Autocratic Control in Social Dilemmas

How much control do autocrats *really* have over the population as a whole in infinitely repeated social dilemma games with arbitrary action spaces?

Bachelor Integration Project (bIEM)

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Abstract

Many governing systems struggle when dealing with social dilemmas as people are reluctant to cooperate for the benefit of the greater good. Research has shown that zero-determinant strategies are effective in allowing an autocrat to have significant control over the group payoff, irrespective of their opponents' strategies. This study aims to take real-life factors such as discounting future payoffs or non-linearity into account to determine how enforceable the strategies are in the context of the public goods game. Building on existing findings regarding infinitely repeated, multiplayer social dilemma games with an arbitrary action space, it asks: how much control do autocrats *really* have over the population as a whole?

Analysis of existing research on zero-determinant strategies derived minimum threshold requirements on several game parameters. Sensitivity analysis then details the effect of varying multiple parameters on the derived thresholds. The results deduce necessary conditions that must be met, which limits the enforceability of each strategy. On this basis, autocrats can control the population if and only if they take notice of those conditions. Further research is needed to test the outcomes of this research in similar social dilemma games to determine the generality of the derived results.

Contents

1 Introduction

Many of the world's most severe problems arise in areas such as economics, political science and evolutionary biology, ranging from climate change to world wars (Dyke 2020). Issues surface due to conflict of interest between individuals and groups, which occurs when collective interests are at odds with private interests, also referred to as social dilemmas (Mcavoy & Hauert 2016).

In social dilemmas, mutual cooperation is the most successful way to prevent free-riding. However, reciprocal cooperation becomes extremely restrictive as group size increases (Boyd & Richerson 1988). Press and Dyson discovered a new set of strategies that prove successful in such social dilemmas, known as zero-determinant (ZD) strategies. They enforce a linear relationship between the players' payoffs in an infinitely repeated game irrespective of the co-player's actual strategy (Hilbe et al. 2018), allowing an autocrat to have significant control over the outcome of the game (Press & Dyson 2012).

Martirosyan et al. (2020) extended on existing research by exploring the new set of probabilistic memory-one strategies; ZD strategies, when applied to multiplayer, infinitely repeated games with an arbitrary action space in the context of the public goods game (PGG). It was then deduced that there are restrictive conditions that must be met in order to successfully enforce the various ZD strategies (Pan et al. 2015), as well as bounds on the value that the autocrat can assign to its opponents in the equaliser strategy case. Considering this, the strategies are further explored in this work in order to investigate the effect of discounting future payoffs as well as non-linearity on the enforceability of the ZD strategies. This is done by exploring constraints on the discount factor and the introduction of a minimum entry threshold requirement in the PGG. Sensitivity analysis is then conducted on the derived restrictive conditions to investigate the enforceability of each strategy under diverse parameter values.

2 Problem Analysis

In this section, the background of the problem is thoroughly analysed. Firstly, existing findings regarding autocratic zero determinant strategies are analysed, and an overview of the social dilemma game focused on throughout this research is provided as a base for the body of knowledge. This preliminary research then deduces the research project's main objective.

2.1 Problem Context

2.1.1 Existing Findings

Past research deduced that cooperation strategies based on trust were found to improve players' performance and overall payoffs. Therefore, systems rely on individuals setting aside their interests for the benefit of the greater good (Martirosyan et al. 2020). However, Press and Dyson's recent discovery of the new class of strategies: Zero-Determinant (ZD) strategies, now allows an autocrat to exert unilateral control over iterated interactions or pin the expected payoff of opponents (Pan et al. 2015). A player can use an autocratic strategy to extort an unfair share of the payoffs by enforcing a linear relationship between their payoff and that of the opponent (Martirosyan et al. 2020). Therefore, one player can have a considerably significant effect on the outcomes of a large population in social dilemma games. This research aims to delve into ZD strategies and explore how one player can control the outcome of a large population in social dilemma games.

In the context of the public goods game, research shows that there are certain restrictive conditions that must be met in order to successfully implement in the game each strategy since different ZD strategies evolve under various evolutionary conditions (Govaert & Cao 2019*a*). Martirosyan et al. (2020) have already given a kick-start to this research by extending existing findings about autocratic strategies in standard infinitely repeated and discounted multiplayer games from only having two action spaces to being arbitrary. Their simulations were then applied in the context of the public goods game (PGG) to explore the required conditions on specific parameters, that allow each sub-strategy to enforce a linear payoff relationship.

2.1.2 Body of Knowledge

Although the discovery of ZD strategies was derived from iterated prisoner's dilemma (IPD), further research shows that ZD strategies are not confined to pairwise games (Hilbe et al. 2018). The two-player, two-strategy (2x2) IPD does not give a realistic representation of how real-world issues are; therefore, other symmetric social dilemma games are more useful for research. Firstly, the PGG is a good reflection as it involves a large number of players with an arbitrary action space. In this game, public goods are considered any resource which benefits everybody regardless of whether they have helped provide it. Here, every player $i \in \{1, ..., N\}$ contributes an amount $x_i \in [0, 1]$ to a common pool, at a contribution cost (c). Total contributions are then multiplied by a known factor (r), which represents the players' productivity level and evenly shared among all group members (N), regardless of whether or not they contributed (Pan et al. 2015). The utility function for player i is given by

$$u_i(x_i, x_{-i}) = \frac{rc\sum_{k=1}^{k=N} x_k}{N} - cx_i \tag{1}$$

with $1 < r < N$ and $c > 0$.

Here, defection is favoured when the other player cooperates, which occurs at the cost of the overall payoff (Kümmerli et al. 2007). Even if, in the short run, it pays off to be selfish, mutual cooperation can be favoured when the individuals encounter each other repeatedly (Govaert & Cao 2019*a*).

2.1.3 Problem Statement

In Martirosyan et al. (2020), necessary conditions on specific parameters for the existence of autocratic strategies in multiplayer games have been derived, mainly in the context of the public goods game. It is still unclear how the element of discounting future rewards or the introduction of a non-linear payoff function influences the enforceability of the ZD strategies and how varying game parameters affect the derived conditions.

2.2 Problem Owner and Stakeholder Analysis

As problem owners and the sole key players in this research project, Cao and Martirosyan are interested in making their research more applicable to real-life scenarios. They have shown interest in exploring further necessary conditions for the existence of autocratic strategies as well as applying their developed theory to games with non-linear payoff functions. Once this is achieved, the findings can then be generalised by being applied to different social dilemma games. This, in turn, allows the application to a vast range of domains, giving rise to crowd stakeholders such as economists, mathematicians, philosophers, and biologists, who would be interested in the possible implementation of those strategies in their governing systems (Bhuiyan 2018).

In the context of the public goods game, the government could be considered a key provider of public goods such as police protection or public health funding. Therefore, governments are also considered stakeholders, as this research would give insight into how they can overcome the free-rider problem, where citizens attempt to use the public good without paying for it (OpenStax 2016).

3 Research Goal

3.1 System Description and Scope

The research will commence by delving into the public goods game to explore the element of discounting future rewards, represented by a discount factor (λ). From this, minimum threshold constraints on λ are derived for each ZD strategy. Then, to prevent the tragedy of the commons in the public goods game, a conditional minimum total donation amount is set to enforce mutual cooperation and compensate for the lack of individual control, making the game return non-linear payoff functions (Hilbe et al. 2014). For each of the conditions derived, the influence of different parameter values such as group sizes and multiplication factors on the derived necessary conditions will be explored.

In-depth analysis and mathematical simulations of games with the following characteristics will be focused on throughout this research;

Most game theory researchers apply their findings in the context of the classic iterated prisoner's dilemma (IPD), limiting research to two-player games with two possible moves: cooperate or defect (Mcavoy & Hauert 2016). This research aims to extend further by investigating multi-player games that are *infinitely repeated* with an *arbitrary action space* rather than only having two possible actions. However, the autocrat will only play two actions throughout the repeated game, while others have a continuous action space to choose from. This two-point autocratic strategy is focused on for simplification since a complex integral for the payoff function is used (Mcavoy & Hauert 2016). In addition, a discount factor will be used to essentially assign lower values to future payoffs, changing the relative weights of payoff functions of different players (Petrosyan & Zenkovic 1996).

According to previous research findings, a symmetric game with a rich action space makes only considering the last previous round of the game enough to support all equilibrium payoffs (Barlo et al. 2009). Also, Press and Dyson explored that a longer memory does not give an advantage if a game is indefinitely repeated (Press & Dyson 2012). This research focuses on infinitely repeated games with an arbitrary action space; therefore, the scope of this research will be limited to memory-one strategies, only taking the previous round into account. Finally, the four main ZD sub-strategies: fair, generous, equaliser and extortionate, will be explored in this research as they are the most prominent (Ichinosea & Masuda 2018) (Hilbe et al. 2014).

