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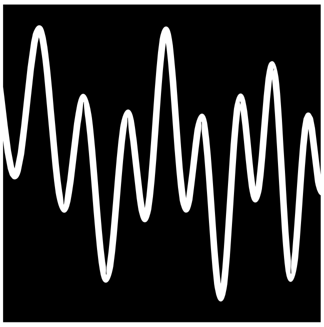

# Zeros of the Zeta Function

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# ABSTRACT

The Riemann Hypothesis states that any non-trivial zero of the Riemann zeta function has real part equal to 1/2. In this bachelor's thesis we study a way to detect such a zero. The zeta function is a complex function and in a part of the complex plane it is given as an infinite sum. By restricting the zeta function to the line $1/2 + i\mathbb{R}$ and using the so-called functional equation, a real function is constructed. A zero of this function corresponds precisely to the imaginary part of a zero of the zeta function. In this way zeta's zeros can be plotted and calculated. The zeta function is a special case of a so-called L-function. We apply the same method to detect zeros of two other L-functions and plot the results.

# Contents

# 1 Introduction

In November 1859, Bernhard Riemann published a paper called *'On the Number of Primes Less Than a Given Magnitude'*[1]. In this paper Riemann introduced methods to study the zeta function [1], which is the analytic function for $z = a + bi$

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \tag{1.1}$$

for $a > 1$ and by analytic continuation for $a \leq 1$, $z \neq 1$. The zeta function could be seen as a product over the prime numbers

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p\in P} \left(1 - \frac{1}{p^z}\right)^{-1},$$

where $P$ denotes the set of all prime numbers. Full knowledge of $\zeta$ behavior should lead to full knowledge of the prime numbers [2]. Riemann studied the complex zeros of $\zeta(z)$ and conjectured that all nontrivial zeros (which are those not of the form $z = m$ for $m$ a negative integer) must lie on the line of complex numbers having real part equal to 1/2. This is known as the Riemann Hypothesis. This has not been proven, but at least we know that all such nontrivial zeros must lie in the critical strip consisting of those complex numbers having real part between 0 and 1.

In this paper we find a way to detect the zeros of the zeta function on the line $1/2 + i\mathbb{R}$. In the book 'The Riemann Zeta-Function', Karatsuba and Voronin write about the Hardy function $Z : \mathbb{R} \to \mathbb{R}$: *"Zeros of $\zeta(z)$ on the critical line are the real zeros of the function $Z(t)$"* [5]. We will investigate the Hardy function and prove that this is indeed the case. We can then detect a zero of $\zeta$ by observing a change of sign in the Hardy function.

On a historical note, the first known paper in which zeros of the zeta function are calculated was published in 1903 by J.P. Gram. He calculated the first 15 non-trivial zeros. The website *ZetaGrid* was a project designed to explore roots of the zeta function [13]. When the project ended in 2005, the first $10^{13}$ zeros were found to have real part equal to 1/2. This final contribution was done by X. Gourdon in 2004 [4].

In the second part of this paper we study a generalized form of the zeta function. The so-called Dirichlet $L$-series. These are analytic functions for $z = a + bi$

$$L(z, \chi_k) = \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^z}$$

for $a > 0$ and by analytic continuation for the complex plane. The functions $\chi_k$ are Dirichlet characters[2] modulo $k$. We will consider the Dirichlet $L$-series for $\chi_3$ and $\chi_4$. In 'Probems in Analytic Number Theory', the functional equations for these functions are studied [12]. Using these, we construct for both complex functions a real function analogous to the Hardy function. In this way we are able to detect their zeros as well.

[1] Originally published in German as *'Über die Anzahl der Primzahlen unter einer gegebenen Grösse'*.

[2] These characters are defined in section 3.1. Note that if we take the constant function mapping $\chi : n \mapsto 1$ for all natural $n > 0$, we obtain the zeta function.

# 2 Finding roots using a real function

## 2.1 The Hardy function

*In this section we introduce the Hardy function Z. This is the function we will be studying throughout the chapter. We will show in the following sections that the function is real and that a real number t is a zero of Z if and only if $1/2+it$ is a zero of the zeta function. We use the Hardy function as defined in Chapter 3 of the book 'The Riemann Zeta-Function' by Karatsuba and Voronin. [5].*

**Definition 2.1.** Let $\Gamma : \mathbb{C} \to \mathbb{C}$ denote the Gamma function. The *Hardy Function* (also known as Riemann-Siegel formula) is defined for a real number $t$ as

$$Z(t) = \pi^{-it/2} \cdot \frac{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}{\left|\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\right|} \cdot \zeta\left(\frac{1}{2}+it\right). \tag{2.1}$$

**Proposition 2.1.** *The Hardy function has the following properties.*

*(i)* $Z(t) = 0$ *if and only if* $\zeta(1/2+it) = 0$.

*(ii)* $Z : \mathbb{R} \to \mathbb{R}$.

The aim of this chapter is proving the two statements above. We first take a closer look at the function and make the following observations.

*Remark.* As a first observation, note that if $\Gamma(1/4+it/2) \neq 0$ and exists for $t \in \mathbb{R}$, then

$$\left|\frac{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}{\left|\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\right|}\right| = \frac{\left|\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\right|}{\left|\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\right|} = 1,$$

so it lies on a circle $\{z \in \mathbb{C} : |z| = 1\}$. Furthermore, for $t \in \mathbb{R}$, we see

$$\left|\pi^{-it/2}\right| = \left|e^{-it\log(\pi)/2}\right| = |e^{ix}| = 1$$

with $x = \frac{t}{2}\log\pi$ a real number. So $\pi^{-it/2}$ also has modulus 1. It then follows that their product

$$\pi^{-it/2} \cdot \frac{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}{\left|\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\right|} \in \left\{e^{i\theta} : \theta \in [0,2\pi)\right\} \subset \mathbb{C}.$$

As a consequence,

$$|Z(t)| = \left|\zeta\left(\frac{1}{2}+it\right)\right|.$$

This shows that if $\zeta(1/2+it) = 0$, then it follows that $Z(t) = 0$ as well, which is one side of Proposition 2.1 (i).

### 2.1.1 Approach

To prove Proposition 2.1, we will be covering the following steps.

1. **Show that the Gamma function has no zeros.** We divide by the absolute value of the Gamma function of $1/4 + it/2$, so we show that this is nonzero. We use Euler's reflection formula to show that the Gamma function has no zeros. The proof of Euler's reflection formula is an application of Cauchy's Residue Theorem.

2. **Show that the function is real.** We will use the functional equation of the zeta function to show that $\overline{Z(t)} = Z(t)$.

## 2.2 Preliminaries

*We will recall some knowledge of Complex Analysis that we will be using throughout the paper. This includes the the definitions of the complex sine and cosine functions and Euler's identity.*

**Definition 2.2.** For a complex number $z$, the functions $\sin z$ and $\cos z$ are defined as

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

**Proposition 2.2.** *(Euler's Identity) For a complex number z we have*

$$e^{iz} = \cos z + i \sin z \tag{2.2}$$

*Proof.* This follows immediately from the definition:

$$\begin{aligned}
\cos z + i \sin z &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) + i \cdot \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\
&= \frac{1}{2}\left(e^{iz} + e^{-iz} + \left(e^{iz} - e^{-iz}\right)\right) \\
&= \frac{1}{2}\left(e^{iz} + e^{iz}\right) \\
&= e^{iz}.
\end{aligned}$$

$\triangle$

**Proposition 2.3.** *For a complex number z and a real number $x > 0$ we have*

$$\left|x^z\right| = x^{Re(z)}.$$

*Proof.* Recall that the modulus of a complex number $z = a + bi$ is given as $|a + bi| = \sqrt{a^2 + b^2}$. Applying Euler's Identity from Proposition 2.2, we see that for a complex number $z$ and a real number $x > 0$ we have

$$\begin{aligned}
\left|x^z\right| &= \left|e^{z\log x}\right| \\
&= \left|e^{\mathrm{Re}(z)\log x + i\mathrm{Im}\,(z)\log x}\right| \\
&= \left|e^{\mathrm{Re}(z)\log x} \cdot e^{i\mathrm{Im}\,(z)\log x}\right| \\
&= \left|e^{\mathrm{Re}(z)\log x}\right| \cdot \left|e^{i\mathrm{Im}\,(z)\log x}\right| \\
&= \left|x^{\mathrm{Re}(z)}\right| \cdot |\cos(\mathrm{Im}\,(z)\log x) + i\sin(\mathrm{Im}\,(z)\log x)| \quad \textit{(Euler's identity)} \\
&= \left|x^{\mathrm{Re}(z)}\right| \cdot \left(\cos^2(\mathrm{Im}\,(z)\log x) + \sin^2(\mathrm{Im}\,(z)\log x)\right) \\
&= \left|x^{\mathrm{Re}(z)}\right| \\
&= x^{\mathrm{Re}(z)}. \qquad (2.3)
\end{aligned}$$

Where we use that $\cos^2(x) + \sin^2(x) = 1$ and in the final step that for $x > 0$, we have $x^{\mathrm{Re}(z)} > 0$. △

## 2.3 The Gamma function

*In the introduction, we saw the appearance of the Gamma function in equation* (2.1)*. We will show that we can use this function for complex values of the form $z = 1/4 + it/2$. Also, since we divide by the absolute value of this function, we will show that for a real number $t$, $\Gamma(1/4 + it/2) \neq 0$.*

**Definition 2.3.** For a complex number $z$ with $\mathrm{Re}(z) > 0$ we define

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx. \qquad (2.4)$$

### 2.3.1 Convergence

**Theorem 2.4.** *Let $a$ and $b$ be real numbers with $a > 0$ and consider $z = a + bi \in \mathbb{C}$. Then $\Gamma(z)$ converges absolutely.*

*Proof.* Substituting $z$ into the definition and then taking the absolute value gives

$$|\Gamma(a + bi)| = \left|\int_0^\infty x^{(a-1)+bi} e^{-x} dx\right| \qquad (2.5)$$

We have

$$|\Gamma(a + bi)| \leq \int_0^\infty \left|x^{(a-1)+bi} e^{-x}\right| dx = \int_0^\infty x^{a-1} e^{-x} dx.$$

We split the interval of integration at $x = 1$ so that we need to prove the convergence of the two integrals in the following sum.

$$|\Gamma(a + bi)| \leq \int_0^1 x^{a-1} e^{-x} dx + \int_1^\infty x^{a-1} e^{-x} dx. \qquad (2.6)$$

- Let us first study the integral for $x \in [0,1]$. For a real value $0 < b < 1$, the integral

$$\int_b^1 x^{a-1} e^{-x} dx$$

converges. That is, we can find a value $G \in \mathbb{R}$, such that $\int_b^1 x^{a-1} e^{-x} dx = G$. The function $e^{-x}$ is monotone decreasing on the interval $[0,1]$. So we can bound this value $G$ by

$$e^{-1} \int_b^1 x^{a-1} dx \leq G \leq \int_b^1 x^{a-1} dx.$$

We calculate the upper bound on the right-hand side explicitly. Since $a > 0$, we have that $a - 1 > -1$ and thus

$$\int_b^1 x^{a-1} dx = \left[ \frac{1}{a} x^a \right]_{x=b}^1 = \frac{1}{a} \left(1 - b^a\right).$$

Note that this is still defined if we send $b \downarrow 0$, again by the fact that $a > 0$. We have

$$\lim_{b \downarrow 0} \frac{1}{a} \left(1 - b^a\right) = \frac{1}{a}.$$

We conclude that

$$\int_0^1 x^{a-1} e^{-x} \leq \frac{1}{a} \quad \text{for } a \in \mathbb{R}_{>0}.$$

- Now we study the integral for $x \in [1, \infty)$. We make use of the indefinite integral of $\int_1^\infty e^{-x} dx$. This integral converges.

$$\int_1^\infty e^{-x} dx = \lim_{k \to \infty} \left[-e^{-x}\right]_{x=1}^k = \lim_{k \to \infty} \left(e^{-1} - e^{-k}\right) = \frac{1}{e}.$$

To prove that the second integral in equation (2.6) converges, we will use the method of mathematical induction to show that for $0 < a < n$, the integral

$$\int_1^\infty x^{a-1} e^{-x} dx < \int_1^\infty e^{-x} dx < \infty$$

for all integers $n \in \mathbb{N}_{>0}$.

