The Geometric Interpretation of Field Theories and Classical Double Copy

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June 2020

General relativity is a non-linear field theory of gravity which implies graviton-graviton interactions. Born-Infeld theory is a modified non-linear electromagnetic field theory with very specific self-interactions as well. Is there a geometric interpretation for the non-linear Born-Infeld theory as there is a natural geometrical origin in general relativity? For some field theories (Dirac Born-Infeld theory and Special Galileon theory) with similar features including scalar theories with self interactions, there is a geometric understanding. In this thesis, the foundation of Born-Infeld theory and general relativity is studied. The physics of branes are studied together with the scalar field theories such as Dirac-Born-Infeld theory (DBI) and Speical Galileon theory (SG) in order to understand their geometric interpretation. Since the electromagnetic field A_μ in Born-Infeld theory only contributes to the antisymmetrical part $F_{\mu\nu}$ of the induced metric, it is found out that there is no nonlinearly realised symmetry and A_μ transforms as a vector field under linearly realised Poincaré symmetry to leave $F_{\mu\nu}$ antisymmetric. So, the geometric interpretation as a brane fluctuating in the transverse direction as in DBI theory and SG theory cannot be applied to Born-Infeld theory. In the study of classical double copy motivated by the color-kinematics duality in scattering amplitude, Kerr-Schild ansatz is used to construct the interconnection bewteen the classical solution of Abelian Maxwell theory and general relativity. Inspired by Kerr-Schild classical double copy, the duality of the classical solution of Born-Infeld theory and Special Galileon theory is investigated by examining the possible classical double copy relations between them. The equation of motion of SG and BI under static and spherically symmetric condition are solved to take the form of $\frac{1}{r}\times$ hypergeometric series. This indicates a possible double copy relation between SG and BI. The difficulties of the extension of this possible classical double copy relation to general relativity is discussed.

What profit hath a man of all his labour which he taketh under the sun? One generation passeth away, and another generation cometh: but the earth abideth for ever. The sun also ariseth, and the sun goeth down, and hasteth to his place where he arose. The wind goeth toward the south, and turneth about unto the north; it whirleth about continually, and the wind returneth again according to his circuits. All the rivers run into the sea; yet the sea is not full; unto the place from whence the rivers come, thither they return again. All things are full of labour; man cannot utter it: the eye is not satisfied with seeing, nor the ear filled with hearing. The thing that hath been, it is that which shall be; and that which is done is that which shall be done: and there is no new thing under the sun. ..... And I gave my heart to know wisdom, and to know madness and folly: I perceived that this also is vexation of spirit. For in much wisdom is much grief: and he that increaseth knowledge increaseth sorrow. -**Ecclesiastes 1** [10].

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Declaration

I declare that some of the derivations are carried out independently in this thesis but no originality is claimed for the results. The main articles or textbooks that I refer to in each chapter is listed below.

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1 Introduction

In the 19th century, English physicist Micheal Faraday discovered experimentally that a changing magnetic field will induce the flow of the electric current. Unlike his contemporaries, Faraday refused to accept the notion that electricity was a material fluid that flows through a wire. Faraday thought electricity as a vibration or transmission of force which was the result of the tension created in the conductor. Faraday proposed his **electrotonic state** to convey this idea. This electrotonic state is considered as a state of tension of the particles in the wire [24]. According to him, the current is appeared as the setting up or the collapse of such a state of tension. The other geometric intuition of Faraday is the so-called **magnetic field lines of force**. This is experimentally seen through sprinkling iron filings in the magnetic field. The elusive geometrical insights of Faraday as an experimentalist lays the foundation of the theory of electromagnetism as a field theory. It was only James Clerk Maxwell who could really describe electromagnetic theory as a field theory in the language of vector calculus. Maxwell learned from reading William Thomson's mathematical paper about the usefulness of the curl equation

$$\vec{B} = \nabla \times \vec{A} \tag{1.1}$$

From this mathematical equation, Maxwell realised that the Faraday's electrotonic intensity can be denoted by $\vec{A}$. His insight leads him to realise that what Faraday has described in so many words can be expressed mathematically as

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} \tag{1.2}$$

$\vec{E}$ is the electric field. Taking the curl on both side of (1.2),

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{A}) \tag{1.3}$$

$$= -\frac{\partial \vec{B}}{\partial t} \tag{1.4}$$

$\vec{B}$ is the magnetic field. The second equality line arises from the identity that $\vec{B} = \nabla \times \vec{A}$ which is realised from the relation $\nabla \cdot \vec{B} = 0$ and the fact that the curl of a vector is always divergenceless. Integrating (1.4) and apply Stoke's theorem,

$$\int (\nabla \times \vec{E}) \cdot dA = \int \vec{E} \cdot dl = -\frac{\partial}{\partial t} \int \vec{B} \cdot dA \tag{1.5}$$

This is the mathematical expression of Faraday's law. The vector $\vec{A}$ which is called as vector potential was initially thought by many (Hertz, Heaviside, etc) as something unnecessary because they thought it as a non-physical quantity and the formulation of electromagnetism in term of $\vec{A}$ should be avoided and eliminated. However, it was realised later with quantum mechanics that this vector potential $\vec{A}$ has physical meaning and cannot be eliminated. The combination of the vector potential $\vec{A}$ and the scalar potential ϕ as A^μ also plays a crucial role as the electromagnetic field in the modern field theory. In physics, the concept of field can be thought of as a quantified physical quantity given at any point of spacetime. In quantum field theory, the particles are the excitation of the quantum field.

Beside electromagnetism, field theory is also important in the theory of gravity. In 1915, Albert Einstein proposed general relativity as a new theory of gravitation to explain gravity as the manifestation of the curvature of the spacetime to replace the explanation provided by the great Issac Newton that gravity is an instantaneous force between two objects with mass. In contrast with the linear gravitational field equation in Newtonian framework which is represented by the Poisson's equation, Einstein's gravitational field equation turns out to be a set of 6 independent non-linear differential equations. Around the same time and even 10 days earlier than Einstein,

German mathematician David Hilbert derived the gravitational field equation independently by figuring out the Lagrangian that leads to Einstein's field equation through the variational principle. In general relativity, the gravitational potential field is the metric tensor $g_{\mu\nu}$ that governs the geometric and causal structure of the spacetime. This revolutionary theory has withstood the test of time. The first success of general relativity was in explaining the anomalous rate of precession of the perihelion of Mercury's orbit. Besides that, the prediction of the deflection of light by the sun given by general relativity is also verified. In 2016, the gravitational wave which is conjectured within the theory of general relativity was detected by the Laser Interferometer Gravitational-Wave Observatory (LIGO).

In 1934 which was before the advent of quantum field theory, Max Born and Leopold Infeld also proposed a new electromagnetic field theory. Their theory leads to a non-linear electromagnetic field equation by setting up a framework using a non-symmetric metric. Born-Infeld theory is a hypothesis of the modified field theory of electromagnetism which is inspired by Maxwell's theory of electromagnetism and Einstein's theory of general relativity. This theory has not been verified by any experiment so far. Born and Infeld argued that principle of finiteness which states that the physical quantities are not allowed to be infinite is a fundamental principle of physics. Since Maxwell theory of electromagnetism fails to satisfy the principle of finiteness, they set up a new framework to improve Maxwell theory. They constructed a theory such that not only the principle of finiteness is satisfied but also the new theory can be approximated to Maxwell's theory in a certain limit. They started by constructing a new Lagrangian. Just like in Special relativity, the Lagrangian is

$$L = mc^2(1 - \sqrt{1 - \frac{v^2}{c^2}}) \tag{1.6}$$

This Lagrangian serves as a modification of the Newtonian action $L = \frac{1}{2}mv^2$. This modification leads to the assumption of an upper limit of velocity c. They apply the same notion to electromagnetism. In Maxwell theory, the Lagrangian is

$$L = \frac{1}{2}(B^2 - E^2) \tag{1.7}$$

To have an upper limit of the electromagnetic field strength, they modified (1.7) to be

$$L = b^2(\sqrt{1 + \frac{1}{b^2}(B^2 - E^2)} - 1) \tag{1.8}$$

where b is a dimensional correction parameter. It is obvious that in the limit $b \to \infty$, (1.8) is approximated to be (1.7). Such kind of arguments are quite convincing but a deeper understanding of the foundation of this new field theory is necessary. In chapter 3, we study the foundation of Born-Infeld theory and general relativity in details.

Since general relativity has a natural geometrical origin, we wonder is there also a geometrical interpretation of Born-Infeld theory. There have been a lot of efforts over the last few decades to geometrize electromagnetism. We also post our question here by asking if Born-Infeld theory is the corrected modified field theory of electromagnetism, is there also a geometrical origin? After all, the Lagrangian density of Born-Infeld theory takes the form as the square root of the determinant of a induced metric $G_{\mu\nu} = \eta_{\mu\nu} + F_{\mu\nu}$ if we fix the background spacetime to be Minkowski spacetime. We turn our attention to the possible extrinsic geometric interpretation as the embedding of a submanifold in a background manifold. One of the main topic in this thesis is the geometric interpretation of the field theories for which their Lagrangian density can be formulated in the expression of the pullback of the ambient metric. We start the thesis with a chapter on the mathematical preliminaries that discuss the relevant concepts of differential geometry used in this thesis.

In chapter 4, we investigate the important concepts in brane theory such as the background dependent of a brane, the symmetric properties of the brane, boundary conditions, D branes ,etc

which are all originated from the study of string theory. After that, we especially study the field theory of the D branes which is called Dirac Born-Infeld theory (DBI). Due to the symmetry breaking of the symmetric Poincare group and the nonlinearly realised symmetry when a D brane is formed, there is a natural geometric interpretation of a D-brane fluctuating in the transverse direction in DBI theory. We also look into another scalar field theory which is called Special Galileon theory and study its geometrical origin using complex geometry with Kahler structure. In the end of chapter 4, we also discuss about whether we can inherit the same methodology to figure out the geometric interpretation of Born-Infeld theory.

The other main topic in this thesis is the classical double copy. The notion of double copy first comes from the study of scattering amplitude in quantum field theory. In the story of scattering amplitude, it was found out that by replacing the color factor to kinematic numerator in the scattering amplitude of Yang-Mills theory, the scattering amplitude of general relativity is obtained. In classical double copy, we study the relevant duality by looking at the classical solutions of the field theories and the map that relates different theories. In chapter 5, We particularly study the known classical double copy between Abelian Maxwell theory and general relativity which is called Kerr-Schild double copy. Then we solve the equation of motion of Born-Infeld theory and Special Galileon theory in perturbative way under the static and spherically symmetric solution to see is it also possible to construct the double copy relations between Special Galileon theory, Born-Infeld theory and general relativity.

2 Mathematical Preliminaries

First of all, we review some mathematical concepts in differential geometry which are important and used throughout the thesis. We will have to understand the concept of the invariant form, maps between manifolds, diffeomorphism and Lie derivative.

2.1 Invariant volume form

2.1.1 Levi-Civita Tensor

In a general D-dimensional Euclidean space $\mathbb{R}^D$, the volume integral, I is

$$\begin{aligned} I &= \int d^D x' \\ &= \int |\frac{\partial x'}{\partial x}| d^D x \end{aligned}$$

For $|\frac{\partial x'}{\partial x}|$ is the Jacobian factor. The imminent task here is to generalize the volume form to any arbitrary space with any specific form of metric $g_{\mu\nu}(x)$. The desired generalized volume form must be covariant with respect to general coordinate transformation. The general coordinate transformation rule discussed in this paper are at least C^1 diffeomorphic. To be able to construct such an invariant volume form, the property of the totally antisymmetric Levi-Civita symbol has to be studied explicitly first. The Levi-Civita symbol is defined as

$$\tilde{\epsilon}_{\mu_1\mu_2.....\mu_D} == \begin{cases} 1, & \text{if } \mu_1....\mu_D \text{ is even permutation of } 12.....D, \\ -1, & \text{if } \mu_1....\mu_D \text{ is odd permutation of } 12.....D, \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

Levi-Civita symbol follows a very nice property when it is combined with the determinant of any matrix M^ν_μ, namely

$$\tilde{\epsilon}_{\mu_1\mu_2.....\mu_D}|M| = \tilde{\epsilon}_{\nu_1\nu_2.....\nu_D} M^{\nu_1}_{\mu_1} M^{\nu_2}_{\mu_2}.....M^{\nu_D}_{\mu_D} \tag{2.2}$$

For M is the determinant of the matrix M^ν_μ. To check whether $\tilde{\epsilon}_{\nu_1\nu_2.....\nu_D}$ is a tensor or not, set M^ν_μ to be the coordinate transformation matrix $\frac{\partial x^\nu}{\partial x'^\mu}$. Consequently,

$$\tilde{\epsilon}_{\mu_1\mu_2.....\mu_D} = |\frac{\partial x}{\partial x'}|^{-1} \tilde{\epsilon}_{\nu_1\nu_2.....\nu_D} \frac{\partial x^{\nu_1}}{\partial x'^{\mu_1}} \frac{\partial x^{\nu_2}}{\partial x'^{\mu_2}} ..... \frac{\partial x^{\nu_D}}{\partial x'^{\mu_D}} \tag{2.3}$$

The above relation shows explicitly that Levi-Civita symbol is not a tenosr as it does not follow the coordinate transformation rule. There is an extra factor $|\frac{\partial x}{\partial x'}|^{-1}$. Levi-Civita symbol is in fact a tensor density with weight $+1$. The weight refers to the order of $|\frac{\partial x'}{\partial x}|$. Apparently, Levi-Civita symbol is not a suitable and qualified candidate to construct the invariant volume form because it is not a tensor. However, $\tilde{\epsilon}_{\mu_1\mu_2.....\mu_D}$ can be combined with another tensor density $\sqrt{|g|}$ with weight -1. For g is the determinant of the metric tensor. To be able to see this, set up a change of the coordinate system and check how this is going to give impact to g

$$g \to g' = det g'_{\mu\nu} \tag{2.4}$$

$$= det(\frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}) \tag{2.5}$$

$$= |\frac{\partial x}{\partial x'}|^2 g \tag{2.6}$$

This indicates that g is not a scalar. Instead it is a tensor density with weight -2. The square root of $|g|$ which is a tensor density with weight -1 can be combined with levi-Civita symbol with

weight +1 to form the so called Levi-Civita tensor.

$$\boxed{\epsilon_{\mu_1\mu_2....\mu_D} = \sqrt{|g|}\tilde{\epsilon}_{\mu_1\mu_2.....\mu_D}} \tag{2.7}$$

The reason that the Levi-Civita tensor can be used to generate invariant volume form is still not explicitly known at this point. It will be clear when the notion of the differential form and wedge product is discussed in the next section. The absolute value of g is taken because for a Lorentzian manifold, g is a negative value.

2.1.2 Differential form and wedge product

Definition 2.1.2.1(Differential form): A special class of tensor of type $(0, p)$,

$$\omega = \omega_{\mu_1\mu_2.....\mu_p} dx^1 \otimes dx^2 \otimes ..... \otimes dx^p \tag{2.8}$$

such that the component $\omega_{\mu_1\mu_2.....\mu_p}$ is totally anti-symmetric.

A typical example of the differential form is the electromagnetic field strength $F_{\mu\nu}$. The diffrential form can be written into the formulation involving wedge product of the bases of the form dx.

$$F_{\mu\nu}dx^\mu \otimes dx^\nu = \frac{1}{2}[F_{\mu\nu} - F_{\nu\mu}]dx^\mu \otimes dx^\nu \tag{2.9}$$

$$= \frac{1}{2}[F_{\mu\nu}dx^\mu \otimes dx^\nu - F_{\nu\mu}dx^\nu \otimes dx^\mu] \tag{2.10}$$

$$= \frac{1}{2}[F_{\mu\nu}dx^\mu \otimes dx^\nu - F_{\mu\nu}dx^\mu \otimes dx^\nu] \tag{2.11}$$

$$= \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu \tag{2.12}$$

for the wedge product $dx^\mu \wedge dx^\nu$ is defined to be $dx^\mu \otimes dx^\nu - dx^\mu \otimes dx^\nu$. In general, the wedge product is defined to be

$$dx^{\mu_1} \wedge ..... \wedge dx^{\mu_p} = \overline{dx^{\mu_1} \otimes ..... \otimes dx^{\mu_p}} \tag{2.13}$$

For $\overline{dx^{\mu_1} \otimes ..... \otimes dx^{\mu_p}}$ denotes the antisymmetrization of $dx^{\mu_1} \otimes ..... \otimes dx^{\mu_p}$

Using the above formulation, the differential p-form can be written as

$$\boxed{\omega = \frac{1}{p!}\omega_{\mu_1\mu_2.....\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge ..... \wedge dx^{\mu_p}} \tag{2.14}$$

Switching back to the Levi-Civita tensor, since the component of Levi-Civita tensor, $\epsilon_{\mu_1\mu_2....\mu_D}$ is totally anti-symmetric, it is actually a differential form. The tensor can be written out explicitly as

$$\epsilon = \frac{1}{D!}\epsilon_{\mu_1\mu_2.....\mu_D} dx^{\mu_1} \wedge dx^{\mu_2} \wedge .....dx^{\mu_D} \tag{2.15}$$

$$= \frac{1}{D!}\sqrt{|g|}\tilde{\epsilon}_{\mu_1....\mu_D} dx^{\mu_1} \wedge dx^{\mu_2} \wedge .....dx^{\mu_D} \tag{2.16}$$

$$= \sqrt{|g|}dx^0 \wedge ..... \wedge dx^{D-1} \tag{2.17}$$

$$\equiv \sqrt{|g|}d^D x \tag{2.18}$$

The third equality line arises by using the fact that the Levi-Civita symbol and the wedge product of dx are both totally antisymmetric. The contraction of them will give rise to a number of $D!$ same term. Thus, the Levi-Civita tensor acts as a volume form. One can check easily in the Euclidean space. If the Cartesian coordinate system with metric δ^ν_μ is used, then $|g| = 1$. This reduces to the familiar volume form $d^D x$.

Note that the combination of the square root of the determinant of any $(0,2)$ tensor with Levi-Civita symbol always forms a tensor which obeys the coordinate transformation rule but only the specific combination of $\sqrt{g}$ with $\tilde{\epsilon}_{\mu 1 \ldots . \mu D}$ serves as the identity of volume form.

Therefore, to compute the volume of any manifold with its structure governed by a metric $g_{\mu\nu}$, one can just simply evaluate the integral

$$I \equiv \int \sqrt{|g|} d^D x \tag{2.19}$$

The obtained volume form can also be used to constructed any kind of action

$$S = \int \Psi(x) \sqrt{|g|} d^D x \tag{2.20}$$

For $\Psi(x)$ is a scalar.

2.2 Maps between manifolds

A manifold M can be linked together with another manifold N by a map ψ, namely $\psi : M \to N$. The property of the map between 2 manifolds is important to be studied for lot of different reasons. For example, if one wants to extract the information of the extrinsic geometry from the intrinsic geometry by embedding a submanifold in a equal or larger dimensional manifold or if someone wants to study the natural formalism of active coordinate transformation of a same manifold, a thorough study of the map between manifolds cannot be avoided.

2.2.1 Pullback and Pushforward

Consider 2 manifolds M and N with dimension m and n respectively such that $n \geq m$ and with coordinate system x^μ and y^μ respectively. Imagine a map $\psi : M \to N$ and a function $f : N \to \mathbb{R}$. With the function f acting on N to give a real number, the **pullback** ψ^* can be defined.

$$\psi^* f = f \circ \psi \tag{2.21}$$

The pullback behaves to pull the operator f from N to M. ψ^* maps the function space $\mathcal{F}(N)$ to the function space $\mathcal{F}(M)$. See figure 1.

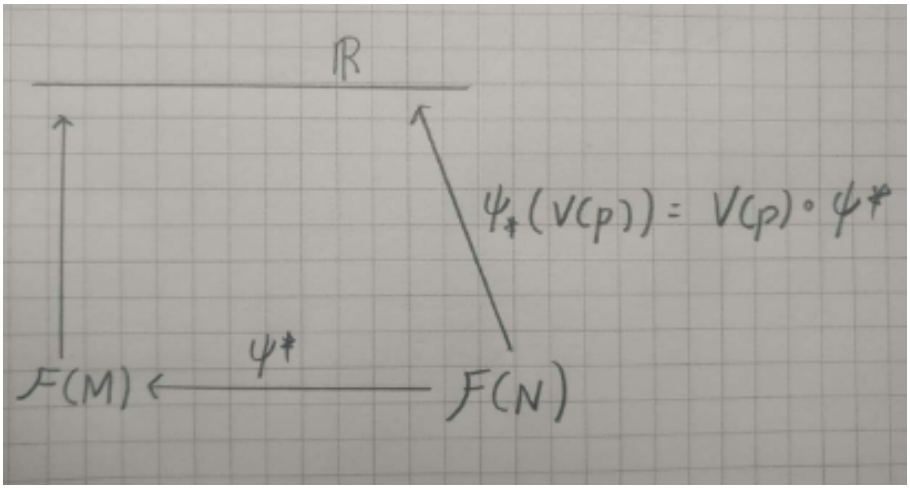

Figure 1: Pullback schematic diagram .

A vector can be thought as a derivative operator in differential geometry acting on a function f to give real number, namely $V : f \to \mathbb{R}$. This is because there exists a one to one correspondence (complete isomorphism) between any vector $\vec{V}$ to the directional partial derivative $\partial_{\vec{V}}$. Therefore a vector $\partial_{\vec{V}}$ can be expressed as

$$\vec{V} = \partial_{\vec{V}} \tag{2.22}$$
$$= V^\mu \partial_\mu \tag{2.23}$$

and

$$V : f = V^\mu \partial_\mu f \to \mathbb{R} \tag{2.24}$$

For $V(p)$ is a vector at a point p on the manifold M, the **pushforward** ψ_* of a vector V can be defined

$$(\psi_* V)f = V(\psi^* f) \tag{2.25}$$

To find the component of the pushforward of a vector $(\psi_* V)^\alpha$, the following deduction is considered

$$(\psi_* V)^\alpha \partial_\alpha f = V^\mu \partial_\mu (\psi^* f) \tag{2.26}$$
$$= V^\mu \partial_\mu [f \circ \psi] \tag{2.27}$$
$$= V^\mu \frac{\partial f}{\partial y^\alpha} \frac{\partial y^\alpha}{\partial x^\mu} \tag{2.28}$$
$$= (V^\mu \frac{\partial y^\alpha}{\partial x^\mu}) \partial_\alpha f \tag{2.29}$$

In the third equality line, chain rule is used. Hence, the component of pushforward of a vector is found.

$$(\psi_* V)^\alpha = V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \tag{2.30}$$

Note that a pushforward maps the tangent space of M at p to the tangent space of N at $\psi(p)$. Also, since a vector acts on a 1-form, ω to give a real number, ω can only be pulled back from N to M. See figure 2

$$(\psi^* \omega)(V)) = \omega(\psi_* V) \tag{2.31}$$

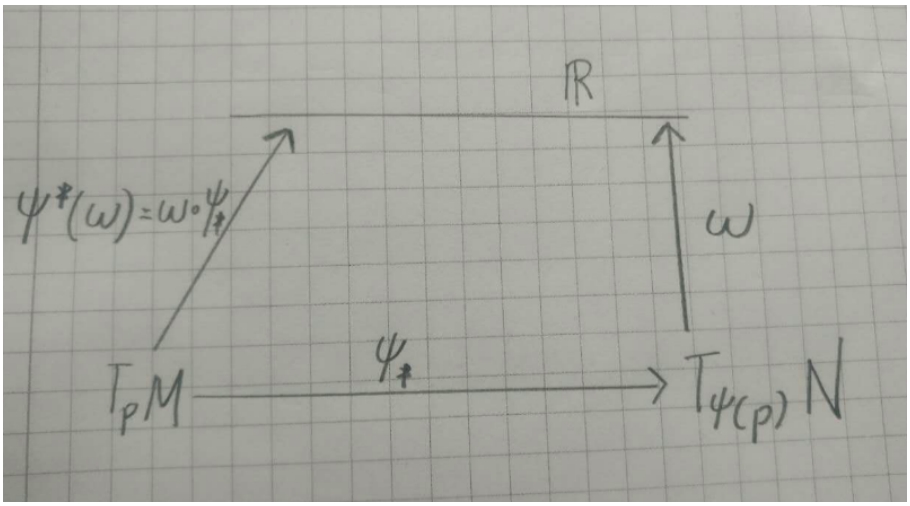

Figure 2: Pushforward schematic diagram [].

If M and N refer to the same manifold, the behaviour of a vector under a pushforward resembles the active coordinate transformation. This is obvious from (2.30). However, if M and N are different manifold (different dimensions,etc), $\frac{\partial y^\alpha}{\partial x^\mu}$ in (2.30) cannot be interpreted as the coordinate transformation matrix because then $\frac{\partial y^\alpha}{\partial x^\mu}$ is not invertible in general.

1-forms are linear maps from vectors to the real numbers. In contrast with vector for which the basis is represented by partial derivative, the basis of 1-form is represented by the gradient dx^μ

$$\omega : V \to \mathbb{R} \tag{2.32}$$

$$\begin{aligned} \omega : V &= \omega_\mu dx^\mu V^\alpha \partial_\alpha && (2.33)\\ &= \omega_\mu V^\alpha \delta^\mu_\alpha && (2.34)\\ &= \omega_\mu V^\mu && (2.35) \end{aligned}$$

1-forms can be pulled back from N to M. The component of the pullback of 1-form, $(\psi^*\omega)_\mu$ can be found by using (2.35)

$$\begin{aligned} (\psi^*\omega)(V) &= \omega(\psi_* V) && (2.36)\\ &= \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu} V^\mu && (2.37)\\ &= (\omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu}) V^\mu && (2.38) \end{aligned}$$

Therefore, the component of pullback of 1-form is

$$(\psi^*\omega)_\mu = (\omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu}) \tag{2.39}$$

Similarly, it is also the case that the behaviour of a 1-form under a pullback represents the active coordinate transformation if M and N are the same manifold.