3.2 Research Objective

This research aims to determine the effect of changing a social dilemma game's rules and parameters on the necessary conditions required to enforce autocratic strategies in the context of the public goods game. This can be done by acquiring inexhaustible knowledge regarding different ZD strategies to mathematically model and simulate each sub-strategy's outcomes under varying states.

3.3 Deliverable

This project requires extensive research due to the recent discovery of ZD strategies and scarcity of information regarding their implementation in social dilemma games other than the iterated prisoner's dilemma. The first aim of this research project is to apply developed theories on ZD strategies to discount future payoffs and introduce non-linearity to a modified PGG and provide an overview of the limitations on certain parameters for ZD strategies to be enforced in the context of PGG.

3.4 Research Questions

The following research questions will be answered during the research to gain the required knowledge to steer research and achieve the final objective.

1. What are the necessary conditions that allow autocratic strategies to enforce the linear payoff function successfully?
   - How does the element of discounting future payoffs, represented by the discount factor, affect the existence of autocratic strategies?
   - How does changing the PGG game parameters influence the necessary conditions?
2. How does a non-linear payoff function influence the enforceability of the strategies?

3.5 Quick Risk Analysis

After careful analysis of how this research project is structured, a few factors impose a risk on the feasibility and applicability of the results in a vast range of domains. Firstly, due to the time constraint, the research is only conducted in the context of the public goods game and therefore, the applicability of the findings for similar games are assumed but not tested. Also, MATLAB is used to conduct sensitivity analysis and determine how some parameters are affected based on changes in other parameters. Due to the range of parameters involved with social dilemma games, some parameters must be fixed to a specific value, limiting the variation of the outcomes.

4 Preliminaries

In this section, the preliminary definitions, notations, and assumptions are formally outlined.

$\mathbb{R}$	Set of real numbers
N	Total number of players in a game
S	Action space
S_i	Action space for player i
$\boldsymbol{x}$	Action profile
π	Combined payoff function
ϵ	Infinitesimal number
λ	Discount factor
p_0	Initial probability to cooperate
χ	Slope of zero-determinant strategy
l	Baseline of zero-determinant strategy

Table 1: Basic symbols and notations used throughout this research project

For each player $i \in \{1, ..., N\}$, the available action space is a measurable space, denoted by S_i. In this research project, the autocrat has a discrete action space with two possible actions; cooperate or defect and is denoted by $s_1 = 0$ and $s_2 = 1$. However, the all the opponents employ continuous action spaces in the unit interval [0,1].

In addition, the action profile generated by N players is denoted by $\boldsymbol{x} = (x_i, x_{-i})$, where x_i is the autocrat's action and x_{-i} represents the action profile chosen by the opponents with $x_{-i} \in S_{-i} = \prod_{j \neq i} S_j$. Finally, let u_i and u_j denote the payoff utility function of the autocrat and of an opponent, respectively.

Note that the normalised cumulative payoff after t rounds to player i is denoted by π_i and defined as

$$\pi_i = (1 - \lambda) \sum_{t=0}^{t=\infty} \lambda^t u_i(x_i^t, x_{-i}^t).$$

Four main sub-classes of ZD strategies have been distinguished; fair strategies ensure that the own payoff (π_i) matches the average payoff of the group (π_{-i}); generous strategies let a player perform below average; extortionate strategies allow a player to perform above average and equaliser strategies allow a player to determine the payoff of the opponent unilaterally (Hilbe et al. 2014) (Ichinosea & Masuda 2018). The basic property for each strategy is mathematically shown in Table 2 below.

ZD strategy	**Property**
Fair	$\pi_i = \pi_{-i}$
Generous	$\pi_i \leq \pi_{-i}$
Extortionate	$\pi_i \geq \pi_{-i}$
Equaliser	$\pi_{-i} = l$

Table 2: Properties of the four main ZD strategies; π_i represents the autocrats own payoff, while π_{-i} matches the average payoff of the group.

4.1 Assumptions

The research focuses on symmetric N-player games, which means that the game's outcome depends only on one's own decision and the number of cooperating co-players, and hence, it does not depend on which of the co-players have cooperated (Govaert & Cao 2019*b*). The following assumptions are consistent throughout the research as they establish the definition of a *social dilemma* where conflict of interest arises;

Assumption 1. If a player fixes their action, but co-players cooperate more in total, the player receives a higher payoff. Therefore, irrespective of their own strategy, a player always prefers others to contribute more.

$$\text{if } x_{-i} > x'_{-i}$$
$$\text{then } u_i(x_i, x_{-i}) > u_i(x_i, x'_{-i})$$

Assumption 2. If all co-players fix their strategies, a player will receive less if they contribute more.

$$\text{if } x_i > x'_i$$
$$\text{then } u_i(x_i, x_{-i}) > u_i(x'_i, x_{-i})$$

Assumption 3. Mutual cooperation is always favoured above mutual defection.

$$\text{if } (x_i, x_{-i}) = (1, ..., 1) \text{ and } (x'_i, x'_{-i}) = (0, ..., 0)$$
$$\text{then } u_i(x_i, x_{-i}) > u_i(x'_i, x'_{-i})$$

The aforementioned assumptions are analogous to those made in (Martirosyan et al. 2020).

5 Public Goods Game

In this section, each of the ZD strategies will be thoroughly analysed and simulated in order to deduce new restrictive conditions and determine the effect, if any, on altering the game's parameters such as the number of players (N), multiplication factor (r) and contribution cost (c) on the derived conditions.

Firstly, the parameters used in the derivations are defined.

Generally, repeating rounds in a social dilemma game influences how agents react in successive rounds, and in many cases, cooperation may prevail in infinitely repeated games (Dal et al. 2009). Repeated interactions may be relevant since reputation, trust, reward, and punishment can then play a role (Pan et al. 2015). Therefore, the average payoff for the repeated game is defined as the average payoff that players obtain over all rounds. The autocrat's (player i) average cumulative is represented by π_i, while π_{-i} represents the weighted cumulative payoff of all opponents.

In order to successfully enforce the linear payoff relationship

$$\alpha\pi_i + \sum_{j\neq i} \beta_j\pi_j + \gamma = 0 \tag{2}$$

for a bounded function $\varphi(.)$ and fixed $\alpha, \gamma \in \mathbb{R}$, $\boldsymbol{\beta} = \{\beta_j \in \mathbb{R} | j \neq i\}$, the following equation must be satisfied

$$\alpha u_i(\boldsymbol{x}) + \sum_{j\neq i} \beta_j u_j(\boldsymbol{x}) + \gamma = \varphi(x_i) - \lambda \int_{s\in S_i} \varphi(s) d\sigma_i[\boldsymbol{x}](s) - (1-\lambda) \int_{s\in S_i} \varphi(s) d\sigma_i^0(s) \tag{3}$$

given an action profile $\boldsymbol{x} = (x_i, x_{-i}) \in S_i \times S_{-i}$, for any strategies of the autocrat's N-1 opponents, where $\sigma_i[\boldsymbol{x}]$ is player i's strategy, and σ_i^0 is their initial strategy. Here, $\sigma_i[\boldsymbol{x}](s)$ gives the probability that player i plays an action s after the action profile $\boldsymbol{x}$ was played, while $\sigma_i^0(s)$ gives the probability to apply action s in the first round of the game.

Due to the complexity of the integrals in (3), Martirosyan et al. (2020) developed a corollary to achieve simplification by limiting the autocrat's action space to two actions throughout the game (cooperate or defect), while co-players have a continuous action space to choose from.

Since an autocrat (player i) is limited to two actions (s_1, s_2) their payoff functions are given by

$$u_i(\boldsymbol{x_1}) = \frac{rc(\sum_{j\neq i} x_j + 1)}{N} - c, \qquad u_i(\boldsymbol{x_2}) = \frac{rc\sum_{j\neq i} x_j}{N} \tag{4}$$

with $1 < r < N$ and $c > 0$.

The first function in (4) portrays the utility function in the case where the autocrat cooperates and the opponents are free to choose any action from the continuous action space $\boldsymbol{x_1} = (1, x_{-i})$. However, the second function applies when the autocrat chooses to defect, given a specific action profile $\boldsymbol{x_2} = (0, x_{-i})$. Here, it is clear that the autocrat prefers others to contribute but not himself, as this yields a higher payoff function.

