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Modified Einsteinian Dynamics

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Abstract

The description of the Universe provided by Einstein's General Relativity has proven extremely successful on solar scales, yet it fails on galactic scales and beyond. Several empirical observations indicate this, where the predictions do not fit the observations. Generally, this problem is approached by adding large amounts of unseen matter or gravity to the General Relativity theory, establishing the so-called Cold Dark Matter theory. Nevertheless, this is not the only possible explanation. As it has been stressed by many physicists, another interesting strategy is to change the point of view: what if the theory needs to be modified on these cosmological scales? From this perspective the Modified Newtonian Dynamics (MOND) was born. In this work we will get initiated in this promising interpretation by explaining the first step towards a relativistic formulation of the MOND algorithm.

Acknowledgements

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En especial, gracias a Guille por hacerme la vida más llevadera creciendo a mi lado. Tu valentía y determinación me animan a seguir.

Contents

Chapter 1

Introduction

Einstein's theory of General Relativity has proved extremely successful over the years at solar scales. In spite of this, its description of the Universe on the largest scales fails, motivating a still open discussion. Not only General Relativity, Newtonian dynamics fails too on these cosmological scales. This failure is manifested, for instance, in the rotation curves for spiral galaxies that are observed to be asymptotically flat in contradiction with the prediction (see Figure 1.1), alongside with the baryonic Tully-Fisher law, which dictates that asymptotic rotational velocity of a galaxy is $L \propto v^4$, as some examples.

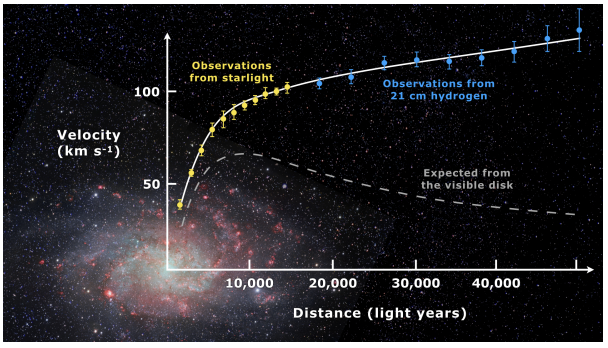

Figure 1.1: Rotation curve of spiral galaxy Messier 33, *source:* [27]. In this graphic it can be seen that the expected velocity curve by the Newtonian dynamics does not correspond with the actual observations. The empirical rotation curve is asymptotically flat.

One of the most popular solutions proposed is the dark matter theory, which is required by the General Relativity (GR) field equations. These equations imply that about 96% of the Universe is formed by energy densities that do not interact electromagnetically, that is to say, they do not couple to light and thus, we cannot see it, hence the name 'dark'. The theory of the *cold dark matter*[1], CDM, arises to solve this problem and many others and it is supposed to consist of weakly interacting cold matter, mostly non-baryonic. Despite its widespread fame, it has several flaws. For instance, some of the predictions of the CDM theory are not observed (as the cuspy halo problem or the missing satellites problem[2].). Additionally, there are several relevant galactic observations, as the Tully-Fisher law for spiral galaxies or the

[1]It is *cold* since it moves slowly compared to the speed of light [25].

[2]Since this is not the topic of this article, here there are some suggestions for further reading about it: [14] and [17]

relation between luminous and dynamical mass, that are not well explained by the standard CDM model. Also, probably one of the most important weak points of this proposal is the fact that even thought there are many candidates, there is no particle for dark matter discovered yet. Consequently, it is reasonable to contemplate other alternatives: instead of looking for a component such as the dark matter for which we do not have evidence yet, we could modify the theory of gravity on the scales where it fails.

From this idea of modifying gravity on these cosmological scales arises the so-called *MOdified Newtonian Dynamics* or MOND for short, by Mordehai Milgrom [6]. This alternative was originally born by implementing an algorithm designed for solving this discrepancy between the Newtonian dynamical mass and the directly observable mass. The algorithm proposed is a deviation from Newton's law (hence the name) that appears in the regime of low acceleration and it can be viewed as a modification of inertia, (1.1), or as a modification of gravity, (1.2):

$$\mathbf{F} = m\mathbf{a}\mu\left(a/a_0\right) \tag{1.1}$$

$$\mathbf{g}_n = \mathbf{g}\mu\left(|g|/a_0\right) \tag{1.2}$$

The above equations include an unknown function, μ, which is required to have an asymptotic behaviour so that it recovers Newtonian dynamics outside from the low acceleration regime:

$$\mu(x) = \begin{cases} x & if \quad x \ll 1 \\ 1 & if \quad x \gg 1 \end{cases}$$

Thus, in the MOND regime, we would get, for the case of the modification of the gravity, the effective gravitational force: $g = \sqrt{g_n a_0}$, where a_0 is a new physical constant with the units of acceleration.

Whereas it is a simple rule, it succeeds at fitting a great number of spiral galaxy observations, as well as predicting the Tully-Fisher law (which is not described by the CDM theory) among many other phenomenological evidences. As some studies about Einsteir-æther theory[3] have shown [18], even though theoretically we cannot assume that the phenomenological MOND theory can reproduce all the systematics of Rotational Curves observations, it is true that the MOND model fits better than CDM based mass models.

Regardless of its remarkably experimental success, there is still a physical basis needed for this rather simple algorithm. This is still an open matter since the various theories proposed fail at some point, and therefore make MOND in its original form clearly incomplete.

In this paper we will investigate some of the proposed trials, explaining their premises and their weak points. Concretely, we are interested in scalar-tensor theories of General Relativity and their application to the MOND formulation. Scalar-tensor theories of gravity are typically presented as the alternative theories of gravity. For the topic we want to study, we will examine a scalar-tensor theory of GR by implementing a conformal change in the Einstein metric as a first step towards a relativistic description of MOND. This approach is known as the *Relativistic AQUAdratic Lagrangian* or *RAQUAL*, yet there are many others, as we will present shortly.

[3]This kind of theory is a generally covariant modification of General Relativity. It describes a spacetime endowed with both a metric and a unit timelike vector field named the aether, presenting a preferred reference frame and thus, violating Lorentz invariance.

Chapter 2

General Relativity

The first step towards the study of the possible theoretical basis of MOND is to understand the standard formulation of General Relativity. One can define three geometrical objects (curvature, torsion and non-metricity) out of the metric tensor $g_{\mu\nu}$ and the connection $\Gamma^{\alpha}_{\mu\nu}$ to classify geometries. As we already mentioned, the standard geometrical formulation of General Relativity by Einstein is actually one of the three possible interpretations of gravity that one can make. In this common formulation, gravity is due to curvature of spacetime, setting to zero the torsion and non-metricity, but one can also consider a flat spacetime with torsion or with non-metricity to describe the same underlying physics. In this work we will first investigate the usual interpretation of General Relativity. Thus, a mathematical framework is needed to define this theory. In order to work with curved spacetime, we need to introduce curved manifolds (Riemann geometry) and to study mathematical objects in them.

2.1 Mathematical framework

A manifold, in general, is a topological space whose importance to us now is that locally it "looks like" a n-dimensional Lorentzian[1] space $\mathbb{R}^n$; that is to say that, for example, even if the manifold is curved, in the neighbourhood of a point it would look flat. In such topological spaces we can define objects such as tensors, whose language is needed in GR. Let us then introduce such objects and see how we can define relevant quantities with them.

2.1.1 Coordinate transformation

One of the main points of tensors is that tensorial equations hold in all coordinate systems. Thus, an important topic to study is how quantities behave under transformation of coordinate systems.