A $(0, l)$ tensor, $T_{\mu_1\mu_2.....\mu_l}$ is a linear map from the direct product of l vectors to $\mathbb{R}$. In general one can pull back tensors with any arbitrary number of lower indices.

$$(\psi^* T)(V^{(1)}, V^{(2)}, ..., V^{(l)}) = T(\psi_* V^{(1)}, \psi_* V^{(2)}, ..., \psi_* V^{(l)}) \tag{2.40}$$

Similarly, one can push forward any $(k, 0)$ tensor, $S^{\alpha_1\alpha_2.....\alpha_k}$ by acting it on pulled back 1-forms

$$(\psi_* S)(\omega^{(1)}, \omega^{(2)}, ..., \omega^{(k)}) = S(\psi^*\omega^{(1)}, \psi^*\omega^{(2)}, ..., \psi^*\omega^{(k)}) \tag{2.41}$$

For the higher-rank tensors, the matrix representation of pullback and pushforward can be extended by assigning one $\frac{\partial y}{\partial x}$ matrix to each index of the tensor

$$\boxed{(\psi^* T)_{\mu_1.....\mu_l} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} ..... \frac{\partial y^{\alpha_l}}{\partial x^{\mu_l}} T_{\alpha_1} .....\alpha_l} \tag{2.42}$$

$$\boxed{(\psi_* S)^{\alpha_1.....\alpha_k} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} ..... \frac{\partial y^{\alpha_k}}{\partial x^{\mu_k}} S^{\mu_1.....\mu_k}} \tag{2.43}$$

2.2.2 An Illuminating Example

There is a very nice example in [3]. Consider the case where a S^2 sphere with spherical coordinate system is embedded in a 3-dimensional Euclidean space $\mathbb{R}^3$ with Cartesian coordinate system. Let M denotes the S^2 manifold with coordinate system (θ, ϕ) and N denotes $\mathbb{R}^3$ with coordinate system (x, y, z). Let ψ be a map, $\psi : M \to N$

$$\psi(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{2.44}$$

ψ will induce an extrinsic metric on S^2, which is the pullback of the flat space Euclidean metric. In Euclidean space,

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{2.45}$$

Using (2.44)

$$dx = \cos\theta\cos\phi d\theta - \sin\theta\sin\phi d\phi \tag{2.46}$$

$$dy = \cos\theta\sin\phi d\theta + \sin\theta\cos\phi d\phi \tag{2.47}$$

$$dz = -\sin\theta d\theta \tag{2.48}$$

It is then easy to deduce that

$$ds^2 = d\theta^2 + sin^2\theta d\phi^2 \tag{2.49}$$

The induced metric, $\bar{g}_{jk}$ is obtained

$$\bar{g}_{jk} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix} \tag{2.50}$$

Now, compute the pullback of metric δ_{jk} in Euclidean space

$$(\psi^*\delta)_{jk} = \frac{\partial y^m}{\partial x^j}\frac{\partial y^n}{\partial x^k}\delta_{mn} \tag{2.51}$$

$$= \bar{g}_{jk} \tag{2.52}$$

for $\frac{\partial y^m}{\partial x^j}$ matrix is

$$\frac{\partial y^m}{\partial x^j} = \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\phi \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{bmatrix} \tag{2.53}$$

Hence, the induced metric used to measure the distance on S^2 is obtained by pulling back the metric from $\mathbb{R}^3$.

2.3 Diffeomorphism and Lie derivatives

2.3.1 Active Coordinate transformation induced by Diffeomorphism

Definition 2.3.1.1 (Diffeomorphism) : Given two manifolds M and N, a map $\psi : M \to N$ is called diffeomorphism if its inverse $\psi^{-1} : N \to M$ exists. If ψ is k times differentiable, ψ is called C^k- diffeomorphism.

The existence of diffeomorphism actually implies that the 2 manifolds M and N are the same. For a diffeomorphic map ψ, one can use both ψ and ψ^{-1} to pull back or push forward any tensors from M to N. Specifically, for a (k,l) tensor field $T^{\mu 1\mu 2.....\mu k}_{\nu 1\nu 2.....\nu l}$ on M, one can define the pushforward

$$(\psi_* T)(\omega^{(1)}, ..., \omega^k, V^{(1)}, ..., V^{(l)}) = T(\psi^*\omega^1, ..., \psi^*\omega^{(k)}, [\psi^{-1}]_* V^{(1)}, ..., [\psi^{-1}]_* V^{(l)}) \tag{2.54}$$

In components form, the pushforward takes the expression

$$(\psi_* T)^{\alpha_1.....\alpha_k}_{\beta_1.....\beta_l} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}}...\frac{\partial y^{\alpha_k}}{\partial x^{\mu_k}}\frac{\partial x^{\nu_1}}{\partial y^{\beta_1}}.....\frac{\partial x^{\nu_l}}{\partial y^{\beta_l}} T^{\mu_1.....\mu_k}_{\nu_1.....\nu_l}. \tag{2.55}$$

For pullback, it is exactly the same story. Therefore, from (2.55), it is now obvious that the pullback and pushforward induced by a diffeomorphic map ψ represents the active coordinate transformations. To change the coordinate system, first one can use a diffeomorphic map ψ and act it on M, $\psi : M \to M$, the new coordinate system is obtained by just pulling the coordinate from the range space to the domain space, $(\psi^* x)^\mu : M \to \mathbb{R}^n$. Similarly, under the active coordinate transformation, the new tensorial value takes the expression $(\psi_* T)^{\mu_1.....\mu_k}_{\nu_1.....\nu_l}$. See figure 3.

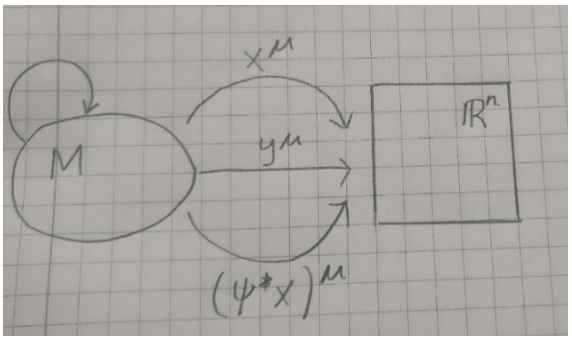

Figure 3: A coordinate change induced by the diffeomorphism .

One can also compute the difference of the tensorial value at 2 different points on the manifold M using the pullback and pushforward induced by a diffeomorphism. To compare the tensorial value at 2 different points, p and ψp on the manifold, the naive way of simple subtracting $T(p)$ and $T(\psi p)$ does not work because $T(p)$ and $T(\psi p)$ lie at different tangent vector space. So, one possible way to compare them is to pull back $T(\psi p)$ to the point p first and then compare them.

$$\Delta T = \psi^*(T(\psi p)) - T(p) \tag{2.56}$$

This method of comparing the difference of the tensorial value at 2 different points suggest a new kind of derivative on tensor fields which is used to identify the rate of change of the tensor field under the flow of diffeomorphism. This derivative is called Lie derivative. We follow the approach in [3] to introduce Lie derivative by at first looking at a particular easy diffeomorphism and then generalize the whole concept.

2.3.2 Lie Derivative

A one parameter family of diffeomorphisms, ψ_t is needed to categorize the rate of change of the tensor under the flow of the diffeomorphism belong to this family. This can be thought as a smooth map $\mathbb{R} \times M \to M$ such that for each $t \in \mathbb{R}$, there exists a diffeomorphism ψ_t which satisfy $\psi_s \circ \psi_t = \psi_{s+t}$ [3]. A vector field $V = \frac{d\psi_t}{dt}$ is then induced by this one parameter family of diffeomorphisms. At a single point p of the manifold M, there is a tangent vector induced there. For a collection of continuous and successive points, a curve is generated. All these curves will fill up the entire manifold M.

Consider the case where $(x^1, x^2, ..., x^n)$ is the coordinate system at the point p on the manifold M. The one-parameter diffeomorphism considered ψ_t is $\psi_t(x^1, x^2, ..., x^n) = (x^1 + t, x^2, ..., x^n)$. The Lie derivative of a tensor T, $\mathcal{L}_V T$ along the vector field generated by ψ_t is defined as

$$\mathcal{L}_V T^{\mu_1....\mu_k}_{\nu_1.....\nu_l} = \lim_{t\to 0}[\frac{\psi_t^*[T^{\mu_1....\mu_k}_{\nu_1.....\nu_l}](\psi_t(p)) - T^{\mu_1....\mu_k}_{\nu_1.....\nu_l}(p)}{t}] \tag{2.57}$$

$\mathcal{L}_V$ maps a (k, l) tensor fields to a (k, l) tensor fields. It obeys the following properties [3]

1) **Linearity**

$$\mathcal{L}_V(aT + bS) = a\mathcal{L}_V T + b\mathcal{L}_V S \tag{2.58}$$

For a and b are constants. T and S are arbitrary tensor.

2) **Leibniz rule**

$$\mathcal{L}_V(T \otimes S) = (\mathcal{L}_v T) \otimes S + T \otimes (\mathcal{L}_v S) \tag{2.59}$$

Lie derivative operator reduces to the directional partial derivative when it acts on a function f

$$\mathcal{L}_V f = V^\mu \partial_\mu f \tag{2.60}$$

In the case that we study, the pullback of a tensor T can be obtained using (2.55)

$$(\psi_* T)^{\alpha_1.....\alpha_k}_{\beta_1.....\beta_l} = \delta^{\alpha_1}_{\mu_1}....\delta^{\alpha_k}_{\mu_k}\delta^{\nu_1}_{\beta_1}....\delta^{\nu_l}_{\beta_l}T^{\mu_1.....\mu_k}_{\nu_1.....\nu_l}(x^1+t, x^2, ..., x^n) \tag{2.61}$$

$$= T^{\alpha_1....\alpha_k}_{\beta_1.....\beta_l}(x^1+t, x^2, ..., x^n) \tag{2.62}$$

Hence, the pullback tensorial value $\psi_t^*[T^{\mu_1....\mu_k}_{\nu_1.....\nu_l}](\psi_t(p)) = T^{\alpha_1....\alpha_k}_{\beta_1.....\beta_l}(x^1+t, x^2, ..., x^n)$. From the definition of the Lie derivative, it is obvious that in such a case, the Lie derivative amounts to

$$\mathcal{L}_V T^{\mu_1....\mu_k}_{\nu_1....\nu_l} = \frac{\partial}{\partial x^1}T^{\mu_1....\mu_k}_{\nu_1....\nu_l} \tag{2.63}$$

In particular, the Lie derivative of a vector field U^μ is

$$\mathcal{L}_V U^\mu = \frac{\partial U^\mu}{\partial x^1} \tag{2.64}$$

The expression of the Lie derivative of a vector field in (2.64) is clearly not in the invariant tensorial form. However, (2.64) can be rexepressed into the form involving commutator $[V, U]$

$$[V, U]^\mu = V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu \tag{2.65}$$

$$= \frac{\partial U^\mu}{\partial x^1} \tag{2.66}$$

$$= \mathcal{L}_V U^\mu \tag{2.67}$$

The commutator relation involving the expression of Lie bracket is a well defined tensor. Therefore, we managed to find a tensorial expression of the Lie derivative generated by a vector field V acting on another vector field U.

To find the expression of the Lie derivative generated by a vector field V acting on a 1-form, use (2.59) and (2.60)

Using (2.59)

$$\mathcal{L}_V(\omega_\mu U^\mu) = (\mathcal{L}_V\omega)_\mu U^\mu + \omega_\mu(\mathcal{L}_V U)^\mu \tag{2.68}$$

$$= (\mathcal{L}_V\omega)_\mu U^\mu + \omega_\mu V^\nu \partial_\nu U^\mu - \omega_\mu U^\nu \partial_\nu V^\mu \tag{2.69}$$

Using (2.60)

$$\mathcal{L}_V(\omega_\mu U^\mu) = V^\nu \partial_\nu(\omega_\mu U^\mu) \tag{2.70}$$

$$= V^\nu(\partial_\nu\omega_\mu)U^\mu + V^\nu\omega_\mu(\partial_\nu U^\mu) \tag{2.71}$$

Equating (2.69) and (2.71) gives rise to the relation of the Lie derivative operator acting on a 1-form ω

$$\mathcal{L}_V\omega_\mu = V^\nu(\partial_\nu\omega_\mu) + (\partial_\mu V^\nu)\omega_\nu \tag{2.72}$$

The Lie derivative of an arbitrary tensor can be obtained by the similar procedure

$$\boxed{\mathcal{L}_V T^{\mu_1\mu_2....\mu_k}_{\nu_1\nu_2.....\nu_l} = V^\sigma\partial_\sigma T^{\mu_1\mu_2....\mu_k}_{\nu_1\nu_2.....\nu_l} - (\partial_\lambda V^{\mu_1})T^{\lambda\mu_2....\mu_k}_{\nu_1\nu_2.....\nu_l} - (\partial_\lambda V^{\mu_2})T^{\mu_1\lambda....\mu_k}_{\nu_1\nu_2.....\nu_l} - ..... + (\partial_{\nu_1}V^\lambda)T^{\mu_1\mu_2....\mu_k}_{\lambda\nu_2.....\nu_l} + ...} \tag{2.73}$$

Note that the partial derivatives ∂ in (2.73) can be replaced by the covariant derivative ∇. It turns out that all the terms involving Levi-Civita connection will cancel each other.

Lie derivative of the metric tensor is a very important relation. Using (2.73), it can be deduced that

$$\mathcal{L}_V g_{\mu\nu} = V^\sigma\nabla_\sigma g_{\mu\nu} + (\nabla_\mu V^\lambda)g_{\lambda\nu} + (\nabla_\nu V^\lambda)g_{\mu\lambda} \tag{2.74}$$

$$= \nabla_\mu V_\nu + \nabla_\nu V_\mu \tag{2.75}$$

In the second equality line, the metric compatibility is considered.

3 Non-linear theory of Electromagnetism and Gravity

In this section, we study the foundation of Born-Infeld theory and general relativity. Born-Infeld theory and general relativity are the non-linear field theories needed to be understood well for both of our research topics − the geometrical interpretation of field theories and classical double copy. Therefore, this chapter serves as the physics preliminaries for this thesis.

3.1 Born-Infeld theory

3.1.1 Brief recap of Maxwell theory

One of the most successful theory in physics is Maxwell's theory of electromagnetism. In the absence of any source, the Lagrangian density of Maxwell theory is known to be

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{3.1}$$

For $F_{\mu\nu}$ is the electromagnetic field strength which is totally anti-symmetric,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.2}$$

Applying the least action principle, the equation of motion is found to be

$$\partial_\mu F^{\mu\nu} = 0 \tag{3.3}$$

In the matrix form, $F_{\mu\nu}$ is written in term of electric and magnetic field as

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \tag{3.4}$$

The equation of motion (3.3) implies two source free Maxwell equations

$$\partial_\mu F^{\mu\nu} = 0 \Leftrightarrow \nabla \cdot \vec{E} = 0, \nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} \tag{3.5}$$

The other two Maxwell equations are obtained from the Bianchi identity

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \Leftrightarrow \nabla \cdot \vec{B} = 0, \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3.6}$$

One important thing worth mentioning is that the Maxwell theory in vacuum possesses Lorentz symmetry. This means that Maxwell action and Maxwell equations in vacuum are covariant with respect to the Lorentz transformation.

3.1.2 Linear vs Non-linear

Maxwell's theory is a linear field theory. There is a very important concept to compare the linear field theory and non-linear field theory. Field equation in Maxwell's theory with source is

$$\partial^2 A_\mu = \kappa J_\mu \tag{3.7}$$

where the 4-current J_μ is (ρ, j_1, j_2, j_3). ρ is the charge density and j_i is the current density. In quantum field theory, the photon γ is the excitation of the electromagnetic field A_μ. The source J_μ is completely independent of the electromagnetic field A_μ. Therefore, there will be no self electromagnetic interaction (photon interacts with photon) in the scattering process.

For non-linear field theory, the self-interaction of the gauge boson is possible. Consider the theory of general relativity which is a non-linear field theory. We will show here briefly how the self

interaction of graviton occurs in general relativity. The concepts discused here can be obtained from other section of this thesis as the theory of general relativity is discussed in details in section 3.2 and linearized gravity is discussed in section 5.3.3. Consider linearized gravity for which the full metric is written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3.8}$$

In linearized gravity, weak field limit is assumed.

$$|h_{\mu\nu}| \ll 1 \tag{3.9}$$

In the weak field limit, the Levi-Civita connection can be well approximated to a form which only carries linear terms of the field $h_{\mu\nu}$

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\eta^{\lambda\rho}}{2}(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\rho\mu} - \partial_\rho h_{\mu\nu}) \tag{3.10}$$

The Ricci tensor which describes the curvature of the spacetime can also be well approximated to $R^{(1)}_{\mu\nu}$ that contains only the linear terms of $h_{\mu\nu}$

$$R^{(1)}_{\mu\nu} = \frac{1}{2}(\partial^2 h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\mu\partial_\lambda h^{\lambda}_{\nu} - \partial_\nu\partial_\lambda h^{\lambda}_{\mu}) \tag{3.11}$$

The gravitational field equation in linearized gravity can be expressed as

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)} = 8\pi G(\tau^{\mu\nu}) \tag{3.12}$$

where $\tau_{\mu\nu}$ is

$$\tau_{\mu\nu} = T_{\mu\nu} + \frac{G^{(1)}_{\mu\nu} - G_{\mu\nu}}{8\pi G} \tag{3.13}$$

$T_{\mu\nu}$ is the energy momentum tensor, $G_{\mu\nu}$ is Einstein tensor and $G^{(1)}_{\mu\nu}$ is the linearized term of Einstein tensor and G is Newton constant. $\tau_{\mu\nu}$ is interpreted to be the source of the field $h_{\mu\nu}$. The excitation of the field $h_{\mu\nu}$ is graviton g. The field equation in linearized gravity will give the expression

$$(\partial^2 + ...)h_{\mu\nu} = 8\pi G\tau_{\mu\nu} \tag{3.14}$$

The term $G_{\mu\nu}$ in the source term $\tau_{\mu\nu}$ carries $h_{\mu\nu}$. This explains why gravitational field can generate gravitational field itself. In other words, the graviton can self-interact with other graviton in the scattering process. See figure 4.

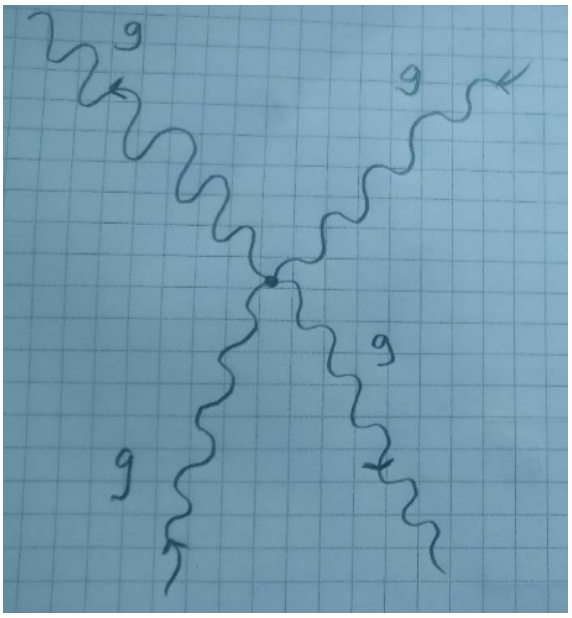

Figure 4: Self-Interaction of gravitons [].

In next scetion, we will discuss Born-Infeld theory which is also a non-linear field theory. Just like in non-linear field theory of gravity the self-interaction of gravitons occurs, the self interaction of the photons also occurs in Born-Infeld theory.

3.1.3 Motivation for Born-Infeld theory and the Postulation of Invariant Action

In 1934, Max Born and Leopold Infeld proposed a non-linear theory of electromagnetism. Note that this was before the advent of quantum field theory. At that time, most of the physicists adopted the dualistic standpoint on the relation of matter and the electromagnetic field. The main idea of the dualistic standpoint is that the particles are the sources of the field, the particles are acted on by the field but they are not a part of the field [2]. The other standpoint which is less popular is the unitarian standpoint that assumes that the only one physical entity is the electromagnetic field. The particles of matter are to be considered as singularities of the field and their mass is the derived notion from the field energy [2]. It is obvious that quantum field theory today as the theoretical framework to construct the standard model takes the unitarian standpoint because the particle is the excitation of the quantum field in this theory. Beside quantum field theory, Born and Infeld actually managed to come out with a modified electromagnetic field theory which is also an unitarian theory. The amazing thing is that their theory satisfies the principle of finiteness. In the electrostatic case of Maxwell's theory, the Coulomb potential, $V \propto -\frac{1}{r}$ does not satisfy the principle of finiteness. So, at the point where the charged particle sits, the Coulomb potential tends to infinity. We will see later in this chapter how Born-Infeld theory overcomes this problem.

Due to the extraordinary success of quantum field theory, there has been a time that Born-Infeld theory was neglected. However, due to the recent development in string theory and other theory like double copy, Born-Infeld theory caught physicists' attention again and a lot of researches were devoted to this theory. Therefore, we discuss Born-Infeld theory in this chapter by reviewing how Born and Infeld constructed this theory.

They started by postulating an invariant action which is covariant with respect to the general spacetime transformations in the same spirit as discussed in section 2.1.

Consider the Lagrangian density

$$\mathcal{L} = \sqrt{det(a_{\mu\nu})} \tag{3.15}$$

For $a_{\mu\nu}$ is a tensor which is neither symmetric nor anti-symmetric. In general, $a_{\mu\nu}$ can be split up into the symmetrical part and antisymmetrical part.

$$a_{\mu\nu} = g_{\mu\nu} + F_{\mu\nu} \tag{3.16}$$

such that

$$g_{\mu\nu} = g_{\nu\mu}; F_{\mu\nu} = -F_{\nu\mu} \tag{3.17}$$

The symmetrical part $g_{\mu\nu}$ is the metrical field while the antisymmetrical part $F_{\mu\nu}$ is the electromagnetic field strength. In last section, it has been shown that one of the possible form of the invariant action with respect to some transformation laws takes the form such that the Lagrangian density is the square root of the determinant of any arbitrary $(0, 2)$ tensor. The simplest assumption for $\mathcal{L}$ is the linear combination

$$\mathcal{L} = \sqrt{-det(g_{\mu\nu} + F_{\mu\nu})} + A\sqrt{-det(g_{\mu\nu})} + B\sqrt{det(F_{\mu\nu})} \tag{3.18}$$

For A and B are arbitrary constant. The minus sign in the first and second term is induced in order to get the real value of the square root due to the fact that the general manifold considered is Lorentzian manifold and the determinant of metric tensor is a negative value.

The last term can be omitted. Note that $F_{\mu\nu}$ is written as $\nabla_\mu A_\nu - \nabla_\nu A_\mu$. The usual partial derivative ∂_μ is replaced by the covariant derivative ∇_μ since the metric used here is not fixed to be constant. The spacetime integral of the last term can be changed to a surface term since it contains only total derivative term. It has no influence on the variational equation of the field since δA_μ is assumed to be zero at the boundary. So, one can take $B = 0$. The remaining task is to determine the coefficient A.

$$\mathcal{L} = \sqrt{-det(g_{\mu\nu} + F_{\mu\nu})} + A\sqrt{-det(g_{\mu\nu})} \tag{3.19}$$

In order to determine the coefficient A, the imposed condition is that in the limiting condition of the flat Minkowski space and in the weak electromagnetic field limit, the classical Maxwell expression for Lagrangian density (3.1) is obtained. This is essential because a good physical theory should be able to be reduced to the well-known theory which has succeeded to describe the world in a certain limit. So, in the limit described above, $\mathcal{L}$ becomes

$$\mathcal{L} = \sqrt{-det(\eta_{\mu\nu} + F_{\mu\nu})} + A\sqrt{-det(\eta_{\mu\nu})} \tag{3.20}$$

For η is Minkowski metric and it takes expression as

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.21}$$

By careful calculation, $-det(\eta_{\mu\nu} + F_{\mu\nu})$ is found to be

$$-det(\eta_{\mu\nu} + F_{\mu\nu}) = 1 + (F_{23}^2 + F_{31}^2 + F_{12}^2 - F_{10}^2 - F_{20}^2 - F_{30}^2) - (F_{23}F_{10} + F_{31}F_{20} + F_{12}F_{30})^2 \tag{3.22}$$

$$= 1 + (F_{23}^2 + F_{31}^2 + F_{12}^2 - F_{10}^2 - F_{20}^2 - F_{30}^2) - det(F_{\mu\nu}) \tag{3.23}$$

In the weak field limit $F_{\mu\nu} \ll 1$, the last term can be safely neglected. So, it reduces to

$$-det(\eta_{\mu\nu} + F_{\mu\nu}) = 1 + B_x^2 + B_y^2 + B_z^2 - E_x^2 - E_y^2 - E_z^2 \tag{3.24}$$

To be able to reduce to the Maxwell form $F_{\mu\nu}F^{\mu\nu} \propto B^2 - E^2$, the term 1 needs to be eliminated. Since $-det(\eta_{\mu\nu}) = 1$, A has to be -1 to cancel the factor 1 after expanding $\sqrt{-det(\eta_{\mu\nu} + F_{\mu\nu})}$ and neglect the higher order term.

$$A = -1 \tag{3.25}$$

Therefore, Lagrangian density takes the form

$$\mathcal{L} = \sqrt{-det(\eta_{\mu\nu} + F_{\mu\nu})} - \sqrt{-det(\eta_{\mu\nu})} \tag{3.26}$$

$$= \sqrt{1 + B^2 - E^2 - B_xE_x - B_yE_y - B_zE_z} - 1 \tag{3.27}$$

$$= \sqrt{1 + B^2 - E^2 - \vec{B}.\vec{E}} - 1 \tag{3.28}$$

$$= \sqrt{1 + F - G^2} - 1 \tag{3.29}$$

such that

$$F = B^2 - E^2 = (F_{23}^2 + F_{31}^2 + F_{12}^2 - F_{10}^2 - F_{20}^2 - F_{30}^2) \tag{3.30}$$

and

$$G^2 = \vec{B}.\vec{E} = det(F_{\mu\nu}) = (F_{23}F_{10} + F_{31}F_{20} + F_{12}F_{30})^2 \tag{3.31}$$

In a general coordinate system for any arbitrary manifold with its structure governed by $g_{\mu\nu}(x)$,

$$det(g_{\mu\nu} + F_{\mu\nu}) = det(g_{\mu\nu}) + \Phi(g_{\mu\nu}, F_{\mu\nu}) + det(F_{\mu\nu}) \tag{3.32}$$

$$= det(g_{\mu\nu})[1 + \frac{\Phi}{det(g_{\mu\nu})} + \frac{det(F_{\mu\nu})}{det(g_{\mu\nu})}] \tag{3.33}$$

In geodetic coordinate system, it has been found out that $\frac{\Phi}{det(g_{\mu\nu})} = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = F$ and $\frac{det(F_{\mu\nu})}{det(g_{\mu\nu})} = -G^2$. Since the form of $\frac{\Phi}{det(g_{\mu\nu})}$ is invariant with respect to the symmetric coordinate transformation rule, it also takes the same form in those coordinate systems.