Re-parameterisation

For the public goods game, it is more convenient to proceed with a slightly different representation of ZD strategies. As done in Hilbe et al. (2014), the parameter transformation is as follows

$$\rho = -\sum_{j\neq i} \beta_j, \qquad \chi = \frac{\alpha}{\rho},$$
$$\omega_{j\neq i} = -\frac{\beta_j}{\rho}, \qquad l = -\frac{\gamma}{\alpha - \rho}. \tag{5}$$

where $\chi, l \in \mathbb{R}$.

This re-parametrisation then shifts the enforceable payoff relationship in (2) to take the form

$$\pi_{-i} = \chi \pi_i + (1 - \chi) l \tag{6}$$

with $\pi_{-i} = \sum_{j\neq i} w_j \pi_j$.

For simplicity, the autocrat puts an equal weight on each co-player in the game and therefore, all $w_j = \frac{1}{N-1}$, $j \neq i$.

PGG among unequals

Generally, social dilemmas occur when individuals are more or less equal, such that they all face similar consequences when they cooperate or defect (Hauser et al. 2019). However, this is not the case for a large population in the context of the PGG as different individuals have different levels of income and motivation, which affects their willingness to cooperate. Hauser et al. (2019) deduced that an unequal distribution of income levels makes it easier for full cooperation to be an equilibrium. They found that cooperation prevails when the two sources of inequality are aligned, that is when more productive individuals receive a higher income. Since this is a likely case in a normal situation, where more motivated members of the population work harder and therefore receive more in return, cooperation can still be achieved. Therefore, for simplicity, it is assumed that all players are equal and that contributions to the public goods are multiplied by a common productivity factor, r. Heterogeneous cases are not considered throughout this research as the main goal is to focus on the necessary conditions to *enforce* the ZD strategies. The derived PGG model can also be adjusted and perfectly applied to accommodate for heterogeneity between individuals by setting different parameters for each player, such as the multiplication factor (r) or contribution cost (c). If heterogeneity was to be taken into account, the updated utility function deduced by Hauser et al. (2019) could be applied, where each player's income is denoted by m_j and each player would then choose to donate a fraction of their income, x_j. Next, the multiplication factor would reflect each individual's motivation level; therefore, each player would have their own factor, r_j. The updated payoff function for the autocrat would then be as follows

$$u_i(\boldsymbol{x}) = \frac{\sum_{j=1}^{N} r_j m_j x_j}{N} + (1 - x_j)m_j.$$

5.1 Pre-derived Necessary Conditions for Each ZD Strategy

Martirosyan et al. (2020) have already derived a set of necessary conditions in order to enforce ZD strategies. First and foremost, a necessary condition for a ZD strategy to be enforced is to set χ less than the value 1. However, from (6) it is clear that in order to enforce the definition of fair strategies ($\pi_i = \pi_{-i}$), it is a requirement that $\chi = 1$, meaning that applying a fair strategy in a discounted continuous PGG is not possible. This conclusion then shifts the focus of this research on the other 3 strategies. The requirements and definitions related to each ZD strategy deduced in Martirosyan et al. (2020) are summarised in Table 3 below.

Autocratic Strategy	Parameters Value	Initial Action Requirement
Extortionate	$l = 0,\ 0 < \chi < 1$	$p_0 = 0$
Generous	$l = rc - c,\ 0 < \chi < 1$	$p_0 = 1$
Equaliser	$\chi = 0$	No requirement

Table 3: Different parameter values for each of the three ZD strategies.

Note that the equaliser strategy requires $\chi = 0$ so that the payoff relation (6) then becomes $\pi_{-i} = l$, where l is bounded by a range of payoffs.

This research will begin by deriving the necessary constraints on the required discount factor, followed by exploring the effect of changing the PGG parameters on the minimum threshold requirement on λ to enforce each ZD strategy. Next, sensitivity analysis will be conducted on the bounds on the value of l that a player i can unilaterally enforce in the equaliser strategy, in order to realise how altering parameters of the game influences the range of payoffs. Finally, non-linearity in the PGG will be explored to discover how it affects the enforceability of ZD strategies. The proofs for each derivation as well as MATLAB code for simulations and sensitivity analysis can be found in the Appendix.

5.2 Minimum Thresholds for the Discount Factor

Govaert & Cao (2019*b*) deduced that certain thresholds exist for the discount factor (λ) above which generous, extortionate, and equaliser payoff relations can be enforced in the infinitely repeated social dilemma games with a finite action space. In this section, thresholds for infinitely repeated games are derived for an *arbitrary* action space in order to find the minimum $\lambda \in [0, 1]$ under which the autocrat can enforce the linear relation with equal weights (w_j) for each ZD strategy. For each strategy, the constraint on λ is tested for a vast range of parameters (r, c, N) to explore how the minimum threshold is influenced.

Note that the autocratic strategist (i) only has possible two actions; cooperating or defecting ($s_1 = 1$ and $s_2 = 0$). After re-parametrisation of the PGG, the following probability functions were derived in Martirosyan et al. (2020)

$$p(\boldsymbol{x_1}) = \frac{1}{\lambda}(\phi\rho(\chi u_i(\boldsymbol{x_1}) - \sum_{j \neq i} \omega_j u_j(\boldsymbol{x_1}) + (1-\chi)l) - (1-\lambda)p_0 + 1) \tag{7}$$

$$p(\boldsymbol{x_2}) = \frac{1}{\lambda}(\phi\rho(\chi u_i(\boldsymbol{x_2}) - \sum_{j \neq i} \omega_j u_j(\boldsymbol{x_2}) + (1-\chi)l) - (1-\lambda)p_0) \tag{8}$$

where $p(\boldsymbol{x_i})$ is the probability of a player playing an action $\boldsymbol{x_i}$ in the next round.

Using the fact that the probability of each action must be within the unit interval and plugging in the utility functions of the PGG, the following minimum thresholds on the discount factor were derived for each of the three ZD strategies.

$$(1-\lambda)(1-p_0) \leq (\phi\rho(-\chi u_i(\boldsymbol{x_1}) + \sum_{j \neq i} \omega_j u_j(\boldsymbol{x_1}) - (1-\chi)l) \leq 1 - (1-\lambda)p_0 \tag{9}$$

$$(1-\lambda)p_0 \leq (\phi\rho(\chi u_i(\boldsymbol{x_2}) - \sum_{j \neq i} \omega_j u_j(\boldsymbol{x_2}) + (1-\chi)l) \leq \lambda + (1-\lambda)p_0 \tag{10}$$

For calculation simplicity, set

$$(-\chi u_i(\boldsymbol{x_1}) + \sum_{j \neq i} \omega_j u_j(\boldsymbol{x_1}) - (1-\chi)l) = \delta_1,$$

$$(\chi u_i(\boldsymbol{x_2}) - \sum_{j \neq i} \omega_j u_j(\boldsymbol{x_2}) + (1-\chi)l) = \delta_2$$

with $u_{i,j}$ being modified versions the utility function (1) from Section 2.1.2.

It must be noted that $\delta_{1,2}$ are different for each ZD strategy, as each strategy has different necessary conditions on the parameters (l, χ, and p_0). The definitions and pre-derived conditions mentioned and summarised in Section 5.1.

5.2.1 Extortionate Strategy

Proposition 1. In the N-player infinitely repeated PGG, the existence of the enforceable extortionate payoff relation requires λ to satisfy the following necessary conditions

$$1 - \frac{\min(\delta_1)}{\max(\delta_1)} \leq \lambda, \tag{11}$$

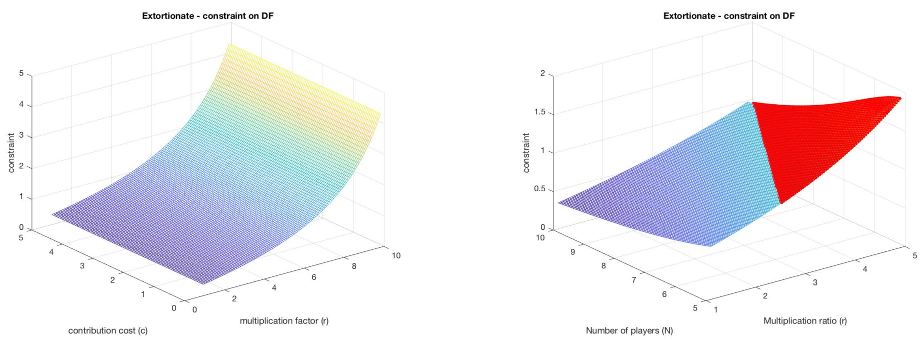
$$\frac{\max(\delta_2)}{\max(\delta_2) + \min(\delta_1)} \leq \lambda. \tag{12}$$

The proof can be found in Appendix A.