If we have a coordinate system u^i so that the covariant coordinate basis[2] is $\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u^i}$ and we transform to a new coordinate system u'^i, then the new coordinate basis can be expressed

[1]This definition works for both Euclidean and Lorentzian spaces, but we will focus on Lorentzian ones

[2]With $\mathbf{r}(u_1, u_2, u_3)$ the position vector of a point P

by the chain rule:

$$\mathbf{e}_j = \frac{\partial \mathbf{r}}{\partial u'^i}\frac{\partial u'^i}{\partial u^j} = \frac{\partial u'^i}{\partial u^j}\mathbf{e}'_i \tag{2.1}$$

for covariant vectors. For contravariant ($\mathbf{e}^i = \nabla u^i$):

$$\mathbf{e}'^j = \frac{\partial u'^j}{\partial u^i}\mathbf{e}^i \tag{2.2}$$

Now, let's define the so-called *tensors*. A tensor of order (p, q) is a multilineal application of $E^p \times (E^*)^q$ in $\mathbb{R}$ and hence it has components:

$$T^{i_1 i_2 \ldots i_q}_{j_1 j_2 \ldots j_p}$$

p times covariant and q times contravariant. Then, analogously to the cases for vectors in (2.1) and (2.2), a second order tensor can transform as:

$$T'^{ij} = \frac{\partial u'^i}{\partial u^k}\frac{\partial u'^j}{u^l}T^{kl} \tag{2.3}$$

$$T'^i_{\ j} = \frac{\partial u'^i}{\partial u^k}\frac{\partial u^l}{u'^j}T^k_l \tag{2.4}$$

$$T'_{ij} = \frac{\partial u^k}{\partial u'^i}\frac{\partial u^l}{\partial u'^j}T_{kl} \tag{2.5}$$

For contravariant, mixed and covariant components respectively. The generalisation to generic (mixed)[3] tensors now follows easily:

$$T'^{ij\ldots k}{}_{lm\ldots n} = \frac{\partial u'^i}{\partial u^a}\frac{\partial u'^j}{\partial u^b}\cdots\frac{\partial u'^k}{\partial u^c}\frac{\partial u^d}{\partial u'^l}\frac{\partial u^e}{\partial u'^m}\cdots\frac{\partial u^f}{\partial u'^n}T^{ab\ldots c}{}_{de\ldots f} \tag{2.6}$$

Let's now define a *relative tensor*. When we perform a general coordinates transformation, as we have seen above, from u^i to u'^i, we can define the transformation matrix, whose determinant is the so-called *Jacobian*:

$$J = \left|\frac{\partial u'}{\partial u}\right|, \qquad \frac{1}{J} = \left|\frac{\partial u}{\partial u'}\right| \tag{2.7}$$

Therefore, a relative tensor of weight w is defined as a tensor which transforms as it follows:

$$T'^{ij\ldots k}{}_{lm\ldots n} = \frac{\partial u'^i}{\partial u^a}\frac{\partial u'^j}{u^b}\cdots\frac{\partial u'^k}{\partial u^c}\frac{\partial u^d}{\partial u'^l}\frac{\partial u^e}{u'^m}\cdots\frac{\partial u^f}{\partial u'^n}T^{ab\ldots c}{}_{de\ldots f}\left|\frac{\partial u}{\partial u'}\right|^w \tag{2.8}$$

Thus, (2.6) is the case where $w = 0$, which are usually called *true* or *absolute tensors*. For $w = -1$ we have a **pseudotensor** and $w = 1$, a tensor density.

[3] A covariant tensor is a "mixed" tensor of order $(p, 0)$ while a contravariant tensor is of order $(0, q)$.

2.1.2 Metric

Now that we've seen what is a tensor and how it transforms, we can define the *metric tensor* as it follows:

$$\mathrm{g}_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \tag{2.9}$$

While the inverse metric would be $\mathrm{g}^{ij} = \mathbf{e}_i\mathbf{e}_j$ assuming that $\det(\mathrm{g}_{ij}) \neq 0$. It can be shown that these quantities, (2.9), are the contravariant components of a symmetric second rank tensor (that is to say, it transforms following (2.6)):

$$\mathrm{g}'_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j$$

Taking the inverse transformation for (2.1), we have:

$$\mathrm{g}'_{ij} = \frac{\partial u^k}{\partial u'^i}\frac{\partial u^l}{\partial u'^j}\mathbf{e}_k\mathbf{e}_l = \frac{\partial u^k}{\partial u'^i}\frac{\partial u^l}{\partial u'^j}\mathrm{g}_{kl}$$

Which proves that the metric $\mathbf{g}$ is a second rank tensor. Also we can see that the mixed components of the metric correspond to the Kronecker delta:

$$\mathrm{g}^i_j = \mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j \tag{2.10}$$

One can also show that another essential property of the covariant and contravariant components of the metric is the raising and lowering of the indices. For a scalar product of two vectors $\mathbf{a}, \mathbf{b}$ we can write:

$$\mathbf{a} \cdot \mathbf{b} = a^i\mathbf{e}_i b^j\mathbf{e}_j = a^i b^j \mathrm{g}_{ij}$$

And also:

$$\mathbf{a} \cdot \mathbf{b} = a_i\mathbf{e}^i b^j\mathbf{e}_j = a_i b^j \delta^i_j = a_i b^i$$

One can see straightforward that $\mathrm{g}_{ij}a^i b^j = a^i b_i$ and similarly for the contravariant case. Then, for arbitrary a^i, one would have $\mathrm{g}_{ij}b^j = b_i$ and $\mathrm{g}^{ij}b_j = b^j$, that is to say, the metric can be used for lowering and raising the indices of a tensor:

$$T^{ij\ldots k}{}_{lm\ldots n} = \mathrm{g}^{ia}\mathrm{g}^{jb}\ldots\mathrm{g}^{kc}\mathrm{g}_{ld}\mathrm{g}_{me}\ldots\mathrm{g}_{nf}T_{ab\ldots c}{}^{de\ldots f}$$

2.1.3 Christoffel symbols

Let's consider the derivative of the coordinate basis, $\frac{\partial \mathbf{e}_i}{\partial u^j}$, which is also a vector i.e., it will be a linear combination of the basis. We can express the coefficients of such a linear combination by:

$$\frac{\partial \mathbf{e}_i}{\partial u^j} = \Gamma^k_{ij}\mathbf{e}_k \tag{2.11}$$

where Γ^k_{ij} are the so-called *Christoffel symbols*. From this definition it can be shown:

$$\Gamma^k_{ij} = \mathbf{e}^k\frac{\partial \mathbf{e}_i}{\partial u^j} \tag{2.12}$$

$$\frac{\partial \mathbf{e}^i}{\partial u^j} = -\Gamma^i_{kj}\mathbf{e}^k \tag{2.13}$$

Even though they look as a $(2,1)$ rank tensor, they are not:

$$\Gamma'^{k}_{ij} = \frac{\partial u'^k}{\partial u^l}\frac{\partial^2 u^l}{\partial u'^j \partial u'^i} + \frac{\partial u'^k}{\partial u^n}\frac{\partial u^l}{\partial u'^i}\frac{\partial u^m}{\partial u'^j}\Gamma^n_{lm} \tag{2.14}$$

where we see that there is an extra term, showing that the Christoffel symbols are not the components of a tensor (and therefore they do depend on the coordinate system). The quantities that transform in this way are called *affine connection*, and so we will refer to the Christoffel symbols as the connection in GR.

We can also define these quantities in terms of derivatives of the metric using the expressions (2.12) and (2.13):

$$\frac{\partial \mathrm{g}_{ij}}{\partial u^k} = \frac{\partial \mathbf{e}_i}{\partial u^k}\mathbf{e}_j + \mathbf{e}_i\frac{\partial \mathbf{e}_j}{\partial u^k} = \Gamma^l_{ik}\mathrm{g}_{lj} + \Gamma^l_{jk}\mathrm{g}_{il}$$

And so, making use of $\mathrm{g}^{ik}\mathrm{g}_{kj} = \delta^i_j$ and permutating the indices one can get[4]:

$$\Gamma^m_{ij} = \frac{1}{2}\mathrm{g}^{mk}\left(\frac{\partial \mathrm{g}_{jk}}{\partial u^i} + \frac{\partial \mathrm{g}_{ki}}{\partial u^j} - \frac{\partial \mathrm{g}_{ij}}{\partial u^k}\right) = \left\{{m \atop ij}\right\} \tag{2.15}$$

It is important to remark that this connection is not a general affine connection, but the one used in a Riemann manifold (the one that we require for the standard formulation of GR). In this case then, the connection (2.15) is known as the Levi-Civita connection ($\left\{{m \atop ij}\right\}$), while a general connection would be written as:

$$\Gamma^m_{ij} = \left\{{m \atop ij}\right\} + L^m_{ij} + K^m_{ij} \tag{2.16}$$

where L^m_{ij} and K^m_{ij} are the disformation tensor and the contorsion tensor respectively[5].

