university of groningen

faculty of science and engineering

mathematics and applied mathematics

Extreme Value Laws for the Generalised Doubling Map Process

Master Project Mathematics

May 2020

Student: N.B-S. Boer

First supervisor: Dr. A.E. Sterk

Second supervisor: Prof.dr. H. Waalkens

Abstract

While it is usually not possible to fully describe the behaviour of a chaotic dynamical system, in certain cases it is possible to derive its asymptotic behaviour. Extreme value theory is the study of looking at extreme values of such a chaotic dynamical system, and prove the asymptotic behaviour of these extreme values. In this paper, we will derive the asymptotic behaviour of the (generalised) doubling map using two methods. First by exactly solving a recursion formula and then using Tannery's theorem, and alternatively by applying properties of generalised Fibonacci sequences to values that satisfy the recursive relation. Finally, we will interpret the obtained results using extreme value theory.

Contents

1. Introduction

In the field of dynamical systems, an often studied phenomenon is the presence of chaos. For dynamical systems, chaos may as well be synonymous with unpredictability, as it is defined by sensitivity to initial conditions, among other criteria. In non-chaotic dynamical systems, it is usually possible to fully describe regular behaviour, for instance when all trajectories eventually converge towards a stable point. Or in particular, to predict the future state of the dynamical system, given the current initial conditions. Whenever a dynamical system is chaotic, it is usually not viable to predict all such exact behaviour. But is it possible to retrieve some information about a chaotic dynamical system, by analysing its limiting behaviour?

In particular, this question has been studied a lot for specific discrete dynamical systems by using extreme value theory. To set up a discrete dynamical system, one only needs an interval I (or more generally, a metric space), and a function $f : I \mapsto I$. Then, for any starting condition $x_0 \in I$, the value is iterated by the function to generate a sequence of values $x_n = f^n(x_0)$. Extreme value theory is concerned with analysing the behaviour of sequences that are governed by probability distributions, and in particular study the behaviour of extreme values in such sequences. This field of study connects well to the research question at hand, namely analysing the limiting behaviour of a chaotic dynamical system. More concretely, in extreme value theory a sequence of random variables (X_i) is studied, which in this case is generated by an iterative map. Then rather than analyse the behaviour of the sequence (X_i), let us consider its extreme values. That is, we define the partial maximum

$$M_n = \max\{X_1, \ldots, X_n\}$$

and try to find its limiting probability distribution, and extreme value law. This will be constructed and explained more elaborately in chapters 2 and 4 of this paper.

Other papers have used extreme value theory in a similar way to analyse other iterative maps before. For instance, both [5] and [6] have analysed the limiting behaviour of the tent map. These papers found that even though the tent map is chaotic, it is possible to analyse and prove the extreme value law of sequences generated by the tent map. Specifically, [5] used that under certain conditions, the limiting probability distribution of such a sequence can be predicted entirely by the similar behaviour of an independently uniformly distributed sequence. Of course, the latter behaviour is easy to deduce, and since it was proven in [5] that the tent map satisfies these specific conditions, its extreme value law could also be derived. In chapter 4, we will discuss whether it is possible to apply this same reasoning to the doubling map, and if not, how this can be explained using extreme value theory.

In [6] on the other hand, a more hands-on approach was taken to derive the limiting behaviour of the tent map. Rather than using existing theorems, which rely on the map in question to satisfy certain conditions, the exact behaviour of the extreme values was derived. The limiting case of the obtained expressions could then be used to derive the extreme value law of the tent map. The specific research method that was used will be discussed more elaborately throughout this paper, as we will be following a similar approach. Namely, the goal of this paper is to reproduce similar results for the doubling map, which is also known as the Bernoulli map or the dyadic map.

In particular, we will first discuss and derive the behaviour of extreme values for the doubling map in chapter 2. Since the exact expressions that will be derived are instrumental in determining the extreme value laws later on, in this chapter we will also focus on providing intuition into the results and their derivation. In fact, a similar study has already been done in [7], where the extreme value law of the doubling map is derived, but no complete proof is given. Moreover, in [7] a another result is deduced, namely the extreme value law of the generalised doubling map, also called the Rényi map or the β-adic map. However, only a derivation of this has been given for a special case of the generalised doubling map. Therefore, in chapter 3 we will generalise the results of the previous chapter in order to obtain and prove the extreme value law of the generalised doubling map.

In both chapters there will also be observations on the occurrence of generalised Fibonacci sequences when studying the behaviour of the extreme values of the (generalised) doubling map. Using this, an alternate proof of both results is given in sections 2.5 and 3.2 which exploits the involvement of generalised Fibonacci sequences. Given the remarkable way in which generalised Fibonacci sequences occur and can be used here, we published a separate article dedicated to proving the results in this paper using a generalised form of the Binet formula, which can be found in [1]. Finally, we will try to extend the results as presented in [7] to another class of the generalised doubling map, in which the coefficient $\beta > 1$ is not necessarily an integer value, which is done in chapter 5.

2. Extreme Values of the Doubling Map Process

The goal of this chapter will be to construct and analyse the so-called doubling map process. In particular, we are interested in finding a pattern in the asymptotic behaviour of extreme values of this doubling map process. To construct this process, let us first introduce the doubling map.

2.1. The Doubling Map Process

The doubling map is given by the iterative map

$$f(x) = 2x \bmod 1 = \begin{cases} 2x & \text{for } 0 \leq x < \frac{1}{2} \\ 2x - 1 & \text{for } \frac{1}{2} \leq x < 1 \end{cases} \tag{2.1}$$

on the interval $x \in [0, 1)$, and is depicted below.

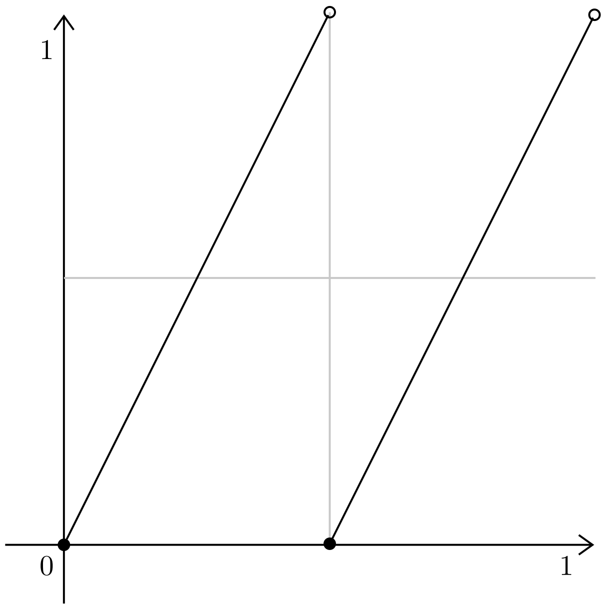

Figure 2.1: The doubling map on $[0, 1)$.

In order to construct the doubling map process, let us start by taking a random variable $X_0 \sim U(0, 1)$, i.e. a uniformly distributed random variable taking any value on the interval $[0, 1)$. Then, we simply keep iterating this random variable through the doubling map in order to obtain an entire sequence of random variables. So let us define

$$X_n = f(X_{n-1}) = f^n(X_0),$$

for $n \in \mathbb{N}$. This is called the doubling map process generated by $X_0 \sim U(0, 1)$, and we will denote the resulting sequence of random variables by (X_i). Clearly, the doubling map $f(x)$ maps any value in the interval $[0, 1)$ to some (other) value on the interval $[0, 1)$. However, it has an even stronger property than this, namely the doubling map can be shown to preserve the distribution of any random variable it maps on the interval $[0, 1)$. This means that not only

$X_1 = f(X_0)$ is equally distributed as X_0, but by repeating this argument every random variable in the resulting sequence (X_i) will share the same distribution on $[0,1)$.

This property will be shown when finding an invariant probability measure for the doubling map. That is, we want to find a measure $\mu(A)$ for intervals $A \subset [0,1)$ that is both

1. invariant with respect to $f(x)$: $\mu(A) = \mu(f^{-1}(A))$ for all $A \subset [0,1)$, and
2. a probability measure: $\mu([0,1)) = 1$, and $\mu(\varnothing) = 0$.

It turns out that the most natural measure for intervals, namely the Lebesgue measure, will be such an invariant probability measure for the doubling map. This measure simply returns the Euclidean length of an interval, i.e. $\mu([a,b)) = b - a$ for any $a, b \in \mathbb{R}$ with $a < b$. Now let us show that the Lebesgue measure indeed satisfies the two properties listed above.

1. First note that by equation (2.1), $f^{-1}(x) = \{\frac{x}{2}, \frac{1}{2} + \frac{x}{2}\}$, which can also be seen in figure 2.1. Let $A = [a,b) \subset [0,1)$, then

   $$f^{-1}(A) = \left[\frac{a}{2}, \frac{b}{2}\right) \cup \left[\frac{1}{2} + \frac{a}{2}, \frac{1}{2} + \frac{b}{2}\right).$$

   Therefore,

   $$\begin{aligned}\mu(f^{-1}(A)) &= \mu\left(\left[\frac{a}{2}, \frac{b}{2}\right)\right) + \mu\left(\left[\frac{1}{2} + \frac{a}{2}, \frac{1}{2} + \frac{b}{2}\right)\right) \\ &= \frac{b}{2} - \frac{a}{2} + \frac{1}{2} + \frac{b}{2} - \left(\frac{1}{2} + \frac{a}{2}\right) = b - a = \mu(A).\end{aligned}$$

2. Clearly, $\mu([0,1)) = 1$, and $\mu(\varnothing) = 0$, by definition of the Lebesgue measure.

So indeed, the Lebesgue measure is an invariant probability measure for the doubling map. Now let us focus on the reason why we are interested in this, and work towards the goal of this chapter. Let (X_i) be the sequence of random variables generated by the doubling map process with $X_0 \sim U(0,1)$. A method to analyse the extreme values in this doubling map process is by looking at the asymptotic statistical behaviour of the maximum

$$M_n = \max\{X_1, \ldots, X_n\},$$

i.e. the random variable that is defined by taking the maximum of the first n random variables generated by the doubling map process. The reason why we wanted to find the invariant probability measure, in this case the Lebesgue measure, is so that

$$\mathbb{P}(f(X) < u) = \mathbb{P}(X \in f^{-1}([0,u))) = \mu(f^{-1}([0,u))) = \mu([0,u)) = \mathbb{P}(X < u).$$

In other words, the probability that any random variable is below some fixed threshold simply corresponds to the probability that the initial random variable is below that threshold. This property will turn out to be vital for analysing the asymptotic behaviour of the maximum value M_n, as will be shown in the following section.

2.2. Asymptotic Behaviour of Extreme Values

The goal of this chapter is to find some sort of asymptotic behaviour of the extreme values of the doubling map process. Using this maximum M_n, this corresponds to finding the statistical behaviour of this random variable whenever we take $n \to \infty$, i.e. expand our sequence (X_i) indefinitely. To do this, we will first look to find an exact expression of some probability involving M_n, before taking $n \to \infty$ to find its asymptotic behaviour. To understand what probability we will be considering to do this, let us first look at the limiting behaviour of a special case of the sequence of random variables (X_i).

Suppose that rather than generate a sequence (X_i) using the the doubling map process with $X_0 \sim U(0,1)$, we instead consider a sequence of independent random variables, that are all identically distributed with $X_i \sim U(0,1)$. In this case,

$$\begin{aligned}\mathbb{P}\left(M_n \leq 1 - \frac{\lambda}{n}\right) &= \mathbb{P}\left(\bigcap_{i=1}^{n}\left\{X_i \leq 1 - \frac{\lambda}{n}\right\}\right) = \prod_{i=1}^{n}\mathbb{P}\left(X_i \leq 1 - \frac{\lambda}{n}\right) \\ &= \left(1 - \frac{\lambda}{n}\right)^n \xrightarrow{n\to\infty} e^{-\lambda}. \end{aligned} \tag{2.2}$$

This not only gives us an insight into the probability that we are interested in taking the limit of, but it also gives a clue of what this limit looks like, in this case the negative exponential of this parameter λ.

So we vary the number of variables (X_i) that we take the maximum of, depending on the rate with which the threshold in the probability above increases. These two rates are connected by the parameter λ. But rather than inserting the parameter λ in the threshold, that is, by considering

$$\lim_{n\to\infty} \mathbb{P}\left(M_n \leq 1 - \frac{\lambda}{n}\right), \tag{2.3}$$

we instead are interested in the probability

$$\lim_{k\to\infty} \mathbb{P}\left(M_{n_k} \leq 1 - 2^{-k}\right),$$

where the parameter λ is now inserted in n_k, the rate with which M_n increases. This way, one can relate n to k using a parameter that can be varied in order to obtain a statistical distribution of this probability, depending on the parameter λ.

Therefore, the goal now will be to prove the result of this chapter, namely that in analogy to the special case above, we get that

$$\lim_{k\to\infty} \mathbb{P}\left(M_{n_k} \leq 1 - 2^{-k}\right) = e^{-\lambda/2}.$$

In order to derive this though, or even provide some intuition into this result, it is necessary to first find an exact expression for the value on the left-hand side. To start with, we are first going to rewrite the probability into a more concrete expression, before deriving its exact value. To this end, let us fix $u = 2^{-k}$ for some $k \in \mathbb{N}$, and define

$$E_i = \{x \in [0,1) \mid X_0 = x \Rightarrow X_i \geq 1 - u\}. \tag{2.4}$$

In other words, we let E_i be the set of values x in the interval $[0,1)$, such that $X_i = f^i(X_0) \geq 1-u$ whenever we set $X_0 = x$. Of course, the initial interval is then simply given by $E_0 = [1-u, 1)$, but note that for the next intervals in this sequence (E_i), we can rewrite the definition to be

$$E_i = \{x \in [0,1) \mid f^i(x) \geq 1-u\}.$$

Then for the next interval E_1, this line simply states

$$E_1 = \{x \in [0,1) \mid f(x) \geq 1-u\} = \{f^{-1}(x) \mid x \in E_0\} = f^{-1}(E_0).$$

Using this, we discover that in general,

$$E_i = \{x \in [0,1) \mid f^i(x) \geq 1-u\} = f^{-1}(E_{i-1}) = f^{-i}(E_0).$$

While this last line gives us a direct formula to obtain an expression for E_i, it is easier to derive using the recursive formula $E_i = f^{-1}(E_{i-1})$, which also suggests that we might want to prove it using induction. In general, we obtain that for any $i \in \mathbb{N}$,

$$E_i = \bigcup_{s=1}^{2^i} \left[\frac{s-u}{2^i}, \frac{s}{2^i}\right). \tag{2.5}$$

Since the proof for this formula is a simple proof by induction, the detailed proof is given in appendix A.1 instead. Now that we have derived an exact expression of E_i, let us observe how this can help us derive an exact expression for a probability related to M_n. To this end, let $n \in \mathbb{N}$, then it follows that

$$\begin{aligned}\mathbb{P}\left(M_n \leq 1-2^{-k}\right) &= \mathbb{P}\left(\bigcap_{i=1}^{n}\left\{X_i \leq 1-2^{-k}\right\}\right) = 1-\mathbb{P}\left(\bigcup_{i=1}^{n}\left\{X_i > 1-2^{-k}\right\}\right)\\ &= 1-\mu\left(\bigcup_{i=1}^{n} E_i\right) = 1-\mu\left(\bigcup_{i=0}^{n-1} E_i\right),\end{aligned} \tag{2.6}$$

where in the last line we used the property that the Lebesgue measure is invariant with respect to the doubling map. Thus, in order to find an exact expression for this probability, we merely have to find the value of this Lebesgue measure, which will be done in the following section.

2.3. Finding the Exact Expression of the Probability

In this section, we are going to derive the exact expression for the probability that was discussed in the last section, by finding the exact value introduced in equation (2.6). Let us start by labelling this sequence of numbers:

$$\mathbb{P}\left(M_n \leq 1-2^{-k}\right) = 1-\mu\left(\bigcup_{i=0}^{n-1} E_i\right) \equiv 1-B_n,$$

for any fixed value of $k \in \mathbb{N}$ and any number $n \in \mathbb{N}$. By finding an exact expression for the values in the sequence (B_n) we will derive the probability above. This is the goal that we will be working towards in this section, and the outcome is stated below.

Theorem 2.1. For any $k \in \mathbb{N}$ and $n = m \cdot (k+1)$ for any $m \in \mathbb{N}$, let (X_i) be the sequence of random variables generated by the doubling map process with $X_0 \sim U(0,1)$ and set

$M_n = \max\{X_1, \dots, X_n\}$. Then

$$\mathbb{P}\left(M_n \le 1 - 2^{-k}\right) = 1 - \sum_{i=0}^{\frac{n}{k+1}-1} (-1)^i \cdot \binom{n - ik + 1}{i+1} \cdot 2^{-(k+1)\cdot(i+1)}$$
$$- \sum_{i=1}^{\frac{n}{k+1}-1} (-1)^{i+1} \cdot \binom{n - ik - 1}{i-1} \cdot 2^{-(k+1)\cdot(i+1)}.$$

The detailed proof of this theorem can be found in appendices A.5 and A.6, but this section will be focusing on deriving this expression by working out detailed observations. We will however first be deriving a lemma that gives us the exact value for the sequence (B_n), which is necessary in order to deduce the theorem above. Also, let us derive this lemma for any $n, k \in \mathbb{N}$, and only restrict the outcome to $n = m \cdot (k+1)$ at the very end of this chapter.

Now, in order to derive a general formula for B_n, let us plot the sets of intervals E_i for different values $k \in \mathbb{N}$ of $u = 2^{-k}$ below, in order to find a pattern in the resulting sequence (B_n) for each such $u = 2^{-k}$.

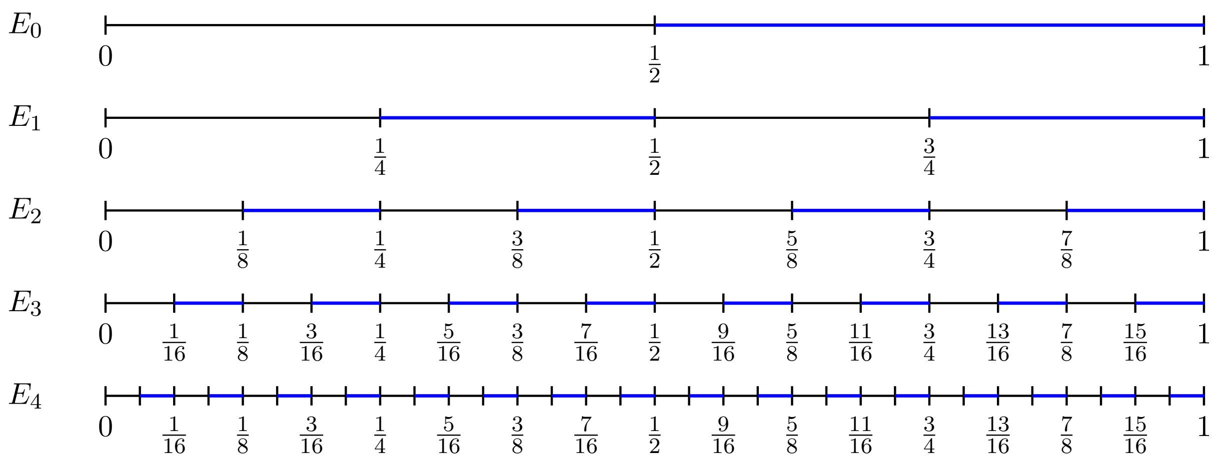

Figure 2.2: Plots for E_i in the case that $u = \frac{1}{2}$.

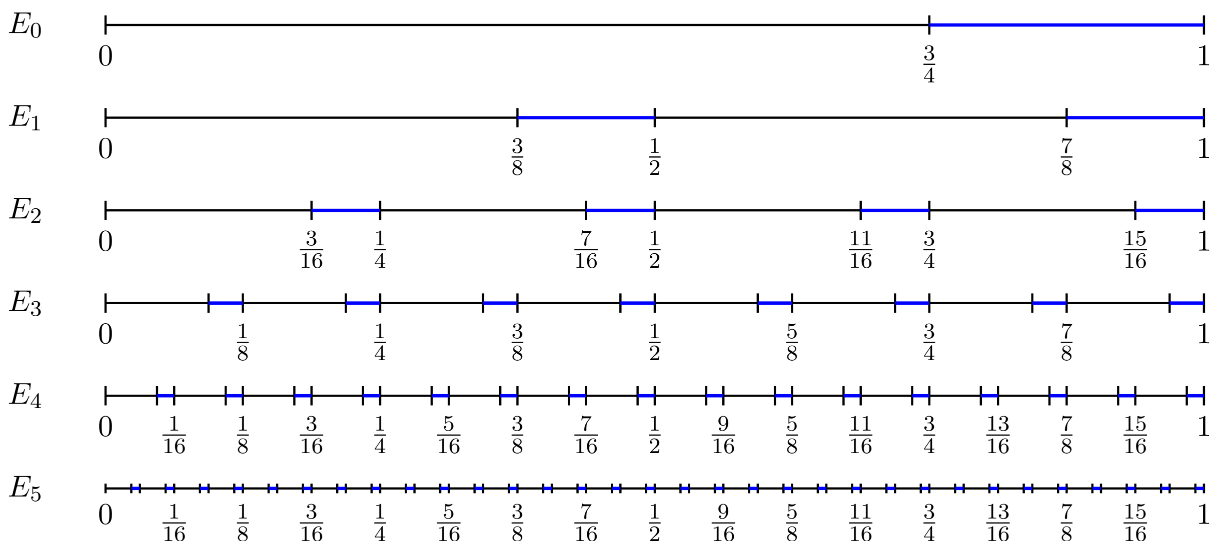

Figure 2.3: Plots for E_i in the case that $u = \frac{1}{4}$.

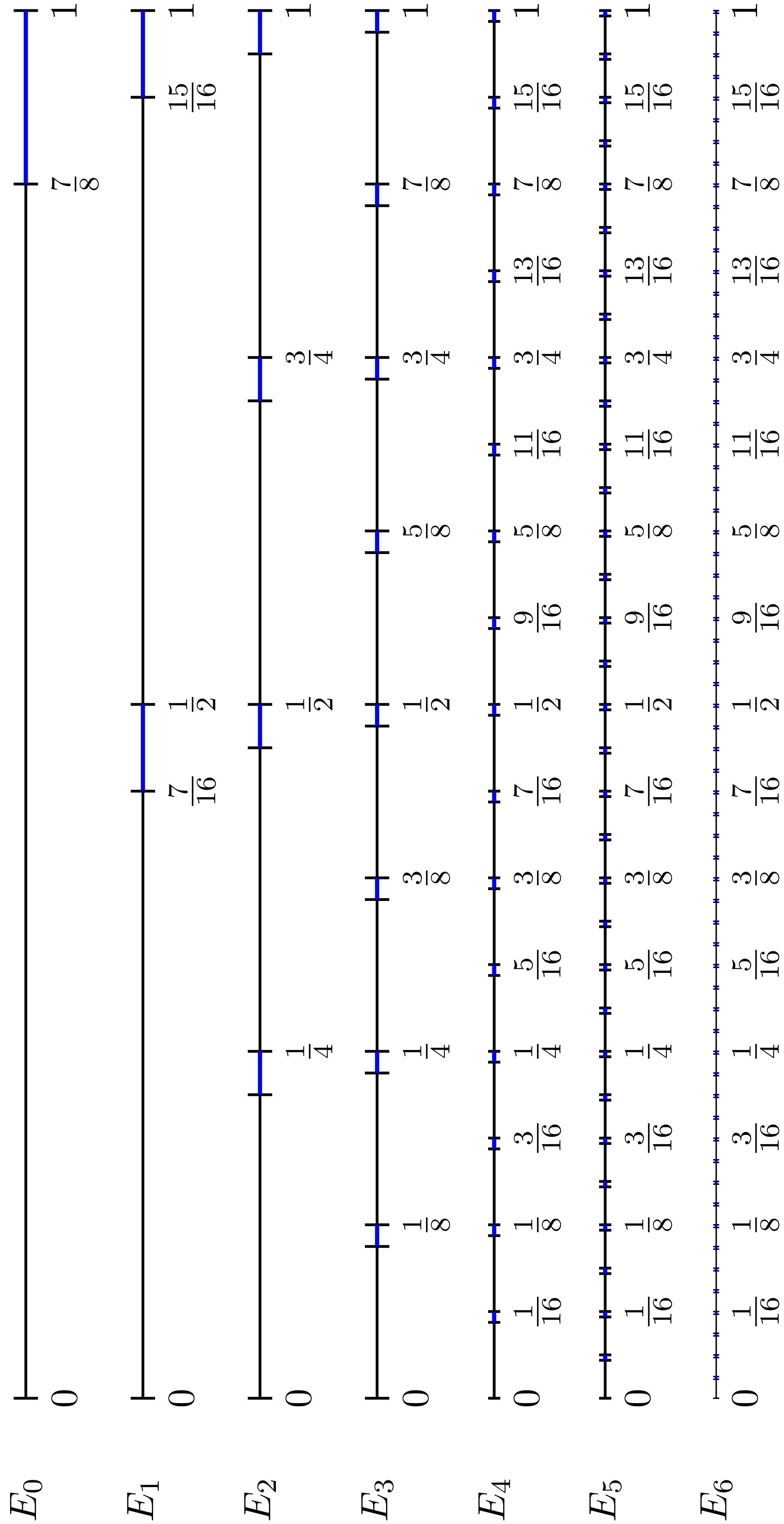

Figure 2.4: Plots for E_i in the case that $u = \frac{1}{8}$.

There are a lot of things to notice about these plots. The most elementary of which is that the length of E_i is preserved and always equal to u, as is to be expected since we have observed that the Lebesgue measure is invariant for the doubling map. Another is that the pattern for the intervals as the sequence (E_i) proceeds can clearly be retraced to the recursion $E_i = f^{-1}(E_{i-1})$, recalling that for the doubling map,

$$f^{-1}([a,b)) = \left[\frac{a}{2}, \frac{b}{2}\right) \cup \left[\frac{1}{2} + \frac{a}{2}, \frac{1}{2} + \frac{b}{2}\right), \text{ for } [a,b) \subset [0,1).$$

Let us now make some more complex observations about the self-similarities of E_i, in order to tie these findings back to our goal; namely to derive an exact expression for the numbers (B_n). To this end, we are going to derive three observations, each of which will be substantiated by several observations regarding the structure of the plots as depicted above.

2.3.1 First Observation

To derive the first of these observations, note that by symmetry of the plots above, it can be seen that the collection of intervals from any E_i onwards is similar to the collection of intervals on the domain $[0, \frac{1}{2})$ from E_{i+1} onwards. To get a visual understanding of this, let us depict this halving similarity below for the case that $u = \frac{1}{4}$.

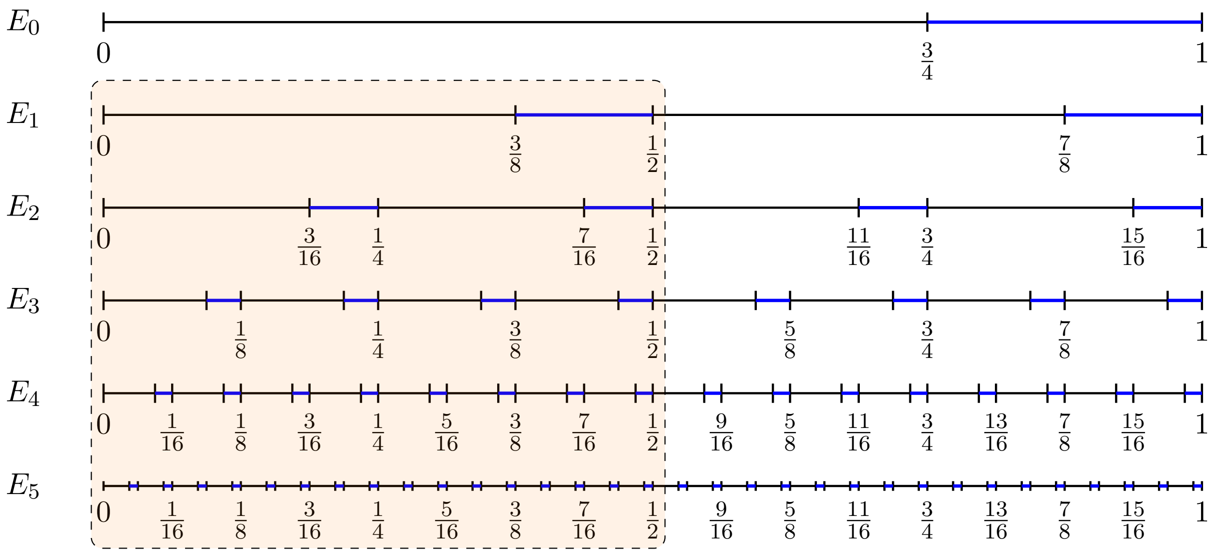

Figure 2.5: Halving similarity of E_i in the case that $u = \frac{1}{4}$.

It can be seen above that the collection of intervals from E_1 onwards on the domain $[0, \frac{1}{2})$, depicted by the orange box, is simply a scaled version of the entire collection of intervals from E_0 onwards. Since the orange box is a copy of the entire collection of values E_i, but scaled with a factor $\frac{1}{2}$, this self-similarity in the intervals E_i is called the halving similarity. The example depicted above can of course be generalised to all cases of this halving similarity. Namely, the collection of all intervals from any E_i onwards is similar to the collection of intervals from E_{i+1} onwards on the domain $[0, \frac{1}{2})$. In other words, one can place this orange box as far down to any level E_{i+1} on the domain $[0, \frac{1}{2})$, and it will always be a copy (scaled by a factor $\frac{1}{2}$) of the collection of intervals starting from the layer above, E_i, and onwards.

This is depicted below, where the orange box has been moved down to include all intervals from E_3 onwards on the domain $[0, \frac{1}{2})$, and the yellow box includes all intervals from E_2 onwards on the entire domain. And again, by the halving similarity described above, the orange box is merely a scaled down copy of the yellow box, by a factor $\frac{1}{2}$.

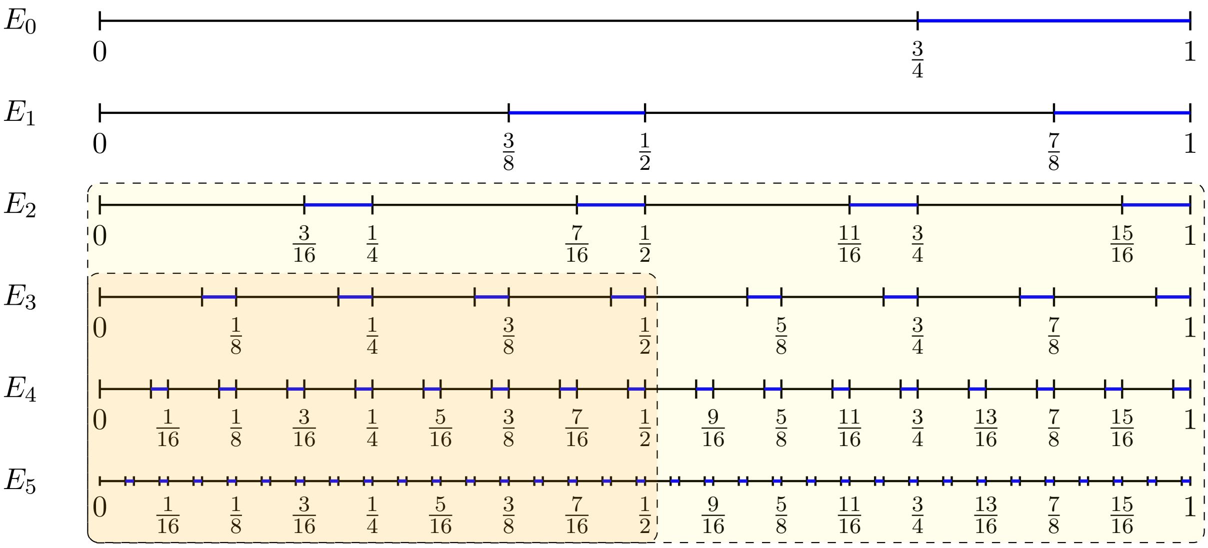

Figure 2.6: Generalised halving similarity of E_i in the case that $u = \frac{1}{4}$.

This final notion, namely that of the halving similarity of E_i, leads us to derive a recursive relation for the sequence (B_n). Namely, let us denote

$$B_n\left(\left[0, \tfrac{1}{2}\right)\right) = \mu\left(\bigcup_{i=0}^{n-1} E_i \cap \left[0, \tfrac{1}{2}\right)\right).$$

This merely means that instead of measuring the length of the union of the entire sequence of intervals (E_i) up to a point $n-1$ and calling the result B_n, we now only measure the contribution of these intervals that lie in the domain $[0, \frac{1}{2})$ and denote the length of the union of all those as $B_n\left([0, \frac{1}{2})\right)$. Similarly, let us define

$$B_n\left(\left[\tfrac{1}{2}, 1\right)\right) = \mu\left(\bigcup_{i=0}^{n-1} E_i \cap \left[\tfrac{1}{2}, 1\right)\right),$$

where by definition $B_n = B_n([0, \frac{1}{2})) + B_n([\frac{1}{2}, 1))$. Let us now turn to the halving symmetry that we observed, namely that the collection of intervals from any E_{i+1} onwards on the domain from $[0, \frac{1}{2})$ is simply the copy of the entire collection of intervals from E_i onwards, scaled by a factor $\frac{1}{2}$. Which means that if we now measure the collection of intervals from any E_{i+1} onwards on the domain from $[0, \frac{1}{2})$, we simply get

$$\text{(Observation 1)} \qquad B_n\left(\left[0, \tfrac{1}{2}\right)\right) = \frac{1}{2} B_{n-1} \quad \text{for } n \geq 2. \tag{2.7}$$

This is the first of three observations that will be used to derive a recursive formula for the entire sequence (B_n). The observation can actually directly be proven from the definition of B_n, which is done in appendix A.2.

2.3.2 Second Observation

The next observation is related to the difference in subsequent values of the sequence (B_n), thus let us recall that it is defined by the sequence (E_i) as

$$B_n = \mu\left(\bigcup_{i=0}^{n-1} E_i\right),$$

for some fixed value $u = 2^{-k}$. So B_1 is simply the length of E_0, which will always be u, i.e. $B_1 = \mu(E_0) = u$. Then B_2 is the length of the union of E_0 and E_1, which means that it is the length of E_0, plus some extra bit. The part that is added to go from B_1 to B_2 is the length of the parts of E_1 that have not been included in E_0 yet, so

$$B_2 = \mu(E_0 \cup E_1) = \mu(E_0) + \mu(E_1 \setminus E_0) = B_1 + \mu(E_1 \setminus E_0).$$

The same can be done for any B_n actually, so

$$B_n = B_{n-1} + \mu\left(E_{n-1} \setminus \bigcup_{i=0}^{n-2} E_i\right).$$

This already gives us some hint that we are actually interested in the total length that is added between B_{n-1} and B_n. To this end, let us list the actual values and differences of subsequent values of B_n for the three values of u that were plotted earlier, namely $u = \frac{1}{2}$, $u = \frac{1}{4}$, and $u = \frac{1}{8}$. The results are listed on the next page, and can easily be derived by looking at the plots in figures 2.2, 2.3, and 2.4, respectively.

$n=1$	$B_1 = u = \frac{1}{2}$	$B_1([0,\frac{1}{2}]) = 0$	$B_1([\frac{1}{2},1]) = u = \frac{1}{2}$
$n=2$	$B_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$	$B_2([0,\frac{1}{2}]) = 0 + \frac{1}{4} = \frac{1}{4}$	$B_2([\frac{1}{2},1]) = \frac{1}{2} + 0 = \frac{1}{2}$
$n=3$	$B_3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$	$B_3([0,\frac{1}{2}]) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$	$B_3([\frac{1}{2},1]) = \frac{1}{2} + 0 = \frac{1}{2}$
$n=4$	$B_4 = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}$	$B_4([0,\frac{1}{2}]) = \frac{3}{8} + \frac{1}{16} = \frac{7}{16}$	$B_4([\frac{1}{2},1]) = \frac{1}{2} + 0 = \frac{1}{2}$
$n=5$	$B_5 = \frac{15}{16} + \frac{1}{32} = \frac{31}{32}$	$B_5([0,\frac{1}{2}]) = \frac{7}{16} + \frac{1}{32} = \frac{15}{32}$	$B_5([\frac{1}{2},1]) = \frac{1}{2} + 0 = \frac{1}{2}$
$n=1$	$B_1 = u = \frac{1}{4}$	$B_1([0,\frac{1}{2}]) = 0$	$B_1([\frac{1}{2},1]) = u = \frac{1}{4}$
$n=2$	$B_2 = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$	$B_2([0,\frac{1}{2}]) = 0 + \frac{1}{8} = \frac{1}{8}$	$B_2([\frac{1}{2},1]) = \frac{1}{4} + 0 = \frac{1}{4}$
$n=3$	$B_3 = \frac{3}{8} + 2\cdot\frac{1}{16} = \frac{1}{2}$	$B_3([0,\frac{1}{2}]) = \frac{1}{8} + \frac{1}{16} = \frac{3}{16}$	$B_3([\frac{1}{2},1]) = \frac{1}{4} + \frac{1}{16} = \frac{5}{16}$
$n=4$	$B_4 = \frac{1}{2} + 3\cdot\frac{1}{32} = \frac{19}{32}$	$B_4([0,\frac{1}{2}]) = \frac{3}{16} + 2\cdot\frac{1}{32} = \frac{1}{4}$	$B_4([\frac{1}{2},1]) = \frac{5}{16} + \frac{1}{32} = \frac{11}{32}$
$n=5$	$B_5 = \frac{19}{32} + 5\cdot\frac{1}{64} = \frac{43}{64}$	$B_5([0,\frac{1}{2}]) = \frac{1}{4} + 3\cdot\frac{1}{64} = \frac{19}{64}$	$B_5([\frac{1}{2},1]) = \frac{11}{32} + 2\cdot\frac{1}{64} = \frac{12}{32}$
$n=6$	$B_6 = \frac{43}{64} + 8\cdot\frac{1}{128} = \frac{47}{64}$	$B_6([0,\frac{1}{2}]) = \frac{19}{64} + 5\cdot\frac{1}{128} = \frac{43}{128}$	$B_6([\frac{1}{2},1]) = \frac{12}{32} + 3\cdot\frac{1}{128} = \frac{51}{128}$
$n=1$	$B_1 = u = \frac{1}{8}$	$B_1([0,\frac{1}{2}]) = 0$	$B_1([\frac{1}{2},1]) = u = \frac{1}{8}$
$n=2$	$B_2 = \frac{1}{8} + \frac{1}{16} = \frac{3}{16}$	$B_2([0,\frac{1}{2}]) = 0 + \frac{1}{16} = \frac{1}{16}$	$B_2([\frac{1}{2},1]) = \frac{1}{8} + 0 = \frac{1}{8}$
$n=3$	$B_3 = \frac{3}{16} + 2\cdot\frac{1}{32} = \frac{1}{4}$	$B_3([0,\frac{1}{2}]) = \frac{1}{16} + \frac{1}{32} = \frac{3}{32}$	$B_3([\frac{1}{2},1]) = \frac{1}{8} + \frac{1}{32} = \frac{5}{32}$
$n=4$	$B_4 = \frac{1}{4} + 4\cdot\frac{1}{64} = \frac{5}{16}$	$B_4([0,\frac{1}{2}]) = \frac{3}{32} + 2\cdot\frac{1}{64} = \frac{1}{8}$	$B_4([\frac{1}{2},1]) = \frac{5}{32} + 2\cdot\frac{1}{64} = \frac{3}{16}$
$n=5$	$B_5 = \frac{5}{16} + 7\cdot\frac{1}{128} = \frac{47}{128}$	$B_5([0,\frac{1}{2}]) = \frac{1}{8} + 4\cdot\frac{1}{128} = \frac{5}{32}$	$B_5([\frac{1}{2},1]) = \frac{3}{16} + 3\cdot\frac{1}{128} = \frac{27}{128}$
$n=6$	$B_6 = \frac{47}{128} + 13\cdot\frac{1}{256} = \frac{107}{256}$	$B_6([0,\frac{1}{2}]) = \frac{5}{32} + 7\cdot\frac{1}{256} = \frac{47}{256}$	$B_6([\frac{1}{2},1]) = \frac{27}{128} + 6\cdot\frac{1}{256} = \frac{15}{64}$
$n=7$	$B_7 = \frac{107}{256} + 24\cdot\frac{1}{512} = \frac{119}{256}$	$B_7([0,\frac{1}{2}]) = \frac{47}{256} + 13\cdot\frac{1}{512} = \frac{107}{512}$	$B_7([\frac{1}{2},1]) = \frac{15}{64} + 11\cdot\frac{1}{512} = \frac{131}{512}$

Table 2.1: Values of B_n for $u = \frac{1}{2}$, $u = \frac{1}{4}$, and $u = \frac{1}{8}$.