Since the discount factor must always satisfy *both* inequalities (11) and (12) for all $\delta_{1,2}$, then the minimum threshold on λ would then be

$$\lambda \geq \max\{1 - \frac{\min(\delta_1)}{\max(\delta_1)}, \frac{\max(\delta_2)}{\max(\delta_2) + \min(\delta_1)}\} \tag{13}$$

Firstly, in order to realise how the parameters r, c and N influence the minimum threshold requirement, (13) was plotted and shown in figure 1 below, fixing $\chi = 0.5$.



(a) Influence of r and c on minimum λ for a 10-player game

(b) Influence of r and N on minimum λ with fixed $c = 2$

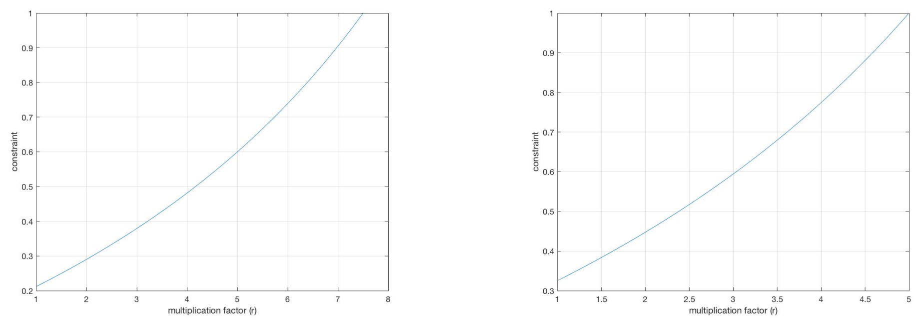
Figure 1: The effect of parameters (r, c and N) on the minimum threshold requirement of λ above which extortionate payoff relations can be enforced.

From Figure 1a, it is clear that the contribution cost does not influence the constraint. This makes sense as c is a common factor of all the terms in (13). In addition, higher values of the multiplication factor (r) yield a higher minimum threshold for λ, making the strategy more difficult to enforce and suppressing the influences of the autocrat. Since the multiplication factor represents the players' productivity level, this observation is plausible as it becomes more challenging to be extortionate when players are more productive. Since the discount factor must be $\in (0, 1)$, Figure 2a shows that when the game involves 15 players, the extortionate strategy cannot be enforced for values of r less than 7.5, while Figure 2b shows that for $N = 10$, r must be greater than 5 in order for the strategy to be enforced.

Remark 1. For an extortionate payoff, Proposition 1 sets the following necessary condition

$$1 < r < \frac{N}{2}$$

for values of $\chi > 0.1$.
Figure 1b then confirms this conditions by portraying that lower values of N cause the threshold to increase and give a smaller feasible region for r for the strategy to be enforced.

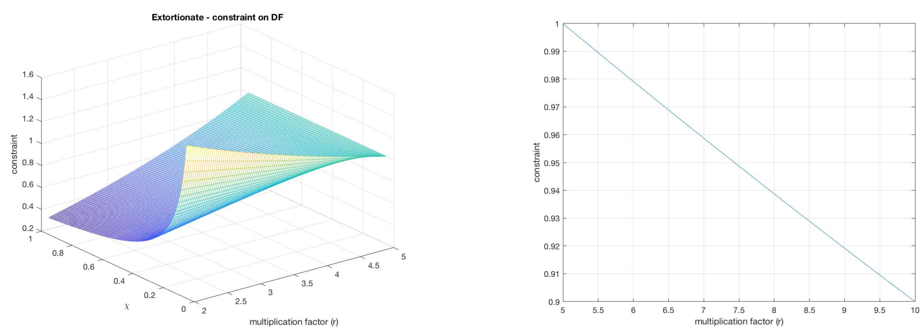


(a) Minimum threshold requirement on λ for a fixed $N = 15$

(b) Minimum threshold requirement on λ for a fixed $N = 10$.

Figure 2: Plot of r against the minimum threshold requirement on λ for a fixed $c = 2$.

The aforementioned relation was tested and deduced for a fixed $\chi = 0.5$. The influence of varying χ on the minimum threshold requirement was then explored, it is shown in Figure 3a. From the figure, it is clear that for higher values of χ, there is no significant influence on the threshold on λ. However, for values of χ very close to zero, the influence becomes more noteworthy. This makes sense as a very low value of χ in turn gives opponents a very small fraction of the total payoff, making the strategy harder to enforce. Figure 3b shows that for $\chi = 0.1$ and $N = 10$, the aforementioned necessary condition on r is reversed to $\frac{N}{2} < r < N$. From this, it is concluded that Remark 1 applies to values of $\chi > 0.1$.



(a) Minimum threshold requirement on λ with varying χ and r

(b) Minimum threshold requirement on λ for a fixed $\chi = 0.1$

Figure 3: Influence of χ and the multiplication factor (r) on the minimum threshold requirement for the discount factor (λ) for a fixed $N = 10$

5.2.2 Generous Strategy

Proposition 2. In the N-player infinitely repeated PGG, the existence of the enforceable generous payoff relation requires λ to satisfy the following necessary conditions

$$\max\{1 - \frac{\min(\delta_2)}{\max(\delta_2)}, \frac{\max(\delta_1)}{\max(\delta_1) + \min(\delta_2)}\} \leq \lambda. \tag{14}$$

The proof can be found in Appendix A.

Again, the constraint was plotted in order to find the varying minimum threshold on λ for a range of values of r, and it is shown in Figure 4 below.

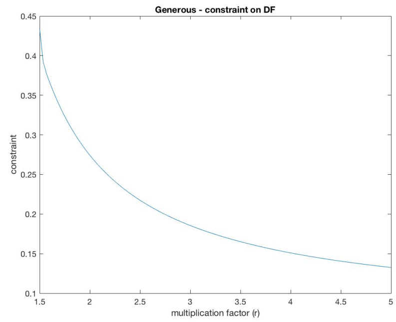
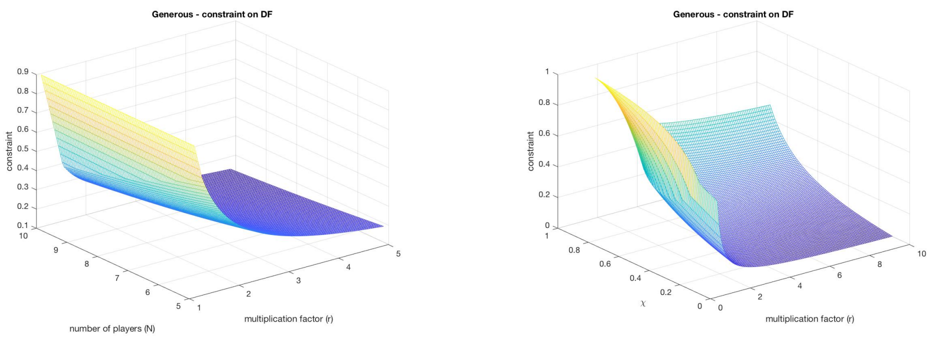


Figure 4: Influence of r on the minimum threshold on λ for $1 < r < N$ above which extortionate payoff relations can be enforced with fixed $c = 3$ and $\chi = 0.5$.

Firstly, it was deduced that the first function in (14) is always greater than the second one and is, therefore, the only function plotted in Figure 4. Next, since c is a common term in the first function, the contribution cost has no influence on the minimum threshold on λ. Therefore, the parameter c is not included in the plot. The plot shows that the generous strategy can be enforced easily for a higher multiplication factor, as the minimum threshold on λ increases as r decreases. Again, this is plausible as an autocrat can be generous and give more to co-players when they have higher productivity levels. However, since the minimum threshold remains significantly low, a generous payoff relation can easily be enforced for a wide range of parameter values.

The effect of N and χ on the minimum threshold of the discount factor are portrayed in Figure 5 below. Considering a range of values of N in Figure 5a, it can be deduced that the minimum threshold increases for higher N values, although the influence of N remains relatively small. This is realistic as it is difficult to be generous to a larger group of people.

The observations mentioned above were made from plots where the parameter χ was fixed to a value of 0.5. Through varying the value of χ, Figure 5b shows that it has a notable effect on the enforceability of the generous strategy, as larger values of χ significantly tighten the constraint on the discount factor. The highest minimum threshold on λ is when the multiplication factor is minimum, and χ is maximum, which limits the enforceability of the strategy. Since higher values of χ denote that the autocrat wants to extort a larger share of the total payoffs, this observation is reasonable as the autocrat cannot enforce the generous strategy when they want to get a large share of the payoff. Therefore, it can be concluded that the generous strategy can be easily enforced for a range of values of the aforementioned parameters, but becomes more difficult once χ values approach 1.



(a) Influence of r and N on minimum λ with fixed $\chi = 0.5$.

