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Variations of the Lorenz-96 Model

Master Thesis in Mathematics:
Mathematics and Complex Dynamical Systems

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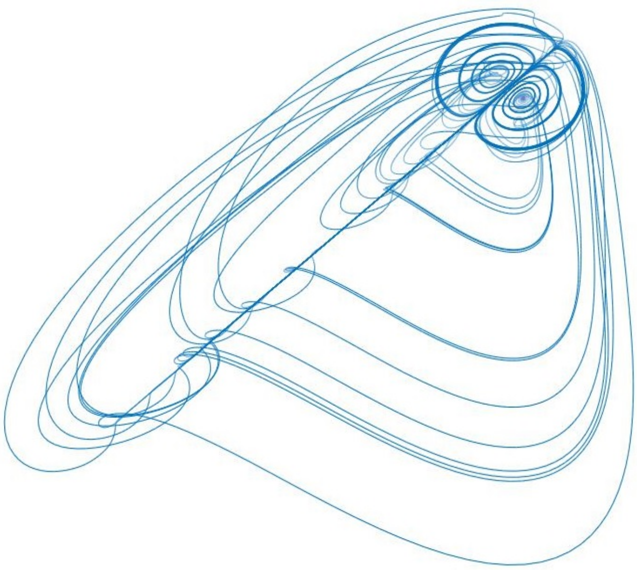

Abstract. In this thesis we will investigate the differences between the monoscale Lorenz-96 model and its modifications. We obtain these modifications by changing the structure of the nonlinear terms in the original Lorenz-96 model. We will treat the general modified model and three specific systems and compare these with the Lorenz-96 model. To determine the dynamics of the models, we will study the eigenvalues of the Jacobian matrix at the trivial equilibrium, Lyapunov coefficients to distinguish between sub- and supercritical Hopf bifurcations, Lyapunov exponents to determine when there is chaos for example etc. Some of the results are that in the Lorenz-96 model there are no escaping orbits, while this is only true under some conditions for the modified systems. Furthermore there are differences in bifurcations. The external forcing F appearing in the equations of all these models can be used as a bifurcation parameter. The trivial equilibrium is always stable when F is zero for every model. For $F < 0$ some modifications has a stable trivial equilibrium, while this is not the case for the original model. For this model there only occur first Hopf bifurcations when F is positive, but for its modifications there can be other (first) bifurcations too, like a Pitchfork bifurcation. There are even bifurcations occuring for these modified systems, while these don't appear in the original Lorenz-96 model case, such as degenerate bifurcations meaning that suddenly there appear simultaneously more than one equilibrium and two eigenvalues of the Jacobian matrix are zero at the same bifurcation parameter F.

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1. Introduction

The Lorenz-96 model, constructed by Edward Norton Lorenz, is used to study the predictability of the atmosphere [Lor06a]. There exists a multiscale version of it besides the monoscale version, the reader can find the information about the multiscale version in the thesis [vK18]. In this thesis we only focus on the monoscale lorenz-96 model, which is a system with dimension $n \in \mathbb{N}$ and is given by

$$\dot{x}_j = x_{j-1}(x_{j+1} - x_{j-2}) - x_j + F, \;\; j \in \{0, 1, \ldots, n-1\},$$

where $x_{j-n} = x_{j+n} = x_j$ and the constant $F \in \mathbb{R}$. Note that the dimension n and the external forcing F are free parameters. The forcing F can be used as a bifurcation parameter.

The variables x_j in the model can be seen as values of atmospheric quantity, such as temperature and pressure. These are measured along a latitude circle of the earth, see Figure 1. As can been seen in this figure, this circle is divided into n equal parts, or sectors, and the j-th part is associated with the variable x_j for all $j = 0, 1, \ldots, n-1$.

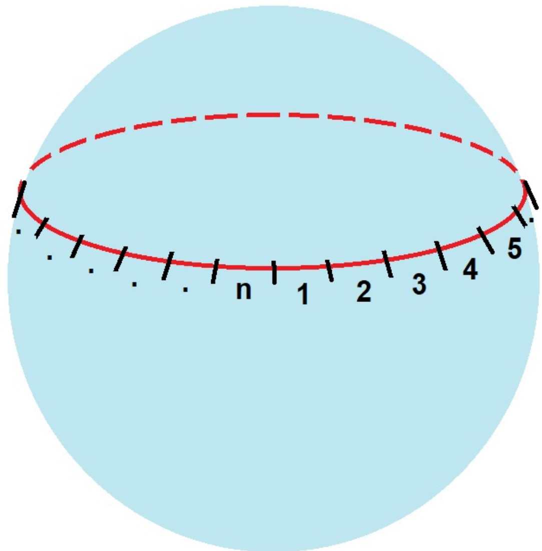

Figure 1. An example of a latitude circle of the earth divided into n equal sectors.

The Lorenz-96 model is not constructed to be a physically realistic model, but it is not complicated and it is easy to use for numerical experiments. The aim of this model is to study the issues about the predictability of the atmosphere and the weather forecasting. However this model has indeed the following physical components. The advection, that conserves the total energy, is determined by the quadratic terms in the model. Moreover the damping, which causes that the energy decreases, is obtained by the linear terms and the external forcing, which keeps the total energy away from zero, is represented by the constant terms, which is F. In Section 3 we will summarize the dynamics of this Lorenz-96 model following the thesis [vK18] and the articles [vKS18a] and [vKS18b]. Before that we will describe the life of Edward Norton Lorenz shortly at the end of this Introduction using the thesis [vK18].

The monoscale Lorenz-96 model has many different applications. It is used for data assimilation and predictability in spatiotemporal chaos and to test new ideas in different areas, like studying the atmosphere and the associated problems in the articles by Lorenz & Emanuel [LE98] and Lorenz [Lor06a] and [Lor06b]. It is also used for the predictibility of extremes in the article by Sterk & Van Kekem [SvK17] and for making and improving forecasts in the papers by Danforth & Yorke [DY06] and by Basnarkov & Kocarev [BK12]. In addition, the model is used for the dynamical systems to study high-dimensional chaos. In conclusion, the Lorenz-96 model is used for many applications in various areas.

The main difference between the use of the Lorenz-96 model in this thesis and in other articles is that we don't only study the dynamics of the Lorenz-96 model, but the dynamics of its modifications too. There are differences between the dynamics of the original Lorenz-96 model and its modifications. Furthermore we discovered some new results about these modified models and this is not done in previous work. So the purpose of this thesis is to modify the Lorenz-96 model and to study the dynamical properties of the modified systems. It is worth to understand these modified models in precise mathematical details and to obtain new good results. There are still open questions about these modified systems, which are described in Section 6. These can be investigate further in the future.

We change the Lorenz-96 model by replacing the indices of the variables x_j. So we introduce the general system for some $\alpha, \beta, \gamma \in \mathbb{Z}$, such that α, β and γ are between $-n$ and n, and this system is

$$\dot{x}_j = x_{j+\alpha}(x_{j+\beta} - x_{j+\gamma}) - x_j + F, \;\; j \in \{0, 1, \dots, n-1\},$$

where again $x_{j-n} = x_{j+n} = x_j$ and $F \in \mathbb{R}$. Thus the main research question is: **What are the dynamical properties of the modified models in comparison with the original monoscale Lorenz-96 model?**

In Section 4 we will treat the dynamical properties of the general system in detail for specific α, β and γ using the references [vK18], [vKS18a] and [vKS18b]. One of the main results is that for any $F \in \mathbb{R}$ the eigenvalues of the Jacobian matrix at the trivial equilibrium $(F, F, \dots, F)$ do not depend on α. Further we will discuss Hopf bifurcations in this situation. At a Hopf bifurcation an equilibrium point loses stability and then there appears a periodic orbit. This periodic orbit can be interpreted as a travelling wave [vKS18a]. One observation from the numerical experiments is that the wave number never decreases as the dimension n increases. In Section 2.1 we will discuss preliminaries on Hopf bifurcations and when this bifurcation is sub- or supercritical for dimension 2 and for higher dimensions using the first Lyapunov coefficient with the aid of the references [Kuz98] and [vKS18a]. Section 2.2 is about some basic definitions and results for the Lyapunov exponents. The Lyapunov exponents are a generalization of the eigenvalues at the equilibrium and of the characteristic multipliers. Lyapunov exponents indicate the rate of the expansion or contraction near an equilibrium point. Moreover we will calculate these exponents for a specific system in Section 5.3.

Before we study some three specific systems, we want to know when there are escaping orbits in Section 4.2 using [Lor80]. This is useful, because then we know how large we can take the constant parameter F and the dimension n to avoid escaping orbits for a system. One result is that

$$|F| < \frac{1}{4C} n^{p-\frac{1}{2}},$$

where $C \in \mathbb{R}_{>0}$ is a constant and $p \approx 1$ as indicated by numerical experiments. This inequality means that if n increases, we are allowed to take F larger.

In Section 5 we will treat the dynamics of three specific systems, namely the systems with $(\alpha, \beta, \gamma) = (-1, 0, -2)$, $(-1, -1, -2)$ and $(-1, 0, -1)$. One result for the system with $(\alpha, \beta, \gamma) = (-1, 0, -1)$ is that there occurs a Pitchfork bifurcation at $F = \frac{1}{2}$ for all $n = 2k$, where $k \in \mathbb{Z}_{>0}$ and a second Pitchfork bifurcation occurs at $F = \frac{3}{5}$ for $n = 4$. In the cases when $(\alpha, \beta, \gamma) = (-1, 0, -2)$ and $(-1, -1, -2)$ there occurs a degenerate bifurcation meaning that there are simultaneously two or more equilibria at the same bifurcation parameter F and two eigenvalues are zero. At the end we will summarize and conclude the research presented in this work. So we will describe what the differences are between our modified systems and the original Lorenz-96 model and discuss the open problems about these modifications in the last section 6.

1.1. **Edward Norton Lorenz.** The original Lorenz-96 model was developed by Edward Norton Lorenz [vK18]. He was born on 23 May 1917 in West Hartford, Connecticut, in USA. In 1938 he received his Bachelor's degree Mathematics at the Dartmouth College in Hanover in New Hamspshire and received his Master's degree Mathematics at Harvard University in 1940. Moreover he got his second Master's degree in Meteorology at Massachusetts Institute of technology (MIT) in 1943. In 1948 he married Jane Loban and he received also his doctorate in Meteorology. He was appointed full Professor in Meteorology in 1962. During his life he awarded a lot of prices, such as Symons Memorial Gold Medal (1973), Crafoord Prize (1983), Kyoto Prize (1991) and Lomosov Gold Medal (2004). He retired in 1987, but he still did a lot for the science. He died on 16 April 2008 in Cambridge, Massachusetts, in USA.

FIGURE 2. Edward Norton Lorenz (1917-2008), reference: [Wik].

Edward Lorenz was an important mathematician and meteorologist, who established theoritical fundaments of weather, climate and atmosphere predictability and he was founder of the modern chaos theory. he discovered the Lorenz attractor and the Butterfly effect in 1963. Besides his Lorenz-96 model he also constructed other models, one of them is the Lorenz-63 model. This model is a three-dimensional model, which has the three equations

$$\begin{aligned}
\dot{x} &= -\sigma x + \sigma y, \\
\dot{y} &= -xz + rx - y, \\
\dot{z} &= xy - \beta z.
\end{aligned}$$

Here x is proportional to the intensity of the convective motion, y is to the difference in temperature between the rising and falling parts of the fluid and z to the deviation of the temperature profile from its equilibrium.

This model simulates the convective motion of the fluid between two parallel infinite horizontal plates in two dimensions. The lower plate is heated and the upper one is cooled. Furthermore Lorenz-63 model has indeed a physical interpretation comparing with the Lorenz-96 model. Although Lorenz-63 model has two disadvantages. It has only three equations and for classical parameter values this model is very dissipative.

Another model is the Lorenz-84 model, which is a three-dimensional model and it describes the long-term atmospheric circulation at the midlatitude. It is given by

$$\begin{aligned}\dot{x} &= -y^2 - z^2 - \alpha x + \alpha F,\\ \dot{y} &= xy - \beta xz - y + G,\\ \dot{z} &= \beta xy + xz - z,\end{aligned}$$

where x is the intensity of the symmetric globe-encircling westerly current wind, y and z are the cosine and sine phases of a chain of superposed waves transporting heat polward.

Edward Norton Lorenz constructed many models. In Section 3 we will treat the dynamical properties of the Lorenz-96 model and later on we will discuss its modification. Before these subjects, we will state some preliminary theory about Hopf bifurcations and Lyapunov exponents.

2. Preliminaries

2.1. **Hopf bifurcations.** First at a Hopf bifurcation the equilibrium loses stability and a periodic orbit appears. A Hopf bifurcation occurs when the complex eigenvalue pair of the Jacobian matrix at an equilibrium crosses the imaginary axis. In this subsection using the references [Kuz98] and [vKS18a] we will describe what sub- and supercritical Hopf bifurcations mean for dimension 2 and higher dimensions. For dimensions larger than 2 we can distinguish between a sub- and supercritical Hopf bifurcation by computing the Lyapunov coefficient. In Appendix C and in the Section 5 we will investigate when there occurs a Hopf bifurcation for some case studies and whether this Hopf bifurcation is sub- or supercritical. First we discuss this for dimension 2 to understand what exactly the difference is between these two.

To explain the supercritical and subcritical cases, we consider the system depending on the parameter α,

$$\begin{aligned}\dot{x}_1 &= \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2),\\ \dot{x}_2 &= x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2).\end{aligned} \tag{2.1}$$

Moreover the eigenvalues of the Jacobian matrix at the equilibrium point $(0, 0)$ are $\alpha + i$ and $\alpha - i$. Let $z = x_1 + ix_2$, then using the equations of the system we get the complex normal form of the Hopf bifurcation

$$\dot{z} = \dot{x}_1 + i\dot{x}_2 = (\alpha + i)z - z|z|^2,$$

where $|z|^2 = z\bar{z}$ and $\bar{z} = x_1 - ix_2$. Take the representation $z = \rho e^{i\phi}$, then the polar form of the system (2.1) is

$$\begin{aligned}\dot{\rho} &= \rho(\alpha - \rho^2),\\ \dot{\phi} &= 1.\end{aligned} \tag{2.2}$$

From this polar form we can analyse the bifurcations of the phase potrait of the system (2.1). The first equation of (2.2) has an equilibrium point $\rho_0 = 0$ for all α and an equilibrium point $\rho_1 = \sqrt{\alpha}$ for only $\alpha > 0$. Moreover the equilibrium ρ_0 is linearly stable for $\alpha < 0$ and linearly unstable for $\alpha > 0$. For $\alpha = 0$ this equilibrium point is still stable, but nonlinear.

Recall that the equilibrium of the system (2.1) is $(0, 0)$. This is stable for $\alpha < 0$ and unstable for $\alpha > 0$. In the last case this equilibrium is surrounded by a limit cycle or also called an isolated closed orbit, which is unique and stable and has radius $\sqrt{\alpha}$. At the critical value $\alpha = 0$ this equilibrium is nonlinearly stable. Therefore we get the bifurcation diagram for the system (2.1) in Figure 3 and for this case the Hopf bifurcation is supercritical. It is called supercritical, because the limit cycle exists after the bifurcation, so when the parameter α is positive.

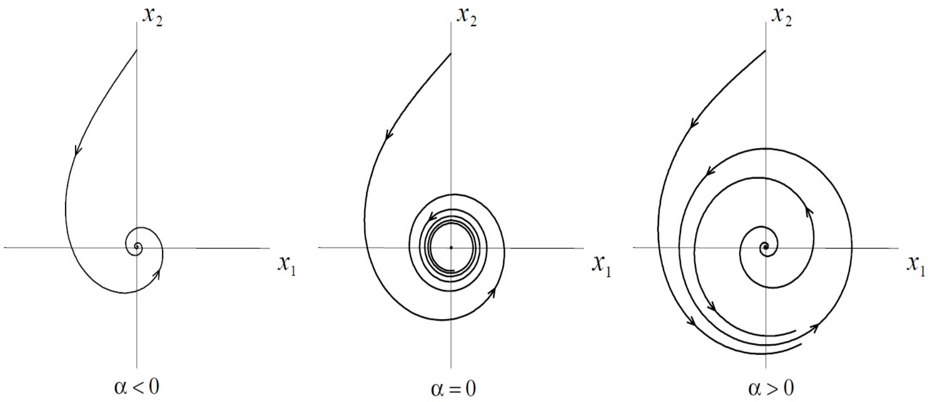

FIGURE 3. The bifurcation diagram for supercritical Hopf bifurcation from [Kuz98].

For the subcritical case we have the system with opposite signed nonlinear terms,

$$\begin{aligned}\dot{x}_1 &= \alpha x_1 - x_2 + x_1(x_1^2 + x_2^2), \\ \dot{x}_2 &= x_1 + \alpha x_2 + x_2(x_1^2 + x_2^2).\end{aligned} \tag{2.3}$$

Reasoning similar to the supercritical case shows that there is an unstable limit cycle for this system (2.3) when $\alpha < 0$. This cycle disappears when α becomes positive and we get the bifurcation diagram in Figure 4 when the Hopf bifurcation is subcritical. We call it subcritical, because the limit cycle appears for $\alpha < 0$.

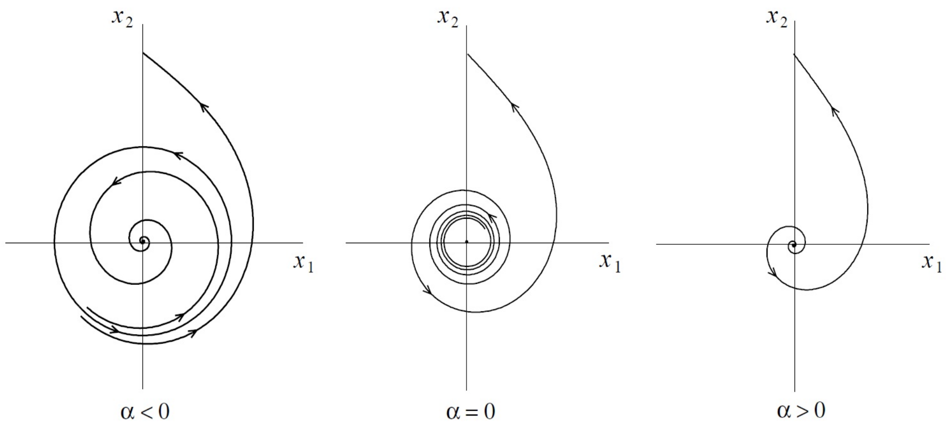

FIGURE 4. The bifurcation diagram for subcritical Hopf bifurcation from [Kuz98].

Now we will explain when the Hopf bifurcation is sub- or supercritical for dimensions higher than 2. In this case we cannot make bifurcation diagrams as we did for dimension 2. We obtain the sub- or supercritical by calculating the Lyapunov coefficient. If this coefficient is negative, then it is supercritical. If it is positive, then we have a subcritical Hopf bifurcation.

The first Lyapunov coefficient can be computed from a center manifold reduction. Here we only sketch the idea how to determine this coefficient, for more details see [Kuz98]. The first Lyapunov coefficient $l_1(F_H)$ corresponding to the Hopf bifurcation at some parameter

$F = F_H$ for the trivial equilibrium $x_F = (F, F, \dots, F)$ and dimension n is given by, see for more information [Kuz98],

$$l_1(F_H) = \frac{1}{2\omega_0}\text{Re}(\langle p, C(q, q, \bar{q})\rangle - 2\langle p, B(q, A^{-1}B(q, \bar{q}))\rangle + \langle p, B(\bar{q}, (2i\omega_0 I_n)^{-1}B(q, q))\rangle),$$

where A is the Jacobian matrix at x_F, B and C are the multilinear function determined by the Taylor expansion of the nonlinear part of the system. Also, p and q are the complex eigenvectors of $A^\top$ and A and ω_0 comes from the eigenvalue $\lambda_l = -\omega_0 i$. Note that the inner product in the last expression is

$$\langle x, y\rangle = \sum_{k=0}^{n-1} \bar{x}_k y_k.$$

To simplify the expression of the first Lyapunov coefficient we change the coordinates, where we translate the equilibrium x_F to the origin using some Taylor transformation. Then we normalised the eigenvectors p and q and eliminate the inverse matrices. Now it is filling in the values in the formula of the first Lyapunov coefficient to determine if it is positive or negative. Moreover if it is zero at F_H, then it means that this doesn't prove the occurrence of a Hopf bifurcation at F_H. In Section 5 and in Appendix C we will calculate the first Lyapunov coefficients for some systems and an example of the complete calculation can be found in Appendix B. Now we will discuss some preliminaries on Lyapunov exponents.

2.2. **Lyapunov Exponents.** To begin with Lyapunov exponents are a generalization of the eigenvalues of the Jacobian matrix at the equilibrium point and of the characteristic multipliers and these both can be used to determine the stability. We will discuss these in this subsection following the books [Kuz98] and [PC89]. We start with the explanation of eigenvalues at the equilibrium point, characteristic multipliers and the Poincaré map.

2.2.1. *Eigenvalues and Characteristic Multipliers.* First consider an equilibrium point $\bar{x}$ of an autonomous system

$$\dot{x} = f(x), \tag{2.4}$$

where f is a vector field. The local behaviour of the flow near the equilibrium point $\bar{x}$ is obtained by the linearization of the vector field at $\bar{x}$ and the linear vector field is

$$\delta x = Df(\bar{x})\delta x,$$

with $\delta x(0) = \delta x_0$. This vector field affects the time evolution of δx_0 in a neighbourhood of $\bar{x}$. Let the eigenvalues of $Df(\bar{x})$ be $\lambda_0, \lambda_1, \dots, \lambda_{n-1} \in \mathbb{C}$. From the book [PC89] an result is that if the eigenvalues are distinct and the initial condition is $\bar{x} + \delta x_0$, then the trajectory, to the first order, is

$$\psi_t(\bar{x} + \delta x_0) = \bar{x} + \delta x(t) = \bar{x} + e^{Df(\bar{x})t}\delta x_0.$$

Furthermore if every $\text{Re}(\lambda_i) < 0$ for all $i = 0, 1, \dots, n-1$, then the equilibrium point $\bar{x}$ is asymptotically stable. If $\text{Re}(\lambda_i) > 0$ for every $i = 0, 1, \dots, n-1$, then $\bar{x}$ is unstable.

The eigenvalues of the Jacobian matrix $Df(\bar{x})$ tell us about the stability of the equilibrium point $\bar{x}$. The stability of a periodic solution is determined by the characteristic multipliers, these are a generalization of the eigenvalues at $\bar{x}$ and a periodic solution corresponds to a fixed point x^* of the Poincaré map. A Poincaré map replaces the flow of a continuous-time system (2.4) of order n with a discrete-time system Σ of order $n-1$.

Let L be a periodic orbit of the system (2.4) and x^* be a point in L. In Figure 5 Σ is the Poincaré section crossing the point x^* and the Poincaré map P is defined by

$$P : \Sigma \to \Sigma,$$
$$x \mapsto \tilde{x} = P(x),$$

where $\tilde{x}$ is some point in Σ near x^*. Moreover $P(x^*) = x^*$ and thus x^* is a fixed point of P.

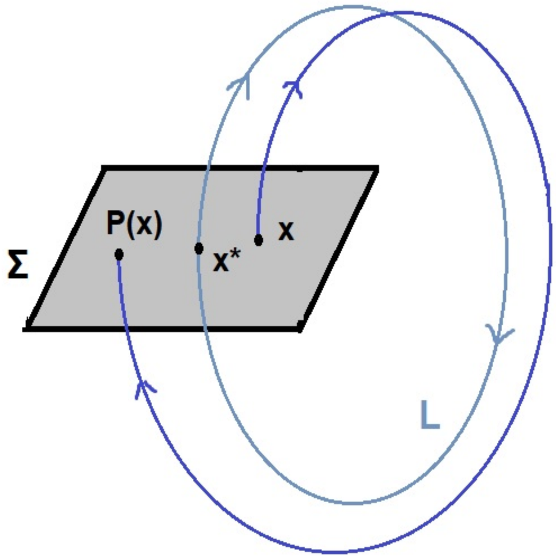

Figure 5. The Poincaré Map with the cycle L.

Going back to the stability, the stability of the periodic solution is equal to the stabilty of the fixed point. The local behaviour of the map P near the fixed point x^* of P is determined by the linear map,

$$\delta x_{k+1} = DP(x^*)\delta x_k,$$

at x^* and this map influences the evolution of a perturbation δx_0 in a neighbourhood of x^*. Let the eigenvalues of $DP(x^*)$ be $m_0, m_1, \ldots, m_p \in \mathbb{C}$. As before, assume that the eigenvalues $m_0, \ldots, m_p$ are distinct and the initial condition is $x^* + \delta x_0$, then the orbit of the Poincaré map P is, to the first order,

$$x_k = x^* + \delta x_k = x^* + DP(x^*)^k \delta x_0.$$

Definition 1. *Let x^* be the fixed point of the Poincaré map P and let the eigenvalues of $DP(x^*)$ be $m_0, m_1, \ldots, m_{n-1}$. These $m_0, m_1, \ldots, m_p$ are called the characteristic multipliers of the periodic solution.*

Using these characteristic multipliers we can say the following about stability of the fixed point x^*, hence about the stability of the periodic solution. If $|m_i| < 1$ for all $i = 0, 1, \ldots, n-1$, then the fixed point x^* is asymptotically stable. If $|m_i| > 1$ for any $i = 0, 1, \ldots, n-1$, then x^* is unstable. If it is the case that for some i, j, $|m_i| < 1$ and $|m_j| > 1$, this implies that the fixed point is a saddle point.

Now we will discuss the relation between the characteristic multipliers and the eigenvalues of the solution of the variational equation. First about the variational equation, consider the system (2.4) of dimension n with initial condition $x(t_0) = x_0$ and this system has the solution ψ such that

$$\dot{\psi}_t(x_0, t_0) = f(\psi_t(x_0, t_0), t), \;\; \psi_{t_0}(x_0, t_0) = x_0.$$

We differentiate the last expression with respect to x_0 to get

$$D_{x_0}\dot{\psi}_t(x_0, t_0) = D_x f(\psi_t(x_0, t_0), t) D_{x_0}\psi_t(x_0, t_0), D_{x_0}\psi_{t_0}(x_0, t_0) = I,$$

where I is the identity matrix. Let $\Phi_t(x_0, t_0) = D_{x_0}\psi_t(x_0, t_0)$, then the last equality is

$$\dot{\Phi}_t(x_0, t_0) = D_x f(\psi_t(x_0, t_0), t)\Phi_t(x_0, t_0), \;\; \Phi_{t_0}(x_0, t_0) = I,$$

and this is the variational equation with solution $\Phi_t(x_0, t_0)$. Since the initial condition is $\Phi_{t_0}(x_0, t_0) = I$, the solution $\Phi_t(x_0, t_0)$ is the state transition matrix of the linear system.

The relation between the characteristic multipliers and the eigenvalues of the state transition matrix $\Phi_T(x^*, t_0)$ is as follows. For the non-autonomous case the characteristic multipliers are the same as the eigenvalues of $\Phi_T(x^*, t_0)$, because $P(x) = \psi_T(x, t_0)$ gives

$$DP(x^*) = D_x\psi_T(x^*, t_0) = \Phi_T(x^*, t_0).$$

From the book [PC89] for the autonomous systems, we have that the eigenvalues of $\Phi_T(x^*)$ are $m_0, m_1, \dots, m_{n-2}, 1$, where $m_0, m_1, \dots, m_{n-2}$ are the characteristic multipliers and T is the minimum period of the limit cycle corresponding to the fixed point x^*.

The characteristic multipliers can be seen as a generalization of the eigenvalues at an equilibrium point and these can be used to determine the stability of the equilibrium point and of the limit cycles. Let any $T > 0$. Consider the autonomous system as a time-periodic, non-autonomous system with period T, because the vector field of an autonomous system is independent of time [PC89]. The Poincaré map is given by $P(x) = \psi_T(x)$ and since $\bar{x}$ is an equilibrium point, $P(\bar{x}) = \bar{x}$. This implies that we can consider $\bar{x}$ as a periodic solution. Now we want to find the characteristic multipliers of this periodic solution $\bar{x}$ and this can be done by calculating $DP(\bar{x}) = \Phi_T(\bar{x})$. The variational equation becomes using $\psi_t(\bar{x}) = \bar{x}$,

$$\dot{\Phi} = Df(\bar{x})\Phi, \;\; \Phi_0 = I.$$

This is a linear system not depending on time t with the state transition matrix

$$\Phi_t(\bar{x}) = e^{Df(\bar{x})t}.$$

Moreover since we have a non-autonomous system, the characteristic multipliers are equal to the eigenvalues of $\Phi_t(\bar{x})$. Therefore by the spectral mapping theorem [PC89] the eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ of $Df(\bar{x})$ and the characteristic multipliers $m_0, m_1, \dots, m_{n-1}$ are connected as

$$m_i = e^{\lambda_i T}, \quad i = 0, 1, \dots, n-1. \tag{2.5}$$

The m_i can be interpreted as the amount of contraction or expansion in T units of time and λ_i as the rate of contraction or expansion. In the next subsubsection we will use these for the Lyapunov exponents.

2.2.2. *Definition.* Recall that Lyapunov exponents are a generalization of the eigenvalues at the equilibrium point and of the characteristic multipliers. The definitions of the Lyapunov exponents are the following.

Definition 2 (Lyapunov exponents for continuous-time systems)**.** *Let x_0 be the initial condition in $\mathbb{R}^n$ and assume that $m_0(t), m_1(t), \dots, m_{n-1}(t)$ are the eigenvalues of $\Phi_t(x_0)$. The Lyapunov exponents of x_0 are for any $i = 0, 1, \dots, n-1$,*

$$\hat{\lambda}_i = \lim_{t\to\infty} \frac{1}{t} \ln |m_i(t)|, \tag{2.6}$$

if the limit exists.

Definition 3 (Lyapunov exponents for discrete-time systems)**.** *Let x_0 be any initial condition and suppose that the orbit of discrete-time system P of dimension p is $\{x_j\}_{j=0}^{\infty}$. Let $m_0(j), m_1(j), \dots, m_{p-1}(j)$ be the eigenvalues of $DP^j(x_0)$. Then the Lyapunov exponents of x_0 are*

$$\hat{\lambda}_i = \lim_{j\to\infty} |m_i(j)|^{\frac{1}{j}}, \ \ i = 0, 1, \dots, p-1,$$

if the limit exists.

Now we want to find the Lyapunov exponents of an equilibrium point $\bar{x}$ to understand these exponents better. Since (2.5), the Lyapunov exponents are given by

$$\hat{\lambda}_i = \lim_{t\to\infty} \frac{1}{t} \ln |e^{\lambda_i t}| = \lim_{t\to\infty} \frac{1}{t} \mathrm{Re}(\lambda_i)t = \mathrm{Re}(\lambda_i),$$

for every $i = 0, 1, \dots, n-1$. So we have expansion when $\hat{\lambda}_i > 0$ and contraction when $\hat{\lambda}_i < 0$ and these Lyapunov exponents indicate the rate of the expansion or contraction near the equilibrium point.