2.1.4 Covariant derivative

Another math tool that we will use is the covariant derivative. Let's consider an arbitrary vector $\mathbf{v} = v^i\mathbf{e}_i$. Its derivative would be:

$$\frac{\partial \mathbf{v}}{\partial u^j} = \frac{\partial v^i}{\partial u^j}\mathbf{e}_i + v^i\frac{\partial \mathbf{e}_i}{\partial u^j} = \left(\frac{\partial v^i}{\partial u^j} + v^k\Gamma^i_{kj}\right)\mathbf{e}_i$$

where we have used the definition of the connection (2.11). Thus, the covariant derivative is defined as:

$$\nabla_j v^i \equiv \frac{\partial v^i}{\partial u^j} + v^k\Gamma^i_{kj} \tag{2.17}$$

[4]It is important to notice that this expression is derived under the assumption that the covariant derivative, defined in (2.17), of the metric is zero. As we will see after, this only holds in a curved, torsion free and non-metric geometry.

[5]These two tensors are important for the alternative geometrical approaches of the General Relativity. Since we will not focus on that, we will not extend more these concepts. However, it is important to note the different between the Levi-Civita connection and the general one.

For the case of a second rank tensor, one would get:

$$\begin{aligned}\nabla_k T^{ij} &= \frac{\partial T^{ij}}{\partial u^k} + \Gamma^i_{lk} T^{lj} + \Gamma^j_{lk} T^{il} \\ \nabla_k T^i_j &= \frac{\partial T^i_j}{\partial u^k} + \Gamma^i_{lk} T^l_j - \Gamma^l_{jk} T^i_l \\ \nabla_k T_{ij} &= \frac{\partial T_{ij}}{\partial u^k} - \Gamma^l_{ik} T_{lj} - \Gamma^l_{jk} T_{il}\end{aligned} \tag{2.18}$$

One can generalise this result for a (p, q) order tensor by adding properly the required $\Gamma^\alpha_{\mu\nu}$.

2.1.5 Riemann tensor

Another important quantity in General Relativity is the so-called *Riemann tensor* or *curvature*, which describes the curvature of the spacetime in Einstein's interpretation of GR. Its definition comes from:

$$\nabla_\mu \nabla_\nu A_\rho - \nabla_\nu \nabla_\nu A_\rho = R^\lambda_{\rho\nu\mu} A_\lambda \tag{2.19}$$

For a generic co-vector A_λ. The quantity $R^\lambda_{\rho\nu\mu}$, the Riemann curvature, is a tensor since $\nabla_\mu \nabla_\nu A_\rho$ and $\nabla_\nu \nabla_\nu A_\rho$ are tensors as well. This Riemann tensor measures how two vectors differ when we parallel transport them, transportation that depends on the path for a curved spacetime.

It can be shown that the Riemann tensor depends on the connection as follows:

$$R^\mu_{\nu\rho\sigma} = \frac{\partial \Gamma^\mu_{\nu\sigma}}{\partial x^\rho} - \frac{\partial \Gamma^\mu_{\nu\rho}}{\partial x^\sigma} + \Gamma^\lambda_{\nu\sigma} \Gamma^\mu_{\rho\lambda} - \Gamma^\lambda_{\nu\rho} \Gamma^\mu_{\sigma\lambda} \tag{2.20}$$

It has some properties (Bianchi's identities), which are:

$$R^\mu_{\nu\rho\sigma} + R^\mu_{\rho\sigma\nu} + R^\mu_{\sigma\nu\rho} = 0 \tag{2.21}$$

$$\nabla_\mu R^\kappa_{\lambda\nu\rho} + \nabla_\nu R^\kappa_{\lambda\rho\mu} + \nabla_\rho R^\kappa_{\lambda\mu\nu} = 0 \tag{2.22}$$

By contracting this tensor one can get the *Ricci tensor* and *Ricci scalar*, respectively:

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = g^{\lambda\rho} R_{\rho\mu\nu\lambda} \tag{2.23}$$

$$R = g^{\mu\nu} R_{\mu\nu} \tag{2.24}$$

This last scalar, R, will be essential in the definition of Einstein-Hilbert action.

In the same way that the Levi-Civita connection is not a general connection, the Riemaniann tensor holds only in this curved spacetime. It should not be confused then with a general curvature tensor.

2.1.6 Geodesic equation

We will derive field equations by imposing the Least Variational Principle, that is to say, extremizing the action and thus, looking for the shortest path connecting two points in a Riemann manifold (remember that Riemann manifolds are curved), the so-called geodesic.

In flat space it is easy to see that such a path would be a straight line, while in a curved geometry it would be a curve. Matter fields, with and action like $S = -mc^2 \int d\tau$ for a point particle, will follow the geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \tag{2.25}$$

where we have chosen as affine parameter $\sigma = \tau$. Otherwise it would have a more generic form:

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\sigma}\frac{dx^\rho}{d\sigma} = \lambda\frac{dx^\mu}{d\sigma} \tag{2.26}$$

2.1.7 Curvature, torsion and non-metricity

Let's define briefly these three geometrical objects that will help us classify geometries.

We already introduced the Riemann curvature, (2.20). As mentioned before, when we parallel transport one vector following two different paths in a curved space, 2.1, the difference between these two paths is regulated by the Riemann tensor.

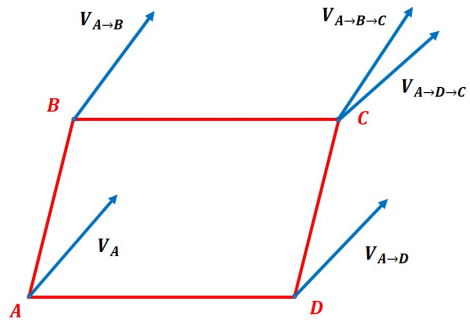

Figure 2.1: Parallel transportation

Apart from the parallel transportation of a vector, the rotation of a vector transported along a closed curve is also given by the curvature, fig.(2.2).

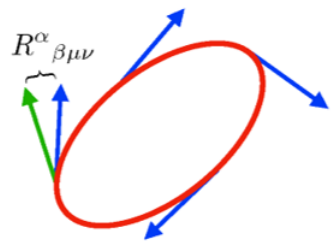

Figure 2.2: Rotation of a vector transported along a closed curve

If we transport two vectors along each other, we would get a parallelogram generally not closed, fig.(2.3). This non-closure is given by the *torsion*. It is defined in terms of the connection:

$$T^\alpha_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} \tag{2.27}$$

As we mentioned before, in the standard GR the connection is symmetric in the two lower indices, that is to say, is a torsion-free geometry.

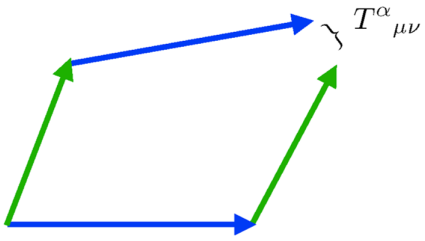

Figure 2.3: Non-closure of the parallelogram formed by two vectors transported along each other

The third geometrical object to define is the non-metricity, which measures the variation of the length of a vector as it is transported, fig.(2.4). Thus, it is defined as the covariant derivative of the metric, which is zero in Einstein's GR:

$$Q_{\alpha\mu\nu} \equiv \nabla_{\alpha}\, g_{\mu\nu} \tag{2.28}$$

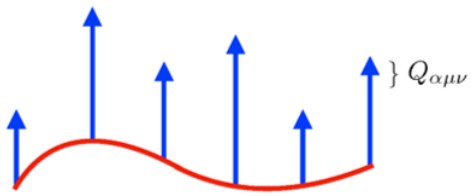

Figure 2.4: Variation of the length of the vector as it is transported[6]

2.2 Standard General Relativity

In order to study the usual formulation of General Relativity, we will derive its field equations. First of all, we must know that this interpretation comes from the assumption that gravity is a result of the curvature of the spacetime. As we will see later, one can also describe the same underlying physics by considering a flat spacetime with torsion or with non-metricity. However, for Einstein's interpretation we will consider that the torsion and non-metricity vanish. This implies that $\nabla_{\rho} g_{\mu\nu} = 0$ and $\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}$.