(b) Influence of r and χ on minimum λ for a 10-player game.

Figure 5: The effect of parameters (r, N and χ) on the minimum threshold requirement of λ above which generous payoff relations can be enforced with fixed $c = 2$.

5.2.3 Equaliser Strategy

Corollary 1. In the N-player discounted PGG with a continuous action space, the existence of the equalisers strategy requires the necessary conditions

$$p_0 \neq 0 \text{ and } p_0 \neq 1.$$

The proof can be found in Appendix B.

Since the equaliser strategy does not impose a specific requirement on the value of the baseline payoff (l), Martirosyan et al. (2020) have deduced that there is a range of values of l for which the strategy can be enforced. Before the constraints on the discount factor can be explored, the restrictions on the range of values of l must first be studied.

Restrictions on the Range of Values of l

For the equaliser strategy, an autocrat can unilaterally set their opponents' total payoff (π_{-i}) to a fixed value, denoted by l. The bounds on the value that the autocrat can assign to their opponents derived in Martirosyan et al. (2020) for finite games were found to be analogous to the case when a game is infinite. In this section, the pre-derived bounds on l are further explored to realise the influence of changing PGG parameters on the region where the equaliser strategy exists.

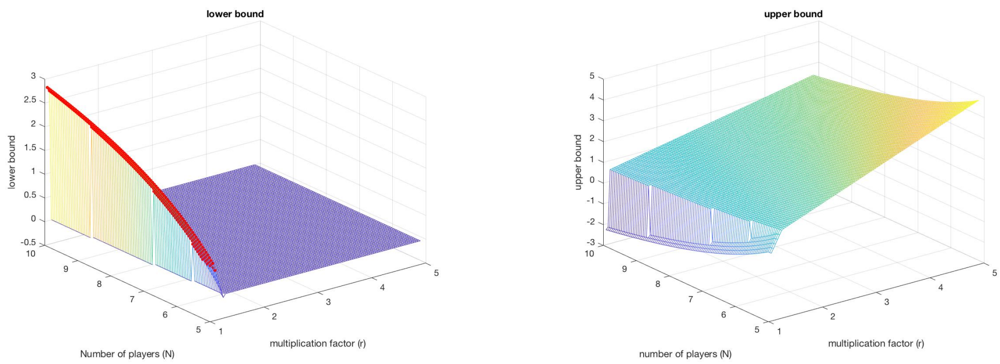
According to Martirosyan et al. (2020), there are upper and lower bounds on l, which are the payoffs to a single player if no one cooperates ($x_k = 0$) and if everyone cooperates $x_k = N$, respectively, for all $k \in [1, ..., N]$. They deduced that the equaliser strategy only exists for the baseline payoff (l) satisfying the following condition

$$\max\{0, c\sum_{j\neq i}(w_j - \frac{r}{N})\mathbb{1}(w_j - \frac{r}{N})\} \leq l \leq \min\{rc - c, c\sum_{j\neq i}(\frac{r}{N} - w_j)\mathbb{1}(w_j - \frac{r}{N}) + \frac{rc}{N}\} \tag{15}$$

Note that the indicator function is used in this section and is defined as

$$\mathbb{1}(y) = \begin{cases} 1 & y \geq 0, \\ 0 & y < 0. \end{cases}$$

Equation (15) was then plotted into MATLAB in order to explore the influence of varying the parameters N and r on the bounds on l and the existence of the strategy. The parameter c was not studied since it is clearly a common term in (15).



(a) Lower bound on l for a range of values of r and N.

(b) Upper bound on l for a range of values of r and N.

Figure 6: Lower and upper bound on the baseline payoff, l. By setting all $w_j = \frac{1}{N-1}$ and fixing $c = 5$, the influence of the parameters r and N on the bounds can be observed.

Logically, l does not exist when the lower bound is strictly greater than the upper bound. From the plots in Figure 6, there is a region where the upper bound on l is lower than the lower bound. The red shaded area in Figure 6a shows the values of r for which the equaliser strategy cannot be enforced. However, this infeasibility only occurs for values of r in the range $(1, \frac{10}{9}]$. Generally, it is for values of r where the indicator function in (15) is positive ($r < w_j \times N$), as the lower bound maximises, limiting the feasibility region. This in turn deduces the following remark

Remark 2. For an equaliser payoff, if all co-players are given weights i.e. $w_j = \frac{1}{N-1}$, the strategy requires the following necessary condition

$$r \geq \frac{N}{N-1}.$$

Through varying the parameters r and N, it was deduced that smaller values of r yield a higher lower bound and therefore decrease the feasible region. The lower bound is at its absolute minimum when the indicator function is negative ($r \geq \frac{N}{N-1}$), as this sets the indicator function on the RHS of the inequality to zero. On the other hand, the upper bound is at its maximum when the multiplication factor increases, and fewer players join the game. This, in turn, widens the range of l that an autocrat can assign to their opponents, giving plenty of space where the equaliser strategy exists.

Minimum Thresholds for the Discount Factor to Enforce the Equaliser Strategy

Proposition 3. For a multitude of p_0 regions in within the unit interval, the following constraints set the minimum threshold discount factor to enforce the equaliser strategy

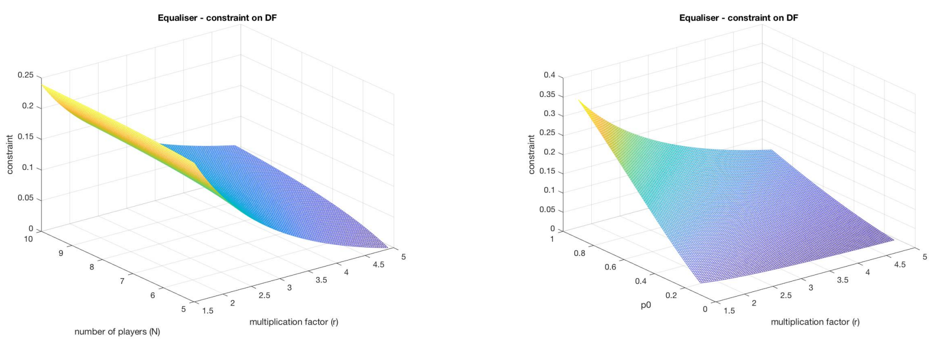
$$\lambda \geq \frac{\max(\delta_1)(1-p_0)+\min(\delta_1)(p_0-1)}{\min(\delta_1)p_0-\max(\delta_1)(p_0-1)}, \qquad \lambda \geq \frac{p_0(\max(\delta_2)-\min(\delta_2))}{\min(\delta_2)+p_0(\max(\delta_2)-\min(\delta_2))}$$

with $p_0 \neq 0$, 1.
The proof can be found in Appendix B.

The parameter l has upper and lower bounds that are also influenced by the parameters of the PGG. Therefore, the minimum threshold on λ is calculated twice, using the minimum and maximum possible values of l, respectively.

Since the lower bound is set to 0 while c is a common term in the upper bound of l in (15), the contribution cost, therefore, does not influence the constraint. Next, Figure 7 shows that when the autocrat is somewhat generous and attempts to give opponents a cumulative payoff equal to the upper bound of l, the strategy is easy to enforce as the minimum threshold on the discount factor remains low for a range of values of the parameters.

From Figure 7a, it is clear that the constraint on λ tightens as the number of players increases, meaning that enforceability of the strategy is more difficult as more players are involved in the game. This agrees with Pan et al. (2015), where it was found that the capacity of a ZD player to either pin or extort other opponents is more strictly limited compared with the two-player games. Figure 7b shows that it becomes easier to give opponents a high payoff for increasing r values, whereas higher p_0 values increase the minimum threshold. Since r represents the players' motivation level, this is plausible since higher motivation makes it easier for the autocrat to assign a large payoff to the opponents.



(a) Influence of r and N on minimum λ with fixed $p_0 = 0.5$

(b) Influence of r and p_0 on minimum λ for a 10-player game

Figure 7: The effect of parameters on the minimum threshold requirement of λ above which equaliser payoff relations can be enforced for the maximum possible values of l.

On the other hand, Figure 8 below shows that the enforceability of the strategy becomes more difficult when the autocrat attempts to set the opponents' payoff to lower values of l. When the autocrat attempts to accumulate their co-players payoff to the lower bound of l i.e., $l = 0$, the minimum threshold for λ is always greater than 1, which means that an equaliser payoff relation becomes unenforceable.