  - If $0 < a < 1$, we have that $x^{a-1} e^{-x} < e^{-x}$ if $x \geq 1$. Therefore

$$\int_1^\infty x^{a-1} e^{-x} dx < \int_1^\infty e^{-x} dx < \infty.$$

  - To show the following steps, we make an observation. If we still assume $0 < a < 1$, we can apply integration by parts to the integral.

$$\begin{aligned} \int_1^\infty x^{a-1} e^{-x} dx &= \lim_{k \to \infty} \left[(a-1) x^{a-2} e^{-x}\right]_{x=1}^k + (a-1) \int_1^\infty x^{a-2} e^{-x} dx \\ &= \frac{(a-1)}{e} + (a-1) \int_1^\infty x^{a-2} e^{-x} dx. \end{aligned} \tag{2.7}$$

  Now, for $0 < a < 2$ we see that $(a-1)/e \in \mathbb{R}$ and

$$x^{a-2} e^{-x} < e^{-x}$$

since $a-2$ is negative. It follows that

$$\int_1^\infty x^{a-2}e^{-x}dx < \int_1^\infty e^{-x}dx < \infty$$

and therefore that for $0 < a < 2$, the integral $\int_1^\infty x^{a-1}e^{-x}dx$ converges. We will generalize this idea. If we continue from equation (2.7) and apply integration by parts again, we see

$$\begin{aligned}\int_1^\infty x^{a-1}e^{-x}dx &= \frac{a-1}{e}+(a-1)\int_1^\infty x^{a-2}e^{-x}dx \\ &= \frac{a-1}{e}+(a-1)\cdot\left(\frac{a-2}{e}+(a-2)\int_1^\infty x^{a-3}e^{-x}dx\right) \\ &= \frac{a-1}{e}+\frac{(a-1)(a-2)}{e}+(a-1)(a-2)\int_1^\infty x^{a-3}e^{-x}dx.\end{aligned}$$

We can iterate these steps. We find that for a real number $a > 0$, we have after applying integration by parts $n$ times,

$$\int_1^\infty x^{a-1}e^{-x}dx = \sum_{m=1}^{n}\left(\frac{1}{e}\prod_{i=1}^{m}(a-i)\right)+\left(\prod_{i=1}^{n}(a-i)\right)\int_1^\infty x^{(a-n-1)}e^{-x}dx, \qquad (2.8)$$

where the latter integral converges by the same argument as before when $a-n-1 < 0$. That happens when $a < n+1$.

Suppose we find a number $\ell \in \mathbb{N}_{>0}$ such that for values $0 < a < \ell$, the integral $\int_1^\infty x^{a-1}e^{-x}dx < \infty$. We apply integration by parts $\ell$ times and obtain equation (2.8) with $n = \ell$ and we see that for $0 < a < \ell+1$, we have

$$\sum_{m=1}^{\ell}\left(\frac{1}{e}\prod_{i=1}^{m}(a-i)\right) < \infty \quad \text{and} \quad \int_1^\infty x^{(a-\ell-1)}e^{-x}dx < \infty.$$

The convergence of the integral follows from the fact that $x^{a-\ell-1} = x^{(a-1)-\ell}$ and $0 < a < \ell$, so $(a-1)-\ell < 0$, and therefore $x^{a-\ell-1}e^{-x} < e^{-x}$. As before we have

$$\int_1^\infty x^{(a-1)-\ell}e^{-x}dx < \int_1^\infty e^{-x}dx < \infty$$

and so the integral $\int_1^\infty x^{(a-1)}e^{-x}dx$ converges for $0 < a < \ell+1$.

By the principle of mathematical induction we conclude that for $a > 0$,

$$\int_1^\infty x^{(a-1)}e^{-x}dx < \infty.$$

That shows us

$$|\Gamma(a+bi)| \le \int_0^1 x^{a-1}e^{-x}dx + \int_1^\infty x^{a-1}e^{-x}dx < \infty.$$

$\triangle\!\!\!\!\times$

### 2.3.2 Continuation of the Gamma function

The previous proof could be used to show that $\Gamma(z)$ converges for all complex values of $z$, as long as $\mathrm{Re}(z) > 0$. We will now prove the following Theorem for complex numbers $z$ having $\mathrm{Re}(z) > 0$ and use it to analytically continue the Gamma function for complex numbers $z$ having $\mathrm{Re}(z) < 0$.

**Theorem 2.5.** *For a complex number $z$ with $Re(z) > 0$, we have $\Gamma(z+1) = z\Gamma(z)$.*

*Proof.* We start with the Gamma function of $z+1$.

$$\Gamma(z+1) = \int_0^\infty e^{-x} x^z dx.$$

If we use the substitution $u(x) = e^{-x}$, implying $du(x) = -e^{-x}dx$, we have

$$\int_0^\infty e^{-x} x^z dt = -\int_0^\infty x^z du(x).$$

Now apply integration by parts:

$$\begin{aligned}
-\int_0^\infty 1 \cdot t^z du(x) &= -\left( \left[u(x) \cdot t^z\right]_{x=0}^\infty - \int_0^\infty u(x) \cdot \frac{d\,[x^z]}{du(x)} \cdot du(x) \right) \\
&= \left[u(x) \cdot x^z\right]_{x=0}^\infty + \int_0^\infty u(x) dx^z \\
&= \underbrace{\left[e^{-x} \cdot x^z\right]_{x=0}^\infty}_{0} + \int_0^\infty e^{-x} dx^z
\end{aligned}$$

Note that

$$\frac{d}{dx} x^z = z \cdot x^{z-1} \qquad \Longrightarrow \qquad dx^z = z x^{z-1} dx$$

Which gives us

$$\Gamma(z+1) = \int_0^\infty e^{-x} dx^z = z \int_0^\infty e^{-x} x^{z-1} dt = z\Gamma(z) \tag{2.9}$$

◬

For $z \neq 0$, it follows from Theorem 2.5 that

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}.$$

This expression can be iterated:

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \cdots = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}.$$

This is what we will use to continue the Gamma function.

**Definition 2.4.** Let $z$ be a complex number. If $\mathrm{Re}(z) \leq 0$ and $z \notin \mathbb{Z}$, then choose an integer $n$ such that $\mathrm{Re}(z) + n > 0$ and define

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}. \tag{2.10}$$

### 2.3.3 Zeros of the Gamma function

It follows from Theorem 2.5 that if we find a value $z$ for which $\Gamma(z) = 0$, then for all natural numbers $n$, also $\Gamma(z+n) = 0$, implying that $\Gamma$ has infinitely many zeroes. We want to show that the Hardy function $Z$ from Definition 2.1 exists, so we will show that no complex number $z$ exists with the property $\Gamma(z) = 0$, since we divide by $\Gamma(z)$. The fact that no such $z$ exists follows from Euler's Reflection Formula.

**Theorem 2.6.** *(Euler's Reflection Formula) For $z$ a complex number such that $z \notin \mathbb{Z}$ we have*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Before we prove this Theorem, we first show that it implies the result about $\Gamma$ we need.

**Corollary 2.7.** *The Gamma function has no zeros.*

*Proof.* Assume for contradiction that there exists a complex number $z$ such that $\Gamma(z) = 0$. Let us investigate what this $z$ could be. First of all, $z$ could be a natural number. But we know that then $\Gamma(z) = (z-1)!$ which is nonzero. So a natural number $z$ gives no zero. Secondly, $z$ could be a negative integer. In that case $\Gamma(z)$ has a pole. So $z$ cannot be a negative integer either. Left over are all complex values of $z$ that are not integers. In that case, $z$ satisfies the conditions of Euler's reflection formula. We substitute $z$ and obtain

$$0 = \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

This is a contradiction, since for no such $z$ the fraction $\pi/\sin(\pi z) = 0$. ◻

We continue with proving Euler's reflection formula from Theorem 2.6. To build the proof, we use Lemmas 2.8, 2.9 and 2.10.

**Lemma 2.8.** *For $z \in \mathbb{C} \setminus \mathbb{Z}$ we have*

$$\sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{n+z} + \frac{1}{n+1-z} \right) = \lim_{m\to\infty} \sum_{m \geq |n|} \frac{(-1)^n}{z-n}. \tag{2.11}$$

*provided that the series converges.*

*Proof.* Let $z \in \mathbb{C} \setminus \mathbb{Z}$. Starting with the right-hand side, we can write out the sum and regroup the terms as follows:

$$\begin{aligned}
\lim_{m\to\infty} \sum_{m\geq|n|} \frac{(-1)^n}{z-n} &= \cdots - \frac{1}{z-3} + \frac{1}{z-2} - \frac{1}{z-1} + \frac{1}{z} - \frac{1}{z+1} + \frac{1}{z+2} - \cdots \\
&= \frac{1}{z} - \frac{1}{z-1} - \frac{1}{z+1} + \frac{1}{z-2} + \frac{1}{z+2} - \frac{1}{z-3} \cdots \\
&= \frac{1}{z} + \frac{1}{-z+1} - \frac{1}{z+1} - \frac{1}{-z+2} + \frac{1}{z+2} + \frac{1}{-z-3} \cdots \\
&= \left(\frac{1}{z} + \frac{1}{-z+1}\right) - \left(\frac{1}{z+1} + \frac{1}{-z+2}\right) + \left(\frac{1}{z+2} + \frac{1}{-z-3}\right) - \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{n+z} + \frac{1}{n-z+1}\right),
\end{aligned}$$

◻

**Lemma 2.9.** *For a real number $t$ in $(0,2\pi)\setminus\{\pi, \pi/2, 3\pi/2\}$ we have*

$$\sin\left(\pi(m+½)\cdot e^{it}\right) \to \pm\infty \quad as \quad m\to\infty. \tag{2.12}$$

*Proof.* We first use Euler's Identity from Proposition 2.2 two times and then we apply the definition of the complex sine function to obtain the following.

$$\begin{aligned}
\sin\left(\pi(m+½)\cdot e^{it}\right) &= \sin\left(\pi(m+½)(\cos t + i\sin t)\right) \\
&= \sin\left(\pi(m+½)\cos t + i\pi(m+½)\sin t\right) \\
&= \sin\left(A\cos t + iA\sin t\right) \\
&= \frac{1}{2i}\left(e^{i(A\cos t+iA\sin t)} - e^{-i(A\cos t+iA\sin t)}\right). \\
&= \frac{e^{iA\cos t}\cdot e^{-A\sin t}}{2i} - \frac{e^{A\sin t}\cdot e^{-iA\cos t}}{2i}.
\end{aligned}$$

where we used $A=\pi(m+½)$ for ease of notation. Applying Euler's Identity to the powers of $e$ that have a cosine function, we rewrite the last line as

$$\begin{aligned}
\sin\left(Ae^{it}\right) = \frac{1}{2i}&\left(e^{-A\sin t}\cos(A\cos t) - e^{A\sin t}\cos(A\cos t)\right) \\
&+\frac{i}{2i}\left(e^{-A\sin t}\sin(A\cos t) + e^{A\sin t}\sin(A\cos t)\right).
\end{aligned} \tag{2.13}$$

Note that $i/2i = ½$, so that the second term is the real part and the first term is the imaginary part of $\sin\left(Ae^{it}\right)$. Note that sending $m\to\infty$ is the same as sending $A\to\infty$. since $A\sim m$. Recall furthermore that for a complex function $w = f_n + g_n i$, the limit

$$\lim_{n\to\infty}(f_n + g_n i) = \infty$$

if either $f_n\to\infty$ or $g_n\to\infty$. Therefore we could just analyze the real term in the equation above and show that is goes to infinity for $t$ in the specified interval. Let us analyze the limit of the real part of (2.13).

$$\lim_{A\to\infty} e^{-A\sin t}\underbrace{\sin(A\cos t)}_{\in[-1,1]} + e^{A\sin t}\underbrace{\sin(A\cos t)}_{\in[-1,1]}. \tag{2.14}$$

Splitting the interval of $t$, we see the following:

- If $t\in(0,\pi)$, then $A\sin t>0$, so $\lim_{A\to\infty} e^{-A\sin t} = 0$ and $\lim_{A\to\infty} e^{A\sin t} = \infty$.
- If $t\in(\pi,2\pi)$, then $A\sin t<0$, so $\lim_{A\to\infty} e^{-A\sin t} = \infty$ and $\lim_{A\to\infty} e^{A\sin t} = 0$.

In conclusion, for $t\in(0,\pi)\cup(\pi,2\pi)$, the limit in equation (2.14) goes to $\pm\infty$ and hence also equation (2.12) is satisfied for $t$ on this interval.

To finish the proof, we study the problem that appears when $A\cos t\in\mathbb{Z}\times\pi$. In that case namely $\sin(A\cos t)=0$ and thus the limit . Let us have a closer look at this expression with $A=\pi(m+1/2)$. Suppose that we have

$$\cos t\in\mathbb{Z}\cdot\frac{\pi}{\pi(m+1/2)}.$$

This would mean that for an integer $k$,

$$\cos t = \frac{2k}{2m+1}$$

Recall that we send $m$ to $\infty$. Under this limit, the fraction is zero. This means that we are only left with the one case in which $\cos t = 0$. For $t$ in the interval $(0, 2\pi)$, the cosine has zeros precisely for $t = \pi/2$ and $t = 3\pi/2$.
This shows that indeed $\sin\left(Ae^{it}\right) \to \pm\infty$ if $t \in (0, 2\pi) \setminus \{\pi, \pi/2, 3\pi/2\}$. △

**Lemma 2.10.** *For $z \in \mathbb{C} \setminus \mathbb{Z}$ we have*

$$\frac{\pi}{\sin(\pi z)} = \lim_{m\to\infty} \sum_{m \geq |n|} \frac{(-1)^n}{z-n}. \tag{2.15}$$

*Outline.* To prove this, we will use a function $g_z$ that has residues $-\pi/\sin(\pi z)$ and $(-1)^n/(z-n)$ at its poles. Then we construct a complex contour depending on $m$ and apply the Residue Theorem. We then have that contour integral of $g_z$ is equal to the sum of its residues. Finally we send $m$ to infinity and show that the contour integral of $g_z$ is zero. That will then directly imply (2.15).