2.3 Einstein-Hilbert action

In a first approach, the basic requirement to get the field equations for a curved spacetime which is torsion-free and with null non-metricity, is that the gravitational field is only

[6]The figures 2.2, 2.3 and 2.4 are originally from the paper [7].

described by the metric tensor. Let's then write the action for the most simple case, which is known as *Einstein-Hilbert action*:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-\mathrm{g}} R \tag{2.29}$$

where R is the Ricci scalar defined in (2.24). We have included the factor $\frac{1}{16\pi G}$ so the action has the correct dimensions. If we vary the action with respect to the metric, following the previous requirement, we would get:

$$\delta S = \frac{1}{16\pi G} \int d^4x \delta \left(\sqrt{-\mathrm{g}}\, \mathrm{g}^{\mu\nu} R_{\mu\nu}\right) = \frac{1}{16\pi G} \int d^4x \sqrt{-\mathrm{g}} \left(-\frac{1}{2}\, \mathrm{g}_{\mu\nu} R + R_{\mu\nu}\right) \delta\, \mathrm{g}^{\mu\nu}$$

Then, since we want to extremise the action:

$$\begin{aligned} G_{\mu\nu} &\equiv -\frac{1}{2}\, \mathrm{g}_{\mu\nu} R + R_{\mu\nu} \\ &= 0 \end{aligned} \tag{2.30}$$

where $G_{\mu\nu}$ is the Einstein tensor. Contracting, one finds:

$$\mathrm{g}^{\mu\nu} G_{\mu\nu} = 0 \rightarrow R = 0 \rightarrow R_{\mu\nu} = 0$$

Thus, in this simple case in absence of matter (vacuum solutions), the Ricci tensor vanishes. The whole derivation can be found in the appendix.

2.4 Coupling to matter

We can go further and add a cosmological constant to the Einstein-Hilbert action. Einstein motivated the introduction of this constant Λ as a way to make the Universe static (which was the common belief in the 20th century) against the action of the gravity [15]:

> "The term is necessary only for the purpose of making possible a quasi-static distribution of matter, as required by the fact of the small velocities of the stars"

One of the most important contributions of Einstein to GR was the equivalence principle, which dictates that the inertial mass, m_I, is equal to the gravitational one, m_G: assumption that has been proved with very little uncertainty. This means that all matter fields couple with the same strength to gravity [16], hence the geometrical nature of gravity.

Then, adding the cosmological constant and considering also matter, the total action would be:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, (R - 2\Lambda) + S_m \tag{2.31}$$

Where S_m is the action corresponding to the matter contribution.

Let's first have a look on S_m. One can define the *energy-momentum tensor* $T_{\mu\nu}$ as it follows:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \tag{2.32}$$

An example of the energy-momentum tensor is given in the appendix. If we now proceed as we did in the former case, we would get:

$$\begin{aligned}\delta S &= \int d^4x\sqrt{-g}\left(G_{\mu\nu}+\Lambda g_{\mu\nu}\right)\delta g^{\mu\nu} - \frac{1}{2}\int d^4x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu} \\ &= \frac{1}{2}\int d^4x\sqrt{-g}\left[\frac{1}{8\pi G}\left(G_{\mu\nu}+\Lambda g_{\mu\nu}\right) - T_{\mu\nu}\right]\delta g^{\mu\nu}\end{aligned}$$

Thus, we get the field equation:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.33}$$

Considering the cosmological constant as another component of the energy-momentum tensor, one can write:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.34}$$

By contracting in a similar way as we did in the previous section, we can write Ricci tensor as:

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}T g^{\mu\nu}\right) \tag{2.35}$$

Where $T = g^{\mu\nu}T_{\mu\nu}$ is the trace of the energy-momentum tensor.

2.5 Other geometrical approaches

As we mentioned before, the standard formulation of General Relativity is the one where the gravity is fully ascribed to curvature. In the other two possibilities, the spacetime is flat and has torsion (TEGR[7]) or non-metricity (STEGR[8]). It is relevant to mention these interpretations since, albeit we will not deepen into the subject, one can implement MOND from a non-linear extension of Coincident General Relativity or CGR, which comes from a gauge choice that makes the connection vanish in the STEGR formulation. This non-linear extension recovers both MOND and General Relativity in the appropriate limits. In this work, however, we will try to explain the underlying physics for MOND by modifying the standard GR.

2.5.1 Teleparallel Equivalent of GR, TEGR

In this other equivalent geometrical interpretation of gravity, the assumptions we made are that curvature and non-metricity are zero. In other words, $\nabla_\alpha g_{\mu\nu} = 0$, as in the standard interpretation, but in this case $\Gamma^\alpha_{\mu\nu} \neq \Gamma^\alpha_{\nu\mu}$. This theory is referred to as parallel since it is formulated in a flat (and metric) space so that vectors do not rotate as they are transported and thus there is a better notion of parallelism at a distance.

As in the standard formulation of GR, the scalar quantity used in this theory is expressed in terms of the torsion and its trace ($T_\alpha = T^\mu_{\alpha\mu}$):

[7]As we will see, TEGR stands for *Teleparallel Equivalent of General Relativity.*
[8]STEGR is an abreviation for *Symmetric Teleparallel Equivalent of General Relativity.*

$$T \equiv -\frac{c_1}{4}T_{\alpha\mu\nu}T^{\alpha\mu\nu} - \frac{c_2}{2}T_{\alpha\mu\nu}T^{\mu\alpha\nu} + c_3 T_\alpha T^\alpha \tag{2.36}$$

With this scalar and making use of the convenient Lagrange multipliers, the general action would be:

$$S = -\int d^4x \left[\frac{1}{16\pi G}\sqrt{-g}T + \lambda_\alpha^{\beta\mu\nu}R^\alpha_{\beta\mu\nu} + \hat{\lambda}^\alpha_{\mu\nu}\nabla_\alpha g^{\mu\nu}\right] \tag{2.37}$$

Where now the tensor $R^\alpha_{\beta\mu\nu}$ is no longer the Riemann tensor due to the fact that we are not in a Riemannian manifold. It is, then, a curvature tensor. Since we are working in a flat space, the curvature vanishes, therefore, the connection is purely inertial[9] so that it can be parametrised by an element $\Lambda^\alpha_\beta \epsilon GL(4,\mathbb{R})$:

$$\Gamma^\alpha_{\mu\nu} = (\Lambda^{-1})^\alpha{}_\lambda \partial_\mu \Lambda^\lambda{}_\nu \tag{2.38}$$

One can recover GR by setting $c_1 = c_2 = c_3 = 1$ in the scalar expression (2.36) (we will express T with this choice of parameter as $\mathbb{T}$), and thus recovering identically the dynamics with the action:

$$S_{TEGR} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\mathbb{T}(g,\Lambda) \tag{2.39}$$

This teleparallel theory equivalent to GR is then called Teleparallel Equivalent of GR or TEGR.

2.5.2 Symmetric Teleparallel Equivalent of GR, STGR

When the geometrical framework has as the fundamental geometrical object the non-metricity, that is to say, gravity is completely due to the non-metricity of the flat and torsion free geometry, it becomes the simplest of the three possibilities. As in the former case, the teleparallel designation arise from the flat geometry. However, this theory has an additional symmetry since, as we will see later, it has an enhanced four-parameter gauge symmetry, hence the name *Symmetric Teleparallel Equivalent of GR.*

The fundamental geometrical object of this theory is $Q_{\alpha\mu\nu} \equiv \nabla_\alpha g_{\mu\nu} \neq 0$. As we did before, we can define the following scalar quantity in terms of the non-metricity tensor and its independent contractions, $Q_\alpha = Q_{\alpha\lambda}{}^\lambda$ and $\tilde{Q}_\alpha = Q^\lambda{}_{\lambda\alpha}$:

$$Q = \frac{c_1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} - \frac{c_2}{2}Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} - \frac{c_3}{4}Q_\alpha Q^\alpha + (c_4-1)\tilde{Q}_\alpha\tilde{Q}^\alpha + \frac{c_5}{2}Q_\alpha\tilde{Q}^\alpha \tag{2.40}$$

The general action for this formulation is constructed analogously to the former case:

$$S = -\int d^4x \left[\frac{1}{16\pi G}\sqrt{-g}Q + \lambda_\alpha{}^{\beta\mu\nu}R^\alpha{}_{\beta\mu\nu} + \lambda_\alpha{}^{\mu\nu}T^\alpha{}_{\mu\nu}\right] \tag{2.41}$$

As in TEGR, the connection is purely inertial since we are also working in a flat space and thus, it can be parametrised by a general element $\Lambda^\alpha_\beta \epsilon GL(4,\mathbb{R})$. However, since

[9] In special relativity, Lorentz connections represent inertial effects present in a given frame.

we are working in a torsion free space, we have an extra constraint that leads us to the parametrisation of the connection as:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial \xi^{\lambda}} \partial_{\mu}\partial_{\nu}\xi^{\lambda} \tag{2.42}$$

From this form of the connection one can show that it can vanish for the gauge choice of coordinates $\xi^{\alpha} = x^{\alpha}$. This is the enhanced four-parameter gauge symmetry that we stressed previously.