The Lagrangian density in general can be written as

$$\mathcal{L} = \sqrt{-det(g_{\mu\nu} + F_{\mu\nu})} - \sqrt{-det(g_{\mu\nu})} \tag{3.34}$$

$$= \sqrt{-det(g_{\mu\nu})}(\sqrt{1 + F - G^2} - 1) \tag{3.35}$$

In some papers and textbooks, there is a constant factor -1 in front of the expression (3.34). The constant factor is trivial because it does not affect the equation of motion obtained from the least action principle. Both F and G are invariant. It is better to convert G into the form for which that its invariance property is obvious. To do this, a $(4, 0)$ tensor j^{abcd} is introduced,

$$j^{abcd} = \begin{cases} \frac{1}{2\sqrt{-det(g_{\mu\nu})}}, & \text{if abcd is even permutation of 1234,} \\ -\frac{1}{2\sqrt{-det(g_{\mu\nu})}}, & \text{if abcd is odd permutation of 1234,} \\ 0, & \text{otherwise} \end{cases} \tag{3.36}$$

After introducing the tensor j^{abcd}, G can be written as

$$G = \frac{1}{4}j^{abcd}F_{ab}F_{cd} \tag{3.37}$$

and the dual of $F_{\mu\nu}$, $F*^{ab}$ can be defined as

$$F*^{ab} = j^{ab\mu\nu}F_{\mu\nu} \tag{3.38}$$

3.1.4 Equations of motion for Born-Infeld theory

One of the most elegant principle in physics is the least action principle. This principle has the power to generate physical theory. Once the relevant action is determined, the equation of motion can be obtained by imposing the constraint of the variation of action with respect to a certain variable to be zero. It is not the exception for Born Infeld theory. Except using least action principle, note that 2 Maxwell equations can be obtained from the Bianchi identity (3.6). The two equations of motion obtained from Bianchi identity are

$$\nabla \cdot \vec{B} = 0; \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3.39}$$

Since the definition $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is inherited as well in the Born Infeld theory, the two equations in (3.39) are also the equation of motions of Born Infeld theory. With help of (3.38), (3.6) and (3.39) can be brought into a more compact form

$$\boxed{\frac{\partial(\sqrt{-det(g_{mn})}F*^{\mu\nu})}{\partial x^\nu} = 0 \Leftrightarrow \nabla \cdot \vec{B} = 0; \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}} \tag{3.40}$$

The divergence operator and the curl operator here is the general one in any general coordinate system. The major difference between Born Infeld theory and Maxwell theory comes from the other two equation of motions obtained from the least action principle. This can be seen by first taking the partial derivative of $\mathcal{L}$ taking the form of (3.34) with respect to $F_{\mu\nu}$. Note that $\mathcal{L}$ can be written as $\sqrt{-\det(g_{\mu\nu})}L$, for L is

$$L = \sqrt{1 + F - G^2} - 1 \tag{3.41}$$

So, $\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} = \sqrt{-\det(g_{\mu\nu})}\frac{\partial L}{\partial F_{\mu\nu}}$ such that $\frac{\partial L}{\partial F_{\mu\nu}} = \frac{\partial L}{\partial F}\frac{\partial F}{\partial F_{\mu\nu}} + \frac{\partial L}{\partial G}\frac{G}{\partial F_{F_{\mu\nu}}}$

$$\frac{\partial L}{\partial F} = \frac{1}{2}(1 + F - G^2)^{-\frac{1}{2}} \tag{3.42}$$

$$\frac{\partial L}{\partial G} = -G(1 + F - G^2)^{-\frac{1}{2}} \tag{3.43}$$

$$\begin{aligned}
\frac{\partial F}{\partial F_{\mu\nu}} &= \frac{1}{2}\frac{\partial(F^{ab}F_{ab})}{\partial F_{\mu\nu}} && (3.44)\\
&= \frac{1}{2}(\delta^\mu_a \delta^\nu_b F^{ab} + g^{am} g^{bn} \delta^\mu_m \delta^\nu_n F_{ab}) && (3.45)\\
&= \frac{1}{2}(2F^{\mu\nu}) && (3.46)\\
&= F^{\mu\nu} && (3.47)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial G}{\partial F_{\mu\nu}} &= \frac{1}{4} j^{abcd}\frac{\partial(F_{ab}F_{cd})}{\partial F_{\mu\nu}} && (3.48)\\
&= \frac{1}{4}(j^{\mu\nu cd}F_{cd} + j^{ab\mu\nu}F_{ab}) && (3.49)\\
&= \frac{1}{4}(j^{\mu\nu cd}F_{cd} + j^{\mu\nu ab}F_{ab}) && (3.50)\\
&= \frac{1}{2}F*^{\mu\nu} && (3.51)
\end{aligned}$$

Therefore,

$$\frac{\partial L}{\partial F_{\mu\nu}} = \frac{1}{2}\frac{F^{\mu\nu} - GF*^{\mu\nu}}{\sqrt{1+F-G^2}} \tag{3.52}$$

$$= \frac{1}{2}P^{\mu\nu} \tag{3.53}$$

For $P^{\mu\nu} = \frac{F^{\mu\nu} - GF*^{\mu\nu}}{\sqrt{1+F-G^2}}$.

$P^{\mu\nu}$ is interpreted to be the electromagnetic field strength in matter. $P^{\mu\nu}$ is related to $F^{\mu\nu}$ in a way to that which, in Maxwell's theory of macroscopic bodies, the dielectric displacement and magnetic induction have to the field strength [2]. In matrix form, $P^{\mu\nu}$ is written as

$$P^{\mu\nu} = \begin{bmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & H_z & -H_y \\ -D_y & -H_z & 0 & H_x \\ -D_z & H_y & -H_x & 0 \end{bmatrix} \tag{3.54}$$

To obtain the remaining two equation of motions, vary the action S with respect to $F_{\mu\nu}$ and require $\delta S = 0$,

$$\delta S = \int \delta\mathcal{L} d^4x \tag{3.55}$$

$$= \int \sqrt{-det(g_{mn})}\frac{\partial L}{\partial F_{\mu\nu}}\delta F_{\mu\nu} d^4x \tag{3.56}$$

$$= \int \sqrt{-det(g_{mn})}\frac{\partial L}{\partial F_{\mu\nu}}(\nabla_\mu \delta A_\nu - \nabla_\nu \delta A_\mu) d^4x \tag{3.57}$$

$$= -\int \partial_\mu(\sqrt{-det(g_{mn})}\frac{\partial L}{\partial F_{\mu\nu}})\delta A_\nu + \partial_\nu(\sqrt{-det(g_{mn})}\frac{\partial L}{\partial F_{\mu\nu}})\delta A_\mu d^4x \tag{3.58}$$

$$= -2\int \partial_\mu(\sqrt{-det(g_{mn})}\frac{\partial L}{\partial F_{\mu\nu}})\delta A_\nu d^4x \tag{3.59}$$

$$= 0 \tag{3.60}$$

In the forth equality line, the relation of covariant divergence $\nabla_\mu K^\mu = \frac{1}{\sqrt{-det(g)}}\partial_\mu(\sqrt{-det(g)}K^\mu)$ is used. Integration by part is also carried out and the total derivative term can be neglected due to the fact that δA_ν is assumed to be zero at the boundary. In the fifth equality, the technique of renaming the dummy indices is used.

This leads to the equation of motion

$$2\partial_\mu(\sqrt{-det(g_{mn})}\frac{\partial L}{\partial F_{\mu\nu}}) = \partial_\mu(\sqrt{-det(g_{mn})}P^{\mu\nu}) = 0 \tag{3.61}$$

In Maxwell theory, $\partial_\mu F^{\mu\nu} = 0 \Leftrightarrow \nabla\cdot\vec{E} = 0; \nabla\times\vec{H} = \frac{\partial\vec{E}}{\partial t}$. In the same spirit

$$\boxed{\partial_\mu(\sqrt{-det(g_{mn})}P^{\mu\nu}) = 0 \Leftrightarrow \nabla\cdot\vec{D} = 0; \nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t}} \tag{3.62}$$

So far all the deductions are carried out using natural unit. In the conventional unit, the electromagnetic field strength has to be divided a dimensional correction constant, b. Then the results obtained before become

$$L = \sqrt{1+F-G^2} \tag{3.63}$$

$$F = \frac{1}{b^2}(B^2 - E^2); G = \frac{1}{b^2}(\vec{B} \cdot \vec{E}) \tag{3.64}$$

$$\vec{H} = b^2 \frac{\partial L}{\partial \vec{B}} = \frac{\vec{B} - G\vec{E}}{\sqrt{1 + F - G^2}} \tag{3.65}$$

$$\vec{D} = b^2 \frac{\partial L}{\partial \vec{E}} = \frac{\vec{E} - G\vec{B}}{\sqrt{1 + F - G^2}} \tag{3.66}$$

3.1.5 The Electrostatic Solution of the Born-Infeld field equation

Consider the electrostatic case (electric field generated by a point charged particle with charge e according to the Coulomb's law) where $B = H = 0$ and all field components are independent of time. Then, the only field equations survived are

$$\nabla \times \vec{E} = 0; \nabla \cdot \vec{D} = 0 \tag{3.67}$$

For the case of central symmetry, it is convenient to work in the spherical coordinate system. It is then easy to solve for D field,

$$\frac{d}{dr}(r^2 D_r) = 0 \Rightarrow D_r = \frac{e}{r^2} \tag{3.68}$$

For D_r indicates the radial component of D field and e is the elementary charge. The surface integral of D field over a Gaussian surface can then be computed. Let the Gaussian surface be a sphere for which the point charged particle is sitting at the centre (origin of the coordinate system) of the sphere.

$$\int D_r r^2 \sin\theta d\theta d\phi = \int e \sin\theta d\theta d\phi \tag{3.69}$$

$$= 4\pi e \tag{3.70}$$

Also, the curl of $\vec{E}$ field vanishes implies that $\vec{E}$ field can be written as the gradient of a potential function, ϕ

$$E = -\nabla\phi \tag{3.71}$$

Since ϕ possesses central symmetry, it only varies with respect to the radial direction. So,

$$E_r = -\frac{d\phi}{dr} = -\phi'(r) \tag{3.72}$$

Use (3.66), D_r can be expressed as

$$D_r = \frac{E_r}{\sqrt{1 - \frac{1}{b^2}E_r^2}} = \frac{e}{r^2} \tag{3.73}$$

Rearranging (3.73), E_r is found to be

$$E_r = \frac{e}{r_0^2\sqrt{1 + (\frac{r}{r_0})^4}} \tag{3.74}$$

For $r_0 = \sqrt{\frac{e}{b}}$. From (3.72), (3.74) is a first order differential equation of electrostatic potential ϕ,

$$\frac{d\phi}{dr} = -\frac{e}{r_0^2\sqrt{1 + (\frac{r}{r_0})^4}} \tag{3.75}$$

Substitute $y = \frac{r}{r_0}$, ϕ can be solved by the integral expression

$$\phi = \frac{e}{r_0}\int_x^\infty \frac{1}{\sqrt{1+y^4}}dy \tag{3.76}$$

Do a further substitution by letting $x = tan\frac{1}{2}\beta$

$$\phi(r) = f(x) = \frac{1}{2}\int_{\alpha(x)}^{\pi} \frac{\sec^2\frac{1}{2}\beta}{\sqrt{1+\tan^4\frac{1}{2}\beta}}d\beta \tag{3.77}$$

$$= \frac{1}{2}\int_{\alpha(x)}^{\pi} \frac{1}{\sqrt{cos^4\frac{1}{2}\beta + \sin^4\frac{1}{2}\beta}}d\beta \tag{3.78}$$

$$= \frac{1}{2}\int_{\alpha(x)}^{\pi} \frac{1}{\sqrt{(\cos^2\frac{1}{2}\beta + \sin^2\frac{1}{2}\beta)^2 - 2\cos^2\frac{1}{2}\beta\sin^2\frac{1}{2}\beta}}d\beta \tag{3.79}$$

$$= \frac{1}{2}\int_{\alpha(x)}^{\pi} \frac{1}{\sqrt{1-\frac{1}{2}\sin^2\beta}}d\beta \tag{3.80}$$

$$= \frac{1}{2}\int_{0}^{\pi} \frac{1}{\sqrt{1-\frac{1}{2}\sin^2\beta}}d\beta - \frac{1}{2}\int_{0}^{\alpha(x)} \frac{1}{\sqrt{1-\frac{1}{2}\sin^2\beta}}d\beta \tag{3.81}$$

$$= f(0) - \frac{1}{2}F(\frac{1}{\sqrt{2}}, \alpha) \tag{3.82}$$

For $\alpha(x) = 2arctan(x)$. $F(\frac{1}{\sqrt{2}}, \alpha) = \int_0^{\alpha(x)} \frac{1}{\sqrt{1-\frac{1}{2}\sin^2\beta}}d\beta$ is recognized as Jacobian elliptic integral. ϕ has its maximum value when $x = 0$. Thus the electrostatic potential has its maximum value in the centre and its value is

$$\phi(0) = \frac{1.8541e}{r_0} \tag{3.83}$$

The curse of the infinity in the old electrostatic theory has been shown to be broken by the modified electrostatic case in Born-Infield theory. This is shown explicitly in figure 5.

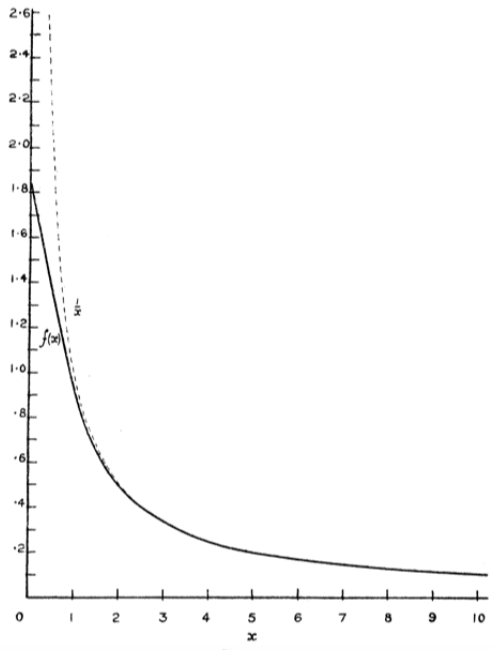

Figure 5: Electrostatic potential graph [2].

3.2 General relativity

3.2.1 Einstein-Hilbert Action and the Einstein Field equation in vacuum

One of the pillar of modern physics is Einstein's general theory of relativity which is used to describe the physics of gravity in term of Riemannian intrinsic geometry. The theory of general relativity can be generated as well by the field theory approach. The action that yields the Einstein field equation in vacuum is Einstein-Hilbert action, S_H

$$S_H = \int \sqrt{-g} R d^4x \tag{3.84}$$

For $\sqrt{-g} = \sqrt{-det(g_{\mu\nu})}$, Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ and $R_{\mu\nu}$ is the Ricci curvature tensor. To obtain the gravitational field equation, applying least action principle by varying S_H with respect to the metrical field $g_{\mu\nu}$ and require that $\delta S_H = 0$.

$$\delta S_H = (\delta S)_1 + (\delta S)_2 + (\delta s)_3 \tag{3.85}$$

where

$$(\delta S)_1 = \int d^4x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \tag{3.86}$$

$$(\delta S)_2 = \int d^4x \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} \tag{3.87}$$

$$(\delta S)_3 = \int d^4x R \delta \sqrt{-g} \tag{3.88}$$

Focus on $(\delta S)_3$ first. One of the relation from linear algebra tells that

$$ln(detW) = Tr(lnW) \tag{3.89}$$

For W is any arbitrary matrix and Tr refers to the trace of a matrix. Then,

$$\delta ln(detW) = \delta Tr(lnW)$$

$$\delta(detW) = detW \cdot Tr(W^{-1}\delta W)$$

Let W be the metric $g_{\mu\nu}$. With the relation above, it follows that

$$\delta g = g(g^{\mu\nu}\delta g_{\mu\nu}) \tag{3.90}$$
$$= -g(g_{\mu\nu}\delta g^{\mu\nu}) \tag{3.91}$$

The relation $\delta(g^{\mu\nu}g_{\mu\nu}) = 0$ is used for the second equality line. Therefore,

$$\delta\sqrt{-g} = -\frac{1}{2}\frac{1}{\sqrt{-g}}\delta g \tag{3.92}$$
$$= \frac{g}{2\sqrt{-g}} g_{\mu\nu}\delta g^{\mu\nu} \tag{3.93}$$
$$= -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu} \tag{3.94}$$

$(\delta S)_3$ in total is

$$(\delta S)_3 = -\int d^4x\sqrt{-g}\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu} \tag{3.95}$$

$(\delta S)_1$ can be shown to be zero. From the definition of Ricci tensor, it can be derived easily that

$$\delta R_{\mu\nu} = \nabla_\lambda(\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu}) \tag{3.96}$$

For Γ is the Levi-Civita connection and $\nabla\Gamma$ is the covariant derivative of the Levi-Civita connection. Using this relation, $(\delta S)_1$ can be shown to be

$$\begin{aligned}(\delta S)_1 &= \int (g^{\mu\nu}\nabla_\lambda(\delta\Gamma^\lambda_{\nu\mu}) - g^{\mu\nu}\nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu}))\sqrt{-g}d^4x &(3.97)\\ &= \int (\nabla_\lambda(g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu})) - \nabla_\nu(g^{\mu\nu}\nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu})))\sqrt{-g}d^4x &(3.98)\\ &= \int \partial_\lambda(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu}) - \partial_\nu(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\mu})d^4x &(3.99)\\ &= 0 &(3.100)\end{aligned}$$

The second equality line arises due to the metric compatibility condition $\nabla_\mu g^{\mu\nu} = 0$. The third equality line arises by using the relation $\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}V^\mu)$. The last equality arises because the total derivative term can be converted to the surface term, since Levi-Civita connection(3.108) is expressed as a function of $\partial g_{\mu\nu}$ and $\delta g_{\mu\nu}$ and $\nabla\delta g_{\mu\nu}$ are assumed to be zero at the boundary, the boundary term can be neglected.

In total, $\delta S_H = (\delta S)_2 + (\delta S)_3$

$$\delta S_H = \int d^4x\sqrt{-g}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu})\delta g^{\mu\nu} \tag{3.101}$$

Imposing $\delta S_H = 0$, Einstein field equation in vacuum is obtained

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = 0} \tag{3.102}$$

3.2.2 Coupling of the gravitational field $g_{\mu\nu}$ with matter field

The left hand side of (3.102) is a geometrical object. It is also called Einstein tensor $G_{\mu\nu}$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} \tag{3.103}$$

One important element is still missing in the theory of gravity. It is the matter. In Newtonian gravity, matter with mass will generate gravitational field. This must also be the case in Einsteinian gravity. In weak field limit, Einstein's theory of gravity can be reduced to the classical Newtonian theory. Consider the static and weak gravitational field which satisfy the relations (3.104) and (3.105)

$$\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau} \tag{3.104}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, ; |h_{\mu\nu}| \ll |\eta_{\mu\nu}| \tag{3.105}$$

For τ is the proper time. (3.104) corresponds to the slow motion condition. Using the geodesic equation(3.106) now

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{3.106}$$

With the constraints (3.104) and (3.105), (3.106) can be approximated to

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}(\frac{dx^0}{d\tau})^2 = 0 \tag{3.107}$$

Considering only the spatial part by replacing μ to i in (3.107). Use the definition of Levi-Civita Connection

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{00}) \tag{3.108}$$

It then can be shown easily that $\Gamma^i_{00} = -\frac{1}{2}g^{i\rho}\partial_\rho g_{00}$ by considering the static condition that the metric is independent of timelike coordinate. Then, the geodesic equation becomes

$$\frac{d^2x^i}{d\tau^2} - \frac{1}{2}\partial^i h_{00} = 0 \tag{3.109}$$

Note that $g^{00} = -1 - h_{00}$. In the Newtonian limit, the geodesic equation takes the form (3.109) by neglecting all the second and higher order term of $h_{\mu\nu}$. Compare (3.109) with the equation of motion in Newtonian theory of gravity

$$\frac{d^2x^i}{dt^2} + \partial^i\phi = 0 \tag{3.110}$$

For ϕ here is the gravitational potential. So,

$$h_{00} = -2\phi \tag{3.111}$$

$$g_{00} = -1 - 2\phi \tag{3.112}$$

In Newtonian gravity, the gravitational field equation is represented by the Poisson equation

$$\nabla^2\phi = 4\pi G\rho \tag{3.113}$$

For G here is the Newton gravitational constant and ρ is the mass density. Using the results obtained before, the Poisson equation can be written into

$$-\nabla^2 g_{00} = 8\pi G T_{00} \tag{3.114}$$

In the classical limit, the gravitational field equation manifests itself in the form of (3.114). It is now known that the left hand side of (3.114) which involves second order derivative of metric tensor is represented by the Einstein tensor. Therefore, the natural reason to couple the energy-momentum tensor $T_{\mu\nu}$ of matter with the geometrical object $G_{\mu\nu}$ is shown here by just simply taking the classical limit of the theory.

In the field theory approach, the Einstein-Hilbert action can be coupled with another action, S_M for which its Lagrangian is a function of the matter field Φ ,

$$S = \frac{1}{16\pi G}S_H + S_M(\Phi, g_{\mu\nu}) \tag{3.115}$$

Applying least action principle again with respect to $g_{\mu\nu}$,

$$\frac{1}{16\pi G}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 0 \tag{3.116}$$

Define the energy-momentum tensor to be

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \tag{3.117}$$

Therefore, the complete Einstein field equation is obtained

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}} \tag{3.118}$$

3.2.3 General Relativity as a General Diffeomorphism Covariant theory

The structure of symmetry of general relativity is general diffeomorphism. This means that the mathematical expression of the theory is invariant with respect to the general diffeomorphism. There is no prior coordinate system which is superior than other. All coordinate system is equivalent and one will always obtain the same physics of gravity no matter what preference of coordinate system is used.

Consider an arbitrary diffeomorphism generated by a vector field V. The full action for general relativity is discussed in the last section which takes the form

$$S = \frac{1}{16\pi G}S_H(g_{\mu\nu}) + S_M(g_{\mu\nu}, \Phi) \tag{3.119}$$

The variation of S under a diffeomorphism is

$$\delta S = \int d^4x \frac{1}{16\pi G}\frac{\delta S_H}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \int d^4x \frac{\delta S_M}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \int d^4x \frac{\delta S_M}{\delta \Phi}\delta\Phi \tag{3.120}$$

$\delta g_{\mu\nu}$ is not arbitrary but it only refers to those induced by diffeomorphism here, $\delta g_{\mu\nu} = \mathcal{L}_V g_{\mu\nu}$. Focus on the variation of Hilbert action term first,

$$\int d^4x \frac{\delta S_H}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = \int d^4x \frac{\delta S_H}{\delta g_{\mu\nu}}(\nabla_\mu V_\nu + \nabla_\nu V_\mu) \tag{3.121}$$

$$= 2\int d^4x \sqrt{-g}(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)\nabla_\mu V_\nu \tag{3.122}$$

$$= -2\int d^4x \sqrt{-g}\nabla_\mu(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)V_\nu \tag{3.123}$$

$$= 0 \tag{3.124}$$

The third equality arises because $\nabla_\mu\sqrt{-g} = 0$ for metric compatible connection. The last equality line arises because the Einstein tensor $G_{\mu\nu}$ obeys the Bianchi identity

$$\nabla_\mu G^{\mu\nu} = \nabla_\mu(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R) = 0 \tag{3.125}$$

The last term in (3.120) also vanishes because the form of matter field Φ which satisfies the matter equation of motion is taken. Then, there is only the second term in (3.120) left which is needed to be worried about.

$$\int d^4x \frac{\delta S_M}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = 2\int d^4x \frac{\delta S_M}{\delta g_{\mu\nu}}\nabla_\mu V_\nu \tag{3.126}$$

$$= -2\int d^4x \sqrt{-g}V_\nu \nabla_\mu(\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{\mu\nu}}) \tag{3.127}$$

$$= -2\int d^4x \sqrt{-g}V_\nu(\nabla_\mu T^{\mu\nu}) \tag{3.128}$$

So, if the general relativity possesses general covariance, energy-momentum has to be conserved

$$\nabla_\mu T^{\mu\nu} = 0 \tag{3.129}$$

The conservation of energy-momentum is one of the deepest consequence of general diffeomorphism covariance.

3.2.4 Schwarzschild's Solution

In 1915, German physicist Karl Schwarzschild provided the first exact solution to the Einstein field equation. He achieved this while serving in German army during world war 1. Schwarzschild's solution is the solution when a spherically symmetric and static spacetime is considered. Furthermore, this solution is also concerned with the situation of the empty space surrounding a spherical body. In this section, we will review Schwarzschild's metric.