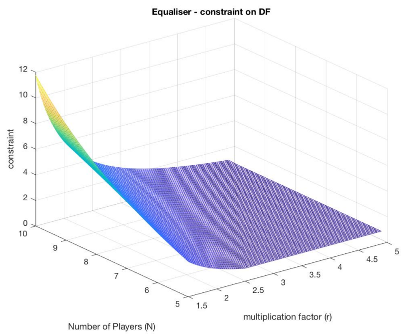


Figure 8: Influence of r and N on the minimum threshold on λ for above which equaliser payoff relations can be enforced with fixed $c = 3$ and $p_0 = 0.1$.

The aforementioned observation can also be deduced analytically. Firstly, the inequality that sets the bounds on l for all ZD strategies (derived from the probability functions) is as follows

$$\max\{u_i(x_2) - \sum_{j \neq i} w_j \frac{u_i(x_2) - u_j(x_2)}{1-\chi}\} \leq l \leq \min\{u_i(x_1) + \sum_{j \neq i} w_j \frac{u_j(x_1) - u_i(x_1)}{1-\chi}\}. \quad (16)$$

Through substituting the definition of χ for the equaliser strategy ($\chi = 0$) into (16), setting the accumulated co-players payoff (l) to 0 in turn yields the following constraint on r

$$r \leq \frac{N}{N-1}. \quad (17)$$

The proof for (16) and (17) can be found in Appendix C.

However, it has already been deduced in Remark 2 that the equaliser strategy is not enforceable for those values of r, which analytically proves that the equaliser payoff relation cannot be enforced if the autocrat attempts make the co-players receive nothing.

6 Non-linear Payoff Functions in the PGG

In this section, the PGG is modified to generate a non-linear payoff function by introducing a minimum entry threshold requirement to motivate more players to cooperate. Here, a public good is provided if and only if the sum of all contributions ($\sum_{j=k}^{N} x_k$) is less than the minimum donation amount (z). Players who did not contribute lose and receive nothing, while players who did contribute will lose their money.

Firstly, the utility functions in Section 5 are updated. If $\sum_{j=k}^{N} x_k \leq z$, the utility functions become

$$\begin{aligned} u_i(\boldsymbol{x_1}) &= -c, & u_j(\boldsymbol{x_1}) &= -cx_j, \\ u_i(\boldsymbol{x_2}) &= 0, & u_j(\boldsymbol{x_2}) &= -cx_j. \end{aligned} \tag{18}$$

However, if $\sum_{j=k}^{N} x_k \geq z$, the standard utility functions can be applied to all players.

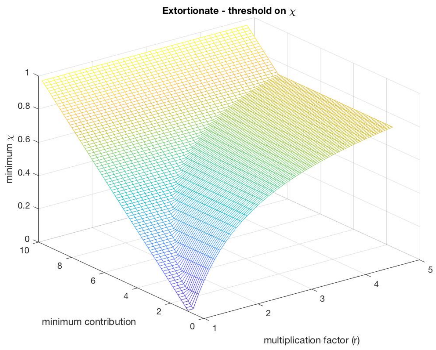
Using the bounds on l in (16) and substituting the definitions of l for each strategy (listed in Table 1) into the functions, minimum thresholds for χ for which each strategy can be enforced were derived. Through substitution of the utility functions into (16) for cases when the minimum cooperator threshold is met or not met, minimum thresholds on χ are derived for each of the ZD strategies and listed in Table 4 below. The proof can be found in Appendix D.

ZD strategy	Minimum threshold on χ
Extortionate	$\chi \geq \max\{\frac{z-1-\epsilon}{N-1} + \epsilon, 1 - \frac{N}{r(N-1)}\}$
Generous	$\chi \geq 1 - \frac{N-z+\epsilon}{r(N-1)}$
Equaliser	Not enforceable

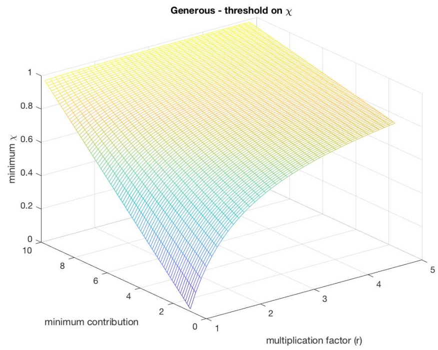
Table 4: Different minimum threshold requirements on χ for each of the ZD strategies to be enforced when the PGG is non-linear. Note that ϵ denotes an infinitesimal number greater than 0.

Since the inequality (16) is analogous to the one derived in Govaert & Cao (2019*b*) but modified to apply to infinite rather than finite games, the requirement that *at least one* inequality must be strict also applies in this situation. For the extortionate strategy, two minimum thresholds for χ were derived, therefore, in order to ensure one is strict, an infinitesimal number, denoted by ϵ, is added to the first inequality. Furthermore, equaliser strategies set a requirement that $\chi = 0$, which in turn generates an unrealistic condition on l to be a negative value. In addition, the deduced upper and lower bounds on l do not intersect. Therefore, equalising ZD strategies do not exist when the PGG is non-linear.

The thresholds listed in Table 4 were plotted in order to deduce the effects of varying r and z on the enforceability of each of the two enforceable ZD strategies. The plots are shown in Figure 9 below.



(a) Minimum threshold for χ to enforce the extortionate strategy



(b) Minimum threshold for χ to enforce the generous strategy

Figure 9: The influence of varying r and z values on the minimum threshold for χ to enforce the extortionate and generous ZD strategies for a fixed 10-player game.

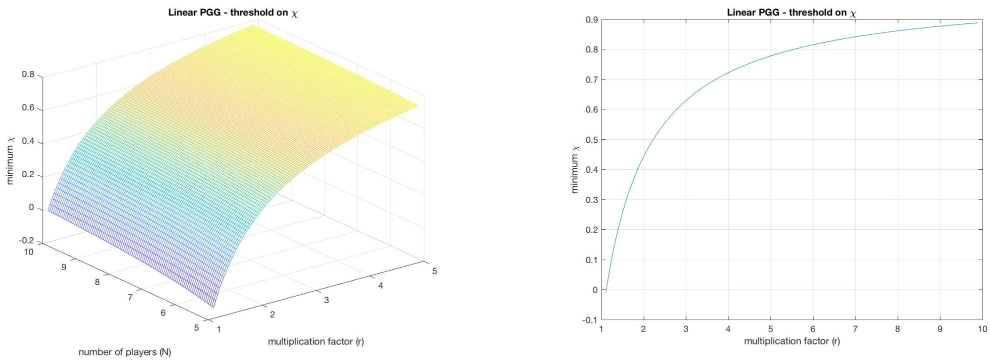
It is clear that there is limited freedom to choose the parameter χ when enforcing both strategies and that the minimum threshold increases as r and z increase. This was expected as the enforceability of the strategies becomes more difficult as the minimum contribution threshold increases. For example, when the minimum contribution is equal to 10 (all players must contribute), the minimum χ approaches the value 1, making the strategies almost unenforceable.

Next, the necessary conditions on χ when the PGG is linear were compared to the case where the PGG is non-linear in order to realise how a non-linear payoff function influences the enforceability of the strategies. Plugging in the linear utility functions into (16) yields the same minimum threshold on χ for both the extortionate and generous strategies; they are shown in Table 5 below. The proof can be found in Appendix D.

ZD strategy	Minimum threshold on χ
Extortionate	$\chi \geq 1 - \frac{N}{r(N-1)}$
Generous	$\chi \geq 1 - \frac{N}{r(N-1)}$
Equaliser	$\chi = 0$

Table 5: Minimum threshold requirement on χ when the PGG is linear for the extortionate and generous ZD strategies.

For the extortionate and generous strategies, the constraint on χ is overall looser and reaches a lower minimum threshold value than the non-linear case. Figure 5a below shows that the minimum threshold increases with higher values of r and N, although the multiplication factor has a more significant influence, further explored in Figure 5b for a 10-player game.



(a) Minimum threshold for χ to enforce the strategies with varying r and N values

(b) Minimum threshold for χ to enforce the strategies for a 10-player game

Figure 10: The influence of varying parameter values on the minimum threshold for χ to enforce the extortionate and generous ZD strategies.

Regarding the equaliser case, it was found to be unenforceable when the PGG is non-linear since it requires the value of χ to be set to 0. However, this can be achieved when the linear PGG utility functions are used, meaning that the equaliser strategy can also be enforced for the linear case. This shows that non-linearity hinders the enforceability of all three ZD strategies. They can be enforced more easily when the PGG is linear, meaning that there is no required minimum donation amount.

7 Discussion and Conclusions

This research project explores zero-determinant strategies to extend on pre-existing research and apply it to multiplayer, infinitely repeated games with an arbitrary action space. It focuses on ZD strategies to determine the influence of discounting future rewards and non-linearity on the enforceability of each of the three main strategies in the context of the public goods game.