As a minor note, observe that as given by the first observation in equation (2.7), it can be seen throughout this table that $B_n([0, \frac{1}{2})) = \frac{1}{2}B_{n-1}$. Let us now resume the goal of finding a recursive relation for the sequence (B_n) using this table of observed data. As emphasised before, there is not necessarily an obvious pattern of the values of B_n itself, but upon closer inspection there is a pattern of the differences between subsequent values of B_n. To see this in a clearer way, let us define the following numbers to measure these difference. Let

$$m_n = \frac{B_n - B_{n-1}}{u/2^{n-1}} \quad \text{for } n \geq 2, \tag{2.8}$$

which is the total number of additional intervals in E_{n-1} (with respect to all previous intervals, $\cup_{i=0}^{n-2} E_i$) whose length contributes to the difference between B_n and B_{n-1}. It divides the difference between B_n and B_{n-1} in the numerator, by the length of any interval in E_{n-1} in the denominator. This yields the number of intervals that has been added to B_{n-1} in order to attain B_n. Visually, one can imagine this as the amount of intervals in E_{n-1} that are not 'covered' by any interval in the layers of E_i above it, for $0 \le i \le n-2$.

$u = 2^{-1}$	$n = 2$	$m_2 = 1$
$(k = 1)$	$n = 3$	$m_3 = 1$
	$n = 4$	$m_4 = 1$
	$n = 5$	$m_5 = 1$
$u = 2^{-2}$	$n = 2$	$m_2 = 1$
$(k = 2)$	$n = 3$	$m_3 = 2$
	$n = 4$	$m_4 = 3$
	$n = 5$	$m_5 = 5$
	$n = 6$	$m_6 = 8$
$u = 2^{-3}$	$n = 2$	$m_2 = 1$
$(k = 3)$	$n = 3$	$m_3 = 2$
	$n = 4$	$m_4 = 4$
	$n = 5$	$m_5 = 7$
	$n = 6$	$m_6 = 13$
	$n = 7$	$m_7 = 24$

Table 2.2: Amount of additional intervals contributing to B_n for $u = \frac{1}{2}$, $u = \frac{1}{4}$, and $u = \frac{1}{8}$.

Let us note that these values m_n can also be seen as coefficients in table 2.1. It is at this point that one may notice a peculiar pattern in the sequence of numbers (m_n). Namely, first note that if we let $u = 2^{-k}$ for $k \in \mathbb{N}$ fixed, then in all cases listed so far,

$$m_n = 2^{n-2} \quad \text{for } 2 \le n \le k+1. \tag{2.9}$$

For example, we can see that in the case that $k = 3$, we see in table 2.2 that $m_2 = 2^0 = 1$, $m_3 = 2^1 = 2$, and $m_4 = 2^2 = 4$. Now that we have conjectured an exact expression for m_n (at least for $2 \le n \le k+1$), it is possible to deduce that

(Observation 2) $$B_n = (n+1) \cdot \frac{u}{2} \quad \text{for } 1 \le n \le k+1. \tag{2.10}$$

A short proof that this observation holds if and only if equation (2.9) holds, is given in appendix A.4. The observation can also be proven directly from the expression of E_i, although that is less insightful than its derivation. Nonetheless, its proof is given in appendix A.3. This gives us the second of three major observations that we use to derive B_n for all $n \in \mathbb{N}$. Now we only need to find an inductive relation for B_n for $n > k+1$.

2.3.3 Third Observation

In order to derive a third and final observation stating such relation, let us instead look to a different self-similarity for E_i that is slightly more complex than the halving similarity discussed before. To derive this, let us look more closely to the plots of E_i in the case that $u = 2^{-3}$, in order to illustrate the property that we will derive.

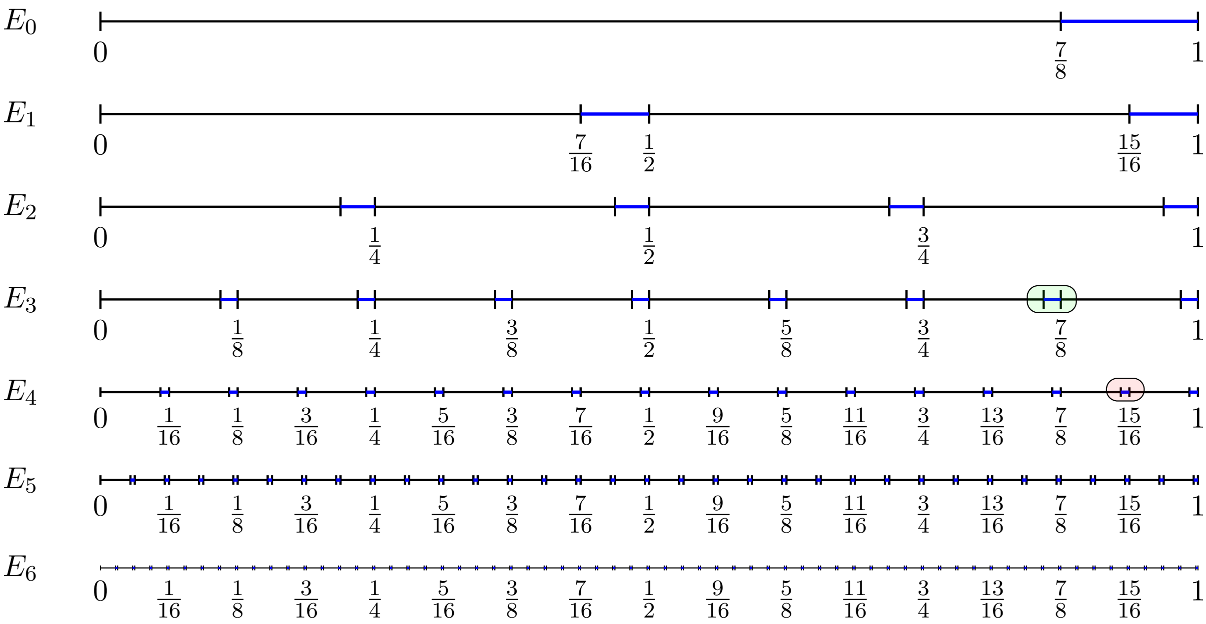

Figure 2.7: First non-trivial interval to be covered by $[1-u, 1)$ in the case that $u = \frac{1}{8}$.

In this plot, let us in particular look at the first interval to be covered by another one, ignoring the most right-hand interval, which will always be covered by $[1-u, 1)$. By the pattern in which the intervals E_i progress, the first such non-trivial interval to be covered by an earlier interval is always being covered by $[1-u, 1)$. In this example, we see that the first non-trivial interval to be covered by an earlier interval, is the interval marked by the red box. Recall that in order to keep the plots clear, only the right-hand side of every interval is denoted after a while. So do not be fooled by the interval marked by the green box, as its right-hand side value of $\frac{7}{8}$ is not part of the interval itself (since it is open on that side), and thus it is not covered by $[1-u, 1) = \left[\frac{7}{8}, 1\right)$. Let us now observe that the interval marked red is part of E_4, i.e. exactly $k + 1 = 4$ 'layers' below E_0.

It turns out that in general it is always the case that the first non-trivial interval to be covered by $[1-u, 1)$ for $u = 2^{-k}$, is part of E_{k+1} and thus $k+1$ layers below E_0. This is a direct result from the second observation as stated in equation (2.10), namely that B_n can be expressed regularly for $n \leq k+1$. It is only after a non-trivial interval is covered by $[1-u, 1)$ that B_n diverges from this regular pattern, thus for $n > k + 1$. This leads us to derive the following self-similarity of E_i, by once again inspecting the example for $u = 2^{-3}$.

Let us define the orange box as in figure 2.8 (shown on the next page) to start from E_3, covering the interval $[1-u, 1)$ on every layer below it. In particular, the top layer included in the box is the layer above the first non-trivial interval covered by $[1-u, 1)$, marked by the red box, which will always be the layer E_k for $u = 2^{-k}$. Then we observe that the orange box is

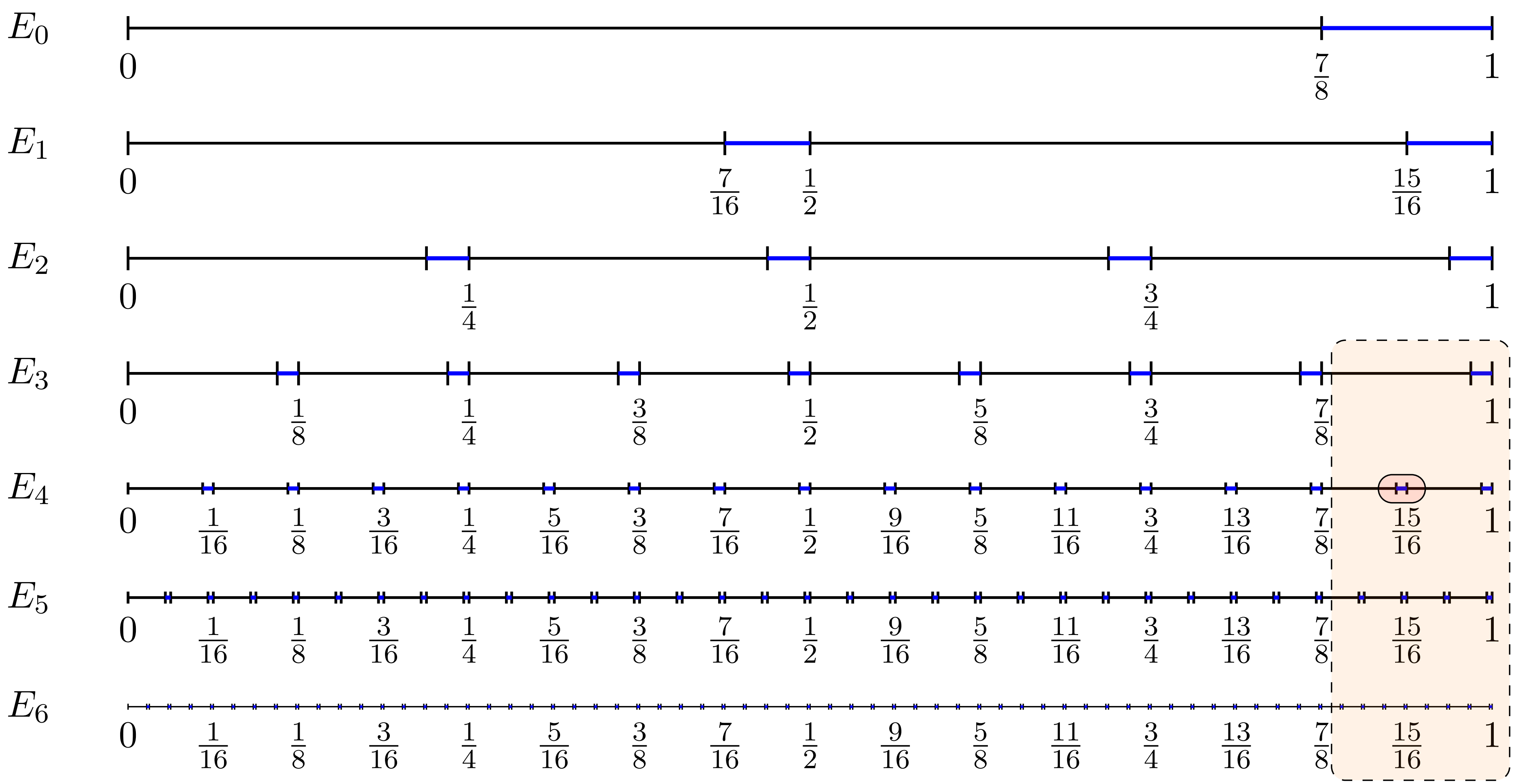

Figure 2.8: Self-similarity of E_i in the case that $u = \frac{1}{8}$.

simply a copy of the entire plot of all E_i on the complete interval [0,1), scaled down by a factor u. Where this factor u is easy to verify by considering that E_0 is similar to the range $[1-u, 1)$ on E_k, comprising the first layer inside the orange box.

Let us now use this self-similarity of E_i in order to derive a recursive relation for B_n. To do this, let us consider any layer of E_i (say E_n) in which an additional interval is covered by $[1-u, 1)$, compared to the amount of intervals covered in E_{n-1} by $[1-u, 1)$. Moreover, let us assume that no other additional interval is covered by an earlier interval, in comparison to that amount in E_{n-1}. So, in this scenario there is an additional difference between $B_{n+1}([\frac{1}{2}, 1))$ and $B_{n+1}([0, \frac{1}{2}))$, that can only be caused by this interval being covered by $[1-u, 1)$. By the self-similarity discussed before, we know that the last time that an additional interval was covered by $[1-u, 1)$ was exactly $k+1$ layers ago. Therefore, we have that

1. the length u of $[1-u, 1)$ contributes fully to $B_{n+1}([\frac{1}{2}, 1))$, but not $B_{n+1}([0, \frac{1}{2}))$ in any case, and

2. the length of this additional interval that is being covered by $[1-u, 1)$ no longer contributes to $B_{n+1}([\frac{1}{2}, 1))$, but its counterpart in $[0, \frac{1}{2})$ still contributes fully to $B_{n+1}([0, \frac{1}{2}))$.

We also know that the length of this additional interval can be derived using the self-similarity, and thus we multiply the total length $k+1$ layers ago (which is $B_{n+1-(k+1)} = B_{n-k}$) by this scaling factor u. The only remaining factor that we need to keep in mind is that, when going from any layer E_n to the next, the length of any separate interval is halved. Therefore, the obtained difference $u - u \cdot B_{n-k}$ derived above, still needs to be scaled down by this factor $\frac{1}{2}$ to compensate for this. Doing that, we get that

$$\text{(Observation 3)} \qquad B_{n+1}\left(\left[\tfrac{1}{2}, 1\right)\right) - B_{n+1}\left(\left[0, \tfrac{1}{2}\right)\right) = \frac{u}{2} - \frac{u}{2} \cdot B_{n-k}, \tag{2.11}$$

which of course only holds for $n \geq k+1$, as otherwise we can not use the self-similarity to look back $k+1$ layers ago, as used in the last step. This resulting relation is our third and final observation, however no proof for it exists, unfortunately. This is discussed more elaborately in section 6.1. Now, note that using the first observation as stated in equation (2.7), we can combine it with this last observation to derive

$$\begin{aligned} B_{n+1}\left(\left[\tfrac{1}{2}, 1\right)\right) &= B_{n+1}\left(\left[0, \tfrac{1}{2}\right)\right) + \frac{u}{2} - \frac{u}{2} \cdot B_{n-k} = \frac{1}{2} B_n + \frac{u}{2} \cdot (1 - B_{n-k}) \\ \Rightarrow \quad B_{n+1} &= B_{n+1}\left(\left[0, \tfrac{1}{2}\right)\right) + B_{n+1}\left(\left[\tfrac{1}{2}, 1\right)\right) = B_{n+1}\left(\left[0, \tfrac{1}{2}\right)\right) + \frac{1}{2} B_n + \frac{u}{2} \cdot (1 - B_{n-k}) \\ &= B_n + \frac{u}{2} \cdot (1 - B_{n-k}). \end{aligned}$$

Again, this holds only for $n \geq k+1$, so we combine it with the second observation as stated in equation (2.10) in order to obtain a full inductive relation for B_n for all $n \in \mathbb{N}$. This is the final result for this section, and is stated in the following lemma.

Lemma 2.2. Let $u = 2^{-k}$ for some $k \in \mathbb{N}$ fixed, then it follows that

1. $B_n = (n+1) \cdot \frac{u}{2}$ for $1 \leq n \leq k+1$, and

2. $B_{n+1} = B_n + \frac{u}{2} \cdot (1 - B_{n-k})$ for $n \geq k+1$.

Whilst no direct proof for the lemma above will be given, note that the lemma has been derived from three observations, two of which have been proven. Therefore, the lemma is shown to hold whenever these hold as well, using the aforementioned derivations given in this section.

2.4. Finding the Asymptotic Limit of the Probability

Now that we have derived lemma 2.2 above, let us use this in order to prove theorem 2.1 as stated in the beginning of last section. Then, we are going to use the result of this theorem in order to find the asymptotic limit, which is the goal of this chapter.

First of all, using the two relations for B_n as stated in lemma 2.2, it is possible to explicitly derive an expression for B_n. To do this, let us first note that by the first part of the lemma, $B_{k+1} = (k+2) \cdot \frac{u}{2}$. We can now use the second part of the lemma to see that

$$B_{k+2} = B_{k+1} + \frac{u}{2} \cdot (1 - B_1) = (k+2) \cdot \frac{u}{2} + \frac{u}{2} \cdot (1-u) = (k+3) \cdot \frac{u}{2} - \frac{u^2}{2}.$$

This can be repeated inductively, using the second part of the lemma in order to obtain $B_{k+1+l+1}$ explicitly, given B_{k+1+l} for $1 \leq l \leq k+1$. The exact derivation and proof for this can be found in appendix A.5, with the result that

$$B_{k+1+l} = \binom{k+2+l}{1} \cdot \frac{u}{2} - \binom{l+2}{2} \cdot \frac{u^2}{4} + \frac{u^2}{4} \quad \text{for } 1 \leq l \leq k+1. \tag{2.12}$$

We have put the first coefficient unnecessarily in the form of a binomial coefficient here, because this allows us to generalise this expression a bit easier. Namely, taking $l = k+1$ in equation (2.12) gives us that

$$B_{2\cdot(k+1)} = \binom{2k+3}{1} \cdot \frac{u}{2} - \binom{k+3}{2} \cdot \frac{u^2}{4} + \frac{u^2}{4}.$$

We can now keep adding $k+1$ to the index inductively in order to obtain $B_{m\cdot(k+1)}$. This is for brevity done in appendix A.6, in which its proof can also be found. The result of this derivation is that

$$\begin{aligned} B_{m\cdot(k+1)} = &\sum_{i=0}^{m-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i+1}{i+1} \cdot \left(\frac{u}{2}\right)^{i+1} \\ &+ \sum_{i=1}^{m-1} (-1)^{i+1} \cdot \binom{(m-i)\cdot(k+1)+i-1}{i-1} \cdot \left(\frac{u}{2}\right)^{i+1}. \end{aligned} \tag{2.13}$$

Now, recall that we wanted to find an expression for

$$\mathbb{P}\left(M_n \leq 1 - 2^{-k}\right) = 1 - \mu\left(\bigcup_{i=0}^{n-1} E_i\right) = 1 - B_n.$$

This is why in theorem 2.1, which we will now derive, we consider a subsequence of $n \in \mathbb{N}$, namely all $n = m \cdot (k+1)$ for $m \in \mathbb{N}$. As then by equation (2.13),

$$\begin{aligned} B_n = &\sum_{i=0}^{\frac{n}{k+1}-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i+1}{i+1} \cdot \left(\frac{u}{2}\right)^{i+1} \\ &+ \sum_{i=1}^{\frac{n}{k+1}-1} (-1)^{i+1} \cdot \binom{(m-i)\cdot(k+1)+i-1}{i-1} \cdot \left(\frac{u}{2}\right)^{i+1}, \end{aligned} \tag{2.14}$$

where it is noted that we can rewrite

$$(m-i)\cdot(k+1)+i\pm 1 = m\cdot(k+1)-ik-i+i\pm 1 = n-ik\pm 1.$$

Therefore, we can rewrite equation (2.14) to

$$B_n = \sum_{i=0}^{\frac{n}{k+1}-1} (-1)^i \cdot \binom{n-ik+1}{i+1}\cdot\left(\frac{u}{2}\right)^{i+1} + \sum_{i=1}^{\frac{n}{k+1}-1} (-1)^{i+1}\cdot\binom{n-ik-1}{i-1}\cdot\left(\frac{u}{2}\right)^{i+1},$$

which only involves n and k, but keep in mind that it only holds for $n = m\cdot(k+1)$ for $m \in \mathbb{N}$, which is the reason for this technicality in theorem 2.1. Now, let us substitute $u = 2^{-k}$ for $k \in \mathbb{N}$ fixed, which yields

$$B_n = \sum_{i=0}^{\frac{n}{k+1}-1} (-1)^i \cdot \binom{n-ik+1}{i+1}\cdot 2^{-(k+1)\cdot(i+1)} + \sum_{i=1}^{\frac{n}{k+1}-1} (-1)^{i+1}\cdot\binom{n-ik-1}{i-1}\cdot 2^{-(k+1)\cdot(i+1)}.$$

This is finally substituted into $\mathbb{P}\left(M_n \leq 1-2^{-k}\right) = 1 - B_n$, in order to obtain theorem 2.1, which is repeated below.

Theorem 2.1. For any $k \in \mathbb{N}$ and $n = m\cdot(k+1)$ for any $m \in \mathbb{N}$, let (X_i) be the sequence of random variables generated by the doubling map process with $X_0 \sim U(0,1)$ and set $M_n = \max\{X_1,\ldots,X_n\}$. Then

$$\begin{aligned}\mathbb{P}\left(M_n \leq 1-2^{-k}\right) = 1 &- \sum_{i=0}^{\frac{n}{k+1}-1} (-1)^i \cdot \binom{n-ik+1}{i+1}\cdot 2^{-(k+1)\cdot(i+1)} \\ &- \sum_{i=1}^{\frac{n}{k+1}-1} (-1)^{i+1}\cdot\binom{n-ik-1}{i-1}\cdot 2^{-(k+1)\cdot(i+1)}.\end{aligned}$$

Now it is possible to use this theorem in order to work towards the goal of this chapter, namely deriving the asymptotic limit of the expression in theorem 2.1. Note that in order to obtain the limit of the expression above, we have to send $k \to \infty$, since $n = m\cdot(k+1)$ will then also go to ∞. However, to get more information on the asymptotic behaviour of the expression above, let us couple n to k using a parameter that can be varied. This will give us an actual statistical distribution of the limit, depending on this introduced parameter, as discussed in section 2.2. To this end, let us define the subsequence

$$n_k = \left\lfloor \frac{\lambda\cdot 2^k}{k+1}\right\rfloor\cdot(k+1),$$

for $k \in \mathbb{N}$ and any fixed $\lambda > 0$. Recall that in theorem 2.1 we need $n = m\cdot(k+1)$ for $m \in \mathbb{N}$, which is why we require the subsequence n_k to be the nearest multiple of $k+1$ below $\lambda\cdot 2^k$. This can be combined with theorem 2.1 in order to derive the final result of this chapter, stated in the following theorem.

Theorem 2.3. Define the sequence $n_k = \left\lfloor \frac{\lambda\cdot 2^k}{k+1}\right\rfloor\cdot(k+1)$ for $k \in \mathbb{N}$ and any fixed $\lambda > 0$, let (X_i) be the sequence of random variables generated by the doubling map process with $X_0 \sim U(0,1)$, and set $M_n = \max\{X_1,\ldots,X_n\}$. Then

$$\lim_{k\to\infty} \mathbb{P}\left(M_{n_k} \leq 1-2^{-k}\right) = e^{-\lambda/2}.$$

The proof and full derivation of this theorem is given in appendix A.7. For some intuition behind this theorem, we take M_{n_k} to be the maximum of an exponentially increasing amount of (X_i), related to k using this parameter $\lambda > 0$. Then the probability that the maximum of this ever increasing amount of (X_i) stays below a threshold converging towards the maximum 1 (namely $1 - 2^{-k}$), will be exponentially distributed with respect to this parameter λ.

So why would this be exponentially distributed, apart from the derivation in appendix A.7 telling us that it is? Of course, we have seen a special case of this expression in section 2.2 that led us down this path in the first place, but let us see whether we can tell from this theorem itself. There are some trivial observations that this limit must adhere to, which we can use to verify some part of this result. First of all, if we take $\lambda > 0$ arbitrarily small and close to 0, then clearly this probability should tend towards 1. This is because we then take the maximum over a very small number of (X_i), or at least a number that is always very small in comparison to 2^k. The fewer random variables we take the maximum of, the likelier it is that this maximum will stay below a threshold that converges towards 1, the maximum of the doubling map. Indeed, our detailed results show us that this probability will be $e^{-\lambda/2} \approx 1$ for $\lambda > 0$ very small. On the other hand, when we choose $\lambda > 0$ to be very large, tending towards ∞, then n_k is going to converge much faster to ∞ than the threshold increases towards the maximum. Therefore, the chance that this maximum will exceed this threshold is very high, even tending towards 1. And indeed, we see that for $\lambda > 0$ very large, the probability of staying below this threshold will tend towards $e^{-\lambda/2} \approx 0$. More insight into the resulting asymptotic behaviour and the reason why this maximum will tend to be exponentially distributed, can be found in appendix A.7.

2.5. Connection with Generalised Fibonacci Sequences

As a final note, it turns out that in subsection 2.3.2 we could have made a final observation of table 2.2, as a certain pattern emerges from the values in this table. Namely, if we look at the part of the table listing m_n for $u = 2^{-2}$, we see that m_n follows a Fibonacci sequence for $n > k + 1$, i.e. after the regular part expressed in equation (2.9). This is no coincidence, as we can see that for $u = 2^{-3}$, a tribonacci sequence can actually be observed for m_n as $n > k + 1$. For those unfamiliar, the tribonacci sequence is a generalised form of the Fibonacci sequence, where rather than starting with $(0, 1)$ and adding every two final numbers of the sequence to generate the next, instead we start with $(0, 1, 2)$ and keep adding every three final numbers of the sequence in order to generate the next.

This leads us to theorise that for $u = 2^{-k}$, (m_n) is a k-bonacci sequence (for $n > k + 1$), which is the generalised form of the Fibonacci sequence in which one keeps adding the last k numbers to generate the next element of the sequence. Recall that we have shown in appendix A.4 that the starting digits of (m_n) are given by $m_n = 2^{n-2}$ for $2 \leq n \leq k + 1$ if and only if the first part of lemma 2.2 holds. It turns out that the second part of lemma 2.2 holds if and only if the sequence (m_n) is k-bonacci. A detailed proof of this can also be found in appendix A.4.

This raises the following question though, could we use this property of the sequence (m_n) being k-bonacci to show the main results in a different way? For we have now proven that from lemma 2.2 the main results of theorems 2.1 and 2.3 hold, and also that (m_n) is k-bonacci whenever lemma 2.2 holds. Hence we can use this equivalence to show the main results using only the property that (m_n) is k-bonacci, which will be done in the final part of this chapter. To start, let us briefly recall that the sequence (m_n) is defined as

$$m_n = \frac{B_n - B_{n-1}}{u/2^{n-1}} \quad \text{for } n \geq 2.$$

This allows us relate the desired probability to the sequence (m_n), in a similar way as was done implicitly in theorem 2.1.

Theorem 2.4. For any $k \in \mathbb{N}$, let (X_i) be the sequence of random variables generated by the doubling map process with $X_0 \sim U(0,1)$, and set $M_n = \max\{X_1, \ldots, X_n\}$. Moreover, let (m_n) be the sequence as defined above, then

$$\mathbb{P}\left(M_n \leq 1 - 2^{-k}\right) = 2^{1-n-k} \cdot m_{n+k+1}, \text{ for all } n \in \mathbb{N}.$$

Now, since the sequence (m_n) turns out to be k-bonacci, we use the approach as done in [4] to find a direct formula for each value of the sequence (m_n). Namely, let us fix some $k \in \mathbb{N}$, and define the values α_i for $1 \leq i \leq k$ as the roots of the polynomial

$$x^k - x^{k-1} - \cdots - 1 = 0.$$

Then by [4] (theorem 1) it follows that the n^{th} value of the k-bonacci sequence can be written as follows.

Theorem 2.5. Fix $k \in \mathbb{N}$, then the n^{th} value of the sequence (m_n) corresponding to $u = 2^{-k}$ is given by

$$m_n = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k+1) \cdot (\alpha_i - 2)} \cdot \alpha_i^{n-1}.$$

This is already a very useful result, however there is an even stronger result that we will be needing, which is derived in [11]. Namely, it turns out that of all the roots α_i for $1 \leq i \leq k$, there is only one root α such that $|\alpha| > 1$, and all others lie within the unit circle. Using this, let us state and then use the main result of [4] (theorem 2) below.

Theorem 2.6. Fix $k \in \mathbb{N}$, then the n^{th} value of the sequence (m_n) corresponding to $u = 2^{-k}$ is given by

$$m_n = \left\lfloor \frac{\alpha - 1}{2 + (k+1) \cdot (\alpha - 2)} \cdot \alpha^{n-1} + \frac{1}{2} \right\rfloor.$$

Now, let us use these direct formulas in order to derive our main result of the chapter, namely theorem 2.3, which is reformulated below.

Theorem 2.7. Define the sequence $n_k = \lfloor \lambda \cdot 2^k \rfloor$ for $k \in \mathbb{N}$ and any fixed $\lambda > 0$, let (X_i) be the sequence of random variables generated by the doubling map process with $X_0 \sim U(0,1)$, and set $M_n = \max\{X_1, \ldots, X_n\}$. Then

$$\lim_{k \to \infty} \mathbb{P}\left(M_{n_k} \leq 1 - 2^{-k}\right) = e^{-\lambda/2}.$$

Note that in this version of theorem 2.3, we require a less strict definition of the sequence (n_k), though the result remains unchanged. It may be noted that this result is simply obtained by substituting the direct formula of (m_n) into the previously obtained result of theorem 2.4, and then taking the desired limit. To see this in detail, the proofs of this theorem and the required result of theorem 2.4 are given in appendix A.8. These proofs are a lot shorter than their counterparts for theorems 2.1 and 2.3 respectively, but also a bit more abstract and perhaps less insightful. However, the largest observation here is that we have now proven the main result of this chapter in an alternative way, using only the property that the sequence (m_n) is k-bonacci.

3. Extreme Values of the Generalised Doubling Map Process

In this chapter, we will consider a more generalised version of the doubling map, where instead of taking

$$f(x) = 2x \bmod 1,$$

we consider a family of functions given by

$$f_\beta(x) = \beta x \bmod 1, \tag{3.1}$$

for any integer $\beta \geq 2$. In the following sections, we will construct and analyse the generalised version of the doubling map process using this family of functions f_β. The goal will be to derive patterns of asymptotic behaviour of the generalised doubling map process that are similar to the case for $\beta = 2$, which was extensively treated in the previous chapter. Therefore, throughout this chapter it may be useful to refer back to the previous chapter in order to recall the intuition behind the various results, in order to generalise them in the following sections.

3.1. The Generalised Doubling Map Process

In a very similar way to how we constructed the doubling map process in section 2.1, let us define the generalised doubling map process by

$$X_n = f_\beta(X_{n-1}) = f_\beta^n(X_0),$$

for any $n \in \mathbb{N}$ and any random variable $X_0 \sim U(0,1)$. Now, note that the Lebesgue measure remains an invariant probability measure for the generalised doubling map. To see this, let us check the two required conditions, namely that μ is both

1. invariant with respect to $f_\beta(x)$: $\mu(A) = \mu(f_\beta^{-1}(A))$ for all $A \subset [0,1)$ and $\beta \geq 2$, and
2. a probability measure: $\mu([0,1)) = 1$, and $\mu(\varnothing) = 0$.

Since only the first condition is dependent on the specific map, let us explicitly check that it holds. To this end, let us first note that by equation (3.1),

$$f_\beta^{-1}([a,b)) = \bigcup_{i=1}^{\beta} \left[\frac{i-1+a}{\beta}, \frac{i-1+b}{\beta}\right).$$

Therefore, for any $A = [a,b) \subset [0,1)$, it follows that

$$\begin{aligned}
\mu(f_\beta^{-1}(A)) &= \mu\left(\bigcup_{i=1}^{\beta} \left[\frac{i-1}{\beta} + \frac{a}{\beta}, \frac{i-1}{\beta} + \frac{b}{\beta}\right)\right) \\
&= \sum_{i=1}^{\beta} \mu\left(\left[\frac{i-1+a}{\beta}, \frac{i-1+b}{\beta}\right)\right)
\end{aligned}$$

$$= \sum_{i=1}^{\beta} \frac{b-a}{\beta} = b - a = \mu(A).$$

Similar to the case in which $\beta = 2$, the goal of this chapter will be to derive the asymptotic statistical behaviour of the maximum

$$M_n = \max\{X_1, \dots, X_n\}.$$

In particular, we are interested in the asymptotic limit of the probability

$$\mathbb{P}\left(M_n \leq 1 - u\right).$$

The reasoning behind inspecting this asymptotic limit is the same as in the previous chapter. Now recall that for $\beta = 2$, we take $n \to \infty$ and simultaneously let $u \to 0$ to obtain this limit. To achieve this behaviour for u, we fixed $u = 2^{-k}$ for $k \in \mathbb{N}$ and considered a subsequence (M_{n_k}) with $n_k \xrightarrow{k\to\infty} \infty$. We will do a similar thing here, where we now define $u = \beta^{-k}$ for any $k \in \mathbb{N}$, and seek to obtain the asymptotic limit of

$$\mathbb{P}\left(M_{n_k} \leq 1 - \beta^{-k}\right),$$

as $k \to \infty$. Before cutting to the result, let us try to truly derive this asymptotic limit, in a very similar way as for $\beta = 2$. To this end let us define, in accordance with equation(2.4) for $\beta = 2$, the sequence

$$E_i = \{x \in [0,1) \mid X_0 = x \Rightarrow X_i \geq 1 - u\}.$$

Using this, we discover that in general,

$$E_i = \{x \in [0,1) \mid f_\beta^i(x) \geq 1 - u\} = f_\beta^{-i}(E_0).$$

Given this relation, we obtain that for any $i \in \mathbb{N}$ and $\beta \geq 2$,

$$E_i = \bigcup_{s=1}^{\beta^i} \left[\frac{s-u}{\beta^i}, \frac{s}{\beta^i}\right). \tag{3.2}$$

Since the proof for this formula is a simple proof by induction, a detailed proof is given in appendix B.1 instead. This sequence can be related back to the desired asymptotic limit by noting that

$$\begin{aligned}
\mathbb{P}\left(M_n \leq 1 - \beta^{-k}\right) &= \mathbb{P}\left(\bigcap_{i=1}^{n} \left\{X_i \leq 1 - \beta^{-k}\right\}\right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{n} \left\{X_i > 1 - \beta^{-k}\right\}\right) \\
&= 1 - \mu\left(\bigcup_{i=1}^{n} E_i\right) = 1 - \mu\left(\bigcup_{i=0}^{n-1} E_i\right) \equiv 1 - B_n. \qquad (3.3)
\end{aligned}$$

3.2. Finding the Exact Expression of the Probability

Now that we have formulated the exact expression for which we seek to evaluate the asymptotic limit, let us start working towards that goal using several intermediate steps. Recall that in section 2.3 we established an intuition behind the defined intervals E_i in the case that $\beta = 2$.

Using this, three major observations were made, which then were derived and discussed at length in the previous chapter. Most importantly though, these observations allowed us to formulate an important lemma, which gives an explicit expression for the sequence B_n, and is repeated below.

Lemma 2.2. Let $u = 2^{-k}$ for some $k \in \mathbb{N}$ fixed, then it follows that

1. $B_n = (n+1) \cdot \frac{u}{2} \quad \text{for } 1 \leq n \leq k+1$, and
2. $B_{n+1} = B_n + \frac{u}{2} \cdot (1 - B_{n-k}) \quad \text{for } n \geq k+1$.

The result of this lemma for $\beta = 2$ can be extended to the generalised doubling map process, which results in the lemma below.