Note that the Lyapunov exponents depend on the initial condition x_0. If it is the case that the equilibrium point $\bar{x} \neq x_0$ and assume that x_0 lies in the basin of the attraction of the equilibrium point $\bar{x}$, then the Lyapunov exponents are the same for $\bar{x}$ and x_0, because in the expression (2.6) the limit $t \to \infty$ is involved. Hence, every point in the basin of the attraction of an attractor has the same Lyapunov exponents as for the initial condition x_0 and therefore these exponents are called the Lyapunov exponents of an attractor.

Looking at the Lyapunov exponents of an attractor, we can classify the non-chaotic attractors. First for the attractor we need that $\sum_{i=0}^{n-1} \hat{\lambda}_i < 0$, which means that contraction must be more than expansion. Therefore the classification is

1. There is an asymptotically stable equilibrium point, if all $\hat{\lambda}_i < 0$ for all $i = 0, 1, \dots, n-1$.
2. There is an asymptotically stable limit cycle, if $\hat{\lambda}_0 = 0$ and for every $i = 1, 2, \dots, n-1$, $\hat{\lambda}_i < 0$.
3. There is an asymptotically stable two-torus, when $\hat{\lambda}_0 = \hat{\lambda}_1 = 0$ and $\hat{\lambda}_i < 0$ for any $i = 2, 3, \dots, n-1$.
4. There is an asymptotically stable K-torus, if $\hat{\lambda}_0 = \hat{\lambda}_1 = \dots = \hat{\lambda}_K$ and for all $i = K+1, \dots, n-1$, $\hat{\lambda}_i < 0$.

Thus we have that every Lyapunov exponent in this situation is never positive.

The Lyapunov exponents of a chaotic attractor must have the following properties. Firstly, for chaotic attractor we need expansion meaning that at least one of the Lyapunov exponents must be positive. So this distinguishes a chaotic and a non-chaotic attractor. For any attractor other than an equilibrium one Lyapunov exponent is zero and lastly, the sum of the Lyapunov exponents of an attractor must be negative, i.e. $\sum_{i=0}^{n-1} \hat{\lambda}_i < 0$. Therefore chaos cannot appear in the first and second dimensional cases. For example when we have 3 dimensional chaos, the Lyapunov exponents are one $\hat{\lambda}_0$ is positive, the other $\hat{\lambda}_1$ is zero and one exponent $\hat{\lambda}_2$ is negative, since $\sum_{i=0}^{n-1} \hat{\lambda}_i < 0$ and $-\hat{\lambda}_0 > \hat{\lambda}_2$.

To summarize we have that the Lyapunov exponents are the same as the expression (2.6), but this is not enough to calculate the Lyapunov exponents of an attractor. Therefore in the next subsubsection we will obtain an algorithm to do this.

2.2.3. *Algorithm for computing the Lyapunov Exponents.* In this subsubsection we will treat the main idea how the algorithm to determine the Lyapunov exponents works and in Section 5.3 we will implement this procedure for the system with $(\alpha, \beta, \gamma) = (-1, 0, -1)$

for some dimensions n in Matlab to obtain the results. It is not workable to use the definition (2.6) of the Lyapunov exponents $\hat{\lambda}_i$ directly, for any $i = 0, 1, \ldots, n-1$. So we need another procedure to obtain and program these Lyapunov exponents. The main idea of the algorithm is to use the Gram-Schmidt Orthonormalization procedure. Thus we estimate all the n Lyapunov exponents by orthonormalization of n vectors.

Let $\{\delta x_1, \delta x_2, \ldots, \delta x_n\}$ be a set of linearly independent n vectors in $\mathbb{R}^n$. This Gram-Schmidt orthonormalization generates an orthonormal set of n vectors, which is denoted by $\{u_1, u_2, \ldots, u_n\}$. This set has the property that the set $\{u_1, \ldots, u_j\}$ spans the same subspace as $\{\delta x_1, \ldots, \delta x_j\}$ for all $j = 1, 2, \ldots, n$. Using this information the following equations hold,

$$\begin{aligned} v_1 &= \delta x_1 \\ u_1 &= \frac{v_1}{||v_1||} \\ v_2 &= \delta x_2 - \langle \delta x_2, u_1 \rangle u_1 \\ u_2 &= \frac{v_2}{||v_2||} \\ &\vdots \\ v_n &= \delta x_n - \langle \delta x_n, u_1 \rangle u_1 - \cdots - \langle \delta x_n, u_{n-1} \rangle u_{n-1} \\ u_n &= \frac{v_n}{||v_n||}. \end{aligned}$$

We can find all n Lyapunov exponents simultaneously, if a set of n linearly independent pertubation vectors $\delta x_1, \delta x_2, \ldots, \delta x_n$ is integrated every T units of time and orthonormalized many times. Thus after k iterations the orthonormalization procedure gives n vectors $v_1^{(k)}, v_2^{(k)}, \ldots, v_n^{(k)}$ and the Lyapunov exponents

$$\begin{aligned} \hat{\lambda}_0 &\approx \frac{1}{kT} \sum_{j=0}^{k} \ln(||v_1^{(j)}||), \\ \hat{\lambda}_1 &\approx \frac{1}{kT} \sum_{j=0}^{k} \ln(||v_2^{(j)}||), \\ &\vdots \\ \hat{\lambda}_{n-1} &\approx \frac{1}{kT} \sum_{j=0}^{k} \ln(||v_n^{(j)}||). \end{aligned}$$

Note that we can choose T, but if it is very small, then it leads to excessive orthonormalization. If it is very large, then it can lead to numerical overflow in the integration. Furthermore if the norms $||v_i||$ are close to one, then we get numerical inaccuracies in the volume calculations. In Section 5.3 we will apply this algorithm to a specific system. Before that we will study the dynamics of the Lorenz-96 model.

3. The Lorenz-96 Model

The original monoscale Lorenz-96 model is defined by the n equations

$$\dot{x}_j = x_{j-1}(x_{j+1} - x_{j-2}) - x_j + F, \quad j \in \{0, 1, \ldots, n-1\}, \tag{3.1}$$

where we have the indices modulo n such that $x_{j-n} = x_{j+n} = x_j$ and the bifurcation parameter $F \in \mathbb{R}$. The main idea is to modify this original model and we are going to work out the dynamics of the modified systems in the Sections 4 and 5. Now we will state the important results of the Lorenz-96 model from the thesis [vK18] and the articles [vKS18a] and [vKS18b].

3.1. **Dynamical Properties.** First the trivial equilibrium of Lorenz-96 model is given by $x_F = (F, F, \dots, F)$ for any $n \geq 1$, where $F \in \mathbb{R}$. Note that for $F = 0$ the equilibrium x_F is stable. Since there is circulant symmetry in this original system, we can determine the eigenvalues of x_F explicitly. Furthermore the Jacobian matrix at x_F is a circulant matrix in any dimension n. This means that each row of the Jacobian matrix is a right cyclic shift of the row above it. Therefore everything of this matrix can be obtained by its first row.

Now assume that the dimension $n \geq 4$. Denote the first row of the Jacobian matrix by

$$(c_0, c_1, \dots, c_{n-1}),$$

where $c_0 = -1, c_1 = F, c_{n-2} = -F$ and for any $k \neq 0, 1, n-2$, $c_k = 0$, because $\dot{x}_0 = x_{-1}(x_1 - x_{-2}) - x_0 + F$. Since the Jacobian matrix is a circulant matrix, we have the following eigenvalues, see Appendix A for more information,

$$\begin{aligned}\lambda_j &= \sum_{k=0}^{n-1} c_k \rho_j^k \\ &= \sum_{k=0}^{n-1} c_k \left(e^{-2\pi i \frac{j}{n}}\right)^k,\end{aligned}$$

for $j = \{0, 1, \dots, n-1\}$. Then

$$\begin{aligned}\lambda_j &= -1 + F\left(e^{-2\pi i \frac{j}{n}} - e^{4\pi i \frac{j}{n}}\right) \\ &= -1 + F\left(\cos\left(\frac{2\pi j}{n}\right) - \cos\left(\frac{4\pi j}{n}\right)\right) + iF\left(-\sin\left(\frac{2\pi j}{n}\right) - \sin\left(\frac{4\pi j}{n}\right)\right) \\ &= -1 + Ff(j,n) + iFg(j,n). \qquad (3.2)\end{aligned}$$

In Figure 6 we see the graphs of $f(j,n)$ and $g(j,n)$. The eigenvector corresponding to the eigenvalue λ_j (3.2) is

$$v_j = \frac{1}{\sqrt{n}}(1, \rho_j, \rho_j^2, \dots, \rho_j^{n-1})^\top.$$

Note that $\lambda_0 = -1$ and if $F = 0$, then we have stability, because every real part of the eigenvalues are -1. If n is even, then all eigenvalues $\lambda_{\frac{n}{2}} = -1 - 2F$ are real. Moreover if n is a multiple of 3, then the eigenvalues $\lambda_{\frac{n}{3}}$, $\lambda_{\frac{2n}{3}} = -1$ and thus these are real.

Especially, we are interested in complex eigenvalue pairs. The j-th eigenvalue pair is defined by $\{\lambda_j, \lambda_{n-j}\}$. Since $\rho_{n-j} = \overline{\rho}_j$, $\lambda_j = \overline{\lambda}_{n-j}$ and $v_j = \overline{v}_{n-j}$, the eigenvalues and eigenvectors form the complex conjugate pairs, except of course for $\lambda_0 = -1$ and for the cases when n is even and when n is a multiple of 3, as we saw before. We need complex eigenvalue pairs to obtain Hopf bifurcations, we will treat this further in the next subsection.

3.2. **Bifurcations.**

3.2.1. *Bifurcations for $F > 0$.*
The complex eigenvalue pair crosses the imaginary axis at some value F, so that it causes a Hopf bifurcation. We can use $F \in \mathbb{R}_{>0}$ as a bifurcation parameter.

Before we go into the results of the Hopf bifurcation, we need the following lemma about eigenvalue crossing.

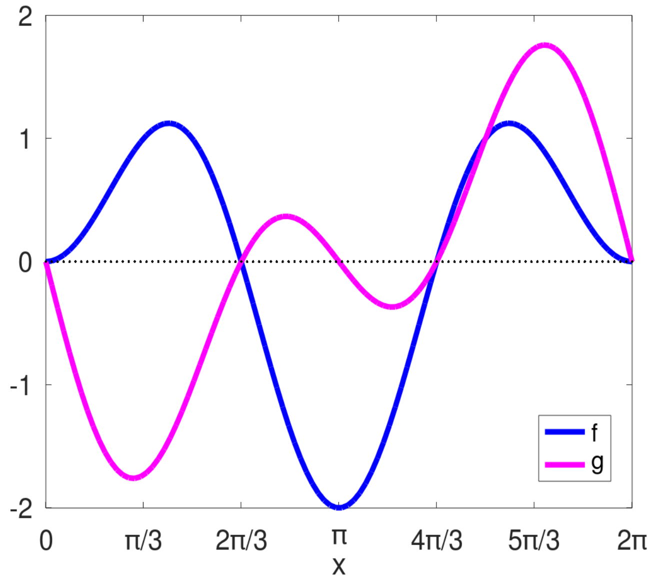

FIGURE 6. The graphs of continuous functions $f(x)$ (blue) and $g(x)$ (purple) defined in equation (3.2) for Lorenz-96 model, where $\frac{2\pi j}{n}$ is replaced by the continuous variable $x \in [0, 2\pi]$.

Lemma 1. *Let $n \geq 4$ and $l \in \mathbb{N}$, where $0 < l < \frac{n}{2}$ and $l \neq \frac{n}{3}$. Then*

1. *the l-th complex eigenvalue pair $\{\lambda_l, \lambda_{n-l}\}$ of the trivial equilibrium $x_F = (F, F, \ldots, F)$ of the system (3.1) crosses the imaginary axis transversally at $F = F_H(l, n) = \frac{1}{f(l,n)}$. This causes change in stability of the equilibrium.*
2. *$F_H(l, n)$ lies in the domain $\left(F_{min}(n), -\frac{1}{2}\right) \cup \left[\frac{8}{9}, F_{\max}(n)\right)$, where*

$$F_{\min}(n) = \begin{cases} -\frac{1}{2} & \text{if} n = 4, 6 \\ \frac{1}{f(r+1,n)} & \text{otherwise} \end{cases}$$

$$F_{\max}(n) = \begin{cases} \frac{1}{f(2,7)} & \text{if} n = 7 \\ \frac{1}{f(1,n)} & \text{otherwise} \end{cases},$$

where r is the quotient of n after division by 3.

The proof of this lemma can be found in the thesis [vK18].

By looking carefully at the graph of $f(j, n)$ in Figure 6, there can be at most two complex eigenvalue pairs that crosses the imaginary axis simultaneously for some value F. This implies that we have a Hopf bifurcation or a Hopf-Hopf bifurcation. In the following theorem it is stated when there is a subcritical Hopf bifurcation or a supercritical one, see [vK18] for the proof of it.

Theorem 1 (Hopf Bifurcation)**.** *Let l and n be as in Lemma 1. If l-th complex eigenvalue pair crosses only once the imaginary axis at parameter value $F = F_H$, then there occurs a Hopf bifurcation at F_H for the trivial equilibrium x_F. The first Lyapunov coefficient for this bifurcation is given by*

$$\begin{aligned} l_1(l, n) &= \frac{4}{n} \tan\left(\frac{\pi l}{n} \sin^2\left(\frac{3\pi l}{n}\right)\right) \\ &= \frac{5\cos\left(\frac{2\pi l}{n}\right) + 8\cos\left(\frac{4\pi l}{n}\right) - 2\cos\left(\frac{6\pi l}{n}\right) - 8}{4\cos\left(\frac{2\pi l}{n}\right) - 4\cos\left(\frac{4\pi l}{n}\right) + 9}. \end{aligned}$$

Fix $y_0 \in (0, \pi)$ such that

$$5\cos y_0 + 8\cos 2y_0 - 2\cos 3y_0 - 8 = 0,$$

then $l_1(l, n)$ is

1 *positive, if l and n satisfy $0 < \frac{l}{n} < \frac{y_0}{2\pi} \approx 0.08825746$. This corresponds to a subcritical Hopf bifurcation.*
2 *negative, if $\frac{l}{n} \in \left(\frac{y_0}{2\pi}, \frac{1}{2}\right) \setminus \{\frac{1}{3}\}$. This gives us a supercritical Hopf bifurcation.*

Moreover we have a Hopf-Hopf bifurcation when the following holds.

Theorem 2. *(Hopf-Hopf bifurcation) Let $n \geq 4$ and $l_1, l_2 \in \mathbb{N}$ satisfying $0 < l_1, l_2 < \frac{n}{2}$ and $l_1, l_2 \neq \frac{n}{3}$. Then for the trivial equilibrium x_F a Hopf-Hopf bifurcation occurs, which means two eigenvalue pairs crosses both the imaginary axis at the same parameter value $F_{HH} = F_H(l_1, n) = F_H(l_2, n)$, if and only if l_1 and l_2 satisfy*

$$\cos\left(\frac{2\pi l_1}{n}\right) + \cos\left(\frac{2\pi l_2}{n}\right) = \frac{1}{2}.$$

If the last expression doesn't hold, then we have a Hopf bifurcation. In conclusion, for $F > 0$ the first bifurcation is either a supercritical Hopf bifurcation or a Hopf-Hopf bifurcation and therefore it is not a Pitchfork bifurcation.

3.2.2. *Bifurcations for $F < 0$.* For $F < 0$ we have first Pitchfork bifurcations in some cases of dimension n. When n is odd, there does not occur a Pitchfork bifurcation. Instead of that the first bifurcation of the trivial equilibrium $x_F = (F, F, \ldots, F)$ is a Hopf bifurcation at $F_H = \frac{1}{\cos\left(\frac{2\pi j}{n}\right) - \cos\left(\frac{4\pi j}{n}\right)}$ with $j = \frac{n-1}{2}$. This bifurcation is supercritical meaning that the equilibrium x_F loses stability and a stable periodic orbit appears after this bifurcation.

For $n = 4k + 2$, for $k \in \mathbb{Z}_{>0}$, we only have one Pitchfork bifurcation. After this bifurcation on both equilibria we get a supercritical Hopf bifurcation simultaneously. So when $n = 6$ we have a Pitchfork bifurcation at $F = -\frac{1}{2}$ with three equilibria $x_F = (F, F, \ldots, F)$, $x_{P1} = (a, b, a, b, a, b)$ and $x_{P2} = (b, a, b, a, b, a)$, where a and b satisfy

$$a = \frac{-1 + \sqrt{-1 - 2F}}{2},$$
$$b = \frac{-1 - \sqrt{-1 - 2F}}{2}.$$

Thereafter a Hopf bifurcation occurs at $F_H = -\frac{7}{2}$.

Moreover when $n = 4k$, for $k \in \mathbb{Z}_{>0}$, there occur two Pitchfork bifurcations consecutively, one at $F = -\frac{1}{2}$ and the other at $F = -3$, and after these Pitchfork bifurcations the four stable equilibria exhibit supercritical Hopf bifurcation at the same time. For example, for $n = 4$ there are two Pitchfork bifurcations at $F = -\frac{1}{2}$ and at $F = -3$, where the four branches of the equilibria are of the form

$$(a, b, c, d), (b, c, d, a), (c, d, a, b) \text{ and } (d, a, b, c).$$

Another result is that for even n one eigenvalue is $\lambda_{\frac{n}{2}} = -1 - 2F = 0$ at $F = -\frac{1}{2}$ and in this case we have a Pitchfork bifurcation. In conclusion, we studied the bifurcations of the Lorenz-96 model shortly. Recall that at a Hopf bifurcation an equilibrium loses stability and gives birth to a periodic orbit. This periodic orbit can be interpreted as a travelling wave [vK18]. In the next subsection we will discuss the wave number and period of the Lorenz-96 model.

3.3. **Wave Number and Period.** For $F > 0$ the first Hopf bifurcation exists with the following index l_1^+.

Proposition 1. *Let $n \geq 4$ be fixed and for $F > 0$ the first Hopf and Hopf-Hopf bifurcation occur for the complex eigenvalue pair $\{\lambda_l, \lambda_{n-l}\}$ with index*

$$l_1^+(n) = \arg \max_{0<j<\frac{n}{3}} f(j, n). \tag{3.3}$$

This index satisfies $\frac{n}{6} \leq l_1^+(n) \leq \frac{n}{4}$ except for $n = 7$. For the case $n = 7$, $l_1^+(n = 7) = 1$.
Furthermore the first Hopf bifurcation of x_F is always supercritical.

A Hopf bifurcation occurs meaning that a birth of a periodic solution from the equilibrium. This changes the stability of this equilibrium. First we treat the general case to get the expression of the periodic orbit and then we apply it to the Lorenz-96 model. Consider a general geophysical model in the following form of a system of ordinary differential equations

$$\dot{x} = f(x, \mu), \tag{3.4}$$

where $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$ is a parameter modelling external circumstances, such as forcing. Let x_0 be an equilibrium for the parameter μ_0, which means that $f(x_0, \mu_0) = 0$. Thus x_0 is a time-independent solution of the system (3.4). Within geophysics this solution x_0 is the steady flow and the eigenvalues of the Jacobian matrix $Df(x_0, \mu_0)$ determine the stability of this equilibrium. Suppose that this Jacobian matrix has two eigenvalues $\pm\omega_0 i$, which indicates the occurrence of a Hopf bifurcation.

If x_0 is stable for $\mu < \mu_0$ and unstable for $\mu > \mu_0$, then the Hopf bifurcation is supercritical under some nondegeneracy conditions [Kuz98]. This means that for $\mu > \mu_0$ a stable periodic orbit exists. Moreover we have an unstable periodic orbit, if the Hopf bifurcations are of unstable equilibrium. For small $\epsilon = \sqrt{(\mu - \mu_0)}$ the general periodic orbit that is born at the Hopf bifurcation is approximation of the expression

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \epsilon \text{Re}((\boldsymbol{u} + i\boldsymbol{v})e^{i\omega t}) + O(\epsilon^2), \tag{3.5}$$

see [vK18]. Without loss of generality assume that the corresponding eigenvectors $\boldsymbol{u} + i\boldsymbol{v}$ of $Df(x_0, \mu_0)$ have unit length.

Back to the Lorenz-96 model, if the conditions of Theorem 1 hold, then we have a Hopf bifurcation meaning that one complex eigenvalue pair crosses the imaginary axis at $F_H = \frac{1}{f(l,n)}$. Let this be the l-th eigenvalue pair $\{\lambda_l, \lambda_{n-l}\}$. By equation (3.2) We can write this into

$$\lambda_l = \frac{g(l, n)}{f(l, n)} i = -\frac{\cos\left(\frac{\pi l}{n}\right)}{\sin\left(\frac{\pi l}{n}\right)} = -\omega_0 i,$$

where ω_0 can be taken to be the absolute value of the imaginary part at the bifurcation value. Using the general expression of the periodic orbit (3.5) and for $\epsilon = \sqrt{F - F_H}$ sufficiently small the approximation of the periodic orbit for the Lorenz-96 model is

$$\boldsymbol{x}(t) = F + \sqrt{F - F_H}\left(v_l e^{i\omega_0 t}\right) + O(\epsilon^2).$$

Because of this expression we are allowed to determine the physical properties of the wave.

Furthermore the j-th compenent of the expression of the periodic orbit is

$$\begin{aligned} x_j(t) &= F + \sqrt{F - F_H} \text{Re}\left(\frac{e^{i\left(\omega_0 t - \frac{2\pi j l}{n}\right)}}{\sqrt{n}}\right) + O(\epsilon^2) \\ &= F + \sqrt{\frac{F - F_H}{n}} \cos\left(\omega_0 t - \frac{2\pi j l}{n}\right) + O(\epsilon^2). \end{aligned}$$

This is indeed the formula of the traveling wave, where l is the wave number and the period is given by

$$T = \frac{2\pi}{\omega_0} = 2\pi \tan\left(\frac{\pi l}{n}\right). \tag{3.6}$$

The most interesting wave number is the one of the first bifurcation for the case when $F > 0$. This is denoted by $l_1^+(n)$ defined as (3.3) and it increases linearly with n. Since f has a maximum at $\frac{2\pi j}{n} = \arccos\left(\frac{1}{4}\right)$, we have the following results as n tends to infinity.

Proposition 2. *As $n \to \infty$, the period of the periodic orbit at the first Hopf bifurcation is given by*

$$\begin{aligned} T_\infty &= \lim_{n\to\infty} 2\pi \tan\left(\frac{\pi l_1^+(n)}{n}\right) \\ &= 2\pi \tan\left(\frac{1}{2}\arccos\left(\frac{1}{4}\right)\right) \approx 4.867. \end{aligned}$$

Moreover the wave number l_1^+ satisfies the limit:

$$\lim_{n\to\infty} \frac{2\pi l_1^+(n)}{n} = \arccos\left(\frac{1}{4}\right).$$

In Figure 7 we see the graphs of the wave number and the period for the Lorenz-96 model. The graph of period tends to a constant as n tends to infinity and the wave number increases linearly as n increases.

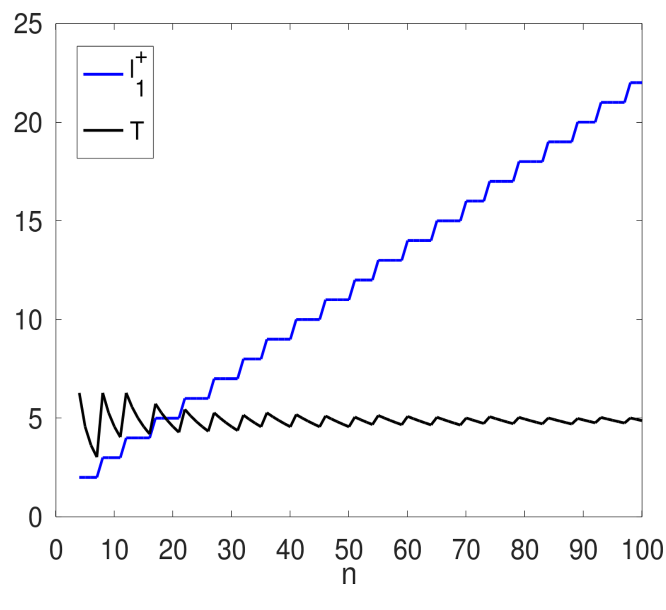

FIGURE 7. Period T (black) given by equation (3.6) and wave number l_1^+ (blue) defined by (3.3) as functions of n for the Lorenz-96 model.

Now we will study the wave number for $F < 0$. Assume that n is odd. Then the index of the first Hopf bifurcation is minimizing the value of the function $f(j, n)$, which is the same as

$$l_1^-(n) = \arg\min f(j, n) = \frac{n-1}{2}. \tag{3.7}$$

Note that in this case the wave number also increases linearly as n increases and it increases faster that for $F > 0$. Here the period is given by

$$T = 2\pi \tan\left(\frac{\pi(n-1)}{2n}\right). \tag{3.8}$$

This period doesn't tend to a limit as n tends to infinity, but it increases, see Figure 8. This is different from the case $F > 0$.

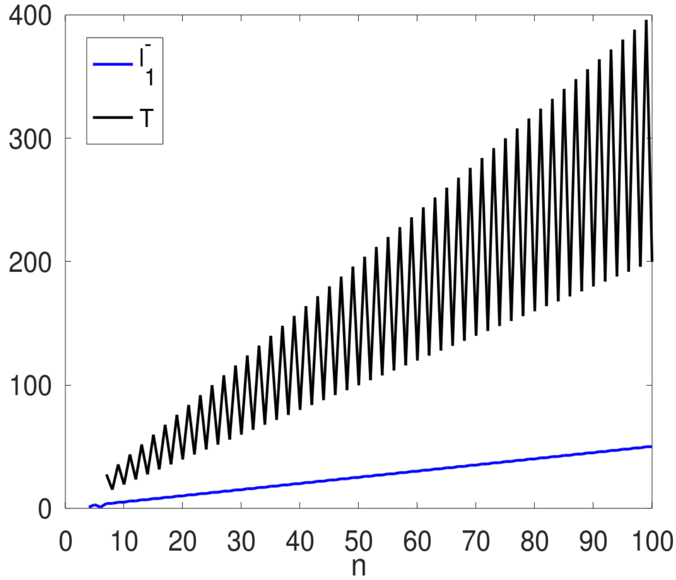

FIGURE 8. Period T (black) given by equation (3.8) and wave number l_1^- (blue) defined by (3.7) as functions of n for the Lorenz-96 model.

These were the important aspects of the Lorenz-96 model. In a later section we will discuss the dynamics of three specific modified systems, but we will first treat some general properties of some modified models.

4. The modified Lorenz-96 Model

The purpose of this section is to investigate what happens in general if we change the indices in the original model using the references [vK18], [vKS18a] and [vKS18b]. We will do this step by step using $\alpha, \beta, \gamma \in \mathbb{Z}$. Let α, β and γ be between $-n$ and n, then the general system is

$$\dot{x}_j = x_{j+\alpha}(x_{j+\beta} - x_{j+\gamma}) - x_j + F, \;\; j \in \{0, 1, \dots, n-1\}, \tag{4.1}$$

where we have the indices modulo n, in other words $x_{j-n} = x_{j+n} = x_j$ and $F \in \mathbb{R}$.

Note that if we put $(\alpha, \beta, \gamma) = (-1, 1, -2)$ into the system (4.1), then we get the original Lorenz-96 model (3.1). First we will discuss the general results for the system (4.1) for some α, β and γ. Moreover we will only look at the case when $n \geq 4$ as for the Lorenz-96 model.

4.1. General Dynamical Properties.

Note that there is circulant symmetry in the general system (4.1), therefore the eigenvalues of the trivial equilibrium $x_F = (F, F, \dots, F)$ can be determined explicitly. As for the original system, we can still obtain everything of the Jacobian matrix of x_F by its first row. So the eigenvalues of the Jacobian matrix at x_F of the general system are defined in the following lemma.

Lemma 2. *Let $n \geq 4$ and let $\alpha, \beta, \gamma \in \mathbb{Z}$. Take α, β and γ between $-n$ and n. The system* (4.1) *has the trivial equilibrium $x_F = (F, F, \ldots, F)$. Then the eigenvalues of the Jacobian matrix at x_F of this system* (4.1) *are for $j \in \{0, 1, \ldots, n-1\}$,*

$$\lambda_j(F, n) = -1 + F\eta(j, n; \beta, \gamma) + iF\xi(j, n; \beta, \gamma), \tag{4.2}$$

where

$$\eta(j, n; \beta, \gamma) = \cos\left(\frac{2\pi j\beta}{n}\right) - \cos\left(\frac{2\pi j\gamma}{n}\right) \quad \text{and}$$
$$\xi(j, n; \beta, \gamma) = \sin\left(\frac{2\pi j\gamma}{n}\right) - \sin\left(\frac{2\pi j\beta}{n}\right).$$

The corresponding eigenvector is given by

$$v_j = \frac{1}{\sqrt{n}}(1, \rho_j, \rho_j^2, \ldots, \rho_j^{n-1})^\top.$$

Proof. Assume that the first row of the Jacobian matrix at x_F is denoted by $(c_0, c_1, \ldots, c_{n-1})$. Moreover the first row of this Jacobian matrix is important, therefore we need the equation

$$\dot{x}_0 = x_\alpha(x_\beta - x_\gamma) - x_0 + F.$$

This equation gives us $c_0 = -1, c_\beta = F, c_\gamma = -F$ and for all $k \neq 0, \beta, \gamma$, $c_k = 0$.

Since the Jacobian matrix is a circulant matrix, the eigenvalues of this matrix are the following, see Appendix A for more information,

$$\lambda_j(F, n) = \sum_{k=0}^{n-1} c_k \rho_j^k,$$

for $j = 0, 1, \ldots, n-1$. Then

$$\begin{aligned}
\lambda_j(F, n) &= -1 + F\rho_j^\beta - F\rho_j^\gamma \\
&= -1 + Fe^{\frac{-2\pi ij\beta}{n}} - Fe^{\frac{-2\pi ij\gamma}{n}} \\
&= -1 + F\left(\cos\left(\frac{-2\pi j\beta}{n}\right) + i\sin\left(\frac{-2\pi j\beta}{n}\right)\right) \\
&\quad - F\left(\cos\left(\frac{-2\pi j\gamma}{n}\right) + i\sin\left(\frac{-2\pi j\gamma}{n}\right)\right) \\
&= -1 + F\left(\cos\left(\frac{2\pi j\beta}{n}\right) - \cos\left(\frac{2\pi j\gamma}{n}\right)\right) \\
&\quad + iF\left(\sin\left(\frac{2\pi j\gamma}{n}\right) - \sin\left(\frac{2\pi j\beta}{n}\right)\right) \\
&= -1 + F\eta(j, n; \beta, \gamma) + iF\xi(j, n; \beta, \gamma).
\end{aligned}$$

This proves the lemma. □

Remark 1. *Note that the expression* (4.2) *of the eigenvalue is independent of α. Therefore we only have to change β and γ in the system* (4.1) *and let α be -1.*

As for the original model, we are interested in the complex eigenvalue pairs. Since $\rho_{n-j} = \overline{\rho}_j$, $\lambda_j = \overline{\lambda}_{n-j}$ and $v_j = \overline{v}_{n-j}$, the eigenvalues and eigenvectors form a complex conjugate pairs, except when the eigenvalues are real. To obtain the Hopf bifurcation we need complex eigenvalue pairs. We go further with Hopf bifurcations in subsection 4.3.