*Proof.* Let $z$ be a fixed complex number. Define function $g_z : \mathbb{C} \setminus (\{z\} \cup \mathbb{Z} \to \mathbb{C})$ as follows:

$$g_z : w \mapsto \frac{\pi}{\sin(\pi w)} \cdot \frac{1}{z-w}.$$

We claim that $g_z$ has residues as described in the outline above. Indeed, consider the poles of this function. Poles appear when the limit of a function goes to infinity. There is a pole when $\sin(\pi w) = 0$. That is, whenever $w = n \in \mathbb{Z}$, we have a pole of order 1. Also, when $w = z$ we obtain a pole; $z$ is a simple pole as well.
We now need the residues of $g_z$ at both poles. For the first pole, let $n \in \mathbb{Z}$. Then

$$\begin{aligned}
\text{Res }(n) &= \lim_{w\to n} (w-n) \cdot g_z(w) \\
&= \lim_{w\to n} \frac{\pi}{\sin(\pi w)} \cdot \frac{w-n}{z-w} \\
&= \pi \cdot \lim_{w\to n} \frac{w-n}{(z-w)\sin(\pi w)}
\end{aligned}$$

As $w \to n$, both numerator and denominator go to zero. Therefore we can make use of 'l Hôspital's rule. Differentiating numerator and denominator with respect to $w$ then gives

$$\begin{aligned}
\text{Res }(n) &= \pi \lim_{w\to n} \frac{1}{-\sin(\pi w) + \pi(z-w)\cos(\pi w)} \\
&= \frac{1}{(z-n)\cos(\pi n)} \\
&= \frac{(-1)^n}{z-n}
\end{aligned}$$

where we use that $1/\cos(\pi n) = (-1)^n$ for $n \in \mathbb{Z}$. We observed there to be a second pole when $w = z$. The residue of $g_z$ at $z$ is

$$\begin{aligned}
\text{Res }(z) &= \lim_{w\to z} (w-z) \cdot g_z(w) \\
&= \lim_{w\to z} \frac{\pi}{\sin(\pi w)} \cdot \frac{w-z}{z-w} \\
&= \lim_{w\to z} \frac{\pi}{\sin(\pi w)} \cdot (-1) \\
&= -\frac{\pi}{\sin(\pi z)}.
\end{aligned}$$

So $g_z$ indeed has the required residues. We will now construct a contour in the complex plane that includes the poles of $g_z$. Pick $m \in \mathbb{N}$ such that $m > |z|$. Consider now the circle $C_m$ with radius $m + \frac{1}{2}$ centered at the origin:

$$C_m := \left\{ (m + 1/2)e^{it} \;\middle|\; 0 \leq t \leq 2\pi \right\}$$

Note that by construction, the pole $z \in C$ and also the poles $-m, -m+1, \cdots, -1, 0, 1 \cdots, m-1, m$ are in the interior of $C_m$.

We now recall Cauchy's Residue Theorem, which states: *If C is a simple closed positively oriented contour and $f$ is analytic inside and on C except at the points $z_1, z_2, \cdots, z_n$ inside C, then*

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} \text{Res }(z_j).$$

Let us apply this Theorem to integrate $g_z(w)$ over $C_m$.

$$\begin{aligned}
\int_{C_m} g_z(w)dw &= \int_C \frac{1}{\sin(\pi w)} \cdot \frac{1}{z-w} dw \\
&= 2\pi i \left( \text{Res }(z) + \sum_{n \in C} \text{Res }(n) \right) \\
&= 2\pi i \left( -\frac{\pi}{\sin(\pi z)} + \sum_{m \geq |n|} \frac{(-1)^n}{z-n} \right), \qquad (2.16)
\end{aligned}$$

Make the observation that if we divide both sides by $2\pi i$ and send $m$ to infinity, we obtain equality (2.15) if we show that $\lim_{m\to\infty} \int_{C_m} g_z(w)dw = 0$ is equal to zero. Our final step in this proof is showing that if $m \to \infty$, then the integral goes to zero.

To integrate $g_z$ over $C_m$ with respect to $w$, we make the substitution $w(t) = (m + 1/2)e^{it}$. We then see $dw = (m + 1/2)ie^{it}dt$ and obtain

$$\lim_{m\to\infty} \int_{C_m} g_z(w)dw = \lim_{m\to\infty} \int_0^{2\pi} g_z\left( (m + 1/2)e^{it} \right) \cdot (m + 1/2) \cdot i \cdot e^{it} dt$$

Consider the value of $g_z\left((m + 1/2)e^{it}\right) \cdot (m + 1/2)$ in the integrand.

$$g_z\left( (m + 1/2)e^{it} \right) \cdot (m + 1/2) = \frac{\pi}{\sin(\pi(m + 1/2)e^{it})} \cdot \frac{m + 1/2}{z - (m + 1/2)e^{it}}$$

Now, if we take the limit of $m \to \infty$ of the second fraction, we obtain

$$\lim_{m\to\infty} \frac{m + 1/2}{z - (m + 1/2)e^{it})} = \lim_{m\to\infty} \frac{1 + \frac{1}{2m}}{\frac{z}{m} - (1 + \frac{1}{2m})e^{it}} = -\frac{1}{e^{it}}.$$

Therefore, we rewrite the integral as

$$\begin{aligned}
\lim_{m\to\infty} \int_C g_z(w)dw &= \lim_{m\to\infty} \int_0^{2\pi} \frac{\pi}{\sin(\pi(m + 1/2)e^{it})} \cdot \frac{m + 1/2}{z - (m + 1/2)e^{it}} \cdot e^{it} \cdot i\, dt \\
&= \lim_{m\to\infty} \int_0^{2\pi} \frac{-i\pi\, dt}{\sin(\pi(m + 1/2)e^{it})}. \qquad (2.17)
\end{aligned}$$

The sine function in the fraction is precisely the function from Lemma 2.9, which tends to infinity for large $m$ for all $t$ in $(0,2\pi)$ except for three points. Splitting the integral on these three points, and use the notation $A=\pi(m+1/2)$ as before, we rewrite (2.17) as

$$-i\pi\cdot\lim_{A\to\infty}\left(\int_0^{\pi/2}\frac{dt}{\sin\left(Ae^{it}\right)}+\int_{\pi/2}^{\pi}\frac{dt}{\sin\left(Ae^{it}\right)}+\int_{\pi}^{3\pi/2}\frac{dt}{\sin\left(Ae^{it}\right)}+\int_{3\pi/2}^{2\pi}\frac{dt}{\sin\left(Ae^{it}\right)}\right)$$

By Lemma 2.9 it now follows that all four integrals go to zero as $A$ goes to infinity. Thus indeed,

$$\lim_{m\to\infty}\int_{C_m} g_z(w)dw = 0$$

and

$$\frac{\pi}{\sin(\pi z)} = \lim_{m\to\infty}\sum_{m\geq|n|}\frac{(-1)^n}{z-n}.$$

$\measuredangle$

We have now proven Lemmas 2.8 and 2.10. These will be used in the proof of Euler's reflection formula below.

*Proof.* (of Theorem 2.6) We use the analytic continuation of the Gamma function defined in Definition 2.4 and prove now for $0<\mathrm{Re}(z)<1$. For such $z$, both $\mathrm{Re}(z)>0$ and $\mathrm{Re}(1-z)>0$. Hence

$$\begin{aligned}\Gamma(z)\Gamma(z-1) &= \left(\int_0^\infty t^{z-1}e^{-t}dt\right)\left(\int_0^\infty s^{-z}e^{-s}ds\right)\\ &= \int_0^\infty\int_0^\infty t^{z-1}s^{-z}e^{-t-s}ds\,dt\\ &= \int_0^\infty\int_0^\infty\left(\frac{t}{s}\right)^z t^{-1}e^{-(s+t)}ds\,dt \qquad (2.18)\end{aligned}$$

Now, let us apply a transformation. Define $u=s+t$ and $v=t/s$. We will now calculate the Jacobian determinant. Solving the equations for $u$ and $v$ for $s$ and $t$, we obtain

$$t=s\cdot v$$

Hence

$$u = s+t = s+sv = s(v+1),$$

which shows

$$s(u,v) = \frac{u}{v+1}.$$

It follows that

$$t(u,v) = s(u,v)\cdot v = \frac{uv}{v+1}$$

The Jacobian determinant then becomes

$$J(u,v)=\left|\begin{vmatrix}s_u & s_v\\ t_u & t_v\end{vmatrix}\right|=\left|\begin{vmatrix}\frac{1}{v+1} & -\frac{u}{(v+1)^2}\\ \frac{v}{v+1} & \frac{u(v+1)-uv}{(v+1^2)}\end{vmatrix}\right|=\left|\frac{u(v+1)-uv}{(v+1)^3}+\frac{uv}{(v+1)^3}\right|=\frac{u}{(v+1)^2}$$

Next we need to determine the boundary values in the new variables.

We see that

$$\begin{aligned}(s,t) &\mapsto \left(u=u+t, v=\frac{t}{s}\right)\\ (u,v) &\mapsto \left(t=\frac{uv}{v+1}, s=\frac{u}{v+1}\right)\end{aligned}$$

are each others inverse as images on the first quadrant. Therefore the boundaries of the new variables must be $u \in (0,\infty)$ and $v \in (0,\infty)$

Now we can apply the transformation to (2.18) by substitution and a multiplication by $J$ as calculated above:

$$\begin{aligned}\Gamma(z)\Gamma(z-1) &= \int_0^\infty \int_0^\infty \left(\frac{t}{s}\right)^z t^{-1} e^{-(s+t)}\, ds\, dt \\ &= \int_0^\infty \int_0^\infty v^z \,(t(u,v))^{-1}\, e^{-v}\, J(u,v)\, du\, dv \\ &= \int_0^\infty \int_0^\infty v^z \,\frac{v+1}{uv}\, e^{-v}\, \frac{u}{(v+1)^2}\, du\, dv \\ &= \int_0^\infty \int_0^\infty \frac{v^{z-1}}{v+1} e^{-v}\, du\, dv \qquad (2.19)\end{aligned}$$

Solving the integral with respect to $u$:

$$\int_0^\infty \frac{v^{z-1}}{v+1} e^{-v}\, du = \lim_{k\to\infty}\left(-\frac{v^{z-1}}{v+1}e^{-k}\right) - \left(-\frac{v^{z-1}}{v+1}e^{0}\right) = \frac{v^{z-1}}{v+1}.$$

Left to solve is the integral that follows from (2.19). If we split the integral into the sum of two integrals running $t$ in $(0,1)$ and $(1,\infty)$ respectively, we obtain

$$\Gamma(z)\Gamma(z-1) = \int_0^1 \frac{v^{z-1}}{1+v}\, dv + \int_1^\infty \frac{v^{z-1}}{1+v}\, dv$$

Now, let us introduce $t = 1/v$, such that $dv = -1/t^2\, dt$, to make a substitution in the second integral. This substitution has the nice property that at the lower boundary $v = 1$ implies $t = 1$ and at the upper boundary, $v \to \infty$ implies $t \to 0$. Thus the integral simplifies to

$$\begin{aligned}\Gamma(z)\Gamma(z-1) &= \int_0^1 \frac{v^{z-1}}{1+v}\, dv + \int_1^\infty \frac{v^{z-1}}{1+v}\, dv \\ &= \int_0^1 \frac{v^{z-1}}{1+v}\, dv + \int_1^0 \frac{t^{1-z}}{t^{-1}+1}\left(-\frac{1}{t^2}\right) dt \\ &= \int_0^1 \frac{v^{z-1}}{1+v}\, dv + \int_0^1 \frac{t^{-z}}{1+t} dt \\ &= \int_0^1 \frac{v^{z-1}}{1+v} + \frac{v^{-z}}{1+v} dv \qquad (2.20) \\ &= \int_0^1 \frac{v^{z-1}+v^{-z}}{1+v}\, dv,\end{aligned}$$

where step (2.20) is justified since the integrals share the same interval of integration. If we write the denominator as $1-(-v)$, then we recognize the fraction as the result of a geometric series:

$$\begin{aligned}\int_0^1 \frac{v^{z-1}+v^{-z}}{1+v}\, dv &= \int_0^1 \left(v^{z-1}+v^{-z}\right) \sum_{n=0}^\infty (-v)^n\, dv \\ &= \sum_{n=0}^\infty (-1)^n \int_0^1 \left(v^{n+z-1}+v^{n-z}\right) dv \qquad (2.21) \\ &= \sum_{n=0}^\infty (-1)^n \left[\frac{v^{n+z}}{n+z} + \frac{v^{n-z+1}}{n-z+1}\right]_{v=0}^{v=1} \\ &= \sum_{n=0}^\infty (-1)^n \left(\frac{1}{n+z} + \frac{1}{n-z+1}\right)\end{aligned}$$

If we now recall the result of Lemma 2.10 it follows[3] that for $z \in \mathbb{R} \setminus \mathbb{Z}$,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

△

# 2.4 The Zeta function

*In this section we study the zeta function. In the introduction we defined the function for complex $z$ with $Re(z) > 1$. Since we need the $\zeta$ function on the line $1/2 + i\mathbb{R}$, we need to analytically extend the function. Apart from that, this section introduces the $\xi$ function — a product of the zeta function and the gamma function, for which we find a functional equation. We need the $\xi$ function and its functional equation in order to prove that the Hardy function maps real numbers to real numbers in the next section.*

## 2.4.1 Continuation of the Zeta function

In the introduction we defined the zeta function for a complex number $z$ with $\mathrm{Re}(z) > 1$ as

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

We want to make sense of $\zeta(1/2 + it)$ for a real number $t$ and that is why we want to find an analytic continuation of the zeta function that is defined for a complex number $z$ having real part equal to 1/2. In the book 'A Course in Arithmetic' by J.P. Serre [9] we find the following.