Lemma 3.1. Fix $\beta \in \mathbb{N}$ with $\beta \geq 2$, and let $u = \beta^{-k}$ for some $k \in \mathbb{N}$ fixed, then it follows that

1. $B_n = (n-1) \cdot \left(1 - \frac{1}{\beta}\right) \cdot u + u \quad \text{for } 1 \leq n \leq k+1$, and
2. $B_{n+1} = B_n + \left(1 - \frac{1}{\beta}\right) \cdot u \cdot (1 - B_{n-k}) \quad \text{for } n \geq k+1$.

First, note that for $\beta = 2$, lemma 3.1 gives the same result as lemma 2.2, which can easily be checked. Moreover, note that the first statement of the lemma can be proven separately, and its proof is given in appendix B.2.
It is difficult to derive this lemma in the same way as we did in section 2.3 for lemma 2.2, however let us try to develop some credible derivation for this lemma, other than its direct proof. To this end, recall that in section 2.2 it was discussed that the sequence

$$m_n = \frac{B_n - B_{n-1}}{u/2^{n-1}} \quad \text{for } n \geq 2,$$

is k-bonacci, with starting digits

$$m_n = 2^{n-2} \quad \text{for } 2 \leq n \leq k+1.$$

This was derived by noting a pattern when computing various values of the sequence (m_n) in table 2.2. In a similar fashion, for $\beta \geq 2$ let us define the sequence

$$m_n = \frac{B_n - B_{n-1}}{u/\beta^{n-1}} \quad \text{for } n \geq 2,$$

recalling that we take $u = \beta^{-k}$ in this case. Then, if one writes out similar sequences (m_n) for various values of β as was done in table 2.2 for $\beta = 2$, one would find similar patterns for this sequence (m_n). In particular, one will find that the sequence (m_n) is in fact $(\beta - 1)$ times the k-bonacci sequence, i.e. every term of the k-bonacci sequence is multiplied by a factor of $(\beta - 1)$. That is,

$$m_{n+1} = \sum_{i=0}^{k-1} (\beta - 1) \cdot m_{n-i} \quad \text{for } n \geq k+1.$$

Of course, in the case that $\beta = 2$, this reduces to the statement that the sequence (m_n) is k-bonacci. Similarly, we get that the starting digits of this sequence (m_n) are given by

$$m_n = (\beta - 1) \cdot \beta^{n-2} \quad \text{for } 2 \leq n \leq k+1. \tag{3.4}$$

So how does this help us derive lemma 3.1? Recall that we have shown in appendix A.4 that lemma 2.2 holds if and only if the sequence (m_n) is k-bonacci, with the associated starting digits. Similarly, we can prove that (m_n) is $(\beta - 1)$ times the k-bonacci sequence, with the starting digits as in equation (3.4), if and only if lemma 3.1 holds. This proof can be found in appendix B.3, and will hopefully provide some sense of credibility that lemma 3.1 indeed holds, other than the proof given in appendix B.2.

3.3. Finding the Asymptotic Limit of the Probability

Going forward, we are going to follow the exact same procedure as in section 2.4 in order to find the statistical behaviour that we are interested in. To this end, let us start with continuously using lemma 3.1 in order to determine the general expression of the sequence (B_n). In the first part of the lemma, such an expression is already given for (B_n), at least whenever $1 \leq n \leq k+1$. So let us note that the last value of (B_n) that can directly be computed is given in lemma 3.1 by

$$B_{k+1} = k \cdot \left(1 - \frac{1}{\beta}\right) \cdot u + u.$$

Then using the second part of the lemma, we get that

$$B_{k+2} = B_{k+1} + \left(1 - \frac{1}{\beta}\right) \cdot u \cdot (1 - B_1) = (k+1) \cdot \left(1 - \frac{1}{\beta}\right) \cdot u + u - \left(1 - \frac{1}{\beta}\right) \cdot u^2.$$

Now we can repeatedly apply the second part of the lemma, in order to obtain a direct expression for $B_{k+1+l+1}$ inductively, given B_{k+1+l} for $1 \leq l \leq k+1$. This process by induction has been explicitly derived in appendix B.4, and proven to result in

$$B_{k+1+l} = \binom{k+l}{1} \cdot \left(1 - \frac{1}{\beta}\right) \cdot u + u - \binom{l}{2} \cdot \left(1 - \frac{1}{\beta}\right)^2 \cdot u^2 - \binom{l}{1} \cdot \left(1 - \frac{1}{\beta}\right) \cdot u^2, \tag{3.5}$$

for $1 \leq l \leq k+1$. Of course, we can keep repeating this process further, and perhaps come up with a generalised expression. To this end, let us compute the final value in the sequence above, by substituting $l = k+1$ into equation (3.5), which yields

$$B_{2\cdot(k+1)} = \binom{2k+1}{1} \cdot \left(1 - \frac{1}{\beta}\right) \cdot u + u - \binom{k+1}{2} \cdot \left(1 - \frac{1}{\beta}\right)^2 \cdot u^2 - \binom{k+1}{1} \cdot \left(1 - \frac{1}{\beta}\right) \cdot u^2.$$

Note that we are putting several coefficients into the form of binomial coefficients, in order to obtain more values for the sequence (B_n) explicitly. Namely, let us keep adding $k+1$ to the index in order to derive an expression for $B_{m\cdot(k+1)}$ inductively. This derivation can be found in appendix B.5, proving the result that

$$\begin{aligned} B_{m\cdot(k+1)} = &\sum_{i=0}^{m-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i-1}{i+1} \cdot \left(1 - \frac{1}{\beta}\right)^{i+1} \cdot u^{i+1} \\ &+ \sum_{i=0}^{m-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i-1}{i} \cdot \left(1 - \frac{1}{\beta}\right)^{i} \cdot u^{i+1}. \end{aligned} \tag{3.6}$$

This result will allow us to find an exact expression for the probability

$$\mathbb{P}\left(M_n \leq 1 - \beta^{-k}\right) = 1 - \mu\left(\bigcup_{i=0}^{n-1} E_i\right) = 1 - B_n.$$

Now, let us specifically consider a subsequence of $n \in \mathbb{N}$, namely all $n = m \cdot (k+1)$ for $m \in \mathbb{N}$. As then by equation (3.6),

$$\begin{aligned} B_n = & \sum_{i=0}^{\frac{n}{k+1}-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i-1}{i+1} \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot u^{i+1} \\ & + \sum_{i=0}^{\frac{n}{k+1}-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i-1}{i} \cdot \left(1-\frac{1}{\beta}\right)^{i} \cdot u^{i+1}. \end{aligned} \tag{3.7}$$

Now, note that the binomial coefficients can be rewritten, as

$$(m-i)\cdot(k+1)+i-1 = m\cdot(k+1) - ik - i + i - 1 = n - ik - 1.$$

Therefore, we can rewrite equation (3.7) to

$$\begin{aligned} B_n = & \sum_{i=0}^{\frac{n}{k+1}-1} (-1)^i \cdot \binom{n-ik-1}{i+1} \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot u^{i+1} \\ & + \sum_{i=0}^{\frac{n}{k+1}-1} (-1)^i \cdot \binom{n-ik-1}{i} \cdot \left(1-\frac{1}{\beta}\right)^{i} \cdot u^{i+1}. \end{aligned}$$

This expression now only involves n and k, but keep in mind that it only holds for $n = m\cdot(k+1)$ for $m \in \mathbb{N}$. Now, let us substitute $u = \beta^{-k}$ for $k \in \mathbb{N}$ fixed, which yields

$$\begin{aligned} B_n = & \sum_{i=0}^{\frac{n}{k+1}-1} (-1)^i \cdot \binom{n-ik-1}{i+1} \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot \beta^{-k\cdot(i+1)} \\ & + \sum_{i=0}^{\frac{n}{k+1}-1} (-1)^i \cdot \binom{n-ik-1}{i} \cdot \left(1-\frac{1}{\beta}\right)^{i} \cdot \beta^{-k\cdot(i+1)}. \end{aligned}$$

This enables us to finally prove an exact expression for the probability above, by substituting this expression of B_n. The result is formulated in the theorem below.

Theorem 3.2. For any $\beta, k \in \mathbb{N}$ with $\beta \geq 2$ and $n = m \cdot (k+1)$ for any $m \in \mathbb{N}$, let (X_i) be the sequence of random variables generated by the doubling map process with $X_0 \sim U(0,1)$ and set $M_n = \max\{X_1, \ldots, X_n\}$. Then

$$\begin{aligned} \mathbb{P}\left(M_n \leq 1-\beta^{-k}\right) = 1 & - \sum_{i=0}^{\frac{n}{k+1}-1} (-1)^i \cdot \binom{n-ik-1}{i+1} \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot \beta^{-k\cdot(i+1)} \\ & - \sum_{i=0}^{\frac{n}{k+1}-1} (-1)^i \cdot \binom{n-ik-1}{i} \cdot \left(1-\frac{1}{\beta}\right)^{i} \cdot \beta^{-k\cdot(i+1)}. \end{aligned}$$

Now in order to derive its asymptotic limit, let us introduce a parameter $\lambda > 0$ in order to couple the two variables n and k, which we both want to send to ∞. This will give us an actual statistical distribution of the limit, depending on this introduced parameter. As in last chapter, let us couple the two variables by introducing the subsequence

$$n_k = \left\lfloor \frac{\lambda \cdot \beta^k}{k+1} \right\rfloor \cdot (k+1),$$

for $k \in \mathbb{N}$ and any fixed $\lambda > 0$. Recall that in theorem 3.2 we need $n = m \cdot (k+1)$ for $m \in \mathbb{N}$, which is why we require the subsequence n_k to be the nearest multiple of $k+1$ below $\lambda \cdot \beta^k$. Just like we did for $\beta = 2$, we will now conclude the chapter by using theorem 3.2 in order to derive the result below.

Theorem 3.3. Define the sequence $n_k = \left\lfloor \frac{\lambda \cdot \beta^k}{k+1} \right\rfloor \cdot (k+1)$ for any $\beta, k \in \mathbb{N}$ with $\beta \geq 2$ and any fixed $\lambda > 0$, let (X_i) be the sequence of random variables generated by the doubling map process with $X_0 \sim U(0,1)$, and set $M_n = \max\{X_1, \ldots, X_n\}$. Then

$$\lim_{k \to \infty} \mathbb{P}\left(M_{n_k} \leq 1 - \beta^{-k}\right) = e^{-(1-\frac{1}{\beta})\cdot\lambda}.$$

The proof and full derivation of this theorem is given in appendix B.6.

4. Theoretical Properties of the Doubling Map

In the previous two chapters, we have shown the asymptotic behaviour of extreme values of the doubling map, and its generalised family of functions with parameter $\beta \geq 2$. In particular, we have studied the probability that the maximum of the doubling map process (X_i) exceeds a certain threshold, and its limiting behaviour as that threshold converges towards 1. Let us now put these concrete results into a more theoretical context in this chapter.

4.1. Extremal Index of the (Generalised) Doubling Map

To start, let us recall that in the result of theorem 2.1, we found that the asymptotic limit of this probability is given by

$$\lim_{k\to\infty} \mathbb{P}\left(M_{n_k} \leq 1 - 2^{-k}\right) = e^{-\lambda/2}, \tag{4.1}$$

where $n_k \xrightarrow{k\to\infty} \infty$. In section 2.2 we constructed this probability as a more mathematically convenient alternative to the original expression, given by equation (2.3) as

$$\mathbb{P}\left(M_n \leq 1 - \frac{\lambda}{n}\right).$$

Hence, we can rewrite the result of equation (4.1) into this form, namely

$$\lim_{n\to\infty} \mathbb{P}\left(M_n \leq 1 - \frac{\lambda}{n}\right) = e^{-\lambda/2}.$$

This can be reformulated as

$$\lim_{n\to\infty} \mathbb{P}\left(n \cdot (M_n - 1) \leq x\right) = e^{x/2},$$

where $x = -\lambda < 0$. If we extend this result by observing that the probability is trivially 1 whenever $x > 0$, we conclude that

$$\lim_{n\to\infty} \mathbb{P}\left(n \cdot (M_n - 1) \leq x\right) = G(x) \equiv \begin{cases} e^{x/2} & \text{for } x < 0 \\ 1 & \text{for } x > 0. \end{cases} \tag{4.2}$$

Hence, we have not merely found the asymptotic behaviour of the doubling map process, but even found that it converges towards an explicit probability distribution. The resulting limiting distribution $G(x)$ is called a generalised extreme value distribution, associated with the random variable $\overline{M}_n = n \cdot (M_n - 1)$. Now, let us also recall from equation (2.2) that if the random variables (X_i^*) were instead chosen to be independently distributed uniformly over $[0, 1)$, we would have that for $M_n^* = \max\{X_1^*, \ldots, X_n^*\}$,

$$\lim_{n\to\infty} \mathbb{P}\left(M_n^* \leq 1 - \frac{\lambda}{n}\right) = e^{-\lambda}.$$

Or equivalently, this can be rewritten with $x = -\lambda$ as

$$\lim_{n\to\infty} \mathbb{P}\left(n \cdot (M_n^* - 1) \leq x\right) = G^*(x) \equiv \begin{cases} e^{x} & \text{for } x < 0 \\ 1 & \text{for } x > 0. \end{cases}$$

The resulting limiting distribution G^* is in this case also a generalised extreme value distribution, for the random variable $\overline{M}_n^* = n \cdot (M_n^* - 1)$. In [8] (theorem 1.4.1), these generalised extreme value distributions are categorised into three separate types. According to that same theorem, the distribution G^* is said to be of type III with parameter $\alpha = 1$, also called a 'Weibull' distribution. Now, let us observe that the distribution in equation (4.2) can be related to G^*, since

$$G(x) = \begin{cases} e^{x/2} & \text{for } x < 0 \\ 1 & \text{for } x > 0. \end{cases} = \left(G^*(x)\right)^{\frac{1}{2}}.$$

This factor $\theta = \frac{1}{2}$ such that $G = \left(G^*(x)\right)^{\theta}$ is called the extremal index of G. This extremal index has to do with clustering, that is, how often subsequent values of the random values exceed a set limit x. In this case, this would mean that it measures how often the random variable M_n exceeds the threshold $u_n = 1 - \frac{\lambda}{n}$ as $n \to \infty$, compared to the case for M_n^*. Whenever the extremal index is $\theta = 1$, such clustering does not appear, which is for instance the case for the tent map, as discussed in [5]. In that paper, the absence of clustering is specifically proven, and we will compare this to the behaviour of the doubling map in section 4.2.

However, in the case that the extremal index gets smaller for $0 < \theta < 1$, more clustering is expected to occur. In other words, more subsequent values of the sequence (M_n) are expected to exceed the threshold $u_n = 1 - \frac{\lambda}{n}$. Such behaviour can clearly be seen in the doubling map. Consider values close to 1 on the interval $[0, 1)$, for instance a value x in some set $U_\delta = [1 - \delta, 1)$ for $\delta > 0$ arbitrarily small. Continuously applying the doubling map f^n to that set U_δ, there will always be values in this set for which $f^n(x)$ is arbitrarily close to 1. Again, to compare to a case for which $\theta = 1$, it can be seen in [6] (figure 2.1) that for the tent map, values in U_δ scatter all over the interval $[0, 1)$ after repeatedly applying the tent map to those values. This confirms that the behaviour of clustering in the doubling map is due to the extremal index $\theta = \frac{1}{2}$ being smaller than 1.

There is another interesting property of the extremal index θ. Namely, it turns out that the mean size of clusters of subsequent values of (M_n) exceeding the threshold u_n is approximately $\frac{1}{\theta}$, as is discussed elaborately in [3] (section 5.2). This property can be seen exceptionally well visually in the doubling map, where random variables exceed the threshold $1 - u$ in pairs, for $u = 2^{-k}$. That is, if we look at the set E_n as defined in equation (2.4), we can see in figures 2.2, 2.3, and 2.4, that the intervals E_n split up into pairs of intervals when applying f^{-1}, that together form $E_{n+1} = f^{-1}(E_n)$. This means that each interval of values X_0 for which $X_n = f^n(X_0)$ exceeds the threshold, is split up into a pair of intervals for which $X_{n+1} = f^{n+1}(X_0)$ exceeds the threshold. This is what is most literally meant by 'exceeding the threshold in pairs', and confirms that the average cluster size for the doubling map is indeed $\frac{1}{\theta} = 2$. Equivalently, consider the interval of values of (X_n) exceeding the threshold, then only half of these values will lead to $X_{n+1} = f(X_n)$ again exceeding the threshold. Let us clarify this view using figure 4.1, shown on the next page.

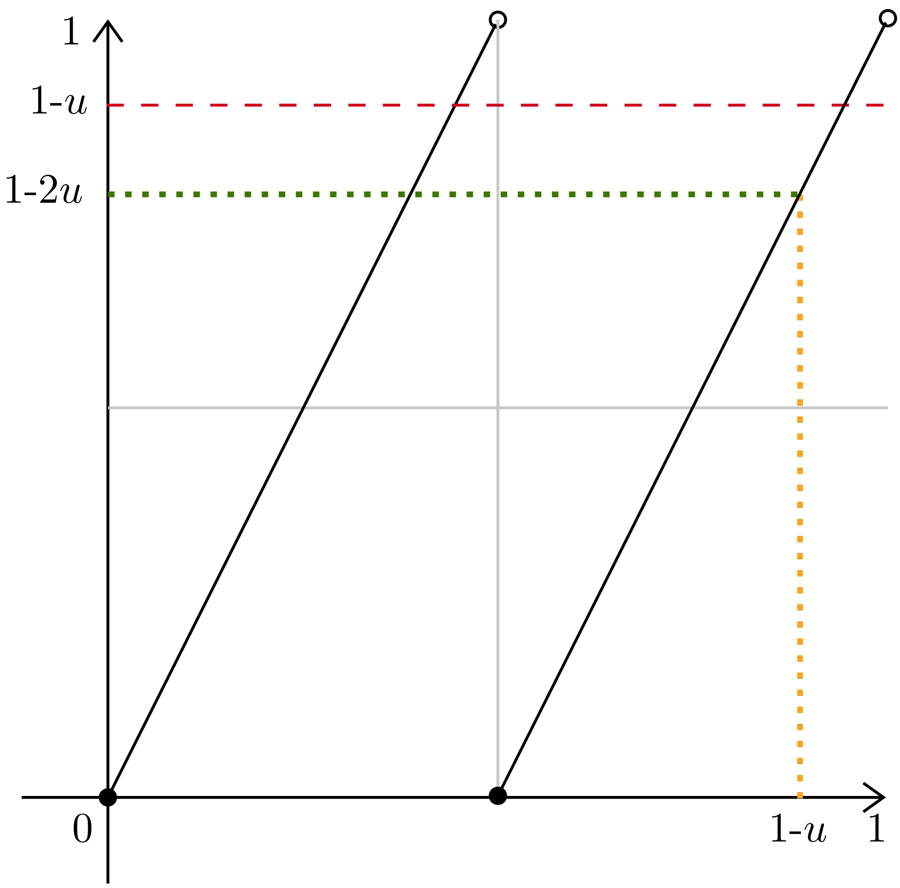

Figure 4.1: Clustering in the doubling map.

In this figure, the values $x \in [1-u, 1)$ for which $X_n = x$ exceeds the threshold $1-u$ are given in orange. Mapping this using the doubling map, we colour the interval $X_{n+1} = f(X_n)$ in green. We clearly see that only half this green interval still exceeds the threshold. Hence, the average cluster size, i.e. the expected number of random variables X_n subsequently exceeding the threshold, is given by

$$\begin{aligned}\mathbb{E}(\text{cluster size}) &= \mathbb{P}(\text{cluster size} = 1) \cdot 1 + \mathbb{P}(\text{cluster size} = 2) \cdot 2 + \mathbb{P}(\text{cluster size} = 3) \cdot 3 + \dots \\ &= \frac{1}{2} + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \dots \\ &= \sum_{n=1}^{\infty} \frac{n}{2^n} = 2 = \frac{1}{\theta}.\end{aligned}$$

This is a more analytical way of showing that the average cluster size is $\frac{1}{\theta}$ for the doubling map. This latter point of view will be more intuitive for the generalised doubling map, which will be discussed next. As a final note though, one may wonder why we are suddenly talking about subsequent values of (X_n) exceeding the threshold, rather than values of (M_n), but let us come back to this in detail in section 4.2.

Let us finish this section by discussing the previously discussed notions for the generalised doubling map. In this case, we see that the result of theorem 3.3 can be rewritten in a similar way to equation 4.2, as

$$\lim_{n\to\infty} \mathbb{P}\left(n \cdot (M_n - 1) \leq x\right) = G_\beta(x) = \begin{cases} e^{(1-\frac{1}{\beta})\cdot x} & \text{for } x < 0 \\ 1 & \text{for } x > 0. \end{cases}$$

We see that in this case, we end up with a distribution G_β, depending on this parameter $\beta \geq 2$. This is again a generalised extreme value distribution, and by noting that $G_\beta = (G^*)^{1-\frac{1}{\beta}}$, we see that the extremal index is $\theta = 1 - \frac{1}{\beta}$ in this case. But what can this adjusted extremal index tell us about the behaviour of the generalised doubling map?

Let us first note that the amount of clustering will decrease whenever $\beta \geq 2$ is increased, since then $1 - \frac{1}{\beta}$ is closer to 1. We can also see this visually for f_β, as fewer values for which X_n exceeds the threshold will lead to $X_{n+1} = f_\beta(X_n)$ again exceeding the threshold. This brings us to the second observation, since this last note compares well with the interpretation for the extremal index that was illustrated by figure 4.1. Let us construct a similar picture below, for $\beta = 3$, in order to confirm this.

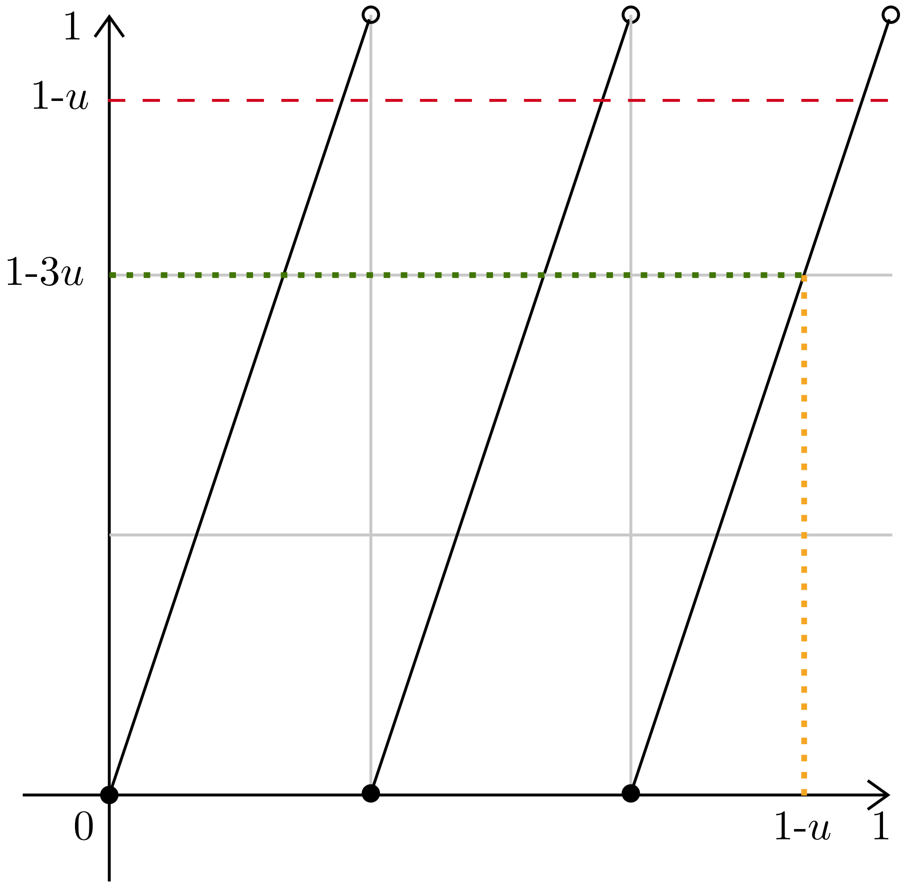

Figure 4.2: Clustering in the generalised doubling map for $\beta = 3$.

In this figure, we again paint the values $x \in [1-u, 1)$ for which $X_n = x$ exceeds the threshold $1 - u$ in orange, and the interval $X_{n+1} = f_\beta(X_n)$ in green. We clearly see that only $\frac{1}{\beta}$ values in this green interval still exceeds the threshold. Hence, the average cluster size, i.e. the expected number of random variables X_n subsequently exceeding the threshold, is given by

$$\mathbb{E}(\text{cluster size}) = \mathbb{P}(\text{cluster size} = 1) \cdot 1 + \mathbb{P}(\text{cluster size} = 2) \cdot 2 + \mathbb{P}(\text{cluster size} = 3) \cdot 3 + \ldots$$

$$
\begin{aligned}
&= 1 - \frac{1}{\beta} + \left(1 - \frac{1}{\beta}\right) \cdot \frac{1}{\beta} \cdot 2 + \left(1 - \frac{1}{\beta}\right) \cdot \frac{1}{\beta^2} \cdot 3 + \dots \\
&= \left(1 - \frac{1}{\beta}\right) \cdot \sum_{n=1}^{\infty} \frac{n}{\beta^{n-1}} = \left(1 - \frac{1}{\beta}\right) \cdot \frac{1}{\left(1 - \frac{1}{\beta}\right)^2} = \frac{1}{1 - \frac{1}{\beta}} = \frac{1}{\theta}.
\end{aligned}
$$

Where the last line follows from the general computation that

$$\sum_{n=1}^{\infty} n \cdot a^{n-1} = \frac{1}{(1-a)^2}, \quad \text{for } |a| < 1.$$

From this, we again see that the average cluster size is $\frac{1}{\theta}$, getting closer to 1 as $\beta \geq 2$ is increased.

4.2. Clustering Behaviour of the (Generalised) Doubling Map

The clustering behaviour that has been discussed in the last section, or rather the absence of such behaviour, can be used to derive the exact generalised extreme value distribution. That is, without the need of having to compute the value of the probability in an exact expression, as we did for the doubling map in chapter 2, and its generalisation in chapter 3. Such an approach has been taken for example in [5] for the tent map, which as mentioned in the last section does not have any clustering behaviour. The goal of this section is to see why this method does not work for the (generalised) doubling map, and explore whether any alternative methods that allow for some clustering may work in this case.

To start, let us explain the procedure with which the generalised extreme value distribution can be found, in the case that there is no clustering. First, we have to check whether two conditions hold for some sequences (u_n), which are given in [8] as follows.

Condition $D(u_n)$. Condition $D(u_n)$ is said to hold for the sequence (X_n) if for any $l, t, n \in \mathbb{N}$,

$$
\begin{aligned}
\Big| \mu\Big(&\{X_0 > u_n\} \cap [\{X_t \leq u_n\} \cap \ldots \cap \{X_{t+l-1} \leq u_n\}]\Big) \\
&- \mu(\{X_0 > u_n\}) \cdot \mu\Big(\{X_0 \leq u_n\} \cap \ldots \cap \{X_{l-1} \leq u_n\}\Big) \Big| \leq \gamma(n,t),
\end{aligned}
$$

for some $\gamma(n,t)$ which is non-increasing with respect to t for all $n \in \mathbb{N}$. Moreover, for all sequences (t_n) with $\frac{t_n}{n} \xrightarrow{n\to\infty} 0$, $\gamma(n,t)$ needs to satisfy $n \cdot \gamma(n, t_n) \xrightarrow{n\to\infty} 0$.

This is a very technical condition, for which it may be difficult to fully understand the purpose of each technicality. The use of this condition is to guarantee that the sequence (X_n) is sufficiently independent. This is a slightly different version of the condition $D(u_n)$ as given in [8], but it suffices for the tent map. Also, as it is satisfied for the (generalised) doubling map, we will not worry too much about this condition. The proof for this can be found in appendix C.1. The second condition does give some trouble for the doubling map though, and is given as follows.

Condition $D'(u_n)$. Condition $D'(u_n)$ is said to hold for the sequence (X_n) if

$$\lim_{k\to\infty} \limsup_{n\to\infty} n \cdot \sum_{j=1}^{\lfloor n/k \rfloor} \mu\left(\{X_0 > u_n\} \cap \{X_j > u_n\}\right) = 0.$$

This condition is much easier on the eye, and hopefully also more intuitive to understand. It can be seen from the definition that it forces the amount of times that subsequent values of

(X_n) exceed the threshold u_n to be insignificant as $n \to \infty$. Or, more loosely interpreted, it guarantees that the amount and size of clusters goes to 0 as $n \to \infty$. In [8] (theorem 3.5.2), the desired asymptotic result is proven to follow whenever the two conditions are met for certain sequences (u_n). In this instance, we will only require (u_n) to satisfy the most strictly necessary condition, as formulated in [5]. Using this, the theorem as in [8] (theorem 3.5.2) is given as follows.

Theorem 4.1. Let (u_n) be such that $n \cdot \mu(\{X_0 > u_n\}) \xrightarrow{n\to\infty} \tau$, for some $\tau \geq 0$. If conditions $D(u_n)$ and $D'(u_n)$ hold for the sequence (X_n), then it follows that for $M_n = \max\{X_1, \ldots, X_n\}$,

$$\lim_{n\to\infty} \mathbb{P}(M_n \leq u_n) = G(e^{-\tau}).$$

Here $G(x)$ is the generalised extreme value distribution as discussed in section 4.1. Using this theorem, we could have skipped all computations and proofs in chapter 2, and simply derived this result for the doubling map, if only condition $D'(u_n)$ were to hold for it. To see that this is not the case, let us give a counterexample to condition $D'(u_n)$, which is done in appendix C.2.

Now that we have shown that condition $D'(u_n)$ does not hold for the doubling map, let us try to find some alternative conditions to $D'(u_n)$, that do allow for (some) clustering, but still give the desired result in the theorem above. Before doing this though, let us come back to an ambiguity from section 4.1, namely why we refer to values of (X_n) exceeding the threshold, rather than those of (M_n). First of all, note that in both conditions $D(u_n)$ and $D'(u_n)$, we only ever measure events for which values of (X_n) exceed a threshold (or stay below it), rather than values of (M_n). This is because the two notions are in fact very closely related. Of course, whenever $M_n > u_n$, it is not necessarily the case that $M_{n+1} > u_{n+1}$. Namely, if the latter holds, then either $M_n > u_{n+1}$ already, or $X_{n+1} > u_{n+1}$. Inductively repeating this argument for $X_n, \ldots, X_1$, we see that the two notions are indeed equivalent. We are also going see this equivalence back in some of the alternative conditions for $D'(u_n)$, which we will be treating now.

All of the alternative definitions that will be discussed are given in [10], and for better comparison, let us first discuss the slightly differently stated condition $D'(u_n)$ in [10] (section 3.2). To start, rather than taking the double limit of $k, n \to \infty$, let us consider sequences (k_n) such that

$$\lim_{n\to\infty} k_n = \infty, \quad \lim_{n\to\infty} k_n \cdot \gamma(n, t_n) = 0, \quad k_n \cdot t_n = \mathcal{O}(n), \tag{4.3}$$

where (t_n) and $\gamma(n, t_n)$ are as in condition $D(u_n)$. Using this, the condition $D'(u_n)$ can be reformulated as follows.

Condition $D'(u_n)$ (As in [10])**.** Condition $D'(u_n)$ is said to hold for the sequence (X_n) if there exists some (k_n) satisfying equation (4.3), such that

$$\lim_{n\to\infty} n \cdot \sum_{j=1}^{\lfloor n/k_n \rfloor} \mu\left(\{X_0 > u_n\} \cap \{X_j > u_n\}\right) = 0.$$

This version of the condition conveys its interpretation perhaps a bit more clearly. Namely, it prevents (X_n) from clustering by guaranteeing that whenever X_0 exceeds the threshold u_n, the chance that the next $\lfloor n/k_n \rfloor$ subsequent values of (X_n) exceed u_n must be arbitrarily small as $n \to \infty$. For the doubling map however, this condition can not hold. No matter how close the threshold u_n is chosen to 1, there is always a measurable set of values for $X_0 > u_n$ such that $X_j > u_n$ for any $1 \leq j \leq \lfloor n/k_n \rfloor$. This is more or less what was shown in the counterexample to condition $D'(u_n)$ in appendix C.2, too. To see this more intuitively for the doubling map, let us suppose that $u_n = 1 - 2^{-k}$, so that $\mu(E_0) = \mu(\{X_0 > u_n\})$ and $\mu(E_j) = \mu(\{X_j > u_n\})$. Then it can be seen that $\mu(E_0 \cap E_j) > 0$ for all $j \in \mathbb{N}$, either visually from figures 2.2, 2.3, and 2.4, or analytically from the explicit expression for E_i, as given in equation (2.5). This analogy has also been worked out more explicitly in appendix C.2.

Let us now discuss a truly alternative condition to $D'(u_n)$, which is given by [10] (section 3.2.2). This condition is designed to allow for some clustering of (X_n), and is formulated as follows.

Condition $D^{(k)}(u_n)$. Condition $D^{(k)}(u_n)$ is said to hold for the sequence (X_n) if there exists some (k_n) satisfying equation (4.3), such that

$$\lim_{n\to\infty} n \cdot \mu\left(\{X_0 > u_n \geq M_{1,k-1}\} \cap \{M_{k,\lfloor n/k_n \rfloor -1} > u_n\}\right) = 0,$$

where

$$M_{i,j} = \begin{cases} \max\{X_i, \dots, X_j\} & \text{for } i \leq j \\ -\infty & \text{for } i > j. \end{cases}$$

This is a more general form of condition $D'(u_n)$, and can easily be verified to be equivalent to it for $k = 1$. It allows for some sequences of length $\lfloor n/k_n \rfloor - k$ to exceed the threshold, as long as $X_1, \dots, X_{k-1}$ do not exceed it, and as long as this behaviour vanishes as $n \to \infty$. For the doubling map, this condition will not work for the same reason that it does not work for condition $D'(u_n)$. This alternative merely allows for more clustering to be present, but also demands that this behaviour must vanish as $n \to \infty$, which is simply not the case for the doubling map. Coming back to the discussion as in section 4.1 though, the appearance of the sequence (M_n) rather than (X_n) can be seen explicitly in this condition.

There is also a slightly weaker version of this previous condition given in [10] (section 4.1.2), that is worth discussing, as follows.

Condition $Д_q(u_n)$. Condition $Д_q(u_n)$ is said to hold for the sequence (X_n) if there exists some (k_n) satisfying equation (4.3), such that for $M_n = \{X_1, \dots, X_n\}$,

$$\lim_{n\to\infty} n \cdot \sum_{j=q+1}^{\lfloor n/k_n \rfloor -1} \mu\left(\{X_0 > u_n \geq M_q\} \cap f^{-j}\left(\{X_0 > u_n \geq M_q\}\right)\right) = 0.$$

At first sight, this may not seem to be comparable to the previous condition. However, note that for any iterative map $f : [0,1) \mapsto [0,1)$ with $X_n = f^n(X_0)$ for all $n \in \mathbb{N}$, it holds that

$$f^{-j}\left(\{X_0 > u_n \geq M_q\}\right) \subset \{X_j > u_n\},$$

for all $j \geq q+1$. This relation is discussed more elaborately in [8] (section 4.1). Using this, it can be seen that condition $Д_q(u_n)$ is virtually the same as condition $D^{q+1}(u_n)$, but leaves a bit more room for clustering. For the doubling map this alteration does not matter however, since the condition still demands that clustering behaviour must vanish as $n \to \infty$, which is not the case for the doubling map.

5. The Generalised Doubling Map Process for Non-Integer Parameters

Recall that in chapter 3, we analysed the asymptotic behaviour of the generalised doubling map process. That is, we considered a family of functions

$$f_\beta(x) = \beta x \bmod 1, \tag{5.1}$$

for any integer $\beta \geq 2$. Then, the generalised doubling map process was defined as

$$X_n = f(X_{n-1}) = f_\beta^n(X_0), \tag{5.2}$$

for $n \in \mathbb{N}$ and any random variable $X_0 \sim U(0,1)$. However, what would happen if we set the parameter $\beta > 1$ to be any real number, rather than only considering integer values? Using the same family of functions as in equation (5.1), we can define the same generalised doubling map process as in equation (5.2). In this chapter, we are going to explore to what end we can still say anything about the asymptotic behaviour of the resulting sequence.

5.1. Constructing an Invariant Probability Measure

The first difference when taking non-integer values of $\beta > 1$, is that we need to construct an invariant probability measure with respect to this new map f_β for every value of $\beta > 1$. Recall that whenever $\beta \geq 2$ was taken to be an integer, the Lebesgue measure was always an invariant probability measure for f_β, but this is not the case whenever $\beta > 1$ is non-integer. In fact, for many values of $\beta > 1$ it is already very difficult to determine this, let alone derive a general formula for any value of $\beta > 1$. Before illustrating this using a couple of examples, let us recall that such invariant probability measure μ needs to be

1. invariant with respect to $f_\beta(x)$: $\mu(A) = \mu(f_\beta^{-1}(A))$ for all $A \subset [0,1)$ and $\beta \geq 2$, and
2. a probability measure: $\mu([0,1)) = 1$, and $\mu(\varnothing) = 0$.