Firstly, the discount factor is used to discount future rewards to deduce a minimum threshold on λ for each strategy. For all three strategies, it was found that for the public goods game, when all players have the same cost of contribution, the said parameter does not influence on the enforceability of the strategies as it is a common term in the derived inequalities.

Regarding the extortionate strategy, the derived constraints on the discount factor deduce that the productivity level (represented by the multiplication ratio), number of players and the slope of the strategy influence the minimum threshold requirement. Firstly, higher productivity levels of the co-players make it harder to extortionate goods. Next, through sensitivity analysis, it was deduced that there is a restrictive condition on the multiplication factor, which is dependant on the number of players. Here, if more players are involved in the game, the range of productivity level values can be greater, and in turn, there are more possibilities to extortionate the players' payoffs. In addition, the slope of the extortionate strategy was found to have a minor influence on the enforceability of the strategy, which becomes more significant as the slope gets closer to 0. When the slope is minimal, the autocrat attempts to get everything, so it becomes difficult to enforce the strategy.

On the contrary, the generous strategy was found to have the opposite characteristics. Here, the strategy becomes harder to enforce as more players join the game, or when players have a lower motivation level. Furthermore, the slope of the generous strategy has a significant impact on the enforceability compared to the other parameters.

Regarding the equaliser strategy, the influence of the parameters on the upper and lower bounds of the baseline payoff that an autocrat can assign to the co-players was initially explored. From this, it was deduced that an equaliser payoff relation is unenforceable for relatively low productivity levels. Enforceability can only be achieved for values of the multiplication factor above a particular value, which is solely dependant on the number of players involved in the game, assuming that all players are given equal weight. Once this is achieved, the enforceability of the strategy is then dependant on the minimum threshold on the discount factor. Similarly to the generous strategy, the strategy is more difficult to enforce as more players join the game or productivity levels of the co-players decreases. Furthermore, it was deduced that if the autocrat wants to equalise the co-players to a large payoff and their productivity level is high, there is no need to contribute much in the beginning. However, the autocrat would need to contribute more if they want to equalise the opponents' payoff to the lower bound of the baseline payoff. Finally, for lower productivity levels, there is more chance that an autocrat would need to initially contribute to compensate for the players' lack of productivity.

Finally, the non-linear PGG is studied through the introduction of a minimum donation amount that must be met so the players can receive a share of the total donation. Results show that it is impossible to enforce the equaliser relationship between players' payoffs when the PGG is non-linear. In addition, the enforceability of the remaining two strategies is compared to the case in which the PGG is linear. The enforceability of non-linear games is more limited due to a relatively high minimum threshold requirement for the slope of both strategies, which depends on the minimum donation amount, the productivity level, and the number of players involved in the game.

Validation

The outcome of this research deduces how much control an autocrat has on their opponents in social dilemmas. It explores the effect of common influences on real-life dilemmas such as discounting and non-linearity, on the enforceability of ZD strategies, and finds that these factors introduce new necessary conditions on the game parameters. In turn, the extent to which an autocrat can still enforce the ZD strategies then depends on the values that are assigned to each of the parameters.

Once background knowledge and pre-derived conditions to enforce ZD strategies are thoroughly analysed, the deliverable is achieved by first delving into the three different ZD strategies, deducing the minimum thresholds on different game parameters in order enforce each strategy successfully. Next, sensitivity analysis explores the influence of the remaining parameters on the enforceability of each strategy to find the range of values that can be assigned to each parameter and ensure that the autocrat remains in control. This results in new necessary conditions and restrictions on the range of parameters in order to enforce the ZD strategies.

Further Research and Application

This research explores the enforceability of the ZD strategies in the context of the multi-player public goods game with an arbitrary action space. One direction of future work would be to explore how a broader set of available actions to the autocrat (rather than limiting it only to two actions) would influence the amount of control they have over their opponents through different methods of discretisation of the autocrat's action space.

Also, the conclusions made in this research could be further validated in another similar social dilemma game, such as the iterated snowdrift game, where a group of drivers are trapped in a snowdrift and have the option of staying in the car or removing the snowdrift (Doebeli 2004). This would allow the generalisation of results to assess compatibility with other dilemmas and, in turn, facilitate applicability in a broader range of domains. Once the findings are generalised, governments, economists, and other autocratic bodies could use these requirements to remain in control and prevent free-riding.

8 Appendix

8.1 Appendix A

Extortionate Strategies

It is known that for the existence of extortionate strategies, it is necessary that $p_0 = 0$. Furthermore, $l = 0$ in $\delta_{1,2}$ since this is an extortionate strategy. The first inequality (9) then simplifies to

$$\frac{1-\lambda}{\delta_1} \leq \phi\rho \leq \frac{1}{\delta_1} \tag{A.1}$$

However, the inequality (A.1) must be satisfied for *all* possible values of δ_1. Therefore, the minima and maxima of the bounds are as follows

$$\frac{1-\lambda}{min(\delta_1)} \leq \phi\rho \leq \frac{1}{max(\delta_1)}$$

In order for such a $\phi\rho\delta_1$ to exist, it needs to hold that

$$\frac{1-\lambda}{min(\delta_1)} \leq \frac{1}{max(\delta_1)} \tag{A.2}$$

By extracting λ, the first condition is therefore

$$1 - \frac{min(\delta_1)}{max(\delta_1)} \leq \lambda$$

The same procedure is done for the second inequality (10), obtaining

$$0 \leq \phi\rho \leq \frac{\lambda}{max(\delta_2)} \tag{A.3}$$

Combining the RHS of (A.3) with the LHS of (A.2) and finding the maximum and minimum bounds gives

$$\frac{1-\lambda}{min(\delta_1)} \leq \phi\rho\delta_2 \leq \frac{\lambda}{max(\delta_2)}$$

Therefore

$$\frac{1-\lambda}{min(\delta_1)} \leq \frac{\lambda}{max(\delta_2)},$$

Giving the second constraint

$$\frac{max(\delta_2)}{max(\delta_2) + min(\delta_1)} \leq \lambda$$

The procedure to find constraints on λ for the other two strategies are analogous to the aforementioned one.

Generous Strategies

Considering generous strategies with $p_0 = 1$ and $l = rc - c$ in $\delta_{1,2}$, the two following inequalities are derived

$$\frac{1-\lambda}{min(\delta_2)} \leq \phi\rho \leq \frac{1}{max(\delta_2)}, \tag{A.4}$$

$$0 \leq \phi\rho \leq \frac{\lambda}{max(\delta_1)} \tag{A.5}$$

By combining the LHS of (A.4) with the RHS (A.5) and finding the inequalities in order for such a $\phi\rho\delta_1$ to exist and , we obtain the two minimum thresholds for λ in (14).

8.2 Appendix B

Equaliser Strategies

Plugging δ_1 into (9) with $\chi = 0$ and $p_0 = [0, 1]$ and rearranging yields

$$(1-\lambda)(1-p_0) \leq \phi\rho\delta_1 \leq 1-(1-\lambda)p_0 \tag{B.1}$$

Focusing on the RHS and LHS respectively and rearranging to extract λ yields the following constraints

$$\frac{\phi\rho\delta_1 - 1 + p_0}{p_0} \leq \lambda,$$

$$1 - \frac{\phi\rho\delta_1}{1-p_0} \leq \lambda \tag{B.2}$$

This in turn shows that for $p_0 = 0$ or $p_0 = 1$, λ must be greater than ∞, therefore an equaliser payoff relation cannot be enforced.

Next, considering equaliser strategies with $\chi = 0$ in $\delta_{1,2}$ and $p_0 = [0, 1]$, (9) gives

$$\frac{(1-\lambda)(1-p_0)}{min(\delta_1)} \leq \phi\rho \leq \frac{1-(1-\lambda)p_0}{max(\delta_1)},$$

$$\frac{(1-\lambda)(1-p_0)}{min(\delta_1)} \leq \frac{1-(1-\lambda)p_0}{max(\delta_1)} \tag{B.3}$$

Similarly, (10) gives

$$\frac{(1-\lambda)p_0}{min(\delta_2)} \leq \frac{\lambda+(1-\lambda)p_0}{max(\delta_2)} \tag{B.4}$$

The discount factor, λ is then extracted from (B.3) and (B.4) in order to find the functions for the minimum threshold.