First we have a stable trivial equilibrium $x_F = (F, F, \ldots, F)$ if in the expression for $\lambda_j(F, n)$ (4.2) the function $\eta(j, n; \beta, \gamma)$ is zero, because then for all $j = 0, 1, \ldots, n-1$, $\lambda_j(F, n)$ has negative real part. This is also stated in the following lemma.

Lemma 3. *Let $n \geq 4$ and let the eigenvalue $\lambda_j(F,n)$ of the Jacobian matrix at x_F be the formula* (4.2) *for $j \in \{0,1,\ldots,n-1\}$.*

1. *For all $j = 0,1,\ldots,n-1$, if $\eta(j,n;\beta,\gamma) = 0$ in* (4.2)*, then the eigenvalue $\lambda_j(F,n)$ cannot cross the imaginary axis for all $F \in \mathbb{R}$. Therefore the trivial eqiulibrium $x_F = (F,F,\ldots,F)$ is stable.*
2. *If $\eta(j,n;\beta,\gamma) \neq 0$, then $\lambda_j(F,n)$ crosses the imaginary axis at $F = F_H = \frac{1}{\eta(j,n;\beta,\gamma)}$.*
3. *For any $j = 0,1,\ldots,n-1$, if $\eta(j,n) \leq 0$, then the trivial equilibrium x_F is stable for all $F > 0$. In particular, this holds when $\gamma = 0$.*
4. *For every $j = 0,1,\ldots,n-1$, if $\eta(j,n) \geq 0$, then x_F is stable for any $F < 0$. In particular, this holds when $\beta = 0$.*

Note that for $F = 0$ the equilibrium x_F is stable.

Proof. The function $\eta(j,n;\beta,\gamma)$ in the expression of eigenvalue $\lambda_j(F,n)$ is $\cos\left(\frac{2\pi j\beta}{n}\right) - \cos\left(\frac{2\pi j\gamma}{n}\right)$. For statement 1 if $\eta(j,n;\beta,\gamma) = 0$, then the eigenvalue $\lambda_j(F,n)$ has a negative real part. Hence, we have stability. We use similar approach for statements 3 and 4. Furthermore if $\gamma = 0$, then $\eta(j,n;\beta,\gamma) \leq 0$ and if $\beta = 0$, then $\eta(j,n;\beta,\gamma) \geq 0$. So this proves the lemma. ☐

Thus from Lemma 3 we only have bifurcation for $\beta = 0$ when $F > 0$ and for $\gamma = 0$ when $F < 0$. An example for a stable equilibrium x_F is taking $(\alpha,\beta,\gamma) = (-1,-1,1)$, then the modified system is

$$\dot{x}_j = x_{j-1}(x_{j-1} - x_{j+1}) - x_j + F, \;\; j \in \{0,1,\ldots,n-1\},$$

where $x_{j-n} = x_{j+n} = x_j$. Moreover two other examples are taking $(\alpha,\beta,\gamma) = (-1,2,-2)$ and $(-1,-1,1)$, then stable x_F is stable.

The next lemma is about when the eigenvalue $\lambda_j(F,n)$ is always real for $j = 0,\ldots,n-1$.

Lemma 4. *Let $n \geq 4$ and we have that the eigenvalue $\lambda_j(F,n)$ of the Jacobian matrix at x_F is the expression* (4.2) *for $j \in \{0,\ldots,n-1\}$. Then $\xi(j,n;\beta,\gamma) = 0$ if and only if $\gamma = \frac{nk}{2j} + (-1)^k\beta$ for $k \in \mathbb{Z}$ and for some $\beta \in \mathbb{Z}$.*

Moreover the eigenvalue $\lambda_j(F,n)$ (4.2) *with $(\beta, \gamma = \frac{nk}{2j} + (-1)^k\beta)$ is always real if and only if $\xi(j,n)$ is zero.*

Proof. The last statement is obvious, so we will show the first statement. Let $A = \frac{2\pi j\beta}{n}$ and $B = \frac{2\pi j\gamma}{n}$. Then $\xi(j,n)$ is zero if and only if using the product-sum rule,

$$\begin{aligned} &\sin B - \sin A = 0 \iff \\ &2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right) = 0, \end{aligned}$$

if and only if for $k \in \mathbb{Z}$,

$$\begin{aligned} &\cos\left(\frac{A+B}{2}\right) = 0 \text{ or } \sin\left(\frac{B-A}{2}\right) = 0 \iff \\ &\frac{A+B}{2} = \left(k - \frac{1}{2}\right)\pi \text{ or } \frac{B-A}{2} = k\pi \iff \\ &B = 2k\pi - \pi - A \text{ or } B = A + 2k\pi. \end{aligned} \tag{4.3}$$

The last two formulas give us the expression

$$B = k\pi + (-1)^k A.$$

So if we fill in A and B, then

$$\frac{2\pi j\gamma}{n} = k\pi + (-1)^k \frac{2\pi j\beta}{n} \iff$$
$$\gamma = \frac{nk}{2j} + (-1)^k\beta, \quad \text{for } k \in \mathbb{Z}.$$

Now fill in A and B in the equations of (4.3), then

$$\gamma = \frac{n}{j}\left(k - \frac{1}{2}\right) - \beta \;\text{ or }\; \gamma = \frac{nk}{j} + \beta.$$

Hence, this proves the lemma. □

Furthermore we have a reflection about the x-axis, if we swap β and γ in the functions $\eta(j,n)$ and $\xi(j,n)$, because

$$\eta(j,n;\beta=\gamma,\gamma=\beta) = (-1)^2\cos\left(\frac{2\pi j\gamma}{n}\right) - \cos\left(\frac{2\pi j\beta}{n}\right)$$
$$= -\eta(j,n;\beta=\beta,\gamma=\gamma).$$

The function $-\eta(j,n)$ reflects $\eta(j,n)$ about the x-axis. Similarly, for the functions $\xi(j,n)$. Thus we get the following result.

Corollary 1. *Let $n \geq 4$ and assume that $\alpha,\beta,\gamma \in \mathbb{Z}$. Suppose that α,β and γ are between $-n$ and n. If we have the transformation swapping β and γ in the general system* (4.1), *then we get a new system with (α,γ,β) given by*

$$\dot{x}_j = x_{j+\alpha}(x_{j+\gamma} - x_{j+\beta}) - x_j + F, \tag{4.4}$$

for $j \in \{0,1,\ldots,n-1\}$. This new system is identical to the general system (4.1) *apart from a reflection of the bifurcation parameter F.*

Proof. Again, the general system (4.1) is

$$\dot{x}_j = x_{j+\alpha}(x_{j+\beta} - x_{j+\gamma}) - x_j + F, \quad j \in \{0,1,\ldots,n-1\}.$$

Take $y_j = -x_j$, then

$$\dot{y}_j = -\dot{x}_j$$
$$= -x_{j+\alpha}(x_{j+\beta} - x_{j+\gamma}) + x_j - F$$
$$= y_{j+\alpha}(y_{j+\gamma} - y_{j+\beta}) - y_j - F.$$

This last system is almost the new system (4.4), but instead of F we have $-F$. This shows that the new system (4.4) is the same as the general system (4.1) apart from the reflection of the parameter F. □

Another fact is that the function $\eta(j,n)$ is symmetric with respect to the straight, vertical line through $(\pi,0)$.

Lemma 5. *Suppose that $n \geq 4$ and that $\alpha,\beta,\gamma \in \mathbb{Z}$ are between $-n$ and n. Let $x = \frac{2\pi j}{n}$ be a continuous variable in the interval $[0,2\pi]$. For all α,β,γ the function $\eta(x)$ given in the expression* (4.2) *is symmetric with respect to the straight, vertical line through the point $(\pi,0)$.*

Proof. To prove this lemma, we need to show that

$$\eta(\pi - x) = \eta(\pi + x).$$

If the last equality holds, then the function $\eta(x)$ is symmetric with respect to the vertical line through the point $(\pi,0)$.

So

$$\begin{aligned}\eta(\pi - x) &= \cos(\beta\pi - \beta x) - \cos(\gamma\pi - \gamma x) \\ &= \cos(\beta\pi)\cos(-\beta x) + \sin(\beta\pi)\sin(\beta x) \\ &\quad - \cos(\gamma\pi)\cos(-\gamma x) - \sin(\gamma\pi)\sin(\gamma x) \\ &= \cos(\beta\pi)\cos(\beta x) - \cos(\gamma\pi)\cos(\gamma x),\end{aligned}$$

because $\beta, \gamma \in \mathbb{Z}$. Also,

$$\begin{aligned}\eta(\pi + x) &= \cos(\beta\pi)\cos(\beta x) - \sin(\beta\pi)\sin(\beta x) \\ &\quad - \cos(\gamma\pi)\cos(\gamma x) + \sin(\gamma\pi)\sin(\gamma x) \\ &= \cos(\beta\pi)\cos(\beta x) - \cos(\gamma\pi)\cos(\gamma x),\end{aligned}$$

since $\beta, \gamma \in \mathbb{Z}$. Hence this proves the lemma. □

Lemma 6. *Let $n \geq 4$ and assume that $\alpha, \beta, \gamma \in \mathbb{Z}$ are between $-n$ and n. Suppose that $x = \frac{2\pi j}{n}$ is a continuous variable in the interval $[0, 2\pi]$. For all α, β, γ the function $\xi(x)$, given in the expression (4.2), is asymmetric with respect to the straight, vertical line throught the point $(\pi, 0)$.*

Proof. For this lemma we need to prove that

$$\xi(\pi - x) = -\xi(\pi + x).$$

If the last expression holds, then we have showed the lemma.

Thus

$$\begin{aligned}\xi(\pi - x) &= \sin(\pi\gamma - x\gamma) - \sin(\pi\beta - x\beta) \\ &= \sin(\pi\gamma)\cos(x\gamma) - \cos(\pi\gamma)\sin(x\gamma) \\ &\quad - \sin(\pi\beta)\cos(x\beta) + \cos(\pi\beta)\sin(x\beta) \\ &= -\cos(\pi\gamma)\sin(x\gamma) + \cos(\pi\beta)\sin(x\beta) \\ &= -\xi(\pi + x).\end{aligned}$$

Hence, $\xi(x)$ is asymmetric. □

4.2. **Attractors and Escaping Orbits.** In this subsection we will study when there are escaping orbits using the article [Lor80]. As before we have the following general modified dynamical system (4.1), where $\alpha = -1$. Define the following functions

$$A = \sum_{j=0}^{n-1} x_j x_{j-1}(x_{j+\beta} - x_{j+\gamma}), \tag{4.5}$$

$$B = \sum_{j=0}^{n-1} x_j^2,$$

$$C = \sum_{j=0}^{n-1} F x_j.$$

Let $A_{\max}$ be the maximum of A, $B_{\min}$ be the minimum of B and $C_{\max}$ be the maximum of C on the unit sphere. Therefore $B_{\min} = 1$. To determine $C_{\max}$, use the Cauchy-Schwarz

(α,β,γ)	$A_{\max}$
$(-1,0,-1)$	decreases, nowhere zero
$(-1,0,-2)$	decreases, nowhere zero
$(-1,4,-2)$	decreases, nowhere zero except at $n=6$
$(-1,-1,-2)$	decreases, nowhere zero

Table 1. The results of $A_{\max}$ for some (α,β,γ).

inequality, then

$$\begin{aligned} C = \sum_{j=0}^{n-1} F x_j &\leq \left| \left\langle \begin{pmatrix} F \\ \vdots \\ F \end{pmatrix}, \begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \end{pmatrix} \right\rangle \right| \\ &\leq \left\| \begin{pmatrix} F \\ \vdots \\ F \end{pmatrix} \right\|_2 \cdot \left\| \begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \end{pmatrix} \right\|_2 \\ &= \sqrt{n}|F| \cdot \sqrt{\sum_{j=0}^{n-1} x_j^2}. \end{aligned}$$

This implies that $C_{max} \leq \sqrt{n}|F|$ and if for $F > 0$ this is equal to

$$\begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \end{pmatrix} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

and if for $F < 0$ it is the same as

$$\begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \end{pmatrix} = -\frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

In this case A is zero, but it doesn't mean that $A_{\max}$ is zero.

So we have found the maximum of C and the minimum of B. Since it is difficult to determine $A_{\max}$ analytically, we will do this numerically in the following way. Choose α, β and γ and as before, assume that $\alpha = -1$. Take a random vector $x = (x_0, x_1, \ldots, x_{n-1})^\top$ with unit norm and calculate A using the formula (4.5). The components of this random vector can be taken between -1 and 1. We do this for thousand x for every dimension n between 4 and 50. Then we plot $A_{\max}$ against the dimension n.

Plotting this, we get the Figure 9 for $(\alpha,\beta,\gamma) = (-1,0,-1)$. After many experiments with this program with different β and γ, the result is that $A_{\max}$ decreases as n increases. In table 1 we have summarized the results. One observation is that for some systems $A_{\max}$ is nowhere zero and that it is zero for the system with $(-1,4,-2)$ for $n = 6$. We get that $A_{\max} = 0$ for $n = 6$, because $A = 0$ for all x with unit norm and $A = 0$ since $j + 4 = j + \beta = j - 2 = (j + \gamma) \mod n$ if $n = 6$. Generally, for the system (4.1) with $\alpha = -1$ and for some n, if $(j + \beta) \mod n = (j + \gamma) \mod n$, then $A_{\max} = 0$ for dimension n.

Moreover in Figure 9 we also see the least squares fit of $A_{\max}$. Here we fit the numerically calculated number $\log(A_{\max})$ with the linear fit, in other words we fit $A_{\max}$ with $f(n) = Cn^p$, where p and C are constants. After many runnings and different values of β and γ

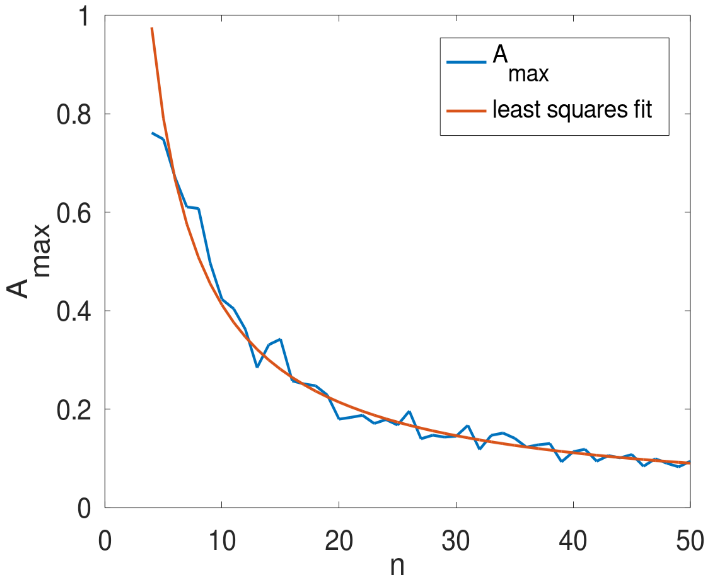

FIGURE 9. The graphs of $A_{\max}$ (blue), defined as the maximum of A (4.5), and of the least squares fit of it (red), from this we obtain that $A_{\max}$ decreases as n increases.

(α, β, γ)	Exponent p
$(-1, 0, -1)$	0.95
$(-1, 0, -2)$	0.78
$(-1, 4, -2)$	0.82
$(-1, -1, -2)$	0.83
$(-1, 20, -5)$	0.88
$(-1, 50, 31)$	0.91

TABLE 2. The values of the exponent p in the formula (4.6) for various choices of (α, β, γ). Notice that in all cases $p \approx 1$.

the same shape of the least squares fit appears as in Figure 9. This implies that

$$A_{\max} \approx \frac{C}{n^p}, \tag{4.6}$$

where $C \in \mathbb{R}$ is a constant and $p \approx 1$ according to the results of the program runnings. These results of the exponent p for different β and γ are summarized in table 2.

We determined $A_{\max}$ numerically and now we will treat the analytic way to obtain $A_{\max}$. We do this by using Lagrange multipliers z for $(\alpha, \beta, \gamma) = (-1, 0, -1)$ and dimension $n = 4$. We want to determine the maximum of $A = \sum_{j=0}^{3} x_j x_{j-1}(x_j - x_{j-1})$ subjected to $g(x_0, x_1, x_2, x_3) = x_0^2 + x_1^2 + x_2^2 + x_3^2 - 1 = 0$. Therefore we get the equation

$$\nabla A = -z \nabla g.$$

This gives us the equations

$$\begin{aligned}
2x_0x_3 - x_3^2 + x_1^2 - 2x_0x_1 + 2zx_0 &= 0,\\
2x_1x_0 - x_0^2 + x_2^2 - 2x_1x_2 + 2zx_1 &= 0,\\
2x_2x_1 - x_1^2 + x_3^2 - 2x_2x_3 + 2zx_2 &= 0,\\
2x_3x_2 - x_2^2 + x_0^2 - 2x_3x_0 + 2zx_3 &= 0
\end{aligned}$$

and these imply that

$$\begin{aligned}
x_0 + x_1 + x_2 + x_3 &= 0,\\
x_0^2 + x_1^2 + x_2^2 + x_3^2 &= 1.
\end{aligned}$$

These last equalities mean that we have a 3-sphere intersected by a plane through the origin in four dimensional space. Thus if x_0, x_1, x_2, x_3 are in this sphere satisfying the equation $x_0 + x_1 + x_2 + x_3 = 0$, then these values give the maximum of A. Further we cannot say something more about this maximum. Fortunately, we could in a numerical way.

Knowing $A_{\max}, B_{\min}$ and $C_{\max}$, we want to obtain the non-escaping orbits analytically. Assume that $R^2 = \sum_{j=0}^{n-1} x_j^2 = B$. Then the following holds,

$$\frac{\mathrm{d}R}{\mathrm{d}t} \leq A_{\max}R^2 - B_{\min}R + C_{\max}.$$

Indeed,

$$\begin{aligned}
\frac{\mathrm{d}R}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\sum_{j=0}^{n-1} x_j^2}\\
&= \frac{1}{2}\left(\sum_{j=0}^{n-1} x_j^2\right)^{-\frac{1}{2}} \cdot \left(\sum_{j=0}^{n-1} 2x_j\dot{x}_j\right)\\
&= \left(\sum_{j=0}^{n-1} x_j^2\right)^{-\frac{1}{2}} \sum_{j=0}^{n-1}(x_j(x_{j-1}x_{j+\beta} - x_{j-1}x_{j+\gamma} - x_j + F))\\
&= \left(\sum_{j=0}^{n-1} x_j^2\right)^{-\frac{1}{2}} (A - B + C)\\
&= R^{-1}(A - B + C)\\
&\leq R^{-1}A_{\max}R^3 - B_{\min}R + R^{-1}C_{\max}R\\
&\leq A_{\max}R^2 - B_{\min}R + C_{\max}.
\end{aligned}$$

The last steps are true, because note that the vector $(\frac{x_1}{R}, \ldots, \frac{x_{n-1}}{R})$ has unit norm since $R = \sqrt{\sum_{j=0}^{n-1} x_j^2}$, so

$$C = \sum_{j=0}^{n-1} Fx_j = R\sum_{j=0}^{n-1} F\frac{x_j}{R} \leq C_{\max}R$$

and similarly,

$$A = \sum_{j=0}^{n-1} x_j x_{j-1}(x_{j+\beta} - x_{j+\gamma}) \leq A_{\max} R^3.$$

Also, recall that $B_{\min} = 1$.

Thus we have the inequality

$$\frac{\mathrm{d}R}{\mathrm{d}t} \leq A_{\max} R^2 - B_{\min} R + C_{\max} \tag{4.7}$$

and $A_{\max} \geq 0$. Now we will look at two cases for $A_{\max}$ to obtain when there are escaping orbits or not.

Consider the first case when $A_{\max} = 0$. This means that the inequality (4.7) becomes $\frac{\mathrm{d}R}{\mathrm{d}t} \leq -B_{\min} R + C_{\max}$. So there exists one solution R_1 to $-B_{\min} R + C_{\max} = 0$ and the last inequality is drawn in Figure 10, where $R > 0$. If $\frac{\mathrm{d}R}{\mathrm{d}t} < 0$, then R decreases. Therefore if R passes R_1, we have that all orbits enter and remain in the interior of the sphere with radius $R_1 + \epsilon$, where $\epsilon > 0$ is arbitrary small, see Figure 10. Note that if $\frac{\mathrm{d}R}{\mathrm{d}t} > 0$, R increases and tend to R_1. So the attractor is enclosed by the sphere with radius $R_1 + \epsilon$.

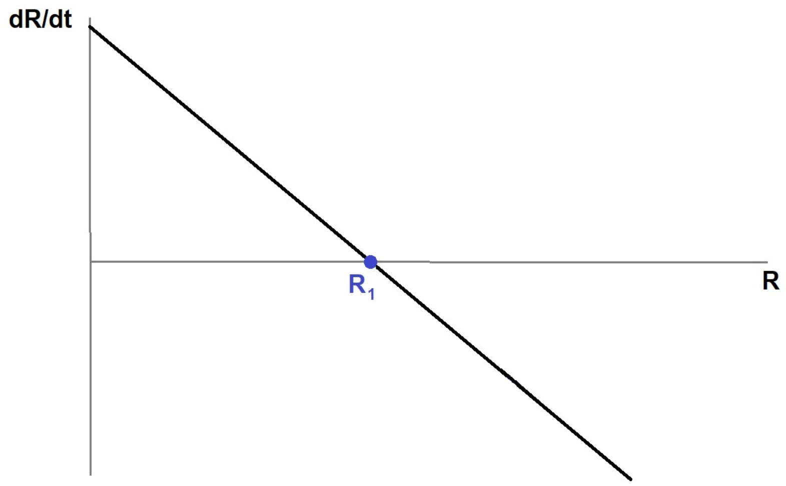

FIGURE 10. The first case when $A_{\max} = 0$.

The second case is when $A_{\max} > 0$. This implies that the graph of $A_{\max} R^2 - B_{\min} R + C_{\max}$ is a parabola opening upward. Assume that the discriminant

$$D = B_{\min}^2 - 4A_{\max} C_{\max}$$

is positive, which means that there exist two distinct solutions R_1 and R_2 to the equation $A_{\max} R^2 - R + C_{\max} = 0$. The corresponding picture of this situation is in Figure 11 on the left. Note that if $\frac{\mathrm{d}R}{\mathrm{d}t} < 0$, then R is between R_1 and R_2. If $\frac{\mathrm{d}R}{\mathrm{d}t} > 0$, then some orbits on or outside the outer sphere with radius R_2 will stay outside or even go to infinity. Some orbits on or outside this outer sphere will go inside this sphere and immediately enter and stay in the interior of the sphere with $R_1 + \epsilon$, see the picture of the spheres on the right in Figure 11. Thus part of the attractor is enclosed by the sphere with $R_1 + \epsilon$.

Moreover if the discriminant $D < 0$, there is no guarantee that some orbits don't escape since $\frac{\mathrm{d}R}{\mathrm{d}t}$ always holds. Similar for $D = 0$, there is still no guarantee that there exists non-escaping orbits, see Figure 12.

To avoid escaping orbits, we only consider the systems such that

$$B_{\min}^2 - 4A_{\max} C_{\max} > 0, \tag{4.8}$$

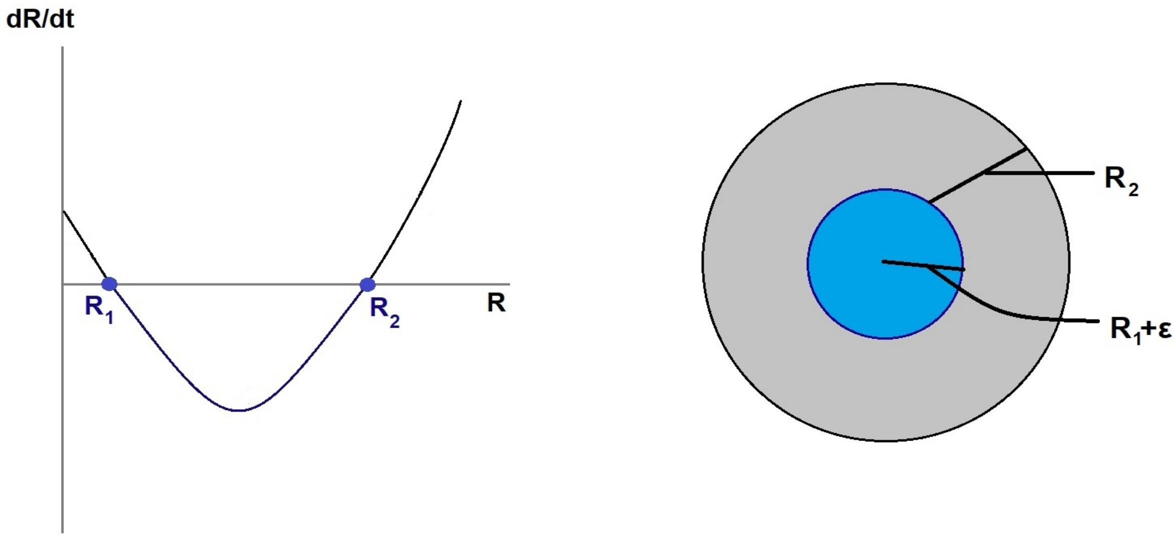

FIGURE 11. On the left the second case when $A_{\max} > 0$ for $D > 0$ and on the right the blue sphere with radius $R_1+\epsilon$ and the outer sphere with radius R_2.

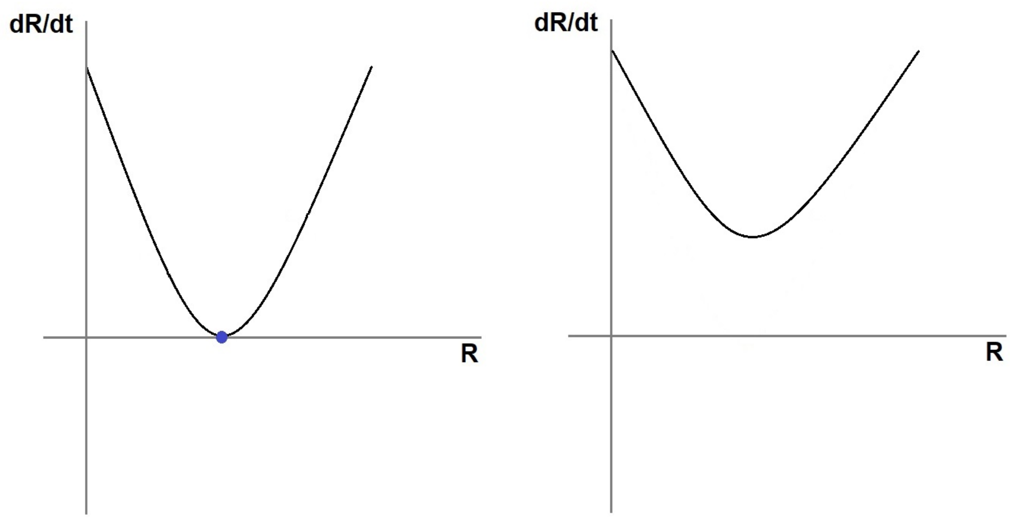

FIGURE 12. This is for the case when $A_{\max} > 0$, where on the left for $D = 0$ and on the right for $D < 0$.

and ignore the orbits on or outside the outer sphere with R_2, which means choose R such that $R < R_2$. Furthermore $B_{\min} = 1$ and

$$\frac{\mathrm{d}R}{\mathrm{d}t} \leq A_{\max}R^2 - R + C_{\max} \leq A_{\max}R^2 - R + \sqrt{n}|F|,$$

gives

$$A_{\max} < \frac{1}{4}\frac{1}{\sqrt{n}|F|}.$$

Using the last inequality and (4.6) for $p \approx 1$ and for $C \in \mathbb{R}$, we get that

$$|F| < \frac{1}{4C} n^{p-\frac{1}{2}} \tag{4.9}$$

gives a limit on $|F|$. This means that if n increases, we are allowed to take F larger.

Moreover $A_{\max}$ and $C_{\max}$ depend on F and n. As said before, $A_{\max}$ decreases and $C_{\max}$ increases as n increases. Thus if $-A_{\max}$ and $C_{\max}$ increase, then to agree condition (4.8) we have to choose n and F not too large. Also, if $A_{\max}$ and $-C_{\max}$ decreases, then the condition (4.8) is satisfied if and only if $A_{\max} C_{\max}$ is small enough.

4.3. **Hopf Bifurcation, Period and Wave Number.**

In this subsection we will discuss the Hopf bifurcation, the period and the wave number for all α, β and γ. The l-th eigenvalue pair $\{\lambda_l, \lambda_{n-l}\}$ of x_F crosses the imaginary axis at $F = F_H = \frac{1}{\eta(l,n;\beta,\gamma)}$ meaning that there occurs a Hopf bifurcation. The general expression of the eigenvalues (4.2) gives $\lambda_l = i\frac{\xi(l,n;\beta,\gamma)}{\eta(l,n;\beta,\gamma)}$ and $\lambda_l = -i\omega_0 = \bar{\lambda}_{n-l}$ with

$$\omega_0 = \frac{\sin\left(\frac{2\pi l\beta}{n}\right) - \sin\left(\frac{2\pi l\gamma}{n}\right)}{\cos\left(\frac{2\pi l\beta}{n}\right) - \cos\left(\frac{2\pi l\gamma}{n}\right)}. \tag{4.10}$$

In subsection 3.3 we discussed the general expression of the periodic orbit

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \epsilon \mathrm{Re}((\boldsymbol{u} + i\boldsymbol{v})e^{i\omega_0 t}) + O(\epsilon^2)$$

and applied it to the Lorenz-96 model. There is no difference between this formula for the Lorenz-96 model and its modifications. As for the original Lorenz-96 model, for sufficiently small $\epsilon = \sqrt{F - F_H}$ a good approximation of the periodic orbit is $\boldsymbol{x}(t)$ and its j-th component is

$$\begin{aligned} x_j(t) &= F + \epsilon \frac{\mathrm{Re}(e^{i\left(\omega_0 t - \frac{2\pi jl}{n}\right)})}{\sqrt{n}} + O(\epsilon^2) \\ &= F + \frac{\epsilon}{\sqrt{n}} \cos\left(\omega_0 t - \frac{2\pi jl}{n}\right) + O(\epsilon^2), \end{aligned} \tag{4.11}$$

for every modified system, where the corresponding eigenvector is $v_j = \frac{1}{\sqrt{n}}(1, \rho_j, \ldots, \rho_j^{n-1})^\top$. This formula of the periodic orbit (4.11) is the expression for a traveling wave in which the wave number is given by l and the period given by

$$T = \frac{2\pi}{\omega_0}. \tag{4.12}$$

If we fill ω_0 in T, then we get the following period.