Now, it might seem difficult to construct a measure from scratch that satisfies even the first requirement. So to assist with that, we are going to make use of the so-called discrete Frobenius-Perron equation. This states that a measure μ is invariant (vis-à-vis the first requirement) on $[0,1)$ with respect to $f : [0,1) \mapsto [0,1)$ whenever

$$\mu(A) = \int_A \rho(x)\ dx, \quad \text{with} \quad \rho(x) = \sum_{y \in f^{-1}(x)} \frac{1}{|f'(y)|} \cdot \rho(y), \quad \text{for all } x \in [0,1), \tag{5.3}$$

for any $A \subset [0,1)$, given that the required measure is absolutely continuous with respect to the Lebesgue measure. Then, the second requirement is satisfied by normalising the resulting probability distribution ρ, that is by setting

$$\mu([0,1)) = \int_0^1 \rho(x)\ dx = 1. \tag{5.4}$$

The most difficult part, of course, is to solve the Frobenius-Perron equation given the map f_β for $\beta > 1$. There are special cases for which it is relatively simple to do this, which we will treat later, but let us first see how far we can get in the general case when $\beta > 1$ is non-integer. First of all, in every case, the slope of the generalised doubling map $f_\beta(x)$ is simply β for all $x \in [0, 1)$ and any $\beta > 1$. Now, let us derive an expression for $f_\beta^{-1}(x)$. Recall that for $\beta \geq 2$ integer, we found that

$$f_\beta^{-1}(x) = \bigcup_{i=1}^{\beta} \left\{ \frac{i-1+x}{\beta} \right\}, \quad \text{for } x \in [0, 1).$$

Similarly, we get that for $\beta > 1$ non-integer,

$$f_\beta^{-1}(x) = \bigcup_{i=1}^{\lfloor \beta \rfloor} \left\{ \frac{i-1+x}{\beta} \right\} \cup \left\{ \frac{\lceil \beta \rceil - 1 + x}{\beta} \right\} \cap [0, 1), \quad \text{for } x \in [0, 1). \tag{5.5}$$

That is, $f_\beta(x)$ has a $\lfloor \beta \rfloor + 1^{\text{st}}$ inverse value if and only if

$$\frac{\lceil \beta \rceil - 1 + x}{\beta} < 1, \quad \text{i.e.} \quad x < \beta \bmod 1,$$

for $\beta > 1$ non-integer. Hence, let us define the indicator function

$$\mathbb{1}_\beta(x) = \begin{cases} 1 & \text{if } 0 \leq x < \beta \bmod 1 \\ 0 & \text{if } \beta \bmod 1 \leq x < 1. \end{cases}$$

Combining this with equation (5.5) and substituting it into the Frobenius-Perron equation, yields

$$\rho(x) = \sum_{i=1}^{\lfloor \beta \rfloor} \frac{1}{\beta} \cdot \rho\left(\frac{i-1+x}{\beta} \right) + \frac{1}{\beta} \cdot \mathbb{1}_\beta(x) \cdot \rho\left(\frac{\lceil \beta \rceil - 1 + x}{\beta} \right), \quad \text{for } x \in [0, 1). \tag{5.6}$$

This is where we run into trouble for most values of $\beta > 1$. Namely, it usually depends on the value of x whenever

$$\frac{i-1+x}{\beta} < \beta \bmod 1.$$

In that case, it becomes very difficult to solve equation (5.6). However, there are specific values of $\beta > 1$ for which there exists some integer $I(\beta)$, such that

$$\frac{i-1+x}{\beta} < \beta \bmod 1 \quad \text{if and only if} \quad 1 \leq i \leq I(\beta), \tag{5.7}$$

independently of $x \in [0, 1)$. In fact, we see that this is the case whenever

$$I(\beta) = \beta \cdot \beta \bmod 1$$

is an integer value. For instance, consider $\beta = \sqrt{3} + 1$, then

$$\begin{aligned} \frac{i-1+x}{\sqrt{3}+1} < \sqrt{3} - 1 \quad &\Leftrightarrow \quad i - 1 + x < 2 \\ &\Leftrightarrow \quad 1 \leq i \leq 2 = I(\beta), \quad \text{for } x \in [0, 1). \end{aligned}$$

In these special cases where $I(\beta)$ is independent with respect to x, it follows from equation (5.6) that

$$\rho(x) = \begin{cases} \dfrac{\beta + 1}{2\beta - \lfloor \beta \rfloor} & \text{if } 0 \leq x < \beta \bmod 1 \\ \dfrac{\beta}{2\beta - \lfloor \beta \rfloor} & \text{if } \beta \bmod 1 \leq x < 1. \end{cases} \tag{5.8}$$

The derivation of this direct formula can be found in appendix D.1. Given this result however, let us give a few examples of probability distributions. For instance, whenever $\beta = \sqrt{2}+1$, it follows that

$$I(\beta) = \left(\sqrt{2}+1\right) \cdot \left(\sqrt{2}-1\right) = 1.$$

Therefore, the formula as in equation (5.8) can be applied, yielding

$$\rho(x) = \begin{cases} \dfrac{\sqrt{2}+1}{2} & \text{if } 0 \leq x < \sqrt{2}-1 \\ \dfrac{\sqrt{2}+1}{2\sqrt{2}} & \text{if } \sqrt{2}-1 \leq x < 1. \end{cases}$$

Let us give a second example that will be elaborated upon next section, namely whenever $\beta = \varphi$, where $\varphi = \frac{\sqrt{5}+1}{2}$ is the golden ratio. As is well known, the golden ratio satisfies $\varphi^2 - \varphi - 1 = 0$, and thus $\varphi - 1 = \frac{1}{\varphi}$. Hence, it follows that

$$I(\beta) = \varphi \cdot (\varphi - 1) = 1,$$

so that equation (5.8) can be applied. Substituting $\beta = \varphi$ into this equation, we get that

$$\rho(x) = \begin{cases} \dfrac{\varphi^3}{\varphi^2+1} & \text{if } 0 \leq x < \varphi - 1 \\ \dfrac{\varphi^2}{\varphi^2+1} & \text{if } \varphi - 1 \leq x < 1. \end{cases} \tag{5.9}$$

5.2. The Limiting Distribution of Generalised Doubling Map Process

We have already seen in the previous section that choosing $\beta > 1$ to be non-integer, leads to many complications compared to $\beta \geq 2$ integer. In the following section, we will show how far it is possible to derive the asymptotic behaviour, even when treating only one value for $\beta > 1$ non-integer. We will do this using the generalised doubling map process with $\beta = \varphi$, i.e. the golden ratio. The resulting generalised doubling map function f_φ is shown below.

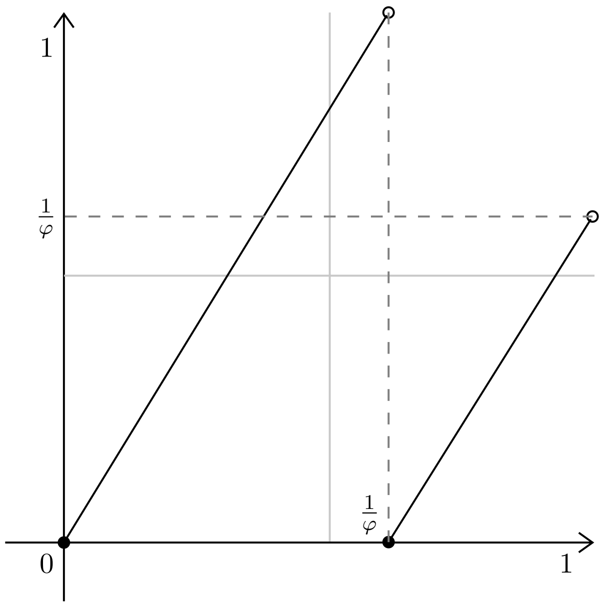

Figure 5.1: The generalised doubling map f_φ on $[0, 1)$.

From this figure, we can see that in line with equation (5.5), the inverse doubling map is given by

$$f_\varphi^{-1}(x) = \left\{\frac{x}{\varphi}\right\} \cup \left\{\frac{1+x}{\varphi}\right\} \cap [0,1), \quad \text{for } x \in [0,1).$$

From this equation (and the figure above), it can be seen that for any connected interval $[a,b) \subset [0,1)$,

$$f_\varphi^{-1}([a,b)) = \begin{cases} \left[\frac{a}{\varphi}, \frac{b}{\varphi}\right) & \text{for } a \geq \frac{1}{\varphi} \\ \left[\frac{a}{\varphi}, \frac{b}{\varphi}\right) \cup \left[\frac{1+a}{\varphi}, 1\right) & \text{for } a < \frac{1}{\varphi} \leq b \\ \left[\frac{a}{\varphi}, \frac{b}{\varphi}\right) \cup \left[\frac{1+a}{\varphi}, \frac{1+b}{\varphi}\right) & \text{for } b < \frac{1}{\varphi}. \end{cases} \tag{5.10}$$

Now, it would be very useful if the condition $D'(u_n)$, as defined in section 4.2, would apply for this generalised doubling map, and some sequence of thresholds (u_n). In that case, we could immediately apply theorem 4.1 in order to determine

$$\lim_{n\to\infty} \mathbb{P}(M_n \leq u_n),$$

i.e. the asymptotic behaviour of the generalised doubling map. However, in a proof very similar to that in appendix C.2, we can show that this condition $D'(u_n)$ will never hold for any sequence (u_n) under the conditions of theorem 4.1. The proof of this in the case that $\beta = \varphi$ is given in appendix D.2. In fact, it turns out that whenever $\beta > 1$ is such that there exists an integer $I(\beta)$ as in equation (5.7) with $I(\beta) = \lfloor\beta\rfloor$, the condition $D'(u_n)$ will never hold. A more generalised proof of appendix D.2 is given for all such values of $\beta > 1$ in appendix D.3.

Of course, in order to apply theorem 4.1, we would first also need to know the generalised extreme value distribution for this specific doubling map. In fact, this can be done for any value of $\beta > 1$ for which there exists an integer $I(\beta)$ as in equation (5.7). That is, whenever it is possible to write the probability density function $\rho(x)$ as

$$\rho(x) = \begin{cases} \dfrac{\beta+1}{2\beta - \lfloor\beta\rfloor} & \text{if } 0 \leq x < \beta \bmod 1 \\ \dfrac{\beta}{2\beta - \lfloor\beta\rfloor} & \text{if } \beta \bmod 1 \leq x < 1, \end{cases}$$

as derived in section 5.1. To derive the generalised extreme value distribution, we consider the independently identically distributed sequence (X_i) with probability density as given above, and its maximum $M_n = \max\{X_1, \dots, X_n\}$, for $n \in \mathbb{N}$. Then, for any $\lambda > 0$ and $n \in \mathbb{N}$ sufficiently large, such that $1 - \frac{\lambda}{n} > \beta \bmod 1$,

$$\begin{aligned} \mathbb{P}\left(X_i \leq 1 - \frac{\lambda}{n}\right) &= \int_0^{1-\lambda/n} \rho(x)\, dx = \int_0^{\beta \bmod 1} \frac{\beta+1}{2\beta - \lfloor\beta\rfloor}\, dx + \int_{\beta \bmod 1}^{1-\lambda/n} \frac{\beta}{2\beta - \lfloor\beta\rfloor}\, dx \\ &= (\beta - \lfloor\beta\rfloor) \cdot \frac{\beta+1}{2\beta - \lfloor\beta\rfloor} + \left(1 - \frac{\lambda}{n} - (\beta - \lfloor\beta\rfloor)\right) \cdot \frac{\beta}{2\beta - \lfloor\beta\rfloor} \\ &= \frac{2\beta - \lfloor\beta\rfloor}{2\beta - \lfloor\beta\rfloor} - \frac{\lambda}{n} \cdot \frac{\beta}{2\beta - \lfloor\beta\rfloor} \\ &= 1 - \frac{\lambda}{n} \cdot \frac{\beta}{2\beta - \lfloor\beta\rfloor}, \end{aligned}$$

for any $1 \leq i \leq n$. Using this, let us note that

$$\begin{aligned}\mathbb{P}\left(M_n \leq 1 - \frac{\lambda}{n}\right) &= \mathbb{P}\left(\bigcap_{i=1}^{n}\left\{X_i \leq 1 - \frac{\lambda}{n}\right\}\right) = \prod_{i=1}^{n}\mathbb{P}\left(X_i \leq 1 - \frac{\lambda}{n}\right) \\ &= \left(1 - \frac{\lambda}{n} \cdot \frac{\beta}{2\beta - \lfloor\beta\rfloor}\right)^n \xrightarrow{n\to\infty} e^{-\lambda \cdot \frac{\beta}{2\beta - \lfloor\beta\rfloor}}.\end{aligned}$$

Therefore, the limiting distribution is given by

$$G^*(x) = \begin{cases} e^{x \cdot \frac{\beta}{2\beta - \lfloor\beta\rfloor}} & \text{for } x < 0 \\ 1 & \text{for } x > 0. \end{cases},$$

for any $\beta > 1$ with $I(\beta) \in \mathbb{N}$. The factor in the exponent is not to be confused with the extremal index, it is merely a scaling factor in the case that $\beta > 1$ is not an integer value. Note that for $\beta \geq 2$ integer, this scaling factor becomes 1, which is in line with the limiting distribution for the generalised doubling map, as stated in equation (4.2). This means that according to [8] (theorem 1.4.1), the limiting distribution in this case is said to be of type III with parameter $\alpha = \frac{\beta}{2\beta - \lfloor\beta\rfloor}$.

However, it is not possible for us to derive the extreme value distribution whenever the sequence (X_n) is not independently identically distributed, as we did for the (generalised) doubling map. At least, not with the methods that were followed in sections 2.4 and 2.5, or section 3.3. For the method that was followed in section 2.4 for the doubling map, and 3.3 for the generalised doubling map, it is not possible to derive a similar result here analytically without using extensive numerical computations. This dead end will be discussed further in the next and final chapter of this paper.

6. Discussion and Future Research

As mentioned in the introduction of this paper, we set out to find the limiting behaviour of the doubling map. The result, which was derived in section 2.4, is not unique as it was published before by [7]. However, the additional goals of this paper were to prove this result, and present it in an intuitive way. Unfortunately, the former of these has not been fully satisfied. As we will discuss more elaborately later on in section 6.1, the proof of the final result in section 2.4 is not airtight, and thus subject to future research.

We also sought to elaborate on the resulting extreme value law of the doubling map, and show a similar result for the generalised doubling map. For this result, no general derivation had been given yet in [7], but it turned out to be very similar to that of the doubling map. However, we discussed in section 5.2 already that it is not possible for us to use a similar approach as in section 3.3 to derive this. Therefore, it is open to further research to show the exact expression for the extreme value law of the generalised doubling map, in the case that $\beta > 1$ is a non-integer value.

Moreover, we have shown that it is possible to derive the extreme value law for the doubling map using an alternative approach involving k-bonacci sequences. It turns out that this is also possible to do for the generalised doubling map. However, it remains to be seen whether this is the case whenever $\beta > 1$ is taken to be non-integer, but this will be discussed more elaborately in section 6.2.

Finally, in the introduction we also mentioned that in [5] it was possible to use extreme value theory to derive the limiting probability distribution of the tent map. In section 4.2, we discussed whether it might be possible to follow a similar approach in order to find the extreme value law of the (generalised) doubling map. Not only could it be shown that the exact method used in [5] did not work here, we even used extreme value theory to prove that no similar derivation could be used to derive this.

6.1. Proof of the Third Observation

Something that may have been forgotten at the end of section 2.4, when the final steps of deriving the expression of the extreme value law of the doubling map are proven, is that the result itself is not proven at all. This is because out of all the intermediate results that are used to derive the final expression, one has not been provided with a proof. That is, of course, the third observation in section 2.3.3, as stated in equation (2.11). This observation, together with the other two observations that are proven in section 2.3, are used to prove lemma 2.2. This lemma is in turn used numerous times in the proofs of theorems 2.1 and 2.3, the latter providing the expression of the extreme value law of the doubling map.

To the extent of my knowledge and researching capabilities, no proof for the third observation exists, or for any similar result. Therefore, this is truly subject to future research, since proving this observation will prove all subsequent results in this paper. However, let us note that the third observation, despite not being proven, was not made out of thin air. In fact, numerical results strongly imply that this observation must hold for the doubling map, and an intuitive derivation of the third observation was made in section 2.3.3. Although both implications do not provide a formal proof, they do strongly suggest that the observation, and hence the subsequent results in this paper, must hold. Moreover, in [6] the presence of this observation is also strongly suggested, even though that paper did not provide a formal proof either.

6.2. $(\beta - 1)$ times k-bonacci Sequences for $\beta > 1$ Non-Integer

In section 2.5, we proved the results of section 2.4 using a different approach. Namely, we observed that the sequence m_n as constructed in section 2.3 are k-bonacci, and proved this as well. Then, we used [4] (theorems 1 and 2) to give a direct expression for the n^{th} k-bonacci number, and used this to directly find an expression for B_n, and prove all subsequent results. A very similar proof holds for the generalised doubling map, for values of $\beta \geq 2$ integer. However, it remains to be seen whether it is possible to use a similar approach for the generalised doubling map, whenever we take $\beta > 1$ to be non-integer. This has been done in the article that we have published, and can be found in [1]. Numerical analysis shows that the proof for $\beta > 1$ integer might be extended to all values of $\beta > \frac{3}{2}$ non-integer, but no proof of this has been given yet. Therefore, any further research on $(\beta - 1)$ times k-bonacci numbers with $\beta > 1$ non-integer would provide a method to extend the results in section 3.3, and expand upon section 5.2.

Additionally, let us remark that the proof as given in appendix B.2 only holds for integer values of $\beta \geq 2$ that are prime. There is no reason to suspect that the result, namely the first part of lemma 3.1, does not hold when β is not prime. But formally, that case has not been proven yet, and can thus also be subject to future research.

As a more general final note, this paper has not touched on the question whether it is possible to apply extreme value theory in a similar way to the generalised doubling map with $\beta < -1$, or even $\beta \in \mathbb{C}$. In particular, it remains to be seen whether a similar generalised Fibonacci sequence would appear for these values of β, and thus this is all open to further research.

Bibliography

[1] Boer, N. B.-S., and Sterk, A. E. Generalized Fibonacci numbers and extreme value laws for the Rényi map. *ArXiv* (2020). https://arxiv.org/abs/2004.05857.

[2] Boyarsky, A., and Góra, P. *Laws of Chaos: Invariant Measures and Dynamical Systems in One Dimension*. Boston: Birkhäuser, 1997.

[3] Coles, S. *An introduction to statistical modeling of extreme values*, vol. 208. Springer, 2001.

[4] Dresden, G. P. B., and Du, Z. A simplified Binet formula for k-generalized Fibonacci numbers. *J. Integer Seq 17*, 4 (2014).

[5] Freitas, A. C. M. Statistics of the maximum for the tent map. *Chaos, Solitons & Fractals 42*, 1 (2009), 604–608.

[6] Haiman, G. Extreme values of the tent map process. *Statistics & Probability Letters 65*, 4 (2003), 451–456.

[7] Haiman, G. Level hitting probabilities and extremal indexes for some particular dynamical systems. *Methodology and Computing in Applied Probability 20*, 2 (2018), 553–562.

[8] Leadbetter, M. R., Lindgren, G., and Rootzén, H. *Extremes and related properties of random sequences and processes*. Springer Science & Business Media, 2012.

[9] Loya, P. *Amazing and aesthetic aspects of analysis*. Springer, 2017.

[10] Lucarini, V., Faranda, D., de Freitas, J. M. M., Holland, M., Kuna, T., Nicol, M., Todd, M., and Vaienti, S. *Extremes and recurrence in dynamical systems*. John Wiley & Sons, 2016.

[11] Wolfram, D. A. Solving generalized Fibonacci recurrences. *The Fibonacci Quarterly 36* (1998), 129–145.

A. Proofs from Chapter 2

In this part of the appendix, some detailed proofs will be treated that have been omitted from chapter 2. These proofs are mostly proofs by induction that, although a bit lengthy at times, are not very mathematically challenging. Neither are these proofs very insightful to the statements that are being proven, which is the primary reason for placing these in the appendix.

A.1. Proof of Equation (2.5)

Let us start by proving the general form of the set E_i, as defined in section 2.2 by

$$E_i = \{x \in [0,1) \mid X_0 = x \Rightarrow X_i \geq 1-u\}.$$

The claim that will be proven is that using the recursion

$$E_i = \{x \in [0,1) \mid f^i(x) \geq 1-u\} = f^{-1}(E_{i-1}) = f^{-i}(E_0),$$

the general form of E_i is given by

$$E_i = \bigcup_{s=1}^{2^i} \left[\frac{s-u}{2^i}, \frac{s}{2^i}\right). \tag{A.1}$$

Note that in the main section of the paper, this is labelled equation (2.5).

Proof. In this proof by induction, we will only be using two basic properties, namely

1. $A = [a,b) \subset [0,1) \Rightarrow f^{-1}(A) = [\frac{a}{2}, \frac{b}{2}) \cup [\frac{1}{2} + \frac{a}{2}, \frac{1}{2} + \frac{b}{2})$, and
2. $A, B \subset [0,1) \Rightarrow f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

The first property is specific for the doubling map and can easily be confirmed by inserting the formula in equation (2.1), or by looking at figure 2.1. Now let us prove the expression for E_i as in equation (A.1) by induction on $i \in \mathbb{N}$.
For $i = 1$, note that

$$E_1 = f^{-1}(E_0) = \left[\frac{1}{2} - \frac{u}{2}, \frac{1}{2}\right) \cup \left[1 - \frac{u}{2}, 1\right) = \bigcup_{s=1}^{2} \left[\frac{s-u}{2^i}, \frac{s}{2^i}\right),$$

in accordance with equation (A.1). Now suppose that for $i \in \mathbb{N}$ fixed, equation (A.1) holds. Then note that

$$\begin{aligned} E_{i+1} = f^{-1}(E_i) &= \bigcup_{s=1}^{2^i} f^{-1}\left(\left[\frac{s-u}{2^i}, \frac{s}{2^i}\right)\right) \\ &= \bigcup_{s=1}^{2^i} \left(\left[\frac{s-u}{2^{i+1}}, \frac{s}{2^{i+1}}\right) \cup \left[\frac{1}{2} + \frac{s-u}{2^{i+1}}, \frac{1}{2} + \frac{s}{2^{i+1}}\right)\right). \end{aligned} \tag{A.2}$$

Note that the second union of intervals in the last line can be rewritten as

$$\bigcup_{s=1}^{2^i} \left[\frac{1}{2} + \frac{s-u}{2^{i+1}}, \frac{1}{2} + \frac{s}{2^{i+1}}\right) = \bigcup_{s=1}^{2^i} \left[\frac{2^i + s - u}{2^{i+1}}, \frac{2^i + s}{2^{i+1}}\right) = \bigcup_{s=2^i+1}^{2^{i+1}} \left[\frac{s-u}{2^{i+1}}, \frac{s}{2^{i+1}}\right).$$

Substituting this back into equation (A.2), we obtain that

$$E_{i+1} = \bigcup_{s=1}^{2^{i+1}} \left[\frac{s-u}{2^{i+1}}, \frac{s}{2^{i+1}} \right),$$

which is equivalent to equation (A.1) for $i+1$, thus completing the proof by induction. $\square$

A.2. Proof of Equation (2.7) (First Observation)

Recall that in section 2.3, we defined

$$B_n\left(\left[0, \tfrac{1}{2}\right)\right) = \mu\left(\bigcup_{i=0}^{n-1} E_i \cap \left[0, \tfrac{1}{2}\right)\right),$$

and observed the following relation to B_n.

$$B_n\left(\left[0, \tfrac{1}{2}\right)\right) = \frac{1}{2} B_{n-1} \quad \text{for } n \geq 2.$$

This claim is labelled equation (2.7) in section 2.3, and will be proven below.

Proof. Recall that

$$E_i = \bigcup_{s=1}^{2^i} \left[\frac{s-u}{2^i}, \frac{s}{2^i} \right).$$

Then, it follows that for all $i \geq 1$,

$$E_i \cap \left[0, \tfrac{1}{2}\right) = \bigcup_{s=1}^{2^{i-1}} \left[\frac{s-u}{2^i}, \frac{s}{2^i} \right) = \frac{1}{2} \bigcup_{s=1}^{2^{i-1}} \left[\frac{s-u}{2^{i-1}}, \frac{s}{2^{i-1}} \right) = \tfrac{1}{2} E_{i-1}.$$

Here we defined that $\frac{1}{2}A = \{\frac{a}{2} \mid a \in A\}$ for any set $A \subset \mathbb{R}$. Note that by this definition, it directly follows that $\mu(\frac{1}{2}A) = \frac{1}{2} \cdot \mu(A)$. Moreover, note that $E_0 \cap \left[0, \frac{1}{2}\right) = \varnothing$. Therefore, for all $n \geq 2$, we have that

$$B_n\left(\left[0, \tfrac{1}{2}\right)\right) = \mu\left(\bigcup_{i=0}^{n-1} E_i \cap \left[0, \tfrac{1}{2}\right)\right) = \mu\left(\bigcup_{i=1}^{n-1} \tfrac{1}{2} E_{i-1}\right) = \frac{1}{2} \cdot \mu\left(\bigcup_{i=0}^{n-2} E_i\right) = \frac{1}{2} \cdot B_{n-1},$$

which proves the claim. $\square$

A.3. Proof of Equation (2.10) (Second Observation)

Recall that in section 2.3, we deduced a pattern for the part of the sequence (B_n), such that $1 \leq n \leq k+1$. This was called the second of three key observations in that section, and is given by

$$B_n = (n+1) \cdot \frac{u}{2} \quad \text{for } 1 \leq n \leq k+1. \tag{A.3}$$

This observation is labelled equation (2.10) in the main part of the paper. Even though in section 2.3 it was deduced, it was never proven. Despite it being less insightful than the deduction, a full proof of the observation is given below.

Proof. Let us first recall that the sequence (E_i) can be written as

$$E_i = \bigcup_{s=1}^{2^i} \left[\frac{s-u}{2^i}, \frac{s}{2^i}\right),$$

for all $i \geq 0$. Let us set $E'_0 = E_0$, and define the sequence

$$E'_i = E_i \setminus \bigcup_{l=0}^{i-1} E_l \text{ for } 1 \leq i \leq k,$$

where $u = 2^{-k}$. Then let us first prove the claim that for $0 \leq i \leq k$,

$$E'_i = \bigcup_{\substack{s=1 \\ \text{odd}}}^{2^i} \left[\frac{s-u}{2^i}, \frac{s}{2^i}\right). \tag{A.4}$$

Proof of equation (A.4). We are going to prove the claim as in equation (A.4) using strong induction on i for $0 \leq i \leq k$. As for the base case $i = 0$, note that

$$E'_0 = E_0 = [1-u, 1) = \left[\frac{1-u}{1}, \frac{1}{1}\right),$$

as desired. Now suppose the claim holds for all integers up and until any $0 \leq i \leq k-1$, then first note that

$$\begin{aligned}
E'_{i+1} &= E_{i+1} \setminus \bigcup_{l=0}^{i} E_l = E_{i+1} \setminus \bigcup_{l=0}^{i} E'_l \\
&= \left(\bigcup_{s=1}^{2^{i+1}} \left[\frac{s-u}{2^{i+1}}, \frac{s}{2^{i+1}}\right)\right) \setminus \left(\bigcup_{l=0}^{i} \bigcup_{\substack{t=1 \\ \text{odd}}}^{2^l} \left[\frac{t-u}{2^l}, \frac{t}{2^l}\right)\right).
\end{aligned} \tag{A.5}$$

Let us note that we can rewrite

$$\begin{aligned}
\bigcup_{s=1}^{2^{i+1}} \left[\frac{s-u}{2^{i+1}}, \frac{s}{2^{i+1}}\right) &= \left(\bigcup_{\substack{s=1 \\ \text{odd}}}^{2^{i+1}} \left[\frac{s-u}{2^{i+1}}, \frac{s}{2^{i+1}}\right)\right) \cup \left(\bigcup_{\substack{s=1 \\ \text{even}}}^{2^{i+1}} \left[\frac{s-u}{2^{i+1}}, \frac{s}{2^{i+1}}\right)\right) \\
&= \left(\bigcup_{\substack{s=1 \\ \text{odd}}}^{2^{i+1}} \left[\frac{s}{2^{i+1}} - \frac{u}{2^{i+1}}, \frac{s}{2^{i+1}}\right)\right) \cup \left(\bigcup_{s=1}^{2^i} \left[\frac{s}{2^i} - \frac{u}{2^{i+1}}, \frac{s}{2^i}\right)\right) \\
&= \left(\bigcup_{\substack{s=1 \\ \text{odd}}}^{2^{i+1}} \left[\frac{s}{2^{i+1}} - \frac{u}{2^{i+1}}, \frac{s}{2^{i+1}}\right)\right) \cup \left(\bigcup_{\substack{s=1 \\ \text{odd}}}^{2^i} \left[\frac{s}{2^i} - \frac{u}{2^{i+1}}, \frac{s}{2^i}\right)\right) \\
&\quad \cup \left(\bigcup_{s=1}^{2^{i-1}} \left[\frac{s}{2^{i-1}} - \frac{u}{2^{i+1}}, \frac{s}{2^{i-1}}\right)\right).
\end{aligned}$$

Repeating the same argument $i-2$ more times, we obtain

$$\begin{aligned}\bigcup_{s=1}^{2^{i+1}}\left[\frac{s-u}{2^{i+1}},\frac{s}{2^{i+1}}\right) &= \left(\bigcup_{\substack{s=1\\ \text{odd}}}^{2^{i+1}}\left[\frac{s}{2^{i+1}}-\frac{u}{2^{i+1}},\frac{s}{2^{i+1}}\right)\right)\cup\left(\bigcup_{\substack{s=1\\ \text{odd}}}^{2^{i}}\left[\frac{s}{2^{i}}-\frac{u}{2^{i+1}},\frac{s}{2^{i}}\right)\right)\\ &\quad\cup\cdots\cup\left(\bigcup_{\substack{s=1\\ \text{odd}}}^{2}\left[\frac{s}{2}-\frac{u}{2^{i+1}},\frac{s}{2}\right)\right)\cup\left(\bigcup_{s=1}^{1}\left[\frac{s}{1}-\frac{u}{2^{i+1}},\frac{s}{1}\right)\right)\\ &= \bigcup_{j=0}^{i+1}\bigcup_{\substack{s=1\\ \text{odd}}}^{2^j}\left[\frac{s}{2^j}-\frac{u}{2^{i+1}},\frac{s}{2^j}\right).\end{aligned}$$

Substituting this result into equation (A.5), we get that

$$E'_{i+1}=\left(\bigcup_{j=0}^{i+1}\bigcup_{\substack{s=1\\ \text{odd}}}^{2^j}\left[\frac{s}{2^j}-\frac{u}{2^{i+1}},\frac{s}{2^j}\right)\right)\setminus\left(\bigcup_{l=0}^{i}\bigcup_{\substack{t=1\\ \text{odd}}}^{2^l}\left[\frac{t-u}{2^l},\frac{t}{2^l}\right)\right).$$

Note that all the intervals of the excluded set on the right-hand side are disjoint by construction of E'_j for $0\le j\le i$, and therefore by the previous equation,

$$E'_{i+1}=\bigcup_{j=0}^{i+1}\bigcup_{\substack{s=1\\ \text{odd}}}^{2^j}\left\{\left[\frac{s}{2^j}-\frac{u}{2^{i+1}},\frac{s}{2^j}\right)\setminus\left(\bigcup_{l=0}^{i}\bigcup_{\substack{t=1\\ \text{odd}}}^{2^l}\left[\frac{t-u}{2^l},\frac{t}{2^l}\right)\right)\right\}. \tag{A.6}$$

Now, we use the following claim, namely that if we consider the part of the union covered by $j=i+1$, we have that

$$\left[\frac{s}{2^{i+1}}-\frac{u}{2^{i+1}},\frac{s}{2^{i+1}}\right)\cap\left(\bigcup_{l=0}^{i}\bigcup_{\substack{t=1\\ \text{odd}}}^{2^l}\left[\frac{t-u}{2^l},\frac{t}{2^l}\right)\right)=\varnothing, \tag{A.7}$$

for any $1\le s\le 2^{i+1}$ odd. This claim will be proven below separately, but if it holds, we can combine it with equation (A.6) to get

$$E'_{i+1}=\bigcup_{j=0}^{i}\bigcup_{\substack{s=1\\ \text{odd}}}^{2^j}\left\{\left[\frac{s}{2^j}-\frac{u}{2^{i+1}},\frac{s}{2^j}\right)\setminus\left(\bigcup_{l=0}^{i}\bigcup_{\substack{t=1\\ \text{odd}}}^{2^l}\left[\frac{t-u}{2^l},\frac{t}{2^l}\right)\right)\right\}\cup\left(\bigcup_{\substack{s=1\\ \text{odd}}}^{2^{i+1}}\left[\frac{s-u}{2^{i+1}},\frac{s}{2^{i+1}}\right)\right). \tag{A.8}$$

Then, note that for all $0\le j\le i$ and all $1\le s\le 2^j$ odd, we have that for $l=j, t=s$,

$$\left[\frac{s}{2^j}-\frac{u}{2^{i+1}},\frac{s}{2^j}\right)\subset\left[\frac{s}{2^j}-\frac{u}{2^j},\frac{s}{2^j}\right)=\left[\frac{t-u}{2^l},\frac{t}{2^l}\right)\subset\bigcup_{l=0}^{i}\bigcup_{\substack{t=1\\ \text{odd}}}^{2^l}\left[\frac{t-u}{2^l},\frac{t}{2^l}\right).$$

Therefore, it follows that

$$\bigcup_{j=0}^{i}\bigcup_{\substack{s=1\\ \text{odd}}}^{2^j}\left\{\left[\frac{s}{2^j}-\frac{u}{2^{i+1}},\frac{s}{2^j}\right)\setminus\left(\bigcup_{l=0}^{i}\bigcup_{\substack{t=1\\ \text{odd}}}^{2^l}\left[\frac{t-u}{2^l},\frac{t}{2^l}\right)\right)\right\}=\varnothing.$$

Substituting this into equation (A.8), we end up with

$$E'_{i+1} = \bigcup_{\substack{s=1 \\ \text{odd}}}^{2^{i+1}} \left[\frac{s-u}{2^{i+1}}, \frac{s}{2^{i+1}}\right),$$

which is equivalent to equation (A.4) for $i+1$, thus completing the proof by induction. $\square$

Before cutting to the end result of the proof, let us first prove the claim made in equation (A.7) below.

Proof of equation (A.7). For a contradiction, let us suppose that there exists some $1 \leq s \leq 2^{i+1}$ odd, such that

$$\left[\frac{s}{2^{i+1}} - \frac{u}{2^{i+1}}, \frac{s}{2^{i+1}}\right) \cap \left(\bigcup_{l=0}^{i} \bigcup_{\substack{t=1 \\ \text{odd}}}^{2^l} \left[\frac{t-u}{2^l}, \frac{t}{2^l}\right)\right) \neq \varnothing.$$

Then there must exist some $0 \leq l \leq i$, and $1 \leq t \leq 2^l$ odd, such that

$$\left[\frac{s}{2^{i+1}} - \frac{u}{2^{i+1}}, \frac{s}{2^{i+1}}\right) \cap \left[\frac{t-u}{2^l}, \frac{t}{2^l}\right) \neq \varnothing.$$

Since both intervals are connected, this means that either

$$\frac{t-u}{2^l} \leq \frac{s-u}{2^{i+1}} < \frac{t}{2^l} \quad \text{or} \quad \frac{s-u}{2^{i+1}} \leq \frac{t-u}{2^l} < \frac{s}{2^{i+1}}, \tag{A.9}$$

or both. In the first case, it follows that since $u = 2^{-k}$,

$$\begin{aligned} & \frac{t-u}{2^l} \leq \frac{s-u}{2^{i+1}} < \frac{t}{2^l} \\ \Rightarrow \quad & (t-u)\cdot 2^{i+1-l} \leq s-u < t \cdot 2^{i+1-l} \\ \Rightarrow \quad & t\cdot 2^{i+1-l} - 2^{i+1-k-l} + 2^{-k} \leq s < t\cdot 2^{i+1-l} + 2^{-k}. \end{aligned}$$

Since $i \leq k-1$ and $l \geq 0$, it follows that $2^{i+1-k-l} \leq 2^{-l} \leq 1$, and thus

$$\begin{aligned} \Rightarrow \quad & t\cdot 2^{i+1-l} - 1 + 2^{-k} \leq s < t\cdot 2^{i+1-l} + 2^{-k} \\ \Rightarrow \quad & t\cdot 2^{i+1-l} - 1 < s < t\cdot 2^{i+1-l} + 2^{-k}. \end{aligned}$$

Since both $t, s \in \mathbb{N}$, it follows that $s = t \cdot 2^{i+1-l}$, which implies that s is even. This contradicts the assumption that s is odd.

Now suppose that the second case presented in equation (A.9) holds. Then,

$$\begin{aligned} & \frac{s-u}{2^{i+1}} \leq \frac{t-u}{2^l} < \frac{s}{2^{i+1}} \\ \Rightarrow \quad & (s-u)\cdot 2^{l-i-1} \leq t-u < s\cdot 2^{l-i-1} \\ \Rightarrow \quad & s\cdot 2^{l-i-1} - 2^{l-k-i-1} + 2^{-k} \leq t < s\cdot 2^{l-i-1} + 2^{-k}. \end{aligned}$$

Since $l < i+1$, it follows that $2^{l-k-i-1} < 2^{-k}$, and thus

$$\Rightarrow \quad s\cdot 2^{l-i-1} < t < s\cdot 2^{l-i-1} + 2^{-k}.$$

Since there are no integers $1 \leq t \leq 2^l$ and $1 \leq s \leq 2^{i+1}$ such that this is satisfied, this also leads to a contradiction. Therefore, in any case of equation (A.9), we are led to a contradiction, thus proving the claim. $\square$

Now that the proof of equation (A.4) is closed, let us use it to prove equation (A.3). Namely, note that for $1 \leq n \leq k+1$,

$$\begin{aligned}
B_n = \mu\left(\bigcup_{i=0}^{n-1} E_i\right) &= \mu\left(\bigcup_{i=0}^{n-1} E_i'\right) = \sum_{i=0}^{n-1} \mu(E_i') \\
&= \mu(E_0) + \sum_{i=1}^{n-1} \mu\left(\bigcup_{\substack{s=1 \\ \text{odd}}}^{2^i} \left[\frac{s-u}{2^i}, \frac{s}{2^i}\right)\right) \\
&= u + \sum_{i=1}^{n-1} \sum_{\substack{s=1 \\ \text{odd}}}^{2^i} \mu\left(\left[\frac{s-u}{2^i}, \frac{s}{2^i}\right)\right) \\
&= u + \sum_{i=1}^{n-1} \sum_{\substack{s=1 \\ \text{odd}}}^{2^i} \frac{u}{2^i} = u + \sum_{i=1}^{n-1} \frac{u}{2^i} \cdot \frac{2^i}{2} \\
&= u + (n-1) \cdot \frac{u}{2} = (n+1) \cdot \frac{u}{2},
\end{aligned}$$

which completes the proof. □

A.4. Proof that m_n is k-bonacci

The goal of this section will be to show that m_n is indeed a k-bonacci sequence (with starting digits as stated in equation (2.9)) if and only if the three observations derived in section 2.3 hold. Let us divide this into two lemmas that will be proven below. The first will regard observation 2 as stated in equation (2.10), using the definitions of B_n and m_n as given in section 2.3. The resulting lemma below gives us the relation between that observation and the starting digits of the sequence (m_n).