**Proposition 2.11.** *One has*

$$\zeta(z) = \frac{1}{z-1} + \phi(z),$$

*where $\phi(z)$ is holomorphic for $Re(z) > 0$.*

In the proof of this Proposition, we will make use of the inequality given in following Lemma.

**Lemma 2.12.** *Let $z$ be a fixed complex number $z$ with $Re(z) > 0$ and consider the function*

$$w : [n, n+1] \to \mathbb{C}, \quad w(t) = \frac{1}{n^z} - \frac{1}{t^z}.$$

*This function satisfies*

$$\sup_{n \le t \le n+1} |w(t)| = |w(n+1)| \le \frac{|z|}{n^{-Re(z)+1}}. \tag{2.22}$$

[3]Note that Lemma 2.10 applies provided that the sum converges. We showed that the gamma function converges, so the sum on the right-hand side converges as well.

*Proof.* The function $w : [n, n+1] \to \mathbb{C} : t \mapsto w(t)$ as above can be seen as a curve in the complex plane. It starts at the origin since $w(n) = 0$. At time $t$ the distance from the origin is

$$|w(t) - w(0)| = |w(t)|.$$

We want to find an upper bound for $|w(t)|$. The velocity vector at time $t$ is $w'(t)$. The 'real' velocity is then

$$|w'(t)| = \frac{|z|}{t^{\mathrm{Re}(z)+1}}.$$

We have that $\mathrm{Re}(z) + 1 > 0$ and $z$ is a fixed number. Therefore the velocity is maximal for $t = n$ (since we have $n \le t \le n+1$). So if we want to find an upper bound for the distance to the origin at time $t$, we use this maximal velocity without changing direction. In that case the change in distance is

$$(t-n)\frac{|z|}{n^{\mathrm{Re}(z)+1}}.$$

Since we have $0 \le t - n \le 1$, the inequality as in (2.22) follows.

*Remark.* Another method of proving the inequality for $n \le t \le n+1$ is showing that

$$\left|\frac{1}{n^z} - \frac{1}{t^z}\right| = \left|\int_n^t \frac{z}{\tau^{z+1}} d\tau\right| \le \int_n^t \frac{|z|}{\tau^{\mathrm{Re}(z)+1}} d\tau \le \frac{|z|}{n^{\mathrm{Re}(z)+1}}.$$

△

We now continue by proving the Proposition. This follows the proof as in the book by Serre.

*Proof. (Proposition 2.26)* Let $z$ be a complex number such that $\mathrm{Re}(z) > 1$. Then we have

$$\int_1^\infty \frac{1}{t^z} dt = \left[\frac{1}{1-z} \cdot t^{1-z}\right]_{t=1}^\infty = \lim_{t\to\infty} \frac{1}{1-z}\left(t^{1-z} - 1\right) = \frac{1}{z-1}.$$

We rewrite the integral as an infinite sum of integrals.

$$\int_1^\infty \frac{1}{t^z} dz = \sum_{n=1}^\infty \int_n^{n+1} \frac{1}{t^z} dz.$$

Using this expression, we rewrite the zeta function.

$$\begin{aligned}
\zeta(z) &= \frac{1}{z-1} - \frac{1}{z-1} + \sum_{n=1}^\infty \frac{1}{n^z} \\
&= \frac{1}{z-1} + \sum_{n=1}^\infty \left(\frac{1}{n^z} - \int_n^{n+1} \frac{1}{t^z} dt\right) \\
&= \frac{1}{z-1} + \sum_{n=1}^\infty \int_n^{n+1} \left(\frac{1}{n^z} - \frac{1}{t^z}\right) dt
\end{aligned}$$

We indeed have that $\zeta(z) = (z-1)^{-1} + \phi(z)$ with

$$\phi_n(z) = \int_n^{n+1} \left(\frac{1}{n^z} - \frac{1}{t^z}\right) dt \qquad \text{and} \qquad \phi(z) = \sum_{n=1}^\infty \phi_n(z).$$

Left to show is that $\phi(z)$ converges and that it is holomorphic for $\mathrm{Re}(z) > 0$. We see that every $\phi_n(z)$ is defined for $\mathrm{Re}(z) > 0$ and that these functions are all holomorphic[4]

[4] A *holomorphic function.* is complex differentiable in a neighborhood of every point of its domain.

Following the proof by Serre, we will show that $\sum_{n=1}^{\infty} \phi_n(z)$ converges normally[5] on all subsets $K \subset \mathbb{C}$ that have the properties

(i) Every $z \in K$ has real part $\mathrm{Re}(z) > 0$

(ii) $K$ a compact subset of $\mathbb{C}$.

That is, We want to show that for every such compact subset $K$,

$$\sum_{n=1}^{\infty} \sup_{z \in K} |\phi_n(z)| = \sum_{n=1}^{\infty} \sup_{z \in K} \left| \int_n^{n+1} \left( \frac{1}{n^z} - \frac{1}{t^z} \right) dt \right| < \infty. \tag{2.23}$$

Note that an upper bound for the integral is given by

$$\left| \int_n^{n+1} \left( \frac{1}{n^z} - \frac{1}{t^z} \right) dt \right| \leq \sup_{n \leq t \leq n+1} \left| \frac{1}{n^z} - \frac{1}{t^z} \right| \tag{2.24}$$

In Lemma 2.12 we found a bound for the right-hand side, namely

$$\left| \frac{1}{n^z} - \frac{1}{t^z} \right| \leq \frac{|z|}{n^{\mathrm{Re}(z)+1}} \quad \text{for } n \leq t \leq n+1.$$

Now consider an arbitrary subset $K \subset \mathbb{C}$ with the properties as defined above. Combining (2.23) and (2.24) with the bound from Lemma 2.12 gives

$$\sum_{n=1}^{\infty} \sup_{z \in K} |\phi_n(z)| = \sum_{n=1}^{\infty} \sup_{z \in K} \left| \int_n^{n+1} \left( \frac{1}{n^z} - \frac{1}{t^z} \right) dt \right| \leq \sum_{n=1}^{\infty} \sup_{z \in K} \frac{|z|}{n^{\mathrm{Re}(z)+1}}. \tag{2.25}$$

Let us analyze the fraction on the right-hand side. We can find a bound for the fraction in the following way.

- $K$ is compact, so it is closed and bounded. Therefore, for every $z \in K$ we have $|z| \leq G$ for a fixed real number $G$.
- The function mapping $z \mapsto \mathrm{Re}(z)$ is continuous, so it has a minimum value on $K$. Denote this minimum by $m$. Note that we have $m > 0$ since every $z \in K$ has $\mathrm{Re}(z) > 0$.

We therefore have for $z \in K$ and $n \in \mathbb{N}$

$$\frac{|z|}{n^{\mathrm{Re}(z)+1}} \leq \frac{G}{n^{m+1}} \quad \text{and thus} \quad \sup_{z \in K} \frac{|z|}{n^{\mathrm{Re}(z)+1}} \leq \frac{G}{n^{m+1}}.$$

We use this upper bound in equation (2.25) and see that

$$\sum_{n=1}^{\infty} \sup_{z \in K} |\phi_n(z)| \leq \sum_{n=1}^{\infty} \frac{G}{n^{m+1}} = G \sum_{n=1}^{\infty} \frac{1}{n^{m+1}}$$

The series on the right-hand is a so-called *p-series* defined for a real number $p$ as

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Such a series is convergent if $p > 1$. We have $m > 0$ and thus $m + 1 > 1$. It follows that the sum converges. So $\sum_{n=1}^{\infty}$ converges normally on the arbitrary chosen subset $K \subset \mathbb{C}$. This finishes the proof. $\triangle$

[5] *Normal convergence* implies absolute and uniform convergence.

### 2.4.2 Functional equation of the Zeta function

We will now have a look at the functional equation for the zeta function. For this we will use section 9.3 of the lecture notes by J.H. Evertse [6].

**Definition 2.5.** Define $\xi : \mathbb{C} \to \mathbb{C}$ as

$$\begin{aligned} \xi(z) &:= \frac{1}{2} z(z-1)\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z) \\ &= (z-1)\pi^{-z/2}\Gamma\left(\frac{z}{2}+1\right)\zeta(z), \end{aligned}$$

where the second line uses the identity $z/2\Gamma(z/2) = \Gamma(z/2+1)$ which follows from Theorem 2.5.

Evertse now states a Theorem, immediately followed by a Corollary that gives an implicit form of the continuation of the zeta function. We will use this continuation to make sense of the Hardy function.

**Theorem 2.13.** *The function $\xi$ has an analytic continuation to $\mathbb{C}$. For this continuation we have for $z \in \mathbb{C}$,*

$$\xi(1-z) = \xi(z).$$

*Proof.* The proof of this Theorem is too involved to include in this text. I would like to refer you to the proof of Theorem 9.7 in the lecture notes by Evertse [6]. △

**Corollary 2.14.** *(Analytic continuation of the Zeta function)*

*(i) The function $\zeta$ has an analytic continuation to $\mathbb{C}\setminus\{1\}$ with a simple pole with residue 1 at $z = 1$.*

*(ii) This analytic continuation of $\zeta$ is given by*

$$\zeta(1-z) = 2^{1-z} \cdot \pi^{-z} \cdot \cos\left(\frac{\pi z}{2}\right) \cdot \Gamma(z) \cdot \zeta(z) \quad \textit{for } z \in \mathbb{C}\setminus\{0,1\} \tag{2.26}$$

*Sketch of Proof.* (i) We define the analytic continuation of $\zeta$ by rewriting $\xi$.

$$\zeta(z) = \frac{\xi(z)\pi^{z/2} \cdot \Gamma\left(\frac{1}{2}z+1\right)^{-1}}{z-1}.$$

All terms in this product are analytic on $\mathbb{C}$. Hence $\zeta$ is analytic on $\mathbb{C}\setminus\{1\}$. Let us investigate what happens if $z = 1$. We know $\Gamma$ has no zeros and also $\pi^{1/2} \neq 0$. Apart from that we see

$$\xi(1) =$$

Theorem 5.2 in Evertse's lecture notes [7] shows that the unique analytic continuation of $\zeta$ has a simple pole with residue 1 at $z = 1$. That finishes this proof.