We work in the spherical coordinate system $x^\mu = (t, r, \theta, \phi)$. Consider a spherically symmetric and static spacetime, the metric becomes

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu && (3.130)\\ &= g_{00}(r,t)dt^2 + 2g_{0r}(r,t)drdt + g_{rr}(r,t)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) && (3.131)\\ &= g_{00}(r)dt^2 + 2g_{0r}(r)drdt + g_{rr}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) && (3.132)\\ &= g_{00}(r)dt^2 + g_{rr}(r)dr^2 + r^2 d\Omega^2 && (3.133) \end{aligned}$$

The second equality line arises due to the fact that a spherically symmetric coordinate system is used. In a rotationally symmetric coordinate system, the terms like $drd\theta$ and $d\theta d\phi$ vanish. The third equality line arises because we consider the spacetime to be stationary. Beside a stationary spacetime, the static spacetime that we imposed also require the condition that the metric is invariant under time reversal. This explains why $drdt$ term is dropped in the forth equality line. Note that $d\Omega^2 = (d\theta^2 + \sin^2\theta d\phi^2)$ in the forth equality line. Therefore, the metric is simplified to

$$ds^2 = g_{00}(r)dt^2 + g_{rr}(r)dr^2 + r^2 d\Omega^2$$

Nothing stops us from writing the metric as

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\Omega^2 \tag{3.134}$$

The scheme that we will use to find Schwarzschild's metric is the following. We will substitute the metric in the expression of (3.134) into Levi-Civita connection Γ. Then, we use Levi-Civita connection to calculate Riemann tensor $R^\alpha_{\beta\mu\nu}$. With Riemann tensor known, we deduce Ricci tensor $R_{\mu\nu}$. Since we are interested in the solution outside a spherical body, we only care about Einstein field equation in vacuum $G_{\mu\nu} = 0$. Equivalently, we want to solve the equations $R_{\mu\nu} = 0$ to obtain the expression of g_{00} and g_{rr}

$$g \to \Gamma \to R^\alpha_{\beta\mu\nu} \to R_{\mu\nu} = 0 \to g_{00}, g_{rr}$$

In the matrix notation, the metric $g_{\mu\nu}$ is

$$g_{\mu\nu} = \begin{bmatrix} -e^{2\nu} & 0 & 0 & 0 \\ 0 & e^{2\lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{bmatrix} \tag{3.135}$$

The inverse of the metric $g^{\mu\nu}$ is

$$g^{\mu\nu} = \begin{bmatrix} -e^{-2\nu} & 0 & 0 & 0 \\ 0 & e^{-2\lambda} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2} sin^{-2}\theta \end{bmatrix} \tag{3.136}$$

Recall that Levi-Civita connection takes the expression as

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu})$$

The independent non-vanishing connection terms are calculated using this expression. For example,

$$\Gamma^1_{00} = \frac{1}{2}g^{1\beta}(-\partial_\beta g_{00}) \tag{3.137}$$
$$= -\frac{1}{2}g^{11}\partial_r g_{00} \tag{3.138}$$
$$= -\frac{1}{2}e^{-2\lambda}\partial_r(-e^{2\nu(r)}) \tag{3.139}$$
$$= e^{2(\nu-\lambda)}(\partial_r\nu) \tag{3.140}$$

The other non-trivial independent connection terms are [3]

$$\Gamma^1_{11} = \partial_r\lambda \tag{3.141}$$
$$\Gamma^0_{10} = \partial_r\nu \tag{3.142}$$
$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r} \tag{3.143}$$
$$\Gamma^1_{22} = -re^{-2\lambda} \tag{3.144}$$
$$\Gamma^3_{23} = \cot\theta \tag{3.145}$$
$$\Gamma^1_{33} = -r\sin^2\theta e^{-2\lambda} \tag{3.146}$$
$$\Gamma^2_{33} = -\sin\theta\cos\theta \tag{3.147}$$

Note that Levi-Civita connections are symmetric under the exchange of the lower 2 indices. The other terms that are not listed are all zero.

Then, we want to calculate Riemann tensor. Recall that Riemann tensor is

$$R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

We will show one example how to calculate Riemann tensor.

$$R^0_{101} = \partial_0\Gamma^0_{11} - \partial_r\Gamma^0_{01} + \Gamma^0_{0\lambda}\Gamma^\lambda_{11} - \Gamma^0_{1\lambda}\Gamma^\lambda_{01} \tag{3.148}$$
$$= -\partial_r(\partial_r\nu) + (\partial_r\nu)(\partial_r\lambda) - (\partial_r\nu)(\partial_r\nu) \tag{3.149}$$
$$= (\partial_r\nu)(\partial_r\lambda) - \partial_r^2\nu - (\partial_r\nu)^2 \tag{3.150}$$

The other non-vanishing independent Riemann tensor terms are [3]

$$R^0_{202} = -re^{-2\lambda}\partial_r\nu \tag{3.151}$$
$$R^0_{303} = -re^{-2\lambda}\sin^2\theta\partial_r\nu \tag{3.152}$$
$$R^1_{212} = re^{-2\lambda}\partial_r\lambda \tag{3.153}$$
$$R^1_{313} = re^{-2\beta}\sin^2\theta\partial_r\lambda \tag{3.154}$$
$$R^2_{323} = (1-e^{-2\lambda})\sin^2\theta \tag{3.155}$$

The other terms that are not listed are either zero or they can be deduced from these independent terms. For example, one of the term that can be deduced from the independent terms is

$$R^1_{010} = g^{1\alpha}R_{\alpha010} \tag{3.156}$$
$$= g^{11}R_{1010} \tag{3.157}$$
$$= g^{11}R_{0101} \tag{3.158}$$
$$= g^{11}g_{00}R^0_{101} \tag{3.159}$$
$$= e^{-2\lambda}(-e^{2\nu})(\partial_r\nu\partial_r\lambda - \partial_r^2\nu - (\partial_r\nu)^2) \tag{3.160}$$

The third equality arises by using the relation $R_{\mu\nu\alpha\beta} = R_{\nu\mu\beta\alpha}$. After we know the Riemann tensor, we want to calculate Ricci tensor

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$$

The non-vanishing independent Ricci tensor terms are [3]

$$R_{00} = e^{2(\nu-\lambda)}[\partial_r^2\nu + (\partial_r\nu)^2 - (\partial_r\nu)(\partial_r\lambda) + \frac{2}{r}\partial_r\nu] \tag{3.161}$$

$$R_{11} = -\partial_r^2\nu - (\partial_r\nu)^2 + (\partial_r\nu)(\partial_r\lambda) + \frac{2}{r}\partial_r\lambda \tag{3.162}$$

$$R_{22} = e^{-2\lambda}[r(\partial_r\lambda - \partial_r\nu) - 1] + 1 \tag{3.163}$$

$$R_{33} = \sin^2\theta R_{22} \tag{3.164}$$

With Ricci tensor known, we want to solve the field equation $R_{\mu\nu} = 0$. We note that

$$R_{00}e^{2(\lambda-\nu)} + R_{11} = 0 \tag{3.165}$$

This implies that

$$\frac{2}{r}(\partial_r\nu + \partial_r\lambda) = 0 \implies \partial_r(\lambda + \nu) = 0 \tag{3.166}$$

So, we can deduce that

$$\lambda = -\nu + a \tag{3.167}$$

where a is a constant. The metric becomes

$$ds^2 = -e^{-2\lambda}e^{2a}dt^2 + e^{2\lambda}dr^2 + r^2d\Omega^2 \tag{3.168}$$

We can absorb e^{2a} into dt^2 by defining $dt'^2 = e^{2a}dt^2$ and again relabel t' as t. So, the metric is

$$ds^2 = -e^{-2\lambda}dt^2 + e^{2\lambda}dr^2 + r^2d\Omega^2 \tag{3.169}$$

We also solve for

$$R_{22} = 0 \tag{3.170}$$

We will obtain

$$e^{2\nu}(-2r\partial_r\nu - 1) + 1 = 0 \tag{3.171}$$

$$e^{2\nu}(2r\partial_r\nu + 1) = 1 \tag{3.172}$$

$$e^{2\nu(r)}[2r\frac{d}{dr}\nu(r) + 1] = 1 \tag{3.173}$$

$$\frac{d}{dr}(re^{2\nu}) = 1 \tag{3.174}$$

This implies that

$$e^{2\nu} = 1 + \frac{b}{r} \tag{3.175}$$

where b is another constant. Then, the metric becomes

$$ds^2 = -(1 + \frac{b}{r})dt^2 + \frac{1}{1 + \frac{b}{r}}dr^2 + r^2d\Omega^2 \tag{3.176}$$

When $r \to \infty$, we should have the weak field limit. In the last section, we have seen that the component of metric g_{00} in the weak field limit is $g_{00} = -(1 + 2\phi)$. In Newtonian gravity, the gravitational potential is $\phi(r) = -\frac{GM}{r}$. So, by taking the weak field limit, we can deduce that

$$b = -2GM \tag{3.177}$$

Eventually, we obtain the Schwarzschild's metric

$$\boxed{-(1 - \frac{2GM}{r})dt^2 + \frac{1}{(1 - \frac{2GM}{r})}dr^2 + r^2 d\Omega^2} \tag{3.178}$$

To understand more physics of Schwarzschild's metric, see [3], [6], [14].

4 The Geometric Interpretation of field theories

The geometrical origin of the field theories can be understood from the viewpoint of background dependent extrinsic geometry if the Lagrangian density is in the form of the square root of the determinant of a induced metric. For example, we have seen that in Born-Infeld theory, the Lagrangian density is $\sqrt{-det(\eta_{\mu\nu} + F_{\mu\nu})}$ if we fix the background spacetime to be Minkowski spacetime. Then we can just interpret the induced metric to be $\eta_{\mu\nu} + F_{\mu\nu}$ and see if we can come out with some geometrical notions of this field theory with the help of the understanding of the concepts in the intrinsic geometric interpretation (background independent) of general relativity that we know. Besides, string theory is also known to be a background- dependent theory by invoking the notion of branes. In the study of string theory, the notion of the point particle has been generalized to higher dimensional object called p-branes. p refers to the spatial dimension. A point particle is a 0 dimensional brane, a string is a 1 dimensional brane, a membrane is a 2 dimensional brane and so on. We will see in this chapter that how the geometrical origin of some field theories that their Lagrangian density can be formulated in the form of the square root of the determinant of the induced metric comes from the interpretation that the branes are embedding within a background space. Therefore, it is necessary to get ourselves familiar with the properties of branes. In this chapter, we first study the physics of branes following the old routine by first studying the simplest case (point particle) and then generalize it to arbitrary dimension. Then, we make use of the properties of the brane to understand the geometrical origin of two scalar field theories. Finally, we see what we can learn and do for the geometrization of Born-Infeld theory.

4.1 The action for brane and its diffeomorphism invariance as gauge symmetry

4.1.1 Point particle as 0-brane

The equation of motion of a relativistic point particle moving through a D-dimensional spacetime is given by the geodesic equation. Its corresponding action S_0 is

$$S_0 = -\alpha \int ds \tag{4.1}$$

α is a constant which has the dimension of Length^{-1}. ds is the infinitesimal path taken by the point particle. The path taken is invariant under spacetime coordinate transformation

$$ds^2 = -g_{\mu\nu}(x)dx^\mu dx^\nu \tag{4.2}$$

The metric $g_{\mu\nu}$ describes the geometry of the background spacetime in which the brane is propagating. In Minkowski flat background spacetime, the action S_0 becomes

$$S_0 = -m \int \sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu} \tag{4.3}$$

$$= -m \int \sqrt{-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2} \tag{4.4}$$

The path taken x^μ of the point particle is parametrized by a real parameter τ which is usually taken to be the proper time. $x^\mu(\tau)$ is called the worldline of the particle. In general, under the parametrization, ds^2 can be written into

$$ds^2 = -g_{\mu\nu}(x)\frac{dx^\mu(\tau)}{d\tau}\frac{dx^\nu(\tau)}{d\tau}d\tau^2 \tag{4.5}$$

Then, the action becomes

$$S_0 = -m\int d\tau\sqrt{-g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu} \tag{4.6}$$

where $\dot{x}^\mu = \frac{dx^\mu(\tau)}{d\tau}$. This action indicates the length of the worldline in the spacetime diagram. See figure 6.

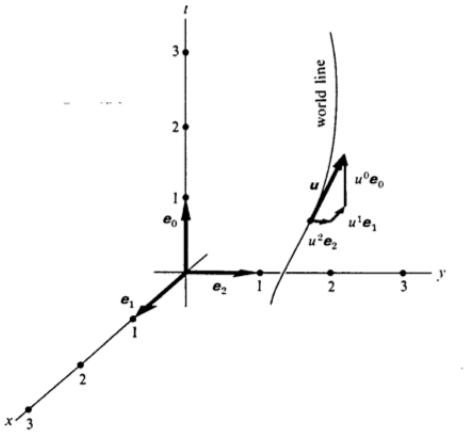

Figure 6: Worldline diagram; u denotes the tangent 4-velocity [14].

Action is invariant under the reparametrization or diffeomorphism of worldline coordinate. This means that the action remains unchanged if replacing the parameter $\tau \to \tau' = f(\tau)$. Consequently, $d\tau \to d\tau' = \frac{\partial f}{\partial \tau}d\tau$. Under this reparametrization, the scalar field $x^\mu(\tau)$ transforms to $x'^\mu(\tau')$ such that

$$x'^\mu(\tau') = x^\mu(\tau) \tag{4.7}$$

and

$$\frac{dx'^\mu(\tau')}{d\tau} = \frac{dx'^\mu(\tau')}{d\tau'}\frac{\partial f(\tau)}{\partial \tau} \tag{4.8}$$

The change of the action S_0' under reparametrization is

$$S_0' = -m\int d\tau'\sqrt{-g'_{\mu\nu}(x')\frac{dx'^\mu(\tau')}{d\tau'}\frac{dx'^\nu(\tau')}{d\tau'}} \tag{4.9}$$

$$= -m\int d\tau'\sqrt{-\frac{dx'^\mu(\tau')}{d\tau'}\frac{dx'_\mu(\tau')}{d\tau'}} \tag{4.10}$$

$$= -m\int \frac{\partial f}{\partial \tau}d\tau\sqrt{-\frac{dx^\mu(\tau)}{d\tau}\frac{dx_\mu(\tau)}{d\tau}(\frac{\partial f}{\partial \tau})^{-2}} \tag{4.11}$$

$$= -m\int d\tau\sqrt{-\frac{dx^\mu(\tau)}{d\tau}\frac{dx_\mu(\tau)}{d\tau}} \tag{4.12}$$

$$= -m\int d\tau\sqrt{-g_{\mu\nu}(x)\frac{dx^\mu(\tau)}{d\tau}\frac{dx^\nu(\tau)}{d\tau}} \tag{4.13}$$

$$= S_0 \tag{4.14}$$

Therefore, one has complete freedom to parametrize the path x^μ.

Since the action S_0 in the form (4.6) contains the Lagrangian density which is a non-linear function as the square root function is involved, it is very hard to extract information (equation of motion, the symmetric property, etc) from the action taking the form (4.6). So, it is easier for us to have an equivalent action which takes the simpler form by invoking the auxiliary field $e(\tau)$.

Consider the equivalent action $\tilde{S}_0$ given by

$$\tilde{S}_0 = \frac{1}{2}\int d\tau (e(\tau)^{-1}\dot{x}^2 - m^2 e(\tau)) \tag{4.15}$$

where $\dot{x}^2 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$. To see that (4.15) is equivalent to (4.6), we have to solve the equation of motion for auxiliary field first. The variation of $\tilde{S}_0$ with respect to e is

$$\delta\tilde{S}_0 = \frac{1}{2}\int d\tau (-\frac{1}{e^2}\dot{x}^2\delta e - m^2\delta e) \tag{4.16}$$

$$= \frac{1}{2}\int d\tau \frac{\delta e}{e^2}(-\dot{x}^2 - m^2 e^2) \tag{4.17}$$

Setting $\delta\tilde{S}_0 = 0$, the auxiliary field equation of motion is obtained and the equation of motion can be solved directly which gives rise to

$$e^2 = \frac{-\dot{x}^2}{m^2} \Rightarrow e = \sqrt{\frac{-\dot{x}^2}{m^2}} \tag{4.18}$$

We then substitute the expression of e in (4.18) into $\tilde{S}_0$

$$\tilde{S}_0 = \frac{1}{2}\int d\tau ((-\frac{\dot{x}^2}{m^2})^{-\frac{1}{2}}\dot{x}^2 - m^2(-\frac{\dot{x}^2}{m^2})^{-\frac{1}{2}}) \tag{4.19}$$

$$= -m\int d\tau (-\dot{x}^2)^{\frac{1}{2}} \tag{4.20}$$

$$= -m\int d\tau \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \tag{4.21}$$

$$= S_0 \tag{4.22}$$

So, $\tilde{S}_0$ is indeed equivalent to S_0 if the auxiliary field equation holds for $e(\tau)$.

It is also essential to show that $\tilde{S}_0$ has reparametrization symmetry. The reparametrization symmetry here refers to the changing of coordinate system of the worldline coordinate τ. The world volume coordinate does not bear any physical meaning. It is just used to parametrize the brane embedded in a target spacetime. So, the reparametrization symmetry is actually a gauge symmetry. It means the redundancy of the description of the same physical state. To see that $\tilde{S}_0$ is invariant under reparametrization of τ, we first have to see how the field $X^\mu(\tau)$ and $e(\tau)$ vary under an infinitesimal change of parametrization $\tau \to \tau' = \tau - \epsilon(\tau)$, for ϵ is infinitesimal.

$$x'^\mu(\tau') = x'^\mu(\tau - \epsilon(\tau)) = x^\mu(\tau) \tag{4.23}$$

$$= x'^\mu(\tau) - \epsilon(\tau)\frac{d}{d\tau}x'^\mu(\tau) = x^\mu(\tau) \tag{4.24}$$

$$= x'^\mu(\tau) - \epsilon(\tau)\frac{d}{d\tau}[x^\mu(\tau+\epsilon)] = x^\mu(\tau) \tag{4.25}$$

$$= x'^\mu(\tau) - \epsilon(\tau)\frac{d}{d\tau}[x^\mu(\tau)] = x^\mu(\tau) \tag{4.26}$$

The forth equality line arises because all the second and higher order term of ϵ is neglected after expanding $\epsilon x^\mu(\tau+\epsilon)$. So, the variation of x^μ, δx^μ is

$$\delta x^\mu = x'^\mu(\tau) - x^\mu(\tau) \tag{4.27}$$

$$= \epsilon(\tau)\dot{x}^\mu \tag{4.28}$$

The auxiliary field $e(\tau)$ transforms under infinitesimal reparametrization as

$$e'(\tau')d\tau' = e'(\tau-\epsilon)(d\tau - \dot{\epsilon}d\tau) = e(\tau)d\tau \tag{4.29}$$
$$= [e'(\tau) - \epsilon\partial_\tau e'(\tau)](d\tau - \dot{\epsilon}d\tau) \tag{4.30}$$
$$= [e'(\tau) - \epsilon\partial_\tau e(\tau+\epsilon)](d\tau - \dot{\epsilon}d\tau) \tag{4.31}$$
$$= [e'(\tau) - \epsilon\partial_\tau e(\tau)](d\tau - \dot{\epsilon}d\tau) \tag{4.32}$$
$$= e'(\tau)d\tau - \dot{\epsilon}e'(\tau)d\tau - \epsilon\partial_\tau e(\tau)d\tau \tag{4.33}$$
$$= e'(\tau)d\tau - \dot{\epsilon}e(\tau+\epsilon)d\tau - \epsilon\partial_\tau e(\tau)d\tau \tag{4.34}$$
$$= e'(\tau)d\tau - \dot{\epsilon}e(\tau)d\tau - \epsilon\partial_\tau e(\tau)d\tau \tag{4.35}$$
$$= e'(\tau)d\tau - \frac{d}{d\tau}(\epsilon\cdot e)d\tau = e(\tau)d\tau \tag{4.36}$$

Note that all second and higher order terms of ϵ is neglected after all expansions in the derivation. (4.36) leads to the expression of the variation of e field under infinitesimal reparametrization is

$$\delta e(\tau) = e'(\tau) - e(\tau) \tag{4.37}$$
$$= \frac{d}{d\tau}(\sigma\cdot e) \tag{4.38}$$

The total variation of $\tilde{S_0}$ with respect to both e field and x^μ field is

$$\delta\tilde{S_0} = \frac{1}{2}\int d\tau(-\frac{\delta e}{e^2}\dot{x}^2 + \frac{2}{e}\dot{x}\delta\dot{x} - m^2\delta e) \tag{4.39}$$

Also note that

$$\delta\dot{x}_\mu = \frac{d}{d\tau}\delta x_\mu = \frac{d}{d\tau}(\epsilon\dot{x}_\mu) \tag{4.40}$$
$$= \dot{\epsilon}\dot{x}_\mu + \epsilon\ddot{x}_\mu \tag{4.41}$$

Substitute (4.41), (4.38) and (4.28) into (4.39)

$$\delta\tilde{S_0} = \frac{1}{2}\int d\tau[\frac{2\dot{x}^\mu}{e}(\dot{\epsilon}\dot{x}_\mu + \epsilon\ddot{x}_\mu) - \frac{\dot{x}^2}{e^2}(\dot{\epsilon}e + e\dot{\epsilon}) - m^2\frac{d(\epsilon e)}{d\tau}] \tag{4.42}$$

The last term can be neglected because it is a total derivative term. So, it becomes

$$\delta\tilde{S_0} = \frac{1}{2}\int d\tau[\frac{2\dot{x}^\mu}{e}(\dot{\epsilon}\dot{x}_\mu + \epsilon\ddot{x}_\mu) - \frac{\dot{x}^2}{e^2}(\dot{\epsilon}e + e\dot{\epsilon})] \tag{4.43}$$
$$= \frac{1}{2}\int d\tau(\dot{\epsilon}e^{-1}\dot{x}^2 - \epsilon e^{-2}\dot{x}^2\dot{e} + 2\dot{x}\ddot{x}\epsilon e^{-1}) \tag{4.44}$$
$$= \frac{1}{2}\int d\tau\frac{d}{d\tau}[\epsilon e^{-1}\dot{x}^2] \tag{4.45}$$
$$= 0 \tag{4.46}$$

Again a total derivative term is obtained and it is dropped as well. Therefore, it is shown explicitly that $\tilde{S_0}$ possesses reparametrization symmetry.

4.1.2 Equation of motion of 0-brane

Due to the reparametrization freedom of the 0-brane action, we can break the gauge freedom by choosing the parameter τ such that the auxiliary field $e(\tau) = 1$. Then, the action $\tilde{S_0}$ becomes

$$\tilde{S_0} = \frac{1}{2}\int d\tau(g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu - m^2) \tag{4.47}$$

Varying $\tilde{S}_0$ with respect to $x^\mu(\tau)$ gives

$$\delta\tilde{S}_0 = \frac{1}{2}\int d\tau(2g_{\mu\nu}(x)\delta\dot{x}^\mu\dot{x}^\nu + \partial_k g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu\delta x^k) \tag{4.48}$$

$$= \frac{1}{2}\int d\tau(-\frac{\partial}{\partial\tau}(2g_{\mu\nu}(x)\dot{x}^\nu)\delta x^\mu + \partial_k g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu\delta x^k) \tag{4.49}$$

$$= \frac{1}{2}\int d\tau(-2\dot{x}^k\partial_k g_{\mu\nu}(x)\delta x^\mu\dot{x}^\nu - 2g_{\mu\nu}\ddot{x}^\nu\delta x^\mu + \delta x^k\partial_k g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu) \tag{4.50}$$

$$= \frac{1}{2}\int d\tau(-2\ddot{x}^\nu g_{\mu\nu}(x) - 2\delta_k g_{\mu\nu}(x)\dot{x}^k\dot{x}^\nu + \partial_\mu g_{k\nu}(x)\dot{x}^k\dot{x}^\nu)\delta x^\mu \tag{4.51}$$

The second equality line can be obtained via integration by part and neglecting the boundary term. The forth equality line arises by renaming the dummy index.

The field equation for $x^\mu(\tau)$ is obtained by setting $\delta\tilde{S}_0 = 0$. The field equation is

$$-2\ddot{x}^\nu g_{\mu\nu}(x) - 2\delta_k g_{\mu\nu}(x)\dot{x}^k\dot{x}^\nu + \partial_\mu g_{k\nu}(x)\dot{x}^k\dot{x}^\nu = 0 \tag{4.52}$$

This can be written as

$$\ddot{x}^\mu + \Gamma^\mu_{k\lambda}\dot{x^k}\dot{x}^\lambda = 0 \tag{4.53}$$

The equation of motion of 0-brane is recognized as the geodesic equation describing the shortest path that the particle travels in the embedding manifold.

4.1.3 Generalization to p-brane

After we are getting ourselves familiar with 0-brane, we ought to extend the whole concept of an action for a point particle (0-brane) to an action for a p-brane. The extension of $S_0 = -m\int ds$ to a p-brane in a $D(\geq p)$ dimensional background spacetime is

$$S_p = -T_p\int d\mu_p \tag{4.54}$$

where T_p is the p-brane tension and it has the unit of mass/ volume, $d\mu_p$ is the $(p+1)$ dimensional volume element given by

$$d\mu_p = \sqrt{-det[G_{\alpha\beta}(x)]}d^{p+1}\sigma \tag{4.55}$$

$G_{\alpha\beta}$ is the induced metric of the worldvolume σ^α which is the pullback of the ambient metric $g_{\mu\nu}(x)$ of the background spacetime. $G_{\alpha\beta}$ is given by

$$G_{\alpha\beta}(x) = \frac{\partial x^\mu}{\partial\sigma^\alpha}\frac{\partial x^\nu}{\partial\sigma^\beta}g_{\mu\nu}(x) \tag{4.56}$$

such that $\alpha, \beta = 0, 1, .., p$. $\sigma^0 = \tau$ while $\sigma^1, \sigma^2, ..., \sigma^p$ are the p spacelike coordinates for the $p+1$ worldvolume embedding in the background spacetime. The role of the induced metric $G_{\alpha\beta}$ is to measure distances on the worldvolume while the metric $g_{\mu\nu}(x)$ plays the role of measuring distances on the background spacetime.

4.2 Bosonic string theory as 1-brane theory

To see how the p-brane works, it is enlightening to see the case of 1-brane which is the simplest non trivial case going beyond the 0-brane(point particle). 1-brane is actually a bosonic string and its action describes the propagation of this string in D dimensional background spacetime. The worldsheet of the string is parametrized by 2 coordinates $\sigma^0 = \tau$ and $\sigma^1 = \sigma$ with τ being timelike and σ being spacelike which is the extension for worldline for point particle. The background spacetime $x^\mu(\tau, \sigma)$ which is parametrized by the worldsheet coordinates can now be viewed as a scalar field. It tells how the string propagates and oscillates through the background spacetime.