Lemma A.1. For any $k \in \mathbb{N}$, let $u = 2^{-k}$, then

$$B_n = (n+1) \cdot \frac{u}{2} \quad \text{for } 1 \leq n \leq k+1 \tag{A.10}$$

if and only if

$$m_n = 2^{n-2} \quad \text{for } 2 \leq n \leq k+1.$$

Proof. For the 'if' part ($\Leftarrow$), first note that for any $u = 2^{-k}$ and $k \in \mathbb{N}$,

$$B_1 = \mu(E_0) = u,$$

by definition of B_1. Let us now prove equation (A.10) by induction on $2 \leq n \leq k+1$ for any $k \in \mathbb{N}$. In the case that $n = 2$, we have that

$$\begin{aligned}
m_2 = \frac{B_2 - B_1}{u/2} &\overset{\text{set}}{=} 2^{2-2} = 1 \\
\Rightarrow \quad B_2 = B_1 + \frac{u}{2} &= \frac{3u}{2}.
\end{aligned}$$

Now suppose that $B_n = (n+1) \cdot \frac{u}{2}$ for some $2 \leq n \leq k$, then

$$\begin{aligned} m_{n+1} &= \frac{B_{n+1} - B_n}{u/2^n} \overset{\text{set}}{=} 2^{n-1} \\ \Rightarrow \quad B_{n+1} &= B_n + 2^{n-1} \cdot \frac{u}{2^n} = (n+1) \cdot \frac{u}{2} + \frac{u}{2} = (n+2) \cdot \frac{u}{2}, \end{aligned}$$

which is equivalent to equation (A.10) for $n+1$, thus completing the proof by induction. For the 'only if' part ($\Rightarrow$), recall that by definition,

$$m_n = \frac{B_n - B_{n-1}}{u/2^{n-1}} \quad \text{for } n \geq 2.$$

Then by equation (A.10), it follows that for $2 \leq n \leq k+1$,

$$m_n = \frac{(n+1) \cdot u/2 - n \cdot u/2}{u/2^{n-1}} = 2^{n-2},$$

which completes the proof. □

The second statement that will be proven shows that observations 1 and 3 hold if and only if the sequence m_n is k-bonacci. Recall that it has been shown in section 2.3 that observations 1 and 3 together are equivalent to the second part of lemma 2.2.

Lemma A.2. For any $k \in \mathbb{N}$, let $u = 2^{-k}$, then

$$B_{n+1} = B_n + \frac{u}{2} \cdot (1 - B_{n-k}) \quad \text{for } n \geq k+1 \tag{A.11}$$

if and only if

$$m_{n+1} = \sum_{i=0}^{k-1} m_{n-i} \quad \text{for } n \geq k+1.$$

Proof. For the 'if' part ($\Leftarrow$), let us prove equation (A.11) using strong induction on $n \geq k+1$. In the case that $n = k+1$, note that by definition of m_n,

$$m_{k+2} = \frac{B_{k+2} - B_{k+1}}{u/2^{k+1}}.$$

On the other hand, m_n is assumed to be k-bonacci, and thus

$$m_{k+2} = \sum_{i=0}^{k-1} m_{k+1-i} = \sum_{i=0}^{k-1} 2^{k-1-i} = 2^k - 1,$$

where the last steps follow from lemma $A.1$. Combining these two equations gives

$$\begin{aligned} \frac{B_{k+2} - B_{k+1}}{u/2^{k+1}} &= 2^k - 1 \\ \Rightarrow \quad B_{k+2} - B_{k+1} &= \frac{u}{2^{k+1}} \cdot \left(2^k - 1\right) = \frac{u}{2} \cdot \left(1 - 2^{-k}\right) = \frac{u}{2} \cdot (1 - u) \\ \Rightarrow \quad B_{k+2} &= B_{k+1} + \frac{u}{2} \cdot (1 - B_1), \end{aligned}$$

recalling that for any $u = 2^{-k}$, $B_1 = u$. Having proven the base step, let us now assume that equation (A.11) holds for all integers up and until any $n \geq k+1$. Again, by definition it holds that

$$m_{n+2} = \frac{B_{n+2} - B_{n+1}}{u/2^{n+1}}.$$

Meanwhile, it is assumed that m_n is k-bonacci, so that

$$m_{n+2} = \sum_{i=0}^{k-1} m_{n+1-i} = m_{n+1} + \sum_{i=0}^{k-2} m_{n-i} = 2m_{n+1} - m_{n-k+1}.$$

Combining these two equations, we get that

$$\begin{aligned}
& \frac{B_{n+2} - B_{n+1}}{u/2^{n+1}} = 2m_{n+1} - m_{n-k+1} \\
\Rightarrow \quad & B_{n+2} - B_{n+1} = \frac{u}{2^n} \cdot m_{n+1} - \frac{u}{2^{n+1}} \cdot m_{n-k+1} \\
& \qquad = \frac{u}{2^n} \cdot \frac{B_{n+1} - B_n}{u/2^n} - \frac{u}{2^{n+1}} \cdot \frac{B_{n-k+1} - B_{n-k}}{u/2^{n-k}} \\
& \qquad = B_{n+1} - B_n - \frac{1}{2^{k+1}} \cdot (B_{n-k+1} - B_{n-k}) \\
& \qquad = \frac{u}{2} \cdot (1 - B_{n-k}) - \frac{u}{2^{k+2}} \cdot (1 - B_{n-2k}) \\
& \qquad = \frac{u}{2} - \frac{u}{2} \cdot \Big(B_{n-k} + \frac{u}{2} \cdot (1 - B_{n-2k})\Big) \\
& \qquad = \frac{u}{2} \cdot (1 - B_{n-k+1}),
\end{aligned}$$

where the last line follows from the assumption that $B_{n-k+1} = B_{n-k} + \frac{u}{2} \cdot (1 - B_{n-2k})$. Rewriting the final equation yields

$$B_{n+2} = B_{n+1} + \frac{u}{2} \cdot (1 - B_{n-k+1}),$$

which proves equation (A.11) for $n+1$, thus completing the proof by induction.

Conversely, for the 'only if' part ($\Rightarrow$), the previous proof by induction is basically reversed. First note that by definition,

$$m_{n+1} = \frac{B_{n+1} - B_n}{u/2^n} = \frac{2^n}{u} \cdot \frac{u}{2} \cdot (1 - B_{n-k}) = 2^{n-1} \cdot (1 - B_{n-k}) \tag{A.12}$$

for any $n \geq k+1$. Thus for the base case $n = k+1$,

$$m_{k+2} = 2^k \cdot (1 - B_1) = 2^k \cdot (1 - u) = 2^k - 1.$$

On the other hand, note that by lemma A.1,

$$\sum_{i=0}^{k-1} m_{k+1-i} = \sum_{i=0}^{k-1} 2^{k-1-i} = 2^k - 1.$$

Combining this gives that

$$m_{k+2} = \sum_{i=0}^{k-1} m_{k+1-i}.$$

For the inductive step, suppose the equation holds for any integer up until any $n \geq k+1$, i.e.

$$m_{n+1} = \sum_{i=0}^{k-1} m_{n-i} \quad \text{for } n \geq k+1.$$

Then it follows by using equation (A.12), that

$$\begin{aligned}\sum_{i=0}^{k-1} m_{n+1-i} &= m_{n+1} + \sum_{i=0}^{k-2} m_{n-i} = 2m_{n+1} - m_{n-k+1} \\ &= 2^n \cdot (1 - B_{n-k}) - 2^{n-k-1} \cdot (1 - B_{n-2k}) \\ &= 2^n \cdot \left(1 - B_{n-k} - \frac{u}{2} \cdot (1 - B_{n-2k})\right) \\ &= 2^n \cdot (1 - B_{n-k+1}) = m_{n+2},\end{aligned}$$

thus completing the proof by induction. □

A.5. Proof of Equation (2.12) (Proof of Theorem 2.1 part 1)

In this section, let us use lemma 2.2 in order to prove equation (2.12). To this end, let us note that by the first part of the lemma, $B_{k+1} = (k+2) \cdot \frac{u}{2}$. We can now use the second part of the lemma to see that

$$B_{k+2} = B_{k+1} + \frac{u}{2} \cdot (1 - B_1) = (k+2) \cdot \frac{u}{2} + \frac{u}{2} \cdot (1-u) = (k+3) \cdot \frac{u}{2} - \frac{u^2}{2}. \tag{A.13}$$

This can be repeated inductively, using the second part of the lemma in order to obtain $B_{k+1+l+1}$ explicitly, given B_{k+1+l} for $1 \leq l \leq k+1$. Doing this will yield the following result, which is labelled equation (2.12) in section 2.4.

$$B_{k+1+l} = \binom{k+2+l}{1} \cdot \frac{u}{2} - \binom{l+2}{2} \cdot \frac{u^2}{4} + \frac{u^2}{4} \quad \text{for } 1 \leq l \leq k+1. \tag{A.14}$$

This is the result that we will now be proving inductively, primarily using lemma 2.2. This is the first of two proofs using induction that together prove theorem 2.1, the other one being shown in appendix A.6.

Proof. First note that in the case that $l = 1$, we have shown in equation (A.13), that the expression in equation (A.14) holds.
Now suppose that equation (A.14) holds for any $1 \leq l \leq k$ with $k \in \mathbb{N}$ fixed. Then note that by lemma 2.2,

$$\begin{aligned}B_{k+1+l+1} &= B_{k+1+l} + \frac{u}{2} \cdot (1 - B_{l+1}) = \binom{k+2+l}{1} \cdot \frac{u}{2} - \binom{l+2}{2} \cdot \frac{u^2}{4} + \frac{u^2}{4} + \frac{u}{2} \cdot \left(1 - (l+2) \cdot \frac{u}{2}\right) \\ &= (k+2+l) \cdot \frac{u}{2} - \frac{1}{2} \cdot (l+1) \cdot (l+2) \cdot \frac{u^2}{4} + \frac{u^2}{4} + \frac{u}{2} - (l+2) \cdot \frac{u^2}{4} \\ &= (k+2+l+1) \cdot \frac{u}{2} - \frac{1}{2}(l+2) \cdot (l+3) \cdot \frac{u^2}{4} + \frac{u^2}{4} \\ &= \binom{k+2+(l+1)}{1} \cdot \frac{u}{2} - \binom{(l+1)+2}{2} \cdot \frac{u^2}{4} + \frac{u^2}{4},\end{aligned}$$

in accordance with equation (A.14) for $l+1$, thus completing the proof by induction. □

A.6. Proof of Equation (2.13) (Proof of Theorem 2.1 part 2)

Let us now prove the general form for B_n, which will form the final step towards proving the expression as stated in theorem 2.1. Recall that in appendix A.5 we derived an expression for

B_{k+1+l} for $1 \le l \le k+1$, as stated in equation (A.14). Now let us substitute $l = k+1$ in that equation, which yields

$$B_{2\cdot(k+1)} = \binom{2k+3}{1} \cdot \frac{u}{2} - \binom{k+3}{2} \cdot \frac{u^2}{4} + \frac{u^2}{4}. \tag{A.15}$$

We can now keep adding $k+1$ to the index inductively in order to obtain $B_{m\cdot(k+1)}$ for any $m \in \mathbb{N}$. The result of this derivation is that for $m \ge 2$,

$$\begin{aligned} B_{m\cdot(k+1)} = &\sum_{i=0}^{m-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i+1}{i+1} \cdot \left(\frac{u}{2}\right)^{i+1} \\ &+ \sum_{i=1}^{m-1} (-1)^{i+1} \cdot \binom{(m-i)\cdot(k+1)+i-1}{i-1} \cdot \left(\frac{u}{2}\right)^{i+1}, \end{aligned} \tag{A.16}$$

which is labelled equation (2.13) in section 2.4, and will be proven below using strong induction on m.

Proof. First note that for $m = 2$, equation (A.16) is satisfied by equation (A.15). Now suppose that equation (A.16) is satisfied for some $m \ge 2$. Then in order to show that it holds for $m+1$, it is necessary to prove by induction on l that

$$\begin{aligned} B_{m\cdot(k+1)+l} = &\sum_{i=0}^{m-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i+1+l}{i+1} \cdot \left(\frac{u}{2}\right)^{i+1} \\ &+ \sum_{i=1}^{m-1} (-1)^{i+1} \cdot \binom{(m-i)\cdot(k+1)+i-1+l}{i-1} \cdot \left(\frac{u}{2}\right)^{i+1} \\ &+ (-1)^m \cdot \binom{m+1+l}{m+1} \cdot \left(\frac{u}{2}\right)^{m+1} + (-1)^{m+1} \cdot \binom{m-1+l}{m-1} \cdot \left(\frac{u}{2}\right)^{m+1}. \end{aligned} \tag{A.17}$$

holds, for all $1 \le l \le k+1$ and for $m \ge 2$ fixed in the assumption.

Proof of equation (A.17). As for the base case, we can simply take $l = 1$ and $m \ge 2$ as fixed in the previous assumption. Thus, we have that by the second part of lemma 2.2,

$$\begin{aligned} &B_{m\cdot(k+1)+1} = B_{m\cdot(k+1)} + \frac{u}{2} \cdot (1 - B_{(m-1)\cdot(k+1)+1}) \\ &= \sum_{i=0}^{m-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i+1}{i+1} \cdot \left(\frac{u}{2}\right)^{i+1} + \sum_{i=1}^{m-1} (-1)^{i+1} \cdot \binom{(m-i)\cdot(k+1)+i-1}{i-1} \cdot \left(\frac{u}{2}\right)^{i+1} \\ &\quad + \frac{u}{2} - \frac{u}{2} \cdot \left(\sum_{i=0}^{m-2} (-1)^i \cdot \binom{(m-1-i)\cdot(k+1)+i+2}{i+1} \cdot \left(\frac{u}{2}\right)^{i+1} \right. \\ &\quad + \sum_{i=1}^{m-2} (-1)^{i+1} \cdot \binom{(m-1-i)\cdot(k+1)+i}{i-1} \cdot \left(\frac{u}{2}\right)^{i+1} + (-1)^{m-1} \cdot \binom{m+1}{m} \cdot \left(\frac{u}{2}\right)^m \\ &\quad \left. + (-1)^m \cdot \binom{m-1}{m-2} \cdot \left(\frac{u}{2}\right)^m \right). \end{aligned}$$

First, let us factor out the brackets;

$$= \sum_{i=0}^{m-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i+1}{i+1} \cdot \left(\frac{u}{2}\right)^{i+1} + \sum_{i=1}^{m-1} (-1)^{i+1} \cdot \binom{(m-i)\cdot(k+1)+i-1}{i-1} \cdot \left(\frac{u}{2}\right)^{i+1}$$

$$+\frac{u}{2}+\sum_{i=0}^{m-2}(-1)^{i+1}\cdot\binom{(m-1-i)\cdot(k+1)+i+2}{i+1}\cdot\left(\frac{u}{2}\right)^{i+2}$$
$$+\sum_{i=1}^{m-2}(-1)^{i+2}\cdot\binom{(m-1-i)\cdot(k+1)+i}{i-1}\cdot\left(\frac{u}{2}\right)^{i+2}+(-1)^{m}\cdot\binom{m+1}{m}\cdot\left(\frac{u}{2}\right)^{m+1}$$
$$+(-1)^{m+1}\cdot\binom{m-1}{m-2}\cdot\left(\frac{u}{2}\right)^{m+1}.$$

Now, we shift the index of the third and fourth sums, to obtain

$$=\sum_{i=0}^{m-1}(-1)^{i}\cdot\binom{(m-i)\cdot(k+1)+i+1}{i+1}\cdot\left(\frac{u}{2}\right)^{i+1}+\sum_{i=1}^{m-1}(-1)^{i+1}\cdot\binom{(m-i)\cdot(k+1)+i-1}{i-1}\cdot\left(\frac{u}{2}\right)^{i+1}$$
$$+\frac{u}{2}+\sum_{i=1}^{m-1}(-1)^{i}\cdot\binom{(m-i)\cdot(k+1)+i+1}{i}\cdot\left(\frac{u}{2}\right)^{i+1}$$
$$+\sum_{i=2}^{m-1}(-1)^{i+1}\cdot\binom{(m-i)\cdot(k+1)+i-1}{i-2}\cdot\left(\frac{u}{2}\right)^{i+1}+(-1)^{m}\cdot\binom{m+1}{m}\cdot\left(\frac{u}{2}\right)^{m+1}$$
$$+(-1)^{m+1}\cdot\binom{m-1}{m-2}\cdot\left(\frac{u}{2}\right)^{m+1}.$$

Combining the first sum with the third, and the second sum with the fourth, gives

$$=\sum_{i=1}^{m-1}(-1)^{i}\cdot\left[\binom{(m-i)\cdot(k+1)+i+1}{i+1}+\binom{(m-i)\cdot(k+1)+i+1}{i}\right]\cdot\left(\frac{u}{2}\right)^{i+1}$$
$$+\sum_{i=2}^{m-1}(-1)^{i+1}\cdot\left[\binom{(m-i)\cdot(k+1)+i-1}{i-1}+\binom{(m-i)\cdot(k+1)+i-1}{i-2}\right]\cdot\left(\frac{u}{2}\right)^{i+1}$$
$$+\frac{u}{2}+\binom{m\cdot(k+1)+1}{1}\cdot\frac{u}{2}+\binom{(m-1)\cdot(k+1)}{0}\cdot\left(\frac{u}{2}\right)^{2}+(-1)^{m}\cdot\binom{m+1}{m}\cdot\left(\frac{u}{2}\right)^{m+1}$$
$$+(-1)^{m+1}\cdot\binom{m-1}{m-2}\cdot\left(\frac{u}{2}\right)^{m+1}.$$

Using a property of the Pascal triangle, which will be elaborated below, we obtain

$$=\sum_{i=1}^{m-1}(-1)^{i}\cdot\binom{(m-i)\cdot(k+1)+i+2}{i+1}\cdot\left(\frac{u}{2}\right)^{i+1}+\sum_{i=2}^{m-1}(-1)^{i+1}\cdot\binom{(m-i)\cdot(k+1)+i}{i-1}\cdot\left(\frac{u}{2}\right)^{i+1}$$
$$+\binom{m\cdot(k+1)+2}{1}\cdot\frac{u}{2}+\binom{(m-1)\cdot(k+1)+1}{0}\cdot\left(\frac{u}{2}\right)^{2}+(-1)^{m}\cdot\binom{m+1}{m}\cdot\left(\frac{u}{2}\right)^{m+1}$$
$$+\left[(-1)^{m}\cdot\binom{m+1}{m+1}\cdot\left(\frac{u}{2}\right)^{m+1}+(-1)^{m+1}\cdot\binom{m-1}{m-1}\cdot\left(\frac{u}{2}\right)^{m+1}\right]+(-1)^{m+1}\cdot\binom{m-1}{m-2}\cdot\left(\frac{u}{2}\right)^{m+1}.$$

Note that between the square brackets the same term has been added and subtracted. Applying the property of the Pascal triangle again, yields

$$=\sum_{i=0}^{m-1}(-1)^{i}\cdot\binom{(m-i)\cdot(k+1)+i+2}{i+1}\cdot\left(\frac{u}{2}\right)^{i+1}+\sum_{i=1}^{m-1}(-1)^{i+1}\cdot\binom{(m-i)\cdot(k+1)+i}{i-1}\cdot\left(\frac{u}{2}\right)^{i+1}$$
$$+(-1)^{m}\cdot\binom{m+2}{m+1}\cdot\left(\frac{u}{2}\right)^{m+1}+(-1)^{m+1}\cdot\binom{m}{m-1}\cdot\left(\frac{u}{2}\right)^{m+1},$$

in accordance with equation (A.17) for $l = 1$, thus completing the proof for the base case. Note that in the last two lines we used a property of the Pascal triangle, namely that

$$\begin{aligned}\binom{p}{q+1} + \binom{p}{q} &= \frac{p!}{(p-q-1)!\,(q+1)!} + \frac{p!}{(p-q)!\,q!} = \frac{(p-q)\,p!}{(p-q)!\,(q+1)!} + \frac{(q+1)\,p!}{(p-q)!\,(q+1)!} \\ &= \frac{(p+1)!}{(p-q)!\,(q+1)!} = \binom{p+1}{q+1}. \end{aligned} \tag{A.18}$$

Now for the inductive part, suppose that equation (A.17) holds for any $1 \le l \le k$, then using the second part of lemma 2.2, we derive that

$$\begin{aligned}
&B_{m\cdot(k+1)+l+1} = B_{m\cdot(k+1)+l} + \frac{u}{2}\cdot\left(1 - B_{(m-1)\cdot(k+1)+l+1}\right) \\
&= \sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i+1+l}{i+1}\cdot\left(\frac{u}{2}\right)^{i+1} \\
&\quad + \sum_{i=1}^{m-1}(-1)^{i+1}\cdot\binom{(m-i)\cdot(k+1)+i-1+l}{i-1}\cdot\left(\frac{u}{2}\right)^{i+1} \\
&\quad + (-1)^m\cdot\binom{m+1+l}{m+1}\cdot\left(\frac{u}{2}\right)^{m+1} + (-1)^{m+1}\cdot\binom{m-1+l}{m-1}\cdot\left(\frac{u}{2}\right)^{m+1} \\
&\quad + \frac{u}{2} - \frac{u}{2}\cdot\left(\sum_{i=0}^{m-2}(-1)^i\cdot\binom{(m-1-i)\cdot(k+1)+i+1+l+1}{i+1}\cdot\left(\frac{u}{2}\right)^{i+1}\right. \\
&\quad + \sum_{i=1}^{m-2}(-1)^{i+1}\cdot\binom{(m-1-i)\cdot(k+1)+i-1+l+1}{i-1}\cdot\left(\frac{u}{2}\right)^{i+1} + (-1)^{m-1}\cdot\binom{m+1+l}{m}\cdot\left(\frac{u}{2}\right)^m \\
&\quad \left. + (-1)^m\cdot\binom{m-1+l}{m-2}\cdot\left(\frac{u}{2}\right)^m\right).
\end{aligned}$$

First, let us factor out the brackets;

$$\begin{aligned}
&= \sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i+1+l}{i+1}\cdot\left(\frac{u}{2}\right)^{i+1} \\
&\quad + \sum_{i=1}^{m-1}(-1)^{i+1}\cdot\binom{(m-i)\cdot(k+1)+i-1+l}{i-1}\cdot\left(\frac{u}{2}\right)^{i+1} + (-1)^m\cdot\binom{m+1+l}{m+1}\cdot\left(\frac{u}{2}\right)^{m+1} \\
&\quad + (-1)^{m+1}\cdot\binom{m-1+l}{m-1}\cdot\left(\frac{u}{2}\right)^{m+1} + \frac{u}{2} + \sum_{i=0}^{m-2}(-1)^{i+1}\cdot\binom{(m-1-i)\cdot(k+1)+i+2+l}{i+1}\cdot\left(\frac{u}{2}\right)^{i+2} \\
&\quad + \sum_{i=1}^{m-2}(-1)^{i+2}\cdot\binom{(m-1-i)\cdot(k+1)+i+l}{i-1}\cdot\left(\frac{u}{2}\right)^{i+2} + (-1)^m\cdot\binom{m+1+l}{m}\cdot\left(\frac{u}{2}\right)^{m+1} \\
&\quad + (-1)^{m+1}\cdot\binom{m-1+l}{m-2}\cdot\left(\frac{u}{2}\right)^{m+1}.
\end{aligned}$$

Now, we shift the index of the third and fourth sums, to obtain

$$\begin{aligned}
&= \sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i+1+l}{i+1}\cdot\left(\frac{u}{2}\right)^{i+1} \\
&\quad + \sum_{i=1}^{m-1}(-1)^{i+1}\cdot\binom{(m-i)\cdot(k+1)+i-1+l}{i-1}\cdot\left(\frac{u}{2}\right)^{i+1} + (-1)^m\cdot\binom{m+1+l}{m+1}\cdot\left(\frac{u}{2}\right)^{m+1}
\end{aligned}$$

$$+(-1)^{m+1}\cdot\binom{m-1+l}{m-1}\cdot\left(\frac{u}{2}\right)^{m+1}+\frac{u}{2}+\sum_{i=1}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i+1+l}{i}\cdot\left(\frac{u}{2}\right)^{i+1}$$
$$+\sum_{i=2}^{m-1}(-1)^{i+1}\cdot\binom{(m-i)\cdot(k+1)+i-1+l}{i-2}\cdot\left(\frac{u}{2}\right)^{i+1}+(-1)^m\cdot\binom{m+1+l}{m}\cdot\left(\frac{u}{2}\right)^{m+1}$$
$$+(-1)^{m+1}\cdot\binom{m-1+l}{m-2}\cdot\left(\frac{u}{2}\right)^{m+1}.$$

Combining the first sum with the third, and the second sum with the fourth, gives

$$=\sum_{i=1}^{m-1}(-1)^i\cdot\left[\binom{(m-i)\cdot(k+1)+i+1+l}{i+1}+\binom{(m-i)\cdot(k+1)+i+1+l}{i}\right]\cdot\left(\frac{u}{2}\right)^{i+1}$$
$$+\sum_{i=2}^{m-1}(-1)^{i+1}\cdot\left[\binom{(m-i)\cdot(k+1)+i-1+l}{i-1}+\binom{(m-i)\cdot(k+1)+i-1+l}{i-2}\right]\cdot\left(\frac{u}{2}\right)^{i+1}$$
$$+\frac{u}{2}+\binom{m\cdot(k+1)+1+l}{1}\cdot\frac{u}{2}+\binom{(m-1)\cdot(k+1)+l}{0}\cdot\left(\frac{u}{2}\right)^2$$
$$+(-1)^m\cdot\left[\binom{m+1+l}{m+1}+\binom{m+1+l}{m}\right]\cdot\left(\frac{u}{2}\right)^{m+1}$$
$$+(-1)^{m+1}\cdot\left[\binom{m-1+l}{m-1}+\binom{m-1+l}{m-2}\right]\cdot\left(\frac{u}{2}\right)^{m+1}.$$

Using the property as in equation (A.18), we obtain

$$=\sum_{i=1}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i+2+l}{i+1}\cdot\left(\frac{u}{2}\right)^{i+1}+\sum_{i=2}^{m-1}(-1)^{i+1}\cdot\binom{(m-i)\cdot(k+1)+i+l}{i-1}\cdot\left(\frac{u}{2}\right)^{i+1}$$
$$+\binom{m\cdot(k+1)+2+l}{1}\cdot\frac{u}{2}+\binom{(m-1)\cdot(k+1)+1+l}{0}\cdot\left(\frac{u}{2}\right)^2$$
$$+(-1)^m\cdot\binom{m+2+l}{m+1}\cdot\left(\frac{u}{2}\right)^{m+1}+(-1)^{m+1}\cdot\binom{m+l}{m-1}\cdot\left(\frac{u}{2}\right)^{m+1}.$$
$$=\sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i+1+(l+1)}{i+1}\cdot\left(\frac{u}{2}\right)^{i+1}$$
$$+\sum_{i=1}^{m-1}(-1)^{i+1}\cdot\binom{(m-i)\cdot(k+1)+i-1+(l+1)}{i-1}\cdot\left(\frac{u}{2}\right)^{i+1}$$
$$+(-1)^m\cdot\binom{m+1+(l+1)}{m+1}\cdot\left(\frac{u}{2}\right)^{m+1}+(-1)^{m+1}\cdot\binom{m-1+(l+1)}{m-1}\cdot\left(\frac{u}{2}\right)^{m+1},$$

Note that we have now obtained the same expression as in equation (A.17), but for $l+1$, thus completing the induction on l. □

Having proven equation (A.17), it now follows that for $l=k+1$,

$$B_{(m+1)\cdot(k+1)}=\sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m+1-i)\cdot(k+1)+i+1}{i+1}\cdot\left(\frac{u}{2}\right)^{i+1}$$
$$+\sum_{i=1}^{m-1}(-1)^{i+1}\cdot\binom{(m+1-i)\cdot(k+1)+i-1}{i-1}\cdot\left(\frac{u}{2}\right)^{i+1}$$
$$+(-1)^m\cdot\binom{m+1+k+1}{m+1}\cdot\left(\frac{u}{2}\right)^{m+1}+(-1)^{m+1}\cdot\binom{m-1+k+1}{m-1}\cdot\left(\frac{u}{2}\right)^{m+1}$$

$$
\begin{aligned}
&= \sum_{i=0}^{m} (-1)^i \cdot \binom{(m+1-i)\cdot(k+1)+i+1}{i+1} \cdot \left(\frac{u}{2}\right)^{i+1} \\
&\quad + \sum_{i=1}^{m} (-1)^{i+1} \cdot \binom{(m+1-i)\cdot(k+1)+i-1}{i-1} \cdot \left(\frac{u}{2}\right)^{i+1}.
\end{aligned}
$$

which is equivalent to equation (A.16) for $m+1$, thus completing the entire proof by induction on m. □

A.7. Proof of Theorem 2.3

Given the result of theorem 2.1, we can now prove theorem 2.3, which will be done below.

Proof. First, let us denote $m_k = \frac{n_k}{k+1} - 1$. Then note that the result of theorem 2.3 can be rewritten, using theorem 2.1, as

$$
\begin{aligned}
\lim_{k\to\infty} \mathbb{P}\left(M_{n_k} \leq 1 - 2^{-k}\right) = \lim_{k\to\infty} 1 &- \sum_{i=0}^{m_k} (-1)^i \cdot \binom{n_k - ik + 1}{i+1} \cdot 2^{-(k+1)\cdot(i+1)} \\
&- \sum_{i=1}^{m_k} (-1)^{i+1} \cdot \binom{n_k - ik - 1}{i-1} \cdot 2^{-(k+1)\cdot(i+1)}.
\end{aligned} \tag{A.19}
$$

This will be proven below, given the assumptions as in theorem 2.3. To do this, we are going to use Tannery's theorem, which is stated in [9] (theorem 3.30) as follows:

Theorem A.3 (Tannery)**.** For all $k \in \mathbb{N}$, let $\sum_{i=1}^{m_k} a_i(k) < \infty$ where $m_k \xrightarrow{k\to\infty} \infty$. If for each $i \in \mathbb{N}$, $\lim_{k\to\infty} a_i(k)$ exists, and there is a series ($C_i \geq 0$) with $\sum_{i=1}^{\infty} C_i < \infty$ such that $|a_i(k)| \leq C_i$ for all $i, k \in \mathbb{N}$, then

$$
\lim_{k\to\infty} \sum_{i=1}^{m_k} a_i(k) = \sum_{i=1}^{\infty} \lim_{k\to\infty} a_i(k),
$$

where both sides are well-defined.

Thus, under the right conditions, this theorem will allow us to insert the limit into the sums given in equation (A.19). To this end, let us define

$$
\begin{aligned}
a_i(k) &= (-1)^i \cdot \binom{n_k - ik + 1}{i+1} \cdot 2^{-(k+1)\cdot(i+1)}, \text{ and} \\
b_i(k) &= (-1)^{i+1} \cdot \binom{n_k - ik - 1}{i-1} \cdot 2^{-(k+1)\cdot(i+1)}.
\end{aligned} \tag{A.20}
$$

Let us now apply the theorem for the first sum in equation (A.19), using the first sequence defined above. For Tannery's theorem to hold in this case, we must thus find a sequence ($C_i \geq 0$) such that $\sum_{i=1}^{\infty} C_i < \infty$ and $|a_i(k)| \leq C_i$ for all $i, k \in \mathbb{N}$. To this end, let us fix any $i \in \mathbb{N}$, then note that

$$
\begin{aligned}
|a_i(k)| &= \binom{n_k - ik + 1}{i+1} \cdot 2^{-(k+1)\cdot(i+1)} \leq \frac{1}{(i+1)!} \cdot \left(\frac{n_k}{2^{k+1}}\right)^{i+1} \\
&\leq \frac{1}{(i+1)!} \cdot \left(\frac{\lambda}{2}\right)^{i+1} \overset{\text{set}}{=} C_i.
\end{aligned}
$$

Also, note that

$$\sum_{i=0}^{\infty} C_i = \sum_{i=0}^{\infty} \frac{1}{(i+1)!} \cdot \left(\frac{\lambda}{2}\right)^{i+1} = \sum_{i=1}^{\infty} \frac{1}{i!} \cdot \left(\frac{\lambda}{2}\right)^{i} = e^{\lambda/2} - 1 < \infty.$$

Now, we only need to check that $\lim_{k\to\infty} a_i(k)$ exists for all $i \in \mathbb{N}$. Therefore, let us rewrite

$$\lim_{k\to\infty} a_i(k) = \lim_{k\to\infty} (-1)^i \cdot \binom{n_k - ik + 1}{i+1} \cdot 2^{-(k+1)\cdot(i+1)}. \tag{A.21}$$

Then we use the identity that for any $p, q \in \mathbb{N}$,

$$\binom{p}{q} \cdot \frac{1}{p^q} = \frac{1}{q!} \cdot \left(1 - \frac{1}{p}\right) \cdot \left(1 - \frac{2}{p}\right) \cdot \ldots \cdot \left(1 - \frac{q-1}{p}\right).$$

Applying this to equation (A.21), we get that

$$\begin{aligned}
\lim_{k\to\infty} a_i(k) &= \lim_{k\to\infty} (-1)^i \cdot \frac{1}{(i+1)!} \cdot \left(1 - \frac{1}{n_k - ik + 1}\right) \cdot \left(1 - \frac{2}{n_k - ik + 1}\right) \cdot \ldots \cdot \left(1 - \frac{i}{n_k - ik + 1}\right) \\
&\quad \cdot \left(\frac{n_k - ik + 1}{2^{k+1}}\right)^{i+1} \\
&= \lim_{k\to\infty} (-1)^i \cdot \frac{1}{(i+1)!} \cdot \left(\frac{n_k - ik + 1}{2^{k+1}}\right)^{i+1}.
\end{aligned} \tag{A.22}$$

Note that

$$\frac{n_k - ik + 1}{2^{k+1}} = \frac{\left\lfloor \frac{\lambda \cdot 2^k}{k+1} \right\rfloor \cdot (k+1) - (ik - 1)}{2^{k+1}} = \frac{\lambda}{2} - \frac{ik - 1}{2^{k+1}} + \mathcal{O}\left(\frac{k+1}{2^{k+1}}\right).$$

To substitute this into equation (A.22), let us first note that

$$\lim_{k\to\infty} \mathcal{O}\left(\frac{k+1}{2^{k+1}}\right) = 0,$$

and we thus get that

$$\lim_{k\to\infty} a_i(k) = \lim_{k\to\infty} (-1)^i \cdot \frac{1}{(i+1)!} \cdot \left(\frac{\lambda}{2} - \frac{ik-1}{2^{k+1}}\right)^{i+1}. \tag{A.23}$$

Now, observe that

$$\left(\frac{\lambda}{2} - \frac{ik-1}{2^{k+1}}\right)^{i+1} = \left(\frac{\lambda}{2}\right)^{i+1} + \mathcal{O}\left(\frac{\lambda \cdot (i+1) \cdot (ik-1)}{2^{k+2}}\right).$$

Since for all $i \in \mathbb{N}$

$$\lim_{k\to\infty} \mathcal{O}\left(\frac{\lambda \cdot (i+1) \cdot (ik-1)}{2^{k+2}}\right) = 0,$$

it follows that substitution into equation (A.23) gives

$$\lim_{k\to\infty} a_i(k) = (-1)^i \cdot \frac{1}{(i+1)!} \cdot \left(\frac{\lambda}{2}\right)^{i+1}. \tag{A.24}$$

Therefore, $\lim_{k\to\infty} a_i(k)$ has been shown to exist for all $i \in \mathbb{N}$, and thus Tannery's theorem is satisfied.