8.3 Appendix C

Bounds on l for All ZD Strategies

Rearranging the inequalities (9) and (10), the following is deduced

$$\begin{aligned}(1-\lambda)(1-p_0) \leq (\phi\rho(\sum_{j\neq i}\omega_j(u_j(\boldsymbol{x_1})-u_i(\boldsymbol{x_1}))-(1-\chi)(l-u_i(\boldsymbol{x_1})) \leq 1-(1-\lambda)p_0 \\ (1-\lambda)p_0 \leq (\phi\rho((1-\chi)(l-u_i(\boldsymbol{x_2}))-\sum_{j\neq i}\omega_j(u_j(\boldsymbol{x_2})-u_i(\boldsymbol{x_2})) \leq \lambda+(1-\lambda)p_0\end{aligned} \tag{C.1}$$

Since $0<\lambda<1$ and $0\leq p_0\leq 1$, it is clear that the LHS of both inequalities in (C.1) are greater than or equal to zero. This in turn yields

$$\begin{aligned}0 &\leq (\phi\rho(\sum_{j\neq i}\omega_j(u_j(\boldsymbol{x_1})-u_i(\boldsymbol{x_1}))-(1-\chi)(l-u_i(\boldsymbol{x_1})) \\ 0 &\leq (\phi\rho((1-\chi)(l-u_i(\boldsymbol{x_2}))-\sum_{j\neq i}\omega_j(u_j(\boldsymbol{x_2})-u_i(\boldsymbol{x_2}))\end{aligned} \tag{C.2}$$

Rearranging both inequalities to extract l then deduces the upper and lower bounds

$$u_i(x_2)-\sum_{j\neq i}w_j\frac{u_i(x_2)-u_j(x_2)}{1-\chi} \leq l \leq u_i(x_1)+\sum_{j\neq i}w_j\frac{u_j(x_1)-u_i(x_1)}{1-\chi} \tag{C.3}$$

Since this must be valid for *all* possible utility functions, l must be greater or equal to the *maximum* of the LHS and less than or equal to the *minimum* possible value of the RHS, deducing the final inequality in (16).

Equaliser Strategy Case

Plugging the utility functions into the RHS of (16) and setting $\chi=0$ with $\omega_j=\frac{1}{N-1}$ yields

$$l \geq \max\{\frac{rc\sum_{j\neq i}x_j}{N}-\frac{c\sum_{j\neq i}x_j}{N-1}\}, \tag{C.4}$$

where the maximum value occurs when $\sum_{j\neq i}x_j$ is maximum i.e. $\sum_{j\neq i}x_j = N-1$. Therefore, (C.4) becomes

$$l \geq \frac{rc(N-1)}{N}-c. \tag{C.5}$$

Setting $l=0$ to set the co-players' payoff to the minimum possible value in turn yields the condition on r in (17).

8.4 Appendix D

Enforceability with Non-linear Payoff Functions

Assuming that the minimum cooperator threshold is not met, substituting the payoff functions into the RHS of (18) yields

$$l \leq \min\{-c + \frac{\frac{-c\sum_{j\neq i} x_j}{N-1} + c}{1-\chi}\} \tag{D.1}$$

minimum value when $\sum_{j\neq i} x_j$ is maximum, i.e. $\sum_{j\neq i} x_j = z - 1 - \epsilon$. Using the definitions of l for the extortionate and generous strategies in turn yields the constraints on χ to enforce the strategies when the PGG is non-linear.

Enforceability with Linear Payoff Functions

Assuming that the minimum cooperator threshold is met, the LHS of (18) then becomes

$$l \geq \max\{\frac{rc\sum_{j\neq i} x_j}{N} + \sum_{j\neq i} \omega_j \frac{\frac{rc\sum_{j\neq i} x_j}{N} - \frac{rc\sum_{j\neq i} x_j}{N} + cx_j}{1-\chi}\}. \tag{D.2}$$

After cancelling out terms, rearranging and using the definition $\omega_j = \frac{1}{N-1}$

$$l \geq \max\{\frac{rc\sum_{j\neq i} x_j}{N} + \frac{\frac{c}{N-1}\sum_{j\neq i} x_j}{1-\chi}\}, \tag{D.3}$$

$$l \geq \max\{\frac{c(\sum_{j\neq i} x_j(r(1-\chi)(N-1)-N)}{N(N-1)(1-\chi)}\}, \tag{D.4}$$

where the maximum value is when $\sum_{j\neq i} x_j$ is maximum, i.e. $\sum_{j\neq i} x_j = z - 1 - \epsilon$. Substituting this value into (D.4) and using the definitions of l in turn yields the constraint for the extortionate strategy when the PGG is linear.

Next, the RHS of (18) becomes

$$l \geq \min\{\frac{rc(\sum_{j\neq i} x_j + 1)}{N} - c + \sum_{j\neq i} \omega_j \frac{\frac{rc(\sum_{j\neq i} x_j+1)}{N} - c\sum_{j\neq i} x_j - \frac{rc(\sum_{j\neq i} x_j+1)}{N} + c}{1-\chi}\}. \tag{D.5}$$

Again, rearranging and using the definition $\omega_j = \frac{1}{N-1}$ yields

$$l \geq \min\{\frac{rc(\sum_{j\neq i} x_j + 1)}{N} - c + \frac{c - \frac{c\sum_{j\neq i} x_j}{N-1}}{1-\chi}\}, \tag{D.6}$$

where the minimum value is when $\sum_{j\neq i} x_j$ is minimum, i.e. $\sum_{j\neq i} x_j = 0$. Substituting this value into (D.6) and using the definitions of l in turn yields the constraint for the generous strategy when the PGG is linear.

8.5 Appendix E

```
%FIX N and Chi, VARY r and c
%l = 0

N=10;
r=linspace(1.01,9.9,100);
c =linspace(0,5,100);
chi=0.5; % 0 < chi < 1
[rr,cc] = meshgrid(r,c);

Term1= (chi.*rr.* cc)/N;
Term2= (rr.*cc)/(N*(N-1));
Term3= cc./(N-1);

xj = ones(1,100);
[xjj] = meshgrid(xj);

mindelta1 = (-Term1 + Term2 - Term3).*xjj - Term1 + chi.*cc + Term2;
maxdelta1 = - Term1 + chi.*cc + Term2;
maxdelta2 = (Term1 - Term2 + Term3).*xjj;

Constraint1 = ones(100,100) - (mindelta1./maxdelta1);
Constraint2 = maxdelta2./(maxdelta2 + mindelta1);

A = max(Constraint1, Constraint2);

figure
mesh(rr,cc,A)
xlabel('multiplication factor (r)')
ylabel('contribution cost (c)')
zlabel('constraint')
title('Extortionate - constraint on DF')
```

Listing 1: MATLAB code to derive and plot the minimum threshold for λ in the extortionate strategy when varying r and c. The code is modified to fix and vary the other parameters and altered to work with the other two strategies.

```
%FIX c, VARY N and r
n = linspace(5,10,100); %number of players
r = linspace(1.05,4.9,100); %multiplication ratio
wj = 1./(n-1); %weighting factor
c = 5; %contribution cost
l=rr.*c-c;

[rr,N] = meshgrid(r,n);

xj=1./(N-1)-rr./N;
xj(xj>=0)=1;
xj(xj<0)=0;

s=0;
for i = 1:N-1
    s = s + (r./N - wj(i));
end
s1=0;
for i = 1:N-1
    s1 = s1 + (wj(i) -r./N);
end

lowerbound = c.*s1.*xj;
upperbound = c.*s.*xj + (rr.*c)./N;
ub = min(upperbound, l);

figure
mesh(rr,N,lowerbound)
xlabel('multiplication factor (r)')
ylabel('Number of players (N)')
zlabel('lower bound')
title('lower bound')
hold on %highlight points where the lower bound is > upper bound (equaliser strategy is
    invalid)
idx = lowerbound>=ub ;
plot3(rr(idx),N(idx),lowerbound(idx),'.r','markersize',10)

figure
mesh(rr,N,ub)
xlabel('multiplication factor (r)')
ylabel('number of players (N)')
zlabel('upper bound')
title('upper bound')
```

Listing 2: MATLAB code to derive and plot the upper and lower bounds on l in the equaliser strategy.

```
%FIX N, VARY r and z

N=10;
r=1.01:0.1:4.9;
z=1.01:0.1:9.9;
e = 0.0001;

[rr,zz] = meshgrid(r,z);

term1 = zz - 1 - e;
term2 = N - 1;
term3 = rr.*(N-1);

chi1 = term1./term2 ;
chi2 = 1 - N./term3 ;
chi = max(chi1,chi2);

figure
mesh(rr,zz,chi)
% set(gca, 'XDir','reverse')
xlabel('multiplication factor (r)')
ylabel('minimum contribution')
zlabel('minimum {\chi}')
title('Extortionate - threshold on {\chi}')
```

Listing 3: MATLAB code to derive and plot the minimum threshold for χ in the non-linear extortionate strategy when varying r and z. The code is modified to work for the generous strategy as well as for the linear case.

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