4.2.1 The string action

Using (4.54), (4.55) and (4.56), we want to construct the string action. We assume that the background spacetime is Minkowski flat spacetime. Then, the metric $g_{\mu\nu}(x)$ becomes $\eta_{\mu\nu}$. The induced metric $G_{\alpha\beta}$ is then given by

$$G_{00} = \frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\nu}{\partial \tau}\eta_{\mu\nu} = \dot{x}^2 \tag{4.57}$$

$$G_{11} = \frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^\nu}{\partial \sigma}\eta_{\mu\nu} = x'^2 \tag{4.58}$$

$$G_{10} = G_{01} = \frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\nu}{\partial \sigma}\eta_{\mu\nu} = \dot{x}x' \tag{4.59}$$

In total,

$$G_{\alpha\beta} = \begin{bmatrix} \dot{x}^2 & \dot{x}x' \\ \dot{x}x' & x'^2 \end{bmatrix} \tag{4.60}$$

The determinant of $G_{\alpha\beta}$ is then given by

$$det(G_{\alpha\beta}) = \dot{x}^2 x'^2 - (\dot{x}x')^2 \tag{4.61}$$

From (4.55), the string action becomes

$$S_{NG} = -T\int d\tau d\sigma \sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2} \tag{4.62}$$

This action is called Nambu Goto action and it can be interpreted as describing the area of the worldsheet mapped out by the string in spacetime. Since the equations of motion is obtained by minimizing the action, the equations of motion for the string can be thought as the smallest area mapped out by the string in the background spacetime [25].

The Nambu Goto action is in the square root form. We can imitate the story of 0-brane here by introducing an auxiliary field $h_{\alpha\beta}(\tau,\sigma)$ to make the action simpler. $h_{\alpha\beta}$ here is another metric living on the worldsheet and it differs from the induced metric $G_{\alpha\beta}$. The equivalent action is given by

$$S_\sigma = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h}h^{\alpha\beta}\frac{\partial x^\mu}{\partial \sigma^\alpha}\frac{\partial x^\nu}{\partial \sigma^\beta}g_{\mu\nu} \tag{4.63}$$

S_σ is called Polyakov action. h in the action refers to the determinant of $h_{\alpha\beta}$. To show the equivalence between S_σ and S_{NG}, we first have to note that varying S_σ with respect to the metrical field $h_{\alpha\beta}$ gives rise to energy momentum tensor as explained in section 3.2.2

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S_\sigma}{\delta h^{\alpha\beta}} \tag{4.64}$$

The equation of motion for the field $h^{\alpha\beta}$ is obtained by setting the variation in the action S_σ with respect to the $h^{\alpha\beta}$ to be zero, $\delta S_\sigma = 0$

$$\delta S_\sigma = \int \frac{\delta S_\sigma}{\delta h^{\alpha\beta}}\delta h^{\alpha\beta} \tag{4.65}$$

$$= -\frac{T}{2}\int d\tau d\sigma \sqrt{-h}\delta h^{\alpha\beta}T_{\alpha\beta} \tag{4.66}$$

$$= 0 \Leftrightarrow T_{\alpha\beta} = 0 \tag{4.67}$$

Using the same technique in the section 3.2.1, $\delta\sqrt{-h}$ is

$$\delta\sqrt{-h} = -\frac{\sqrt{-h}}{2}h_{\alpha\beta}\delta h^{\alpha\beta} \tag{4.68}$$

So, the variation of S_σ with respect to $h^{\alpha\beta}$, δS_σ is

$$\delta S_\sigma = -\frac{T}{2}\int d\tau d\sigma((\delta\sqrt{-h}h^{\lambda a}\partial_\lambda x\cdot\partial_a x) + (\sqrt{-h}\delta h^{\alpha\beta}\partial_\alpha x\cdot\partial_\beta x)) \tag{4.69}$$

$$= -\frac{T}{2}\int d\tau d\sigma\sqrt{-h}\delta h^{\alpha\beta}(-\frac{1}{2}h_{\alpha\beta}h^{\lambda a}\partial_\lambda x\cdot\partial_a x + \partial_\alpha x\cdot\partial_\beta x) \tag{4.70}$$

For $T_{\alpha\beta}$ is

$$T_{\alpha\beta} = -\frac{1}{2}h_{\alpha\beta}h^{\lambda a}\partial_\lambda x\cdot\partial_a x + \partial_\alpha x\cdot\partial_\beta x \tag{4.71}$$

The condition that $T_{\alpha\beta} = 0$ implies

$$\frac{1}{2}h_{\alpha\beta}h^{\lambda a}\partial_\lambda x\cdot\partial_a x = \partial_\alpha x\cdot\partial_\beta x = G_{\alpha\beta} \tag{4.72}$$

Taking the square root of minus of the determinant of the tensor with indices $\alpha\beta$ gives rise to

$$\frac{1}{2}\sqrt{-h}h^{\lambda a}\partial_\lambda x\cdot\partial_a x = \sqrt{-det(G_{\alpha\beta})} \tag{4.73}$$

Thus, S_σ is shown to be equivalent to S_{NG} when considering the equation of motion for $h^{\alpha\beta}$ of S_σ.

4.2.2 Symmetries of 1-brane

The Polyakov action (4.63) possesses one global Poincaré symmetry and 2 local symmetries which are reparametrization symmetry and Weyl symmetry. The invariance of a theory under global transformation gives rise to conserved current via Noether's theorem while the invariance of the theory under local transformations in this case is a sign of absent degree of freedom pointing to the gauge symmetry. Reparametrization symmetry has been shown explicitly for 0-brane in section 4.1.1. It is intuitively obvious to expect the extended action of p-brane should also have parametrization freedom because the propagating of p-brane in the background spacetime should not be affected by the way to parametrize the worldvolume. Therefore, we are left with Poincaré transformation and Weyl transformation.

Poincaré global transformation

Poincaré transformation is the translational extension of Lorentz transformation. The infinitesimal form of Poincaré transformation is written as

$$\delta x^\mu(\tau,\sigma) = a^\mu_\nu x^\nu(\tau,\sigma) + b^\mu \tag{4.74}$$

For Polyakov action, the transformation solely contributed by Poincaré transformation also requires that the field $h_{\alpha\beta}$ remains unchanged, $\delta h_{\alpha\beta}(\tau,\sigma) = 0$. The scalar field $x^\mu(\tau,\sigma)$ are defined on the worldsheet. b^μ comes from the translation. The coefficient a^μ_ν originates from Lorentz transformation, with both indices down, $a_{\mu\nu}$ is antisymmetric,

$$a_{\mu\nu} = -a_{\nu\mu} \tag{4.75}$$

To show that Polyakov action is invariant under Poincaré transformation, vary S_σ with respect to x^μ. Using (4.74), it gives

$$\delta S_\sigma = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h} h^{\alpha\beta}(\partial_\alpha(\delta x^\mu)\partial_\beta x^\nu + \partial_\alpha x^\nu \partial_\beta(\delta x^\mu))g_{\mu\nu} \tag{4.76}$$

$$= -T\int d\tau d\sigma \sqrt{-h} h^{\alpha\beta}\partial_\alpha(a^\mu_k x^k + b^\mu)\partial_\beta x^\nu g_{\mu\nu} \tag{4.77}$$

$$= -T\int d\tau d\sigma \sqrt{-h}(a_{\nu k})(h^{\alpha\beta}\partial_\alpha x^k \partial_\beta x^\nu) \tag{4.78}$$

$$= 0 \tag{4.79}$$

The last equality arises because $a_{\nu k}$ is antisymmetric while $h^{\alpha\beta}\partial_\alpha x^k \partial_\beta x^\nu$ is symmetric. Their contraction gives zero. So, the Polyakov action is indeed invariant under Poincaré transformation.

Weyl local transformation

Weyl transformations are transformations that change the scale of the metric $h_{\alpha\beta}$

$$h_{\alpha\beta}(\tau,\sigma) \to h'_{\alpha\beta}(\tau,\sigma) = e^{2\phi(\sigma)}h_{\alpha\beta}(\tau,\sigma) \tag{4.80}$$

while leaving $x^\mu(\tau,\sigma)$ unchanged. To show that Polyakov action is invariant under Weyl transformation, we first have to observe how $\sqrt{-h}$ transforms

$$\sqrt{-h'} = \sqrt{-det(h'_{\alpha\beta})} \tag{4.81}$$

$$= \sqrt{-e^{4\phi(\sigma)}det(h_{\alpha\beta})} \tag{4.82}$$

$$= e^{2\phi(\sigma)}\sqrt{-h} \tag{4.83}$$

Then, observe that $\sqrt{-h}h^{\alpha\beta}$ transforms as

$$\sqrt{h'}h'^{\alpha\beta} = \sqrt{-h}e^{2\phi(\sigma)}e^{-2\phi(\sigma)}h^{\alpha\beta} \tag{4.84}$$

$$= \sqrt{-h}h^{\alpha\beta} \tag{4.85}$$

Thus, Polyakov action is invariant under Weyl transformation.

Reparametrization symmetry

As mentioned before, reparametrization symmetry is a local symmetry for the worldsheet. Under the changing of the parameter, σ to $\sigma' = f(\sigma)$, the Polyakov action is invariant. The field x^μ and $h_{\alpha\beta}$ transforms under reparamnetrization as

$$x^\mu(\tau,\sigma) = x'^\mu(\tau,\sigma') \tag{4.86}$$

$$h'_{\alpha\beta}(\tau,\sigma') = \frac{\partial f^\gamma}{\partial\sigma^\alpha}\frac{\partial f^a}{\partial\sigma^\beta}h_{\gamma a}(\tau,\sigma') \tag{4.87}$$

Gauge fixing using Weyl symmetry and reparametrization symmetry to make the intrinsic metric $h_{\alpha\beta}$ flat.

Since the theory of 1 brane is invariant under reparametrization and Weyl transformations, a gauge fixing can be made such that the intrinsic metric $h_{\alpha\beta}$ becomes flat. The metric $h_{\alpha\beta}$ is symmetric and it has 3 independent components which are $h_{00}(\sigma), h_{11}(\sigma)$ and $h_{10}(\sigma) = h_{01}(\sigma)$. Since the theory has reparametrization freedom, use a set of parameters $\sigma' = f(\sigma)$ such that the metric $h_{\alpha\beta}$ is brought into the form $h(\sigma)\eta_{\alpha\beta}$, for $h(\sigma)$ is a scalar function. Then, one can use the property of Weyl symmetry to eliminate the function $h(\sigma)$, namely

$$e^{2\phi(\sigma)}h(\lambda) = 1 \tag{4.88}$$

Then, under gauge fixing, $h_{\alpha\beta} = \eta_{\alpha\beta}$. The combination of reparametrization and Weyl transformation is called conformal transformation. Since gauge symmetries are local symmetries, so the metric $h_{\alpha\beta}$ can only be brought into a flat metric locally. One can only bring $h_{\alpha\beta}$ into $\eta_{\alpha\beta}$ in the whole world sheet if the worldsheet is free of topological obstructions which means that the Euler characteristic is zero.

4.2.3 Field equation for the Polyakov Action and the Boundary Conditions

Suppose that the worldsheet topology allows gauge fixed flat intrinsic metric $h_{\alpha\beta} = \eta_{\alpha\beta}$ to be extended globally, the Polyakov action becomes

$$S_\sigma = \frac{T}{2}\int d\tau d\sigma(\dot{x}^2 - (x')^2) \tag{4.89}$$

To derive the equation of motion for the field x^μ, setting the variation of S_σ with respect to the field x^μ equal to zero. The variation δS_σ is

$$\begin{aligned}
\delta S_\sigma &= \frac{T}{2}\int d\tau d\sigma(2\dot{x}\delta\dot{x} - 2x'\delta x') && (4.90)\\
&= T\int d\tau d\sigma(\frac{\partial x^\mu}{\partial\tau}\frac{\partial}{\partial\tau}(\delta x_\mu) - \frac{\partial x^\mu}{\partial\sigma}\frac{\partial}{\partial\sigma}(\delta\delta x_\mu)) && (4.91)\\
&= T\int d\tau d\sigma(-\frac{\partial^2 x^\mu}{\partial\tau^2}\delta x_\mu + \frac{\partial}{\partial\tau}(\frac{\partial x^\mu}{\partial\tau}\delta x_\mu) + \frac{\partial^2 x^\mu}{\partial\sigma^2}\delta x_\mu - \frac{\partial}{\partial\sigma}(\frac{\partial x^\mu}{\partial\sigma}\delta x^\mu)) && (4.92)\\
&= T\int d\tau d\sigma((-\partial_\tau^2 + \partial_\sigma^2)x^\mu)\delta x^\mu + T\int d\sigma[\dot{x}^\mu\delta x_\mu]_{\tau_i}^{\tau_f} - T\int d\tau[x'\delta x_\mu]_{\sigma=0}^{\sigma=\pi} && (4.93)
\end{aligned}$$

The third equality arises via using integration by part. Let us look at the boundary term first

$$T\int d\sigma[\dot{x}^\mu\delta x_\mu]_{\tau_i}^{\tau_f} - T\int d\tau[x'\delta x_\mu]_{\sigma=0}^{\sigma=\pi} \tag{4.94}$$

The second term is familiar. The equation of motions are derived by requiring that $\delta x_\mu = 0$ at $\tau = \tau_f$ and τ_i. So, the second term varnishes. The third term vanishes under 3 conditions

Condition 1 : Closed string

For closed strings, σ is taken to have the periodic condition

$$x^\mu(\tau, \sigma + \pi) = x^\mu(\tau, \sigma) \tag{4.95}$$

This implies that

$$\delta x_\mu(\tau, \sigma = 0) = \delta x_\mu(\tau, \sigma = \pi) \tag{4.96}$$

This makes the second boundary term vanishes. The equation of motion for this case is

$$(\partial_\tau^2 - \partial_\sigma^2)x^\mu(\tau, \sigma) = 0 \tag{4.97}$$

with boundary condition (4.95).

Condition 2 : Open string (Neumann Boundary Condition)

In this condition, we set the derivative of x^μ with respect to σ varnishes at the σ boundary

$$\partial_\sigma x^\mu(\tau, \sigma = 0) = \partial_\sigma x^\mu(\tau, \sigma = \pi) = 0 \tag{4.98}$$

The equation of motion is again the one in (4.97) with Neumann boundary condition (4.98). Note that Neumann boundary condition preserves Poincaré symmetry because

$$\partial_\sigma(x^\mu)|_{\sigma=0,\pi} = \partial_\sigma[a^\mu_\nu x^\nu + b^\mu]|_{\sigma=0,\pi} = 0 \tag{4.99}$$

Condition 3: Open string (Dirichlet Boundary Condition)

In this condition, we kill the σ boundary term by setting the value of x^μ to be constant at the σ boundary

$$x^\mu(\tau, \sigma = 0) = x^\mu(\tau, \sigma = \pi) = c^\mu \tag{4.100}$$

The equation of motion is then the one in (4.97) with Dirichlet boundary condition (4.100). The Dirichlet boundary condition condition does not preserve Poincaré symmetry since

$$a^\mu_\nu x^\nu|_{\sigma=0,\pi} + b^\mu \neq c^\mu \tag{4.101}$$

Therefore, under a Poincaré transformation, the end of the open string changes.

4.3 Dirac-Born-Infeld theory

4.3.1 D-brane

From the last section, it is shown that the p-brane action is expressed as an invariant volume form with the induced metric involved. This immediately implies the natural geometrical origin of p-brane as a $p+1$-dimensional hypersurface embedded in a $D(\geq p)$-dimensional manifold. In this section, a specific type of brane which is called D-brane is studied. The action of the D-brane is the so called Dirac Born-Infeld action and again D-brane theory has a very nice geometric interpretation. We recall that there exist 2 type of boundary conditions for closed string in 1-brane theory. One is Neumann boundary condition. For Neumann boundary condition, the string can oscillate and its endpoints can still move along the boundaries as long as their derivatives vanish at the boundaries. The other one is Dirichlet boundary condition. For Dirichlet boundary condition, the string can oscillate but the endpoints are fixed at the boundary. One can of course extend these 2 boundary conditions to p-brane. Now, we consider the situation that Dirichlet boundary condition holds for some coordinates while Neumann boundary condition holds for the others. So, at the end points of the brane, we have

$$\partial_\alpha x^\mu = 0 \qquad \text{for } \mu = 0, 1, ..., p \tag{4.102}$$

$$x^\nu = c^\nu \qquad \text{for } \nu = p+1, ..., D-1 \tag{4.103}$$

So, the end points of the brane are fixed and lied in a $p+1$-dimensional hypersurface embedding in a D-dimensional spacetime. See figure 5. This $p+1$ dimensional hypersurface is called D-brane. D here stands for Dirichlet condition while p is the spatial dimension of the brane. The global Poincare symmetric group is also broken into

$$ISO(1, D-1) \to ISO(1, p-1) \times SO(D-p-1) \tag{4.104}$$

$SO(D-p-1)$ stands for the rotation in the direction imposed by Dirichlet boundary condition.

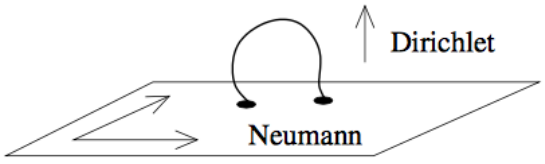

Figure 7: D-brane [22].

Note that the timelike coordinate x^0 cannot have the Dirichlet boundary condition. D-brane hypersurface can be thought as a dynamical object on its own. This will become obvious when the action of D-brane under a gauge fixing is introduced in the next section.

4.3.2 DBI action

In natural unit, the action for a brane in the form of invariant volume for which the background spacetime is flat Minkowski spacetime is

$$S = -\int d^{p+1}\sigma\sqrt{-det[\frac{\partial x^M}{\partial\sigma^\alpha}\frac{\partial x^N}{\partial\sigma^\beta}\eta_{MN}]} \tag{4.105}$$

Let the background spacetime coordinates be $N = (J, K)$, for J denotes the coordinates imposed by the Neumann boundary condition and K denotes the one imposed by Dirichlet boundary condition. In conventional unit, there will be a dimensional correction parameter $\frac{\Lambda^D}{\kappa}$ which has the meaning of the brane tension but here we crank it to 1. Since the action (4.105) has reparametrization freedom, we fix the gauge freedom by imposing static gauge

$$\sigma^\alpha = x^\alpha \tag{4.106}$$

This implies the induced metric becomes

$$\frac{\partial x^M}{\partial\sigma^\alpha}\frac{\partial x^N}{\partial\sigma^\beta}\eta_{MN} = \frac{\partial\sigma_J}{\partial\sigma^\alpha}\frac{\partial\sigma^J}{\partial\sigma^\beta} + \frac{\partial x_K}{\partial\sigma^\alpha}\frac{\partial x^K}{\partial\sigma^\beta} \tag{4.107}$$

$$= \frac{\partial\sigma^a}{\partial\sigma^\alpha}\frac{\partial\sigma^J}{\partial\sigma^\beta}\eta_{aJ} + \partial_\alpha x_K\partial_\beta x^K \tag{4.108}$$

$$= \delta^a_\alpha\eta^J_\beta\eta_{aJ} + \partial_\alpha x_K\partial_\beta x^K \tag{4.109}$$

$$= \eta_{\alpha\beta} + \partial_\alpha x_K\partial_\beta x^K \tag{4.110}$$

Define the scalar field x^K to be ϕ^K. Then the action (4.105) becomes

$$S_{DBI} = -\int d^{p+1}x\sqrt{-det[\eta_{\alpha\beta} + \partial_\alpha\phi^K\partial_\beta\phi_K]} \tag{4.111}$$

If we are interested in the situation with small partial derivative $\partial_\alpha\phi^K$, we can expand the determinant form to leading order term

$$S_{DBI} = -\int d^{p+1}x\sqrt{1 + \partial_\alpha\phi^K\partial^\alpha\phi_K} \tag{4.112}$$

Doing a further expansion to get rid of square root gives

$$S_{DBI} = \int d^{p+1}x(1 - \frac{1}{2}\partial_\alpha\phi^K\partial^\alpha\phi_K + ...) \tag{4.113}$$

The DBI action has a geometric interpretation as the world volume action of a $p+1$-dimensional D-brane embedded in D-dimensional Minkowski space. The scalar field ϕ is interpreted as the fluctuation of the D-brane in the transverse direction.

4.3.3 Nonlinearly realised symmetries of DBI theory and minimal coupling to matter

We consider the DBI action describing a D_4 brane embedded in 5-dimensional Minkowski space. The action is

$$S_{DBI} = -\int d^4x\sqrt{1 + (\partial\phi)^2} \tag{4.114}$$

For $(\partial\phi)^2 = \partial^\alpha\phi\partial_\alpha\phi$. This DBI action is protected by a nonlinearly realised 5 dimensional Poincare invariance $ISO(1,4)$ for an unwarped brane. This action is obviously invariant under the linearly realised $ISO(1,3)$ subgroup $(M_{\alpha\beta}, P_\alpha)$. One nonlinearly realised symmetric group is P_4 corresponding to the translation in the fifth dimension. This shift transformation is

$$\phi \to \phi + b \tag{4.115}$$

For b is a constant. This symmetric transformation is obvious because the action takes the form of partial derivative of ϕ.

The other nonlinearly realised symmetric group is $M_{4\alpha}$ corresponding to the Lorentz transformation in the 5th dimension. If the signature of the metric is chosen to be $(-,+,+,+)$, the infinitesimal variation under this transformation is

$$\delta_v \phi(x') = \phi'(x') - \phi(x) = v_\alpha x^\alpha \tag{4.116}$$

$$\delta_v x^\alpha = -v^\alpha \phi(x) \tag{4.117}$$

Therefore, under the infinitesimal fifth dimensional Lorentz transformation,

$$\begin{aligned}
\phi(x^\alpha) \rightarrow & \phi'(x'^\alpha) = \phi(x^\alpha) + v_\alpha x^\alpha && (4.118)\\
& \phi'(x^\alpha - v^\alpha \phi(x)) = \phi(x^\alpha) + v_\alpha x^\alpha && (4.119)\\
& \phi'(x^\alpha) - \phi(x) v^\alpha \partial_\alpha \phi'(x) = \phi(x^\alpha) + v_\alpha x^\alpha && (4.120)\\
& \phi'(x^\alpha) - \phi(x) v^\alpha \partial_\alpha \phi(x + v^\alpha \phi(x)) = \phi(x^\alpha) + v_\alpha x^\alpha && (4.121)\\
& \phi'(x^\alpha) - \phi(x) v^\alpha \partial_\alpha \phi(x) = \phi(x^\alpha) + v_\alpha x^\alpha && (4.122)
\end{aligned}$$

The fifth line arises by further expanding and neglect all the second and higher order term of v^α since v^α is infinitesimal. This leads to the relation that

$$\begin{aligned}
\delta_v \phi(x) &= \phi'(x) - \phi(x) && (4.123)\\
&= v_\alpha x^\alpha + \phi(x) v^\alpha \partial_\alpha \phi(x) && (4.124)
\end{aligned}$$

Since the fifth dimensional Lorentz transformation takes the theory out of the static gauge, so a compensating world-volume reparametrization is needed to restore the static gauge. This explains the variation (4.124). One can check by direct substitution that the Lagrangian of the DBI action shifts by a total derivative term which can be neglected under this transformation. This transformation can also be viewed from another more inspiring perspective. Recall that the induced action for DBI theory is $G_{\alpha\beta} = \eta_{\alpha\beta} + \partial_\alpha \phi \partial_\beta \phi$. A careful calculation using (4.124) shows that the infinitesimal variation of the induced metric under this transformation is

$$\begin{aligned}
\delta_v G_{\alpha\beta} &= \partial_\alpha \phi \partial_\beta \delta_v \phi + \partial_\beta \phi \partial_\alpha \delta_v \phi && (4.125)\\
&= \partial_\alpha \phi (v_\beta + \partial_\nu (\phi v^\kappa \partial_\kappa \phi) + \partial_\beta \phi (v_\alpha + \phi v^\kappa \partial_\kappa \phi)) && (4.126)\\
&= \epsilon_v^\kappa \partial_\kappa G_{\alpha\beta} + \partial_\alpha \epsilon_v^\kappa G_{\kappa\beta} + \partial_\beta \epsilon_v^\kappa G_{\kappa\alpha} && (4.127)\\
&= \mathcal{L}_v G_{\alpha\beta} && (4.128)
\end{aligned}$$

For $\epsilon_v^\kappa = v^\kappa \phi(x)$. (4.127) is recognized as Lie derivative on the induced metric. The corresponding diffeomorphism is induced by the vector field ϵ_v^κ. Hence, the transformation on ϕ can be viewed as to induce a field dependent diffeomorphic coordinate transformation. Then, we can imitate the story in general relativity to use this induced diffeomorphism to couple the scalar field ϕ with matter field Φ. For the minimally coupling matter field Φ with respect to the induced metric $G_{\alpha\beta}$, its matter action S_m must also transform under the induced diffeomorphism symmetry. For example, scalar matter field transforms as

$$\delta_v \Phi = \mathcal{L}_v \Phi = v^\alpha \phi \partial_\alpha \Phi \tag{4.129}$$

Then, the full action $S_{DBI} + S_m$ is invariant under the DBI symmetries. A simple example for S_m can be

$$S_m = \int d^4x \sqrt{-G} (-\frac{1}{2} G^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi - \frac{m_\Phi^2}{2} \Phi^2) \tag{4.130}$$

For m_Φ is the mass of Φ scalar.