Then by its result, and using equation (A.24), it follows that

$$\begin{aligned}
\lim_{k\to\infty}\sum_{i=0}^{m_k}(-1)^i\cdot\binom{n_k-ik+1}{i+1}\cdot 2^{-(k+1)\cdot(i+1)} &= \sum_{i=0}^{\infty}\lim_{k\to\infty}(-1)^i\cdot\binom{n_k-ik+1}{i+1}\cdot 2^{-(k+1)\cdot(i+1)} \\
&= \sum_{i=0}^{\infty}(-1)^i\cdot\frac{1}{(i+1)!}\cdot\left(\frac{\lambda}{2}\right)^{i+1} \\
&= 1-\sum_{i=0}^{\infty}(-1)^i\cdot\frac{1}{i!}\cdot\left(\frac{\lambda}{2}\right)^{i} = 1-e^{-\lambda/2}. \quad \text{(A.25)}
\end{aligned}$$

Now, let us similarly apply the theorem for the second sum in equation (A.19), using the sequence $(b_i(k))$ as defined in equation (A.20). First, us derive a sequence $(\bar{C}_i \geq 0)$ such that $\sum_{i=1}^{\infty}\bar{C}_i < \infty$ and $|b_i(k)| \leq \bar{C}_i$ for all $i,k\in\mathbb{N}$. So, fix $i\in\mathbb{N}$, then

$$\begin{aligned}
|b_i(k)| &= \binom{n_k-ik-1}{i-1}\cdot 2^{-(k+1)\cdot(i+1)} \leq \frac{1}{(i-1)!}\cdot\left(\frac{n_k}{2^{k+1}}\right)^{i-1} \\
&\leq \frac{1}{(i-1)!}\cdot\left(\frac{\lambda}{2}\right)^{i-1} \overset{\text{set}}{=} \bar{C}_i.
\end{aligned}$$

Where it follows that

$$\sum_{i=1}^{\infty}\bar{C}_i = \sum_{i=1}^{\infty}\frac{1}{(i-1)!}\cdot\left(\frac{\lambda}{2}\right)^{i-1} = \sum_{i=0}^{\infty}\frac{1}{i!}\left(\frac{\lambda}{2}\right)^{i} = e^{\lambda/2} < \infty.$$

However, now observe that

$$\begin{aligned}
\lim_{k\to\infty} b_i(k) &= \lim_{k\to\infty}(-1)^{i+1}\cdot\binom{n_k-ik-1}{i-1}\cdot 2^{-(k+1)\cdot(i+1)} \\
&= \lim_{k\to\infty}(-1)^{i+1}\cdot\frac{1}{(i-1)!}\cdot\left(1-\frac{1}{n_k-ik-1}\right)\cdot\left(1-\frac{2}{n_k-ik-1}\right)\cdot\ldots\cdot\left(1-\frac{i-2}{n_k-ik-1}\right) \\
&\quad\cdot\left(\frac{n_k-ik-1}{2^{k+1}}\right)^{i-1}\cdot 2^{-2\cdot(k+1)} \\
&= \lim_{k\to\infty}(-1)^{i+1}\cdot\frac{1}{(i-1)!}\cdot\left(\frac{\lambda}{2}\right)^{i-1}\cdot 2^{-2\cdot(k+1)} = 0.
\end{aligned}$$

Here the last line was derived similarly as for $a_i(k)$, leading to equation (A.24). Applying Tannery's theorem to this result, we get that

$$\lim_{k\to\infty}\sum_{i=1}^{m_k}(-1)^{i+1}\cdot\binom{n_k-ik-1}{i-1}\cdot 2^{-(k+1)\cdot(i+1)} = 0.$$

Substituting this and equation (A.25) into equation (A.19), we get that

$$\begin{aligned}
\lim_{k\to\infty}\mathbb{P}\left(M_{n_k}\leq 1-2^{-k}\right) &= \lim_{k\to\infty} 1-\sum_{i=0}^{m_k}(-1)^i\cdot\binom{n_k-ik+1}{i+1}\cdot 2^{-(k+1)\cdot(i+1)} \\
&\quad -\sum_{i=1}^{m_k}(-1)^{i+1}\cdot\binom{n_k-ik-1}{i-1}\cdot 2^{-(k+1)\cdot(i+1)} \\
&= 1-\left(1-e^{-\lambda/2}\right) = e^{-\lambda/2},
\end{aligned}$$

which proves theorem 2.3. $\square$

A.8. Proof of Theorems 2.4 and 2.7

In this proof, we are going to use the main results in [4] in order to prove theorem 2.7. To do this, let us first prove theorem 2.4, which can be done briefly.

Proof of theorem 2.4. Let us recall that by lemma 2.2, we can rewrite m_n as

$$m_n = \frac{B_n - B_{n-1}}{u/2^{n-1}} = 2^{n-2} \cdot (1 - B_{n-k-1}),$$

for $n \geq k+2$. Rewriting this equation, we get that

$$B_{n-k-1} = 1 - 2^{2-n} \cdot m_n \quad \Rightarrow \quad B_n = 1 - 2^{1-n-k} \cdot m_{n+k+1},$$

which proves theorem 2.4. $\square$

Now, let us define α_i for $1 \leq i \leq k$ as the roots of the polynomial

$$p_k(x) = x^k - x^{k-1} - \cdots - 1. \tag{A.26}$$

Recall that of the roots α_i for $1 \leq i \leq k$, there is exactly one α such that $|\alpha| > 1$, for any $k \in \mathbb{N}$. Using theorem 2.6 which is derived in [4] (theorem 2), and combining it with theorem 2.4, we get that

$$\mathbb{P}\left(M_n \leq 1 - 2^{-k}\right) = 1 - B_n = 2^{1-n-k} \cdot \left\lfloor \frac{\alpha - 1}{2 + (k+1)\cdot(\alpha - 2)} \cdot \alpha^{n+k} + \frac{1}{2} \right\rfloor, \tag{A.27}$$

In the rest of the proof, let us show that using this result, we can prove theorem 2.3 in an alternative way. Moreover, the proof will result in a slightly more general formulation of the theorem, as presented in theorem 2.7.

Proof of theorem 2.7. First note that if we substitute the sequence $n_k = \lfloor \lambda \cdot 2^k \rfloor$ into equation (A.27), we get that

$$\mathbb{P}\left(M_{n_k} \leq 1 - 2^{-k}\right) = 2^{1-n_k-k} \cdot \left\lfloor \frac{\alpha - 1}{2 + (k+1)\cdot(\alpha - 2)} \cdot \alpha^{n_k+k} + \frac{1}{2} \right\rfloor \tag{A.28}$$

To evaluate the limit as $k \to \infty$, let us first observe that for any sequence $(x_k) \in \mathbb{R}$,

$$\begin{aligned}
& & x_k - \frac{1}{2} \leq \left\lfloor x_k + \frac{1}{2} \right\rfloor \leq x_k + \frac{1}{2} \\
\Rightarrow & \lim_{k\to\infty} 2^{1-n_k-k} \cdot \left(x_k - \frac{1}{2}\right) \leq & \lim_{k\to\infty} 2^{1-n_k-k} \cdot \left\lfloor x_k + \frac{1}{2} \right\rfloor \leq \lim_{k\to\infty} 2^{1-n_k-k} \cdot \left(x_k + \frac{1}{2}\right) \\
\Rightarrow & \lim_{k\to\infty} 2^{1-n_k-k} \cdot x_k \leq & \lim_{k\to\infty} 2^{1-n_k-k} \cdot \left\lfloor x_k + \frac{1}{2} \right\rfloor \leq \lim_{k\to\infty} 2^{1-n_k-k} \cdot x_k \\
\Rightarrow & & \lim_{k\to\infty} 2^{1-n_k-k} \cdot \left\lfloor x_k + \frac{1}{2} \right\rfloor = \lim_{k\to\infty} 2^{1-n_k-k} \cdot x_k,
\end{aligned}$$

given that this limit exists. Applying this to equation (A.28), we get that

$$\begin{aligned}
\lim_{k\to\infty} \mathbb{P}\left(M_{n_k} \leq 1 - 2^{-k}\right) &= \lim_{k\to\infty} 2^{1-n_k-k} \cdot \frac{\alpha - 1}{2 + (k+1)\cdot(\alpha - 2)} \cdot \alpha^{n_k+k} \\
&= \lim_{k\to\infty} 2 \cdot \frac{\alpha - 1}{2 + (k+1)\cdot(\alpha - 2)} \cdot \left(\frac{\alpha}{2}\right)^{n_k+k}
\end{aligned} \tag{A.29}$$

Now, let us use the following result, which will be proven separately below.

Corollary A.7. Let α be the distinct root of the polynomial $p_k(x)$, as given in equation (A.26), with $|\alpha| > 1$ for any $k \in \mathbb{N}$. Then, for $n_k = \lfloor \lambda \cdot 2^k \rfloor$, it holds that

$$\lim_{k\to\infty} \left(\frac{\alpha}{2}\right)^{n_k+k} = e^{-\lambda/2}.$$

For now, let us assume that the corollary holds. Then, let us recall that $\lim_{k\to\infty} \alpha = 2$, and use that $\lim_{k\to\infty}(k+1) \cdot (\alpha - 2) = 0$. This second limit will be proven near the end of this section, in equation (A.36). Using these two limits, we see that

$$\lim_{k\to\infty} 2 \cdot \frac{\alpha - 1}{2 + (k+1) \cdot (\alpha - 2)} = 1. \tag{A.30}$$

Substituting this into equation (A.29), and using the result of the corollary, we conclude that

$$\lim_{k\to\infty} \mathbb{P}\left(M_{n_k} \leq 1 - 2^{-k}\right) = e^{-\lambda/2},$$

which proves theorem 2.7. □

However, we used a corollary in the proof, which still needs to be proven. In order to do this, let us first introduce a lemma which will be used throughout the proof.

Lemma A.4. Let $a > 1$ and (b_k) be a positive sequence such that $\lim_{k\to\infty} a^k \cdot b_k = c$ for some $c > 0$. Then,

$$\lim_{k\to\infty} \frac{a^k - (a - b_k)^k}{k} = \frac{c}{a}.$$

Proof. Consider the identity

$$x^k - y^k = (x - y) \cdot \sum_{i=0}^{k-1} x^{k-1-i} \cdot y^i,$$

for any $x, y > 0$. Substituting $x = a$ and $y = a - b_k$, yields

$$\frac{a^k - (a - b_k)^k}{k} = \frac{a^k \cdot b_k}{a} \cdot S_k, \tag{A.31}$$

where for

$$S_k = \frac{1}{k} \cdot \sum_{i=0}^{k-1} \left(1 - \frac{b_k}{a}\right)^i,$$

we claim that $\lim_{k\to\infty} S_k = 1$. To show this, let us note that by assumption, $\lim_{k\to\infty} b_k = 0$, and thus for k sufficiently large, $-1 < \frac{b_k}{a} < 0$. Then, by Bernoulli's inequality.

$$1 - i \cdot \frac{b_k}{a} \leq \left(1 - \frac{b_k}{a}\right)^i < 1.$$

Combining this with the observation that $\lim_{k\to\infty} k \cdot b_k = 0$, we get that

$$\begin{aligned}
& \frac{1}{k} \cdot \sum_{i=0}^{k-1} \left(1 - i \cdot \frac{b_k}{a}\right) \leq S_k && < \frac{1}{k} \cdot \sum_{i=0}^{k-1} 1^i \\
\Rightarrow \quad & 1 - \frac{k-1}{2} \cdot \frac{b_k}{a} \leq S_k && < 1 \\
\Rightarrow \quad & \lim_{k\to\infty} 1 - \frac{k-1}{2} \cdot \frac{b_k}{a} \leq \lim_{k\to\infty} S_k \leq 1 && \\
\Rightarrow \quad & \lim_{k\to\infty} S_k = 1. &&
\end{aligned}$$

Taking the limit of equation (A.31) completes the proof. □

Before proving corollary A.7, it is useful to determine a strict bound on α first. This is done in the lemma below, after which the proof of corollary A.7 will be given.

Lemma A.5. Let α be the distinct root of the polynomial $p_k(x)$, as given in equation (A.26), with $|\alpha| > 1$ for any $k \in \mathbb{N}$. Define

$$\alpha_{\max} = 2 - \frac{1}{2^k - 1}, \quad \text{and} \quad \alpha_{\min} = 2 - \frac{1 + 2^{-k/2}}{2^k - 1},$$

then the following hold for k sufficiently large:

I. $p_k(\alpha_{\max}) > 0$.

II. $p_k(\alpha_{\min}) < 0$.

III. $\alpha_{\min} < \alpha < \alpha_{\max}$.

Proof. Let us prove each of the statements individually below.

I. Proof that $p_k(\alpha_{\max}) > 0$.
Let us first use that for any $k \in \mathbb{N}$,

$$\sum_{i=0}^{k-1} x^i = \frac{x^k - 1}{x - 1},$$

so that

$$p_k(x) = x^k - \sum_{i=0}^{k-1} x^i = x^k \cdot \frac{x-2}{x-1} + \frac{1}{x-1}. \tag{A.32}$$

Then, it follows that for $x = \alpha_{\max}$,

$$\begin{aligned} p_k(\alpha_{\max}) &= \left(2 - \frac{1}{2^k - 1}\right)^k \cdot \frac{\frac{1}{2^k-1}}{\frac{1}{2^k-1} - 1} - \frac{1}{\frac{1}{2^k-1} - 1} \\ &= \left(2 - \frac{1}{2^k - 1}\right)^k \cdot \frac{-1}{2^k - 2} + \frac{2^k - 1}{2^k - 2}. \end{aligned} \tag{A.33}$$

Clearly, the denominator is positive for $k \geq 2$. As for the numerator, let us show that

$$2^k - 1 - \left(2 - \frac{1}{2^k - 1}\right)^k > 0,$$

for k sufficiently large. To do this, let us use lemma A.4, substituting $a = 2$ and

$$b_k = \frac{1}{2^k - 1}, \quad \text{where} \quad \lim_{k \to \infty} a^k \cdot b_k = 1 = c.$$

Then by lemma A.4,

$$\lim_{k \to \infty} \frac{1}{k} \cdot \left(2^k - \left(2 - \frac{1}{2^k - 1}\right)^k\right) = \frac{1}{2}.$$

Thus, for k sufficiently large, we have that

$$2^k - \left(2 - \frac{1}{2^k - 1}\right)^k - 1 \geq \frac{k}{4} - 1 > 0,$$

for $k > 4$. Hence, the numerator of equation (A.33) is also positive for k sufficiently large, and thus $p_k(\alpha_{\max}) > 0$, proving the claim.

II. Proof that $p_k(\alpha_{\text{min}}) < 0$.
Let us again use equation (A.32), then substituting $x = \alpha_{\text{min}}$, gives that

$$p_k(\alpha_{\text{min}}) = \left(2 - \frac{1+2^{-k/2}}{2^k-1}\right)^k \cdot \frac{\frac{1+2^{-k/2}}{2^k-1}}{\frac{1+2^{-k/2}}{2^k-1} - 1} - \frac{1}{\frac{1+2^{-k/2}}{2^k-1} - 1}$$
$$= \left(2 - \frac{1+2^{-k/2}}{2^k-1}\right)^k \cdot \frac{1+2^{-k/2}}{2+2^{-k/2}-2^k} - \frac{2^k-1}{2+2^{-k/2}-2^k}. \tag{A.34}$$

For $k \geq 2$, the denominator has a negative sign. As for the numerator, let us show that

$$\left(2 - \frac{1+2^{-k/2}}{2^k-1}\right)^k \cdot \left(1+2^{-k/2}\right) - 2^k + 1 > 0,$$

for k sufficiently large. To do this, let us use lemma A.4, substituting $a = 2$ and

$$b_k = \frac{1+2^{-k/2}}{2^k-1}, \quad \text{where} \quad \lim_{k\to\infty} a^k \cdot b_k = 1 = c.$$

Then by lemma A.4,

$$\lim_{k\to\infty} \frac{1}{k} \cdot \left(2^k - \left(2 - \frac{1+2^{-k/2}}{2^k-1}\right)^k\right) = \frac{1}{2}.$$

Thus, for k sufficiently large, we have that

$$2^k - \left(2 - \frac{1+2^{-k/2}}{2^k-1}\right)^k \leq k.$$

From this, it follows that

$$\left(2 - \frac{1+2^{-k/2}}{2^k-1}\right)^k \cdot \left(1+2^{-k/2}\right) - 2^k + 1 = 2^{-k/2} + 1 - \left(1+2^{-k/2}\right) \cdot \left(2^k - \left(2 - \frac{1+2^{-k/2}}{2^k-1}\right)^k\right)$$
$$\geq 2^{-k/2} + 1 - \left(1+2^{-k/2}\right) \cdot k > 0,$$

for k sufficiently large. Hence, the numerator of equation (A.34) is also positive for k sufficiently large, and thus $p_k(\alpha_{\text{max}}) > 0$, proving the claim.

III. Proof that $\alpha_{\text{min}} < \alpha < \alpha_{\text{max}}$.
First, let us simply note that for k sufficiently large,

$$2 - \frac{1}{k} < \alpha_{\text{min}} < \alpha_{\text{max}} < 2.$$

Recall that $p_k(\alpha) = 0$, and that α is the only root of p_k with $|\alpha| > 1$. Since by I. $p_k(\alpha_{\text{min}}) < 0$ and by II. $p_k(\alpha_{\text{max}}) > 0$, it follows from the intermediate value theorem that there exists some value $\alpha_{\text{min}} < c < \alpha_{\text{max}}$ such that $p_k(c) = 0$. Since $c > 1$, it follows by the observation above that $c = \alpha$. Therefore, $\alpha_{\text{min}} < \alpha < \alpha_{\text{max}}$, for k sufficiently large. □

In order to prove corollary A.7, let us introduce a commonly used lemma, below.

Lemma A.6. Let (a_k) be a sequence such that $\lim_{k\to\infty} k \cdot a_k = c$ for any constant $c \in \mathbb{R}$, then

$$\lim_{k\to\infty} (1 - a_k)^k = e^{-c}.$$

Proof. Let us first note that by the condition of the lemma, for all $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|k \cdot a_k - 1| < \varepsilon$. From this, it follows that

$$\left(1 - \frac{1+\varepsilon}{k}\right)^k \leq (1 - a_k)^k \leq \left(1 - \frac{1-\varepsilon}{k}\right)^k,$$

for all $k \geq N$. Hence, we get that

$$e^{-1-\varepsilon} \leq \liminf_{k\to\infty} (1 - a_k)^k \leq \limsup_{k\to\infty} (1 - a_k)^k \leq e^{-1+\varepsilon}.$$

Since this holds for any $\varepsilon > 0$, it follows that the desired limit exists and is equal to

$$\lim_{k\to\infty} (1 - a_k)^k = e^{-1},$$

proving the lemma. □

Corollary A.7. Let α be the distinct root of the polynomial $p_k(x)$, as given in equation (A.26), with $|\alpha| > 1$ for any $k \in \mathbb{N}$. Then, for $n_k = \lfloor \lambda \cdot 2^k \rfloor$, it holds that

$$\lim_{k\to\infty} \left(\frac{\alpha}{2}\right)^{n_k + k} = e^{-\lambda/2}.$$

Proof. By the last result of lemma A.5, it follows that for k sufficiently large,

$$2 - \frac{1 + 2^{-k/2}}{2^k - 1} < \alpha < 2 - \frac{1}{2^k - 1} \tag{A.35}$$

From this, let us first note that

$$\begin{aligned} 0 = -\lim_{k\to\infty} (k+1) \cdot \frac{1 + 2^{-k}}{2^k - 1} < \lim_{k\to\infty} (k+1) \cdot (\alpha - 2) < -\lim_{k\to\infty} (k+1) \cdot \frac{1}{2^k - 1} = 0 \\ \Rightarrow \lim_{k\to\infty} (k+1) \cdot (\alpha - 2) = 0, \end{aligned} \tag{A.36}$$

which proves the limit as in equation (A.30). As for the result in the corollary, let us note that by equation (A.35), it follows that

$$1 - \frac{1 + 2^{-k/2}}{2^{k+1} - 2} < \frac{\alpha}{2} < 1 - \frac{1}{2^{k+1} - 2}. \tag{A.37}$$

Let us define the sequences

$$a_k = \frac{1 + 2^{-k/2}}{2^{k+1} - 2} \quad \text{and} \quad b_k = \frac{1}{2^{k+1} - 2},$$

corresponding to the expressions on the left-hand side and right-hand side of the inequality above, respectively. It can easily be seen that

$$\lim_{k\to\infty} a_k \cdot 2^{k+1} = \lim_{k\to\infty} b_k \cdot 2^{k+1} = 1.$$

Hence, it follows from lemma A.6 that

$$\lim_{k\to\infty}(1-a_k)^{\lambda\cdot 2^k} = \lim_{k\to\infty}(1-b_k)^{\lambda\cdot 2^k} = e^{-\lambda/2},$$

and

$$\lim_{k\to\infty}(1-a_k)^{k-1} = \lim_{k\to\infty}(1-b_k)^{k} = e^{0} = 1,$$

Applying these limits to equation (A.37), and using that $n_k = \lfloor \lambda \cdot 2^k \rfloor$, we get that

$$\begin{aligned}
& & 1-a_k \leq \frac{\alpha}{2} \leq 1-b_k \\
\Rightarrow & & \lim_{k\to\infty}(1-a_k)^{n_k+k} \leq \lim_{k\to\infty}\left(\frac{\alpha}{2}\right)^{n_k+k} \leq \lim_{k\to\infty}(1-b_k)^{n_k+k} \\
\Rightarrow & & \lim_{k\to\infty}(1-a_k)^{\lambda\cdot 2^k+k-1} \leq \lim_{k\to\infty}\left(\frac{\alpha}{2}\right)^{n_k+k} \leq \lim_{k\to\infty}(1-b_k)^{\lambda\cdot 2^k+k} \\
\Rightarrow & & \lim_{k\to\infty}\left(\frac{\alpha}{2}\right)^{n^k+k} = e^{-\lambda/2},
\end{aligned}$$

proving the corollary. □

B. Proofs from Chapter 3

In this part of the appendix, some detailed proofs will be treated that have been omitted from chapter 3. These proofs are mostly proofs by induction that, although a bit lengthy at times, are not very mathematically challenging. Neither are these proofs very insightful to the statements that are being proven, which is the primary reason for placing these in the appendix.

B.1. Proof of Equation (3.2)

Let us start by proving the general form of the set E_i, as defined in section 3.1 by

$$E_i = \{x \in [0,1) \mid X_0 = x \Rightarrow X_i \geq 1-u\},$$

is given by

$$E_i = \bigcup_{s=1}^{\beta^i} \left[\frac{s-u}{\beta^i}, \frac{s}{\beta^i}\right). \tag{B.1}$$

Note that in the main section of the paper, this is labelled equation (3.2).

Proof. In this proof by induction, we will only be using two basic properties, namely

1. $A = [a,b) \subset [0,1) \Rightarrow f_\beta^{-1}(A) = \bigcup_{i=1}^{\beta} \left[\frac{i-1}{\beta} + \frac{a}{\beta}, \frac{i-1}{\beta} + \frac{b}{\beta}\right)$, and
2. $A, B \subset [0,1) \Rightarrow f_\beta^{-1}(A \cup B) = f_\beta^{-1}(A) \cup f_\beta^{-1}(B)$.

The first property is specific for the doubling map and can easily be confirmed by inserting the formula in equation (3.1). Now let us prove the expression for E_i as in equation (B.1) by induction on $i \in \mathbb{N}$.

For $i = 1$, note that by this first property, we get that

$$E_1 = f_\beta^{-1}(E_0) = \bigcup_{s=1}^{\beta} \left[\frac{s-u}{\beta}, \frac{s}{\beta}\right),$$

in accordance with equation (B.1). Now suppose that for $i \in \mathbb{N}$ fixed, equation (B.1) holds. Then note that

$$\begin{aligned}
E_{i+1} = f_\beta^{-1}(E_i) &= \bigcup_{s=1}^{\beta^i} f_\beta^{-1}\left(\left[\frac{s-u}{\beta^i}, \frac{s}{\beta^i}\right)\right) \\
&= \bigcup_{s=1}^{\beta^i} \bigcup_{t=0}^{\beta-1} \left[\frac{t}{\beta} + \frac{s-u}{\beta^{i+1}}, \frac{t}{\beta} + \frac{s}{\beta^{i+1}}\right) \\
&= \bigcup_{t=0}^{\beta-1} \bigcup_{s=1}^{\beta^i} \left[\frac{t\cdot\beta^i + s - u}{\beta^{i+1}}, \frac{t\cdot\beta^i + s}{\beta^{i+1}}\right) \\
&= \bigcup_{t=0}^{\beta-1} \bigcup_{s=t\cdot\beta^i+1}^{(t+1)\cdot\beta^i} \left[\frac{s-u}{\beta^{i+1}}, \frac{s}{\beta^{i+1}}\right) \\
&= \bigcup_{s=1}^{\beta^{i+1}} \left[\frac{s-u}{\beta^{i+1}}, \frac{s}{\beta^{i+1}}\right),
\end{aligned}$$

which is equivalent to equation (B.1) for $i+1$, thus completing the proof by induction. □

B.2. Proof of the First Statement in Lemma 3.1

Recall that in section 3.2, we deduced a pattern for the part of the sequence (B_n), such that $1 \leq n \leq k+1$. This was formulated in the first statement of lemma 3.1, and is given by

$$B_n = (n-1) \cdot \left(1 - \frac{1}{\beta}\right) \cdot u + u \quad \text{for } 1 \leq n \leq k+1. \tag{B.2}$$

This statement can be proven separately, which is done below.

Proof. Let us first recall that the sequence (E_i) can be written as

$$E_i = \bigcup_{s=1}^{\beta^i} \left[\frac{s-u}{\beta^i}, \frac{s}{\beta^i}\right),$$

for all $i \geq 0$. Let us set $E'_0 = E_0$, and define the sequence

$$E'_i = E_i \setminus \bigcup_{l=0}^{i-1} E_l \text{ for } 1 \leq i \leq k,$$

where $u = 2^{-k}$. Then let us first prove the claim that for $0 \leq i \leq k$,

$$E'_i = \bigcup_{\substack{s=1 \\ \beta \nmid s}}^{\beta^i} \left[\frac{s-u}{\beta^i}, \frac{s}{\beta^i}\right), \tag{B.3}$$

where $\beta \nmid s$ denotes that β does not divide s.

Proof of equation (B.3). We are going to prove the claim as in equation (B.3) using strong induction on i for $0 \leq i \leq k$. Note that we will only prove the claim for the case that β is prime. For the base case $i = 0$, note that

$$E'_0 = E_0 = [1-u, 1) = \left[\frac{1-u}{1}, \frac{1}{1}\right),$$

as desired. Now suppose the claim holds for all integers up and until any $0 \leq i \leq k-1$, then first note that

$$\begin{aligned} E'_{i+1} &= E_{i+1} \setminus \bigcup_{l=0}^{i} E_l = E_{i+1} \setminus \bigcup_{l=0}^{i} E'_l \\ &= \left(\bigcup_{s=1}^{\beta^{i+1}} \left[\frac{s-u}{\beta^{i+1}}, \frac{s}{\beta^{i+1}}\right)\right) \setminus \left(\bigcup_{l=0}^{i} \bigcup_{\substack{t=1 \\ \beta \nmid t}}^{\beta^l} \left[\frac{t-u}{\beta^l}, \frac{t}{\beta^l}\right)\right). \end{aligned} \tag{B.4}$$

Let us note that we can rewrite

$$\begin{aligned} \bigcup_{s=1}^{\beta^{i+1}} \left[\frac{s-u}{\beta^{i+1}}, \frac{s}{\beta^{i+1}}\right) &= \left(\bigcup_{\substack{s=1 \\ \beta \nmid s}}^{\beta^{i+1}} \left[\frac{s-u}{\beta^{i+1}}, \frac{s}{\beta^{i+1}}\right)\right) \cup \left(\bigcup_{\substack{s=1 \\ \beta \mid s}}^{\beta^{i+1}} \left[\frac{s-u}{\beta^{i+1}}, \frac{s}{\beta^{i+1}}\right)\right) \\ &= \left(\bigcup_{\substack{s=1 \\ \beta \nmid s}}^{\beta^{i+1}} \left[\frac{s}{\beta^{i+1}} - \frac{u}{\beta^{i+1}}, \frac{s}{\beta^{i+1}}\right)\right) \cup \left(\bigcup_{s=1}^{\beta^i} \left[\frac{s}{\beta^i} - \frac{u}{\beta^{i+1}}, \frac{s}{\beta^i}\right)\right) \end{aligned}$$

$$= \left(\bigcup_{\substack{s=1 \\ \beta \nmid s}}^{\beta^{i+1}} \left[\frac{s}{\beta^{i+1}} - \frac{u}{\beta^{i+1}}, \frac{s}{\beta^{i+1}} \right) \right) \cup \left(\bigcup_{\substack{s=1 \\ \beta \nmid s}}^{\beta^{i}} \left[\frac{s}{\beta^{i}} - \frac{u}{\beta^{i+1}}, \frac{s}{\beta^{i}} \right) \right)$$
$$\cup \left(\bigcup_{s=1}^{\beta^{i-1}} \left[\frac{s}{\beta^{i-1}} - \frac{u}{\beta^{i+1}}, \frac{s}{\beta^{i-1}} \right) \right).$$

Repeating the same argument $i-2$ more times, we obtain

$$\bigcup_{s=1}^{\beta^{i+1}} \left[\frac{s-u}{\beta^{i+1}}, \frac{s}{\beta^{i+1}} \right) = \left(\bigcup_{\substack{s=1 \\ \beta \nmid s}}^{\beta^{i+1}} \left[\frac{s}{\beta^{i+1}} - \frac{u}{\beta^{i+1}}, \frac{s}{\beta^{i+1}} \right) \right) \cup \left(\bigcup_{\substack{s=1 \\ \beta \nmid s}}^{\beta^{i}} \left[\frac{s}{\beta^{i}} - \frac{u}{\beta^{i+1}}, \frac{s}{\beta^{i}} \right) \right)$$
$$\cup \cdots \cup \left(\bigcup_{\substack{s=1 \\ \beta \nmid s}}^{\beta} \left[\frac{s}{\beta} - \frac{u}{\beta^{i+1}}, \frac{s}{\beta} \right) \right) \cup \left(\bigcup_{s=1}^{1} \left[\frac{s}{1} - \frac{u}{\beta^{i+1}}, \frac{s}{1} \right) \right)$$
$$= \bigcup_{j=0}^{i+1} \bigcup_{\substack{s=1 \\ \beta \nmid s}}^{\beta^{j}} \left[\frac{s}{\beta^{j}} - \frac{u}{\beta^{i+1}}, \frac{s}{\beta^{j}} \right).$$

Substituting this result into equation (B.4), we get that

$$E'_{i+1} = \left(\bigcup_{j=0}^{i+1} \bigcup_{\substack{s=1 \\ \beta \nmid s}}^{\beta^{j}} \left[\frac{s}{\beta^{j}} - \frac{u}{\beta^{i+1}}, \frac{s}{\beta^{j}} \right) \right) \setminus \left(\bigcup_{l=0}^{i} \bigcup_{\substack{t=1 \\ \beta \nmid t}}^{\beta^{l}} \left[\frac{t-u}{\beta^{l}}, \frac{t}{\beta^{l}} \right) \right).$$

Note that all the intervals of the excluded set on the right-hand side are disjoint by construction of E'_j for $0 \le j \le i$, and therefore by the previous equation,

$$E'_{i+1} = \bigcup_{j=0}^{i+1} \bigcup_{\substack{s=1 \\ \beta \nmid s}}^{\beta^{j}} \left\{ \left[\frac{s}{\beta^{j}} - \frac{u}{\beta^{i+1}}, \frac{s}{\beta^{j}} \right) \setminus \left(\bigcup_{l=0}^{i} \bigcup_{\substack{t=1 \\ \beta \nmid t}}^{\beta^{l}} \left[\frac{t-u}{\beta^{l}}, \frac{t}{\beta^{l}} \right) \right) \right\}. \tag{B.5}$$

Now, we use the following claim, namely that if we consider the part of the union covered by $j = i+1$, we have that

$$\left[\frac{s}{\beta^{i+1}} - \frac{u}{\beta^{i+1}}, \frac{s}{\beta^{i+1}} \right) \cap \left(\bigcup_{l=0}^{i} \bigcup_{\substack{t=1 \\ \beta \nmid t}}^{\beta^{l}} \left[\frac{t-u}{\beta^{l}}, \frac{t}{\beta^{l}} \right) \right) = \varnothing, \tag{B.6}$$

for any $1 \le s \le \beta^{i+1}$ with $\beta \nmid s$. This claim will be proven below separately, but if it holds, we can combine it with equation (B.5) to get

$$E'_{i+1} = \bigcup_{j=0}^{i} \bigcup_{\substack{s=1 \\ \beta \nmid s}}^{\beta^{j}} \left\{ \left[\frac{s}{\beta^{j}} - \frac{u}{\beta^{i+1}}, \frac{s}{\beta^{j}} \right) \setminus \left(\bigcup_{l=0}^{i} \bigcup_{\substack{t=1 \\ \beta \nmid t}}^{\beta^{l}} \left[\frac{t-u}{\beta^{l}}, \frac{t}{\beta^{l}} \right) \right) \right\} \cup \left(\bigcup_{\substack{s=1 \\ \beta \nmid s}}^{\beta^{i+1}} \left[\frac{s-u}{\beta^{i+1}}, \frac{s}{\beta^{i+1}} \right) \right). \tag{B.7}$$

Then, note that for all $0 \le j \le i$ and all $1 \le s \le \beta^j$ with $\beta \nmid s$, we have that for $l = j, t = s$,

$$\left[\frac{s}{\beta^j} - \frac{u}{\beta^{i+1}}, \frac{s}{\beta^j}\right) \subset \left[\frac{s}{\beta^j} - \frac{u}{\beta^j}, \frac{s}{\beta^j}\right) = \left[\frac{t-u}{\beta^l}, \frac{t}{\beta^l}\right) \subset \bigcup_{l=0}^{i} \bigcup_{\substack{t=1 \\ \beta \nmid t}}^{\beta^l} \left[\frac{t-u}{\beta^l}, \frac{t}{\beta^l}\right).$$

Therefore, it follows that

$$\bigcup_{j=0}^{i} \bigcup_{\substack{s=1 \\ \beta \nmid s}}^{\beta^j} \left\{ \left[\frac{s}{\beta^j} - \frac{u}{\beta^{i+1}}, \frac{s}{\beta^j}\right) \setminus \left(\bigcup_{l=0}^{i} \bigcup_{\substack{t=1 \\ \beta \nmid t}}^{\beta^l} \left[\frac{t-u}{\beta^l}, \frac{t}{\beta^l}\right) \right) \right\} = \varnothing.$$

Substituting this into equation (B.7), we end up with

$$E'_{i+1} = \bigcup_{\substack{s=1 \\ \beta \nmid s}}^{\beta^{i+1}} \left[\frac{s-u}{\beta^{i+1}}, \frac{s}{\beta^{i+1}}\right),$$

which is equivalent to equation (B.3) for $i+1$, thus completing the proof by induction. □

Before cutting to the end result of the proof, let us first prove the claim made in equation (B.6) below.

Proof of equation (B.6). For a contradiction, let us suppose that there exists some $1 \le s \le \beta^{i+1}$ with $\beta \nmid s$, such that

$$\left[\frac{s}{\beta^{i+1}} - \frac{u}{\beta^{i+1}}, \frac{s}{\beta^{i+1}}\right) \cap \left(\bigcup_{l=0}^{i} \bigcup_{\substack{t=1 \\ \beta \nmid t}}^{\beta^l} \left[\frac{t-u}{\beta^l}, \frac{t}{\beta^l}\right) \right) \neq \varnothing.$$

Then there must exist some $0 \le l \le i$, and $1 \le t \le \beta^l$ with $\beta \nmid t$, such that

$$\left[\frac{s}{\beta^{i+1}} - \frac{u}{\beta^{i+1}}, \frac{s}{\beta^{i+1}}\right) \cap \left[\frac{t-u}{\beta^l}, \frac{t}{\beta^l}\right) \neq \varnothing.$$

Since both intervals are connected, this means that either

$$\frac{t-u}{\beta^l} \le \frac{s-u}{\beta^{i+1}} < \frac{t}{\beta^l} \quad \text{or} \quad \frac{s-u}{\beta^{i+1}} \le \frac{t-u}{\beta^l} < \frac{s}{\beta^{i+1}}, \tag{B.8}$$

or both. In the first case, it follows that since $u = \beta^{-k}$,

$$\begin{aligned}
& & \frac{t-u}{\beta^l} \le \frac{s-u}{\beta^{i+1}} &< \frac{t}{\beta^l} \\
&\Rightarrow & (t-u)\cdot\beta^{i+1-l} \le s-u &< t\cdot\beta^{i+1-l} \\
&\Rightarrow & t\cdot\beta^{i+1-l} - \beta^{i+1-k-l} + \beta^{-k} \le s &< t\cdot\beta^{i+1-l} + \beta^{-k}.
\end{aligned}$$

Since $i \le k-1$ and $l \ge 0$, it follows that $\beta^{i+1-k-l} \le \beta^{-l} \le 1$, and thus

$$\begin{aligned}
&\Rightarrow & t\cdot\beta^{i+1-l} - 1 + \beta^{-k} \le s &< t\cdot\beta^{i+1-l} + \beta^{-k} \\
&\Rightarrow & t\cdot\beta^{i+1-l} - 1 < s &< t\cdot\beta^{i+1-l} + \beta^{-k}.
\end{aligned}$$

Since both $t, s \in \mathbb{N}$, it follows that $s = t \cdot \beta^{i+1-l}$, which implies that $\beta \mid s$. This contradicts the assumption that $\beta \nmid s$.
Now suppose that the second case presented in equation (B.8) holds. Then,

$$\begin{aligned}
& & \frac{s-u}{\beta^{i+1}} \leq \frac{t-u}{\beta^l} &< \frac{s}{\beta^{i+1}} \\
&\Rightarrow & (s-u)\cdot \beta^{l-i-1} \leq t-u &< s\cdot \beta^{l-i-1} \\
&\Rightarrow & s\cdot\beta^{l-i-1} - \beta^{l-k-i-1} + \beta^{-k} \leq t &< s\cdot\beta^{l-i-1} + \beta^{-k}.
\end{aligned}$$

Since $l < i+1$, it follows that $\beta^{l-k-i-1} < \beta^{-k}$, and thus

$$\Rightarrow \qquad s\cdot\beta^{l-i-1} < t \qquad < s\cdot\beta^{l-i-1} + \beta^{-k}.$$

Since there are no integers $1 \leq t \leq \beta^l$ and $1 \leq s \leq \beta^{i+1}$ such that this is satisfied, this also leads to a contradiction. Therefore, in any case of equation (B.8), we are led to a contradiction, thus proving the claim. □

Now that the proof of equation (B.3) is closed, let us use it to prove equation (B.2). Namely, note that for $1 \leq n \leq k+1$,

$$B_n = \mu\left(\bigcup_{i=0}^{n-1} E_i\right) = \mu\left(\bigcup_{i=0}^{n-1} E_i'\right) = \sum_{i=0}^{n-1}\mu(E_i'). \tag{B.9}$$

Where, for $1 \leq i \leq n-1$, we have that

$$\mu(E_i') = \sum_{\substack{s=1\\ \beta\nmid s}}^{\beta^i} \mu\left(\left[\frac{s-u}{\beta^i}, \frac{s}{\beta^i}\right)\right) = \sum_{\substack{s=1\\ \beta\nmid s}}^{\beta^i} \frac{u}{\beta^i}. \tag{B.10}$$

Since all of these terms are independent of s, we simply have to count the number of terms in the sum. To this end, note that for any integer p, the number of integers between 1 and p that are coprime to p is given by Euler's totient function, denoted by $\varphi(p)$. There is no direct formula for this function, however in the case that p is prime, it follows by Euler's product formula that

$$\varphi(p^i) = \left(1 - \frac{1}{p}\right)\cdot p^i, \tag{B.11}$$

for any integer i. Applying this to the sum in equation (B.10), we get that for β prime,

$$\mu(E_i') = \varphi(\beta^i)\cdot\frac{u}{\beta^i} = \left(1-\frac{1}{\beta}\right)\cdot u,$$

for any $1 \leq i \leq n-1$. Substituting this into equation (B.9), we get that

$$B_n = \sum_{i=0}^{n-1}\mu(E_i') = \mu(E_0') + \sum_{i=1}^{n-1}\mu(E_i') = (n-1)\cdot\left(1-\frac{1}{\beta}\right)\cdot u + u,$$

which completes the proof. □

B.3. Proof that m_n is $(\beta - 1)$ times the k-bonacci Sequence

The goal of this section will be to show that m_n is indeed $(\beta - 1)$ times the k-bonacci sequence (with starting digits as stated in equation (3.4)) if and only if lemma 3.1 holds. Let us divide this into two lemmas that will be proven below. The first will regard the first statement of lemma 3.1, using the definitions of B_n and m_n as given in section 3.2. The resulting lemma below gives us the relation between that statement and the starting digits of the sequence (m_n).