When we choose all the parameters in (2.40) to be 1, we get the scalar quantity $\mathbb{Q}$. General Relativity is recovered in that case by the action:

$$S_{STGR} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \mathbb{Q}(g, \xi) \tag{2.43}$$

This is the action for the Symmetric Teleparallel Equivalent of GR theory, reproducing completely the dynamics of GR. With the gauge choice we mentioned before (where the connection vanishes), one can write this action, (2.43), as:

$$S_{CGR} = S_{STGR}[\Gamma = 0] = \frac{1}{16\pi G} \int d^4x \sqrt{-g} g^{\mu\nu} \left(\left\{ {\alpha \atop \beta\mu} \right\} \left\{ {\beta \atop \nu\alpha} \right\} - \left\{ {\alpha \atop \beta\alpha} \right\} \left\{ {\beta \atop \mu\nu} \right\} \right) \tag{2.44}$$

Where CGR stands for Coincident General Relativity (since it reproduces the Einstein-Hilbert action devoid of boundary terms) and $\left\{ {\alpha \atop \beta\mu} \right\}$ are the Levi-Civita connection[10].

This is the simplest geometrical formulation of the three possibilities. Through considering non-linear extensions of it, it is possible to provide a theory that can be viewed as the first relativistic and covariant formulation of MOND. Nevertheless, as we mentioned previously, the approach presented in this work will be focused in the standard GR and its scalar-tensor theories.

[10]This connection, symmetric and metric-compatible is given by the Christoffel symbols of the metric, like we saw in equation (2.15)

Chapter 3

Modified Newtonian Dynamics

3.1 Introduction

After understanding briefly the formulation of Einstein's General Relativity and how to work with it, we can now introduce the Modified Newtonian Dynamics. As we shortly stressed in the introduction, Milgrom's first proposal consisted of essentially an algorithm formulated to solve *ad hoc* some discrepancies as asymptotically flatness of the rotation curves for spiral galaxies. An earlier idea to solve this problem was to modify the inverse dependence $1/r^2$ that was leading to the expected (but empirically incorrect) velocity of the spiral galaxies. However, Milgrom noticed that any modification attached to a length scale would lead to larger discrepancies in larger galaxies, in disagreement with the observations, [3].

That is why Milgrom focused on the acceleration scale and proposed a modification of the Newtonian dynamics that would appear below a critical acceleration, $a_0 \approx cH_0/6$. This modification can be seen as a modification of inertia:

$$\mathbf{F} = m\mathbf{a}\mu\left(a/a_0\right) \tag{3.1}$$

Or as a modification of gravity:

$$\mathbf{g}_n = \mathbf{g}\mu\left(|g|/a_0\right) \tag{3.2}$$

Then, the equality for a test particles's acceleration $\mathbf{a} = -\nabla\Phi_N$ would be modified:

$$\mu(|\mathbf{a}|/a_0)\mathbf{a} = -\nabla\Phi_N \tag{3.3}$$

with Φ_N the Newtonian potential. As we mentioned in the introduction, in all the previous expressions, the function introduced, μ, is not known *a priori* but it is required to have an asymptotic behaviour to recover the Newtonian case at the scales where it works while correcting its dynamics at the low acceleration regime.

This MOND first proposal makes use of a new physical parameter, a_0, which is an acceleration scale of order 10^{-8} cm s^{-2}. This is thus the constant that indicates the limit below which the effective gravitational attraction approaches $\sqrt{g_N a_0}$. It is remarkable to notice that $a_0 \approx cH_0/6$. This apparent coincidence could suggest that MOND might exhibit the effect of cosmology on local particle dynamics, [21].

Milgrom's formulation presented a great empirical success[1], being of special interest the prediction of the Tully-Fisher law that is not predicted by the CDM theory. Not only that, when fitting rotation curves that are adjusted by CDM models using three parameters, MOND does it with just one free parameter. This shows that MOND is more compact than the dark matter proposal when fitting this data.

Despite its striking empirical success, this prescription is far from being complete. For instance, this formulation does not verify the conservation laws, as it is shown in the appendix. In addition to the problems that it presents, an underlying theory is needed for MOND not to be a merely phenomenological summary of galaxy phenomenology but a consequence of a modification of the current physics.

There have been many attempts to derive a valid physical basis for MOND, but this still remains an open question. Nevertheless, there are some compelling theoretical proposals. In this work we will study the first steps towards a relativistic modified gravity as a possible theory for MOND, starting with the nonrelativistic formulation by Bekenstein and Milgrom in 1984.

3.2 Nonrelativistic AQUAL

Bekenstein and Milgrom's first proposal consisted of a modified nonrelativistic gravity theory. This theory is Lagrangian-based[2] and so, it does conserve the momentum, energy and angular momentum, unlike the original MOND scheme. Thus, this trial starts from a nonrelativistic Lagrangian that maintains the Galilean and rotational invariance of the Lagrangian that generates the Poisson's equation while including an unknown function that gives to it an aquadratic sense (and then, breaking the linearity requirement). The name comes from this aquadratic character: AQUAdratic Lagrangian or *AQUAL* for short.

Therefore, instead of having the quadratic Lagrangian $\mathcal{L} = -\frac{1}{8\pi G}(\nabla\Phi)^2 - \rho\Phi$, we will have an aquadratic Lagrangian density as it follows:

$$\mathcal{L} = -\frac{a_0^2}{8\pi G}\tilde{f}\left(\frac{|\nabla\Phi|^2}{a_0^2}\right) - \rho\Phi \tag{3.4}$$

where ρ is the total mass density and Φ is generally not the Newtonian potential but the true gravitational one, so that the acceleration of a test particle would be $\mathbf{a} = -\nabla\Phi$. The function $\tilde{f}$ is unknown but it has an asymptotic behaviour so it recovers the Newtonian formulation when it is not in the low acceleration regime (MOND regime):

$$\tilde{f}(x) = \begin{cases} x & x \gg 1 \\ \frac{2}{3}x^{3/2} & x \ll 1 \end{cases}$$

One can construct an action of the form:

$$S = -\int d^3r \left[\rho\phi + \frac{a_0^2}{8\pi G}\tilde{f}\left(\frac{|\nabla\Phi|^2}{a_0^2}\right)\right] \tag{3.5}$$

[1]For further reading, see: [21]

[2]Deriving a theory from a Lagrangian with the suitable symmetries ensures that the conservation laws are verified, according to Noether's theorem.

We now vary the action with respect to the potential Φ:

$$\begin{aligned}\delta S &= -\int d^3r \left[\rho\delta\Phi + \frac{1}{4\pi G}\left(\frac{d\tilde{f}}{dx}|\nabla\Phi|\delta\Phi\right)\right] \\ &= -\int d^3r \left[\rho\delta\Phi + \frac{1}{4\pi G}\delta\Phi\nabla\left(\frac{d\tilde{f}}{dx}|\nabla\Phi|\right)\right] \\ &= -\int d^3r \left[\rho + \frac{1}{4\pi G}\nabla\left(\tilde{\mu}(x)|\nabla\Phi|\right)\right]\delta\Phi\end{aligned}$$

where we have integrated by parts from the first line to the second and we defined the function $\tilde{\mu}(\sqrt{x}) \equiv \frac{d\tilde{f}}{dx}$. Imposing that the variation of the action is zero, we get a non-linear Poisson's equation:

$$\nabla \cdot \left[\tilde{\mu}\left(\frac{|\nabla\Phi|}{a_0}\right) \cdot |\nabla\Phi|\right] = 4\pi G\rho \tag{3.6}$$

We must identify the function $\tilde{\mu}$ in this field equation with the one used in the equations (3.1) and (3.2) for this theory to satisfy the assumptions of MOND. One can see that, when the function $\tilde{\mu}(x)$ tends to the unity (Newtonian regime), Poisson's equation (3.7) is recovered.

$$\nabla \cdot \nabla\Phi_N = 4\pi G\rho \tag{3.7}$$

By comparing this AQUAL field equation with the ususal Poisson's equation, one can write it in terms of the unmodified Newtonian field:

$$\nabla \cdot [\tilde{\mu}\left(|\nabla\Phi|/a_0\right) \cdot |\nabla\Phi| - \nabla\Phi_N] = 0 \tag{3.8}$$

$$\tilde{\mu}(|\nabla\Phi|/a_0) \cdot \nabla\Phi = \nabla\Phi_N - \nabla \times \mathbf{h} \tag{3.9}$$

Where we introduced the vector field $\mathbf{h}$ to ensure that both sides of (3.6) have the same nonvanishing curl. One can see that this curl term is then a correction from the original MOND equation (3.3), and thus, it is the term that guarantees that the conservation laws are fulfilled.