Definition 4. *In general, for all α, β and γ the period is given by*

$$T(\beta, \gamma) = T = 2\pi \frac{\cos\left(\frac{2\pi l\beta}{n}\right) - \cos\left(\frac{2\pi l\gamma}{n}\right)}{\sin\left(\frac{2\pi l\beta}{n}\right) - \sin\left(\frac{2\pi l\gamma}{n}\right)}. \tag{4.13}$$

As for the Lorenz-96 model, we are interested in the wave number of the first Hopf bifurcation given by l_1^+ for $F > 0$ or by l_1^- for $F < 0$.

Definition 5. *For $F > 0$ and $\gamma \neq 0$, from $F_H = \frac{1}{\eta(j,n;\beta,\gamma)}$ it follows that the first Hopf bifurcation occurs for the complex eigenvalue pair $\{\lambda_l, \lambda_{n-l}\}$ with the index*

$$l_1^+(n, \beta, \gamma) = l_1^+(n) = \arg \max_{0<j\leq \frac{n}{2}} \eta(j, n; \beta, \gamma). \tag{4.14}$$

This l_1^+ is the wave number for $F > 0$ and for all α, β, γ.

If $F < 0$ and $\beta \neq 0$, then we have the index

$$l_1^-(n, \beta, \gamma) = l_1^-(n) = \arg \min_{0<j\leq \frac{n}{2}} \eta(j, n; \beta, \gamma). \tag{4.15}$$

This l_1^- is the wave number for $F < 0$ and for any α, β, γ.

Note that $\gamma \neq 0$ holds for the wave number when $F > 0$, because $\gamma = 0$ gives that using Lemma 3 there is no bifurcation and thus no wave number. Furthermore we only need $0 < j \leq \frac{n}{2}$ in the expression for the wave number and not $0 < j \leq n$, because the function $\eta(j, n)$ is symmetric according to Lemma 5.

In Figure 13 we see that the wave number l_1^+ increases as n tends to infinity for the system with $(\alpha, \beta, \gamma) = (-1, 0, -2)$. Strictly speaking, we can only prove that this wave number never decreases for every $n \geq 4$. The idea is as follows. Let $x = \frac{2\pi j}{n}$. For this system the function $\eta(x) = 1 - \cos(2x)$ has its maximum at $\frac{\pi}{2}$. In the interval $[0, \frac{\pi}{2})$ this function increases and decreases in $(\frac{\pi}{2}, \pi]$. If for some dimension n the function η has its maximum at $\frac{2\pi l_1^+}{n}$ in the interval $[0, \frac{\pi}{2})$, then the point $\frac{2\pi l_1^+}{n+1}$ shifts to the left with respect to the point $\frac{2\pi l_1^+}{n}$. This yields η decreases, but in this interval η increases. Therefore for $n + 1$ the wave number l_1^+ must increase. Now the point $\frac{2\pi l_1^+}{n}$ is in the interval $(\frac{\pi}{2}, \pi]$, then there are two options. The first case is when $\frac{2\pi l_1^+}{n+1} \in (\frac{\pi}{2}, \pi]$, therefore we get closer to the maximum of η. So the wave number stays the same. Secondly, when the point $\frac{2\pi l_1^+}{n+1} \in [0, \frac{\pi}{2})$ the wave number l_1^+ increases or remains the same.

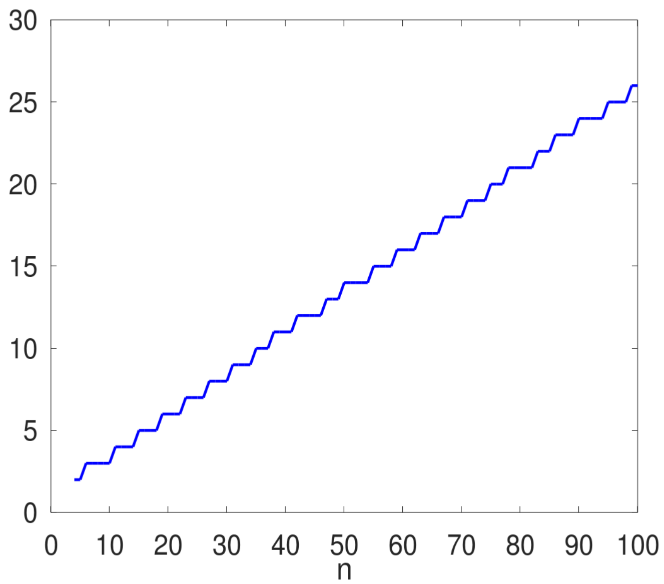

FIGURE 13. The increasing wave number as function of n for the system with $(\alpha, \beta, \gamma) = (-1, 0, -2)$.

Hence, for the system with $(-1, 0, -2)$ the wave number never decreases as n increases and when j always starts with 1 and increases. If j doesn't start at 1, then the wave number can behave differently. This is similar to the general case.

Lemma 7. *Let $n \geq 4$ and let the wave number be given by* (4.14) *or* (4.15)*. Assume that j starts with* 1 *and increases with one by one. If $n \to \infty$, then the wave number increases for all α, β, γ and for any $j = 0, 1, ..., n - 1$. Strictly speaking, the wave number never decreases for every $n \geq 4$.*

Proof. The proof of this lemma is similar as for the specific case when $(\alpha, \beta, \gamma) = (-1, 0, -2)$. Therefore the wave number never decreases for all α, β, γ, for any $j = 0, 1, \ldots, n-1$ and for every dimension $n \geq 4$. □

Lastly, let $\eta(x)$ have its maximum at $x_{\max}$. Therefore we have the following limit:

$$\lim_{n\to\infty} \frac{2\pi l_1^+(n, \beta, \gamma)}{n} = x_{\max}. \tag{4.16}$$

Using these general properties and results we will study the dynamics of the three systems with $(\alpha, \beta, \gamma) = (-1, 0, -2)$, $(-1, -1, -2)$ and $(-1, 0, -1)$ in the next section. Later in section 6 we will use this to compare the Lorenz-96 model with its modifications.

5. Case Studies

5.1. The Case $(\alpha, \beta, \gamma) = (-1, 0, -2)$.

5.1.1. *Eigenvalues and Graphs of Solutions.* In this subsection we will investigate the dynamical properties of the system with $(\alpha, \beta, \gamma) = (-1, 0, -2)$. Using Lemma 2 the eigenvalues for this system are

$$\begin{aligned} \lambda_j(F, n) &= -1 + F\left(1 - \cos\left(\frac{4\pi j}{n}\right)\right) + iF\left(-\sin\left(\frac{4\pi j}{n}\right)\right) \\ &= -1 + F\eta(j, n; \beta = 0, \gamma = -2) + iF\xi(j, n; \beta = 0, \gamma = -2), \end{aligned} \tag{5.1}$$

for $j = 0, 1, \ldots, n-1$. The graphs of $\eta(j, n)$ and $\xi(j, n)$ are shown in Figure 14, where $\frac{2\pi j}{n}$ is replaced by the continuous variable $x \in [0, 2\pi]$.

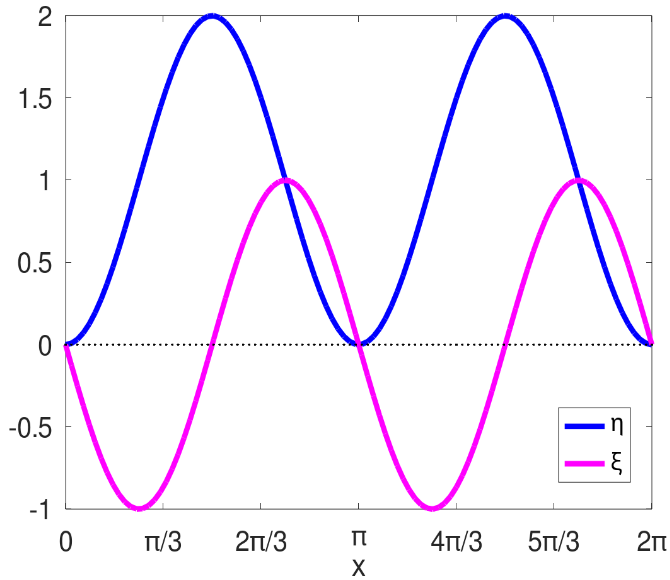

FIGURE 14. The graphs of the continuous functions $\eta(x)$ (blue) and $\xi(x)$ (purple) defined in equation (5.1) for $(\alpha, \beta, \gamma) = (-1, 0, 2)$, where the continuous variable $x = \frac{2\pi j}{n}$ and $x \in [0, 2\pi]$.

Moreover for $n \geq 4$ and $F = \frac{1}{2}$ take $a = \frac{n}{4}$ and $b = 3a = \frac{3n}{4}$. If n is a multiple of 4, then the eigenvalue λ_a is

$$\begin{aligned}\lambda_a &= -1 + F\left(1 - \cos\left(\frac{4\pi a}{n}\right)\right) + iF\left(-\sin\left(\frac{4\pi a}{n}\right)\right)\\ &= -1 + F(1 - \cos\pi) + iF(-\sin\pi)\\ &= -1 + 2F = 0,\end{aligned}$$

Also,

$$\lambda_b = -1 + F(1 - \cos 3\pi) + iF(-\sin 3\pi) = 0.$$

Thus at $F = \frac{1}{2}$ two eigenvalues are zero. Therefore the trivial equilibrium becomes unstable. Now we want to know what happens with the solutions of this system at $F = \frac{1}{2}$, $F < \frac{1}{2}$ and $F > \frac{1}{2}$ by doing some numerical experiments for dimension $n = 4$. It turns out that we have to be careful with initial conditions x_C, which one we choose for x. If the initial conditions are all the same, for example 0, then every solution converges to the trivial equilibrium x_F, no matter what F is.

Let the initial condition be $x_C = (0, 0.1, 0.2, 0.3)$, then the pictures of the solutions for some parameter F are in Figures 15 and 16. On the left in Figure 15 the solution

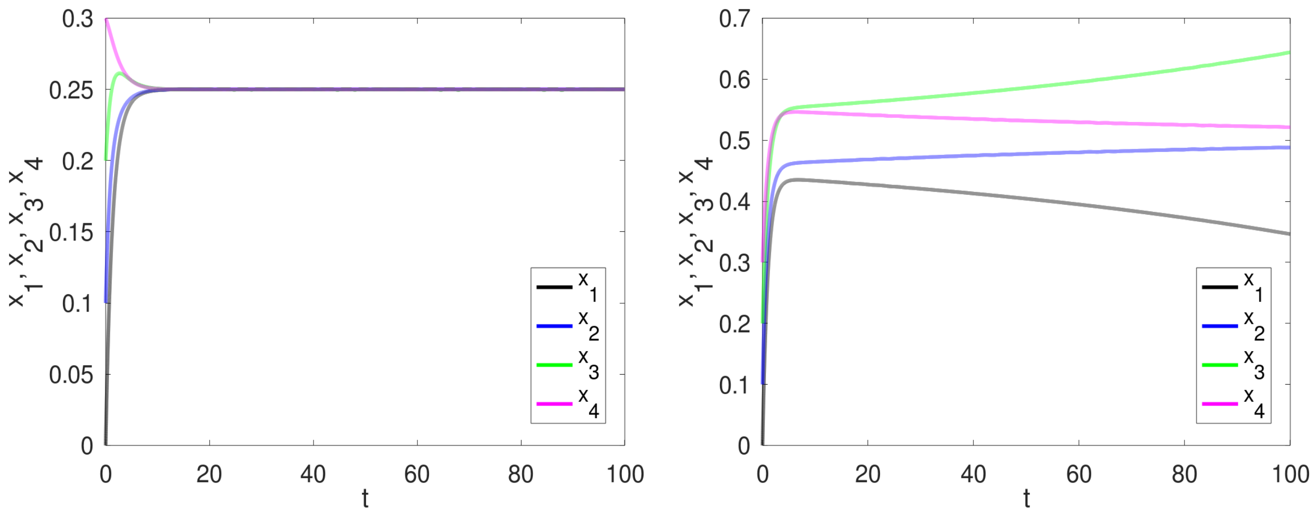

FIGURE 15. For $n = 4$ the solutions of the system with $(\alpha, \beta, \gamma) = (-1, 0, -2)$ for $F = \frac{1}{4}$ on the left and on the right for $F = \frac{1}{2}$, where initial condition $x_C = (0, 0.1, 0.2, 0.3)$.

converges to the equilibrium x_F for $F = \frac{1}{4} < \frac{1}{2}$. When $F = \frac{1}{2}$ the solution doesn't converge. The orbit escapes to infinity if $F > \frac{1}{2}$ obtained from Figure 16.

Moreover take $c = \frac{n}{2}$ and if n is a multiple of 4, then $\lambda_c(F, n) = -1$. Indeed,

$$\lambda_c = -1 + F(1 - \cos 2\pi) + iF(-\sin 2\pi) = -1,$$

for any $F > 0$. Note that if $F < 0$, then from Lemma 3 we have no bifurcation.

5.1.2. *Wave Number and Period.* As for the general case, we have a Hopf bifurcation, when the l-th pair $\{\lambda_l, \lambda_{n-l}\}$ of x_F crosses the imaginary axis at F. So $\lambda_l = -i\omega_0$, where from (4.10) ω_0 is

$$\omega_0 = \frac{\cos\left(\frac{2\pi l}{n}\right)}{\sin\left(\frac{2\pi l}{n}\right)}.$$

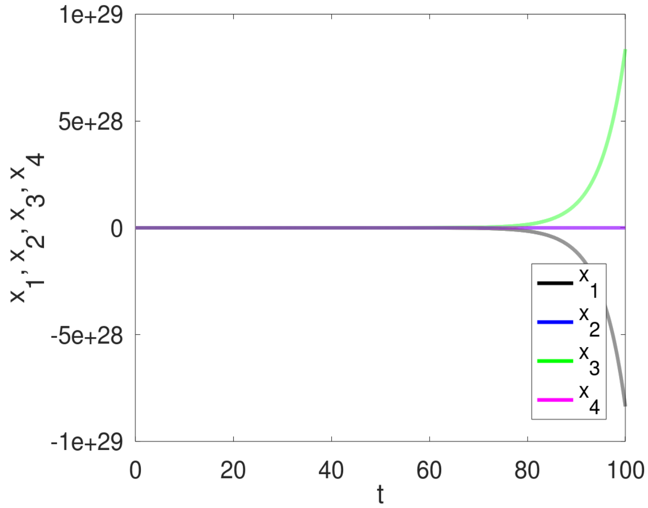

FIGURE 16. Similar as in Figure 15, but now for $F = \frac{3}{5}$.

Moreover in Figure 17 we have plotted the wave number and the period. Note that there is no wave number l_1^- for $F < 0$, because the function $\eta(j, n)$ is not negative and thus there is no bifurcation. The difference between this graph of the period and the

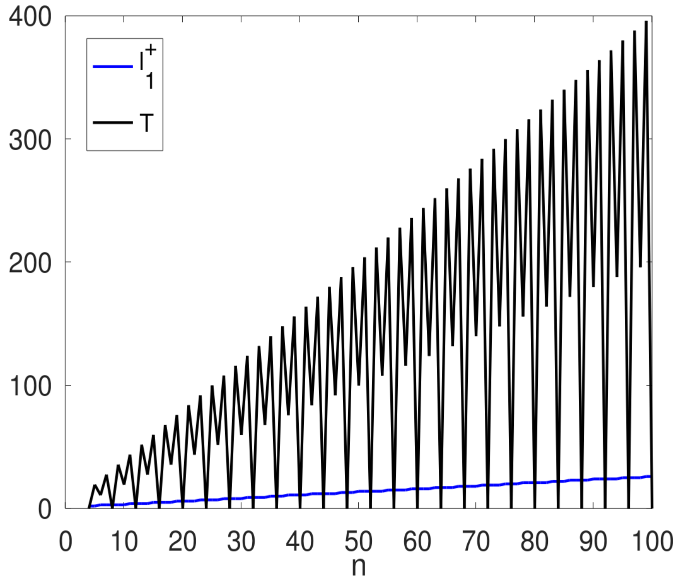

FIGURE 17. Period T (black) given by equation (4.13) and wave number l_1^+ (blue) defined by (4.14) as functions of n for $(\alpha, \beta, \gamma) = (-1, 0, -2)$.

period of the Lorenz-96 model is that the period is not tending to a constant number in this case, while it is for the original model for $F > 0$. The wave number increases, which is also the case for the original system. If we look carefully how fast the wave number increases for this system with $(\alpha, \beta, \gamma) = (-1, 0, -2)$, then this is $\frac{1}{4}$ and for the original system it is $\frac{5}{24} \approx 0.2083$.

In Figure 17 the period is zero at when n is a multiple of 4. Actually, the period cannot be zero, only positive. Although a zero period means in this case that we don't have a first Hopf bifurcation. If we look again closely to the graph of the period in Figure 17, then we see that at $n = 6, 10, 14, 18, 22, \ldots, 98$ there are (downward) points (for example in Figure 18 it is point b) between the two other points, where there is no first Hopf bifurcation and at $n = 5, 7, 9, 11, \dots, 99$ we have peaks (in Figure 18 these are the points a and c). In Figure 18 if n becomes very large, then the period at point a is almost twice as big than the period at point b. If n becomes very large, then the period at point c is at least twice as big than the period at point b.

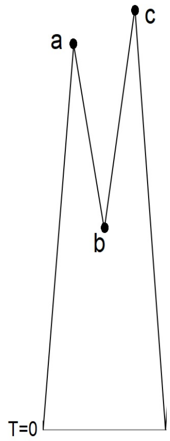

FIGURE 18. Repeated structure of part of the graph of the period T, where a, c are the peaks and b the downward point. $T = 0$ means no first Hopf bifurcation.

5.1.3. *Bifurcations.* Since for $n = 4$ the eigenvalues of the Jacobian matrix at the trivial equilibrium $x_F = (F, F, F, F)$ are $\lambda_0 = \lambda_2 = -1$ and $\lambda_1 = \lambda_3 = -1 + 2F = 0$ at $F = \frac{1}{2}$, there is no Hopf bifurcation, but we have a first subcritical Hopf bifurcation for $n = 5$ at $F_H \approx \frac{1}{1.81}$, see Appendix C and table 4 for more details. It is subcritical, because the Lyapunov coefficient is $l_1 \approx 0.091740 > 0$. For dimension $n = 6$ there occurs a Hopf-Hopf bifurcation at $F = \frac{2}{3}$.

Now we are interested in what kind of bifurcation there is for dimension $n = 4$. It is not a saddle-node bifurcation, neither is it a Pitchfork nor a Transcritical bifurcation. We have two zero eigenvalues, but this is not a Bogdanov-Takens bifurcation, because the Jacobian matrix is diagonalizable. If it was not the case, then it could be a Bogdanov-Takens bifurcation. We have spontaneously, simultaneously two equilibria "lines" for all $z \in \mathbb{R}$ at $F = \frac{1}{2}$ and these are

$$\left(\frac{1}{2}, z, \frac{1}{2}, 1 - z\right) \text{ and } \left(z, \frac{1}{2}, 1 - z, \frac{1}{2}\right).$$

Therefore we have a degenerate bifurcation. Actually, this is true for all $n = 4k$, for $k \in \mathbb{Z}_{>0}$.

Theorem 3. *Consider the system with* $(\alpha, \beta, \gamma) = (-1, 0, -2)$ *and for* $k \in \mathbb{Z}_{>0}$, *for any* $n = 4k$ *there are two equilibria of the form*

$$L_1 = \left(\frac{1}{2}, z, \frac{1}{2}, 1 - z, \dots, \frac{1}{2}, z, \frac{1}{2}, 1 - z\right) \quad \textit{and}$$
$$L_2 = \left(z, \frac{1}{2}, 1 - z, \frac{1}{2}, \dots, z, \frac{1}{2}, 1 - z, \frac{1}{2}\right),$$

for all $z \in \mathbb{R}$ *and each of them has* n *components. Moreover two eigenvalues are zero at* $F = \frac{1}{2}$, *then there occurs a degenerate bifurcation at* $F = \frac{1}{2}$.

Proof. This proof can be done by checking if the expressions of these "lines" are equilibria for any $z \in \mathbb{R}$. Thus for $j = 0, 1, \ldots, n-1$ at $F = \frac{1}{2}$,

$$\begin{aligned}
\dot{x}_j &= x_{j-1}(x_j - x_{j-2}) - x_j + F \quad \Longrightarrow \\
\dot{x}_0 &= x_{n-1}(x_0 - x_{n-2}) - x_0 + F = (1-z)\left(\frac{1}{2} - \frac{1}{2}\right) - \frac{1}{2} + F = 0, \\
\dot{x}_1 &= x_0(x_1 - x_{n-1}) - x_1 + F = \frac{1}{2}(z - (1-z)) - z + F = 0, \\
&\vdots \\
\dot{x}_{n-1} &= x_{n-2}(x_{n-1} - x_{n-3}) - x_{n-1} + F \\
&= \frac{1}{2}((1-z) - z) - (1-z) + F = 0,
\end{aligned}$$

for every $z \in \mathbb{R}$. This yields we have two equilibria of the form L_1 and L_2 and two zero eigenvalues at $F = \frac{1}{2}$ for all $n = 4k$, $k \in \mathbb{Z}_{>0}$. Therefore there occurs a degenerate bifurcation. □

In the next subsection we will discuss the dynamical properties of the system with $(\alpha, \beta, \gamma) = (-1, -1, -2)$. For example, for that system we have Hopf, Pitchfork and also degenerate bifurcations.

5.2. The Case $(\alpha, \beta, \gamma) = (-1, -1, -2)$.

5.2.1. *Eigenvalues, Period and Wave Number.* In this subsection we will study the dynamics of the system with $(\alpha, \beta, \gamma) = (-1, -1, -2)$. To begin with the eigenvalues for this system by Lemma 2,

$$\begin{aligned}
\lambda_j(F, n) &= -1 + F\left(\cos\left(\frac{2\pi j}{n}\right) - \cos\left(\frac{4\pi j}{n}\right)\right) + iF\left(\sin\left(\frac{2\pi j}{n}\right) - \sin\left(\frac{4\pi j}{n}\right)\right) \qquad (5.2) \\
&= -1 + F\eta(j, n) + iF\xi(j, n),
\end{aligned}$$

where the graphs of $\eta(x = \frac{2\pi j}{n})$ and $\xi(x = \frac{2\pi j}{n})$ are in Figure 19. The wave number and the period are shown in Figure 20. Using the expression of the eigenvalues, we will treat the bifurcations in the next subsubsection.

5.2.2. *Bifurcations.* For this system for $n = 4$ we calculate the first Lyapunov coefficient analytically instead of numerically to show how it works in practice and we will investigate further the kind of bifurcations. First for $n = 4$ we have a subcritical Hopf bifurcation at $F = 1$.

Theorem 4. *Let $n = 4$ and let the system be with $(\alpha, \beta, \gamma) = (-1, -1, -2)$. There occurs a Hopf bifurcation at $F_H = 1$, which means that the l-th eigenvalue pair with $\omega_0 = 1$ and $l = 3$ crosses the imaginary axis at this parameter value $F_H = 1$. Then the first Lyapunov coefficient is given by*

$$l_1(F_H = 1) = \frac{3}{13} > 0.$$

Thus the Hopf bifurcation is subcritical.

The proof of this theorem is in the Appendix B, because it is a long proof.

Furthermore for $n = 5$ we also have a Hopf bifurcation at $F \approx \frac{1}{1.12}$, which is subcritical and one at $F \approx -\frac{1}{1.12}$, which is supercritical, see Appendix C and table 4. For dimension $n = 6$ we have neither a Hopf bifurcation nor a Hopf-Hopf bifurcation. To discover the

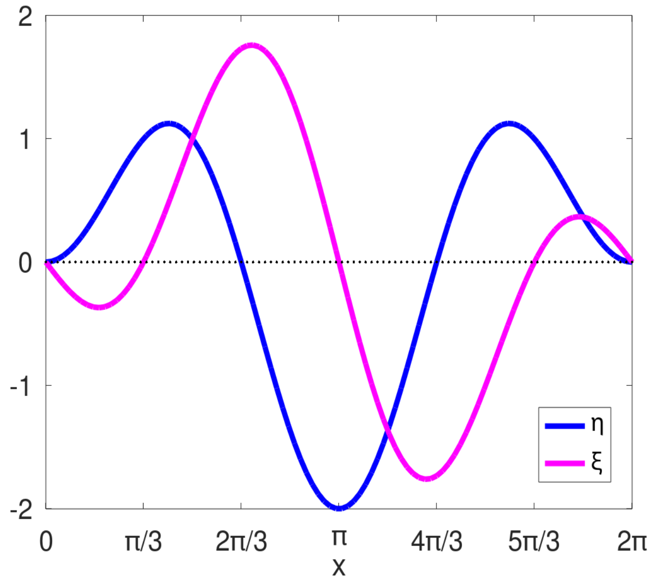

FIGURE 19. The graphs of the continuous functions $\eta(x)$ (blue) and $\xi(x)$ (purple) defined in equation (5.2) for $(\alpha, \beta, \gamma) = (-1, -1, -2)$, where $x = \frac{2\pi j}{n}$ and $x \in [0, 2\pi]$.

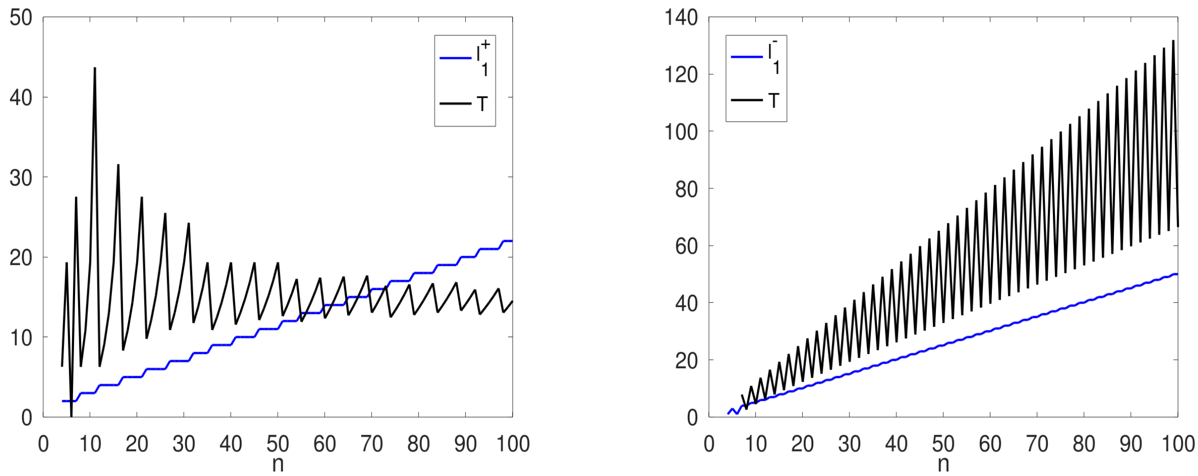

FIGURE 20. Period T (black) defined by equation (4.13) and wave number (blue) as functions of n for $(\alpha, \beta, \gamma) = (-1, -1, -2)$, on the left for $F > 0$ the wave number l_1^+ is given by (4.14) and on the right for $F < 0$ the wave number l_1^- is (4.15).

kind of bifurcation, we need to know what the eigenvalues are exactly for $n = 6$. These are

$$\begin{aligned}\lambda_0 &= -1,\\ \lambda_1 &= -1+F,\\ \lambda_2 &= -1+iF\sqrt{3},\\ \lambda_3 &= -1-2F,\\ \lambda_4 &= -1-iF\sqrt{3},\\ \lambda_5 &= -1+F,\end{aligned}$$

this yields two zero eigenvalues at $F=1$ and one zero eigenvalue at $F=-\frac{1}{2}$.

At $F=1$ we have a degenerate bifurcation, because there are two zero eigenvalues and besides the trivial equilibrium $x_{F=1}=(1,1,1,1,1,1)$ there are six other equilibria of the forms by using Mathematica

$$\begin{aligned}&(a,b,c,d,e,f),\\ &(b,c,d,e,f,a),\\ &(c,d,e,f,a,b),\\ &(d,e,f,a,b,c),\\ &(e,f,a,b,c,d),\\ &(f,a,b,c,d,e),\end{aligned}$$

where

$$\begin{aligned}&a\approx 1.856,\quad b\approx 2.708,\\ &c\approx 3.306,\quad d\approx 2.977,\\ &e\approx 0.022\ \text{ and }\ f\approx 0.936.\end{aligned}$$

Using Mathematica for $F=0.9, 0.99$ and 1.01 there are 13 equilibria and when $F=1.1$ we only have one equilibrium, the trivial x_F, and this is also true for $F=12$. One observation is that if F is very close to 1, e.g. $0.99, 1.01$, then the components of the 6 of the 13 equilibria are very close to the six equilibria of the form (a,b,c,d,e,f) for the case when $F=1$. Note that it cannot be a Bogdanov-Takens, since the Jacobian matrix is diagonalizable.