4.4 Special Galileon theory

4.4.1 The structure of Special Galileon

The other scalar field theory that we desire to study is the special galileon theory. It is found out by Jiri Novotny that the speicial galileon theory also has a very similar extrinsic geometric interpretation with the Dirac Born Infeld theory but a complex manifold is needed. The special galileon is the sum of all the galileon terms with even numbers of fields in D-dimension. The special galileon action is given as

$$S_{sgal} = -\frac{1}{2}\int d^D x \sum_{n=1}^{\lfloor \frac{D+1}{2} \rfloor} \frac{\alpha^{n-1}}{(2n-1)!\Lambda^{(D+2)(n-1)}} (\partial\phi)^2 \mathcal{L}^{TD}_{2n-2} \tag{4.131}$$

where Λ is an energy scale and α is a dimensionless parameter. Note that in the section of DBI theory, we work in the unit such that $\frac{\Lambda^D}{\alpha} = 1$ so that we do not need to keep track of these parameters in DBI theory. We show the explicit action of special galileon with fixed relative coefficients here. $\mathcal{L}^{TD}_n$ are defined by

$$\mathcal{L}^{TD}_n = \sum_p (-1)^p \eta^{\mu_1 p(\nu_1)} ..... \eta^{\mu_n p(\nu_n)} \phi_{\mu_1\nu_1} ... \phi_{\mu_n\nu_n} \tag{4.132}$$

for $\phi_{\mu\nu} = \partial_\mu\partial_\nu\phi$. The sum runs over all permutations of ν indices with the sign of permutation $(-1)^p$. We will work out explicitly for the case $D = 4$ as an example. The building blocks needed to construct the action in $D = 4$ case are

$$\mathcal{L}^{TD}_0 = (-1)^0 = 1 \tag{4.133}$$

$$\begin{aligned}
\mathcal{L}^{TD}_2 &= \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\phi_{\mu_1\nu_1}\phi_{\mu_2\nu_2} - \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1}\phi_{\mu_1\nu_1}\phi_{\mu_2\nu_2} && (4.134)\\
&= \phi^{\nu_1}_{\nu_1}\phi^{\nu_2}_{\nu_2} - \phi^{\nu_2}_{\nu_1}\phi^{\nu_1}_{\nu_2} && (4.135)\\
&= (\Box\phi)^2 - (\partial_\mu\partial_\nu\phi)^2 && (4.136)
\end{aligned}$$

To put everything together with the coefficients, the action for $D = 4$ is

$$S = -\frac{1}{2}\int d^4x((\partial\phi)^2 + \frac{\alpha}{6\Lambda^6}(\partial\phi)^2[(\Box\phi)^2 - (\partial_\mu\partial_\nu\phi)^2]) \tag{4.137}$$

The individual galileon terms have the Galileon symmetry. The infinitesimal variation of the galileon field ϕ under Galileon transformation is

$$\delta\phi = c + b_\mu x^\mu \tag{4.138}$$

The special galileon (sum of the even galileon terms) enjoy another higher-order shift symmetry

$$\delta\phi = s_{\mu\nu}(x^\mu x^\nu - \frac{\alpha}{\Lambda^{D+2}}\partial^\mu\phi\partial^\nu\phi) \tag{4.139}$$

At first sight, the special galileon theory seems unrelated to the story of the extrinsic geometry of brane embedded in a background manifold. However, the geometric construction of the special Galileon action using D dimensional brane propagating in $2D$ dimensional flat pseudo-riemannian space is shown by Jiri Novotny.

4.4.2 The geometrical origin of the Special Galileon

Let us assume the target space to be a D-dimensional complex space $M^D_{\mathbb{C}}$ with coordinates

$$Z^\mu = X^\mu + \frac{i}{\alpha}L^\mu \tag{4.140}$$

Its complex conjugate is

$$\bar{Z}^\mu = X - \frac{i}{\alpha}L^\mu \tag{4.141}$$

where X^μ and L^μ are real coordinates. $M^D_{\mathbb{C}}$ is equipped with a hermitian form h defined as

$$\begin{aligned} h &= \eta_{\mu\nu} dZ^\mu \otimes d\bar{Z}^\nu && (4.142)\\ &= \eta_{\mu\nu}([dX^\mu + \frac{i}{\alpha}dL^\mu] \otimes [dX^\nu - \frac{i}{\alpha}dL^\nu]) && (4.143)\\ &= \eta_{\mu\nu}[dX^\mu \otimes dX^\nu + \frac{1}{\alpha^2}dL^\mu \otimes dL^\nu] + \eta_{\mu\nu}(\frac{-i}{\alpha})[dX^\mu \otimes dL^\nu - dL^\mu \otimes dL^\nu] && (4.144) \end{aligned}$$

The real part of this form defines a metric with signature $(2, 2(D-1))$ on $M^D_{\mathbb{C}}$ and it can be treated as a real $2D$ dimensional space

$$\begin{aligned} ds^2 &= \eta_{\mu\nu}[dX^\mu \otimes dX^\nu + \frac{1}{\alpha^2}dL^\mu \otimes dL^\nu] && (4.145)\\ &= \eta_{\mu\nu}[dX^\mu \otimes dX^\nu - \frac{i}{\alpha}dX^\mu \otimes dL^\nu + \frac{i}{\alpha}dL^\mu \otimes dX^\nu + \frac{1}{\alpha^2}dL^\mu \otimes dL^\nu] && (4.146)\\ &= \eta_{\mu\nu}[dX^\mu + \frac{i}{\alpha}dL^\mu][dX^\nu - \frac{i}{\alpha}dL^\nu] && (4.147)\\ &= dZ \cdot d\bar{Z} && (4.148) \end{aligned}$$

The imaginary part of h, ω produces a symplectic Kahler form

$$\begin{aligned} \omega &= \eta_{\mu\nu}(\frac{-1}{\alpha})[dX^\mu \otimes dL^\nu - dL^\mu \otimes dL^\nu] && (4.149)\\ &= \frac{1}{\alpha}\eta_{\mu\nu} dX^\mu \wedge dL^\nu && (4.150)\\ &= \frac{1}{2\alpha}\eta_{\mu\nu}[dX^\mu \wedge dL^\nu - dL^\mu \wedge dX^\nu] && (4.151)\\ &= \frac{i}{2}\eta_{\mu\nu}[dX^\mu \wedge dX^\nu - \frac{i}{\alpha}dX^\mu \wedge dL^\nu + \frac{i}{\alpha}dL^\mu \wedge dX^\nu + \frac{1}{\alpha^2}dL^\mu \wedge dL^\nu] && (4.152)\\ &= \frac{i}{2}\eta_{\mu\nu}(dX^\mu + \frac{i}{\alpha}dL^\mu) \wedge (dX^\nu - \frac{i}{\alpha}dL^\nu) && (4.153)\\ &= \frac{i}{2}\eta_{\mu\nu} dZ^\mu \wedge d\bar{Z}^\nu && (4.154) \end{aligned}$$

The antisymmetric property of the wedge product is used in the third equality line. The term $dX \wedge dX$ and $dL \wedge dL$ is added in the forth equality line because these terms contract with $\eta_{\mu\nu}$ to give zero.

The forms (4.148), (4.150) and (4.154) are all invariant with respect to the transformations

$$Z'\mu = R^\mu_\nu Z^\nu + A^\mu \tag{4.155}$$

This can be viewed as the complex version of the Poincaré transformations where the rotation matrix $R^\mu_\nu \in U(1, D-1)$ satisfies the relation of Lorentz transformation

$$R^+ \cdot \eta \cdot R = \eta \tag{4.156}$$

The complex vector $A^\mu = c + \frac{i}{\alpha}b$ is the translation in $M^D_{\mathbb{C}}$. This transformations generate a group which can be interpreted as the complex generalization of the Poincaré group $ISO(1, D-1)$.

To imitate the story in DBI theory, we assume a D-dimensional real Minkowski manifold $M^D_{\mathbb{R}}$ embedded in $M^D_{\mathbb{C}}$. The embedding is parametrized by real parameters σ^μ with $\mu = 0, ..., D-1$

$$Z^\mu = Z^\mu(\sigma) = X^\mu(\sigma) + \frac{i}{\alpha}L^\mu(\sigma) \tag{4.157}$$

We choose a gauge such that the Kahler form vanishes on the brane $M^D_{\mathbb{R}}$

$$\omega|_{M^D_{\mathbb{R}}} = 0 \tag{4.158}$$

This constraint implies that

$$\eta_{\mu\nu} dZ^\mu \wedge d\bar{Z}^\nu = \eta_{\mu\nu}[dZ^\mu \otimes d\bar{Z}^\nu - d\bar{Z}^\nu \otimes dZ^\mu] \tag{4.159}$$

$$= \eta_{\mu\nu}[\frac{\partial Z^\mu}{\partial \sigma^\alpha}\frac{\partial \bar{Z}^\nu}{\partial \sigma^\beta} - \frac{\partial \bar{Z}^\nu}{\partial \sigma^\alpha}\frac{\partial Z^\mu}{\partial \sigma^\beta}] d\sigma^\alpha \otimes d\sigma^\beta \tag{4.160}$$

$$= [\frac{\partial Z}{\partial \sigma^\alpha} \cdot \frac{\partial \bar{Z}}{\partial \sigma^\beta} - \frac{\partial \bar{Z}}{\partial \sigma^\alpha} \cdot \frac{\partial z}{\partial \sigma^\beta}] d\sigma^\alpha \otimes d\sigma^\beta \tag{4.161}$$

$$= 0 \tag{4.162}$$

This implies that

$$\frac{\partial Z}{\partial \sigma^\mu} \cdot \frac{\partial \bar{Z}}{\partial \sigma^\nu} - \frac{\partial \bar{Z}}{\partial \sigma^\mu} \cdot \frac{\partial Z}{\partial \sigma^\nu} = 0 \tag{4.163}$$

Alternatively, the constraint induced by the gauge satisfying (4.158) induces the relation in the form of real coordinates equivalent to (4.163) is

$$\frac{\partial X}{\partial \sigma^\mu} \cdot \frac{\partial L}{\partial \sigma^\nu} - \frac{\partial L}{\partial \sigma^\mu} \cdot \frac{\partial X}{\partial \sigma^\nu} = 0 \tag{4.164}$$

The derivation of (4.164) is similar to the the derivation of (4.163) but one should start with (4.150) instead of (4.154). This constraint is invarint with respect to the transformation (4.155).On the brane $M^D_{\mathbb{R}}$, there is a real induced metric

$$G_{\alpha\beta} = \frac{\partial Z^\mu}{\partial \sigma^\alpha}\frac{\partial \bar{Z}^\nu}{\partial \sigma^\beta}\eta_{\mu\nu} \tag{4.165}$$

Our next step is to proceed in the way analogous to the construction of DBI-like action. In order to do that, we have to further fix the gauge freedom. This is always allowed due to reparametrization freedom. We introduce a new coordinates x^μ on the brane as

$$x^\mu = X^\mu(\sigma) \tag{4.166}$$

In this new parametrization, the embedding is

$$X^\mu(x) = x^\mu, L^\mu = L^\mu(x) \tag{4.167}$$

In Special Galileon theory, the fluctuation of the brane is effectively described by the fields $L^\mu(x)$. We first have to consider both the constraints (4.164) and (4.167)

$$\frac{\partial X^\mu}{\partial x^\alpha}\frac{\partial L_\mu}{\partial x^\beta} = \frac{\partial L^\nu}{\partial x^\alpha}\frac{\partial X_\nu}{\partial x^\beta} \tag{4.168}$$

$$\frac{\partial x^\mu}{\partial x^\alpha}\frac{\partial L_\mu}{\partial x^\beta} = \frac{\partial L^\nu}{\partial x^\alpha}\frac{\partial x_\nu}{\partial x^\beta} \tag{4.169}$$

$$\delta^\mu_\alpha \frac{\partial L_\mu}{\partial x^\beta} = \delta_{\nu\beta}\frac{\partial L^\nu}{\partial x^\alpha} \tag{4.170}$$

$$\boxed{\partial_\beta L_\alpha = \partial_\alpha L_\beta} \tag{4.171}$$

(4.171) implies that

$$L_\alpha = \partial_\alpha \phi(x) \tag{4.172}$$

It is seen that the additional constraint reduces the number of effective degree of freedom of the special Galileon field ϕ. The explicit induced metric can then be found as

$$G_{\alpha\beta} = \eta_{\alpha\beta} - \frac{i}{\alpha}\partial_\beta L_\alpha + \frac{i}{\alpha}\partial_\alpha L_\beta + \frac{1}{\alpha^2}\eta_{\mu\nu}\partial_\alpha L^\mu \partial_\beta L^\nu \tag{4.173}$$

$$= \eta_{\alpha\beta} - \frac{i}{\alpha}\partial_\beta\partial_\alpha\phi + \frac{i}{\alpha}\partial_\alpha\partial_\beta\phi + \frac{1}{\alpha^2}\eta_{\mu\nu}\partial_\alpha\partial^\mu\phi\partial_\beta\partial^\nu\phi \tag{4.174}$$

$$= \eta_{\alpha\beta} + \frac{1}{\alpha^2}\partial_\alpha\partial_\nu\phi\partial^\nu\partial_\beta\phi \tag{4.175}$$

In this gauge, we therefore have the embedding of the brane as

$$X^\mu(x) = x^\mu, L^\mu(x) = \eta^{\mu\nu}\partial_\nu\phi \tag{4.176}$$

The gauge condition (4.167) is not invariant under the transformation (4.155). So, it is necessary to combine the traget space transformation (4.155) with compensated reparametrization in order to preserve the gauge condition (4.176). As a result, the field $\phi(x)$ will transform nonlinearly under (4.155).

Let us consider the complex translation first,

$$Z'^\mu = Z^\mu + (c^\mu + \frac{i}{\alpha}b^\mu) \tag{4.177}$$

x^μ is shifted by c^μ while $L^\mu(x)$ is shifted by b^μ

$$X'^\mu(x) = x^\mu + c^\mu, L'^\mu(x) = \eta^{\mu\nu}\partial_\nu\phi(x) + b^\mu = \eta^{\mu\nu}\partial_\nu[\phi(x) + b\cdot x] \tag{4.178}$$

Due to gauge freedom, we define a new parameter x'^μ as

$$x'^\mu = x^\mu + b\cdot x \tag{4.179}$$

Then we can redefine the field ϕ using this new parameter,

$$\phi'(x') = \phi(x) + b\cdot x \tag{4.180}$$

Since it is a translation, $\partial'_\nu = \partial_\nu$, we have

$$L''(X') = L'^\mu(x(x')) = \eta^{\mu\nu}\partial'_\nu\phi'(x') \tag{4.181}$$

Hence the gauge (4.176) is preserved using the redefined field $\phi'(x')$. (4.180) corresponds to the symmetric Galileon transformation (4.138) for each individual Galileon term. So, the complex translation can be viewed as the combination of spacetime translation and the Galileon transformation of the Galileon field.

For the complex Lorentz transformation $R^\mu_\nu \in U(1, D-1)$, it is written as

$$R = e^{\mathcal{M}+i\mathcal{G}} = \Lambda + iU \tag{4.182}$$

where Λ and U are real matrices, $\mathcal{M}$ and $\mathcal{G}$ are real generators. The generators satisfy the algebra

$$\eta_{\mu\rho}\mathcal{M}^\rho_\nu + \eta_{\nu\rho}\mathcal{M}^\rho_\mu = 0 \tag{4.183}$$

$$\eta_{\mu\rho}\mathcal{G}^\rho_\nu - \eta_{\nu\rho}\mathcal{G}^\rho_\mu = 0 \tag{4.184}$$

The transformation due to R^μ_ν leads to

$$X'^\mu(x) = \Lambda^\mu_\nu x^\nu \ - \frac{1}{\alpha}U^\mu_\nu L^\nu(x) = \Lambda^\mu_\nu x^\nu - \frac{1}{\alpha}U^\mu_\nu\eta^{\nu\rho}\partial_\rho\phi(x) \tag{4.185}$$

$$L'(x) = \frac{1}{\alpha}\Lambda^\mu_\nu L^\nu(x) + U^\mu_\nu x^\nu = \frac{1}{\alpha}\Lambda^\mu_\nu \partial^\nu \phi(x) + U^\mu_\nu x^\nu \tag{4.186}$$

If $U = 0$, $\lambda = e^{\mathcal{M}} \in O(1, D-1)$ form a subgroup of $U(1, D-1)$. The transformation generated by $O(1, D-1)$ is realised to be linear

$$x'^\mu = \Lambda^\mu_\nu x^\nu \tag{4.187}$$

Then, from (4.186), it is obvious that the new redefined field using the parameter (4.187) is

$$\phi'(x') = \phi(x) \tag{4.188}$$

which preserves the gauge condition (4.176). The remaining transformations generated by $\mathcal{G}$ are identified as hidden duality transformation discussed in [17]. The compensating gauge transformation for the case $U \neq 0$ is

$$x'^\mu(x) = \Lambda^\mu_\nu x^\nu \ - \frac{1}{\alpha}U^\mu_\nu L^\nu(x) = \Lambda^\mu_\nu x^\nu - \frac{1}{\alpha}U^\mu_\nu \eta^{\nu\rho}\partial_\rho \phi(x) \tag{4.189}$$

Consequently, the redefined field after the transformation to preserve the gauge condition is [17]

$$\phi'(x') = \phi(x) - \frac{1}{2}x \cdot \partial\phi(x) + \frac{1}{2}\eta_{\mu\nu}(\Lambda^\mu_\rho x^\rho - \frac{1}{\alpha}U^\mu_\rho \eta^{\rho\alpha}\partial_\alpha\phi(x))(\frac{1}{\alpha}\Lambda^\nu_\sigma \eta^{\sigma\beta}\partial_\beta\phi(x) + U^\nu_\sigma x^\sigma) \tag{4.190}$$

(4.190) corresponds to the nonlinearly realised special Galileon transformation (4.139). The action $\int d^D x\sqrt{det(G_{\mu\nu})}$ is also shown in [17] to be equivalent to the quartic formulation of Special Galileon action through a complicated process which is not enlightening to show here.

As a summary, in term of geometric interpretation, the Galileon field ϕ is interpreted as the scalar degree of freedom depicting the fluctuation of a D-dimensional brane embedded in $2D$-dimensional $\mathbb{R}^{2,2D-2}$ Kahler manifold. The nonlinearly realised symmetry of Special Galileon can be explained as the non-linear realization of the target space symmetry group $U(1, D-1)$.

4.4.3 Minimal coupling of the Special Galileon with Matter field

Within the formalism of brane construction, the induced metric of the Special Galileon theory is found to be

$$G_{\alpha\beta} = \eta_{\alpha\beta} + \frac{1}{\alpha^2}\partial_\alpha\partial_\nu\phi\partial^\nu\partial_\beta\phi \tag{4.191}$$

The higher order shift symmetry is given in (4.139). With the induced metric and the higher order shift symmetry known, we are well equipped with all the essential elements to do the matter coupling as what we have shown in DBI theory. It is easy to checked that under the higher order shift, the infinitesimal variation of the induced metric can be expressed as a Lie derivative along the following vector field v^μ

$$\delta G_{\alpha\beta} = \mathcal{L}_v G_{\alpha\beta}, v^\mu = -\frac{2\alpha}{\Lambda^{D+2}}S^{\mu\nu}\partial_\nu\phi \tag{4.192}$$

To do the minimal coupling with matter in the diffeomorphism invariant way, we follow the same prescription in DBI theory by letting the matter fields transform under the special galileon higher order shift symmetry as a Lie derivative induced by the vector field v^μ For instance, we can couple S_{sgal} with a spin-1 particle A_μ with mass m_A this time.

$$S_A = \int d^D x\sqrt{-G}(-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{m_A^2}{2}A_\alpha A^\alpha) \tag{4.193}$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ and its indices are manipulated by the induced metric. The vector field A_α transforms under special Galileon symmetry as

$$\delta A_\alpha = \mathcal{L}_v A_\mu = -\frac{2\alpha}{\Lambda^{D+2}}s^{\mu\nu}(\partial_\nu\phi\partial_\mu A_\alpha + \partial_\nu\partial_\alpha\phi A_\mu) \tag{4.194}$$

4.5 The geometric interpretation of Born-Infeld theory ?

We have seen the geometric interpretation of two scalar field theories as D-brane embedded in a background spacetime. What about Born-Infeld theory for which the field is the electromagnetic field A_μ? If we fix the background spacetime to be Minkowski spacetime, we note that the Born-Infeld Lagrangian is invariant under Lorentz trnasformation. Born-Infeld Lagrangian is shown before to be

$$\mathcal{L} = \sqrt{-det(\eta_{\mu\nu} + F_{\mu\nu}) - 1} \tag{4.195}$$

To see Lorentz invariance of $det(\eta_{\mu\nu} + F_{\mu\nu})$, we first realize that the determinant of any arbitrary matrix M with components $M_{\mu\nu}$ is the same with the determinant of the matrix $\bar{M}$ with component $M^{\mu\nu}$.

$$M_{\mu\nu} = \eta_{\mu a}\eta_{\nu b}M^{ab} \tag{4.196}$$

$$= \eta_{\mu a}M^{ab}\eta_{\nu b} \tag{4.197}$$

In matrix notation

$$M = \eta\bar{M}\eta \tag{4.198}$$

Taking the determinant of both sides, we obtain

$$detM = det(\eta\bar{M}\eta) = (det(\bar{M}))(det\eta)^2 = det\bar{M} \tag{4.199}$$

We consider a Lorentz transformation $x'^\mu = \Lambda^\mu_\nu x^\nu$ with $(det\Lambda)^2 = 1$. After Lorentz transformation, we see that

$$\eta'^{\mu\nu} + F'^{\mu\nu} = \Lambda^\mu_\rho\Lambda^\nu_\sigma(\eta^{\rho\sigma} + F^{\rho\sigma}) = \Lambda^\mu_\rho(\eta^{\rho\sigma} + F^{\rho\sigma})\Lambda^\nu_\sigma \tag{4.200}$$

Again, in the matrix notation

$$\bar{\eta}' + \bar{F}' = \Lambda(\bar{\eta} + \bar{F})\Lambda^T \tag{4.201}$$

Taking determinant of both side,

$$det(\bar{\eta}' + \bar{F}') = det(\bar{\eta} + \bar{F}) \tag{4.202}$$

This proves the Lorentz invariance of the Born Infeld theory for the background spacetime to be Minkowski spacetime. The induced metric in this case can be written as

$$G_{\mu\nu} = \eta_{\mu\nu} + F_{\mu\nu} \tag{4.203}$$

The Born-Infeld vector A_μ only contributes to the antisymmerical part of the induced metric. We have seen before that both DBI theory and speial Galileon theory tranform covariantly under the induced diffeomorphism. The induced diffeomorphism is originated from the nonlinearly realised symmetry for which that the scalar fields mix with the spacetime coordinates. For BI theory, if there exist an induced diffeomorphism, both symmetrical part $\eta_{\mu\nu}$ and antisymmetrical part $F_{\mu\nu}$ have to transform covariantly separately under the induced diffeomorphism. This leaves only the linearised Poincaré symmetry. This can be seen by considering the following arguments. Consider a tensor $a_{\mu\nu}$. Decompose $a_{\mu\nu}$ into its symmetrical part and antisymmetrical part

$$a_{\mu\nu} = \frac{1}{2}(a_{\mu\nu} + a_{\nu\mu}) + \frac{1}{2}(a_{\mu\nu} - a_{\nu\mu}) \tag{4.204}$$

$$= S_{\mu\nu} + A_{\mu\nu} \tag{4.205}$$

The symmetrical part is $S_{\mu\nu} = \frac{1}{2}(a_{\mu\nu}+a_{\nu\mu})$ and the antisymmetrical part is $A_{\mu\nu} = \frac{1}{2}(a_{\mu\nu}-\frac{1}{2}a_{\nu\mu})$. Under a general transformation D. S_{ab} transforms as

$$S_{\mu\nu} = D^a_\mu D^b_\nu S_{ab} \tag{4.206}$$
$$= D^a_\mu D^b_\nu S_{ba} \tag{4.207}$$
$$= S_{\nu\mu} \tag{4.208}$$

Under transformation, $S_{\mu\nu}$ is still symmetric. Similarly, for the antisymmetrical part A_{ab},

$$A_{\mu\nu} = D^a_\mu D^b_\nu A_{ab} \tag{4.209}$$
$$= -D^a_\mu D^b_\nu A_{ba} \tag{4.210}$$
$$= -A_{\nu\mu} \tag{4.211}$$

$A_{\mu\nu}$ is still antisymmetric under the transformation. Hence, $S_{\mu\nu}$ and $A_{\mu\nu}$ form invariant subspace of the representations $D \in G$ where G is the group of the transformations. This arguments also hold to the case $a_{\mu\nu} = \eta_{\mu\nu} + F_{\mu\nu}$. So, under induced diffeomorphism, $\eta_{\mu\nu}$ will not interact with $F_{\mu\nu}$. The only realised transformation that will leave the theory invariant and also $F_{\mu\nu}$ to be antisymmetric with the fixed Minkowski spacetime background is Lorentz trnsformation. This implies the components of the field A_μ are bonded together to transform as a vector field and it will not mix with the spacetime coordinates.

We have seen that for scalar field theories the infinitesimal variation of the extrinsic metric induced by the diffeomorphism caused by the nonlinearly realised symmetry can be formulated as a Lie derivative and the induced metric can be used to couple the scalar field with matter field in a diffeomorphism invariant way with respect to the non-linearly realised symmetry. Since there is no nonlinearly realised symmetry for Born-Infeld theory, this implies the similar geometrical interpretation as happened in DBI theory and special Galileon theory cannot be applied to BI theory.

5 Classical Double Copy

5.1 Introduction to Double Copy

In physics, scattering amplitude is a function of momenta and spin describing the probability that a given scattering process occurs. Quantum field theory (QFT) is the theoretical framework used to predict the scattering amplitude. In it, Feynman diagrams which are a diagrammatic organization of the perturbative expansion of scattering amplitudes are used to calculate the scattering amplitude for a given process. Our topic in this chapter is mainly motivated by a kind of duality in the study of the scattering amplitude. This duality is called color-kinematics duality or Bern-Carrasco-Johansson (BCJ) duality. Basically, BCJ duality has 2 elements [4]:

1) Amplitudes can be rearranged in a way that their kinematic structures satisfy a kinematic analog of Jacobi identity.

2) Amplitudes in the dual form can be double copied to generate the new amplitude in other theories. Such a story produces a web of relations of the form $graviton = gluon^2$ and $special\ galileon = pion^2$.

Gluon is the gauge boson in strong interaction and it is the excitation of $SU(3)$ gauge field. The pion is one of the particle mediating the interaction between a pair of nucleons and it is the scalar field in non-linear sigma model (NLSM). Jacobi identity in element **1**) is the binary operation describing how the result of the operation is affected by the order of evaluation. For example, the color factor Jacobi identity in gauge theory is [13]

$$c_s + c_t + c_u = 0 \tag{5.1}$$

where $c_s = f^{a_1 a_2 b} f^{b a_3 a_4}$, $c_t = f^{a_1 a_3 b} f^{b a_4 a_2}$, $c_u = f^{a_1 a_4 b} f^{b a_2 a_3}$. The numerical factor f^{abc} is the structure constant of the gauge Lie group in non-Abelian Yang Mills theory. The gauge field $A_\mu(x)$ is a traceless hermitian matrix of fields and it can be expanded in the following way

$$A_\mu(x) = A^a_\mu(x) T^a \tag{5.2}$$

for T^a matrices are the generator of the gauge group. The generator matrices obey commutation relations

$$[T^a, T^b] = i f^{abc} T^c \tag{5.3}$$

What it means for element **1**) of BCJ duality is that one can always find a representation such that the parallel relations hold for color and kinematic factors. [13]

$$c_i + c_j + c_k = 0 \Leftrightarrow n_i + n_j + n_k = 0 \tag{5.4}$$

The study of this duality in scattering amplitude is beyond the scope of this thesis. The discussion of the scattering amplitude in the introduction section of this chapter only serves for the purpose of motivation for another relevant duality in the level of the classical solutions of different field theories and the details will not be discussed. For a thorough discussion, see [4] and [13]. We only outline the general idea here and we will not go deep to discuss the concepts in scattering amplitude.