For spherical, plane and cylindrical symmetries, the term $\nabla \times \mathbf{h}$ vanishes, and then the expression (3.9) is exactly the equation (3.3), that is to say, for these symmetries, AQUAL reduces exactly to MOND formula.

This nonrelativistic formulation solves many of the problems and paradoxes of the original MOND formula while showing a great empirical description. However, both MOND and AQUAL formulation have the same defect: they do not have the means to calculate gravitational lensing by extragalactic objects. Therefore, one cannot study properly the mass discrepancies exhibited by the lensing of distant galaxies by clusters, [3].

Notwithstanding these flaws, this nonrelativistic AQUAL is an important first step towards the underlying theoretical basis of MOND while doing really well phenomenologically. In fact, a significant advantage of this formulation is that it can be taken as the nonrelativistic theory on which one can base the relativistic theoretical approach to MOND.

3.3 Relativistic AQUAL, RAQUAL

Now, we need to derive a relativistic version of this theory. A relativistic description is essential for many key subjects, among others, to enable investigation of relativistic systems that are of relevance (for instance, binary pulsars or cosmology) and to compute the gravitational lensing. We will search then for a relativistic theory to describe MOND that conserves the benefits of AQUAL.

3.3.1 Some considerations

In addition to fitting of the experimental evidences that are well established, if one wants to propose a relativistic theory for the MOND formula, there are some theoretical requirements to be considered in order to make it a valid theory.

It is indispensable for a theory to verify all the conservation laws of linear and angular momenta and energy. To do so, the theory must be derivable from an **action principle** as an integral over a local Lagrangian density as the simplest choice to make.

Since the special relativity has been proved through direct evidence in numerous occasions, the theory that we are looking for should ensure that all the equations derived from it are **relativistically invariant**. Therefore, the action should be a relativistic scalar.

Einstein's Equivalence Principle dictates that a gravitational field cannot be distinguished from a suitably chosen accelerated reference frame, being this demonstrated with great accuracy. Thus, the theory must be a metric theory: all nongravitational laws of physics must be expressed replacing the Lorentz metric with the curved metric $g_{\mu\nu}$ in their usual laboratory forms, so that they account for the gravitation effects.

Another needed key to take into account is to derive a theory that verifies the **causality principle**. In order to do so, the derived equations should not permit superluminal (exceeding the speed invariant under the Lorentz transformations) propagation of any measurable field or of linear and angular momenta and energy.

To avoid instabilities of the vacuum, any bound system of the theory must present **positive energy**, taking the requirement in terms of global positivity.

Deviation from Newtonian gravity must appear below a preferred scale of acceleration (that the wanted theory must present), even at low velocities.

Consequently, these are the theoretical aspects that will indicate the validity of the theory. As we will see, the relativistic version of AQUAL implies superluminal propagation of the scalar field, and thus it is not an acceptable final version of the theory.

3.3.2 Scalar-tensor theory of GR: why?

As we mentioned when studying AQUAL, this nonrelativistic theory can lead to a good candidate for a relativistic gravitational field theory for MOND. The question is *how?*. In the nonrelativistic case we replaced the quadratic $(\nabla\Phi_N)^2$ for a general function that was, in general, aquadratic $\tilde{f}\left(|\nabla\Phi|^2/a_0^2\right)$. Following the same argument we could try to take the prescription of General Relativity, namely, the Einstein-Hilbert action (2.29), and replace the Ricci scalar R by a generic function $f(R)$.

Despite its logic, this approach would not work. We can see why by studying the expression of the Ricci scalar. As we studied in the section 2.1.5, the Ricci scalar in General

Relativity takes the schematic form $R \sim \partial\Gamma + \Gamma\Gamma$ while the Levi-Civita connection has the form $\Gamma \sim \partial\,\mathrm{g}$, [23]. Then, the Ricci scalar has a part that is quadratic in derivatives of the metric components and a term cointaining second derivatives of it, $R \sim \partial^2\,\mathrm{g} + \partial\,\mathrm{g}\,\partial\,\mathrm{g}$. The quadratic part would lead to a nonrelativistic limit similar to the Lagrangian for AQUAL, (3.4). If we recall the derivation of the field equations shown in the appendix, the term of the Ricci scalar involving the second derivatives led to a boundary term on the action that we could ignore. In this case, however, there would be a contribution that will not be dismissable and, in fact, would dominate over the previous term. Consequently, we would get the expression that we are looking for plus an undesired term.

We can also refer to the *Soussa-Woodard theorem*, [22]:

> "No purely metric-based, relativistic formulation of MOND whose energy functional is stable (in the sense of being quadratic in perturbations) can be consistent with the observed amount of gravitational lensing from galaxies."

Or, in other words, if a gravitational theory is built exclusively on the metric, it cannot reproduce the anomalously large gravitational lensing observed *and* recover a nonrelativistic limit with the form of MOND, [3]. Therefore, it is clear that we need to add another degree to the theory. That is the motivation for a scalar-tensor theory of GR, by promoting the potential in AQUAL Φ to a scalar field, ψ.

3.3.3 Formulation of RAQUAL

The physical metric in this relativistic theory is taken as conformal to the Einstein metric (the one that we used in the previous sections):

$$\tilde{\mathrm{g}}_{\alpha\beta} = e^{2\psi}\,\mathrm{g}_{\alpha\beta} \tag{3.10}$$

The introduction of the scalar ψ has attached its Lagrangian density, that is written as the linear Lagrangian density for a scalar field $\mathcal{L}_\psi = -\frac{1}{8\pi G}\,\mathrm{g}^{\alpha\beta}\,\partial_\alpha\psi\partial_\beta\psi$ but in a, generally, aquadratic form:

$$\mathcal{L}_\psi = -\frac{a_0^2}{8\pi G} f\left(\frac{\mathrm{g}^{\alpha\beta}\,\partial_\alpha\psi\partial_\beta\psi}{a_0^2}\right) \tag{3.11}$$

This function f might be different from the function used in the nonrelativistic formulation, $\tilde{f}$.

Then, the Lagrangian density for this theory consists of the Ricci scalar in addition to (3.11) and a matter Lagrangian density, $\mathcal{L}_m$. To derive a expression for the matter Lagrangian with these metrics, we need to replace $\mathrm{g}_{\alpha\beta}$ by our physical metrics, $\tilde{g}_{\alpha\beta}$[3].

[3]For example, the matter action for a particle of mass m would have the form:

$$S_m = -m\int e^{\psi}\left(-\,\mathrm{g}_{\alpha\beta}\,dx^\alpha dx^\beta\right)^{1/2}$$

By replacing Einstein metric by its conformal transformed metric. Another possibility could be to take matter as a Klein-Gordon scalar.

One can derive the field equations for this relativistic theory RAQUAL following the methods that we used in the previous sections. From the action:

$$S = \int d^4x\sqrt{-\mathrm{g}}\,(R + \mathcal{L}_\psi) + S_m \tag{3.12}$$

We make the variation of this action with respect to the scalar field, taking into account that the matter action is defined in terms of the physical metric $\tilde{g}_{\alpha\beta}$:

$$\begin{aligned}\delta S &= \int d^4x\sqrt{-\mathrm{g}}\left[-\frac{a_0^2}{8\pi G}\delta f\left(\frac{\mathrm{g}^{\alpha\beta}\,\partial_\alpha\psi\partial_\beta\psi}{a_0^2}\right)\right] - \frac{1}{2}\int d^4x\sqrt{-\tilde{\mathrm{g}}}\tilde{T}_{\alpha\beta}\delta\tilde{\mathrm{g}}^{\alpha\beta}\\ &= \int d^4x\sqrt{-\mathrm{g}}\left[-\frac{a_0^2}{8\pi G}\frac{df}{dx}2\,\mathrm{g}^{\rho\nu}\,\frac{\partial_\nu\psi}{a_0^2}\partial_\rho\delta\psi\right] - \frac{1}{2}\int d^4x\sqrt{-\tilde{\mathrm{g}}}\tilde{T}_{\alpha\beta}\delta\tilde{\mathrm{g}}^{\alpha\beta}\end{aligned}$$