At $F=-\frac{1}{2}$ we only have one zero eigenvalue λ_3. We will check if this bifurcation is a Pitchfork bifurcation or not. Notice that it cannot be a saddle-node bifurcation, because the trivial equilibrium x_F still exists when $F<-\frac{1}{2}$. Take an equilibrium solution of the form (a,b,a,b,a,b) such that a and b satisfy the following equations

$$\begin{aligned}b(b-a)-a+F&=0,\\ a(a-b)-b+F&=0.\end{aligned}$$

One solution of these last equations is $a=b=F$ and now we want to find the other solution by rewriting the last expressions such that

$$\begin{aligned}&a=\frac{b^2+F}{b+1}\ \text{ and }\ a^2-ab-b+F=0 \iff\\ &\left(\frac{b^2+F}{b+1}\right)^2-b\left(\frac{b^2+F}{b+1}\right)-b+F=0 \iff\\ &-2b^3+b^2(2F-2)+b(-1+F)+F(1+F)=0.\end{aligned}$$

We can solve this last equation by Vieta's formulas, these are

$$p+q+r = F-1,$$
$$pq+qr+rp = \frac{1}{2}-\frac{1}{2}F,$$
$$pqr = \frac{1}{2}F(1+F),$$

where $p = F$ such that $(b-p)(b-q)(b-r) = 0$ and after solving these we get

$$r_{+/-} = -\frac{1}{2} \mp \frac{1}{2}\sqrt{-1-2F} \text{ and}$$
$$q_{+/-} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-1-2F}.$$

Take

$$b = b_1 = -\frac{1}{2}+\frac{1}{2}\sqrt{-1-2F},$$
$$b = b_2 = -\frac{1}{2}-\frac{1}{2}\sqrt{-1-2F}.$$

Then for b_1 we get

$$a_1 = \frac{b_1^2+F}{b_1+1} = -\frac{1}{2}-\frac{1}{2}\sqrt{-1-2F}$$

and for b_2

$$a_2 = \frac{b_2^2+F}{b_2+1} = -\frac{1}{2}+\frac{1}{2}\sqrt{-1-2F}.$$

Therefore we have three equilibria $x_F = (F, F, \dots, F)$, $x_{P1} = (a, b, a, b, a, b)$ and $x_{P2} = (b, a, b, a, b, a)$ for $F < -\frac{1}{2}$, where

$$a = -\frac{1}{2}+\frac{1}{2}\sqrt{-1-2F},$$
$$b = -\frac{1}{2}-\frac{1}{2}\sqrt{-1-2F}.$$

Hence, there occurs a Pitchfork bifurcation at $F = -\frac{1}{2}$ for $n = 6$. Moreover we have the following result.

Theorem 5. *Consider the system with $(\alpha, \beta, \gamma) = (-1, -1, -2)$. Then, for $k \in \mathbb{Z}_{>0}$, for every $n = 6k$ there occurs a Pitchfork bifurcation at $F = -\frac{1}{2}$ and the eigenvalue $\lambda_{\frac{n}{2}} = -1-2F$ is zero for all $n = 6k$.*

Proof. First by filling in $j = \frac{n}{2}$ the eigenvalue

$$\lambda_{(j=\frac{n}{2})} = -1-2F,$$

which is zero at $F = -\frac{1}{2}$. So there is a change in stability at $F = -\frac{1}{2}$. To prove that this is a Pitchfork bifurcation for all $n = 6k$, for $k \in \mathbb{Z}_{>0}$ is similar to the case when $n = 6$, which we saw before. In this case we get the three equilibria $x_F = (F, F, \dots, F)$, $x_{P1} = (a, b, \dots, a, b)$ and $x_{P2} = (b, a, \dots, b, a)$, where

$$a = -\frac{1}{2}+\frac{1}{2}\sqrt{-1-2F},$$
$$b = -\frac{1}{2}-\frac{1}{2}\sqrt{-1-2F}$$

for $F < -\frac{1}{2}$. Thus there occurs a Pitchfork bifurcation at $F = -\frac{1}{2}$ for all $n = 6k$, for $k \in \mathbb{Z}_{>0}$. This proves the theorem. □

In the next subsection we will treat another interesting system with $(\alpha, \beta, \gamma) = (-1, 0, -1)$, where we study its dynamics using also Lyapunov exponents.

5.3. **The Case** $(\alpha, \beta, \gamma) = (-1, 0, -1)$.

5.3.1. *The Solutions and Attractors.* In this subsection we will study the dynamical properties of the system with $(\alpha, \beta, \gamma) = (-1, 0, -1)$. For the expression of eigenvalues at the trivial equilibrium $x_F = (F, F, \ldots, F)$ (4.2) the graphs of the functions $\eta(x)$ and $\xi(x)$ are shown in Figure 21. The wave number and period can be seen in Figure 22. Note that in this case we only have the wave number and thus bifurcations for $F > 0$.

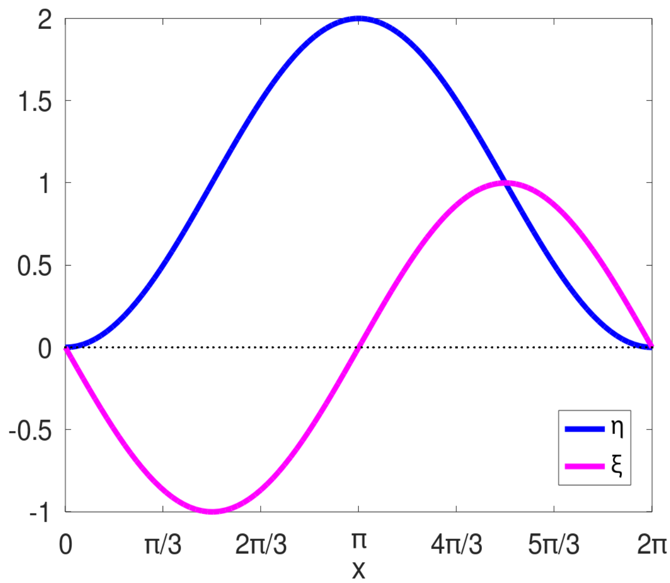

FIGURE 21. The graphs of the continuous functions $\eta(x)$ (blue) and $\xi(x)$ (purple) given in equation (4.2) for $(\alpha, \beta, \gamma) = (-1, 0, -1)$, where $x = \frac{2\pi j}{n}$ and $x \in [0, 2\pi]$.

Now we are interested in the plots of the solutions for this system and these are the Figures 23, 24, 25 and 26, where x_1, x_2, x_3 and x_4 are functions of time t. In Figure 23 the solution converge to the trivial equilibrium, which is similar as for other systems. Although in the Figures 25 and 26 there is a different structure than for other case studies. In some other case studies the orbit escapes to infinity, but for this system not. It doesn't converge either. Observe that from these plots this structure has a pattern. This structure is also to be seen in Figure 27 when $n = 8$.

We saw the plots of the solutions and now we will look at the attractors of this system. The Figures 28, 29, 30, 31, 32, 33 and 34 are the plots of these attractors for the dimensions $n = 4, 5, 6, 7, 8$ and 12. In these figures we plot the solutions against each other, especially x_1 against x_4. Furthermore we see that there occurs a nice shap, which looks like a butterfly, where the solutions are repelling or attracting. From these figures for even n we have some two attracting spirals and for odd n not. By looking at the 3D plots of the attractor for $n = 4$ in the Figures 35, 36 and 37 it seems that every solution atracts to or repels from some "line".

To understand better the pictures of the attractors, we plot the distance between this "line" and the solution of the system with $(-1, 0, -1)$, where "line" is $L = \text{span}\{\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\}$. In other words, the distance is between the solution $x(t)$ and the orthogonal projection

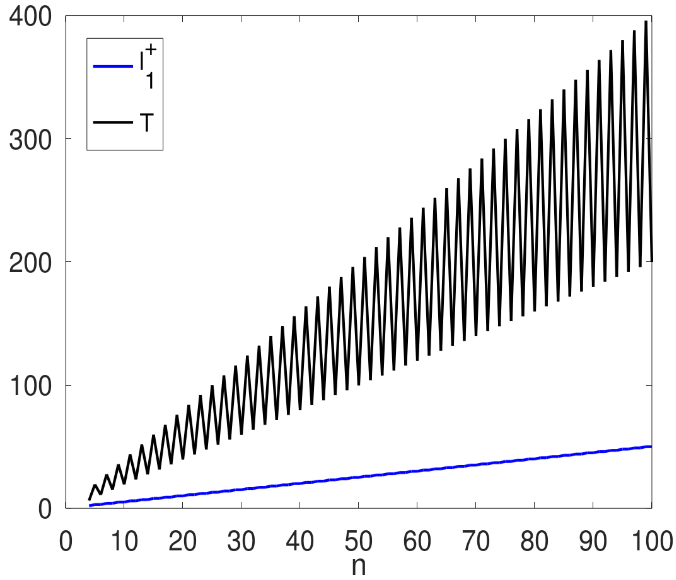

FIGURE 22. Period T (black) given by (4.13) and wave number $l_1^+(n)$ (blue) defined by (4.14) as functions of n for $(\alpha, \beta, \gamma) = (-1, 0, -1)$.

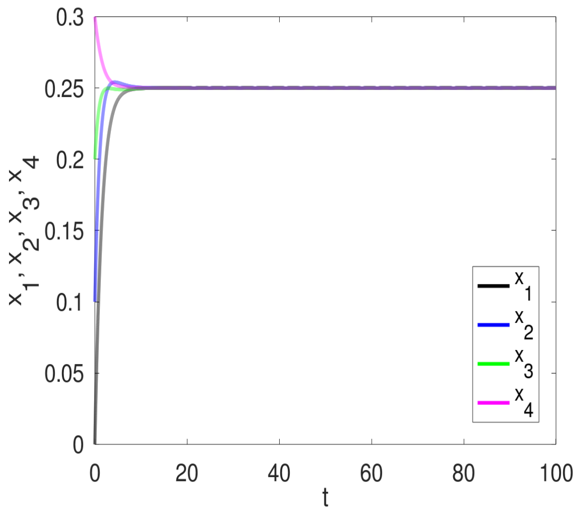

FIGURE 23. For $n = 4$ the solutions of the system with $(\alpha, \beta, \gamma) = (-1, 0, -1)$ for $F = \frac{1}{4}$, where the initial condition $x_C = (0, 0.1, 0.2, 0.3)$.

from $x(t)$ onto $L = \text{span}\{\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\} = \text{span}\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$ for $n = 4$. For now we only look at the case when the dimension $n = 4$. The orthogonal projection P is

$$P = \left\langle x(t), \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\rangle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

To determine the distance, we use the expression of the distance D

$$D = ||x(t) - P||_2,$$

which is the 2-norm of the difference between the orthogonal projection P and the solution $x(t)$. Therefore we get the Figure 38 for the distance against time t for different F, but

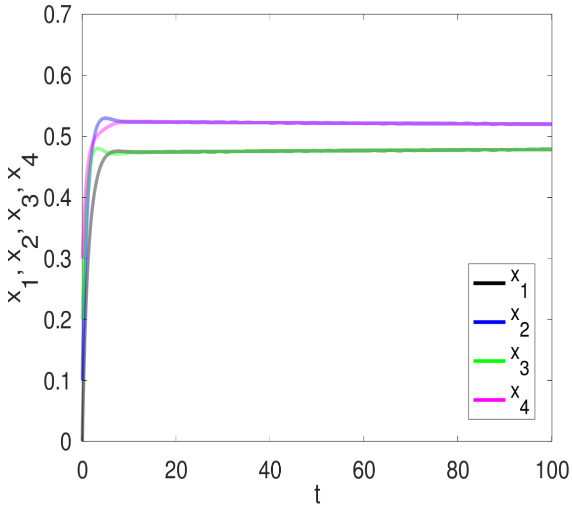

FIGURE 24. Similar as for Figure 23, but now for $F = \frac{1}{2}$.

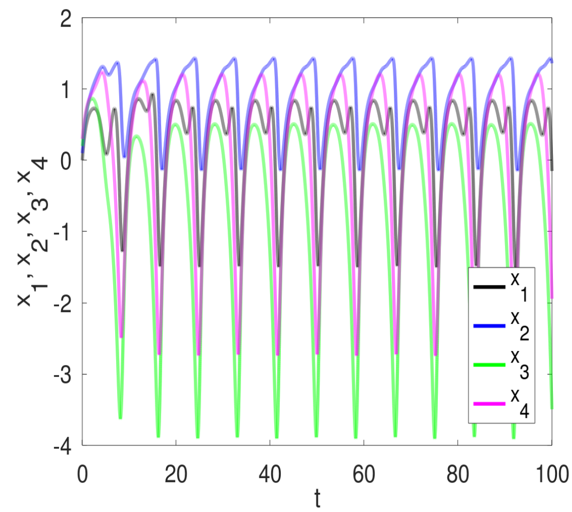

FIGURE 25. As for Figure 23, but now for $F = 1$.

we cannot say much about the attractors looking at these plots. So it is still an open problem, what exactly happens in the figures of these attractors.

5.3.2. *The Eigenvalues.* First we study the eigenvalues of the Jacobian matrix at x_F for different dimensions n for this system with $(\alpha, \beta, \gamma) = (-1, 0, -1)$. This system is denoted by the following n equations for $j = 0, \ldots, n-1$ and $F \in \mathbb{R}$,

$$\dot{x}_j = x_{j-1}(x_j - x_{j-1}) - x_j + F.$$

For $n = 1$ the Jacobian matrix is -1. For $n = 2$ the Jacobian matrix at x_F is

$$\begin{pmatrix} F-1 & -F \\ -F & F-1 \end{pmatrix}$$

and its eigenvalues are $\lambda_0 = -1$ and $\lambda_1 = -1 + 2F$. At $F = \frac{1}{2}$ there is a change in stability of the trivial equilibrium, since one eigenvalue $\lambda_1 = 0$. In the case when $n = 3$

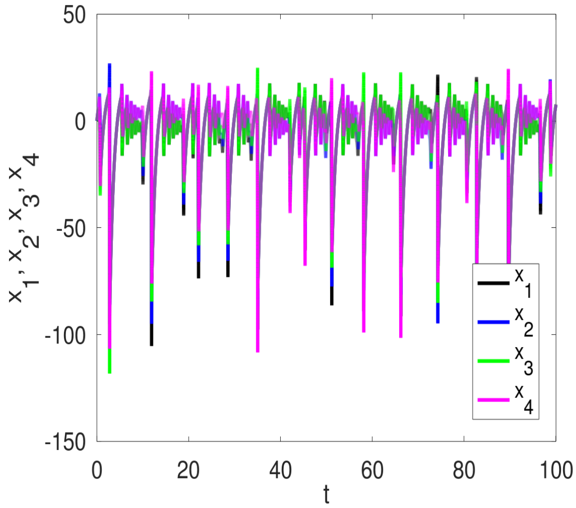

FIGURE 26. Similar as for Figure 23, but now for $F = 20$.

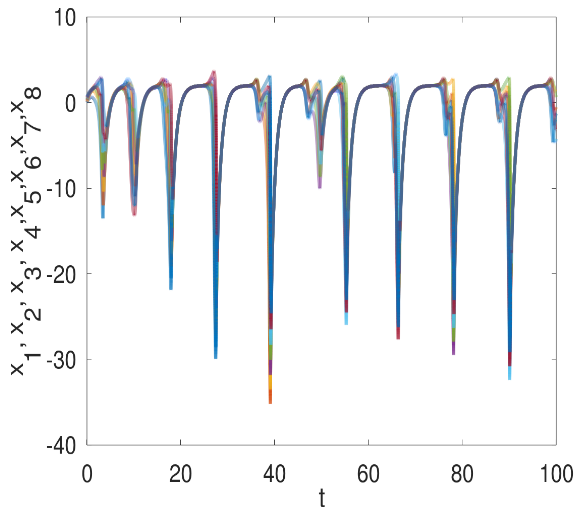

FIGURE 27. For $n = 8$ the solutions of the system with $(\alpha, \beta, \gamma) = (-1, 0, -1)$ for $F = 2$, where initial condition $x_C = (0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7)$.

the Jacobian matrix is

$$\begin{pmatrix} F-1 & 0 & -F \\ -F & F-1 & 0 \\ 0 & -F & F-1 \end{pmatrix}.$$

Moreover its eigenvalues are

$$\lambda_0 = -1, \lambda_1 = -1 + 1\frac{1}{2}F - iF\frac{1}{2}\sqrt{3} \text{ and } \lambda_2 = -1 + 1\frac{1}{2}F + iF\frac{1}{2}\sqrt{3}.$$

This gives us that at $F_H = \frac{2}{3}$ there occurs a Hopf bifurcation, where $\lambda_1 = -i\frac{1}{3}\sqrt{3}$ and $\lambda_2 = i\frac{1}{3}\sqrt{3}$.

Now let the dimension $n \geq 4$. Denote the first row of the Jacobian matrix by $(c_0, c_1, \ldots, c_{n-1})$, where $c_0 = F - 1, c_{n-1} = -F$ and $c_k = 0$ for any $k \neq 0$. Therefore the eigenvalue for

$j = 0, \dots, n-1$ is

$$\lambda_j(F,n) = -1 + F\left(1 - \cos\left(\frac{2\pi j}{n}\right)\right) + iF\left(-\sin\left(\frac{2\pi j}{n}\right)\right).$$

The next step is to write every eigenvalue out for some different n. First take $n = 4$, then

$$\begin{aligned}\lambda_0 &= -1,\\ \lambda_1 &= -1 + F - iF,\\ \lambda_2 &= -1 + 2F,\\ \lambda_3 &= -1 + F + iF.\end{aligned}$$

This gives us that for $F = \frac{1}{2}$ one eigenvalue is zero and the rest has negative real part. When $F = 1$, then we have a Hopf bifurcation, because λ_1 and λ_3 forms a complex eigenvalue pair.

Now for $n = 5$ we have a eigenvalue pair $\{\lambda_2, \lambda_3\}$ at $F = \frac{1}{1-\cos\left(\frac{4\pi}{5}\right)}$ and a eigenvalue pair $\{\lambda_1, \lambda_4\}$ at $F = \frac{1}{1-\cos\left(\frac{2\pi}{5}\right)}$. So we have a Hopf bifurcation at two different F, where at $F_H = \frac{1}{1-\cos\left(\frac{4\pi}{5}\right)}$ the Hopf bifurcation is supercritical, see Appendix C and table 4.

For $n = 6$ we have

$$\begin{aligned}\lambda_0 &= -1,\\ \lambda_1 &= -1 + \frac{1}{2}F + iF\left(-\frac{1}{2}\sqrt{3}\right),\\ \lambda_2 &= -1 + 1\frac{1}{2}F + iF\left(-\frac{1}{2}\sqrt{3}\right),\\ \lambda_3 &= -1 + 2F,\\ \lambda_4 &= -1 + 1\frac{1}{2}F + iF\left(\frac{1}{2}\sqrt{3}\right),\\ \lambda_5 &= -1 + \frac{1}{2}F + iF\left(\frac{1}{2}\sqrt{3}\right).\end{aligned}$$

Here we can conclude that for $F = \frac{1}{2}$ there is a change in stability. At $F = 2$ the complex eigenvalue pair is $\{\lambda_1, \lambda_5\}$ and at $F = \frac{2}{3}$ there is the complex eigenvalue pair $\{\lambda_2, \lambda_4\}$.

When $n = 7$ we have a Hopf bifurcation at three different F. At these three different F we have the eigenvalue pairs $\{\lambda_1, \lambda_6\}$, $\{\lambda_2, \lambda_5\}$ or $\{\lambda_3, \lambda_4\}$. For $n = 8$ we have that $\lambda_4 = 0$ at $F = \frac{1}{2}$. Moreover if $F = \frac{1}{1+\frac{1}{2}\sqrt{2}}$, then the complex eigenvalue pair is $\{\lambda_3, \lambda_5\}$ and if $F = \frac{1}{1-\frac{1}{2}\sqrt{2}}$, then we have the complex eigenvalue pair $\{\lambda_1, \lambda_7\}$. In the next subsubsection we will treat the bifurcations of this system.

5.3.3. *Bifurcations.* First we will investigate what happens with this system when $n = 4$. According to the pictures we see that something different happens for this system than the other systems. It seems that there occurs chaos at some point. As said before, we have a Hopf bifurcation at $F_H = 1$. If we calculate the first Lyapunov coefficient for this situation, then this is -1. Therefore this Hopf bifurcation is supercritical.

Although at $F = F_H = 1$ one eigenvalue has positive real part and the rest negative. This means that we have a change in stability before $F_H = 1$, that is when $F = \frac{1}{2}$. Then the eigenvalue $\lambda_2 = 0$. So before the Hopf bifurcation there occurs a bifurcation, which is not a saddle-node bifurcation, because the trivial equilibrium $x_F = (F, F, F, F)$

exists when $F > \frac{1}{2}$. Thus we have two options, it can be a Pitchfork bifurcation or a Transcritical bifurcation.

To check this, try an equilibrium solution of the form $x_P = (a, b, a, b)$, where a and b satisfy the equations

$$\begin{aligned} b(a-b) - a + F &= 0 \\ a(b-a) - b + F &= 0. \end{aligned}$$

One of the solutions of the equilibrium is when $a = b = F$, but this is already the trivial equilibrium x_F. To obtain other forms of solutions we rewrite the last equations into

$$\begin{aligned} a &= \frac{b^2 - F}{b-1} \implies \\ b&\frac{b^2-F}{b-1} - \frac{(b^2-F)^2}{(b-1)^2} - b + F = 0, \end{aligned}$$

solving this we get

$$-2b^3 + b^2(2+2F) + b(-1-F) + F(1-F) = 0.$$

This expression we can rewrite into the form $(b-p)(b-q)(b-r) = 0$, where take $p = F$ and q, r have to be calculated using Vieta's formulas, which are

$$\begin{aligned} p + q + r &= 1 + F, \\ pq + qr + rp &= \frac{1}{2} + \frac{1}{2}F, \\ pqr &= \frac{1}{2}F(1-F). \end{aligned}$$

After many calculations we get the result that

$$\begin{aligned} r_{+/-} &= \frac{-1 \pm \sqrt{-1+2F}}{-2} \quad \text{and} \\ q_{+/-} &= \frac{-1 \mp \sqrt{-1+2F}}{-2}. \end{aligned}$$

If we do this also for a, then we get the same result as for b. Hence, another solutions of a and b are

$$\begin{aligned} a &= \frac{1}{2} + \frac{1}{2}\sqrt{-1+2F}, \\ b &= \frac{1}{2} - \frac{1}{2}\sqrt{-1+2F}. \end{aligned}$$

With these a and b we have two new equilibria, these are $x_{P1} = (a, b, a, b)$ and $x_{P2} = (b, a, b, a)$. Therefore there are these two equilibria and x_F when $F > \frac{1}{2}$ and hence, there occurs a Pitchfork bifurcation at $F = \frac{1}{2}$.

Now the interesting part is what happens at these two equilibria x_{P1} and x_{P2} further. First the Jacobian matrix at $x_{P1} = (a, b, a, b)$ is

$$J_{P1} = \begin{pmatrix} -1+b & 0 & 0 & a-2b \\ b-2a & -1+a & 0 & 0 \\ 0 & a-2b & -1+b & 0 \\ 0 & 0 & b-2a & -1+a \end{pmatrix}.$$

Swap in the last matrix a and b, then we have the Jacobian matrix J_{P2} at x_{P2}. Note that these Jacobian matrices are not circulant matrices. Thus we cannot use the expression of the eigenvalues of a circulant Jacobian matrix in this case.

We can calculate the eigenvalues by using the equation $\det(J_{P1} - \lambda I) = 0$ and this is the same as

$$
\begin{aligned}
&(-1+b-\lambda)^2(-1+a-\lambda)^2 - (a-2b)^2(b-2a)^2 = 0 \iff \\
&((-1+b-\lambda)(-1+a-\lambda))^2 - ((a-2b)(b-2a))^2 = 0 \iff \\
&((-1+b-\lambda)(-1+a-\lambda) + (a-2b)(b-2a))\times \\
&\qquad ((-1+b-\lambda)(-1+a-\lambda) - (a-2b)(b-2a)) = 0 \implies \\
&(-1+b-\lambda)(-1+a-\lambda) + (a-2b)(b-2a) = 0, \\
&(-1+b-\lambda)(-1+a-\lambda) - (a-2b)(b-2a) = 0.
\end{aligned}
$$

Solving these equations, the eigenvalues of the matrix J_{P1} are

$$
\begin{aligned}
\lambda_0 &= -1 + \frac{1}{2}b + \frac{1}{2}a + 1\frac{1}{2}\sqrt{a^2 + b^2 - 2\frac{4}{9}ab}, \\
\lambda_1 &= -1 + \frac{1}{2}b + \frac{1}{2}a - 1\frac{1}{2}\sqrt{a^2 + b^2 - 2\frac{4}{9}ab}, \\
\lambda_2 &= -1 + \frac{1}{2}b + \frac{1}{2}a + \sqrt{-1\frac{3}{4}(a^2 + b^2) + 4\frac{1}{2}ab}, \\
\lambda_3 &= -1 + \frac{1}{2}b + \frac{1}{2}a - \sqrt{-1\frac{3}{4}(a^2 + b^2) + 4\frac{1}{2}ab}.
\end{aligned}
$$

Filling in a and b, we get

$$
\begin{aligned}
\lambda_0 &= -\frac{1}{2} + \sqrt{2\frac{1}{4}F - 1\frac{3}{8} + 1\frac{3}{8}\sqrt{-1+2F}}, \\
\lambda_1 &= -\frac{1}{2} - \sqrt{2\frac{1}{4}F - 1\frac{3}{8} + 1\frac{3}{8}\sqrt{-1+2F}}, \\
\lambda_2 &= -\frac{1}{2} + \sqrt{-1\frac{3}{4}F + 1\frac{1}{8} - 1\frac{1}{8}\sqrt{-1+2F}}, \\
\lambda_3 &= -\frac{1}{2} - \sqrt{-1\frac{3}{4}F + 1\frac{1}{8} - 1\frac{1}{8}\sqrt{-1+2F}}.
\end{aligned}
$$

If we swap a and b in the expressions of the eigenvalues of the Jacobian matrix J_{P1}, we get that these eigenvalues are eigenvalues of J_{P2}. Moreover at $F = \frac{3}{5}$, λ_0 is zero. This means that we have change in stability.

From the previous observations, we can come up with a theorem for this system with $(-1, 0, -1)$, that shows us that there always occurs a Pitchfork bifurcation at $F = \frac{1}{2}$ for any $n = 2k$, where $k \in \mathbb{Z}_{>0}$.

Theorem 6. *Consider the system with $(\alpha, \beta, \gamma) = (-1, 0, -1)$. For $k \in \mathbb{Z}_{>0}$, for all $n = 2k$, there occurs a Pitchfork bifurcation at $F = \frac{1}{2}$. Furthermore, at $F = \frac{1}{2}$ the eigenvalue $\lambda_{\frac{n}{2}} = -1 + 2F$ is zero, for any $n = 2k$.*

Proof. First we will show that for any $n = 2k$, $k \in \mathbb{Z}_{>0}$, $\lambda_{\frac{n}{2}} = -1 + 2F$. Thus, the eigenvalue for this system is

$$
\begin{aligned}
\lambda_{(j=\frac{n}{2})}(F, n = 2k) &= -1 + F\left(1 - \cos\left(\frac{2\pi j}{n}\right)\right) - iF\sin\left(\frac{2\pi j}{n}\right) \\
&= -1 + F(1 - \cos(\pi)) - i\sin(\pi) \\
&= -1 + 2F.
\end{aligned}
$$

Therefore if $F = \frac{1}{2}$, then $\lambda_{\frac{n}{2}} = 0$, for all $n = 2k$.

So at $F = \frac{1}{2}$ we have a change in stability. For $k \in \mathbb{Z}_{>0}$ and $\forall n = 2k$, it is not a saddle-node bifurcation, because the trivial equilibrium x_F still exists when $F > \frac{1}{2}$. Now we will prove that we have a Pitchfork bifurcation and not a Transcritical bifurcation.

Try an equilibrium solution of the form $x_{P,n} = (a, b, a, b, a, b, \dots, a, b)$, where a and b satisfy

$$\begin{aligned} b(a-b) - a + F &= 0, \\ a(b-a) - b + F &= 0. \end{aligned} \tag{5.3}$$

Note that if the first point of the equilibrium is a, then the last point is b and vica versa, because the dimension n is a multiple of 2. Since we have the same equations (5.3) as for the case when $n = 4$, we get the same as before using Vieta's formulas,

$$\begin{aligned} a &= \frac{1}{2} + \frac{1}{2}\sqrt{-1+2F}, \\ b &= \frac{1}{2} - \frac{1}{2}\sqrt{-1+2F}. \end{aligned}$$

With these values of a and b, we have three equilibria, namely x_F, $x_{P1,n} = (a, b, a, b, a, b, \dots, a, b)$ and $x_{P2,n} = (b, a, b, a, b, a, \dots, b, a)$. This shows us that we indeed have a Pitchfork bifurcation at $F = \frac{1}{2}$, for all $n = 2k$, where $k \in \mathbb{Z}_{>0}$. ☐

Note that the Jacobian matrix of $x_{P1,n}$ is for any $n = 2k$,

$$J_{P1,n} = \begin{pmatrix} -1+b & 0 & \dots & 0 & a-2b \\ b-2a & -1+a & \dots & 0 & 0 \\ 0 & a-2b & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & b-2a & -1+a \end{pmatrix}. \tag{5.4}$$

As before, if we swap a and b, we get the Jacobian matrix $J_{P2,n}$ of the equilibrium $x_{P2,n}$. This matrix has the following properties.

Theorem 7. *Consider the matrix $J_{P1,n}$ (5.4) for given a and b and for any $n = 2k$, where $k \in \mathbb{Z}_{>0}$. Then the eigenvalues of this matrix satisfy*

$$(-1+b-\lambda)^k(-1+a-\lambda)^k - (a-2b)^k(b-2a)^k = 0$$

and therefore these eigenvalues are given by

$$\lambda_{\pm} = -1 + \frac{1}{2}(a+b) \pm \frac{1}{2}\sqrt{(2-a-b)^2 - 4C},$$

where $C = 1 - a - b + ab - ((a-2b)^k(b-2a)^k)^{\frac{1}{k}}$.

Proof. To prove that $(-1+b-\lambda)^k(-1+a-\lambda)^k - (a-2b)^k(b-2a)^k = 0$ holds, we use Induction on k. The first, Base step is to check that it holds for $k = 1$, so for $n = 2$. Then the matrix $J_{P1,2}$ is

$$\begin{pmatrix} -1+b & a-2b \\ b-2a & -1+a \end{pmatrix}.$$

This has eigenvalues, that satisy

$$\begin{aligned} &(-1+b-\lambda)(-1+a-\lambda) - (a-2b)(b-2a) = 0, \\ &\lambda_{\pm} = -1 + \frac{1}{2}(a+b) \pm \frac{1}{2}\sqrt{(2-a-b)^2 - 4(1-a-b-4ab+2(a^2+b^2))}. \end{aligned}$$

This shows the Base step.