*Proof.* (ii) Now we substitute $1-z$ into the equation and then we use the continuation of $\xi$ as stated in Theorem 2.13. For $z \in \mathbb{C}\setminus\{0,1\}$ we have

$$\begin{aligned} \zeta(1-z) &= \frac{\xi(1-z)}{\frac{1}{2}(1-z)(-z)\pi^{-(1-z)/2}\Gamma\left(\frac{1}{2}(1-z)\right)} \\ &= \frac{\xi(z)}{\frac{1}{2}z(z-1)\pi^{-(1-z)/2}\Gamma\left(\frac{1}{2}(1-z)\right)} \\ &= \frac{\frac{1}{2}z(z-1)\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)}{\frac{1}{2}z(z-1)\pi^{-(1-z)/2}\Gamma\left(\frac{1}{2}(1-z)\right)} \cdot \zeta(z). \\ &= F(z)\zeta(z) \end{aligned}$$

Here we have

$$
\begin{aligned}
F(z) &= \pi^{(1/2)-z} \cdot \frac{\Gamma\left(\frac{1}{2}z\right)\Gamma\left(\frac{1}{2}z+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}z\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}z\right)} \\
&= \pi^{(1/2)-z} \frac{2^{1-z}\sqrt{\pi}\,\Gamma(z)}{\pi/\sin\left(\pi\left(\frac{1}{2}-\frac{1}{2}z\right)\right)} \\
&= \pi^{-z}\, 2^{1-z} \cos\left(\frac{1}{2}\pi z\right)\Gamma(z),
\end{aligned}
$$

from which Corollary 2.14 (ii) follows directly.
The second step above is obtained by using Euler's Reflection Formula (Theorem 2.18) and the so-called 'duplication formula' for the Gamma function, which states:

*Let $z \in \mathbb{C}$, such that $z \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, -2, -\frac{5}{2}, \cdots$. Then*

$$\sqrt{\pi} \cdot \Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right).$$

This formula is included as Corollary 8.12 in Evertse's lecture notes [8], in which the proof can be found. $\triangle$

## 2.5 Real Function

*So far we have been building towards this section. We want to prove that the Hardy function $Z(t)$ as defined in (2.1) maps real numbers to real numbers. The $\xi$ function that we just defined will be of importance in proving this. To show the function is indeed real, we prove that, $Z(t) = \overline{Z(t)}$ in Theorem 2.16.*

### 2.5.1 Conjugating $Z(t)$

In Theorem 2.16, we conjugate some of the functions we covered so far. These functions have the properties as in Lemma 2.15. In proving this Lemma, we will make use of Definition 2.2 in the form

$$2i\sin x = e^{ix} - e^{-ix}. \tag{2.27}$$

**Lemma 2.15.** *Recall $\xi$ from Definition 2.5. Let $x > 0$ and $t$ be real numbers and $z$ a complex number. Then the following five things hold.*

*(i)* $\overline{x^z} = x^{\overline{z}}$,

*(ii)* $\overline{\Gamma(z)} = \Gamma(\overline{z})$ *for* $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$,

*(iii)* $\overline{\zeta\left(\frac{1}{2}+it\right)} = \zeta\left(\overline{\frac{1}{2}+it}\right) = \zeta(\frac{1}{2}-it)$ *for $z$ with $Re(z) > 0$,*

*(iv)* $\xi\left(\frac{1}{2}+it\right) = \xi\left(\overline{\frac{1}{2}+it}\right) = \xi(\frac{1}{2}-it)$.

*(v)* $A(z) := \frac{1}{2}z(z-1)$ *is real for* $z = \frac{1}{2}+it$.

*Proof.* (i) Let us write $z = a + bi$ for real $a$ and $b$ and rewrite $x^z$ as a power of $e$ and then calculate its conjugate.

$$\begin{aligned}
\overline{x^{a+bi}} &= \overline{e^{(a+bi)\log x}} \\
&= e^{\overline{(a+bi)\log x}} \\
&= e^{\overline{(a+bi)}\log x} \\
&= x^{\overline{a+bi}}.
\end{aligned}$$

(ii) Let $z \in \mathbb{C}$. We can write $z = (a+1)+bi$. Recall that the Gamma function for $\mathrm{Re}(z) > 0$ is given as

$$\int_0^\infty x^{z-1} e^{-x} dx.$$

We first rewrite the integrand of the Gamma function. Using the fact that for $z \in \mathbb{C}$, we can write $x^z = e^{z \log x}$.

$$\begin{aligned}
x^{z-1}\, e^{-x} &= x^{a+bi}\, e^{-x} \\
&= e^{(\log x)(a+bi)}\, e^{-x} \\
&= e^{a\log x - x}\, e^{bi(\log x)}.
\end{aligned}$$

Now using Euler's identity (2.2) from Proposition 2.2, we have

$$\begin{aligned}
e^{a\log x - x}\, e^{bi(\log x)} &= e^{a\log x} e^{-x} \left(\cos(b\log x) + i\sin(b\log x)\right) \\
&= x^a\, e^{-x} \left(\cos(b\log x) + i\sin(b\log x)\right). \qquad (2.28)
\end{aligned}$$

An integral of a complex function over a real domain can be split into a real and an imaginary part. It therefore has the following property regarding conjugation.

$$\begin{aligned}
\overline{\int_{\mathbb{R}} f(x)dx} &= \overline{\int_{\mathbb{R}} a(x)dx + i\int_{\mathbb{R}} b(x)dx} \\
&= \overline{\int_{\mathbb{R}} a(x)dx} - i\,\overline{\int_{\mathbb{R}} b(x)dx} \\
&= \int_{\mathbb{R}} \overline{a(x)}dx - i\int_{\mathbb{R}} \overline{b(x)}dx \\
&= \int_{\mathbb{R}} \overline{f(x)}dx.
\end{aligned}$$

With the integrand as in 2.28, we continue. For convenience we denote $P := b\log x$ in the following calculations. First we conjugate, then we divide the term $x^a e^{-x}$ out. We make use of the fact that for two complex numbers $u$ and $v$, we have $\overline{u \cdot v} = \overline{u} \cdot \overline{v}$.

$$\begin{aligned}
\overline{\Gamma(z)} &= \overline{\int_0^\infty (\cos P + i\sin P) x^a e^{-x}\, dx} \\
&= \int_0^\infty \overline{(\cos P + i\cdot\sin P) x^a e^{-x}}\, dx \\
&= \int_0^\infty (\cos P - i\cdot\sin P) x^a e^{-x}\, dx \\
&= \int_0^\infty (\cos P + [i\sin P - 2i\sin P]) \cdot x^a e^{-x}\, dx \\
&= \int_0^\infty ([\cos P + i\sin P] - 2i\sin P) \cdot x^a e^{-x}\, dx.
\end{aligned}$$

Now we recognize both Euler's identity and the identity from Definition 2.2. We can now rewrite all terms using powers of $e$ and finish the proof.

$$\begin{aligned}
\overline{\Gamma(z)} &= \int_0^\infty \left(e^{bi\log x} - \frac{2i}{2i}\cdot\left(e^{bi\log x} - e^{-bi\log x}\right)\right)\cdot x^a e^{-x}\,dx \\
&= \int_0^\infty \left(x^{bi} - \frac{2i}{2i}\cdot\left(x^{bi} - x^{-bi}\right)\right)\cdot x^a e^{-x}\,dx \\
&= \int_0^\infty x^{-bi} x^a e^{-x}\,dx \\
&= \int_0^\infty x^{a-bi} e^{-x}\,dx \\
&= \Gamma(\overline{z}).
\end{aligned}$$

In Definition 2.4, we extended $\Gamma$ to $z \in \mathbb{C}\setminus\mathbb{Z}_{<0}$ as follows. If $\mathrm{Re}(z) < 0$, choose an integer $n$ such that $\mathrm{Re}(z) + n > 0$ and define

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}. \tag{2.29}$$

Conjugating equation 2.29 gives

$$\begin{aligned}
\overline{\Gamma(z)} &= \frac{\overline{\Gamma(z+1)}}{\overline{z\cdot(z+1)\cdots(z+n-1)}} \\
&= \frac{\Gamma(\overline{z+n})}{\overline{z}\cdot\overline{(z+1)}\cdots\overline{(z+n-1)}} \\
&= \frac{\Gamma(\overline{z}+n)}{\overline{z}\cdot(\overline{z}+1)\cdots(\overline{z}+n-1)} \\
&= \Gamma(\overline{z}).
\end{aligned}$$

(iii) Recall the analytic continuation of the zeta function as in Proposition 2.26. For a complex number $z$ with $\mathrm{Re}(z) > 0$,

$$\begin{aligned}
\overline{\zeta(z)} &= \overline{\frac{1}{z+1} + \sum_{n=1}^{\infty}\int_n^{n+1}\left(\frac{1}{n^z} - \frac{1}{t^z}\right)dt.} \\
&= \overline{\frac{1}{z+1}} + \sum_{n=1}^{\infty}\overline{\int_n^{n+1}\left(\frac{1}{n^z} - \frac{1}{t^z}\right)dt.} \\
&= \frac{1}{\overline{z}+1} + \sum_{n=1}^{\infty}\int_n^{n+1}\overline{\left(\frac{1}{n^z} - \frac{1}{t^z}\right)}dt. \\
&= \frac{1}{\overline{z}+1} + \sum_{n=1}^{\infty}\int_n^{n+1}\left(\frac{1}{n^{\overline{z}}} - \frac{1}{t^{\overline{z}}}\right)dt. \\
&= \zeta(\overline{z}).
\end{aligned}$$

Here we used (i) of this Lemma and the fact that for complex $u$ and $w$ we have $\overline{u+w} = \overline{u} + \overline{w}$. Furthermore, we applied conjugation to an integral again as in the proof of (ii).

(iv) This follows immediately after plugging in $z = 1/2 + it$ into the functional equality of $\xi$ from Theorem 2.13.

(v) If we take $z = \frac{1}{2} + it$, with $t$ real, we have $A(z) = -\frac{1}{8} - \frac{t^2}{2} \in \mathbb{R}$.

△

**Theorem 2.16.** *The Hardy Function as defined in Definition 2.1 satisfies $Z(t) \in \mathbb{R}$ for all $t \in \mathbb{R}$.*

*Proof.* We first rewrite $Z$ using the $\xi$ function from Definition 2.5 and solve it for $\zeta$. Let $z = \frac{1}{2} + it$ and denote $A(z) = \frac{1}{2}z(z-1)$. Then

$$\zeta(z) = \frac{\xi(z)}{\left(\frac{1}{2}z(z-1)\right) \cdot \Gamma(z/2) \cdot \pi^{-z/2}} = \frac{1}{A(z)\pi^{-1/4}} \cdot \frac{\xi\left(\frac{1}{2} + it\right)}{\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\pi^{-it/2}},$$

with $A(z)\pi^{1/4}$ a real number. We used that $\pi^{-z/2} = \pi^{-1/4}\pi^{-it/2}$. If we substitute this into $Z$, some terms divide out.

$$\begin{aligned} Z(t) &= \frac{\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)}{\left|\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right|} \cdot \pi^{-it/2} \cdot \left(A(z)\pi^{1/4}\right)^{-1} \cdot \frac{\xi\left(\frac{1}{2} + it\right)}{\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\pi^{-it/2}} \\ &= \frac{\left(A(z)\pi^{1/4}\right)^{-1}}{\left|\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right|} \cdot \xi\left(\frac{1}{2} + it\right) \end{aligned}$$

Observe that $Z(t) = \overline{Z(t)}$ if and only if $\xi\left(\frac{1}{2} + it\right) = \overline{\xi\left(\frac{1}{2} + it\right)}$, which is is indeed the case. Using Definition 2.5, we have

$$\overline{\xi\left(\frac{1}{2} + it\right)} = \overline{\frac{A(z)}{\pi^{1/4}} \cdot \pi^{-it/2} \cdot \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) \cdot \zeta\left(\frac{1}{2} + it\right)} = \frac{A(z)}{\pi^{1/4}} \cdot \overline{\pi^{-it/2}} \cdot \overline{\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)} \cdot \overline{\zeta\left(\frac{1}{2} + it\right)}.$$

Now Lemma 2.15 (i) to (iii) implies that this expression is precisely equal to $\xi\left(\overline{\frac{1}{2} + it}\right)$, which is equal to $\xi\left(\frac{1}{2} + it\right)$ by (iv) of this Lemma. That shows

$$\overline{Z(t)} = \frac{\left(A(z)\pi^{1/4}\right)^{-1}}{\left|\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right|} \cdot \overline{\xi\left(\frac{1}{2} + it\right)} = \frac{\left(A(z)\pi^{1/4}\right)^{-1}}{\left|\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right|} \cdot \xi\left(\frac{1}{2} + it\right) = Z(t)$$

so $Z(t)$ is a real number.

△

## 2.6 Calculating zeros

*Now that we know Z is a continuous real function, we can use software to calculate its zeros. Every sign change of Z locates a zero of the zeta function.*

Using Maple, we can plot the function. A plot for $0 \le t \le 100$ is included in Figure 1. The code can be found in the appendix.