Since the physical metric includes the scalar, the variation of $\delta\tilde{\mathrm{g}}^{\alpha\beta}$ can be written as:

$$\delta\tilde{\mathrm{g}}^{\alpha\beta} = \delta\left(e^{-2\psi}\,\mathrm{g}^{\alpha\beta}\right) = -2e^{2\psi}\,\mathrm{g}^{\alpha\beta}\,\delta\psi = -2\tilde{\mathrm{g}}^{\alpha\beta}\delta\psi$$

So then, with this expression and integrating by parts the first term:

$$\begin{aligned}\delta S &= \frac{1}{4\pi G}\int d^4x\sqrt{-\mathrm{g}}\left[\nabla_\rho\left(f'\,\mathrm{g}^{\rho\nu}\,\partial_\nu\psi\right)\right]\delta\psi + \int d^4x\sqrt{-\tilde{\mathrm{g}}}\tilde{T}_{\alpha\beta}\tilde{\mathrm{g}}^{\alpha\beta}\delta\psi\\ &= 0\end{aligned}$$

The field equations, defining[4] $\mu(x) \equiv \frac{df}{dx}$, are:

$$\nabla_\rho\left[\mu\left(\frac{\mathrm{g}^{\alpha\beta}\,\partial_\alpha\psi\partial_\beta\psi}{a_0^2}\right)\mathrm{g}^{\rho\nu}\,\partial_\nu\psi\right] = -4\pi G\tilde{\mathrm{g}}^{\alpha\beta}\tilde{T}_{\alpha\beta} \tag{3.13}$$

To make it coincide with AQUAL, we must make the function f:

$$f(x) = \begin{cases} x & x \gg 1 \\ \frac{2}{3}x^{3/2} & 0 < x \ll 1 \end{cases}$$

We can now study the limits of this formulation to show if it recovers properly Newtonian and MOND limits. First of all, we need to take the nonrelativistic limit for General Relativity (the derivation is shown in the appendix). In the limit for slow motion in a quasistatic situation with nearly flat metric $\mathrm{g}_{\alpha\beta}$, we see that we can approximate the component $\mathrm{g}_{00} \approx -(1 + 2\Phi_N(x))$. Then, in a weak field ψ, we get for the physical metric:

$$\begin{aligned}\tilde{\mathrm{g}}_{\alpha\beta} &= e^{2\psi}\,\mathrm{g}_{\alpha\beta}\\ \tilde{\mathrm{g}}_{00} &= e^{2\psi}\,\mathrm{g}_{00} \approx -(1+2\psi)(1+2\Phi_N)\\ &\approx -(1+2\Phi_N+2\psi)\end{aligned}$$

[4]Notice that, unlike the nonrelativistic case, here $\mu(x) \equiv \frac{df}{dx}$, while in the former case was $\tilde{\mu}(\sqrt{x}) \equiv \frac{d\tilde{f}}{dx}$

Since $\tilde{g}_{00} \approx -(1 + 2\Phi_N + 2\psi)$, the nonrelativistic gravitational potential is $\Phi = \Phi_N + \psi$, so that the acceleration of a test particle within this description would be:

$$\mathbf{a} = -\nabla\Phi = -\nabla(\Phi_N + \psi) \tag{3.14}$$

Then, this extra scalar will replace the gravitational potential of the dark matter[5], [3], being attached to a modification of the gravitational theory in the suitable limit rather than an "artificial" addition.

Let's show how, in this limit, the field equation (3.13) reduces to its nonrelativistic form, (3.6):

- In stationary weak fields $\partial_\mu \equiv \partial_t + \nabla = \nabla$. Then:

$$g^{\alpha\beta}\,\partial_\alpha\psi\partial_\beta\psi = (\nabla\psi)^2 \tag{3.15}$$

$$g^{\mu\nu}\,\partial_\nu\psi = \nabla\psi \tag{3.16}$$

- In this stationary configuration, the energy-momentum tensor has the components:

$$T_{00} = \rho(x) \tag{3.17}$$

$$T_{ij} = 0 \qquad \forall i, j \tag{3.18}$$

  Then, the scalar $\tilde{g}^{\alpha\beta}\tilde{T}_{\alpha\beta}$ will reduce to the mass density ρ.

Putting all together, one would get:

$$\nabla \cdot \left[\mu(|\nabla\psi|^2/a_0^2) \cdot \nabla\psi\right] = 4\pi G\rho \tag{3.19}$$

Comparing with the AQUAL field equation (3.6), it is easy to see that both equations have the same form.

Let us now study the behaviour of this theory at different scales in this previous limit:

- Let's suppose that $|\nabla\psi| \ll a_0$, so that:

$$\text{since } x \ll 1 \rightarrow f(x) = \frac{2}{3}x^{3/2}$$

$$\mu\,(x) = x^{1/2} \ll 1, \qquad x \equiv \frac{|\nabla\psi|^2}{a_0^2}$$

  If one compares the equation (3.19) with Poisson's equation (3.7), one can see that $\nabla\Phi_N/\nabla\psi = \mu \ll 1$, that is to say, $\nabla\Phi_N \ll \nabla\psi$. Then, according to equation (3.14):

$$\nabla\Phi \approx \nabla\psi \tag{3.20}$$

  This is the confirmation that, indeed, RAQUAL and AQUAL coincide in the appropiate limit (and, by doing so, **we can attribute AQUAL's achievements to RAQUAL**).

[5]In General Relativity, $\Phi = \Phi_N$ comes from the Poisson's equation requiring dark matter in the source. However, in this theory with $\Phi = \Phi_N + \psi$, both Φ_N and ψ have only visible matter as sources.

- If now $|\nabla\psi| \gg a_0$:

$$\text{since } x \gg 1 \rightarrow f(x) = x$$
$$\mu(x) \approx 1, \qquad x \equiv \frac{|\nabla\psi|^2}{a_0^2}$$

Following the same process as before, one can see that $\nabla\Phi_N \approx \nabla\psi$, and thus:

$$\nabla\Phi \approx \nabla(\Phi_N + \Phi_N) \tag{3.21}$$
$$\nabla\Phi \approx 2\nabla\Phi_N \tag{3.22}$$

In this case it looks like the particle's acceleration in this theory is twice the Newtonian one. However, this actually means that the measurable Newtonian constant G_N is twice the bare constant G that we have in the Lagrangian density for the scalar field, (3.11).

3.3.4 Failure of RAQUAL and proposed solutions

Although this RAQUAL theory correctly recovers both its nonrelativistic version and Newtonian dynamics, it has serious problems that prevent it from being a valid theory or, at least, the last version of it. As we mentioned before, an acceptable relativistic theory must avoid superluminal propagation since it violates the causality principle. It can be shown that, due to the aquadratic form of the Lagrangian density of the scalar field, the ψ waves can propagate faster than the speed of light.

Another relevant problem that is found in RAQUAL is the gravitational lensing. It has been observed an anomalously strong light bending that is commonly attributed to the presence of dark matter. If we want our alternative theory to describe this strong light deflection *without* invoking dark matter, the metric must be significantly different from that in General Relativity without dark matter [3]. Since in this proposal the metrics are conformally related, (3.10), light paths cannot be distinguished in the two metrics: null cones of the physical metric coincide with the lightcones of the Einstein metric. Thus, this anomalously strong light bending cannot be obtained.

It is clear then that the RAQUAL formulation is not the final answer for the physical basis matter. The next question must be then: what can be done to solve this? As we saw in section 3.3.2, one of the reasons for which we proposed a scalar-tensor theory was that we are looking for a theory that reproduces MOND dynamics *while* describing the anomalously large gravitational lensing observed. Then, since RAQUAL fails in this matter, it has been shown that adding just one scalar field is not enough. Therefore, we could try to add extra degrees to the formulation and to break the conformal relation of the metrics.

From this arise some proposed solutions, for instance, the *Phase Coupling Gravitation* or *PCG*. This two-scalar field theory comes from changing the real scalar field ψ in RAQUAL by a complex field in order to avoid the acausal propagation:

$$\chi = Ae^{i\phi} \tag{3.23}$$

For this complex field, the Lagrangian density would take again an aquadratic form:

$$\mathcal{L}_s = \frac{1}{2}A^2\partial_\alpha\phi\partial^\alpha\phi + \partial_\alpha A\partial^\alpha A + V(A^2) \tag{3.24}$$

with $V(A^2)$ a potential associated with the scalar field [21]. In this theory, only the phase couples to matter, hence its name.

However, this theory also suffers from the gravitational lensing problem along with inducing instabilities in bound systems [5]. PCG is not then the correct theory for MOND formulation.

