$.
Having the notations set, we define a sequence $\{\|\cdot\|_m\}_{m=0}^{\infty}$ on $\mathbb{R}^{(\mathbb{N})}$ for any fixed $x \in \mathbb{R}^{(\mathbb{N})}$ and $m \ge 0$ by

$$\begin{cases} \|x\|_0 = \max_{n\in\mathbb{N}} |a_n|, \text{ and} \\ \|x\|_{m+1} = \max\left\{ \|x\|_m, \frac{1}{2} \max_{E_k} \left[\sum_{j=1}^{k} \|E_j x\|_m \right] \right\}. \end{cases}$$

The maximum over E_k is the maximum over all finite subsets of $\mathbb{N}$ with $k \le E_1 < E_2 < \cdots < E_k$. We can see that $\|x\|_m = \sum_{n\in\mathbb{N}} |a_n|$ for all m as for all k, we have that n' with $|a_{n'}| = \max_n |a_n|$ can only be in at most one of the E_k since we have strict inequalities between them. For iterations $m \ge 2$, note that in the inner brackets, $E_j x$ does restrict the $x = \sum_{n\in\mathbb{N}} a_n t_n$ to $\sum_{n\in E_j} a_n t_n$, so again, the maximal scalar $|a_n|$ can only appear in on of those terms in the sum. Also, by the definition of the norm, the elements of the sequence increase with increasing m, as each $\|x\|_{m+1}$ is either the previous $\|x\|_m$ or a new, higher value. Therefore, for each $x \in \mathbb{R}^{(\mathbb{N})}$, we have convergence with limit $\|x\| := \lim_{m\to\infty} \|x\|_m$ bounded above by the 1-norm $\|x\|_1$.
Then, the Tsirelson space given by Figiel and Johnson is the completion of $(\mathbb{R}^{(\mathbb{N})}, \|\cdot\|)$.

While we omit validating that this construct does indeed have the properties of a Tsirelson space, we explicitly remark that by the definition of the norm above, said norm is possible to be computed which makes the space better describable in an analytical sense. Also, the original space by Tsirelson even turns out to be the dual of Figiel and Johnson's space with unit vectors $\{t_n^*\}_{n=1}^{\infty}$.

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