Lemma B.1. For any $k \in \mathbb{N}$, let $u = \beta^{-k}$, then

$$B_n = (n-1) \cdot \left(1 - \frac{1}{\beta}\right) \cdot u + u \quad \text{for } 1 \leq n \leq k+1 \tag{B.12}$$

if and only if

$$m_n = (\beta - 1) \cdot \beta^{n-2} \quad \text{for } 2 \leq n \leq k+1.$$

Proof. For the 'if' part ($\Leftarrow$), first note that for any $u = \beta^{-k}$ and $k \in \mathbb{N}$,

$$B_1 = \mu(E_0) = u,$$

by definition of B_1. Let us now prove equation (B.12) by induction on $2 \leq n \leq k+1$ for any $k \in \mathbb{N}$. In the case that $n = 2$, we have that

$$\begin{aligned} m_2 &= \frac{B_2 - B_1}{u/\beta} \stackrel{\text{set}}{=} \beta - 1 \\ \Rightarrow \quad B_2 &= B_1 + \frac{\beta - 1}{\beta} \cdot u = \left(1 - \frac{1}{\beta}\right) \cdot u + u. \end{aligned}$$

Now suppose that equation (B.12) holds for some $2 \leq n \leq k$, then

$$\begin{aligned} m_{n+1} &= \frac{B_{n+1} - B_n}{u/\beta^n} \stackrel{\text{set}}{=} (\beta - 1) \cdot \beta^{n-1} \\ \Rightarrow \quad B_{n+1} &= B_n + \frac{\beta^n - \beta^{n-1}}{\beta^n} \cdot u = B_n + \left(1 - \frac{1}{\beta}\right) \cdot u \\ &= n \cdot \left(1 - \frac{1}{\beta}\right) \cdot u + u, \end{aligned}$$

which is equivalent to equation (B.12) for $n+1$, thus completing the proof by induction. For the 'only if' part ($\Rightarrow$), recall that by definition,

$$m_n = \frac{B_n - B_{n-1}}{u/\beta^{n-1}} \quad \text{for } n \geq 2.$$

Then by equation (B.12), it follows that for $2 \leq n \leq k+1$,

$$m_n = \frac{(n - n + 1) \cdot \left(1 - \frac{1}{\beta}\right) \cdot u + u - u}{u/\beta^{n-1}} = \frac{1 - \frac{1}{\beta}}{1/\beta^{n-1}} = (\beta - 1) \cdot \beta^{n-2},$$

which completes the proof. □

The second lemma that will be proven shows that the second statement of lemma 3.1 holds if and only if the sequence m_n is $(\beta - 1)$ times the k-bonacci sequence.

Lemma B.2. For any $k \in \mathbb{N}$, let $u = \beta^{-k}$, then

$$B_{n+1} = B_n + \left(1 - \frac{1}{\beta}\right) \cdot u \cdot (1 - B_{n-k}) \quad \text{for } n \geq k+1 \tag{B.13}$$

if and only if

$$m_{n+1} = \sum_{i=0}^{k-1} (\beta - 1) \cdot m_{n-i} \quad \text{for } n \geq k+1.$$

Proof. For the 'if' part ($\Leftarrow$), let us prove the identity for B_n using strong induction on $n \geq k+1$. In the case that $n = k+1$, note that by definition of m_n,

$$m_{k+2} = \frac{B_{k+2} - B_{k+1}}{u/\beta^{k+1}}.$$

On the other hand, m_n is assumed to be k-bonacci, and thus

$$\begin{aligned} m_{k+2} &= \sum_{i=0}^{k-1} (\beta - 1) \cdot m_{k+1-i} = \sum_{i=0}^{k-1} (\beta - 1)^2 \cdot \beta^{k-1-i} \\ &= (\beta - 1)^2 \cdot \sum_{i=0}^{k-1} \beta^{k-1-i} = (\beta - 1)^2 \cdot \frac{\beta^k - 1}{\beta - 1} \\ &= (\beta - 1) \cdot (\beta^k - 1), \end{aligned}$$

where the first steps follow from lemma $B.1$. Combining these two equations gives

$$\begin{aligned} & \frac{B_{k+2} - B_{k+1}}{u/\beta^{k+1}} = (\beta - 1) \cdot (\beta^k - 1) \\ \Rightarrow \quad & B_{k+2} - B_{k+1} = \frac{u}{\beta^{k+1}} \cdot (\beta - 1) \cdot (\beta^k - 1) = \left(1 - \frac{1}{\beta}\right) \cdot u \cdot \frac{\beta^k - 1}{\beta^k} \\ & \qquad = \left(1 - \frac{1}{\beta}\right) \cdot u \cdot (1 - u) \\ \Rightarrow \quad & B_{k+2} = B_{k+1} + \left(1 - \frac{1}{\beta}\right) \cdot u \cdot (1 - B_1), \end{aligned}$$

recalling that for any $u = \beta^{-k}$, $B_1 = u$. Having proven the base step, let us now assume that equation (B.13) holds up until any $n \geq k+1$. Again, by definition it holds that

$$m_{n+2} = \frac{B_{n+2} - B_{n+1}}{u/\beta^{n+1}}.$$

Meanwhile, it is assumed that m_n is $(\beta - 1)$ times the k-bonacci sequence, so that

$$\begin{aligned} m_{n+2} &= \sum_{i=0}^{k-1} (\beta - 1) \cdot m_{n+1-i} = (\beta - 1) \cdot m_{n+1} + \sum_{i=0}^{k-2} (\beta - 1) \cdot m_{n-i} \\ &= (\beta - 1) \cdot m_{n+1} + \sum_{i=0}^{k-1} (\beta - 1) \cdot m_{n-i} - (\beta - 1) \cdot m_{n-k+1} \\ &= \beta \cdot m_{n+1} - (\beta - 1) \cdot m_{n-k+1}. \end{aligned}$$

Combining these two equations, we get that

$$\begin{aligned}
\frac{B_{n+2}-B_{n+1}}{u/\beta^{n+1}} &= \beta \cdot m_{n+1} - (\beta-1)\cdot m_{n-k+1} \\
\Rightarrow \quad B_{n+2}-B_{n+1} &= \frac{u}{\beta^n}\cdot m_{n+1} - \left(1-\frac{1}{\beta}\right)\cdot \frac{u}{\beta^n}\cdot m_{n-k+1} \\
&= \frac{u}{\beta^n}\cdot \frac{B_{n+1}-B_n}{u/\beta^n} - \left(1-\frac{1}{\beta}\right)\cdot \frac{u}{\beta^n}\cdot \frac{B_{n-k+1}-B_{n-k}}{u/\beta^{n-k}} \\
&= B_{n+1}-B_n - \left(1-\frac{1}{\beta}\right)\cdot \frac{1}{\beta^k}\cdot (B_{n-k+1}-B_{n-k}) \\
&= \left(1-\frac{1}{\beta}\right)\cdot u\cdot (1-B_{n-k}) - \left(1-\frac{1}{\beta}\right)^2\cdot \frac{u}{\beta^k}\cdot (1-B_{n-2k}) \\
&= \left(1-\frac{1}{\beta}\right)\cdot u - \left(1-\frac{1}{\beta}\right)\cdot u\cdot \left(B_{n-k} + \left(1-\frac{1}{\beta}\right)\cdot u\cdot (1-B_{n-2k})\right) \\
&= \left(1-\frac{1}{\beta}\right)\cdot u\cdot (1-B_{n-k+1}),
\end{aligned}$$

where the last line follows from the assumption that $B_{n-k+1} = B_{n-k} + \left(1-\frac{1}{\beta}\right)\cdot u\cdot (1-B_{n-2k})$. Rewriting the final equation yields

$$B_{n+2} = B_{n+1} + \left(1-\frac{1}{\beta}\right)\cdot u\cdot (1-B_{n-k+1}),$$

which proves equation (B.13) for $n+1$, thus completing the proof by induction.
Conversely, for the 'only if' part ($\Rightarrow$), the previous proof by induction is basically reversed. First note that by definition,

$$\begin{aligned}
m_{n+1} = \frac{B_{n+1}-B_n}{u/\beta^n} &= \frac{\beta^n}{u}\cdot \left(1-\frac{1}{\beta}\right)\cdot u\cdot (1-B_{n-k}) \\
&= \left(1-\frac{1}{\beta}\right)\cdot \beta^n\cdot (1-B_{n-k})
\end{aligned} \tag{B.14}$$

for any $n \geq k+1$. Thus for the base case $n = k+1$,

$$m_{k+2} = \left(1-\frac{1}{\beta}\right)\cdot \beta^{k+1}\cdot (1-u) = (\beta-1)\cdot(\beta^k-1).$$

On the other hand, note that by lemma B.1,

$$\begin{aligned}
\sum_{i=0}^{k-1}(\beta-1)\cdot m_{k+1-i} = \sum_{i=0}^{k-1}(\beta-1)^2\cdot \beta^{k-1-i} &= (\beta-1)^2\cdot \frac{\beta^k-1}{\beta-1} \\
&= (\beta-1)\cdot(\beta^k-1).
\end{aligned}$$

Combining this gives that

$$m_{k+2} = \sum_{i=0}^{k-1}(\beta-1)\cdot m_{k+1-i}.$$

For the inductive step, suppose the equation holds for all integers up and until any $n \geq k+1$, i.e.

$$m_{n+1} = \sum_{i=0}^{k-1}(\beta-1)\cdot m_{n-i} \quad \text{for } n \geq k+1.$$

Then it follows by using equation (B.14), that

$$\begin{aligned}
\sum_{i=0}^{k-1}(\beta-1)\cdot m_{n+1-i} &= (\beta-1)\cdot m_{n+1}+\sum_{i=0}^{k-2}(\beta-1)\cdot m_{n-i}\\
&= \beta\cdot m_{n+1}-(\beta-1)\cdot m_{n-k+1}\\
&= \left(1-\frac{1}{\beta}\right)\cdot\beta^{n+1}\cdot(1-B_{n-k})-(\beta-1)\cdot\left(1-\frac{1}{\beta}\right)\cdot\beta^{n-k}\cdot(1-B_{n-2k})\\
&= \left(1-\frac{1}{\beta}\right)\cdot\beta^{n+1}\cdot\left(1-B_{n-k}-\left(1-\frac{1}{\beta}\right)\cdot u\cdot(1-B_{n-2k})\right)\\
&= \left(1-\frac{1}{\beta}\right)\cdot\beta^{n+1}\cdot(1-B_{n-k+1})=m_{n+2},
\end{aligned}$$

thus completing the proof by induction. □

B.4. Proof of Equation (3.5) (Proof of Theorem 3.2 part 1)

In this section, we are going to use lemma 3.1 in order to prove equation (3.5). To this end, let us note that by the first part of the lemma, $B_{k+1}=k\cdot(1-\frac{1}{\beta})\cdot u+u$. We can now use the second part of the lemma to see that

$$B_{k+2}=B_{k+1}+\left(1-\frac{1}{\beta}\right)\cdot u\cdot(1-B_1)=(k+1)\cdot\left(1-\frac{1}{\beta}\right)\cdot u+u-\left(1-\frac{1}{\beta}\right)\cdot u^2. \quad \text{(B.15)}$$

This can be repeated inductively, using the second part of the lemma in order to obtain $B_{k+1+l+1}$ explicitly, given B_{k+1+l} for $1\le l\le k+1$. Doing this will yield the following result, which is labelled equation (3.5) in section 3.3.

$$B_{k+1+l}=\binom{k+l}{1}\cdot\left(1-\frac{1}{\beta}\right)\cdot u+u-\binom{l}{2}\cdot\left(1-\frac{1}{\beta}\right)^2\cdot u^2-\binom{l}{1}\cdot\left(1-\frac{1}{\beta}\right)\cdot u^2, \quad \text{(B.16)}$$

for $1\le l\le k+1$. This is the result that we will now be proving inductively, primarily using lemma 3.1. This is the first of two proofs using induction that together prove theorem 3.2, the other one being shown in appendix B.5.

Proof. First note that in the case that $l=1$, we have shown in equation (B.15), that the expression in equation (B.16) holds, using that $\binom{1}{2}=0$.
Now suppose that equation (B.16) holds for any $1\le l\le k$ with $k\in\mathbb{N}$ fixed. Then note that by lemma 3.1,

$$\begin{aligned}
B_{k+1+l+1} &= B_{k+1+l}+\left(1-\frac{1}{\beta}\right)\cdot u\cdot(1-B_{l+1})\\
&= \binom{k+l}{1}\cdot\left(1-\frac{1}{\beta}\right)\cdot u+u-\binom{l}{2}\cdot\left(1-\frac{1}{\beta}\right)^2\cdot u^2-\binom{l}{1}\cdot\left(1-\frac{1}{\beta}\right)\cdot u^2\\
&\quad+\left(1-\frac{1}{\beta}\right)\cdot u\cdot\left(1-l\cdot\left(1-\frac{1}{\beta}\right)\cdot u-u\right)\\
&= (k+l)\cdot\left(1-\frac{1}{\beta}\right)\cdot u+u-\frac{1}{2}\cdot l\cdot(l-1)\cdot\left(1-\frac{1}{\beta}\right)^2\cdot u^2-l\cdot\left(1-\frac{1}{\beta}\right)\cdot u^2\\
&\quad+\left(1-\frac{1}{\beta}\right)\cdot u-l\cdot\left(1-\frac{1}{\beta}\right)^2\cdot u^2-\left(1-\frac{1}{\beta}\right)\cdot u^2
\end{aligned}$$

$$= (k+l+1)\cdot\left(1-\frac{1}{\beta}\right)\cdot u + u - \frac{1}{2}\cdot(l+1)\cdot l\cdot\left(1-\frac{1}{\beta}\right)^2\cdot u^2 - (l+1)\cdot\left(1-\frac{1}{\beta}\right)\cdot u^2$$
$$= \binom{k+l+1}{1}\cdot\left(1-\frac{1}{\beta}\right)\cdot u + u - \binom{l+1}{2}\cdot\left(1-\frac{1}{\beta}\right)^2\cdot u^2 - \binom{l+1}{1}\cdot\left(1-\frac{1}{\beta}\right)\cdot u^2,$$

in accordance with equation (B.16) for $l+1$, thus completing the proof by induction. □

B.5. Proof of Equation (3.6) (Proof of Theorem 3.2 part 2)

Let us now prove the general form for B_n, which will form the final step towards proving the expression as stated in theorem 3.2. Recall that in appendix B.4 we derived an expression for B_{k+1+l} for $1 \leq l \leq k+1$, as stated in equation (B.16). Now let us substitute $l = k+1$ in that equation, which yields

$$B_{2\cdot(k+1)} = \binom{2k+1}{1}\cdot\left(1-\frac{1}{\beta}\right)\cdot u + u - \binom{k+1}{2}\cdot\left(1-\frac{1}{\beta}\right)^2\cdot u^2 - \binom{k+1}{1}\cdot\left(1-\frac{1}{\beta}\right)\cdot u^2. \quad \text{(B.17)}$$

We can now keep adding $k+1$ to the index inductively in order to obtain B_m for any $m \in \mathbb{N}$. The result of this derivation is that for $m \geq 2$,

$$\begin{aligned} B_{m\cdot(k+1)} = &\sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i-1}{i+1}\cdot\left(1-\frac{1}{\beta}\right)^{i+1}\cdot u^{i+1} \\ &+\sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i-1}{i}\cdot\left(1-\frac{1}{\beta}\right)^{i}\cdot u^{i+1}, \end{aligned} \quad \text{(B.18)}$$

which is labelled equation (3.6) in section 3.3, and will be proven below using strong induction on m.

Proof. First note that for $m = 2$, equation (B.18) is satisfied by equation (B.17). Now suppose that equation (B.18) is satisfied for some $m \geq 2$. Then in order to show that it holds for $m+1$, it is necessary to prove by induction on l that

$$\begin{aligned} B_{m\cdot(k+1)+l} = &\sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i-1+l}{i+1}\cdot\left(1-\frac{1}{\beta}\right)^{i+1}\cdot u^{i+1} \\ &+\sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i-1+l}{i}\cdot\left(1-\frac{1}{\beta}\right)^{i}\cdot u^{i+1} \\ &+(-1)^m\cdot\binom{m-1+l}{m+1}\cdot\left(1-\frac{1}{\beta}\right)^{m+1}\cdot u^{m+1} \\ &+(-1)^m\cdot\binom{m-1+l}{m}\cdot\left(1-\frac{1}{\beta}\right)^{m}\cdot u^{m+1}. \end{aligned} \quad \text{(B.19)}$$

holds, for all $1 \leq l \leq k+1$ and for $m \geq 2$ fixed in the assumption.

Proof of equation (B.19). As for the base case, we can simply take $l = 1$ and $m \geq 2$ as fixed in the previous assumption.

Thus, we have that by the second part of lemma 3.1,

$$
\begin{aligned}
&B_{m\cdot(k+1)+1} = B_{m\cdot(k+1)} + \left(1-\frac{1}{\beta}\right)\cdot u\cdot(1-B_{(m-1)\cdot(k+1)+1})\\
&= \sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i-1}{i+1}\cdot\left(1-\frac{1}{\beta}\right)^{i+1}\cdot u^{i+1}\\
&\quad+\sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i-1}{i}\cdot\left(1-\frac{1}{\beta}\right)^{i}\cdot u^{i+1}\\
&\quad+\left(1-\frac{1}{\beta}\right)\cdot u-\left(1-\frac{1}{\beta}\right)\cdot u\cdot\left(\sum_{i=0}^{m-2}(-1)^i\cdot\binom{(m-1-i)\cdot(k+1)+i}{i+1}\cdot\left(1-\frac{1}{\beta}\right)^{i+1}\cdot u^{i+1}\right.\\
&\quad+\sum_{i=0}^{m-2}(-1)^i\cdot\binom{(m-1-i)\cdot(k+1)+i}{i}\cdot\left(1-\frac{1}{\beta}\right)^{i}\cdot u^{i+1}\\
&\quad\left.+(-1)^{m-1}\cdot\binom{m-1}{m}\cdot\left(1-\frac{1}{\beta}\right)^{m}\cdot u^{m}+(-1)^{m-1}\cdot\binom{m-1}{m-1}\cdot\left(1-\frac{1}{\beta}\right)^{m-1}\cdot u^{m}\right).
\end{aligned}
$$

First, let us factor out the brackets;

$$
\begin{aligned}
&= \sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i-1}{i+1}\cdot\left(1-\frac{1}{\beta}\right)^{i+1}\cdot u^{i+1}\\
&\quad+\sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i-1}{i}\cdot\left(1-\frac{1}{\beta}\right)^{i}\cdot u^{i+1}\\
&\quad+\left(1-\frac{1}{\beta}\right)\cdot u+\sum_{i=0}^{m-2}(-1)^{i+1}\cdot\binom{(m-1-i)\cdot(k+1)+i}{i+1}\cdot\left(1-\frac{1}{\beta}\right)^{i+2}\cdot u^{i+2}\\
&\quad+\sum_{i=0}^{m-2}(-1)^{i+1}\cdot\binom{(m-1-i)\cdot(k+1)+i}{i}\cdot\left(1-\frac{1}{\beta}\right)^{i+1}\cdot u^{i+2}\\
&\quad+(-1)^{m}\cdot\binom{m-1}{m}\cdot\left(1-\frac{1}{\beta}\right)^{m+1}\cdot u^{m+1}+(-1)^{m}\cdot\binom{m-1}{m-1}\cdot\left(1-\frac{1}{\beta}\right)^{m}\cdot u^{m+1}.
\end{aligned}
$$

Now, we shift the index of the third and fourth sums, to obtain

$$
\begin{aligned}
&= \sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i-1}{i+1}\cdot\left(1-\frac{1}{\beta}\right)^{i+1}\cdot u^{i+1}\\
&\quad+\sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i-1}{i}\cdot\left(1-\frac{1}{\beta}\right)^{i}\cdot u^{i+1}\\
&\quad+\left(1-\frac{1}{\beta}\right)\cdot u+\sum_{i=1}^{m-1}(-1)^{i}\cdot\binom{(m-i)\cdot(k+1)+i-1}{i}\cdot\left(1-\frac{1}{\beta}\right)^{i+1}\cdot u^{i+1}\\
&\quad+\sum_{i=1}^{m-1}(-1)^{i}\cdot\binom{(m-i)\cdot(k+1)+i-1}{i-1}\cdot\left(1-\frac{1}{\beta}\right)^{i}\cdot u^{i+1}\\
&\quad+(-1)^{m}\cdot\binom{m-1}{m}\cdot\left(1-\frac{1}{\beta}\right)^{m+1}\cdot u^{m+1}+(-1)^{m}\cdot\binom{m-1}{m-1}\cdot\left(1-\frac{1}{\beta}\right)^{m}\cdot u^{m+1}.
\end{aligned}
$$

Combining the first sum with the third, and the second sum with the fourth, gives

$$
= \sum_{i=1}^{m-1}(-1)^i\cdot\left[\binom{(m-i)\cdot(k+1)+i-1}{i+1}+\binom{(m-i)\cdot(k+1)+i-1}{i}\right]\cdot\left(1-\frac{1}{\beta}\right)^{i+1}\cdot u^{i+1}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{m-1}(-1)^i\cdot\left[\binom{(m-i)\cdot(k+1)+i-1}{i}+\binom{(m-i)\cdot(k+1)+i-1}{i-1}\right]\cdot\left(1-\frac{1}{\beta}\right)^i\cdot u^{i+1}\\
&+\left(1-\frac{1}{\beta}\right)\cdot u+\binom{m\cdot(k+1)-1}{1}\cdot\left(1-\frac{1}{\beta}\right)\cdot u+\binom{m\cdot(k+1)-1}{0}\cdot u\\
&+(-1)^m\cdot\binom{m-1}{m}\cdot\left(1-\frac{1}{\beta}\right)^{m+1}\cdot u^{m+1}+(-1)^m\cdot\binom{m-1}{m-1}\cdot\left(1-\frac{1}{\beta}\right)^{m}\cdot u^{m+1}.
\end{aligned}
$$

Using the property as in equation (A.18), we obtain

$$
\begin{aligned}
&=\sum_{i=1}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i}{i+1}\cdot\left(1-\frac{1}{\beta}\right)^{i+1}\cdot u^{i+1}+\binom{m\cdot(k+1)}{1}\cdot\left(1-\frac{1}{\beta}\right)\cdot u\\
&+\sum_{i=1}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i}{i}\cdot\left(1-\frac{1}{\beta}\right)^{i}\cdot u^{i+1}+\binom{m\cdot(k+1)}{0}\cdot u\\
&+(-1)^m\cdot\binom{m}{m+1}\cdot\left(1-\frac{1}{\beta}\right)^{m+1}\cdot u^{m+1}+(-1)^m\cdot\binom{m}{m}\cdot\left(1-\frac{1}{\beta}\right)^{m}\cdot u^{m+1}.
\end{aligned}
$$

Inserting the leftover terms into the sums, we get that

$$
\begin{aligned}
&=\sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i}{i+1}\cdot\left(1-\frac{1}{\beta}\right)^{i+1}\cdot u^{i+1}\\
&+\sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i}{i}\cdot\left(1-\frac{1}{\beta}\right)^{i}\cdot u^{i+1}\\
&+(-1)^m\cdot\binom{m}{m+1}\cdot\left(1-\frac{1}{\beta}\right)^{m+1}\cdot u^{m+1}+(-1)^m\cdot\binom{m}{m}\cdot\left(1-\frac{1}{\beta}\right)^{m}\cdot u^{m+1},
\end{aligned}
$$

in accordance with equation (B.19) for $l=1$, thus completing the proof for the base case. Now for the inductive part, suppose that equation (B.19) holds for any $1\le l\le k$, then using the second part of lemma 3.1, we derive that

$$
\begin{aligned}
&B_{m\cdot(k+1)+l+1}=B_{m\cdot(k+1)+l}+\frac{u}{2}\cdot(1-B_{(m-1)\cdot(k+1)+l+1})\\
&=\sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i-1+l}{i+1}\cdot\left(1-\frac{1}{\beta}\right)^{i+1}\cdot u^{i+1}\\
&+\sum_{i=0}^{m-1}(-1)^i\cdot\binom{(m-i)\cdot(k+1)+i-1+l}{i}\cdot\left(1-\frac{1}{\beta}\right)^{i}\cdot u^{i+1}\\
&+(-1)^m\cdot\binom{m-1+l}{m+1}\cdot\left(1-\frac{1}{\beta}\right)^{m+1}\cdot u^{m+1}+(-1)^m\cdot\binom{m-1+l}{m}\cdot\left(1-\frac{1}{\beta}\right)^{m}\cdot u^{m+1}\\
&+\left(1-\frac{1}{\beta}\right)\cdot u-\left(1-\frac{1}{\beta}\right)\cdot u\cdot\left(\sum_{i=0}^{m-2}(-1)^i\cdot\binom{(m-1-i)\cdot(k+1)+i+l}{i+1}\cdot\left(1-\frac{1}{\beta}\right)^{i+1}\cdot u^{i+1}\right.\\
&+\sum_{i=0}^{m-2}(-1)^i\cdot\binom{(m-1-i)\cdot(k+1)+i+l}{i}\cdot\left(1-\frac{1}{\beta}\right)^{i}\cdot u^{i+1}\\
&\left.+(-1)^{m-1}\cdot\binom{m-1+l}{m}\cdot\left(1-\frac{1}{\beta}\right)^{m}\cdot u^{m}+(-1)^{m-1}\cdot\binom{m-1+l}{m-1}\cdot\left(1-\frac{1}{\beta}\right)^{m-1}\cdot u^{m}\right).
\end{aligned}
$$

First, let us factor out the brackets;

$$
\begin{aligned}
&= \sum_{i=0}^{m-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i-1+l}{i+1} \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot u^{i+1} \\
&\quad + \sum_{i=0}^{m-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i-1+l}{i} \cdot \left(1-\frac{1}{\beta}\right)^{i} \cdot u^{i+1} \\
&\quad + (-1)^m \cdot \binom{m-1+l}{m+1} \cdot \left(1-\frac{1}{\beta}\right)^{m+1} \cdot u^{m+1} + (-1)^m \cdot \binom{m-1+l}{m} \cdot \left(1-\frac{1}{\beta}\right)^{m} \cdot u^{m+1} \\
&\quad + \left(1-\frac{1}{\beta}\right) \cdot u + \sum_{i=0}^{m-2} (-1)^{i+1} \cdot \binom{(m-1-i)\cdot(k+1)+i+l}{i+1} \cdot \left(1-\frac{1}{\beta}\right)^{i+2} \cdot u^{i+2} \\
&\quad + \sum_{i=0}^{m-2} (-1)^{i+1} \cdot \binom{(m-1-i)\cdot(k+1)+i+l}{i} \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot u^{i+2} \\
&\quad + (-1)^m \cdot \binom{m-1+l}{m} \cdot \left(1-\frac{1}{\beta}\right)^{m+1} \cdot u^{m+1} + (-1)^m \cdot \binom{m-1+l}{m-1} \cdot \left(1-\frac{1}{\beta}\right)^{m} \cdot u^{m+1}.
\end{aligned}
$$

Now, we shift the index of the third and fourth sums, to obtain

$$
\begin{aligned}
&= \sum_{i=0}^{m-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i-1+l}{i+1} \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot u^{i+1} \\
&\quad + \sum_{i=0}^{m-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i-1+l}{i} \cdot \left(1-\frac{1}{\beta}\right)^{i} \cdot u^{i+1} \\
&\quad + (-1)^m \cdot \binom{m-1+l}{m+1} \cdot \left(1-\frac{1}{\beta}\right)^{m+1} \cdot u^{m+1} + (-1)^m \cdot \binom{m-1+l}{m} \cdot \left(1-\frac{1}{\beta}\right)^{m} \cdot u^{m+1} \\
&\quad + \left(1-\frac{1}{\beta}\right) \cdot u + \sum_{i=1}^{m-1} (-1)^{i} \cdot \binom{(m-i)\cdot(k+1)+i-1+l}{i} \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot u^{i+1} \\
&\quad + \sum_{i=1}^{m-1} (-1)^{i} \cdot \binom{(m-i)\cdot(k+1)+i-1+l}{i-1} \cdot \left(1-\frac{1}{\beta}\right)^{i} \cdot u^{i+1} \\
&\quad + (-1)^m \cdot \binom{m-1+l}{m} \cdot \left(1-\frac{1}{\beta}\right)^{m+1} \cdot u^{m+1} + (-1)^m \cdot \binom{m-1+l}{m-1} \cdot \left(1-\frac{1}{\beta}\right)^{m} \cdot u^{m+1}.
\end{aligned}
$$

Combining the first sum with the third, and the second sum with the fourth, gives

$$
\begin{aligned}
&= \sum_{i=1}^{m-1} (-1)^i \cdot \left[\binom{(m-i)\cdot(k+1)+i-1+l}{i+1} + \binom{(m-i)\cdot(k+1)+i-1+l}{i}\right] \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot u^{i+1} \\
&\quad + \sum_{i=1}^{m-1} (-1)^i \cdot \left[\binom{(m-i)\cdot(k+1)+i-1+l}{i} + \binom{(m-i)\cdot(k+1)+i-1+l}{i-1}\right] \cdot \left(1-\frac{1}{\beta}\right)^{i} \cdot u^{i+1} \\
&\quad + \left(1-\frac{1}{\beta}\right) \cdot u + \binom{m\cdot(k+1)-1+l}{1} \cdot \left(1-\frac{1}{\beta}\right) \cdot u + \binom{m\cdot(k+1)+-1+l}{0} \cdot u \\
&\quad + (-1)^m \cdot \left[\binom{m-1+l}{m+1} + \binom{m-1+l}{m}\right] \cdot \left(1-\frac{1}{\beta}\right)^{m+1} \cdot u^{m+1} \\
&\quad + (-1)^m \cdot \left[\binom{m-1+l}{m} + \binom{m-1+l}{m-1}\right] \cdot \left(1-\frac{1}{\beta}\right)^{m} \cdot u^{m+1}.
\end{aligned}
$$

Using the property as in equation (A.18), we obtain

$$
\begin{aligned}
&= \sum_{i=1}^{m-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i+l}{i+1} \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot u^{i+1} \\
&\quad + \sum_{i=1}^{m-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i+l}{i} \cdot \left(1-\frac{1}{\beta}\right)^{i} \cdot u^{i+1} \\
&\quad + \binom{m\cdot(k+1)+l}{1} \cdot \left(1-\frac{1}{\beta}\right) \cdot u + \binom{m\cdot(k+1)+l}{0} \cdot u \\
&\quad + (-1)^m \cdot \binom{m+l}{m+1} \cdot \left(1-\frac{1}{\beta}\right)^{m+1} \cdot u^{m+1} + (-1)^m \cdot \binom{m+l}{m} \cdot \left(1-\frac{1}{\beta}\right)^{m} \cdot u^{m+1}.
\end{aligned}
$$

Inserting the leftover terms into the sums, we get that

$$
\begin{aligned}
&= \sum_{i=0}^{m-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i+l}{i+1} \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot u^{i+1} \\
&\quad + \sum_{i=0}^{m-1} (-1)^i \cdot \binom{(m-i)\cdot(k+1)+i+l}{i} \cdot \left(1-\frac{1}{\beta}\right)^{i} \cdot u^{i+1} \\
&\quad + (-1)^m \cdot \binom{m+l}{m+1} \cdot \left(1-\frac{1}{\beta}\right)^{m+1} \cdot u^{m+1} + (-1)^m \cdot \binom{m+l}{m} \cdot \left(1-\frac{1}{\beta}\right)^{m} \cdot u^{m+1},
\end{aligned}
$$

Note that we have now obtained the same expression as in equation (B.19), but for $l+1$, thus completing the induction on l. $\square$

Having proven (B.19), it now follows that for $l = k+1$,

$$
\begin{aligned}
B_{(m+1)\cdot(k+1)} &= \sum_{i=0}^{m-1} (-1)^i \cdot \binom{(m+1-i)\cdot(k+1)+i-1}{i+1} \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot u^{i+1} \\
&\quad + \sum_{i=0}^{m-1} (-1)^i \cdot \binom{(m+1-i)\cdot(k+1)+i-1}{i} \cdot \left(1-\frac{1}{\beta}\right)^{i} \cdot u^{i+1} \\
&\quad + (-1)^m \cdot \binom{m+k}{m+1} \cdot \left(1-\frac{1}{\beta}\right)^{m+1} \cdot u^{m+1} + (-1)^m \cdot \binom{m+k}{m} \cdot \left(1-\frac{1}{\beta}\right)^{m} \cdot u^{m+1} \\
&= \sum_{i=0}^{m} (-1)^i \cdot \binom{(m+1-i)\cdot(k+1)+i-1}{i+1} \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot u^{i+1} \\
&\quad + \sum_{i=0}^{m} (-1)^i \cdot \binom{(m+1-i)\cdot(k+1)+i-1}{i} \cdot \left(1-\frac{1}{\beta}\right)^{i} \cdot u^{i+1},
\end{aligned}
$$

which is equivalent to (B.18) for $m+1$, thus completing the entire proof by induction on m. $\square$

B.6. Proof of Theorem 3.3

We can use the result of theorem 3.2 in order to prove theorem 3.3, which will be done below.

Proof. First, let us denote $m_k = \frac{n_k}{k+1} - 1$. Then note that the result of theorem 3.3 can be rewritten using theorem 3.2.