In BCJ duality, the theory of a perturbative duality between gauge theory and gravity is called double copy. The theory of double copy states that color numerator in the scattering amplitude of gauge theories can be replaced by kinematic numerator in a well defined way to give the gravity amplitudes. The general form of an m-point, L-loop amplitude in non-Abelian gauge theory may be written as [15]

$$\mathcal{A}^{(L)}_m = i^L g^{m-2+2L} \sum_{i\in\Gamma} \int \prod_{l=1}^{L} \frac{d^D p_l}{2\pi^D} \frac{1}{S_i} \frac{n_i c_i}{\pi_{\alpha_i} p^2_{\alpha_i}} \tag{5.5}$$

where the sum is over all cubic topologies Γ ; n_i and c_i are kinematic numerators and color factors respectively. g is the coupling constant. The numerators n_i are chosen to satisfy the similar Jacobi identities to the color factors.

The double copy states that the gravity amplitude can be obtained in the following way [15]

$$\mathcal{M}^{(L)}_m = i^{L+1} (\frac{\kappa}{2})^{m-2+2L} \sum_{i\in\Gamma} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{\pi_{\alpha_i} p^2_{\alpha_i}} \tag{5.6}$$

Comparing (5.5) with (5.6), the coupling constant g is replaced by the gravitational coupling constant for $\kappa = \sqrt{16\pi G_N}$; the color factor c_i is replaced by the kinematic numerators $\tilde{n}_i$.

Similarly one can start with (5.5) and replace the kinematic numerators n_i by a second set of color factor $\tilde{c}_i$. The corresponding scattering amplitude is

$$\mathcal{T}^{(L)}_m = i^L y^{m-2+2L} \sum_{i\in\Gamma} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{c_i \tilde{c}_i}{\pi_{\alpha_i} p^2_{\alpha_i}} \tag{5.7}$$

y is another appropriate coupling constant. This scattering amplitude corresponds to the bi-adjoint scalar field theory. Its field equation is [15]

$$\partial^2 \Phi^{aa'} - y f^{abc} \tilde{f}^{a'b'c'} \Phi^{bb'} \Phi^{cc'} = 0 \tag{5.8}$$

f^{abc} and $\tilde{f}^{a'b'c'}$ are structure constants of two Lie algebras as two color factors are involved in the scattering amplitude.

It was found out that this prescription can actually be extended to more theories and actually there is a web of theories whose amplitudes are the product of color (c) or kinematic (n, r) factors subjected to Jacobi-like identity. See figure 6.

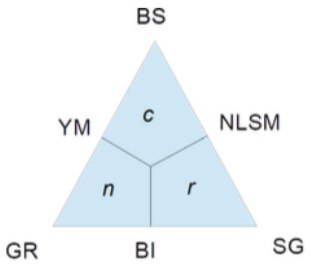

Figure 8: A web of theories of double copy [21].

In figure 6, BS refers to bi-adjoint scalar field theory; YM refers to Yang-Mills theory; GR refers to general relativity; BI refers to Born Infeld theory; SG refers to special galileon theory and NLSM refers to non-linear sigma model.

The double copy is intrinsically perturbative. In this thesis, instead of the double copy in term of scattering amplitude, we will focus on the related notion which is called classical double copy. Classical double copy is the theory that intends to find out the map between the classical solution of the theories within the web in figure 6.

5.2 Kerr-Schild Double Copy

5.2.1 The Duality between General Relativity and Maxwell's Electromagnetism

In this subsection, we study the classical double copy of (BS $\leftrightarrow$ Abelian YM $\leftrightarrow$ GR). Abelian YM theory here refers to $U(1)$ Maxwell theory of electromagnetism. We will see how these theories are related to each other through Kerr- Schild double copy.

Consider general relativity in Kerr-Schild coordinate system. The full metric is

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{5.9}$$
$$= \eta_{\mu\nu} + \kappa k_\mu k_\nu \phi \tag{5.10}$$

κ is a dimensional correction constant. The tensor field $h_{\mu\nu}$ is graviton. ϕ is a scalar field. The vector k_μ has the property that it is null with respect to both the flat metric $\eta_{\mu\nu}$ and the full metric $g_{\mu\nu}$

$$k_\mu \eta^{\mu\nu} k_\nu = 0 = k_\mu g^{\mu\nu} k_\nu \tag{5.11}$$

The vector field k_μ is also geodetic

$$k^\mu \partial_\mu k_\nu = 0 \tag{5.12}$$

Due to the properties of k_μ vector, the inverse of the full metric is

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa k^\mu k^\nu \phi \tag{5.13}$$

It can be checked easily that the inverse metric defined in (5.13) satisfies the relation

$$g_{\mu\nu} g^{\nu\alpha} = \delta^\alpha_\mu \tag{5.14}$$

Given this full metric, we can compute the Christoffel symbol, Ricci curvature tensor and Ricci scalar.

The metric-compatible and torsion-free connection takes the form in (3.108). The Ricci tensor is

$$R_{\sigma\nu} = R^{\rho}_{\sigma\rho\nu} = \Gamma^{\rho}_{\nu\sigma,\rho} - \Gamma^{\rho}_{\rho\sigma,\nu} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\rho\sigma} \tag{5.15}$$

The Ricci scalar is

$$R = g^{\nu\mu}R_{\mu\nu} \tag{5.16}$$

By careful calculation and making use of the properties of k_μ vector, the Ricci tensor with one index raising up using full metric takes the expression

$$R^{\mu}_{\nu} = \frac{1}{2}(\partial^{\mu}\partial_{\alpha}(\phi k^{\alpha}k_{\nu}) + \partial_{\nu}\partial^{\alpha}(\phi k_{\alpha}k^{\mu}) - \partial^2(\phi k^{\mu}k_{\nu})) \tag{5.17}$$

The corresponding Ricci scalar is

$$R = \partial_{\mu}\partial_{\nu}(\phi k^{\mu}k^{\nu}) \tag{5.18}$$

The Ricci tensor is solved exactly to be linear in this case. We consider the stationary case in which all the times derivative vanish. We also set $k^0 = 1$ without any loss of generality. In such a condition, the Ricci tensor and Ricci scalar can be solved

$$R^0_0 = \frac{1}{2}\partial^2\phi \tag{5.19}$$

$$R^i_0 = \frac{1}{2}[\partial^i\partial_{\alpha}(\phi k^{\alpha}) + \partial^2(\phi k^i)] \tag{5.20}$$

$$= -\frac{1}{2}\partial_j[\partial^i(\phi k^j) - \partial^j(\phi k^i)] \tag{5.21}$$

$$R^i_j = \frac{1}{2}\partial_l[\partial^i(\rho k^l k_j) + \partial_j(\rho k^l k^i) - \partial^l(\rho k^i k_j)] \tag{5.22}$$

$$R = \partial_i\partial_j(\phi k^l k^i) \tag{5.23}$$

The vacuum solution for Einstein equation is just $R^{\mu}_{\nu} = 0$. To interpret the result in the spirit of double copy,we define a vector field A_μ

$$A_{\mu} = k_{\mu}\phi \tag{5.24}$$

We also define the field strength as

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \tag{5.25}$$

$$= \partial_{\mu}\partial_{\nu}\phi - \partial_{\nu}\partial_{\mu}\phi \tag{5.26}$$

The definition (5.24) and (5.26) is called Kerr-Schild ansatz. The vacuum Einstein equation $R^{\mu}_{\nu} = 0$ implies that $\partial_{\mu}F^{\mu\nu} = 0$. To see this, we explicitly works out $\partial_{\mu}F^{\mu\nu} = 0$

$$\partial_{\mu}F^{\mu\nu} = \partial^2(\phi k^{\nu}) - \partial_j\partial^{\nu}(\phi k^j) = 0 \tag{5.27}$$

When $\nu = 0$, (5.27) becomes

$$\partial^2\phi = 0 \tag{5.28}$$

When $\nu = i$, (5.27) becomes

$$\partial^2(\phi k^i) - \partial_j \partial^i(\phi k^j) = 0 \tag{5.29}$$

Therefore, it is shown that $\partial_\mu F^{\mu\nu} = 0$ coincides with $R^0_0 = 0$ and $R^i_0 = 0$. If we interpret A_μ to be the electromagnetic field and $F_{\mu\nu}$ to be the electromagnetic field strength, then the duality between Maxwell theory and general relativity is constructed in the stationary case. The graviton $h_{\mu\nu}$ is obtained by adding a factor of k_ν to the gauge field A_μ. We can also interpret the scalar field ϕ in the spirit of zeroth copy. In the stationary case, ϕ satisfies the equation

$$\partial^2 \phi = 0 \tag{5.30}$$

This coincides with the field equation of bi-adjoint scalar field in the abelian case where all the color structure constants vanish. Hence,ϕ can be interpreted as bi-adjoint scalar field.

As a summary of this subsubsection, the vector k_μ plays an important role in the mapping relation of (BS $\leftrightarrow$ Abelian YM $\leftrightarrow$ GR). Adding the factor k_μ to the bi-adjoint scalar field gives the Abelian gauge field A_μ. Adding the factors k_ν and k_μ to the bi-adjoint scalar field gives the graviton $h_{\mu\nu}$. The mapping relation can be extended to the non-Abelian self dual YM in a very similar way by promoting k_μ to an operator which is shown in [15].

5.2.2 The duality between Schwarzschild's solution and Coulomb's electrostatic solution

We will see now how the Schwarzschild's solution in general relativity corresponds to Coulomb's electrostatic solution in Maxwell's electromagnetism by Kerr-Schild double copy. We first recall that in a spherically symmetric and static universe, the Schwarzschild's metric is read as

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{1}{1 - \frac{2GM}{r}}dr^2 + f(r)^2 d\Omega^2 \tag{5.31}$$

When $r \to \infty$, (5.31) is approximated to the metric in the Newtoninan limit

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 + \frac{2GM}{r})dr^2 + f(r)^2 d\Omega^2 \tag{5.32}$$

For the Schwarzschild's solution, the universe is flat everywhere except at a point where the source with a pointlike mass M is located. Hence, the energy-momentum tensor is represented by a Dirac delta function

$$T^{\mu\nu} = M v^\mu v^\nu \delta^{(3)}(x) \tag{5.33}$$

where $v^\mu = (1, 0, 0, 0)$ is a vector merely in the timelike direction. The gravitational field equation is just

$$G_{\mu\nu} = \frac{\kappa^2}{2} T_{\mu\nu}; \kappa^2 = 16\pi G \tag{5.34}$$

with $T_{\mu\nu}$ represented by (5.33). The Kerr-Schild form of the metric $g_{\mu\nu}$ exists because general relativity is invariant under general diffeomorphism indicating there is freedom of choice of coordinate system. The Kerr-Schild form of the exterior Schwarzschild metric is

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2GM}{r} k_\mu k_\nu \tag{5.35}$$

with

$$k^\mu = (1, \frac{x^i}{r}), r^2 = x^i x_i; i = 1...3 \tag{5.36}$$

Since $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, the graviton $h_{\mu\nu}$ in this case is

$$h_{\mu\nu} = \frac{\kappa}{2}\phi k_\mu k_\nu \tag{5.37}$$

with the scalar field ϕ as

$$\phi = \frac{M}{4\pi r} \tag{5.38}$$

From the viewpoint of double copy, we can reproduce the electromagnetic counterpart by removing one k_ν vector and replacing the coupling constant and the source

$$\frac{\kappa}{2} \to g, M \to c_a T^a, k_\mu k_\nu \to k_\mu \tag{5.39}$$

where $c_a T^a$ is the superposition of the color charge. Then, we obtain the vector gauge field A_μ

$$A_\mu = \frac{g c_a T^a}{4\pi r} k_\mu \tag{5.40}$$

The Abelian Maxwell equation with source j^ν is

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{5.41}$$

We substitute (5.40) into the left hand side of (5.41) to check what is the source j^ν.

For the timelike component,

$$\partial_\mu F^{\mu 0} = \partial_\mu(\partial^\mu A^0 - \partial^0 A^\mu) \tag{5.42}$$
$$= \partial_\mu(\partial^\mu A^0) \tag{5.43}$$
$$= \partial^2[\frac{g c_a T^a}{4\pi r}] \tag{5.44}$$
$$= \frac{g c_a T^a}{4\pi}\nabla^2[\frac{1}{r}] \tag{5.45}$$
$$= -\frac{g c_a T^a}{4\pi}\delta^3(x) \tag{5.46}$$

Note that we are working in the static and spherically symmetric case where A^μ does not depend on the timelike coordinate. For the spacelike component,

$$\partial_\mu F^{\mu i} \propto \partial_j\partial_j(\frac{x^i}{r^2}) - \partial_i\partial_j(\frac{x^j}{r^2}) \tag{5.47}$$
$$= 0 \tag{5.48}$$

Thus, the source j^ν is

$$j^\nu = -g(c_a T^a)v^\nu \delta^{(3)}(x) \tag{5.49}$$

The source in both Schwarzschild's solution and Abelian Maxwell's solution is represented by the dirac delta function. So, we see that the classical double copy relation is still well constructed in this case. Since Abelian Maxwell theory has gauge symmetry, we can perform the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \lambda \tag{5.50}$$

For λ is any arbitrary function. We choose λ to be

$$\lambda = -\frac{g c_a}{4\pi} log(\frac{r}{r_0}) \tag{5.51}$$

For such a gauge fixing, the spatial part of A^μ vanishes.

$$A^\mu = (\frac{g c_a T^a}{4\pi r}, 0, 0, 0) \tag{5.52}$$

This solution is recognized as the Coulomb's electrostatic solution with a point color charge located at the origin. Therefore, upon a particlar gauge fixing, the double copy between Schwarzschild's solution and Coulomb's solution is constructed.

5.3 Classical Double Copy between SG, BI and GR ?

From figure 6, it is realised that there is also double copy relation between Special Galileon theory (SG), Born Infeld theory (BI) and General Relativity (GR) in term of scattering amplitude. The n-particle tree level amplitude for SG is [4]

$$\mathcal{A}_{SG} = \sum_i \frac{r_i^2}{d_i} \tag{5.53}$$

where the sum rums over all cubic topologies and d_i are the associated products of propagator denominators. Replacing one NLSM kinematic numerator r_i to YM kinematic numerator n_i, the amplitude of BI is obtained

$$\mathcal{A}_{BI} = \sum_i \frac{r_i n_i}{d_i} \tag{5.54}$$

If replacing another r_i to n_i, the scattering amplitude of GR is obtained

$$\mathcal{A}_{GR} = \sum_i \frac{n_i^2}{d_i} \tag{5.55}$$

We ask the question now whether there is also a double copy relation of the classical solution between these theories like what we have seen before for (BS-YM-GR). In this section, we discuss the possible route implying the classical double copy of (SG-BI-GR).

5.3.1 The Schematic solution of BI for Static and Spherically symmetric condition

Before, we have seen the classical double copy between the Schwarzschild's solution in GR and Coulomb's electrostatic solution in $U(1)$ Maxwell electromagnetism for which that both theories have static and spherically symmetric conditions. In searching the possible classical double copy relation between SG and BI, we ought to see the pattern of the solutions in simple case by imposing the static and spherically symmetric condition. By static we mean that the field does not depend on the timelike coordinate. By spherically symmetric we mean the field only depends only on the radial direction r. In section 2, we have actually solved for BI theory. The $\vec{E}$ field is proportional to $\frac{1}{\sqrt{1+r^4}}$ and the timelike component of the vector potential A^0 is expressed as Jacobian elliptical integral. In this section, we will use the perturbative approach to find the solution of the field equations of BI and SG schematically. The goal here is not to solve the field equations exactly but is to find the pattern of the solutions and see the similarity between them.

Recall that the action of BI is

$$S_{BI} = \frac{1}{b^2}\int d^4x\sqrt{-det(\eta_{\mu\nu} + bF_{\mu\nu})} \tag{5.56}$$

$$= \frac{1}{b^2}\int d^4x\sqrt{1 + b^2F_{\mu\nu}F^{\mu\nu}} \tag{5.57}$$

$$= \frac{1}{b^2}\int d^4x[\frac{1}{b^2} + \frac{1}{2}F^2 - \frac{b^2}{8}F^4 + \frac{b^4}{16}F^6 + ...] \tag{5.58}$$

Note that the term $det(F_{\mu\nu})$ after expanding out the determinant form is neglected because $det(F_{\mu\nu}) \propto \vec{B}\cdot\vec{E} = 0$ since there is no magnetic field $\vec{B}$ in electrostatic case. We let the dimensional correction parameter b^2 to be $\frac{\alpha}{\Lambda^4}$. The dimension of Λ , $[\Lambda] = 1$. The dimension here refers to the dimension of mass. We also note that $[\partial] = 1$ and $[A^\nu] = 1$. The field strength $F_{\mu\nu}$ scales as ∂A, so $[F_{\mu\nu}] = 2$. In the schematic way, the BI action is

$$S_{BI} = \int d^4x[\frac{\Lambda^4}{\alpha} + \frac{1}{2}(\partial A)^2 - \frac{\alpha}{8\Lambda^4}(\partial A)^4 + \frac{\alpha^2}{16\Lambda^8}(\partial A)^6 + ...] \tag{5.59}$$

To find the equation of motion, we vary the action with respect to the vector potential field A^μ and require that $\delta S = 0$

$$\delta S = \int d^4x[\partial A \partial \delta A - \frac{4\alpha}{8\Lambda^4}\partial(\partial A)^3 \delta A + ...] \tag{5.60}$$

The corresponding schematic equation of motion is

$$\partial^2 A - \frac{4\alpha}{8\Lambda^4}\partial(\partial A)^3 + ... = 0 \tag{5.61}$$

We let the solution of (5.61) to be a perturbative series

$$A \sim A^{(0)} + \frac{1}{\Lambda^4}A^{(1)} + \frac{1}{\Lambda^8}A^{(2)} + ... \tag{5.62}$$

There is actually coefficient assigned to each terms for the full solution. Since we are only interested about the pattern of the solution, we do not pay attention to the coefficients now. Substituting (5.62) into (5.61) and collect terms by terms with respect to the parameter $\frac{1}{\Lambda^n}$. We show here the first two terms corresponding to $\frac{1}{\Lambda^0}$ and $\frac{1}{\Lambda^4}$

$$\partial^2 A^{(0)} = 0 \tag{5.63}$$

$$\partial^2 A^{(1)} = -\frac{\alpha}{2}\partial(\partial A^{(0)})^3 + ... \tag{5.64}$$

Note that since the metric used here is the standard Minkowski metric, we fix the background coordinate to be Cartesian-like. Although we are working in the schematic way, the exact form (5.63) is actually $\partial^2 A^{(0)} = 0$ with only one d'Alembertian involved. This can be checked easily by using the exact form of action to derive the equation of motion. Since the static condition is considered, d'Alembertian operator can be reduced to Laplacian operator. So, the leading term of the solution $A^{(0)}$ is

$$A^{(0)} \sim \frac{1}{r} \tag{5.65}$$

The subleading terms can be found by considering dimensional analysis. In (5.64), the term $\partial(\partial A^{(0)})^3 \sim \frac{1}{r^7}$, so, $A^{(1)}$ on the left hand side has to scale with $\frac{1}{r^5}$. Doing the same for the other subleading terms, we find the pattern of the solution which is the alternative expression of the scalar potential $A^0 = \phi$ as Jacobian elliptic integral provided in section 3.1.5.

$$A^0 \sim \frac{1}{r} + \frac{1}{\Lambda^4}\frac{1}{r^5} + \frac{1}{\Lambda^8}\frac{1}{r^9} + ... \tag{5.66}$$

In the electrostatic case, the gauge fixing is such that the spatial component of A^μ vanish. So, the only non trivial component is A^0. The full solution of (5.66) with the correct coefficient for each term is an asymptotic expansion of the exact solution expressed in the integral form. In the weak field limit for $r \to \infty$, (5.66) is well approximated to $A^0 \sim \frac{1}{r}$ in Maxwell theory.

5.3.2 The Schematic solution of SG for Static and Spherically symmetric condition

Now, we turn our attention to the Special Galileon theory (SG). We want to do the same as to find the pattern of the field equation in static and spherically condition. Schematically, the Lagrangian density for special Galileon in $D = 4$ is the sum of the quadratic and quartic Galileon term

$$S_{SG} = \int d^4x[(\partial\phi)^2 - \frac{\alpha}{6\Lambda^6}(\partial\phi)^2(\partial\partial\phi)^2] \tag{5.67}$$

Vary the action with respect to the Galileon field ϕ

$$\delta S_{SG} = \int d^4x[2(\partial\phi)\partial\delta\phi - \frac{\alpha}{12\Lambda^6}(2\partial\phi\partial\delta\phi)(\partial\partial\phi)^2 - \frac{\alpha}{12\Lambda^6}(\partial\phi)^2 2.\partial\partial\phi\partial\partial\delta\phi] \tag{5.68}$$

This leads to the schematic equation of motion by requiring $\frac{\delta S_{SG}}{\partial\phi} = 0$

$$\partial^2\phi - \frac{\alpha}{6\Lambda^6}(\partial^2\phi)^3 = 0 \tag{5.69}$$

The first term is exactly the d'Alembertian in the full solution. However, the second term in (5.69) contains a lot of different terms with the same form but with the indices contracted in different ways. We again assume the solution ϕ takes the perturbative form

$$\phi \sim \phi^{(0)} + \frac{1}{\Lambda^6}\phi^{(1)} + \frac{1}{\Lambda^{12}}\phi^{(2)} + ... \tag{5.70}$$

Of coarse there are coefficients for each term in (5.70) for the full and exact solution. Substituting (5.70) into (5.69) and we collect term by with respect to the parameter $\frac{1}{\Lambda^n}$. We show three terms here.

For $\frac{1}{\Lambda^0}$:

$$\partial^2\phi^{(0)} = 0 \tag{5.71}$$

For $\frac{1}{\Lambda^6}$:

$$\partial^2\phi^{(1)} = \alpha(\partial^2\phi^{(0)})^3 \tag{5.72}$$

For $\frac{1}{\Lambda^{12}}$:

$$\partial^2\phi^{(2)} = \alpha(\partial^2\phi^{(0)})^2\partial^2\phi^{(1)} \tag{5.73}$$

(5.71) implies that $\phi^{(0)} \sim \frac{1}{r}$. By considering dimensional analysis, we find that $\phi^{(1)} \sim \frac{1}{r^7}$ and $\phi^{(2)} \sim \frac{1}{r^{13}}$. Therefore, the pattern of the solution is found to be

$$\phi \sim \frac{1}{r} + \frac{1}{\Lambda^6}\frac{1}{r^7} + \frac{1}{\Lambda^{12}}\frac{1}{r^{13}} + ... \tag{5.74}$$

5.3.3 Perturbative Gravity

We would also like to see the pattern of the solution of general relativity by considering the perturbative aspect of gravity. The main benefit of expressing gravity in perturbative way is that the role of graviton $h_{\mu\nu}$ is obvious. Here, we fix the background coordinate system to be Cartesian-like. First, we write the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{5.75}$$

We will at some point crank the dimensional correction parameter κ to 1 so we do not need to keep track of it and we can simply recover κ later by considering dimensional analysis after we obtain the desired form of the gravity. Since general relativity is invariant under general coordinate transformation. In infinitesimal form, the transformation is written as

$$x^\mu \to x^\mu + \kappa\epsilon^\mu(x) \tag{5.76}$$

For λ is infinitesimal. From section 2.3.2, we have seen that the variation of the metric induced by diffeomorphism is expressed as the form of Lie derivative

$$g_{\mu\nu} \to g_{\mu\nu} + \kappa(\epsilon^\alpha\partial_\alpha g_{\mu\nu} + g_{\alpha\nu}\partial_\mu\epsilon^\alpha + g_{\mu\alpha}\partial_\nu\epsilon^\alpha) \tag{5.77}$$

$$= g_{\mu\nu} + \kappa\partial_\mu\epsilon_\nu + \kappa\partial_\nu\epsilon_\mu \tag{5.78}$$

In the second equality line, all second and higher order terms of κ is neglected. Since Minkowski metric $\eta_{\mu\nu}$ is constant, (5.78) implies that

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \tag{5.79}$$

(5.79) is the gauge transformation in general relativity and the gauge symmetry of general relativity is induced by diffeomorphism invariance of the theory. The inverse of the metric takes the exact form as

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^\mu_\lambda h^{\lambda\nu} + ... \tag{5.80}$$

One can check that $g_{\mu\nu}g^{\nu\alpha} = \delta^\alpha_\mu$. Now, we want to expand Einstein-Hilbert action in powers of $h_{\mu\nu}$. At this point, we will crank κ to 1. The Lagrangian density of Einstein-Hilbert action is

$$\mathcal{L}_{EH} = \sqrt{-g}R \tag{5.81}$$

for g is the determinant of the metric tensor and R is Ricci scalar. Using (5.75) and (5.80), the expansion of $\sqrt{-g}R$ can be found schematically to be

$$\sqrt{-g}R = (1 + h + h^2 + ...)(\partial\partial h + h\partial\partial h + ...) \tag{5.82}$$

$$= \partial\partial h + h\partial\partial h + h^2\partial\partial h + h^3\partial\partial h + ... \tag{5.83}$$

The action can then be expressed as

$$S = \int d^4x[(\partial h)^2 + \kappa h(\partial h)^2 + \kappa^2 h^2(\partial h)^2 + ...] \tag{5.84}$$