Now for the Induction step we assume that it holds for $n = 2k$, which means the eigenvalues of the matrix $J_{P1,n}$ satisfy $(-1+b-\lambda)^k(-1+a-\lambda)^k - (a-2b)^k(b-2a)^k = 0$. We have to show it for $n = 2(k+1) = 2k+2$. Consider the matrix $J_{P1,n=2k}$

$$\begin{pmatrix} -1+b & 0 & \dots & 0 & a-2b \\ b-2a & -1+a & \dots & 0 & 0 \\ 0 & a-2b & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & b-2a & -1+a \end{pmatrix}.$$

Then using this, the matrix $J_{P1,n=2k+2}$ is almost the same as $J_{P1,n=2k}$, but with two extra rows and columns. So $J_{P1,n=2k+2}$ equals

$$\begin{pmatrix} -1+b & 0 & \dots & 0 & 0 & 0 & a-2b \\ b-2a & -1+a & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b-2a & -1+a & 0 & 0 \\ 0 & 0 & \dots & 0 & a-2b & -1+b & 0 \\ 0 & 0 & \dots & 0 & 0 & b-2a & -1+a \end{pmatrix}.$$

Further, the determinant of $(J_{P1,n=2k+2} - \lambda I)$ is

$$\det\begin{pmatrix} -1+b-\lambda & 0 & \dots & 0 & 0 & 0 & a-2b \\ b-2a & -1+a-\lambda & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b-2a & -1+a-\lambda & 0 & 0 \\ 0 & 0 & \dots & 0 & a-2b & -1+b-\lambda & 0 \\ 0 & 0 & \dots & 0 & 0 & b-2a & -1+a-\lambda \end{pmatrix} = 0.$$

Looking at the blue part, the last equality is the same as

$$(-1+b-\lambda)^k(-1+a-\lambda)^k \cdot \det\begin{pmatrix} -1+b-\lambda & 0 \\ b-2a & -1+a-\lambda \end{pmatrix}$$
$$-(a-2b)^k(b-2a)^k \cdot \det\begin{pmatrix} a-2b & -1+b-\lambda \\ 0 & b-2a \end{pmatrix} = 0 \iff$$
$$(-1+b-\lambda)^k(-1+a-\lambda)^{k+1} - (a-2b)^k(b-2a)^{k+1} = 0.$$

This gives that the eigenvalues of $J_{P1,n=2k+2}$ satisfy $(-1+b-\lambda)^k(-1+a-\lambda)^{k+1} - (a-2b)^k(b-2a)^{k+1} = 0$. Therefore we have showed the Induction step. Hence, the eigenvalues satisfy for all $k \in \mathbb{Z} > 0$,

$$(-1+b-\lambda)^k(-1+a-\lambda)^k - (a-2b)^k(b-2a)^k = 0.$$

If we solve the last equation, we get

$$(-1+b-\lambda)(-1+a-\lambda) = ((a-2b)^k(b-2a)^k)^{\frac{1}{k}}.$$

Then the eigenvalues satisfy

$$\lambda_\pm = -1 + \frac{1}{2}(a+b) \pm \frac{1}{2}\sqrt{(2-a-b)^2 - 4C},$$

where $C = 1 - a - b + ab - ((a-2b)^k(b-2a)^k)^{\frac{1}{k}}$. Hence, this proves the theorem. □

Definition 6. *Since the matrix J_{P1} has a special form and a special form of eigenvalues, we call this matrix a Double Row Circulant matrix. Note that this matrix is a square $n \times n$-matrix.*

Dimension n	Important eigenvalues	F	Bifurcation
4	$0, -1, -0.5+0.387i, -0.5-0.387i$	0.6	Pitchfork
6	$0.236i, -0.236i$	0.6161	Hopf
8	0	0.6	Pitchfork?
8	$0.419i, -0.419i$	0.6486	Hopf
10	$0.13i, -0.13i$	0.605	Hopf
12	0	0.6	Pitchfork?
12	.	.	Hopf

Table 3. Certain bifurcations and important eigenvalues for some dimensions n for the system with $(\alpha, \beta, \gamma) = (-1, 0, -1)$.

Furthermore if we go further numerically, we see the following observations, which is summarized in table 3. In the first column of this table the dimension n is stated, the third the eigenvalues, which we get at some value F, which is in the fourth column. In the last column we have the kind of the bifurcation. One observation is that for $n = 4$ at $F = \frac{3}{5}$ we have that one of the eigenvalues is zero and the rest has negative real part and if F becomes larger, than only one eigenvalue is positive and becomes larger and the rest eigenvalues still have negative real part. For $n = 4, 8, 12$ a first Pitchfork bifurcation always occurs at $F = \frac{1}{2}$ and maybe a second one occurs at $F = \frac{3}{5}$ for $n = 8, 12$.

For $n = 4$ we will show that we have a second Pitchfork at $F = \frac{3}{5}$. To show this we need seven equilibria, we already have the equilibria $x_F = (F, F, F, F), x_{P1} = (a, b, a, b)$ and $x_{P2} = (b, a, b, a)$. Try an equilibrium solution of the form $x_{PP} = (c, d, e, f)$, where c, d, e and f satisfy the following

$$
\begin{aligned}
f(c-f) - c + F &= 0,\\
c(d-c) - d + F &= 0,\\
d(e-d) - e + F &= 0,\\
e(f-e) - f + F &= 0.
\end{aligned}
\tag{5.5}
$$

We want to know what the values of c, d, e and f are and to prove immediately that it is a Pitchfork bifurcation by using the Newton's method in Matlab. For this we use a random initial condition. Afer many runnings with different random initial conditions in Matlab, we get the result $(c, d, e, f) = (0.75, 0.18, 0.71, 0.38)$. Since we have the values of c, d, e and f, we can form four other equilibria, which are $x_{PP1} = (c, d, e, f), x_{PP2} = (f, c, d, e), x_{PP3} = (e, f, c, d)$ and $x_{PP4} = (d, e, f, c)$. Therefore there are seven equilibria when $F > \frac{3}{5}$, so there occurs a Pitchfork bifurcation at $F = \frac{3}{5}$ for $n = 4$.

In conclusion, there occurs a first Pitchfork bifurcation at $F = \frac{1}{2}$ for every $n = 2k$, for $k \in \mathbb{Z}$, and a second one at $F = \frac{3}{5}$ for $n = 4$. Moreover for $n = 4$ there is a supercritical Hopf bifurcation at $F = 1$ and the schematic bifurcation diagram is shown in Figure 39.

5.3.4. *The Lyapunov Exponents for this System.* We apply the algorithm of determining the Lyapunov exponents to the system with $(\alpha, \beta, \gamma) = (-1, 0, -1)$ for dimension $n = 4, 5, 6, 7$ and 8. Thus in the case $n = 4$ we will have four Lyapunov exponents. For programming these Lyapunov exponents we use the Matlab codes from the reference [Gov04].

First we plot the Lyapunov exponents against time t in the Figures 40 and 41 for $F = 1$ and $F = 7$ for dimension $n = 4$. We see that for $F = 1$ two Lyapunov exponents becomes zero and the rest are negative. For $F = 7$ we have one positive Lyapunov exponent and one is zero. In the Figures 42, 43 and 44 we see the plots of the Lyapunov exponents against the parameter value F for $n = 4$. For these plots we use the initial condition $x_C = (0, 0.1, 0.2, 0.3)$.

Moreover according to the Figures 42 and 43 we have a change in stability at $F = \frac{1}{2}$ and at $F = \frac{3}{5}$, because one of the Lyapunov exponents is zero. So at these values F we have a Pitchfork bifurcation, which we have proved. If F is between 0.85 and 0.86 one of the Lyapunov exponents becomes even positive, which means that there begins chaos.

In Figures 44 and 45 the Lyapunov exponents is plotted against F for $n = 5$. Here we indeed have a change in stability at $F \approx \frac{1}{1.81}$, where a supercritical Hopf bifurcation occurs. If F is around $\frac{7}{10}$, then chaos starts, because we have a positive Lyapunov exponent. For dimension $n = 6$ we have the pictures 46 and 47 and indeed, we have a Pitchfork at $F = \frac{1}{2}$. Furthermore we have a supercritical Hopf bifurcation for $F = \frac{2}{3}$. Since one of the Lyapunov exponents is positive when F is between 0.65 and 0.66, the chaos begins.

The Figures 48 and 49 gives the graphs of the Lyapunov exponents for $n = 7$. Here around $F = 0.55$ we see a change in stability, because one of the Lyapunov exponents is zero. Further if F is around 0.66, then chaos starts, since one Lyapunov exponent becomes then positive. For the last dimension $n = 8$ we have the pictures 50 and 51 and at $F = \frac{1}{2}$ a Pitchfork bifurcation occurs as expected. At $F \approx 0.6486$ we have a Hopf bifurcation and when F is between 0.64 and 0.65 there starts chaos.

An observation is that for $n = 4$ and $n = 8$ one part, i.e. half of the number of the Lyapunov exponents, and the other part of the Lyapunov exponents come together at two points, which are approx -0.33 and -0.66, at $F \approx 0.556$.

We discussed the dynamical properties of this system and of the two previous systems with $(\alpha, \beta, \gamma) = (-1, 0, -2)$ and $(-1, -1, -2)$. In the next section we will summarize this and give the conclusion.

6. Conclusion and Open Problems

Here we will summarize and conclude what the differences are between our modified models, especially the three systems with $(\alpha, \beta, \gamma) = (-1, 0, -2)$, $(-1, -1, -2)$ and $(-1, 0, -1)$, and the original monoscale Lorenz-96 model. At the end we will state some observations and ideas for future research.

6.1. Summary.

Recall that the main purpose is to compare the dynamical properties of the modified systems with the original monoscale Lorenz-96 model (3.1). Its modifications can be obtained by changing the structure of the nonlinear terms in the original Lorenz-96 model and these modified systems are denoted by (4.1) with (α, β, γ), where we can use F as a bifurcation parameter. Before we described their dynamics, we started with the preliminary theory about sub- and supercritical Hopf bifurcations and the Lyapunov exponents. Thereafter we stated the main results about the dynamics of the Lorenz-96 model. Then we discussed the general properties of the modified system (4.1) for some α, β, γ, such as for all $j = 0, 1, \ldots, n-1$ the eigenvalues $\lambda_j(F, n)$ (4.2) of the Jacobian matrix at the trivial equilibrium $x_F = (F, F, \ldots, F)$ don't depend on α and the wave number never decreases as the dimension n increases obtained from the numerical experiments for many different systems with (α, β, γ).

The important result of Section 4.2 is a concerning escaping orbits. From the inequality (4.8) there exist non-escaping orbits and this still holds when the parameter F satisfies the inequality (4.9) meaning that we are allowed to take F larger as n increases. In Section 5 we studied the dynamical properties of three specific systems with $(\alpha, \beta, \gamma) = (-1, 0, -2), (-1, -1, -2)$ and $(-1, 0, -1)$. This dynamics and the bifurcations of these three systems will be summarize in the following subsection and we will obtain the main differences between the Lorenz-96 model and its modifications to answer our main purpose.

6.2. **Differences.**

6.2.1. *Escaping Orbits.* Now we will discuss the main differences between our modified systems and the original Lorenz-96 model. To determine whether there are the escaping orbits, we defined the functions A, B and C by (4.5). We had that the maximum of A is $A_{\max}$ given by (4.6) and the minimum of B is one for the modified systems. For these systems there only exist non-escaping orbits when the inequality (4.8) holds. Since $A_{\max} = 0$ for the Lorenz-96 model, there are only non-escaping orbits, while there can be escaping orbits for some modified systems.

6.2.2. *Wave Number and Period.* Moreover the wave number for the Lorenz-96 model and our systems increases as n tends to infinity. However the period tends to a constant for the Lorenz-96 model as n increases for $F > 0$, while this is not the case for some systems with (α, β, γ). For example, the period increases for the system with $(-1, 0, -1)$ and $(-1, 0, -2)$, and this is the case for only $F < 0$ with $(-1, -1, -2)$. Furthermore when $F < 0$ the period increases as n increases for the Lorenz-96 model. Thus the period tends to a constant or increases as n becomes larger for every general modified system.

6.2.3. *Bifurcations.* The eigenvalues of the Jacobian matrix at the trivial equilibrium are given by (4.2) for any general modified system. Notice that if $(\alpha, \beta, \gamma) = (-1, 1, -2)$, then we have the eigenvalues for the Lorenz-96 model. By studying these eigenvalues too, we obtained the bifurcations of the modified systems and the original model.
6.2.3.1 *Case* $(\alpha, \beta, \gamma) = (-1, 0, -1)$. Note that for $F < 0$ the trivial equilibrium $x_F = (F, F, \ldots, F)$ is stable and we only have bifurcations for $F > 0$ for this system. This is not the case for the Lorenz-96 model, there are bifurcations for $F > 0$ and $F < 0$. Furthermore for $F < 0$ there occurs a first Pitchfork bifurcation for $n \geq 4$ is even and no one for $F > 0$, only a first Hopf bifurcation or Hopf-Hopf bifurcation. While for this system there are indeed first Pitchfork bifurcations and Hopf bifurcations for $F > 0$. Notice that in the case when n is even, for this case there occurs the first Pitchfork bifurcation at $F = \frac{1}{2}$ and for the Lorenz-96 model at $F = -\frac{1}{2}$.
6.2.3.2 *Case* $(\alpha, \beta, \gamma) = (-1, 0, -2)$. Similar as for the system with $(-1, 0, -1)$, the system with $(-1, 0, -2)$ there are only bifurcations for $F > 0$, while there are bifurcations for the Lorenz-96 model for all $F \neq 0$. Moreover if n is a multiple of 4, then we have a degenerate bifurcation. This means that two equilibria "lines" appear simultaneously at $F = \frac{1}{2}$ and two eigenvalues are zero. However this is not the case for the Lorenz-96 model for every F.
6.2.3.3 *Case* $(\alpha, \beta, \gamma) = (-1, -1, -2)$. For the system with $(-1, -1, -2)$ there occurs a Pitchfork bifurcation at $F = -\frac{1}{2}$ when n is a multiple of 6. This is also true for the Lorenz-96 model with the same equilibria x_F, $x_{P1} = (a, b, a, b, \ldots, a, b)$ and $x_{P2} = (b, a, b, a, \ldots, b, a)$, where

$$a = -\frac{1}{2} + \frac{1}{2}\sqrt{-1 - 2F},$$
$$b = -\frac{1}{2} - \frac{1}{2}\sqrt{-1 - 2F}.$$

Actually, for the Lorenz-96 model we have a Pitchfork bifurcation at $F = -\frac{1}{2}$ for even $n \geq 4$, which is not always true for this system, only if n is a multiple of 6. Moreover at $F = 1$ there occurs a degenerate bifurcation, which does not appear for the Lorenz-96 model. In this case for $n = 4$ and $F > 0$ the first Hopf bifurcation is subcritical, while for the Lorenz-96 model if the first bifurcation is a Hopf bifurcation, then this is always supercritical [vK18]. These were the main differences between our modified systems and the original Lorenz-96 model.

6.3. **Observations and Ideas.**

6.3.1. *Period.* Now we discuss some observations and ideas, which can be used for future research. First we focus on the period (4.13) for the wave number $l_1^+(n)$ (4.14), namely for $F > 0$. As for the system with $(\alpha, \beta, \gamma) = (-1, 0, -2)$ we see that system with $(-1, 4, -2)$ has the same graph of the period and the same values for the period, except at $n = 6$. Then the difference between these two is about 1.00922, but the rest are identically the same. If we take $(\alpha, \beta, \gamma) = (-1, 8, -2)$, $(-1, 12, -2)$ and $(-1, 16, -2)$, then these systems has zero period at $n = 4, 8, 12, \ldots, 96, 100$ as for the system with $(-1, 0, -2)$. Actually the period can never be zero, only positive. Here with zero period we mean that there is no first Hopf bifurcation.

Thus if β is a multiple of 4 and $\gamma = 4k + 2$ for $k \in \mathbb{Z}$, then for the system with $(\alpha = -1, \beta, \gamma)$ there is no first Hopf bifurcation when n is a multiple of 4. One observation is that for some systems when n is not only a multiple of 4 there are no first Hopf bifurcations. This can happen if β and/or γ become larger. So the idea is as follows.

Idea 1. *Let $n \geq 4$ and let the wave number be $l_1^+(n)$ defined as* (4.14)*. Let $\beta = 4m$ and $\gamma = 4k + 2$ for $m, k \in \mathbb{Z}$, then for the system with $(\alpha, \beta, \gamma) = (-1, 4m, 4k + 2)$ there is no first Hopf bifurcation when n is a multiple of* 4.

Moreover if we take $\beta = \pm 1$, then it turns out that the period for $F > 0$, tends to a constant as n tends to infinity. How larger $\gamma > 0$ is, how longer it takes that the period reaches that constant and similarly for γ is negative and becomes very small.

Idea 2. *Let $n \geq 4$ and let the wave number be $l_1^+(n)$ defined as* (4.14)*. Take $\beta = \pm 1$ and any γ, then the system with $(\alpha, \beta, \gamma) = (-1, \pm 1, \gamma)$ has the period T given by* (4.13)*, that converges to a constant as $n \to \infty$.*

In the next subsubsection we will discussed an open problem about escaping orbits.

6.3.2. *Escaping Orbits.* In Section 4.2 we studied when there are non-escaping orbits for the general modified models, where we defined the function A as (4.5). This has maximum $A_{\max}$ (4.6). As mentioned in Section 4.2, we determined this maximum by numerical experiments, but why does $A_{\max}$ decrease as n increases? We obtained no results by studying the components of the random vector x, because these don't have a pattern in the way they are chosen. We tried to get an expression for $A_{\max}$ using Lagrange multipliers, but no result. Another option for future work is to use Lagrange multipliers numerical methods to obtain a formula for $A_{\max}$ such that we can know why $A_{\max}$ decreases as n becomes larger.

6.3.3. *Attractors.* For a specific system with $(\alpha, \beta, \gamma) = (-1, 0, -1)$ in Section 5.3 we plotted the attractors for some bifurcation parameter F and some dimension n, see Figures 28, 29, 33, 30, 31, 32, 34, 35, 36 and 37. We discussed why we got such figures. It seems that every solution attracts to or repels from some "line" and we have showned the plots of the distances between this "line" and the solution of this system. This "line" is $\text{span}\{\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\}$ and to be more specific, the distance is between the solution and the orthogonal projection from this solution onto this "line". Although we couldn't much obtain from the plots of these distances to clarify the figures of the attractors. So the open problems are why do these attractors look like a butterfly and do every solution attracts to or repels from some "line" in the plots of these attractors? What exactly do these solutions in the figures of the attractors?

In conclusion, we summarized and discussed the main differences between our modified systems with (α, β, γ) and the original Lorenz-96 model. We treated some open problems and ideas for further research. So for some aspects we could answer our main research question, what the dynamical properties of the modified systems are in comparison with the original monoscale Lorenz-96 model, in great detail, but not for all aspects.

Appendix A. Eigenvalues and Eigenvectors of Circulant Matrices

In this section we discuss how to determine the eigenvalues and the eigenvectors of the circulant matrix C following the article [Gra06]. A circulant matrix is of the following form:

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{(n-1)} \\ c_{(n-1)} & c_0 & c_1 & \cdots & c_{(n-2)} \\ c_{(n-2)} & c_{(n-1)} & c_0 & \cdots & c_{(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix}$$

and every entry $C_{i,j}$ of C is given by $C_{i,j} = c_{(j-i) \mod n}$. Note that every row of this matrix C is a right cyclic shift of the row above it.

Now the eigenvalues and the corresponding eigenvectors of C are the solutions of the equation

$$Cy = \lambda y,$$

which is the same as

$$\begin{pmatrix} c_0 y_0 + ... + c_{n-1} y_{n-1} \\ c_{n-1} y_0 + ... + c_{n-2} y_{n-1} \\ \vdots \\ c_1 y_0 + ... + c_0 y_{n-1} \end{pmatrix} = \lambda \begin{pmatrix} y_0 \\ \vdots \\ y_{n-1} \end{pmatrix}.$$

This gives us n difference equations

$$\sum_{k=m}^{n-1} c_{k-m} y_k + \sum_{k=0}^{m-1} c_{n-m+k} y_k = \lambda y_m, \ \ m = 0, 1, ..., n-1, \tag{A.1}$$

and the eigenvalues and the corresponding eigenvectors are the solutions of these equations too. Further, the n equations (A.1) are equivalent to

$$\sum_{k=0}^{n-1-m} c_k y_{k+m} + \sum_{k=n-m}^{n-1} c_k y_{k-n-m} = \lambda y_m, \tag{A.2}$$

for $m = 0, 1, ..., n-1$.

A good guess for a solution to the equations is $y_k = \rho^k$, because the equations are linear with constant coefficients. First we obtain what happens, if we make this choice and later on we will prove that this is the right guess. Substitute $y_k = \rho^k$ into (A.2) and divide these by ρ^m, we get for $m = 0, 1, ..., n-1$,

$$\sum_{k=0}^{n-1-m} c_k \rho^k + \rho^{-n} \sum_{k=n-m}^{n-1} c_k \rho^k = \lambda.$$

Now take ρ to be one of the n distinct complex n-th roots of unity, in other words $\rho^{-n} = 1$, then an eigenvalue is given by

$$\lambda = \sum_{k=0}^{n-1} c_k \rho^k$$

and the corresponding eigenvector is

$$y = \frac{1}{\sqrt{n}}(1, \rho, \rho^2, ..., \rho^{n-1})^\top.$$

Note that this is a unit eigenvector and it has Euclidean norm equal to 1. Let $\rho_m = e^{-2\pi im/n}$, which means that ρ_m is the complex n-th root of unity. This implies that the eigenvalue is given by

$$\lambda_m = \sum_{k=0}^{n-1} c_k e^{-2\pi imk/n} \tag{A.3}$$

and the corresponding eigenvector is

$$y^{(m)} = \frac{1}{\sqrt{n}}(1, e^{-2\pi im/n}, ..., e^{-2\pi im(n-1)/n})^\top,$$

where $m = 0, 1, ..., n-1$.

Now we show that our guess is the right one. First we want to obtain the sequence $\{c_k\}$ from λ_k by using the Fourier inversion formula, which is the same as using the expression of λ_m (A.3)

$$\begin{aligned} \frac{1}{n}\sum_{m=0}^{n-1} \lambda_m e^{2\pi ilm/n} &= \frac{1}{n}\sum_{m=0}^{n-1}\sum_{k=0}^{n-1}(c_k e^{-2\pi imk/n})e^{2\pi ilm/n} \\ &= \sum_{k=0}^{n-1} c_k \; \frac{1}{n}\sum_{m=0}^{n-1} e^{2\pi i(l-k)m/n} \\ &= c_l, \end{aligned}$$

because

$$\sum_{m=0}^{n-1} e^{2\pi imk/n} = \begin{cases} n & \text{if } k \mod n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

This gives as the sequence $\{c_k\}$, so that the first row of a circulant matrix is the Fourier inversion formula, or also called the inverse discrete Fourier transform, of the eigenvalues. The eigenvalues of a circulant matrix is the same as the discrete Fourier transform of the first row of the circulant matrix. Here the discrete Fourier transform is of the form $\lambda_m = \sum_{l=0}^{n-1} c_l e^{-2\pi iml/n}$. Thus our guess is right. Hence, we indeed have the eigenvalue λ_m and the corresponding eigenvector $y^{(m)}$.

Furthermore, take

$$\begin{aligned} U &= [y^{(0)}|y^{(1)}|...|y^{(n-1)}] \\ &= \frac{1}{\sqrt{n}}[e^{-2\pi imk/n}; m, k = 0, 1, ..., n-1] \end{aligned}$$

and let

$$\Lambda = \begin{pmatrix} \lambda_0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} \end{pmatrix}$$

is the diagonal matrix, also denoted by $\Lambda = \text{diag}(\lambda_k)$.
This implies that equation $Cy^{(m)} = \lambda_m y^{(m)}$ is the same as

$$CU = Y\Lambda. \tag{A.4}$$

Now we claim that U is unitary. Indeed, let $u_{i,j}$ be the (k,j)-th element of the matrix UU^*, where $*$ means the conjugate transpose and $U^* = \overline{U}^\top$. Note that $u_{k,j}$ is the k-th row of U times the j-th column of U^*. So

$$
\begin{aligned}
u_{k,j} &= \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & e^{-2\pi ik/n} & e^{-2\pi i2k/n} & \cdots & e^{-2\pi i(n-1)k/n} \end{pmatrix} \times \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ e^{2\pi ij/n} \\ \vdots \\ e^{2\pi i(n-1)j/n} \end{pmatrix} \\
&= \frac{1}{n} \sum_{m=0}^{n-1} e^{2\pi im(j-k)/n} \\
&= \begin{cases} 1 & \text{if } (k-j) \mod n = 0 \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}
$$

Therefore $UU^* = I$. Similarly, it follows that $U^*U = I$.

Hence, equation (A.4) gives us

$$C = U\Lambda U^* \text{ and } \Lambda = U^*CU.$$

It is obvious that Λ is normal. Also, C is normal, i.e. $C^*C = CC^*$, because using $U^*U = UU^* = I$,

$$
\begin{aligned}
C^*C &= U\Lambda^*U^*U\Lambda U^* \\
&= U\Lambda^*\Lambda U^* \\
&= U\Lambda\Lambda^*U^* \\
&= U\Lambda U^*U\Lambda^*U^* \\
&= CC^*.
\end{aligned}
$$

So we have the following theorem, which we already proved.

Theorem 8. *The eigenvalues of any circulant matrix C is given by*

$$\lambda_m = \sum_{k=0}^{n-1} c_k e^{-2\pi imk/n},$$

for $m = 0, 1, ..., n-1$ and the corresponding eigenvector is for $m = 0, 1, ..., n-1$

$$y^{(m)} = \frac{1}{\sqrt{n}} (1, e^{-2\pi im/n}, ..., e^{-2\pi im(n-1)/n})^\top.$$

Let $U = [y^{(0)}|y^{(1)}|...|y^{(n-1)}]$ and $\Lambda = diag(\lambda_k)$, then

$$C = U\Lambda U^*.$$

Appendix B. Proof of Theorem 4

In this section we will prove Theorem 4 about the positive first Lyapunov coefficient for the system with $(\alpha, \beta, \gamma) = (-1, -1, -2)$ and $n = 4$. Recall that this means that the first Hopf bifurcation for this case is subcritical.

Proof. First, the eigenvalues of the Jacobian matrix at the trivial equilibrium $x_F = (F, F, F, F)$ for this system with $(-1, -1, -2)$ and $n = 4$ are $\lambda_0 = -1, \lambda_1 = i, \lambda_2 = -3$ and $\lambda_3 = -i$ for $F = F_H = 1$. In this case there occurs a Hopf bifurcation at $F_H = 1$ a.nd $\omega_0 = 1$

Now the question is, is this bifurcation sub- or supercritical? To obtain this, we need to calculate the first Lyapunov coefficient $l_1(F_H = 1) = l_1$. The expression of the first

Lyapunov coefficient corresponding to the Hopf bifurcation at x_F for the l-th eigenvalue pair is as follows in general using [Kuz98] and [vKS18a],

$$\begin{aligned} l_1(F_H) = \frac{1}{2\omega_0}\mathrm{Re}(\langle p, C(q,q,\bar{q})\rangle - 2\langle p, B(q, A^{-1}B(q,\bar{q}))\rangle \\ + \langle p, B(\bar{q}, (2i\omega_0 I_n)^{-1}B(q,q))\rangle), \end{aligned}$$

where A is the Jacobian matrix at x_F, B and C are the multilinear function determined by the Taylor expansion of the nonlinear part of the system. Further, the vectors p and q are the complex eigenvectors of $A^\top$ and A. Note that we take the inner product as

$$\langle x, y\rangle = \sum_{k=0}^{n-1} \bar{x}_k y_k.$$

Our system with $(\alpha, \beta, \gamma) = (-1, -1, -2)$ can be rewritten in the form using change in coordinates, $y_j = x_j - F$,

$$\dot{y} = Ay + \frac{1}{2}B(y,y), \;\; y \in \mathbb{R}^4.$$

Here A is the Jacobian matrix, which is for $F = F_H = 1$,

$$A = \begin{pmatrix} -1 & 0 & -1 & 1 \\ 1 & -1 & 0 & -1 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix}$$

and $B_j(x,y) = x_{j-1}(y_{j-1} - y_{j-2}) + y_{j-1}(x_{j-1} - x_{j-2})$.

Since the cubic terms are not in this system, the first Lyapunov coefficient is

$$\begin{aligned} l_1(F_H) &= \frac{1}{2\omega_0}\mathrm{Re}(-2\langle p, B(q, A^{-1}B(q,\bar{q}))\rangle + \langle p, B(\bar{q}, (2i\omega_0 I_n)^{-1}B(q,q))\rangle) \\ &= \frac{1}{2\omega_0}\mathrm{Re}(-2l_{1,1} + l_{1,2}). \end{aligned}$$

Note that A is a circulant matrix. Moreover A is real, therefore $Av = \lambda v \iff A^\top v = \bar{\lambda} v$, where λ is an eigenvalue and v the corresponding eigenvector of A.

In this case $l = 3$, so $Aq = \lambda_1 q = iq$ and $A^\top p = -ip$, because p and q are the corresponding eigenvectors as said before. Take $p = q = \frac{1}{2}(1, \rho, \rho^2, \rho^3)^\top = \frac{1}{2}(1, -i, -1, i)^\top$, then p and q satisfy the previous equations and $\langle p, q\rangle = 1$. Remember that $\rho_j = e^{-2\pi i \frac{j}{n}}$.