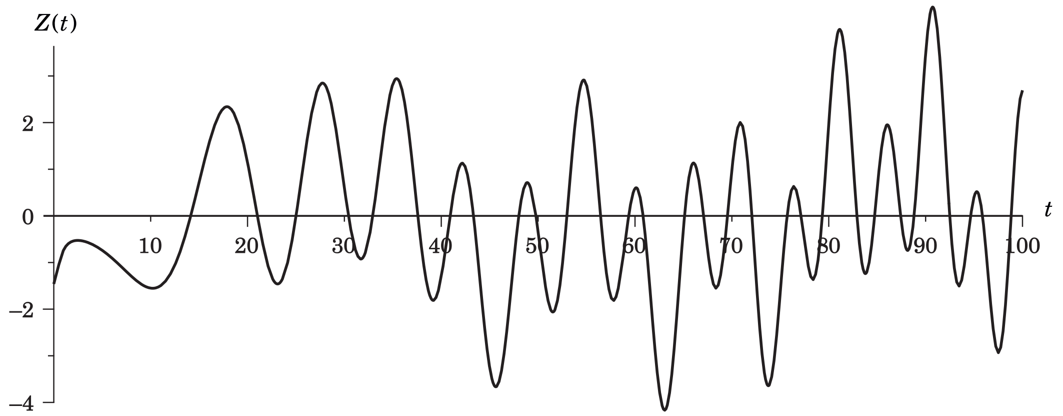


Figure 1: A plot of $Z(t)$

To give another illustration, we will use the `fsolve`[6] function of Maple to produce a table containing the zeroes of $Z$. The results are included in Table 1.

| range of $t$ | zero | range of $t$ | zero | range of $t$ | zero |
|---|---|---|---|---|---|
| (10,20) | 14.13472514 | (51,55) | 52.97032148 | (78,80) | 79.33737502 |
| (20,23) | 21.02203964 | (55,58) | 56.44624770 | (80,83) | 82.91038085 |
| (23,30) | 25.01085758 | (58,60) | 59.34704400 | (83,85) | 84.73549298 |
| (30,32) | 30.42487613 | (60,64) | 60.83177852 | (85,88) | 87.4252746 |
| (32,35) | 32.93506159 | (64,66) | 65.11254405 | (88,90) | 88.80911121 |
| (35,40) | 37.58617816 | (66,68) | 67.07981053 | (90,93) | 92.49189927 |
| (40,42) | 40.91871901 | (68,70) | 69.54640171 | (93,95) | 94.65134404 |
| (42,45) | 43.32707328 | (70,75) | 72.06715767 | (95,97) | 95.87063423 |
| (45,49) | 48.00515088 | (75,76) | 75.70469070 | (97,100) | 98.83119422 |
| (49,51) | 49.77383248 | (76,78) | 77.14484007 | | |

Table 1: Zeroes of $Z(t)$ in the interval $(0,100)$ calculated using Maple

[6] More details can be found in the appendix.

# 3 EXTENTION TO OTHER L-SERIES

## 3.1 Dirichlet L-series

*The $\zeta$ function is a special case of the so-called Dirichlet L-series. In fact, the $\zeta$ is the Dirichlet L-series for a constant function. The aim of this section is to find zeroes of two other L-series using a similar method to that of Chapter 2.*

**Definition 3.1.** [10] A *Dirichlet character* modulo $k$ is a function $\chi_k : \mathbb{N} \to \mathbb{C}$ satisfying the following four conditions for all positive integers $m, n$:

1. $\chi_k(1) = 1$
2. $\chi_k(n) = \chi_k(n+k)$
3. $\chi_k(m)\chi_k(n) = \chi_k(mn)$
4. $\chi_k(n) = 0$ if $k$ and $n$ have a common divisor greater than one.

**Lemma 3.1.** *Let $k$ be a positive integer. Then $\chi_k$ has the following properties.*

*(i) For all positive integers $a$ and $n$ we have $\chi_k(a+kn) = \chi_k(a)$.*

*(ii) The map $\mathbb{Z}/k\mathbb{Z} \to \mathbb{C}$ that sends $(a \text{ mod } k) \mapsto \chi_k(a)$ is well-defined for $a \geq 0$.*

*(iii) If $\overline{a} \in (\mathbb{Z}/k\mathbb{Z})^\times$, then $\chi_k(a) \in \mathbb{C}^\times$.*

*Proof.* (i) Let $a, k, n$ be strictly positive integers. Then $a + kn = a + (k + \cdots + k)$, with $k$ appearing $n$ times. Applying Definition 3.1 part 2 $n$ times shows

$$\chi_k(a+kn) = \chi_k(a + (\underbrace{k + \cdots + k}_{n-\text{times}})) = \chi_k(a + (\underbrace{k + \cdots + k}_{n-1-\text{times}})) = \cdots = \chi_k(a).$$

(ii) We have $(a \text{ mod } k) \mapsto \chi_k(a)$. Now consider another positive integer $b$ such that $b \equiv a \text{ mod } k$. Then there is an integer $n$ such that $b = a + kn$. By (i) of this Lemma it follows that

$$(b \text{ mod } k) \mapsto \chi_k(b) = \chi_k(a+kn) = \chi_k(a).$$

So $\chi_k$ is well-defined.

(iii) Let $\overline{a} \in (\mathbb{Z}/k\mathbb{Z})^\times$. Then there exists an integer $m$ such that $(a \text{ mod } k)^m = \overline{1}$. By using (1) and (3) of Definition 3.1 we obtain

$$\chi_k(a)^m = \chi_k\left(a^m\right) = \chi_k(1) = 1.$$

So $\chi_k(a) \in \mathbb{C}^\times$.

△

For a positive integer $k$, we can construct a new map (that we also call $\chi_k$), mapping from $(\mathbb{Z}/k\mathbb{Z})^\times \to \mathbb{C}^\times$ in such a way that it is a Dirichlet character modulo $k$. Namely if it sends $(1 \bmod k) \mapsto 1$ and $(a \bmod k) \mapsto a$, it satisfies conditions 2 and 3 of Definition 3.1. Note that for all $(a \bmod k) \in (\mathbb{Z}/k\mathbb{Z})^\times$ we have $\gcd(a,k) = 1$. By (4) of Definition 3.1, $\chi_k(a) \neq 0$. The map is well-defined by part (ii) of the previous Lemma. In fact, $\chi_k$ is a homomorphism of groups since for $\overline{a}, \overline{b} \in (\mathbb{Z}/k\mathbb{Z})^\times$, we have $\chi_k(\overline{a})\chi_k(\overline{b}) = \chi_k(\overline{ab})$.

To finish the construction of a Dirichlet character, we use this homomorphism to create an image $\mathbb{N} \to \mathbb{C}$ in the following way.

$$\begin{aligned} \chi_k \ &: \ \mathbb{N} \to \mathbb{C} \\ &: \ n \mapsto \begin{cases} \chi_k(\overline{n}) & \text{if} \quad n \bmod k \in (\mathbb{Z}/k\mathbb{Z})^\times \\ 0 & \text{if} \quad n \bmod k \notin (\mathbb{Z}/k\mathbb{Z})^\times \end{cases} \end{aligned}$$

We will now look at two Dirichlet characters of this type. Namely $\chi_3$ and $\chi_4$. They are illustrated in the following two examples.

**Example 3.2.** For $k = 3$, the group $(\mathbb{Z}/3\mathbb{Z})^\times = \{\overline{1}, \overline{2}\}$ has two elements. We can therefore write down $\chi_k$ as an explicit mapping.

$$\begin{aligned} \chi_3 : (\mathbb{Z}/3\mathbb{Z})^\times &\to \{\pm 1\} \\ \overline{2} &\mapsto -1 \\ \overline{1} &\mapsto 1 \end{aligned}$$

Note that it is indeed a homomorphism; $\chi_3(\overline{1})\chi_3(\overline{2}) = \chi_3(\overline{2})$ and $\chi_3(\overline{2})\chi_3(\overline{2}) = \chi_3(\overline{4})$. As a map from $\mathbb{N}$ to $\mathbb{C}$ we have

$$\chi_3 : n \mapsto \begin{cases} 1 & \text{if} \quad \overline{n} \equiv 1 \bmod 3 \\ -1 & \text{if} \quad \overline{n} \equiv 2 \bmod 3 \\ 0 & \text{otherwise} \end{cases}$$

**Example 3.3.** If we now let $k = 4$, we see $(\mathbb{Z}/4\mathbb{Z})^\times = \{\overline{1}, \overline{3}\}$. We can do the same as in the previous example.

$$\begin{aligned} \chi_4 : (\mathbb{Z}/4\mathbb{Z})^\times &\to \{\pm 1\} \\ \overline{3} &\mapsto -1 \\ \overline{1} &\mapsto 1 \end{aligned}$$

This is also a homomorphism since $\chi_4(\overline{1})\chi_4(\overline{3}) = \chi_4(\overline{3})$ and $\chi_4(\overline{3})\chi_4(\overline{3}) = \chi_4(\overline{9})$ and finally, $\chi_4(1)\chi_3(1) = \chi_3(1)$. As a map from $\mathbb{N}$ to $\mathbb{C}$ we have

$$\chi_4 : n \mapsto \begin{cases} 1 & \text{if} \quad \overline{n} \equiv 1 \bmod 4 \\ -1 & \text{if} \quad \overline{n} \equiv 3 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

The Dirichlet characters play an important role in the so-called Dirichlet $L$-series.

**Definition 3.2.** A *Dirichlet L-series* is a series of the form

$$L_k(z, \chi) = \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^z} \tag{3.1}$$

where $z$ is a complex number with $\mathrm{Re}(z) > 0$ and $\chi_k(n) : \mathbb{N} \to \mathbb{C}$ is a Dirichlet character modulo $k$. [11]

**Example 3.4.** The Riemann $\zeta$-function we discussed in the previous chapter is a Dirichlet $L$-series using the constant function $\chi_1(n) = 1$.

**Example 3.5.** We can use Examples 3.2 an 3.3 and obtain the Dirichlet $L$-series for $k = 3$ and for $k = 4$.

$$\begin{aligned} L(z,\chi_3) &= \sum_{n=1}^{\infty} \frac{\chi_3(n)}{n^z} = \sum_{3\nmid n}^{\infty} \frac{(-1)^{n-1}}{n^z} \\ L(z,\chi_4) &= \sum_{n=1}^{\infty} \frac{\chi_4(n)}{n^z} = \sum_{n\,\mathrm{odd}}^{\infty} \frac{(-1)^{(n-1)/2}}{n^z} \end{aligned}$$

**Theorem 3.6.** *For Re$(z) > 1$, the functions $L(z,\chi_3)$ and $L(z,\chi_4)$ converge absolutely.*

*Proof.* First consider $L(z,\chi_4)$ for $\mathrm{Re}(z) > 1$. Taking absolute values, we see

$$\left|L(z,\chi_4)\right| = \sum_{n\,\mathrm{odd}}^{\infty} \left|\frac{(-1)^{(n-1)/2}}{n^z}\right| = \sum_{n\,\mathrm{odd}}^{\infty} \frac{1}{n^{\mathrm{Re}(z)}}$$

where we used $|n^z| = n^{\mathrm{Re}(z)}$ from Proposition 2.3. Let us define $a_n = 1/n^z + 1/(n+1)^z$. We can write

$$|\zeta(z)| = \sum_{n\,\mathrm{odd}}^{\infty} a_n.$$

Since $\mathrm{Re}(z) > 1$, we have for all natural numbers $n > 0$,

$$\begin{aligned} 0 \le \left|\frac{1}{n^z}\right| = \frac{1}{n^{\mathrm{Re}(z)}} &\le |a_n| \\ &= \left|\frac{1}{n^z} + \frac{1}{(n+1)^z}\right| \\ &\le \left|\frac{1}{n^z}\right| + \left|\frac{1}{(n+1)^z}\right| \\ &= \frac{1}{n^{\mathrm{Re}(z)}} + \frac{1}{(n+1)^{\mathrm{Re}(z)}}. \end{aligned}$$

Since $\zeta(z)$ converges absolutely, it follows that $\sum_{n\,\mathrm{odd}}^{\infty} |1/n^z|$ converges by the Direct Comparison Test. That shows $L(z,\chi_4)$ converges absolutely.

Now consider $L(z,\chi_3)$ for $\mathrm{Re}(z) > 1$. We have

$$\left|L(z,\chi_3)\right| = \sum_{3\nmid n}^{\infty} \left|\frac{(-1)^{n-1}}{n^z}\right| = \sum_{3\nmid n}^{\infty} \frac{1}{n^{\mathrm{Re}(z)}} = |\zeta(z)| - \sum_{3\mid n}^{\infty} \frac{1}{n^{\mathrm{Re}(z)}}. \tag{3.2}$$

We will show that the latter sum[7] converges. so that the right-hand side converges absolutely. Since $\mathrm{Re}(z) > 1$, we can write

$$\begin{aligned} \sum_{3|n}^{\infty} \frac{1}{n^{\mathrm{Re}(z)}} &= \sum_{m=1}^{\infty} \frac{1}{3m^{\mathrm{Re}(z)}} \\ &= \frac{1}{3^{\mathrm{Re}(z)}} \sum_{m=1}^{\infty} \frac{1}{m^{\mathrm{Re}(z)}} \\ &= \frac{1}{3^{\mathrm{Re}(z)}} |\zeta(z)|. \end{aligned}$$

Since $\zeta(z)$ converges absolutely, we see that $\sum_{3\mid n}^{\infty} \frac{1}{n^z}$ converges absolutely as well. From equation (3.2) we see that $L(z,\chi_3)$ converges absolutely. △

[7] Note that here summation is taken over values of $n$ that *do have* 3 as a divisor.