Namely,

$$\lim_{k\to\infty} \mathbb{P}\left(M_{n_k} \leq 1-\beta^{-k}\right) = \lim_{k\to\infty} 1 - \sum_{i=0}^{m_k} (-1)^i \cdot \binom{n_k - ik - 1}{i+1} \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot \beta^{-k\cdot(i+1)}$$
$$- \sum_{i=0}^{m_k} (-1)^i \cdot \binom{n_k - ik - 1}{i} \cdot \left(1-\frac{1}{\beta}\right)^{i} \cdot \beta^{-k\cdot(i+1)}. \tag{B.20}$$

This will be proven below, given the assumptions as in theorem 3.3. To do this, let us recall Tannery's theorem, which had already been treated in appendix A.7. In a nutshell, under the right conditions, this theorem will allow us to insert the limit into the sums given in equation (B.20). To this end, let us define

$$a_i(k) = (-1)^i \cdot \binom{n_k - ik - 1}{i+1} \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot \beta^{-k\cdot(i+1)}, \text{ and}$$
$$b_i(k) = (-1)^i \cdot \binom{n_k - ik - 1}{i} \cdot \left(1-\frac{1}{\beta}\right)^{i} \cdot \beta^{-k\cdot(i+1)}. \tag{B.21}$$

Let us now apply the theorem for the first sum in equation (B.20), using the first sequence defined above. For Tannery's theorem to hold in this case, we must thus find a sequence ($C_i \geq 0$) such that $\sum_{i=1}^{\infty} C_i < \infty$ and $|a_i(k)| \leq C_i$ for all $i, k \in \mathbb{N}$. To this end, let us fix any $i \in \mathbb{N}$, then note that

$$\begin{aligned} |a_i(k)| &= \binom{n_k - ik - 1}{i+1} \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot \beta^{-k\cdot(i+1)} \\ &\leq \frac{1}{(i+1)!} \cdot \left(\frac{n_k}{\beta^k}\right)^{i+1} \\ &\leq \frac{1}{(i+1)!} \cdot \lambda^{i+1} \overset{\text{set}}{=} C_i. \end{aligned}$$

Also, note that

$$\sum_{i=0}^{\infty} C_i = \sum_{i=0}^{\infty} \frac{1}{(i+1)!} \cdot \lambda^{i+1} = \sum_{i=1}^{\infty} \frac{1}{i!}\lambda^i = e^{\lambda} - 1 < \infty.$$

Now, we only need to check that $\lim_{k\to\infty} a_i(k)$ exists for all $i \in \mathbb{N}$. Therefore, let us rewrite

$$\lim_{k\to\infty} a_i(k) = \lim_{k\to\infty} (-1)^i \cdot \binom{n_k - ik - 1}{i+1} \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot \beta^{-k\cdot(i+1)}. \tag{B.22}$$

Then we use the identity that for any $p, q \in \mathbb{N}$,

$$\binom{p}{q} \cdot \frac{1}{p^q} = \frac{1}{q!} \cdot \left(1-\frac{1}{p}\right) \cdot \left(1-\frac{2}{p}\right) \cdot \ldots \cdot \left(1-\frac{q-1}{p}\right).$$

Applying this to equation (B.22), we get that

$$\begin{aligned} \lim_{k\to\infty} a_i(k) &= \lim_{k\to\infty} (-1)^i \cdot \frac{1}{(i+1)!} \cdot \left(1-\frac{1}{n_k - ik - 1}\right) \cdot \left(1-\frac{2}{n_k - ik - 1}\right) \cdot \ldots \cdot \left(1-\frac{i}{n_k - ik - 1}\right) \\ &\quad \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot \left(\frac{n_k - ik - 1}{\beta^k}\right)^{i+1} \\ &= \lim_{k\to\infty} (-1)^i \cdot \frac{1}{(i+1)!} \cdot \left(1-\frac{1}{\beta}\right)^{i+1} \cdot \left(\frac{n_k - ik - 1}{\beta^k}\right)^{i+1}. \end{aligned} \tag{B.23}$$

Note that

$$\frac{n_k - ik - 1}{\beta^k} = \frac{\left\lfloor \frac{\lambda \cdot \beta^k}{k+1} \right\rfloor \cdot (k+1) - (ik+1)}{\beta^k} = \lambda - \frac{ik+1}{\beta^k} + \mathcal{O}\left(\frac{k+1}{\beta^k}\right).$$

To substitute this into equation (B.23), let us first note that

$$\lim_{k\to\infty} \mathcal{O}\left(\frac{k+1}{\beta^k}\right) = 0,$$

and we thus get that

$$\lim_{k\to\infty} a_i(k) = \lim_{k\to\infty} (-1)^i \cdot \frac{1}{(i+1)!} \cdot \left(1 - \frac{1}{\beta}\right)^{i+1} \cdot \left(\lambda - \frac{ik+1}{\beta^k}\right)^{i+1}. \tag{B.24}$$

Now, observe that

$$\left(\lambda - \frac{ik+1}{\beta^k}\right)^{i+1} = \lambda^{i+1} + \mathcal{O}\left(\frac{\lambda \cdot (i+1) \cdot (ik+1)}{\beta^k}\right).$$

Since for all $i \in \mathbb{N}$

$$\lim_{k\to\infty} \mathcal{O}\left(\frac{\lambda \cdot (i+1) \cdot (ik+1)}{\beta^k}\right) = 0,$$

it follows that substitution into equation (B.24) gives

$$\lim_{k\to\infty} a_i(k) = (-1)^i \cdot \frac{1}{(i+1)!} \cdot \left(1 - \frac{1}{\beta}\right)^{i+1} \cdot \lambda^{i+1}. \tag{B.25}$$

Therefore, $\lim_{k\to\infty} a_i(k)$ has been shown to exist for all $i \in \mathbb{N}$, and thus Tannery's theorem is satisfied. Then by its result, and using equation (B.25), it follows that

$$\begin{aligned}
&\lim_{k\to\infty} \sum_{i=0}^{m_k} (-1)^i \cdot \binom{n_k - ik - 1}{i+1} \cdot \left(1 - \frac{1}{\beta}\right)^{i+1} \cdot \beta^{-k\cdot(i+1)} &\text{(B.26)}\\
&= \sum_{i=0}^{\infty} \lim_{k\to\infty} (-1)^i \cdot \binom{n_k - ik - 1}{i+1} \cdot \left(1 - \frac{1}{\beta}\right)^{i+1} \cdot \beta^{-k\cdot(i+1)}\\
&= \sum_{i=0}^{\infty} (-1)^i \cdot \frac{1}{(i+1)!} \cdot \left(1 - \frac{1}{\beta}\right)^{i+1} \cdot \lambda^{i+1}\\
&= 1 - \sum_{i=0}^{\infty} (-1)^i \cdot \frac{1}{i!} \cdot \left(\left(1 - \frac{1}{\beta}\right) \cdot \lambda\right)^i = 1 - e^{-(1-\frac{1}{\beta})\cdot\lambda}. &\text{(B.27)}
\end{aligned}$$

Now, let us similarly apply the theorem for the second sum in equation (B.20), using the sequence $(b_i(k))$ as defined in equation (B.21). First, us derive a sequence $(\bar{C}_i \geq 0)$ such that $\sum_{i=1}^{\infty} \bar{C}_i < \infty$ and $|b_i(k)| \leq \bar{C}_i$ for all $i, k \in \mathbb{N}$. So, fix $i \in \mathbb{N}$, then

$$\begin{aligned}
|b_i(k)| &= \binom{n_k - ik - 1}{i} \cdot \left(1 - \frac{1}{\beta}\right)^i \cdot \beta^{-k\cdot(i+1)}\\
&\leq \frac{1}{i!} \cdot \left(1 - \frac{1}{\beta}\right)^i \cdot \left(\frac{n_k}{\beta^k}\right)^i\\
&\leq \frac{1}{i!} \cdot \lambda^i \overset{\text{set}}{=} \bar{C}_i.
\end{aligned}$$

Where it follows that

$$\sum_{i=0}^{\infty} \bar{C}_i = \sum_{i=0}^{\infty} \frac{1}{i!} \cdot \lambda^i = e^{\lambda} < \infty.$$

However, now observe that

$$\begin{aligned}
\lim_{k\to\infty} b_i(k) &= \lim_{k\to\infty} (-1)^i \cdot \binom{n_k - ik - 1}{i} \cdot \left(1 - \frac{1}{\beta}\right)^i \cdot \beta^{-k\cdot(i+1)} \\
&= \lim_{k\to\infty} (-1)^i \cdot \frac{1}{i!} \cdot \left(1 - \frac{1}{n_k - ik - 1}\right) \cdot \left(1 - \frac{2}{n_k - ik - 1}\right) \cdot \ldots \cdot \left(1 - \frac{i-1}{n_k - ik - 1}\right) \\
&\quad \cdot \left(1 - \frac{1}{\beta}\right)^i \cdot \left(\frac{n_k - ik - 1}{\beta^k}\right)^i \cdot \beta^{-k} \\
&= \lim_{k\to\infty} (-1)^i \cdot \frac{1}{i!} \cdot \left(1 - \frac{1}{\beta}\right)^i \cdot \lambda^i \cdot \beta^{-k} = 0.
\end{aligned}$$

Here the last line was derived similarly as for $a_i(k)$, leading to equation (B.25). Applying Tannery's theorem to this result, we get that

$$\lim_{k\to\infty} \sum_{i=1}^{m_k} (-1)^i \cdot \binom{n_k - ik - 1}{i} \cdot \left(1 - \frac{1}{\beta}\right)^i \cdot \beta^{-k\cdot(i+1)} = 0.$$

Substituting this and equation (B.27) into equation (B.20), we get that

$$\begin{aligned}
\lim_{k\to\infty} \mathbb{P}\left(M_{n_k} \leq 1 - \beta^{-k}\right) &= \lim_{k\to\infty} 1 - \sum_{i=0}^{m_k} (-1)^i \cdot \binom{n_k - ik - 1}{i+1} \cdot \left(1 - \frac{1}{\beta}\right)^{i+1} \cdot \beta^{-k\cdot(i+1)} \\
&\quad - \sum_{i=0}^{m_k} (-1)^i \cdot \binom{n_k - ik - 1}{i} \cdot \left(1 - \frac{1}{\beta}\right)^i \cdot \beta^{-k\cdot(i+1)} \\
&= 1 - \left(1 - e^{-(1-\frac{1}{\beta})\cdot\lambda}\right) = e^{-(1-\frac{1}{\beta})\cdot\lambda},
\end{aligned}$$

which proves theorem 2.3. □

C. Proofs from Chapter 4

C.1. Proof of Condition $D(u_n)$ for the Generalised Doubling Map

In this section, we will prove that condition $D(u_n)$ holds for the generalised doubling map. Of course, condition $D(u_n)$ holds with respect to some sequence (u_n), and below we will prove that it holds for any sequence (u_n) as specified in theorem 4.1.

Proof. In this proof, we are going to make use of [2] (theorem 8.3.2), which is given as follows.

Theorem C.1. For any domain $I \subset \mathbb{R}$ and maps $g_1 \in \mathcal{L}^1(I)$, $g_2 \in \mathcal{L}^\infty(I)$ with bounded total variation, if μ is an invariant measure with respect to $f : I \mapsto I$, such that (f, μ) is weakly mixing, then there exist $C \geq 0$ and $0 < r < 1$ such that

$$\left| \int_I g_1 \cdot \left(g_2 \circ f^t\right) \, d\mu - \int_I g_1 \, d\mu \cdot \int_I g_2 \, d\mu \right| \leq C r^t \cdot V(g_1, I) \cdot \|g_2\|_\infty,$$

where $V(g_1, I)$ is the total variation of g_2 on the domain I.

To apply this theorem here, let us define

$$g_1 = \mathbb{1}_{\{X_0 > u_n\}}, \quad g_2 = \mathbb{1}_{\{X_0 \leq u_n\} \cap \ldots \cap \{X_{l-1} \leq u_n\}},$$

and $I = [0, 1)$. Clearly, both maps have bounded variation, and in fact

$$V(g_1, I) \leq 2\beta - 1,$$

for any value of $\beta \geq 2$. Moreover, since g_2 is an indicator function, it follows that $\|g_2\|_\infty = 1$, so that $g_2 \in \mathcal{L}^\infty([0, 1))$. Let us also assume that (f, μ) is weakly mixing, and save the proof for this claim for the end. Then, theorem C.1 can be applied here, yielding

$$\begin{aligned}
&\Big| \mu\Big(\{X_0 > u_n\} \cap [\{X_t \leq u_n\} \cap \ldots \cap \{X_{t+l-1} \leq u_n\}]\Big) \\
&\qquad - \mu(\{X_0 > u_n\}) \cdot \mu\Big(\{X_0 \leq u_n\} \cap \ldots \cap \{X_{l-1} \leq u_n\}\Big) \Big| \\
&= \left| \int_I g_1 \cdot \left(g_2 \circ f^t\right) \, d\mu - \int_I g_1 \, d\mu \cdot \int_I g_2 \, d\mu \right| \\
&\leq C r^t \cdot V(g_1, I) \cdot \|g_2\|_\infty \leq (2\beta - 1) \cdot C r^t \equiv \gamma(n, t).
\end{aligned}$$

Since $0 < r < 1$, $\gamma(n, t)$ is non-increasing with respect to t for all $n \in \mathbb{N}$. And since $0 < r < 1$, it follows that for all sequences (t_n) with $\frac{t_n}{n} \xrightarrow{n\to\infty} 0$, $n \cdot \gamma(n, t_n) \xrightarrow{n\to\infty} 0$. Thus, condition $D(u_n)$ is satisfied for f_β, for all values of $\beta \geq 2$ and all sequences (u_n) as in theorem 4.1. □

However, recall that we still need to prove that (f, μ) is weakly mixing. To do this, we are actually going to prove that (f, μ) is strongly mixing, which implies weak mixing and is defined in [2] (definition 3.4.1) as follows.

Definition C.2. Let $f : I \mapsto I$ with measure μ, and σ-algebra $\mathcal{A}$ of I. Then, (f, μ) is strongly mixing if for all $A, B \in \mathcal{A}$,

$$\mu\left(f^{-n}(A) \cap B\right) \xrightarrow{n\to\infty} \mu(A) \cdot \mu(B).$$

Here, recall that we let $I = [0,1)$, and take the σ-algebra $\mathcal{A}$ to be the collection of all subsets of $[0,1)$. Now, let us prove that (f,μ) is strongly mixing below, which will conclude the entire proof.

Proof. First, let us recall that for any $[a,b) \subset [0,1)$,

$$f^{-n}([a,b)) = \bigcup_{s=1}^{\beta^n} \left[\frac{s-1+a}{\beta^n}, \frac{s-1+b}{\beta^n}\right),$$

for any $\beta \geq 2$. Now suppose that $A = [a,b) \subset [0,1)$ and $B = [c,d) \subset [0,1)$. Then,

$$\begin{aligned}
\mu\left(f^{-n}(A) \cap B\right) &= \sum_{s=1}^{2^n} \mu\left(\left[\frac{s-1+a}{\beta^n}, \frac{s-1+b}{\beta^n}\right) \cap [c,d)\right) \\
&= \sum_{s=t_1(n)+1}^{t_2(n)-1} \mu\left(\left[\frac{s-1+a}{\beta^n}, \frac{s-1+b}{\beta^n}\right)\right) + \mu\left(\left[c, \frac{t_1(n)-1+b}{\beta^n}\right)\right) \\
&\quad + \mu\left(\left[\frac{t_2(n)-1+a}{\beta^n}, d\right)\right), \qquad \text{(C.1)}
\end{aligned}$$

where $1 \leq t_1(n) < t_2(n) \leq 2^n$ are defined to be such that

$$c \in \left[\frac{t_1(n)-1+a}{\beta^n}, \frac{t_1(n)-1+b}{\beta^n}\right) \quad \text{and} \quad d \in \left[\frac{t_2(n)-1+a}{\beta^n}, \frac{t_2(n)-1+b}{\beta^n}\right),$$

for all $n \in \mathbb{N}$. By this definition, it also follows that

$$\begin{aligned}
&\frac{t_1(n)-1+a}{\beta^n} \leq c \leq \frac{t_1(n)-1+b}{\beta^n} \\
\Rightarrow\quad &\lim_{n\to\infty} \frac{t_1(n)-1+a}{\beta^n} \leq c \leq \lim_{n\to\infty} \frac{t_1(n)-1+b}{\beta^n} \\
\Rightarrow\quad &\lim_{n\to\infty} \frac{t_1(n)}{\beta^n} \leq c \leq \lim_{n\to\infty} \frac{t_1(n)}{\beta^n} \\
\Rightarrow\quad &\lim_{n\to\infty} \frac{t_1(n)}{\beta^n} = c,
\end{aligned}$$

and similarly,

$$\lim_{n\to\infty} \frac{t_2(n)}{\beta^n} = d.$$

Then it follows from equation (C.1) that

$$\begin{aligned}
\lim_{n\to\infty} \mu\left(f^{-n}(A) \cap B\right) &= \lim_{n\to\infty} \sum_{s=t_1(n)+1}^{t_2(n)-1} \mu\left(\left[\frac{s-1+a}{\beta^n}, \frac{s-1+b}{\beta^n}\right)\right) \\
&\quad + \lim_{n\to\infty} \mu\left(\left[c, \frac{t_1(n)-1+b}{\beta^n}\right)\right) + \lim_{n\to\infty} \mu\left(\left[\frac{t_2(n)-1+a}{\beta^n}, d\right)\right) \\
&= \lim_{n\to\infty} \frac{t_2(n)-1-t_1(n)-1}{\beta^n} \cdot \sum_{s=1}^{\beta^n} \mu\left(\left[\frac{s-1+a}{\beta^n}, \frac{s-1+b}{\beta^n}\right)\right) \\
&= \lim_{n\to\infty} (d-c) \cdot \sum_{s=1}^{\beta^n} \mu\left(\left[\frac{s-1+a}{\beta^n}, \frac{s-1+b}{\beta^n}\right)\right) \\
&= \mu(f^{-n}([a,b))) \cdot \mu([c,d)) = \mu(A) \cdot \mu(B),
\end{aligned}$$

as desired. $\square$

C.2. Counterexample of Condition $D'(u_n)$ for the Doubling Map

In this section, we are going to prove that for the doubling map, the condition $D'(u_n)$ does not hold, by giving a counterexample. In fact, we are not only going to give a single counterexample, but show that for almost all sequences (u_n) as in theorem 4.1, the condition $D'(u_n)$ does not hold. Recall that by theorem 4.1, the sequence (u_n) has to satisfy

$$\lim_{n\to\infty} n \cdot \mu(\{X_0 > u_n\}) = \tau, \tag{C.2}$$

for some $\tau \geq 0$. In this proof, we will show that for all sequences (u_n) satisfying equation (C.2) with $\tau > 0$, the condition $D'(u_n)$ does not hold.

Proof. First, let us recall that $X_0 \sim U(0,1)$, so that

$$\mu(\{X_0 > u_n\}) = \mu([u_n, 1)) = 1 - u_n.$$

Then by equation (C.2), it follows that

$$\lim_{n\to\infty} n \cdot (1 - u_n) = \tau \quad \Rightarrow \quad \lim_{n\to\infty} u_n = 1. \tag{C.3}$$

Let us define $U_n = [u_n, 1)$, and let us note that

$$\begin{aligned}
\mu\left(\{X_0 > u_n\} \cap \{X_j > u_n\}\right) &= \mu\left(\{X_0 \in U_n\} \cap \{X_j \in U_n\}\right) \\
&= \mu\left(\{X_0 \in U_n\} \cap \{X_0 \in f^{-j}(U_n)\}\right) \\
&= \mu\left(\{X_0 \in \left(U_n \cap f^{-j}(U_n)\right)\}\right) \\
&= \mu\left(U_n \cap f^{-j}(U_n)\right).
\end{aligned} \tag{C.4}$$

For the doubling map, it holds that

$$U_n \cap f^{-j}(U_n) = [u_n, 1) \cap f^{-j}([u_n, 1)) \supset \left[1 - \frac{1-u_n}{2^j}, 1\right), \tag{C.5}$$

for any $j, n \in \mathbb{N}$. To see this, consider the case that $u_n = 1 - 2^{-m}$ for some $m \in \mathbb{N}$, then $U_n = E_0 = [1 - 2^{-m}, 1)$, as defined in section 2.2, and $f^{-j}(U_n) = E_j$. Then for any $j \in \mathbb{N}$, it follows that

$$E_0 \cap E_j \supset \left[1 - \frac{2^{-m}}{2^j}, 1\right) = \left[1 - \frac{1-u_n}{2^j}, 1\right).$$

For any (u_n) satisfying equation (C.2), the proof of equation (C.5) is very similar to that of this special case. Inserting this into equation (C.4), yields

$$\mu\left(\{X_0 > u_n\} \cap \{X_j > u_n\}\right) \geq \mu\left(\left[1 - \frac{1-u_n}{2^j}, 1\right)\right) = \frac{1-u_n}{2^j},$$

for any $j, n \in \mathbb{N}$. So for any $k, n \in \mathbb{N}$, we get that

$$n \cdot \sum_{j=1}^{\lfloor n/k \rfloor} \mu\left(\{X_0 > u_n\} \cap \{X_j > u_n\}\right) \geq n \cdot \sum_{j=1}^{\lfloor n/k \rfloor} \frac{1-u_n}{2^j} = n \cdot (1-u_n) \cdot \left(1 - 2^{\lfloor n/k \rfloor}\right).$$

By the limit on the left-hand side in equation (C.3), it then follows that

$$\begin{aligned}
\lim_{k\to\infty} \limsup_{n\to\infty} n \cdot \sum_{j=1}^{\lfloor n/k \rfloor} \mu\left(\{X_0 > u_n\} \cap \{X_j > u_n\}\right) &\geq \lim_{k\to\infty} \limsup_{n\to\infty} n \cdot (1-u_n) \cdot \left(1 - 2^{\lfloor n/k \rfloor}\right) \\
&= \lim_{k\to\infty} \limsup_{n\to\infty} \tau \cdot \left(1 - 2^{\lfloor n/k \rfloor}\right) = \tau > 0.
\end{aligned}$$

Hence, the condition $D'(u_n)$ does not hold for (u_n). □

D. Proofs from Chapter 5

D.1. Proof of Equation (5.8)

In this section, we will elaborate on the derivation of equation (5.8), which is given as follows:

$$\rho(x) = \begin{cases} \dfrac{\beta+1}{2\beta - \lfloor \beta \rfloor} & \text{if } 0 \leq x < \beta \bmod 1 \\ \dfrac{\beta}{2\beta - \lfloor \beta \rfloor} & \text{if } \beta \bmod 1 \leq x < 1. \end{cases}$$

Recall that this formula only holds for non-integer values of $\beta > 1$ for which there exists an integer $I(\beta)$ as in equation (5.7). To show this, let us note that it follows directly from equation (5.6) that

$$\rho(x) = \begin{cases} \displaystyle\sum_{i=1}^{\lceil \beta \rceil} \frac{1}{\beta} \cdot \rho\left(\frac{i-1+x}{\beta}\right) & \text{if } 0 \leq x < \beta \bmod 1 \\ \displaystyle\sum_{i=1}^{\lfloor \beta \rfloor} \frac{1}{\beta} \cdot \rho\left(\frac{i-1+x}{\beta}\right) & \text{if } \beta \bmod 1 \leq x < 1. \end{cases} \tag{D.1}$$

Since in these cases equation (5.7) holds for some integer $I(\beta)$, it follows that $\rho(x)$ is constant on both intervals, i.e.

$$\rho(x) = \begin{cases} g(\beta) & \text{if } 0 \leq x < \beta \bmod 1 \\ h(\beta) & \text{if } \beta \bmod 1 \leq x < 1, \end{cases} \tag{D.2}$$

for some functions $g, h : \mathbb{R} \mapsto [0, 1)$. Then by equation (D.1), it follows that

$$\begin{aligned} &\begin{cases} g(\beta) &= \displaystyle\sum_{i=1}^{I(\beta)} \frac{1}{\beta} \cdot g(\beta) + \sum_{i=I(\beta)+1}^{\lceil \beta \rceil} \frac{1}{\beta} \cdot h(\beta) \\ h(\beta) &= \displaystyle\sum_{i=1}^{I(\beta)} \frac{1}{\beta} \cdot g(\beta) + \sum_{i=I(\beta)+1}^{\lfloor \beta \rfloor} \frac{1}{\beta} \cdot h(\beta) \end{cases} \\ \Rightarrow\ &\begin{cases} g(\beta) &= \dfrac{I(\beta)}{\beta} \cdot g(\beta) + \dfrac{\lfloor \beta \rfloor - I(\beta) + 1}{\beta} \cdot h(\beta) \\ h(\beta) &= \dfrac{I(\beta)}{\beta} \cdot g(\beta) + \dfrac{\lfloor \beta \rfloor - I(\beta)}{\beta} \cdot h(\beta) \end{cases} \\ \Rightarrow\ & g(\beta) = \frac{\beta+1}{\beta} \cdot h(\beta). \end{aligned} \tag{D.3}$$

From this it is possible to obtain the functions $g(\beta)$ and $h(\beta)$ and reconstruct $\rho(x)$ using equation (D.2). Of course, the system of equations above is still satisfied whenever both $g(\beta)$ and $h(\beta)$ are multiplied by some constant. So in order to uniquely determine $\rho(x)$, let us use the second requirement of an invariant probability measure, and normalise $\rho(x)$ using equation (5.4);

$$\begin{aligned} 1 &= \int_0^1 \rho(x)\ dx = \int_0^{\beta \bmod 1} g(\beta)\ dx + \int_{\beta \bmod 1}^1 h(\beta)\ dx \\ &= (\beta \bmod 1) \cdot g(\beta) + (1 - \beta \bmod 1) \cdot h(\beta). \end{aligned}$$

Now substituting the expression found in equation (D.3) into the equation above, yields

$$\begin{aligned}1 &= \left((\beta \bmod 1)\cdot\frac{\beta+1}{\beta}+1-\beta \bmod 1\right)\cdot h(\beta)\\ &= \left((\beta-\lfloor\beta\rfloor)\cdot\frac{1}{\beta}\right)\cdot h(\beta)\\ &= \left(2-\frac{\lfloor\beta\rfloor}{\beta}\right)\cdot h(\beta).\end{aligned}$$

Hence, it follows that

$$\begin{aligned} & h(\beta) = \frac{\beta}{2\beta-\lfloor\beta\rfloor}\\ \Rightarrow\quad & g(\beta) = \frac{\beta+1}{2\beta-\lfloor\beta\rfloor},\end{aligned}$$

and thus

$$\rho(x) = \begin{cases}\dfrac{\beta+1}{2\beta-\lfloor\beta\rfloor} & \text{if } 0\le x<\beta \bmod 1\\ \dfrac{\beta}{2\beta-\lfloor\beta\rfloor} & \text{if } \beta \bmod 1\le x<1,\end{cases}$$

as desired.

D.2. Counterexample of Condition $D'(u_n)$ for $\beta=\varphi$

In this section, we are going to prove that for the generalised doubling map with $\beta=\varphi$, the condition $D'(u_n)$ does not hold, in a similar way as the proof in section C.2.

Proof. First, let us recall that X_0 has the probability distribution as given in equation (5.9). Then, for any sequence (u_n) with $u_n\ge\varphi-1$ for all $n\in\mathbb{N}$,

$$\mu(\{X_0>u_n\})=\mu([u_n,1))=\frac{\varphi^2}{\varphi^2+1}\cdot(1-u_n).$$

Then by equation (C.2), it follows that

$$\lim_{n\to\infty} n\cdot\frac{\varphi^2}{\varphi^2+1}\cdot(1-u_n)=\tau \quad\Rightarrow\quad \lim_{n\to\infty}u_n=1. \tag{D.4}$$

Let us define $U_n=[u_n,1)$, and let us note that

$$\begin{aligned}\mu\left(\{X_0>u_n\}\cap\{X_j>u_n\}\right) &= \mu\left(\{X_0\in U_n\}\cap\{X_j\in U_n\}\right)\\ &= \mu\left(\{X_0\in U_n\}\cap\{X_0\in f_\varphi^{-j}(U_n)\}\right)\\ &= \mu\left(\{X_0\in\left(U_n\cap f_\varphi^{-j}(U_n)\right)\}\right)\\ &= \mu\left(U_n\cap f_\varphi^{-j}(U_n)\right).\end{aligned} \tag{D.5}$$

Let us claim that for the generalised doubling map, it holds that

$$U_n\cap f_\varphi^{-2j}(U_n)=[u_n,1)\cap f_\varphi^{-2j}([u_n,1))\supset\left[1-\frac{1-u_n}{\varphi^{2j}},1\right), \tag{D.6}$$

for any $j,n\in\mathbb{N}$. To see this, let us note that by equation (5.10),

$$f_\varphi^{-1}([u_n,1))=\left[\frac{u_n}{\varphi},\frac{1}{\varphi}\right),$$

so that

$$f_\varphi^{-2}([u_n,1)) = \left[\frac{u_n}{\varphi^2},\frac{1}{\varphi^2}\right) \cup \left[\frac{1+\frac{u_n}{\varphi}}{\varphi},1\right) = \left[\frac{u_n}{\varphi^2},\frac{1}{\varphi^2}\right) \cup \left[1-\frac{1-u_n}{\varphi^2},1\right)$$
$$\supset \left[1-\frac{1-u_n}{\varphi^2},1\right).$$

This proofs equation (D.6) for $j=1$. To prove it using induction on j, let us assume that equation (D.6) holds, and note that

$$f_\varphi^{-1}\left(f_\varphi^{-2j}([u_n,1))\right) \supset f_\varphi^{-1}\left(\left[1-\frac{1-u_n}{\varphi^{2j}},1\right)\right) = \left[\frac{1}{\varphi}-\frac{1-u_n}{\varphi^{2j+1}},\frac{1}{\varphi}\right).$$

Then, it follows that

$$\begin{aligned} f_\varphi^{-2}\left(f_\varphi^{-2j}([u_n,1))\right) &\supset f_\varphi^{-1}\left(\left[\frac{1}{\varphi}-\frac{1-u_n}{\varphi^{2j+1}},\frac{1}{\varphi}\right)\right) \\ &= \left[\frac{1}{\varphi^2}-\frac{1-u_n}{\varphi^{2j+2}},\frac{1}{\varphi^2}\right) \cup \left[\frac{1+\frac{1}{\varphi}-\frac{1-u_n}{\varphi^{2j+1}}}{\varphi},1\right) \\ &\supset \left[1-\frac{1-u_n}{\varphi^{2(j+1)}},1\right). \end{aligned}$$

This is equivalent to equation (D.6) for $j+1$, thus completing its proof by induction. Inserting equation (D.6) into equation (D.5), yields

$$\mu\left(\{X_0>u_n\}\cap\{X_j>u_n\}\right) \geq \mu\left(\left[1-\frac{1-u_n}{\varphi^j},1\right)\right) = \frac{\varphi^2}{\varphi^2+1}\cdot\frac{1-u_n}{\varphi^j},$$

for any $j,n\in\mathbb{N}$. So for any $k,n\in\mathbb{N}$, we get that

$$\begin{aligned} n\cdot\sum_{j=1}^{\lfloor n/k\rfloor}\mu\left(\{X_0>u_n\}\cap\{X_j>u_n\}\right) &\geq n\cdot\frac{\varphi^2}{\varphi^2+1}\cdot\sum_{\substack{j=1\\ \text{even}}}^{\lfloor n/k\rfloor}\frac{1-u_n}{\varphi^j} \\ &= n\cdot\frac{\varphi^2}{\varphi^2+1}\cdot\sum_{j=1}^{\lfloor n/(2k)\rfloor}\frac{1-u_n}{\varphi^{2j}} \\ &= n\cdot\frac{\varphi^2}{\varphi^2+1}\cdot(1-u_n)\cdot\left(1-\varphi^{\lfloor n/(2k)\rfloor}\right). \end{aligned}$$

By the limit on the left-hand side in equation (D.4), it then follows that

$$\begin{aligned} &\lim_{k\to\infty}\limsup_{n\to\infty} n\cdot\sum_{j=1}^{\lfloor n/k\rfloor}\mu\left(\{X_0>u_n\}\cap\{X_j>u_n\}\right) \\ &\geq \lim_{k\to\infty}\limsup_{n\to\infty} n\cdot\frac{\varphi^2}{\varphi^2+1}\cdot(1-u_n)\cdot\left(1-\varphi^{\lfloor n/(2k)\rfloor}\right) \\ &= \lim_{k\to\infty}\limsup_{n\to\infty} \tau\cdot\left(1-\varphi^{\lfloor n/(2k)\rfloor}\right) = \tau > 0. \end{aligned}$$

Hence, the condition $D'(u_n)$ does not hold for (u_n). □

D.3. Generalised Counterexample of Condition $D'(u_n)$

Previous section, we have shown that condition $D'(u_n)$ does not hold for the generalised doubling map with $\beta = \varphi$. It turns out that the specific counterexample given can be generalised to hold for any $\beta > 1$ such that $I(\beta) = \lfloor \beta \rfloor$, where $I(\beta)$ is defined as in equation (5.7) to be $I(\beta) = \beta \cdot \beta \bmod 1$. Hence, the condition on $\beta > 1$ for which the following proof will hold, is given by

$$I(\beta) = \beta \cdot \beta \bmod 1 = \lfloor \beta \rfloor. \tag{D.7}$$

There are infinitely many values of $\beta > 1$ that satisfy this condition, for example $\beta = \varphi$ and $\beta = \sqrt{3} + 1$, thus showing the significance of this special case.

Proof. First, let us recall that X_0 has the probability distribution as given in equation (5.8). Then, for any sequence (u_n) with $u_n \geq \beta \bmod 1$ for all $n \in \mathbb{N}$,

$$\mu(\{X_0 > u_n\}) = \mu([u_n, 1)) = \frac{\beta}{2\beta - \lfloor \beta \rfloor} \cdot (1 - u_n).$$

Then by equation (C.2), it follows that

$$\lim_{n \to \infty} n \cdot \frac{\beta}{2\beta - \lfloor \beta \rfloor} \cdot (1 - u_n) = \tau \quad \Rightarrow \quad \lim_{n \to \infty} u_n = 1. \tag{D.8}$$

Let us define $U_n = [u_n, 1)$, and let us note that

$$\begin{aligned}
\mu\left(\{X_0 > u_n\} \cap \{X_j > u_n\}\right) &= \mu\left(\{X_0 \in U_n\} \cap \{X_j \in U_n\}\right) \\
&= \mu\left(\{X_0 \in U_n\} \cap \{X_0 \in f_\beta^{-j}(U_n)\}\right) \\
&= \mu\left(\{X_0 \in \left(U_n \cap f_\beta^{-j}(U_n)\right)\}\right) \\
&= \mu\left(U_n \cap f_\beta^{-j}(U_n)\right).
\end{aligned} \tag{D.9}$$

Let us claim that for the generalised doubling map, it holds that

$$U_n \cap f_\beta^{-2j}(U_n) = [u_n, 1) \cap f_\beta^{-2j}([u_n, 1)) \supset \left[1 - \frac{1 - u_n}{\beta^{2j}}, 1\right), \tag{D.10}$$

for any $j, n \in \mathbb{N}$. To see this, let us note that by equation (5.5),

$$f_\beta^{-1}([u_n, 1)) = \bigcup_{i=1}^{\lfloor \beta \rfloor} \left[\frac{i - 1 + u_n}{\beta}, \frac{i}{\beta}\right) \supset \left[\frac{\lfloor \beta \rfloor - 1 + u_n}{\beta}, \frac{\lfloor \beta \rfloor}{\beta}\right)$$

Then by equation (D.7), we see that $\frac{\lfloor \beta \rfloor}{\beta} = \beta \bmod 1$, so that

$$\begin{aligned}
f_\beta^{-2}([u_n, 1)) &\supset f_\beta^{-1}\left(\left[\frac{\lfloor \beta \rfloor - 1 + u_n}{\beta}, \frac{\lfloor \beta \rfloor}{\beta}\right)\right) \\
&= \bigcup_{i=1}^{\lfloor \beta \rfloor} \left[\frac{i - 1 + \frac{\lfloor \beta \rfloor - 1 + u_n}{\beta}}{\beta}, \frac{i - 1 + \frac{\lfloor \beta \rfloor}{\beta}}{\beta}\right) \cup \left[\frac{\lceil \beta \rceil - 1 + \frac{\lfloor \beta \rfloor - 1 + u_n}{\beta}}{\beta}, 1\right) \\
&\supset \left[\frac{\beta \cdot \lfloor \beta \rfloor + \lfloor \beta \rfloor - 1 + u_n}{\beta^2}, 1\right) = \left[1 - \frac{\beta^2 - \beta \cdot \lfloor \beta \rfloor - \lfloor \beta \rfloor + 1 - u_n}{\beta^2}, 1\right)
\end{aligned}$$

$$= \left[1 - \frac{\beta \cdot \beta \bmod 1 - \lfloor \beta \rfloor + 1 - u_n}{\beta^2}, 1\right) = \left[1 - \frac{1 - u_n}{\beta^2}, 1\right),$$

where in the last line equation (D.7) was used. This proofs equation (D.10) for $j = 1$. To prove it using induction on j, let us assume that equation (D.10) holds, and note that

$$\begin{aligned} f_\beta^{-1}\left(f_\beta^{-2j}([u_n, 1))\right) &\supset f_\beta^{-1}\left(\left[1 - \frac{1 - u_n}{\beta^{2j}}, 1\right)\right) = \bigcup_{i=1}^{\lfloor \beta \rfloor} \left[\frac{i - \frac{1-u_n}{\beta^{2j}}}{\beta}, \frac{i}{\beta}\right) \\ &\supset \left[\frac{\lfloor \beta \rfloor}{\beta} - \frac{1 - u_n}{\beta^{2j+1}}, \frac{\lfloor \beta \rfloor}{\beta}\right). \end{aligned}$$

Again using equation (D.7), we get that

$$\begin{aligned} f_\beta^{-2}\left(f_\beta^{-2j}([u_n, 1))\right) &\supset f_\beta^{-1}\left(\left[\frac{\lfloor \beta \rfloor}{\beta} - \frac{1 - u_n}{\beta^{2j+1}}, \frac{\lfloor \beta \rfloor}{\beta}\right)\right) \\ &= \bigcup_{i=1}^{\lfloor \beta \rfloor} \left[\frac{i - 1 + \frac{\lfloor \beta \rfloor}{\beta} - \frac{1-u_n}{\beta^{2j+1}}}{\beta}, \frac{i - 1 + \frac{\lfloor \beta \rfloor}{\beta}}{\beta}\right) \cup \left[\frac{\lceil \beta \rceil - 1 + \frac{\lfloor \beta \rfloor}{\beta} - \frac{1-u_n}{\beta^{2j+1}}}{\beta}, 1\right) \\ &\supset \left[\frac{\beta \cdot \lfloor \beta \rfloor + \lfloor \beta \rfloor - \frac{1-u_n}{\beta^{2j}}}{\beta^2}, 1\right) = \left[\frac{\beta^{2j} \cdot (\beta \cdot \lfloor \beta \rfloor + \lfloor \beta \rfloor) - 1 + u_n}{\beta^{2j+2}}, 1\right) \\ &= \left[1 - \frac{\beta \cdot \beta \bmod 1 - \lfloor \beta \rfloor}{\beta^2} - \frac{1 - u_n}{\beta^{2j+2}}, 1\right) = \left[1 - \frac{1 - u_n}{\beta^{2j+2}}, 1\right). \end{aligned}$$

This is equivalent to equation (D.10) for $j+1$, thus completing its proof by induction. Inserting equation (D.10) into equation (D.9), yields

$$\mu\left(\{X_0 > u_n\} \cap \{X_j > u_n\}\right) \geq \mu\left(\left[1 - \frac{1 - u_n}{\beta^j}, 1\right)\right) = \frac{\beta}{2\beta - \lfloor \beta \rfloor} \cdot \frac{1 - u_n}{\beta^j},$$

for any $j, n \in \mathbb{N}$. So for any $k, n \in \mathbb{N}$, we get that

$$\begin{aligned} n \cdot \sum_{j=1}^{\lfloor n/k \rfloor} \mu\left(\{X_0 > u_n\} \cap \{X_j > u_n\}\right) &\geq n \cdot \frac{\beta}{2\beta - \lfloor \beta \rfloor} \cdot \sum_{\substack{j=1 \\ \text{even}}}^{\lfloor n/k \rfloor} \frac{1 - u_n}{\beta^j} \\ &= n \cdot \frac{\beta}{2\beta - \lfloor \beta \rfloor} \cdot \sum_{j=1}^{\lfloor n/(2k) \rfloor} \frac{1 - u_n}{\beta^{2j}} \\ &= n \cdot \frac{\beta}{2\beta - \lfloor \beta \rfloor} \cdot (1 - u_n) \cdot \left(1 - \beta^{\lfloor n/(2k) \rfloor}\right). \end{aligned}$$

By the limit on the left-hand side in equation (D.8), it then follows that

$$\begin{aligned} &\lim_{k \to \infty} \limsup_{n \to \infty} n \cdot \sum_{j=1}^{\lfloor n/k \rfloor} \mu\left(\{X_0 > u_n\} \cap \{X_j > u_n\}\right) \\ &\geq \lim_{k \to \infty} \limsup_{n \to \infty} n \cdot \frac{\beta}{2\beta - \lfloor \beta \rfloor} \cdot (1 - u_n) \cdot \left(1 - \beta^{\lfloor n/(2k) \rfloor}\right) \\ &= \lim_{k \to \infty} \limsup_{n \to \infty} \tau \cdot \left(1 - \beta^{\lfloor n/(2k) \rfloor}\right) = \tau > 0. \end{aligned}$$

Hence, the condition $D'(u_n)$ does not hold for (u_n). $\square$