Now we will calculate the first term $l_{1,1}$ in the first Lyapunov coefficient. First,

$$A^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

and for $j = 0,1,2,3$, $B_j(q,\bar{q}) = q_{j-1}(\bar{q}_{j-1} - \bar{q}_{j-2}) + \bar{q}_{j-1}(q_{j-1} - q_{j-2})$. Since $q_{-1} = q_3$, $q_{-2} = q_2$ and $\bar{q} = \frac{1}{2}(1, i, -1, -i)^\top$,

$$B(q,\bar{q}) = \frac{1}{2}(1,1,1,1)^\top.$$

Moreover, $A^{-1}B(q,\bar{q}) = \frac{1}{2}(-1,-1,-1,-1)^\top$ and for $j = 0,1,2,3$, $B_j(q, A^{-1}B(q,\bar{q})) = -\frac{1}{2}(q_{j-1} - q_{j-2})$, because $A^{-1}B(q,\bar{q})_{j-1} - A^{-1}B(q,\bar{q})_{j-2} = 0$. Thus, $B(q, A^{-1}B(q,\bar{q})) =$

$\frac{1}{4}(-1-i,-1+i,1+i,1-i)^\top$ and hence,

$$l_{1,1} = \sum_{j=0}^{3} \bar{p}_j B_j(q, A^{-1}B(q,\bar{q})) = -\frac{1}{2} - \frac{1}{2}i.$$

Now we will calculate the second term

$$\begin{aligned} l_{1,2} &= \langle p, B(\bar{q}, (2i\omega_0 I_4 - A)^{-1}B(q,q))\rangle \\ &= \sum_{j=0}^{3} \frac{1}{2i\omega_0 - \lambda_j}\langle v_j, B(q,q)\rangle\langle p, B(\bar{q}, v_j)\rangle, \end{aligned}$$

using the fact that for any $x \in \mathbb{C}^n$ by a standard Fourier decomposition, $x = \sum_{j=0}^{n-1}\langle v_j, x\rangle v_j$, because the eigenvectors of A, the v_j, form a unitary matrix. Then,

$$\begin{aligned} &B(q,q) = 2q_{j-1}(q_{j-1} - q_{j-2}) = \frac{1}{2}(-1+i, 1-i, -1+i, 1-i)^\top, \text{ and} \\ &v_0 = \frac{1}{2}(1,1,1,1)^\top, \ \ v_1 = q = \frac{1}{2}(1,-i,-1,i)^\top, \\ &v_2 = \frac{1}{2}(1,-1,1,-1)^\top \text{ and } v_3 = \bar{q} = \frac{1}{2}(1,i,-1,-i)^\top. \end{aligned}$$

Also,

$$\begin{aligned} B_j(\bar{q}, v_0) &= \frac{1}{2}(\bar{q}_{j-1} - \bar{q}_{j-2})^\top, \text{ for } j = 0,1,2,3, \text{ yields} \\ B(\bar{q}, v_0) &= \frac{1}{4}(1-i, 1+i, -1+i, -1-i)^\top. \end{aligned}$$

We have that $B_j(\bar{q}, q) = B_j(q, \bar{q})$, this implies that $B(\bar{q}, v_1) = \frac{1}{2}(1,1,1,1)^\top$. Furthermore,

$$\begin{aligned} B(\bar{q}, v_2) &= \frac{1}{4}(-1+3i, 3+i, -3i+1, -3-i)^\top \text{ and} \\ B(\bar{q}, v_3) &= \frac{1}{2}(-1-i, 1+i, -1-i, 1+i)^\top. \end{aligned}$$

Since $\langle v_0, B(q,q)\rangle = \langle p, B(\bar{q}, v_1)\rangle = \langle v_3, B(q,q)\rangle = 0$, the second term is

$$\begin{aligned} l_{1,2} &= \sum_{j=0}^{3} \frac{1}{2i\omega_0 - \lambda_j}\langle v_j, B(q,q)\rangle\langle p, B(\bar{q}, v_j)\rangle \\ &= \frac{1}{104}(-4i-2)(-2i+3) = -\frac{7}{13} - \frac{4}{13}i. \end{aligned}$$

If we combine the results of the two terms, we get that the first Lyapunov coefficient is

$$\begin{aligned} l_1 &= \frac{1}{2\omega_0}\mathrm{Re}(-2l_{1,1} + l_{1,2}) \\ &= \frac{3}{13} > 0. \end{aligned}$$

This means that the Hopf bifurcation at $F_H = 1$ is subcritical. □

Appendix C. Hopf bifurcation for some Case Studies

We know that the Hopf bifurcation is subcritical for the system with $(\alpha, \beta, \gamma) = (-1,-1,-2)$ and for $n = 4$, but what happens in the case when $n = 5$ or $n = 6$ and for some other systems. We have stated the result in the table 4. In this table we are given which bifurcation we have or none. Also, what the l-th eigenvalue pair is with some ω_0 and what the first Lyapunov coefficient is. If this is positive, then we have a subcritical

Systems with	n	Bifurcation	F_H	l	ω_0	l_1	Re(eigenvalue)
$(-1,-1,-2)$	4	Hopf bif.	1	3	1	3/13	all < 0
$(-1,-1,-2)$	5	Hopf bif.	$-1/1.12$	3	1.37	-0.76732	all < 0
$(-1,-1,-2)$	5	Hopf bif.	$1/1.12$	4	0.32	0.10818	all < 0
$(-1,-1,-2)$	6	No (Hopf) bif.	-	-	-	-	-
$(-1,0,-2)$	4	No (Hopf) bif.	-	-	-	-	-
$(-1,0,-2)$	5	Hopf bif.	$1/1.81$	1	0.32	0.091740	all < 0
$(-1,0,-2)$	6	Hopf-Hopf bif.	$F_{HH} = 2/3$	-	-	-	-
$(-1,4,-2)$	4	No (Hopf) bif.	-	-	-	-	-
$(-1,4,-2)$	5	Hopf bif.	$-1/1.12$	3	1.38	-0.76135	all < 0
$(-1,4,-2)$	5	Hopf bif.	$1/1.12$	4	0.32	0.10818	all < 0
$(-1,4,-2)$	6	No (Hopf) bif.	-	-	-	-	-
$(-1,0,-1)$	4	Hopf bif.	1	3	1	-1	one > 0, one < 0
$(-1,0,-1)$	5	Hopf bif.	$1/1.81$	2	0.33	-5.2644	all < 0
$(-1,0,-1)$	6	Hopf bif.	2/3	4	$\frac{1}{3}\sqrt{3}$	-1.2990	one > 0, three < 0

TABLE 4. For different dimensions n and for some systems when a Hopf bifurcation or a Hopf-Hopf bifurcation occurs at some F and dash means neither a Hopf nor a Hopf-Hopf bifurcation. Obtain a sub- or supercritical Hopf bifurcation by the first Lyapunov coefficient l_1, if it is positive, then subcritical and if negative, then supercritical.

Hopf bifurcation. Otherwise, it is supercritical. Moreover in the table there is stated how many eigenvalues have positive real part and how many have negative real part.

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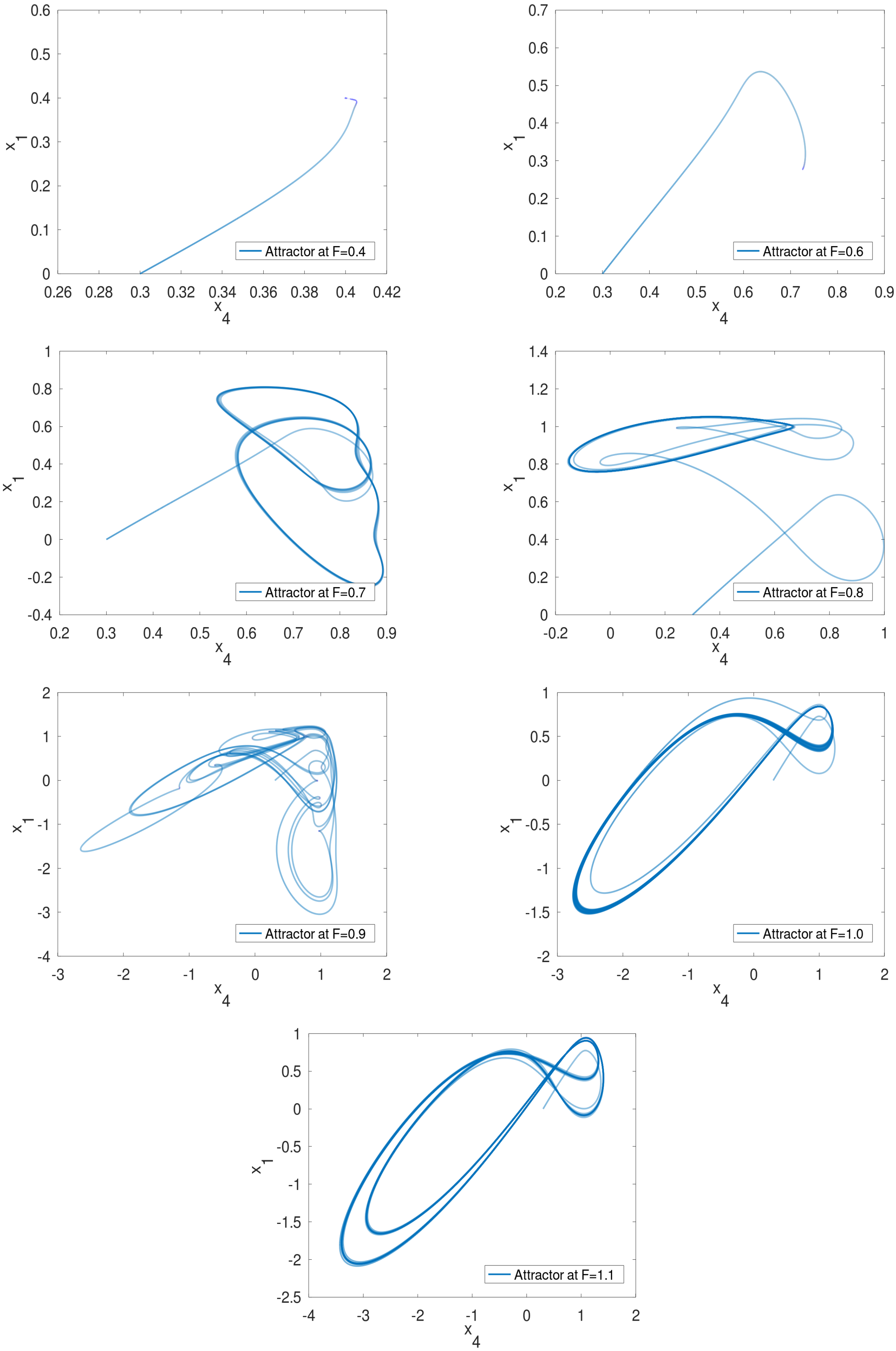

FIGURE 28. The attractors of the system with $(\alpha, \beta, \gamma) = (-1, 0, -1)$ for $n = 4$ at F from $\frac{2}{5}$ up to $1\frac{1}{10}$, where we plot x_1 against x_4 and the initial condition $x_C = (0, 0.1, 0.2, 0.3)$.

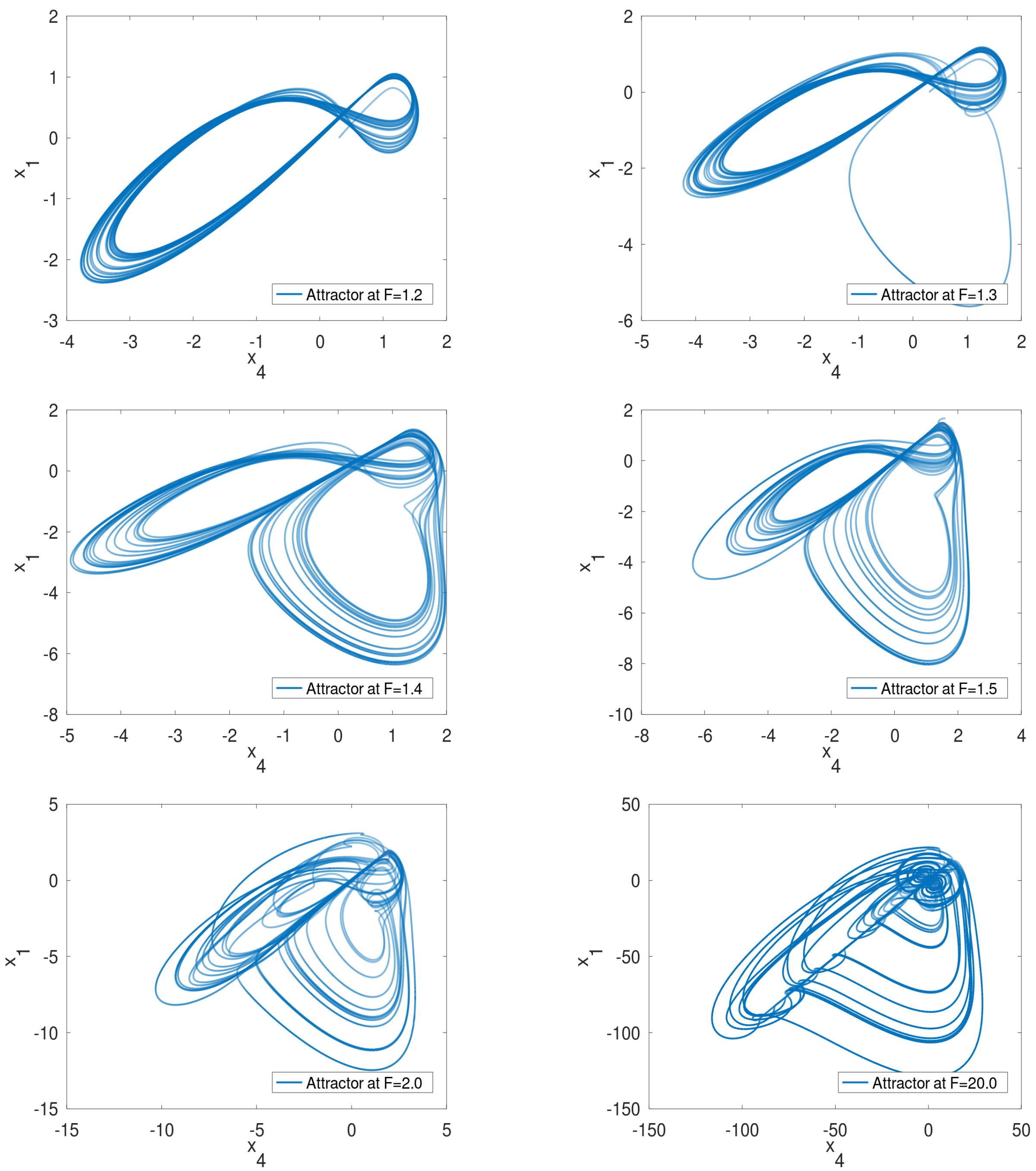

FIGURE 29. Similar as in Figure 28, but now at F from $1\frac{1}{5}$ up to 20.

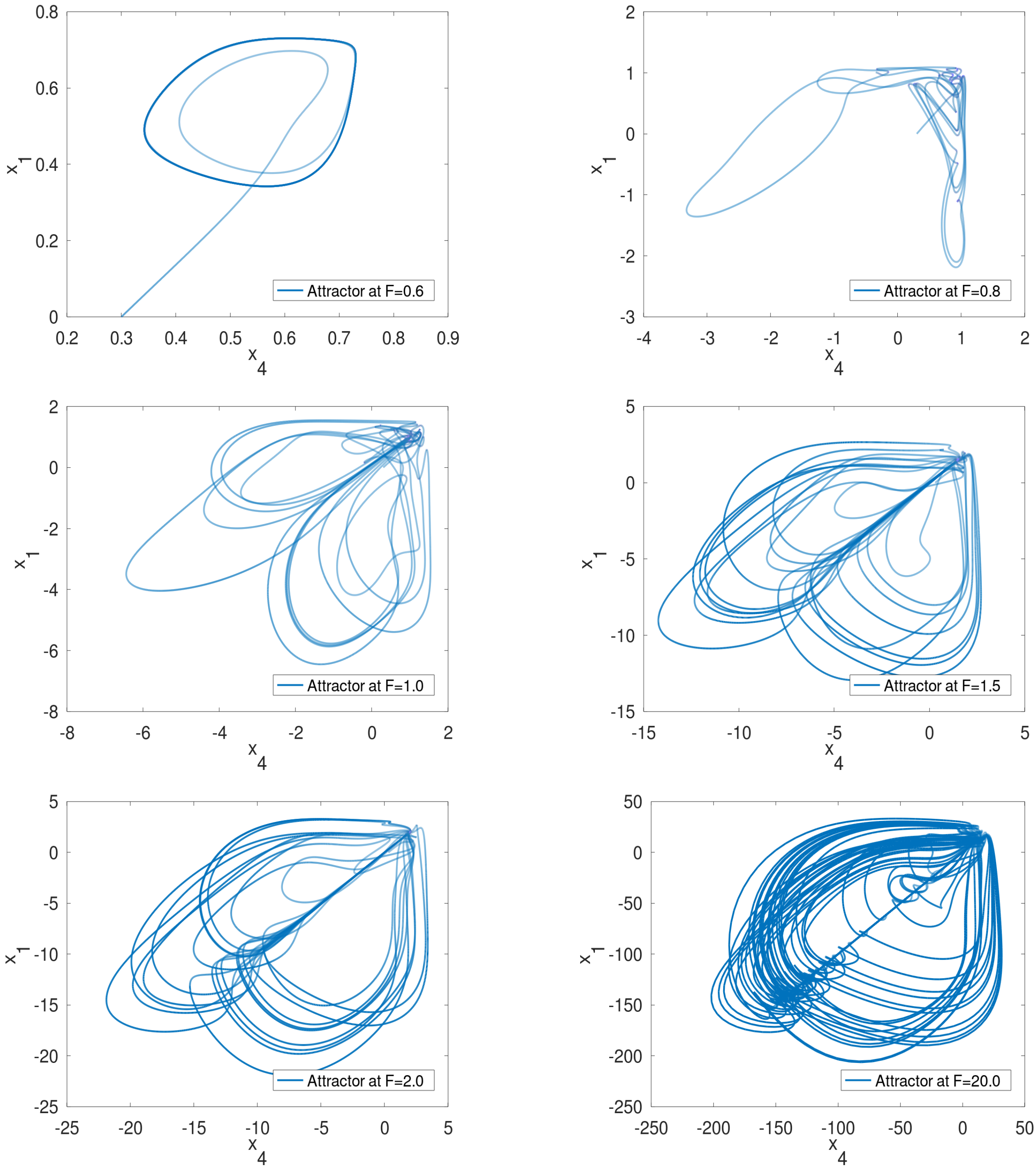

FIGURE 30. The attractors of the system with $(\alpha, \beta, \gamma) = (-1, 0, -1)$ for $n = 5$ at $F = \frac{3}{5}, \frac{4}{5}, 1, 1\frac{1}{2}, 2, 20$, where we plot x_1 against x_4 and the initial condition $x_C = (0, 0.1, 0.2, 0.3, 0.4)$.

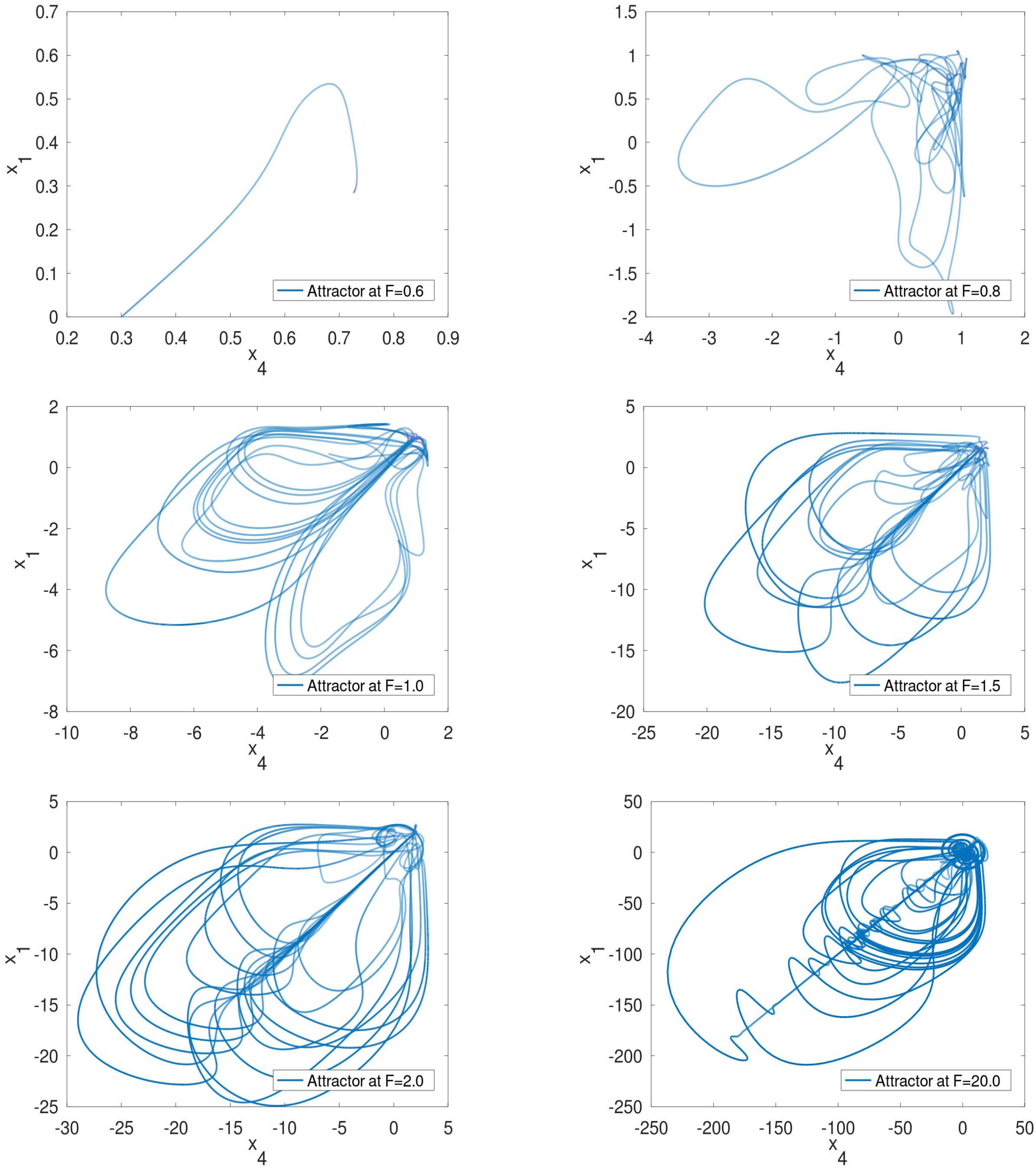

FIGURE 31. Similar as in Figure 30, but for $n = 6$ and $x_C = (0, 0.1, 0.2, 0.3, 0.4, 0.5)$.

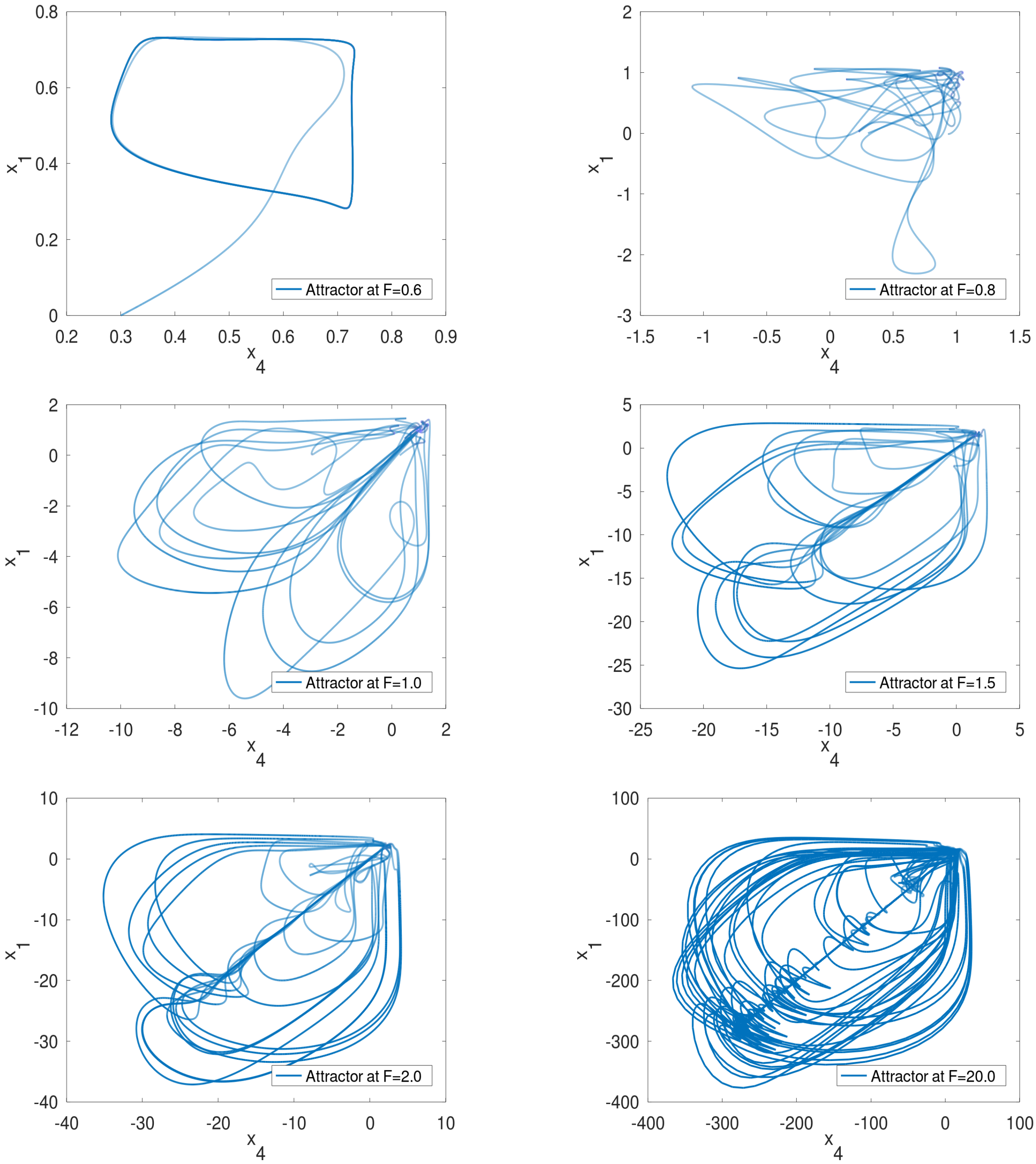

FIGURE 32. Similar as in Figure 30, but for $n = 7$ and $x_C = (0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6)$.

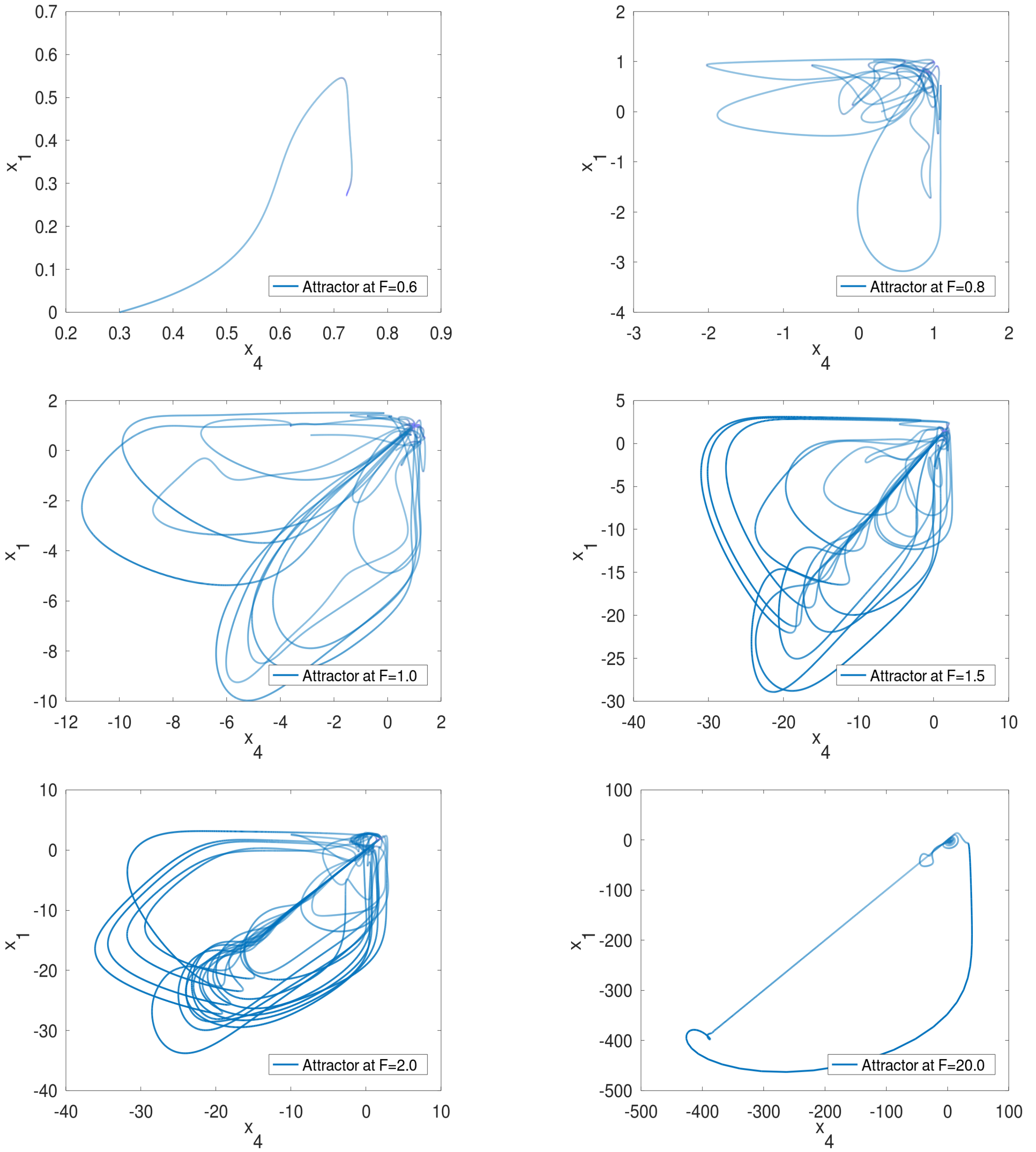

FIGURE 33. Similar as in Figure 30, but now for $n = 8$ and $x_C = (0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7)$.

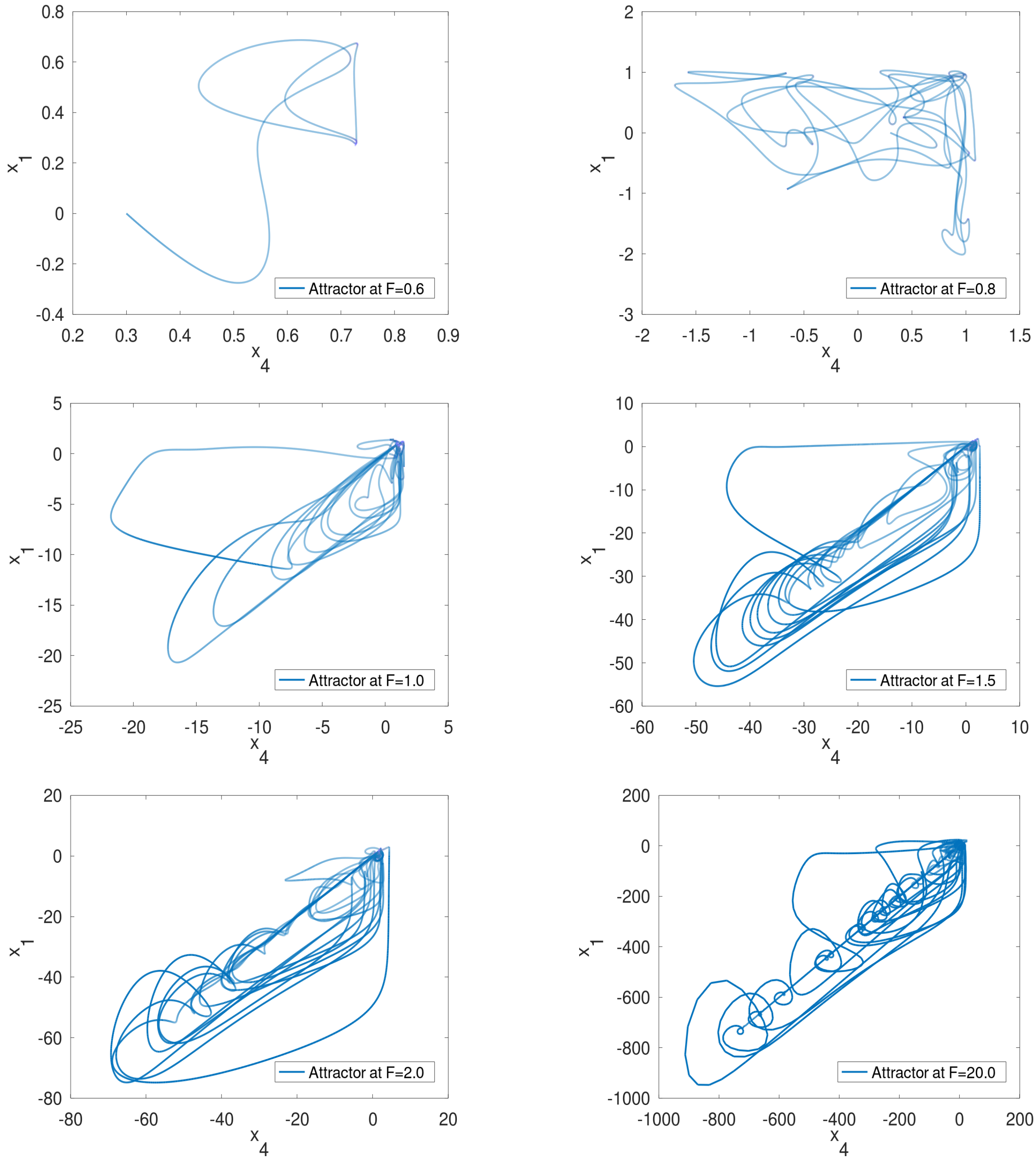

FIGURE 34. Similar as in Figure 30, but for $n = 12$ and $x_C = (0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1)$.