## 3.2 Functional equations of $L$-series

*Just as with the zeta function, we will study implicit functional equations for $L(z,\chi_3)$ and $L(z,\chi_4)$ that we call $\xi$. These equations will be used to construct 'Hardy functions' (i.e. functions analogous to the Hardy function) in the next section.*

**Definition 3.3.** [12] For a Dirichlet character $\chi_k$ the *Gauss sum* $\tau(\chi)$ is defined by

$$\tau\left(\chi_k\right) = \sum_{m=1}^{k} \chi_k(m) e^{2\pi m i/k}$$

**Definition 3.4.** A Dirichlet character $\chi_k$ is called *even* if $\chi_k(k-1) = 1$ and *odd* if $\chi_k(k-1) = -1$.

**Theorem 3.7.** *(Functional equation of $L(z,\chi_k)$) Consider a positive integer $k$, and an odd Dirichlet character $\chi_k$. If for a complex number $z$ having $Re(z) > 1$,*

$$\xi(z,\chi_k) = \pi^{-z/2} k^{z/2} \Gamma\left(\frac{z+1}{2}\right) L(z,\chi_k) \tag{3.3}$$

*then $\xi(z,\chi_k)$ is an entire function and*

$$\xi(z,\chi_k) = w_{\chi_k} \xi(1-z,\overline{\chi_k}) \tag{3.4}$$

*where $w_{\chi_k} = \tau\left(\chi_k\right)/ik^{1/2}$ and $\overline{\chi_k}$ is the complex conjugate of $\chi_k$.*

*Proof.* The derivation of the $\xi(z,\chi_k)$ is included as Exercise 5.4.5 in the book by M. Ram Murty [12]. In the solutions section the derivation can be found on page 310. The $q$ used in that proof is $k = 3,4$ in our case. ⚠

Let us elaborate on some terms that appear in equations (3.3) and (3.4). We are interested in the Dirichlet $L$-series for $k = 3$ and $k = 4$. So let us calculate $w_{\chi_3}$ and $w_{\chi_4}$. Note that the Gauss sum $\tau(\chi_k)$ from Definition 3.3 appears in the product.
In the calculation we will use that

$$\begin{aligned}
e^{2\pi i/3} &= -1/2 + 1/2\sqrt{3}\cdot i \\
e^{4\pi i/3} &= -1/2 - 1/2\sqrt{3}\cdot i \\
e^{2\pi i/4} &= = i \\
e^{6\pi i/4} &= -i.
\end{aligned}$$

Using Theorem 3.7 we have

$$\begin{aligned}
w_{\chi_3} &= \frac{1}{i\sqrt{3}} \tau\left(\chi_3\right) \\
&= \frac{1}{i\sqrt{3}} \sum_{m=1}^{3} \chi_3(m) e^{2\pi m i/3} \\
&= \frac{1}{i\sqrt{3}} \left(e^{2\pi i/3} - e^{4\pi i/3}\right) \\
&= \frac{1}{i\sqrt{3}} i\sqrt{3} \\
&= 1.
\end{aligned}$$

and in the same way,

$$\begin{aligned} w_{\chi_4} &= \frac{1}{2i}\tau\left(\chi_4\right) \\ &= \frac{1}{2i}\sum_{m=1}^{4}\chi_4(m)e^{2\pi mi/4} \\ &= \frac{1}{2i}\left(e^{2\pi i/4}-e^{6\pi i/4}\right) \\ &= \frac{1}{2i}2i \\ &= 1. \end{aligned}$$

Now we almost have the functional equations for the $L$-series for $k=3$ and $k=4$. We saw in equation (3.4) the complex conjugate $\overline{\chi_k}$. We have the following.

**Lemma 3.8.** *For $k=3,4$, we have $\chi_k=\overline{\chi_k}:\mathbb{N}\to\mathbb{C}$.*

*Proof.* For a natural number $k$ we can display the map $\chi_k$ in two steps as following:

$$\overline{\chi_k}:\mathbb{N}\xrightarrow{\chi_k}\mathbb{C}\xrightarrow{\text{conj.}}\mathbb{C},$$

such that a natural number $n$ maps to $\chi_k(n)$ and then to its conjugate $\overline{\chi_k(n)}$ using the conjugation map. We know explicitly what the images of $\chi_3$ and $\chi_4$ onto $\mathbb{C}$ are from Examples 3.2 and 3.3. Namely for $k=3,4$ we have $\chi_k(\mathbb{N})=\{0,1,-1\}$. All elements in the image are real and thus invariant under the conjugation map. Hence $\chi_k=\overline{\chi_k}$. △

In conclusion, we have, using (3.3) and (3.4).

$$\xi(z,\chi_3)=\pi^{-z/2}3^{z/2}\Gamma\left(\frac{z+1}{2}\right)L(z,\chi_3) \tag{3.5}$$

such that $\xi(z,\chi_3)=\xi(1-z,\chi_3)$. And

$$\xi(z,\chi_4)=\pi^{-z/2}4^{z/2}\Gamma\left(\frac{z+1}{2}\right)L(z,\chi_4) \tag{3.6}$$

such that $\xi(z,\chi_4)=\xi(1-z,\chi_4)$.

## 3.3 Constructing 'Hardy Functions' for $L$-series

*Now that we have found equations for $\xi(z,\chi_3)$ and $\xi(z,\chi_4)$ and their functional equations, we can try to construct functions $Z(t,\chi_k):\mathbb{R}\to\mathbb{R}$ for $L(z,\chi_k)$ that have the same properties as the Hardy function from Definition 2.1 has. That is, finding a zero of $Z(t,\chi_k)$ is equivalent to finding a zero of $L(z,\chi_k)$. We make use of the generalized Riemann hypothesis, which postulates that non-trivial zeros of L-series have real part equal to* 1/2.

Let us first consider $Z(t,\chi_3)$. We want $Z(t,\chi_3)$ to be a function mapping real numbers to real numbers. We start off with $\xi(z,\chi_3)$ as in equation (3.5) and bring $L(z,\chi_3)$ to the left-hand side. The generalized Riemann hypothesis postulates that if $L(z,\chi_k)$ is a Dirichlet $L$-series with $\chi_k$ a character modulo $k$, a zero $z$ of $L(z,\chi_k)$ which is not $z=m$ for some negative integer $m$, has real part equal to 1/2. This is what we will use. Analogous to the Hardy function, we insert the complex value $z = 1/2 + it$ for a real variable $t$ into the $L(z, \chi_3)$.

$$\begin{aligned} L(z,\chi_3) &= \frac{\xi(1/2+it,\chi_3)}{\pi^{-z/2}3^{z/2}\Gamma\left(\frac{z+1}{2}\right)} \\ L(1/2+it,\chi_3) &= \frac{\xi(1/2+it,\chi_3)}{\Gamma(3/4+it/2)}\cdot\left(\frac{\pi}{3}\right)^{it/2}\left(\frac{\pi}{3}\right)^{1/4} \end{aligned}$$

Now we modify the equality such that we obtain a real number on the right-hand side. If we first multiply by $(3/\pi)^{it/2}$ and then by $\Gamma(3/4+it/2)/|\Gamma(3/4+it/2)|$ (analogue to the way the Hardy function has been obtained), terms will be divided out such that we have the following equality:

$$\frac{\Gamma(3/4+it/2)}{|\Gamma(3/4+it/2)|}\left(\frac{3}{\pi}\right)^{it/2}\cdot L(1/2+it,\chi_3) = \xi(1/2+it,\chi_3)\cdot\underbrace{\frac{\left(\frac{\pi}{3}\right)^{1/4}}{|\Gamma(3/4+it/2)|}}_{\in\mathbb{R}} \tag{3.7}$$

We make two observations about this equality.

(i) Suppose we find a value for $t$ such that the left-hand side of this equation is zero. We have shown before that $\Gamma$ has no zeroes. Furthermore, $(3/\pi)^{it/2}$ cannot be zero. The conclusion is then that $L(1/2+it,\chi_3) = 0$.

(ii) We see that the equality is real if and only if $\xi(1/2+it,\chi_3)$ is real. We prove that in Lemma 3.10.

Before we can prove Lemma 3.10, we will quickly recall Lemma 2.15 (i) and (ii) from the previous section. We saw that for complex $z$ and real positive number $x$ we have $\overline{\Gamma(z)} = \Gamma(\overline{z})$ and $\overline{x^z} = x^{\overline{z}}$. We also need the following

**Lemma 3.9.** *For $k = 3,4$, $z$ a complex number and $\chi_k$ a Dirichlet character modulo $k$ we have $\overline{L(z,\chi_k)} = L(\overline{z},\overline{\chi_k}) = L(\overline{z},\chi_k)$.*

*Proof.* The proof is similar to that of Lemma 2.15 (iii). Take a complex number $z$ and $k = 3,4$. We have

$$\begin{aligned} \overline{L(z,\chi_k)} &= \overline{\sum_{n=1}^{\infty}\frac{\chi_k(n)}{n^z}} \\ &= \sum_{n=1}^{\infty}\overline{\left(\frac{\chi_k(n)}{e^{z\log n}}\right)} \\ &= \sum_{n=1}^{\infty}\frac{\overline{\chi_k(n)}}{\overline{e^{z\log n}}} \\ &= \sum_{n=1}^{\infty}\frac{\overline{\chi_k(n)}}{e^{\overline{z}\log n}} \\ &= \sum_{n=1}^{\infty}\frac{\overline{\chi_k(n)}}{n^{\overline{z}}} \\ &= L(\overline{z},\overline{\chi_k}) \end{aligned} \tag{3.8}$$

Using Lemma 3.8, we see $L(\overline{z},\overline{\chi_k}) = L(\overline{z},\chi_k)$, which finishes the proof. △

**Lemma 3.10.** *Let $t$ be a real number, $k = 3,4$ and consider $\xi$ as in* (3.3)*. Then $\xi(1/2+it,\chi_k)$ is a real number.*

*Proof.* We will show that $\overline{\xi(1/2+it,\chi_k)} = \xi(1/2+it,\chi_k)$. We use equation (3.3) and use Lemma 2.15 and 3.9 to obtain the following.

$$\begin{aligned}
\overline{\xi(1/2+it,\chi_k)} &= \overline{\pi^{-(1/2+it)/2}\cdot k^{(1/2+it)/2}\cdot\Gamma\left(\frac{(1/2+it)+1}{2}\right)\cdot L(1/2+it,\chi_k)} \\
&= \overline{\pi^{-(1/2+it)/2}}\cdot\overline{k^{(1/2+it)/2}}\cdot\overline{\Gamma\left(\frac{(1/2+it)+1}{2}\right)}\cdot\overline{L(1/2+it,\chi_k)} \\
&= \pi^{\overline{-(1/2+it)/2}}\cdot k^{\overline{(1/2+it)/2}}\cdot\Gamma\left(\frac{\overline{(1/2+it)+1}}{2}\right)\cdot L(\overline{1/2+it},\chi_k) \\
&= \pi^{-(1/2-it)/2}\cdot k^{(1/2-it)/2}\cdot\Gamma\left(\frac{(1/2-it)+1}{2}\right)\cdot L(1/2-it,\chi_k) \\
&= \xi(1/2-it,\chi_k)
\end{aligned}$$

By the functional equality of $\xi$ it follows that $\overline{\xi(1/2+it,\chi_k)} = \xi(1/2-it,\chi_k) = \xi(1/2+it,\chi_k)$. This shows that $\xi(1/2+it,\chi_k)$ is a real number for $t\in\mathbb{R}$. △

Now we know that $\xi(1/2+it,\chi_k)$ is a real number, we recall equation (3.7):

$$\frac{\Gamma(3/4+it/2)}{|\Gamma(3/4+it/2)|}\left(\frac{3}{\pi}\right)^{it/2}\cdot L(1/2+it,\chi_3) = \xi(1/2+it,\chi_3)\cdot\underbrace{\frac{\left(\frac{\pi}{3}\right)^{1/4}}{|\Gamma(3/4+it/2)|}}_{\in\mathbb{R}}$$

We now see that the right-hand side is a product of real numbers and hence real. This finishes the construction of $Z(t,\chi_3)$. We define it as follows.

**Definition 3.5.** Define the function $Z(t,\chi_3):\mathbb{R}\to\mathbb{R}$ as

$$Z(t,\chi_3) = \frac{\Gamma(3/4+it/2)}{|\Gamma(3/4+it/2)|}\left(\frac{3}{\pi}\right)^{it/2}\cdot L(1/2+it,\chi_3)$$

where $L(1/2+it,\chi_3)$ is the Dirichlet $L$-function for $\chi_3$.

If we would have started our reasoning with $L(z,\chi_4)$, this would have yielded an analogous result which is the following.