Note that we have recovered κ in (5.84) to correct the dimension in each term. Note also that each term is just a schematic representation and one will find a mess of indices contracted in different way if deriving explicitly. The first term $\partial\partial h$ in (5.83) can be neglected in the action because it represents a total derivative term. (5.84) is obtained from (5.83) by simply integrating by part and neglect the total derivative terms. κ is defined as

$$\kappa = \frac{1}{M_{pl}}; M_{pl} \approx 2 \times 10^{18} GeV \tag{5.85}$$

M_{pl} is the Planck mass. In order to get the pattern of the solution of the equation of motion from (5.84), we need to at least know explicitly the first leading term (quadratic of h). We first expand $\sqrt{-g}$ first

$$\sqrt{-g} = \sqrt{-det(\eta_{\mu\nu} + h_{\mu\nu})} \tag{5.86}$$

$$= exp\frac{1}{2}ln[-det(\eta_{\mu\nu} + h_{\mu\nu})] \tag{5.87}$$

$$= exp[\frac{1}{2}ln(det(1 + h^\mu_\nu))] \tag{5.88}$$

$$= exp[\frac{1}{2}trln(1 + h^\mu_\nu)] \tag{5.89}$$

$$= exp[\frac{1}{2}tr(h^\mu_\nu - \frac{1}{2}h^2 + ...)] \tag{5.90}$$

$$= 1 + \frac{1}{2}tr(h^\mu_\nu) - \frac{1}{4}tr(h^2) + \frac{1}{2}(\frac{1}{2}tr(h^\mu_\nu) - \frac{1}{4}tr(h^2)^2) + \mathcal{O}(h^3) \tag{5.91}$$

$$= 1 + \frac{1}{2}h^\mu_\mu - \frac{1}{4}h^{\mu\nu}h_{\mu\nu} + \frac{1}{8}(h^\mu_\mu)^2 + \mathcal{O}(h^3) \tag{5.92}$$

The linearized term of Ricci tensor is found to be

$$R^{(1)}_{\mu\nu} = \partial_\mu\partial_\nu h + \partial_\lambda\partial^\lambda h_{\mu\nu} - \partial_\mu\partial_\lambda h^\lambda_\nu - \partial_\lambda\partial_\nu h^\lambda_\mu \tag{5.93}$$

Therefore, the first leading term can be obtained explicitly by careful derivation using (5.80), (5.92) and (5.93)

$$\sqrt{-g}g^{\mu\nu}R^{(1)}_{\mu\nu} = \frac{1}{2}\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu} - \frac{1}{2}\partial^\alpha h\partial_\alpha h - \partial_\mu h^{\mu\nu}\partial_\nu h - \partial_\mu h^{\mu\nu}\partial_\lambda h^\lambda_\nu + ... \tag{5.94}$$

If we are only interested about linearized gravity, the corresponding action is

$$S_{FP} = \int d^4x(\frac{1}{2}\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu} - \frac{1}{2}\partial^\alpha h\partial_\mu h^{\mu\nu}\partial_\nu h - \partial_\mu h^{\mu\nu}\partial_\nu h - \partial_\mu h^{\mu\nu}\partial_\lambda h^\lambda_\nu) \tag{5.95}$$

(5.95) is the massless Fierz-Pauli action. h in (5.95) refers to the trace of $h_{\mu\nu}$ This action describes the linearized gravity with massless graviton in the weak field limit. One can then just simply use the variational technique to obtain the equation of motion. The equation of motion is found to be

$$\frac{\delta S}{\delta h^{\mu\nu}} = \partial^2 h_{\mu\nu} - \partial_\lambda\partial_\mu h^\lambda_\nu - \partial_\lambda\partial_\nu h^\lambda_\mu + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^2 h = 0 \tag{5.96}$$

To simplify the equation of motion, we use a constraint to fix the gauge freedom. The constraint is

$$\partial^\mu h_{\mu\nu} - \partial_\nu h = 0 \tag{5.97}$$

Plugging (5.97) into (5.96), the equation of motion is simplified to

$$\partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h = 0 \tag{5.98}$$

To further simplify the equation of motion, we can further fix the gauge subjected to (5.97), namely transverse traceless gauge

$$\partial^\mu h_{\mu\nu} = 0; h = 0 \tag{5.99}$$

So, we obtain the simple equation of motion after fixing the gauge

$$\partial^2 h_{\mu\nu} = 0 \tag{5.100}$$

In the static and spherically symmetric condition, the solution of (5.100) takes the form $h_{\mu\nu} \sim \frac{1}{r}$. Now, we return to the perturbative gravity case where the action takes the form (5.84). We can again vary this action with respect to $h_{\mu\nu}$ to obtain the schematic equation of motion. We assume the solution takes the perturbative form

$$h_{\mu\nu} = h^{(0)}_{\mu\nu} + \kappa h^{(1)}_{\mu\nu} + \kappa^2 h^{(2)}_{\mu\nu} + \kappa^3 h^{(3)}_{\mu\nu}.... \tag{5.101}$$

Since we have found that $h^{(0)}_{\mu\nu} \sim \frac{1}{r}$, we can follow the same procedure as in the case for BI and SG to obtain the pattern of the subleading terms by considering dimensional analysis. Therefore, the pattern of the solution is found to be

$$h_{\mu\nu} \sim \frac{1}{r} + \frac{1}{M_{pl}}\frac{1}{r^2} + \frac{1}{M^2_{pl}}\frac{1}{r^3} + ... \tag{5.102}$$

If we assume parity symmetry, the terms $h^n(\partial h)^2$ in (5.84), for n is odd will not contribute and can be neglected in the action. Then, the pattern of the solution becomes

$$h_{\mu\nu} \sim \frac{1}{r} + \frac{1}{M^2_{pl}}\frac{1}{r^3} + \frac{1}{M^4_{pl}}\frac{1}{r^5} + ... \tag{5.103}$$

With this argument, we see that the series expression of the solution of perturbative gravity in static and spherically symmetric condition resembles the one in BI theory and SG theory. The solution of these 3 theories can be expressed as the summation of the $\frac{1}{r^k}$ terms with fixed coefficients where k has to be odd. So, with rough calculation by considering dimensional analysis, we see that the solution of these 3 theories are very similar in term of their mathematical expression. Since these 3 theories are connected together by the double copy relation in the level of scattering amplitude and also due to the similarity between their classical solutions, we have enough motivation and excuse to figure out the possible underlying structure that connects the classical solutions of these 3 theories. The first step we can do is to solve the equation of motions exactly to obtain the coefficients for each $\frac{1}{r^k}$ term.

5.3.4 Getting the right coefficients and Hypergeometric Series

The standard procedure to obtain the coefficients in this thesis follows the method explained in the last section. We let the solution to be a series of $\frac{1}{r^k}$ and substitute this series into the equation of motion. Then, we collect term by term with respect to the order of the dimensional correction parameter $\frac{1}{\Lambda^n}$. From here each coefficient can be calculated. In principle, this method is always valid and the computation process can be simplified a lot with the help of computer-aided software (Wofram Mathematica, Matlab, Python, etc). After we have found the coefficients, we need to look for the structure of the series to gain some ideas on the classical double copy relation.

Born Infeld theory

We will start with Born-Infeld theory. For BI, there is a much more simple way to derive the coefficients. We have seen in section 3.1.4 that the Electric field E_r in electrostatic case of BI theory takes the form as (3.74). We do a Taylor expansion of E_r around $\frac{r_0^4}{r^4}$

$$E_r = \frac{-e}{\sqrt{r_0^4 + r^4}} \tag{5.104}$$

$$= -\frac{e}{r^2}[1 + \frac{r_0^4}{r^4}]^{-\frac{1}{2}} \tag{5.105}$$

$$= -\frac{e}{r^2}[1 - \frac{r_0^4}{2r^4} + \frac{3}{8}(\frac{r_0^4}{r^4})^2 - \frac{5}{16}(\frac{r_0^4}{r^4})^3 + ...] \tag{5.106}$$

$$= -\frac{e}{r^2} + \frac{er_0^4}{2r^6} - \frac{3}{8}\frac{er_0^8}{r^{10}} + ... \tag{5.107}$$

In the electrostatic case, the gauge fixing of the electromagnetic field A^μ is such that the spatial components A^i vanish.A^0 which corresponds to the point charge source can be obtained by integrating (5.107) with respect to the radial coordinate r

$$A^0 = \int_\infty^r E_r dr \tag{5.108}$$

$$= \int_\infty^r dr(-\frac{e}{r^2} + \frac{er_0^4}{2r^6} - \frac{3}{8}\frac{er_0^8}{r^{10}} + ...) \tag{5.109}$$

$$= \frac{e}{r} - \frac{er_0^4}{10r^5} + \frac{1}{24}\frac{r_0^8}{r^9} - ..... \tag{5.110}$$

$$= \frac{e}{r}[1 - \frac{r_0^4}{10r^4} + \frac{1}{24}\frac{r_0^8}{r^8} - .....] \tag{5.111}$$

Therefore, we have found the coefficients for the series. $c_0 = 1, c_1 = -\frac{1}{10}, c_2 = \frac{1}{24}$ and so on. It is realised that the parentheses in (5.111) takes the form of Gauss hypergeometric function

$$1 - \frac{r_0^4}{10r^4} + \frac{1}{24}\frac{r_0^8}{r^8} - ..... = {}_2F_1(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}, -\frac{r_0^4}{r^4}) \tag{5.112}$$

The subscript in front of F denotes the number of Pochhammer symbol $(\alpha)_n$ used in the numerator while the subscript behind F denotes the number of Pochhammer symbol used in the denominator. The Pochhammer symbol is defined as

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)....(\alpha+n-1) \tag{5.113}$$

The Pochhammer symbol can also be expressed in term of gamma function

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)....(\alpha+n-1) \times \frac{1 \cdot 2 \cdot 3.....(\alpha-1)}{1 \cdot 2 \cdot 3.....(\alpha-1)} \tag{5.114}$$

$$= \frac{(\alpha+n-1)!}{(\alpha-1)!} \tag{5.115}$$

$$= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \tag{5.116}$$

The Hypergeometric function is defined as

$$F(\alpha,\beta;\gamma;x)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{x^k}{k!} \tag{5.117}$$

$$=1+\frac{\alpha\beta}{\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!}+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!}+.... \tag{5.118}$$

Using (5.114), one can confirm that the structure of the series in (5.111) is indeed governed by Gauss hypergeometric series. Thus, the exact solution of A^0 is determined. We know that Born-Infeld theory can be approximated to Maxwell theory ($A^0\sim\frac{1}{r}$) in the weak field limit. We can see this by considering $r\to\infty$, the series is indeed dictated by the first leading term $\sim\frac{1}{r}$. With all the correction terms involved, BI theory turns out to be non-singular as discussed in section 2.2.3.

Special Galileon theory

We will follow the standard procedure to derive the coefficient of the solution of Special Galileon theory in static and spherically symmetric condition. In $D=4$, the action of special galileon is the sum of the quadratic and quartic Galileon terms taking the expression as (4.137). Using variational technique and least action principle, the equation of motion is [16]

$$\partial^2\phi+\frac{1}{\Lambda^6}[(\partial^2\phi)^3-3(\partial^2\phi)\partial_\alpha\partial_\beta\phi\partial^\alpha\partial^\beta\phi+2\partial_\alpha\partial^\beta\phi\partial_\beta\partial^\gamma\phi\partial_\gamma\partial^\alpha\phi]=0 \tag{5.119}$$

(5.119) can be simplified further to a differential equation with only radial coordinate, $r=\sqrt{x^ix_i}$ involved under static and spherically symmetric condition where $\phi=\phi(r)$. This can be seen by using tensor calculus. We note that

$$\frac{\partial r}{\partial x^\alpha}=\frac{x_\alpha}{r};\frac{\partial r}{\partial x_\alpha}=\frac{x^\alpha}{r};x^\alpha x_\alpha=r^2 \tag{5.120}$$

In this case all indices take $1,2,3$ because we are working under static condition as ϕ only depends on spatial coordinates. We calculate $\partial^2\phi$ first

$$\partial^\alpha\partial_\alpha\phi(r)=\partial^\alpha[\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x^\alpha}] \tag{5.121}$$

$$=\partial^\alpha(\frac{\partial\phi}{\partial r})\frac{x_\alpha}{r}+\frac{\partial\phi}{\partial r}\partial^\alpha(\frac{x_\alpha}{r}) \tag{5.122}$$

$$=\frac{\partial^2\phi}{\partial r}\frac{\partial r}{\partial x_\alpha}\frac{x_\alpha}{r}+\frac{\partial\phi}{\partial r}[\frac{3}{r}-\frac{1}{r}] \tag{5.123}$$

$$=\frac{\partial^2\phi}{\partial r^2}\frac{x^\alpha x_\alpha}{r^2}+\frac{\partial\phi}{\partial r}[\frac{2}{r}] \tag{5.124}$$

$$=\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r} \tag{5.125}$$

$$=\phi''(r)+\frac{2}{r}\phi'(r) \tag{5.126}$$

Then, we calculate $(\partial^2\phi)^3$

$$(\partial^\alpha\partial_\alpha\phi)^3=[\frac{r\phi''+2\phi'}{r}]^3 \tag{5.127}$$

$$=(\phi'')^3+\frac{6}{r}\phi'(\phi'')^2+\frac{12\phi'^2\phi''}{r^2}+\frac{8\phi'^3}{r^3} \tag{5.128}$$

To calculate the third term $-3(\partial^2\phi)\partial_\alpha\partial_\beta\phi\partial^\alpha\partial^\beta\phi$, we first notice that

$$\partial_\alpha\partial_\beta\phi = \partial_\alpha[\frac{\partial\phi}{\partial r}\frac{x_\beta}{r}] \tag{5.129}$$

$$= \frac{\partial^2\phi}{\partial r^2}\frac{x_\alpha x_\beta}{r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}\delta_{\beta\alpha} - \frac{x_\alpha x_\beta}{r^3}\frac{\partial\phi}{\partial r} \tag{5.130}$$

For $\partial^\alpha\partial^\beta\phi$, we just raise the indices on the right hand side of (5.130). Then, we can compute

$$\partial_\alpha\partial_\beta\phi\partial^\alpha\partial^\beta\phi = \frac{2(\phi')^2}{r} + (\phi'')^2 \tag{5.131}$$

Using (5.126) and (5.131), we can calculate

$$-3(\partial^2\phi)\partial_\alpha\partial_\beta\phi\partial^\alpha\partial^\beta\phi = -3[\phi'' + \frac{2}{r}\phi'][(\phi'')^2 + \frac{2}{r}(\phi')^2] \tag{5.132}$$

$$= -\frac{12(\phi')^3}{r^3} - \frac{6\phi'(\phi'')^2}{r^2} - \frac{6\phi'(\phi'')^2}{r} - 3(\phi'')^2 \tag{5.133}$$

For the last term $2\partial_\alpha\partial^\beta\phi\partial_\beta\partial^\gamma\phi\partial_\gamma\partial^\alpha\phi$, it is simplified into

$$2\partial_\alpha\partial^\beta\phi\partial_\beta\partial^\gamma\phi\partial_\gamma\partial^\alpha\phi = 2(\phi'')^3 + \frac{4(\phi')^3}{r^3} \tag{5.134}$$

We arrange everything in the order of (5.119) and we find the simplified equation of motion

$$\phi'' + \frac{2}{r}\phi' + \frac{1}{\Lambda^6}[\frac{6(\phi')^2\phi''}{r^2}] = 0 \tag{5.135}$$

With this equation of motion, we can figure out the coefficients of the solution as a series. We let the solution be $\phi = \sum_{n=0}^{\infty}\frac{\phi^{(n)}}{\Lambda^{6n}}$ and we substitute this into (5.135). We then collect the terms with respect to $\frac{1}{\Lambda^{6n}}$.

For $\frac{1}{\Lambda^0}$:

$$\frac{d^2\phi^{(0)}}{dr^2} + \frac{2}{r}\frac{d\phi^{(0)}}{dr} = \nabla^2\phi = 0 \implies \phi^{(0)} = \frac{1}{r} \tag{5.136}$$

For $\frac{1}{\Lambda^6}$:

$$\frac{2}{r}\frac{d}{dr}[\phi^{(1)}] + \frac{d^2}{dr^2}[\phi^{(1)}] = -\frac{6}{r^2}[\frac{d}{dr}(\phi^{(0)})]^2\frac{d^2}{dr^2}[\phi^{(0)}] \tag{5.137}$$

Since we know that $\phi^{(0)} = \frac{1}{r}$ and by dimensional analysis, we know that $\phi^{(1)} = \frac{c_1}{r^7}$. Substituting these expression into (5.137), we have

$$\frac{2c_1}{r}[-\frac{7}{r^8}] + c_1\frac{56}{r^9} = -\frac{12}{r^9} \tag{5.138}$$

$$\frac{42c_1}{r^9} = -\frac{12}{r^9} \implies c_1 = -\frac{2}{7} \tag{5.139}$$

For $\frac{1}{\Lambda^{12}}$:

$$\frac{2}{r}\frac{d}{dr}[\frac{c_2}{r^{13}}] + \frac{d^2}{dr^2}[\frac{c_2}{r^{13}}] = -\frac{12}{7}[\frac{d}{dr}(\frac{1}{r})\frac{d}{dr}(-\frac{2}{7r^7})][\frac{d^2}{dr^2}(\frac{1}{r})] - \frac{6}{r^2}[\frac{d}{dr}(\frac{1}{r})]^2\frac{d^2}{dr^2}[-\frac{2}{7r^7}] \tag{5.140}$$

$$\frac{156c_2}{r^{15}} = \frac{144}{r^{15}} \implies c_2 = \frac{12}{13} \tag{5.141}$$

Iteratively, we can use this procedure to find all coefficients. The coefficients are found to be $c_0 = 1, c_1 = -\frac{2}{7}, c_2 = \frac{12}{13}, c_3 = -\frac{96}{19}, c_4 = \frac{880}{25}, ...$ It is realised again that this series can be expressed in term of a generalized hypergeometric series

$$\phi = \frac{1}{r} - \frac{2}{7\Lambda^6}\frac{1}{r^7} + \frac{12}{13\Lambda^{12}}\frac{1}{r^{13}} + ... \tag{5.142}$$

$$= \frac{1}{r}{}_3F_2(\frac{1}{6}, \frac{1}{3}, \frac{2}{3}; \frac{7}{6}, \frac{3}{2}; \frac{-27}{2\Lambda^6}\frac{1}{r^6}) \tag{5.143}$$

General Relativity and Remarks

The usual algorithm used to find the coefficients is not very effective for the solution of general relativity. The main reason is that the counterpart of the solution of general relativity in classical double copy is not known and obvious yet. There is also technical difficulty due to the highly non-linear structure that general relativity is based on. One can already see that in perturbative gravity, the indices mess even in the linear term. In spite of the difficulties, the solution of the field in BI and SG which inherits the same structure as $\frac{1}{r}\times$ hypergeometric series in static and spherically symmetric condition shed light on the possible classical double copy solution of the general relativity. The correction terms in hypergeometric series help to eliminate the singularity in SG and BI in a coordinate system that is isotropic and static. This implies that the counterpart solution of general relativity in an isotropic and static coordinate system should also take the form of $\frac{1}{r}\times$ hypergeometric series for which the curvature singularity at $r = 0$ is eliminated. The exact mapping relation is also worth to study and one should look into this from the perspective of double copy in term of scattering amplitude.

6 Discussion

As a summary, the volume of a manifold with its geometry governed by the metric $g_{\mu\nu}$ can be calculated using the coordinate integral of the invariant volume form, $\int d^D x\sqrt{-g}$. For a map ψ that links together 2 manifolds M and N, the pushforward of a vector $\psi_* V$ and the pullback of a one-form $\psi^*\omega$ can be defined. For the case that a submanifold is embedding within a background manifold, the pullback of the metric tensor of the background manifold defines the induced metric which plays the role to measure the distance on the submanifold. A map is a diffeomorphism if its inverse exists. For M and N be the same manifold, the pullback and the pushforward induced by a diffeomorphic map represents the active coordinate transformation. The infinitesimal variation of any tensor induced by diffeomorphism can be expressed in the form of Lie derivative.

The Lagrangian density of the modified electromagnetism proposed by Born and Infeld taking the form $\mathcal{L} = \sqrt{-det(g_{\mu\nu} + F_{\mu\nu})} - \sqrt{-det(g_{\mu\nu})}$ satisfies the principle of finiteness and it can be approximated to the Maxwell's theory if the spacetime is the Minkowski space and the field strength is weak. Born-Infeld theory serves as a non-linear theory of electromagnetism with specific self-interactions of photons. The action for Einstein general relativity is Hilbert-Einstein action for which its Lagrangian takes the form $\sqrt{-g}R$. Using the idea that gravity is the manifestation of the curvature of the spacetime, the appropriate geometrical tensor in the gravitational field equation which is known to have the second derivative of metric involved from the Newtonian limit has to be the Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ since it obeys Bianchi identity. This requirement is essential for the conservation of energy-momentum to hold. Schwarzschild's metric describes a static and spherically symmetric spacetime with the situation of the empty space surrounding a spherical body.

The point particle can be viewed as a 0-brane and this concept can be generalized to higher dimensional object which is called p-brane. p-brane is parametrized by the worldvolume coordinate. The action of p-brane which is called Nambu-Goto action which takes the form of the square root of the induced metric is invariant under any choice of the worldvolume coordinate system. The

symmetry due to the freedom of the choice of coordinate system is called reparametrization symmetry and it is a gauge symmetry. 1-brane acts as a bosonic string and its action is invariant under global Poincaré trnasformation, local Weyl transformation and local reparametrization transformation. By invoking an auxiliary field $h_{\alpha\beta}$, Nambu Goto action can be shown to be equivalent to Polyakov action. In the simple case that assuming the topology of worldsheet allows gauge fixed flat intrinsic metric $h_{\alpha\beta} = \eta_{\alpha\beta}$ to be extended globally, the equation of motion can be derived under 3 boundary conditions which are closed string case, open string Neumann condition and open string Dirichlet condition. In Neumann boundary condition, the endpoints of the brane are free to move. In Dirichlet boundary condition, the endpoints of the brane are fixed. For a D-dimensional background spacetime, the $D+1$ hypersurface is called D-brane in the situation that p worldvolume spacelike coordinates and the timelike coordinate obey Dirichlet boundary condition while the other $D-P-1$ worldvolume coordinates obey Neumann boundary condition. The action of D-brane is described by Dirac Born Infeld action. The formation of D-brane leads to the symmetry breaking of Poincaré group. Dirac Born- Infeld theory has a natural geometric interpretation as a D-brane fluctuating in the transverse direction. For the case that a D_4 brane embedded in a 5-dimensional Minkowski space, the DBI action is protected by a nonlinearly realised 5 dimensional Poincaré group. The DBI action is found to be invariant under the nonlinearly realised shift and Lorentz transformation in the fifth dimension. The higher dimensional rotation will bring DBI theory out of the static gauge and a compensating world-volume reparametrization is needed to restore the static gauge. The infinitesimal variation of the induced metric can be formulated into Lie derivative induced by a vector field. Thus, this transformation can be viewed as to be induced by diffeomorphism and the scalar field in DBI theory can be minimally coupled with matter field in diffeomorphism invariant way with respect to DBI nonlinearly realised symmetry. The special Galileon theory is formulated as the sum of the galileon terms with even number of field. Beside the galileon symmetry enjoyed by each galileon term, the special galileon action is protected by a higher order shift symmetry. There is also a geometrical origin for Special Galileon theory as a D-brane fluctuation in the transverse direction but a complex geometry with Kahler structure is needed. Thus, the action of Special Galileon can also be written as the square root of the determinant of an induced metric which is the typical form used to study the extrinsic geometry. The infinitesimal variation of this induced metric which is caused by the higher order shift symmetry can also be formulated as a Lie derivative induced by a vector field. Therefore, the galileon field can also be coupled with matter field in a diffeomorphism invariant way with respect to special Galileon symmetry. For Born-Infeld theory, the vector field A_μ only contributes to the antisymmetric part of the induced metric. Since both the symmetrical part and the antisymmetrical part of the induced metric has to transform covariantly under the induced diffeomorphism, there is no nonlinearly realised symmetry. This suggests that the same way to give geometrical interpretation as in DBI theory and special Galileon theory cannot be applied to BI theory.

Under Kerr-Schild geometry, Einstein field equation is solved exactly to be linear. Kerr-Schild ansatz is used to construct the classical double copy relation between biadjoint scalar field theory, Abelian Maxwell theory and general relativity. There is also a classical double copy relation between Schwarzschild's solution and Coulomb's electrostatic solution using Kerr-Schild ansatz. The solution of the Born-Infeld theory and special Galileon theory under spherically symmetric and static condition are solved to take take the form $\frac{1}{r}\times$ hypergeometric series. The similarity between the structure of these 2 solutions suggest a possible classical double relation between them and also general relativity as there is double copy relation between them in scattering amplitude.

7 Conclusion

Conclusively, under the fixed Minkowski background spacetime, the electromagnetic field A_μ in Born-Infeld theory transforms as a vector field under linearly realised Poincare symmetry because A_μ contributes only to the antisymmetrical field strength $F_{\mu\nu}$ of the induced metric and there is only linearly realised Poincaré symmetry that will leave $\eta_{\mu\nu}$ and $F_{\mu\nu}$ to transform covariantly in

a separate way. Therefore, there is no nonlinearly realised symmetry and the interpretation of the field as a brane fluctuating in the transverse direction does not hold in Born-Infeld theory.

Besides that, the classical solution of Born-Infeld theory and special Galileon theory under static and spherically symmetric condition are solved to be expressed as a hypergeometric series. For BI, the solution is $\frac{1}{r}{}_2F_1(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}, -\frac{r_0^4}{r^4})$. For SG, the solution is $\frac{1}{r}{}_3F_2(\frac{1}{6}, \frac{1}{3}, \frac{2}{3}; \frac{7}{6}, \frac{3}{2}; \frac{-27}{2\Lambda^6}\frac{1}{r^6})$. The similarity between the structure of these 2 solution implies a possible classical double copy relation between them. A thorough investigation of the double copy relation from the viewpoint of scattering amplitude and also the classical solutions are needed to understand the duality between Special Galileon theory, Born-Infeld theory and general relativity in a deeper way.

8 Acknowledgement

I want to express my gratitude towards God for sustaining my life up to this point. I am grateful for my parents who work very hard to raise me up. I want to thank my supervisor Professor Diederik Roest for his kindness and guidance throughout the research period. I also want to thank Ivan Kolar and Shubham Maheshwari fot the useful discussion.

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