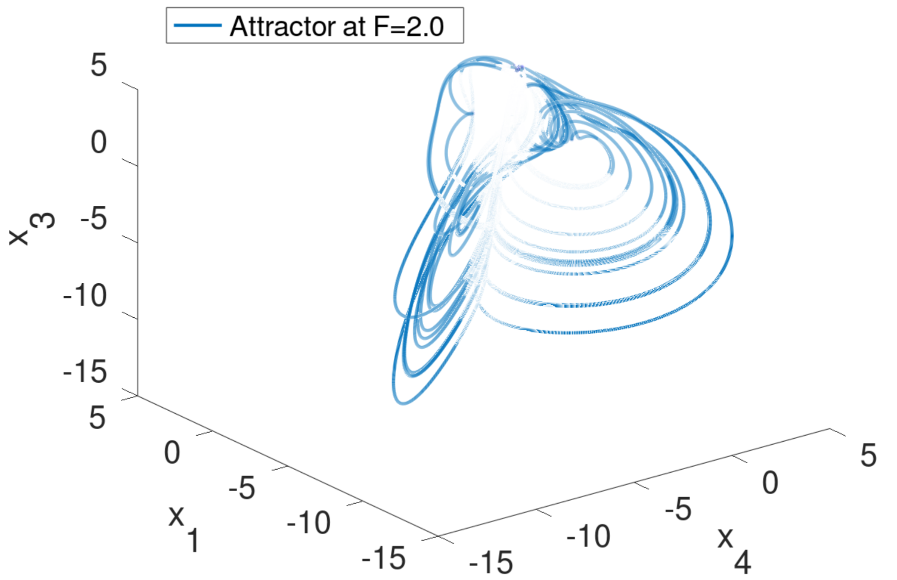

FIGURE 35. The 3D plot of the attractor of the system with $(\alpha, \beta, \gamma) = (-1, 0, -1)$ for $n = 4$ at $F = 2$, where we plot x_1, x_4 and x_3 and the initial condition $x_C = (0, 0.1, 0.2, 0.3)$.

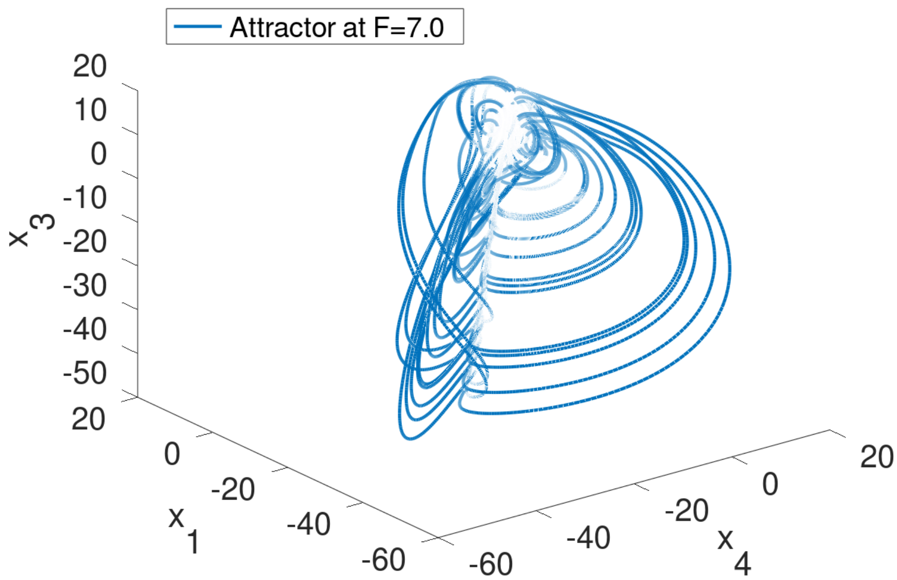

FIGURE 36. Similar as in Figure 35, but at $F = 7$.

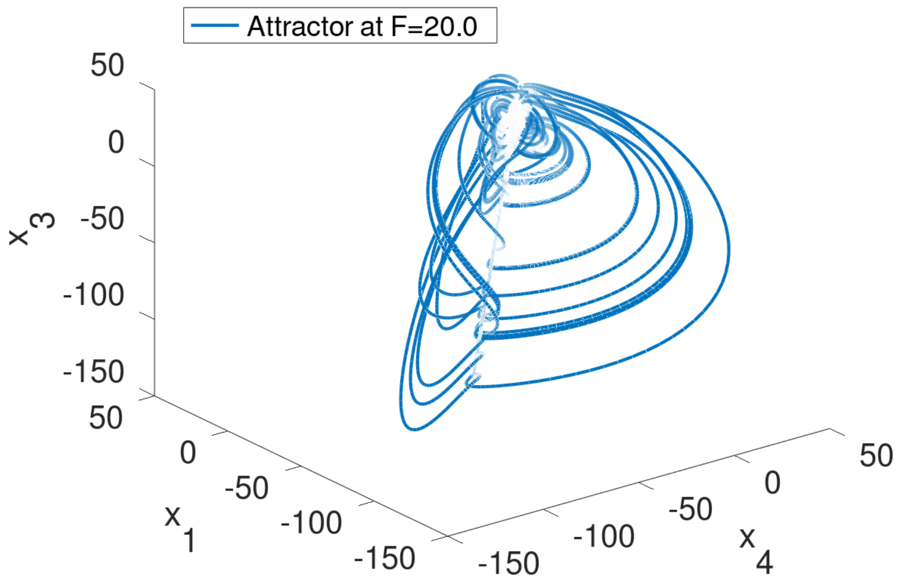

FIGURE 37. Similar as in Figure 35, but at $F = 20$.

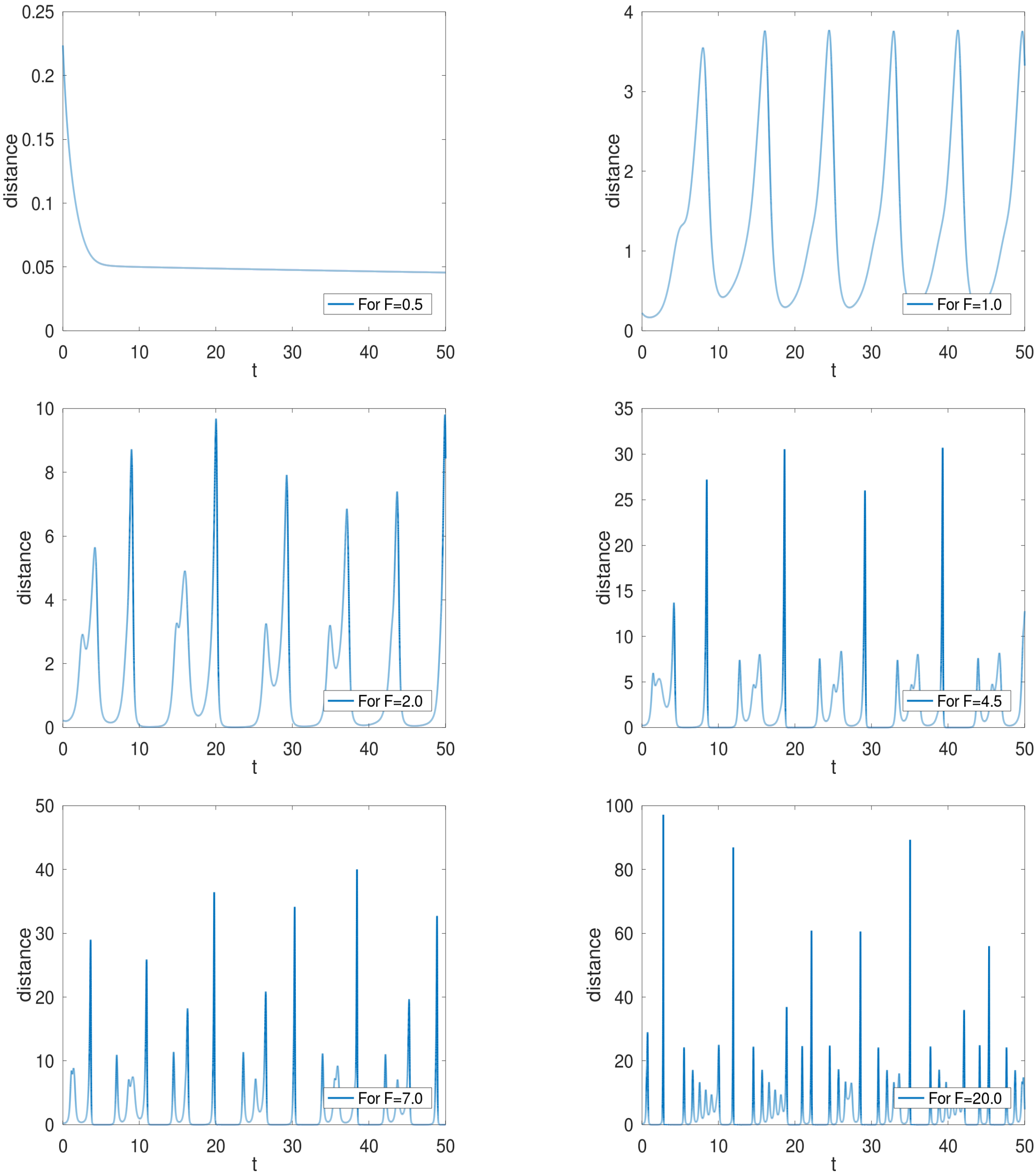

FIGURE 38. The plots of the distances between $x(t)$ and the orthogonal projection P with respect to t for $F = \frac{1}{2}, 1, 2, 4\frac{1}{2}, 7, 20$ for $n = 4$ and the system with $(\alpha, \beta, \gamma) = (-1, 0, -1)$.

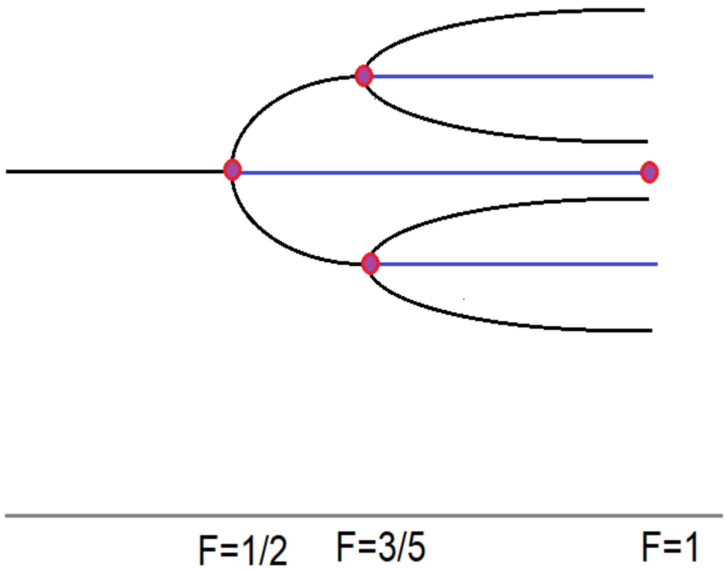

FIGURE 39. For $n = 4$ the schematic bifurcation diagram for the system with $(\alpha, \beta, \gamma) = (-1, 0, -1)$, where at $F = \frac{1}{2}$ a first Pitchfork bifurcation occurs and a second one at $F = \frac{3}{5}$. At $F = 1$ we have a supercritical Hopf bifurcation.

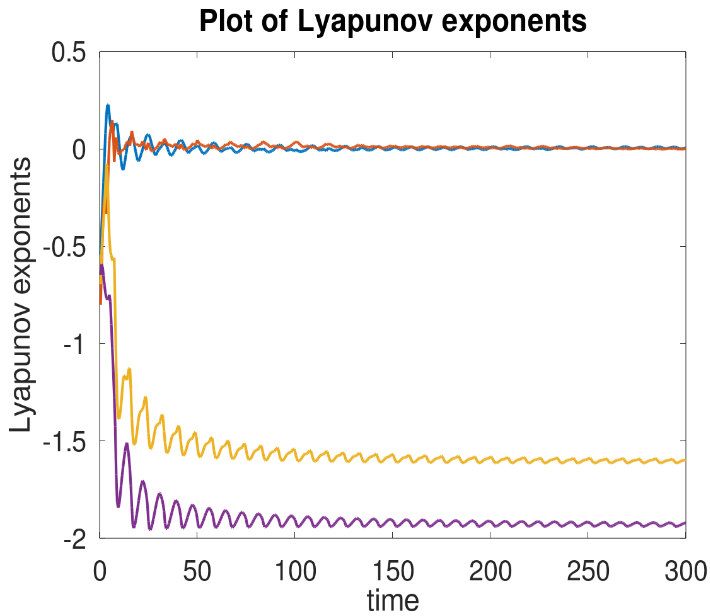

FIGURE 40. Estimated Lyapunov exponents as a function of time for $F = 1$ and the system with $(\alpha, \beta, \gamma) = (-1, 0, -1)$.

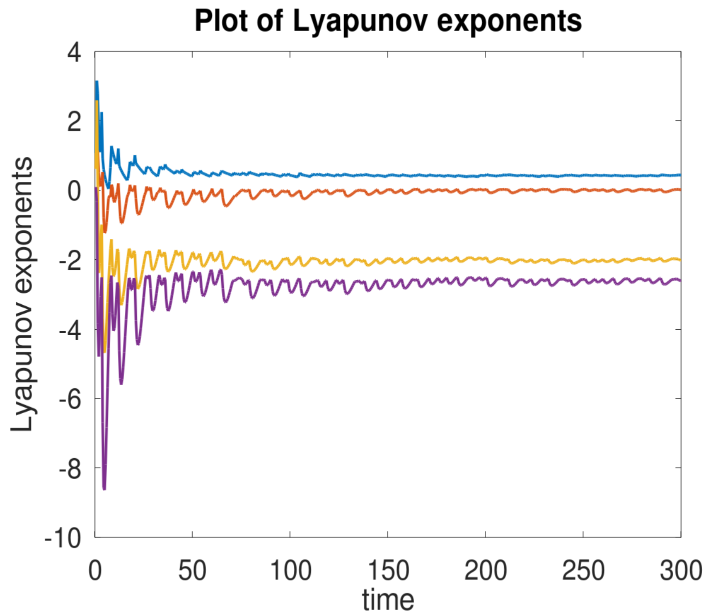

FIGURE 41. Similar as the Figure 40, but for $F = 7$.

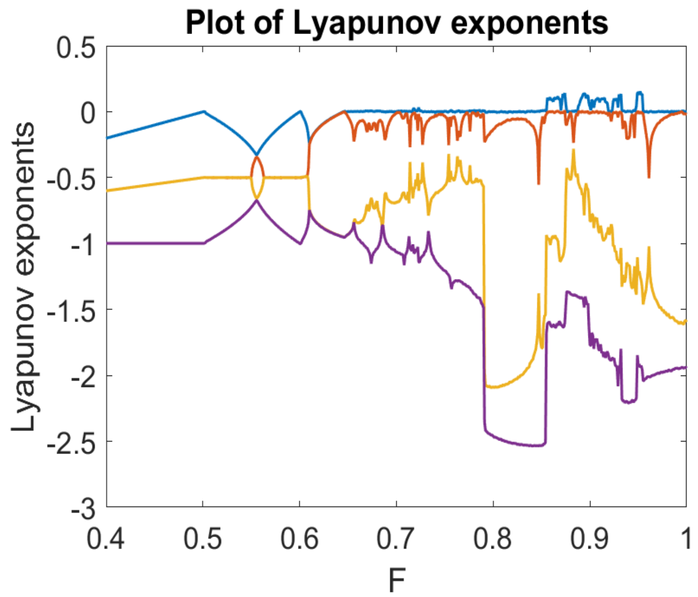

FIGURE 42. The Lyapunov exponents against F, where F is from 0.4 up to 1 for $n = 4$ and $(\alpha, \beta, \gamma) = (-1, 0, -1)$.

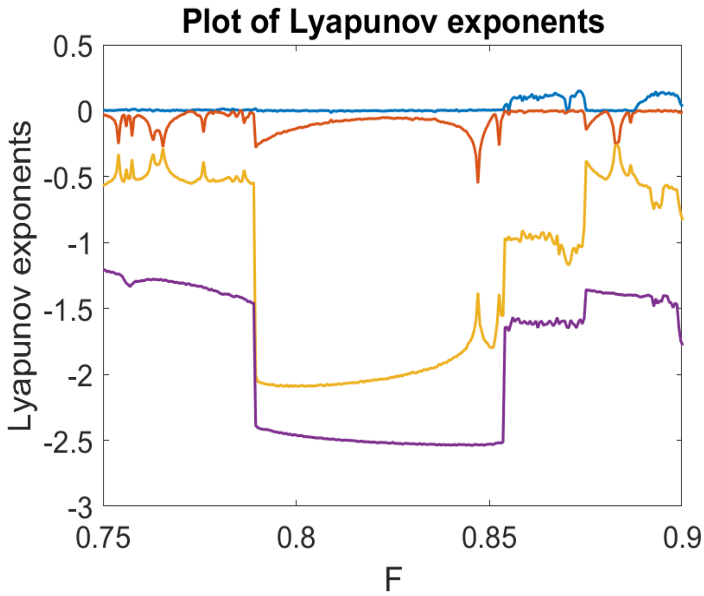

FIGURE 43. Similar as in Figure 42, but F between 0.75 and 0.9 to zoom in on this area.

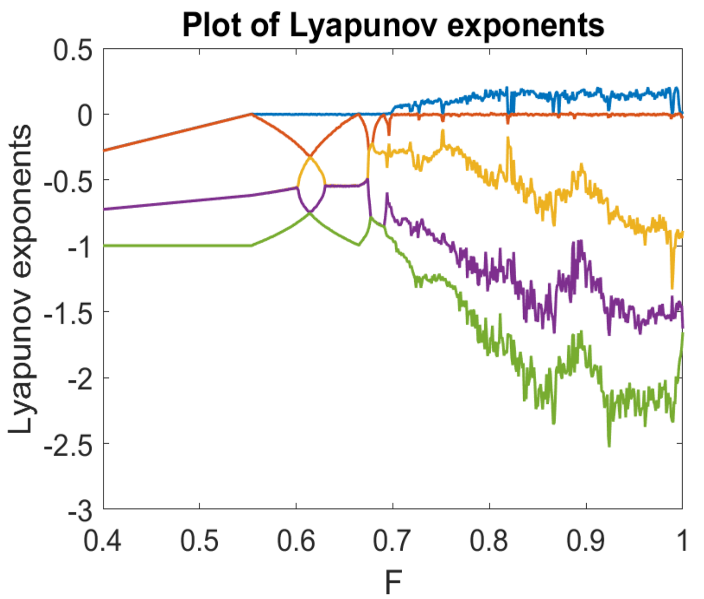

FIGURE 44. Similar as in Figure 42, but for $n = 5$.

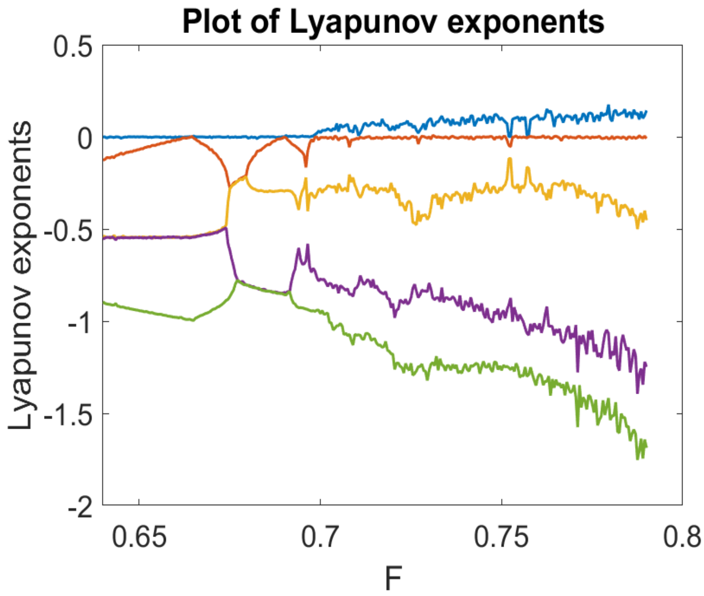

FIGURE 45. Similar as in Figure 42, but for F between 0.64 and 0.79 to zoom in and for $n = 5$.

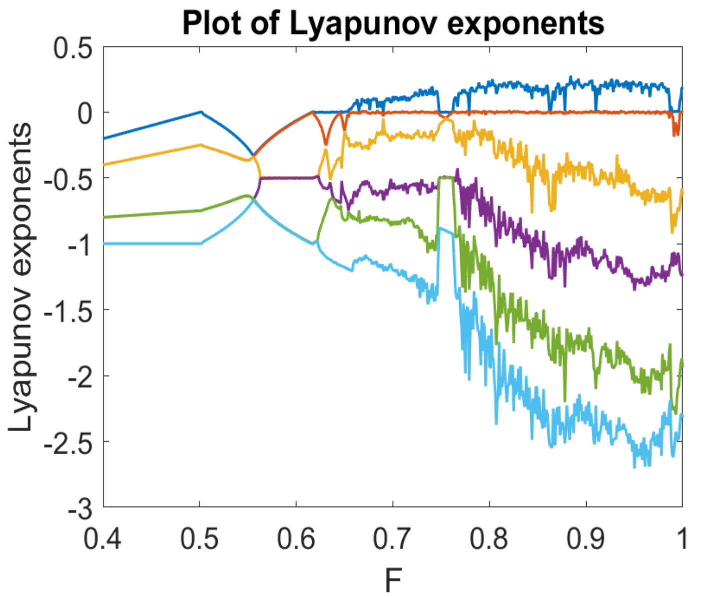

FIGURE 46. Similar as in Figure 42, but for $n = 6$.

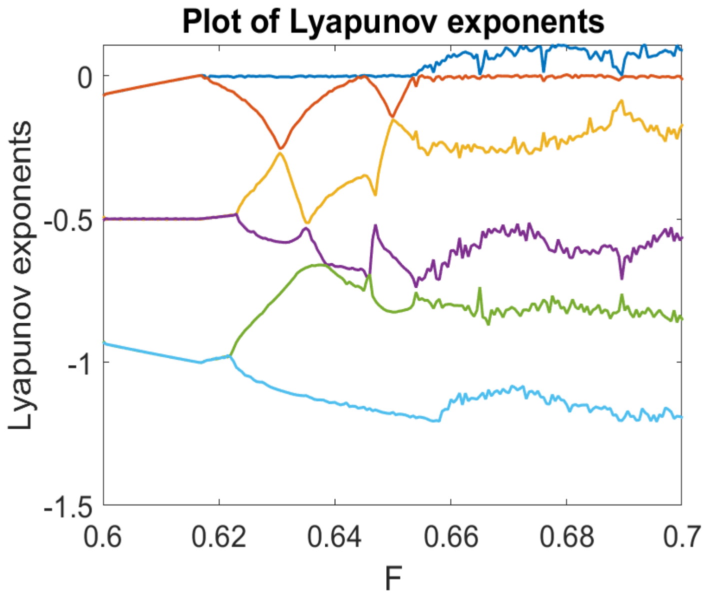

FIGURE 47. Similar as in Figure 42, but for $n = 6$ and F between 0.6 and 0.7 to zoom in.

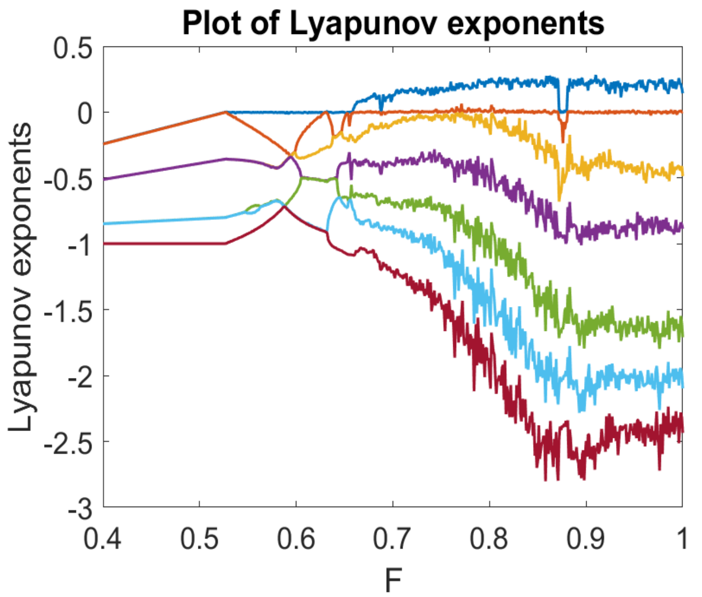

FIGURE 48. Similar as in Figure 42, but now for $n = 7$.

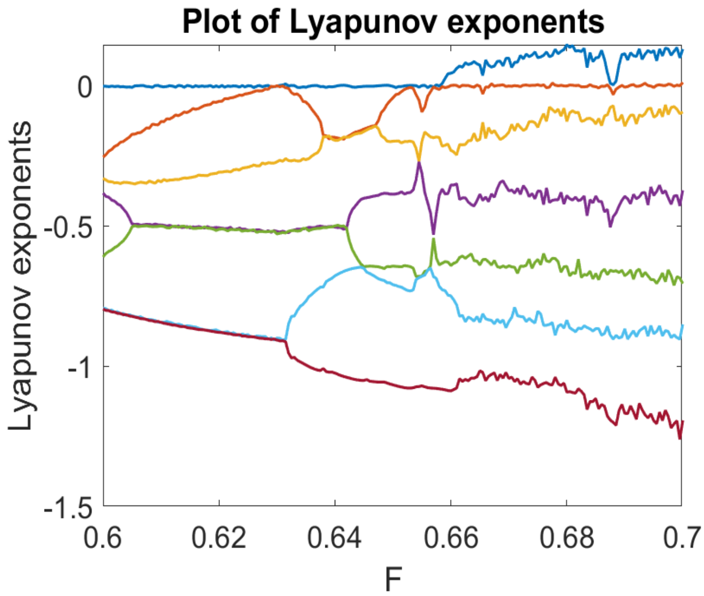

FIGURE 49. Similar as in Figure 42, but for $n = 7$ and F between 0.6 and 0.7 to zoom in.

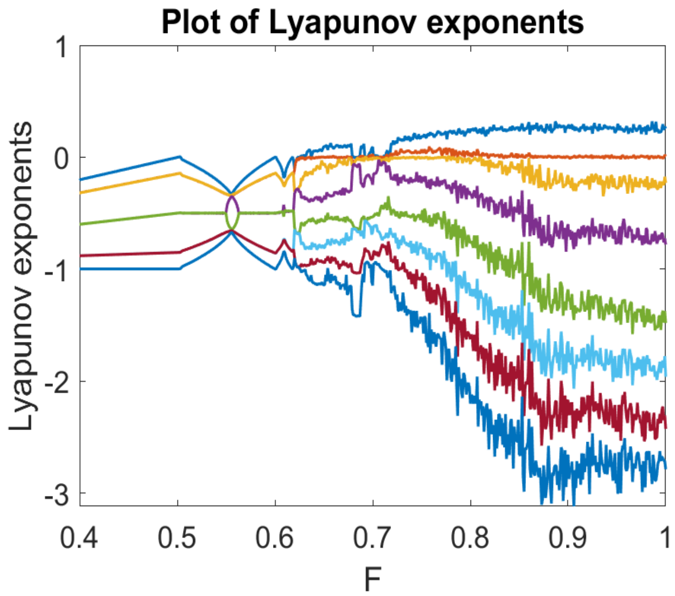

FIGURE 50. Similar as in Figure 42, but for $n = 8$.

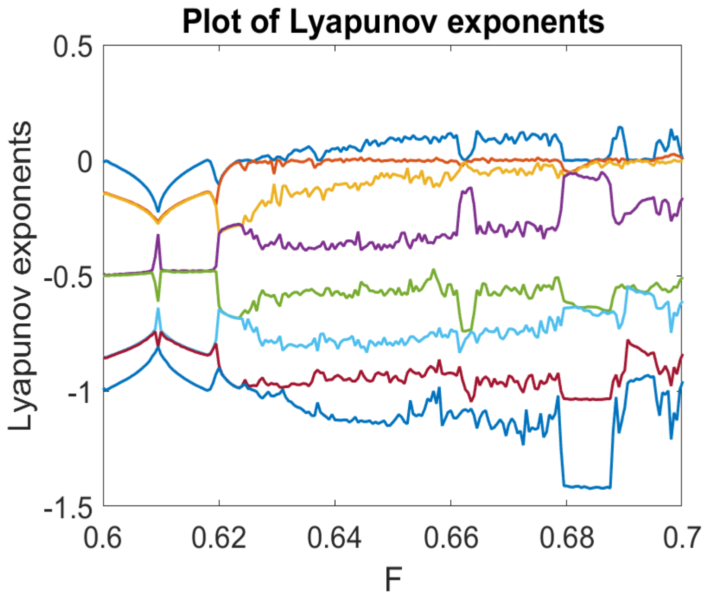

Figure 51. Similar as in Figure 42, but for F between 0.6 and 0.7 to zoom in and for $n = 8$.