**Definition 3.6.** Define the function $Z(t,\chi_4):\mathbb{R}\to\mathbb{R}$ as

$$Z(t,\chi_4) = \frac{\Gamma(3/4+it/2)}{|\Gamma(3/4+it/2)|}\left(\frac{4}{\pi}\right)^{it/2}\cdot L(1/2+it,\chi_4)$$

where $L(1/2+it,\chi_3)$ is the Dirichlet $L$-function for $\chi_3$.

## 3.4 Calculating zeros

*Now that we know $Z(t,\chi_3)$ and $Z(t,\chi_4)$, we can just as with the Hardy function, use software to calculate zeros.*

Python supports both the complex Gamma function and Dirichlet $L$-series. The code that was used to calculate their function values can be found in the appendix. Plots for $0 \le t \le 100$ for $Z(t,\chi_3)$ and $Z(t,\chi_4)$ are included in Figures 2 and 3. The function as

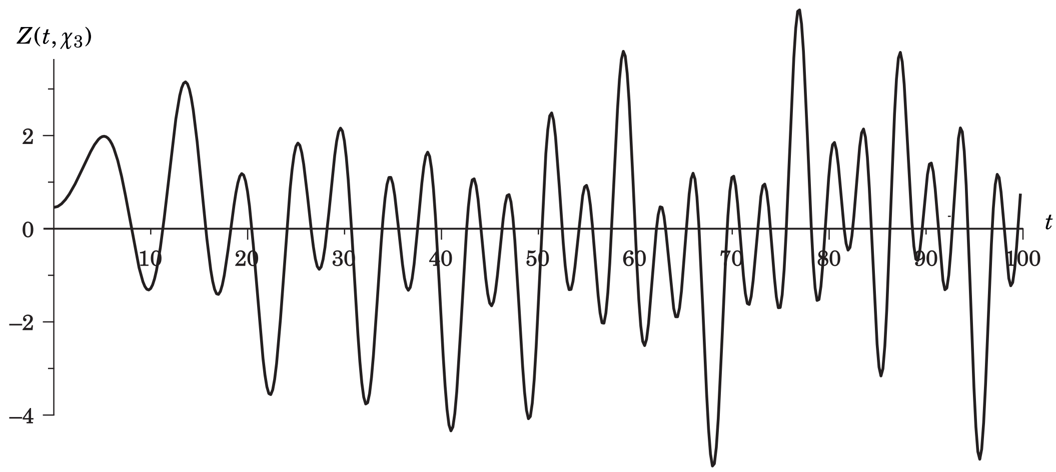


Figure 2: A plot of $Z(t,\chi_3)$

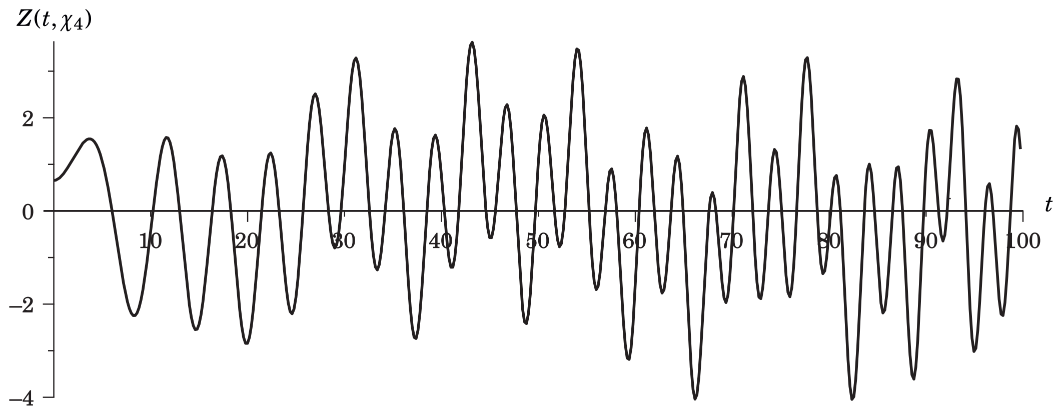


Figure 3: A plot of $Z(t,\chi_4)$

defined in Python can also be used to approximate the zeroes. In the interval $t \in (0,100)$ we find that $Z(t,\chi_4)$ has a root at the following values for $t$. To calculate these values, the `optimize.brentq`-function in Python was used.

| | | |
|---|---|---|
| 6.02094890469759 | 18.291993196123535 | 28.359634343025327 |
| 10.243770304167027 | 21.450611343983805 | 32.59218652711716 |
| 12.988098012312422 | 23.27837652045953 | 34.19995750921315 |
| 16.34260710458749 | 25.728756425088605 | 36.14288045830314 |

| | | |
|---|---|---|
| 38.51192314171866 | 60.42171394900784 | 81.21395162688314 |
| 40.322674066690546 | 62.00863228576777 | 83.66665601447087 |
| 41.80708462000456 | 63.71464111878544 | 84.73174036378119 |
| 44.6178910586623 | 64.976170573096 | 86.57766016839027 |
| 45.59958439679156 | 67.63692086354608 | 87.6297181195879 |
| 47.74156228093914 | 70.18587990880211 | 89.80113161669584 |
| 49.72312932378259 | 72.15548497438188 | 91.34970381469758 |
| 51.68609345287053 | 73.7676355214859 | 92.23749991045423 |
| 52.768820767804726 | 75.1431216474331 | 94.1666195859601 |
| 55.26754358469923 | 76.6963032034302 | 96.13601116178056 |
| 56.934374055202426 | 78.80999831432092 | 96.96174157941749 |
| 58.11670711067392 | 80.21013123836664 | 98.75530041575453 |

# 4 Conclusion and Discussion

For the Riemann zeta function, and the two Dirichlet $L$-series $L(z, \chi_3)$ and $L(z, \chi_4)$ we found a way to detect their zeros. The generalized Riemann hypothesis postulates that zeros $z$ of these three functions that are not of the form $z = m$ for a negative integer $m$, have real part equal to 1/2. Therefore we looked for zeros of this form. The functions we found to detect such zeros are the following.

- Zeros of the Riemann zeta function $\zeta$ can be detected using the Hardy function, that is defined as $Z(t) : \mathbb{R} \to \mathbb{R}$:

  $$Z(t) = \pi^{-it/2} \cdot \frac{\Gamma(1/4 + it/2)}{|\Gamma(1/4 + it/2)|} \cdot \zeta(1/2 + it).$$

  It has the property that $Z : t \mapsto 0$ if and only if $\zeta : 1/2 + it \mapsto 0$.

- Zeros of the $L$-series $L(z, \chi_3)$ can be detected using the function $Z(t, \chi_3) : \mathbb{R} \to \mathbb{R}$:

  $$Z(t, \chi_3) = \frac{\Gamma(3/4 + it/2)}{|\Gamma(3/4 + it/2)|} \left(\frac{3}{\pi}\right)^{it/2} \cdot L(1/2 + it, \chi_3).$$

  It has the property that $Z(t, \chi_3) : t \mapsto 0$ if and only if $L(z, \chi_3) : 1/2 + it \mapsto 0$.

- Zeros of the $L$-series $L(z, \chi_4)$ can be detected using the function $Z(t, \chi_4) : \mathbb{R} \to \mathbb{R}$:

  $$Z(t, \chi_4) = \frac{\Gamma(3/4 + it/2)}{|\Gamma(3/4 + it/2)|} \left(\frac{4}{\pi}\right)^{it/2} \cdot L(1/2 + it, \chi_4).$$

  It has the property that $Z(t, \chi_3) : t \mapsto 0$ if and only if $L(z, \chi_3) : 1/2 + it \mapsto 0$.

We found analytic continuation of the zeta function to make sense of complex numbers having real part equal to 1/2. However, no such continuation of the both $L$ functions is included. Finding such a continuation would complete the section about $L$-series. Also, it might be interesting to study how to calculate the zeros of $L$-functions with character $\chi_k$ for an integer $k$. In particular, given the similarities of the Hardy functions, there may be a way to write a computer program that calculates the zeros of these functions in a given interval for a given natural number $k$.

# REFERENCES


[1] Harold M. Edwards, *Riemann's Zeta Function*, Courier Corporation, 2001.

[2] J.M. Borwein, D. M. Bradley, R.E. Crandall, *Computational Strategies for the Riemann Zeta Function*, Journal of Computational and Applied Mathematics 121, 2000.

[3] J. P. Gram, *Note sur les zéros de la fonction de Riemann.* Acta Mathematica, 1903.

[4] Xavier Gourdon, *The* $10^13$ *first zeros of the Riemann Zeta function, and zeros computation at very large height.* Source: `http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeros1e13-1e24.pdf`

[5] Anatoly A. Karatsuba and S. M. Voronin, *The Riemann Zeta-Function*. De Gruyter, 1992.

[6] Dr. Jan-Hendrik Evertse, *Course notes 'Analytic Number Theory'*, Chapter 9. University of Leiden, 2013.
Source: `http://www.math.leidenuniv.nl/~evertse/ant13-9.pdf`

[7] Dr. Jan-Hendrik Evertse, *Course notes 'Analytic Number Theory'*, Chapter 5. University of Leiden, 2013.
Source: `http://www.math.leidenuniv.nl/~evertse/ant13-5.pdf`

[8] Dr. Jan-Hendrik Evertse, *Course notes 'Analytic Number Theory'*, Chapter 8. University of Leiden, 2013.
Source: `http://www.math.leidenuniv.nl/~evertse/ant13-8.pdf`

[9] Jean Pierre Serre, *A Course in Arithmetic*, Chapter 3.2. Springer-Verlag, 1973.

[10] Jonathan Sondow and Eric W. Weisstein, *Number Theoretic Character*, MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/NumberTheoreticCharacter.html`

[11] Eric W. Weisstein, *Dirichlet L-Series*, MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/DirichletL-Series.html`

[12] M. Ram Murty, *Probems in Analytic Number Theory*, Graduate Texts in Mathematics, Springer-Verlag, 2001.

[13] *ZetaGrid*, Wikimedia Foundation, last updated in December 2019. Source: `https://en.wikipedia.org/wiki/ZetaGrid`

# 5 APPENDIX

## 5.1 Code in Maple

The function values of the Hardy function were calculated using Maple using the code

```
Z:=t->Pi^(-I*t/2)*GAMMA(1/4+I*t/2)/abs(GAMMA(1/4+I*t/2))*Zeta(1/2+I*t)
plot(Z(t),t=0..100)
```

To find zeros of the Hardy function, we also use Maple. To do so, the command `fsolve`, wants us to define a range for $t$ in which a zero of $Z$ can be found. For example, from the plot we see that the first zero $t_1 \in (10,20)$. We give the command

```
t1:=fsolve(Z,10..20)
```

and Maple returns the value of $t_1 = 14.13472514$. The results are included in Table 1 of section 2.

## 5.2 Code in Python

For the function values of the $L$-series, we used Python. In the Python code we make use of the `dirichlet`-function to generate $L(z,\chi_k)$. The code uses two input arguments; a complex number $z$ and the list $[\chi_k(0),\chi_k(1),\cdots,\chi_k(k-1)]$, consisting of the function values of the Dirichlet character $\chi_k$. In the case $k = 3,4$ we use

$$\begin{aligned} L(z,\chi_3) &= \texttt{dirichlet(z,[0,1,-1])} \\ L(z,\chi_4) &= \texttt{dirichlet(z,[0,1,0,-1])}. \end{aligned}$$

This function could be used to plot the Zeta function as well In that case

$$\zeta(z) = \texttt{dirichlet(z,[1])}$$

The Python code for plotting $Z(t,\chi_3)$ is displayed below. Note that in defining `Z3` there are two splits in the code line to fit the page.

```
from mpmath import *
import math
import matplotlib.pyplot as plt
import numpy as np

def Z3(t):
    return np.real((3/math.pi)**(t/2*j)*
    dirichlet(1/2+t*j,[0,1,-1])*
    gamma(3/4+t/2*j)/abs(gamma(3/4+t/2*j)))
t = np.arange(0.0, 100.0, 1)

y3=[0 for i in range(500)]
for i in range(500):
 y3[i]=Z3(i/5)

t=np.arange(0.0, 100.0, 0.2)
```

```
plt.plot(t, y3, 'k')
plt.savefig('Z3plot.pdf')
```

A similar Python code plots $Z(t, \chi_4)$. Note that in defining `Z4` there are two splits in the code line to fit the page.

```
\begin{verbatim}
from mpmath import *
import math
import matplotlib.pyplot as plt
import numpy as np

def Z4(t):
    return np.real((4/math.pi)**(t/2*j)*
    dirichlet(1/2+t*j,[0,1,0,-1])*
    gamma(3/4+t/2*j)/abs(gamma(3/4+t/2*j)))
t = np.arange(0.0, 100.0, 1)

y4=[0 for i in range(500)]
for i in range(500):
 y4[i]=Z4(i/5)

t=np.arange(0.0, 100.0, 0.2)

plt.plot(t, y4, 'k')
plt.savefig('Z4plot.pdf')
```