Another more promising proposal is the so-called *Tensor-Vector-Scalar theory* or *TeVeS*. This theory starts from a different definition for the physical metric [20]:

$$\tilde{\mathrm{g}}_{\alpha\beta} = e^{-2\phi}\left(\mathrm{g}_{\alpha\beta} + \mathfrak{U}_\alpha \mathfrak{U}_\beta\right) - e^{2\phi}\mathfrak{U}_\alpha \mathfrak{U}_\beta \tag{3.25}$$

It introduces one nondynamical scalar field σ and three dynamical gravitational fields:

- The Einstein metric $\mathrm{g}_{\alpha\beta}$ so that $\mathrm{g}_{\mu\rho}\,\mathrm{g}^{\rho\nu} = \delta^\nu_\mu$ (the inverse is well defined)
- A timelike 4-vector field $\mathfrak{U}_\mu$ that verifies $\mathrm{g}^{\alpha\beta}\,\mathfrak{U}_\alpha \mathfrak{U}_\beta = -1$
- A scalar field ϕ

Due to its complexity, we will not deepen on its formulation. Nevertheless, it is important to mention its advantages over the previous theories. As we stressed before, the main two recurrent problems perceived are the gravitational lensing and the acausal propagation. TeVeS, on the contrary, predicts gravitational lensing of the correct magnitude in the low acceleration regime in agreement with the intergalatic lensing observations *without* invoking dark matter [6], while reproducing the MOND formulation in the appropiate limit. For the acausal problem (previously approached with PCG but showing other unsatisfactory results), the way in which TeVeS is constructed enables that ϕ waves can propagate causally [3], being consistent with causality.

Nonetheless, and despite its remarkable benefits, it presents some problems that remain to be fixed. For instance, it does not perfectly recover Newtonian limit in the outer solar system.

In any case, this theory does really well both phenomenologically and conceptually, solving the main problems that invalidate the previous theories.

Chapter 4

Conclusions

As we have seen, it is clear that the Cold Dark Matter theory is currently far from being the answer to the problem of the missing mass. It is convenient then to expand the study of this problem to other points of view, such as modifying the gravitational theories for the scales where they fail instead of adding an unknown component to them. Then, even though MOND is not the final answer neither, its phenomenological accomplishments show that this is a particularly interesting interpretation. In fact, its main problem (the absence of a physical basis that describes it) has been approached in several occasions with reasonable success: the proposed theories are able to recover this MOND formula in the proper limit, providing a physical explanation to the algorithm. Despite some observed theoretical problems, it has been shown that the first relativistic proposal for MOND and its later proposed solutions have indeed a promising future. It is then clear that there is still a lot to investigate and discover from this MOND formulation.

We have also seen that modifying Einstein's General Relativity by adding a scalar field is a compelling first step towards the theoretical formulation of MOND. This relativistic theory, RAQUAL, has very satisfactory empirical results while solving many of the conceptual problems of the original MOND formula. It is true that, as we stressed before, it is just the first step since it has important theoretical problems: the acausality and the gravitational lensing. However, the TeVeS theory reinforces the validity of modifying General Relativity as a proposal since it successfully solves the mentioned problems.

Appendix A

Appendix

A.1 Einstein's field equation

When we vary the action (2.29), we would get:

$$\begin{aligned}\delta S_{EH} &= \int d^4x \delta \left(\sqrt{-\mathrm{g}}\, \mathrm{g}^{\mu\nu}\, R_{\mu\nu}\right) \\ &= \int d^4x \left[\delta\sqrt{-\mathrm{g}}\, \mathrm{g}^{\mu\nu}\, R_{\mu\nu}\sqrt{-\mathrm{g}}\delta\, \mathrm{g}^{\mu\nu}\, R_{\mu\nu} + \sqrt{-\mathrm{g}}\, \mathrm{g}^{\mu\nu}\, \delta R_{\mu\nu}\right] \end{aligned} \tag{A.1}$$

Let's now work on $\delta\sqrt{-g}$ and $\delta R_{\mu\nu}$:

- $\delta\sqrt{-g}$: For any diagonalisable matrix it is verified:

  $$\log(\det A) = \mathrm{tr}(\log A)$$

  So that:

  $$\begin{aligned}\frac{1}{\det A}\delta \det A &= \mathrm{tr}(A^{-1}\delta A) \\ \delta \det A &= \det A\, \mathrm{tr}(A^{-1}\delta A)\end{aligned}$$

  Then, for $\delta\sqrt{-\mathrm{g}}$ we can compute:

  $$\begin{aligned}\delta\sqrt{-\mathrm{g}} &= \frac{1}{2\sqrt{-\mathrm{g}}}\delta \det(-\mathrm{g}_{\mu\nu}) \\ &= \frac{1}{2}\sqrt{-\mathrm{g}}(\mathrm{g}^{\mu\nu}\, \delta\, \mathrm{g}_{\mu\nu})\end{aligned}$$

  Since it is convenient to express it in terms of $\delta\, \mathrm{g}^{\mu\nu}$, we must first show:

  $$\begin{aligned}\mathrm{g}_{\mu\nu}\, \mathrm{g}^{\mu\nu} &= 1 \\ \delta\, \mathrm{g}_{\mu\nu}\, \mathrm{g}^{\mu\nu} + \mathrm{g}_{\mu\nu}\, \delta\, \mathrm{g}^{\mu\nu} &= 0 \\ \mathrm{g}^{\mu\nu}\, \delta\, \mathrm{g}_{\mu\nu} &= -\mathrm{g}_{\mu\nu}\, \delta\, \mathrm{g}^{\mu\nu}\end{aligned} \tag{A.2}$$

  Thus, with this expression. (A.2), we can write:

  $$\delta\sqrt{-\mathrm{g}} = -\frac{1}{2}\sqrt{-\mathrm{g}}\, \mathrm{g}_{\mu\nu}\, \delta\, \mathrm{g}^{\mu\nu} \tag{A.3}$$

- $\delta R_{\mu\nu}$: As we saw in (2.20), the Riemann curvature in terms of the connection is:

$$R^{\mu}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\mu}_{\rho\lambda} - \Gamma^{\lambda}_{\nu\rho}\Gamma^{\mu}_{\sigma\lambda}$$

We must now remember that the Christoffel symbols are not tensors, that is to say, they do depend on the coordinate system. Thus, we can always choose to work in normal coordinates for any point $p \,\epsilon\, \mathcal{M}$ of the manifold so that $\partial_\rho g_{\mu\nu} = 0$ and then $\Gamma^{\rho}_{\mu\nu} = 0$. Also, if we compute $\delta\Gamma^{\rho}_{\mu\nu}$ evaluated in p, we can write the partial derivatives as covariant derivatives since they only differ by a $\Gamma^{\rho}_{\mu\nu}$ which, in this point, is zero ($\partial_\mu = \nabla_\mu$ in p):

$$\delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\,\mathrm{g}^{\rho\sigma}\left(\nabla_\mu \delta\,\mathrm{g}_{\sigma\nu} + \nabla_\nu \delta\,\mathrm{g}_{\sigma\mu} - \nabla_\sigma \delta\,\mathrm{g}_{\mu\nu}\right) \tag{A.4}$$

It is important to notice that in this expression, both sides are tensors[1], that is to say, it holds in any coordinate system (so it is a general expression). Thus, knowing that $\delta\Gamma^{\rho}_{\mu\nu}$ is a tensor, (A.4), and $\Gamma^{\rho}_{\mu\nu} = 0$ in normal coordinates, one can write evaluated in p:

$$R^{\mu}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho}$$

If we now vary it:

$$\begin{aligned} \delta R^{\mu}_{\nu\rho\sigma} &= \partial_\rho \delta\Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \delta\Gamma^{\mu}_{\nu\rho} \\ &= \nabla_\rho \delta\Gamma^{\mu}_{\nu\sigma} - \nabla_\sigma \delta\Gamma^{\mu}_{\nu\rho} \end{aligned}$$

As we saw before, both sides of this expression are tensors, so it holds generally. Therefore, one can write the expression for the variation of the Ricci tensor as:

$$\delta R^{\mu}_{\nu\rho\sigma} = \nabla_\rho \delta\Gamma^{\mu}_{\nu\sigma} - \nabla_\sigma \delta\Gamma^{\mu}_{\nu\rho} \tag{A.5}$$

Then, computing the Ricci tensor, (2.23):

$$\delta R^{\rho}_{\nu\rho\sigma} \equiv \delta R_{\nu\sigma} = \nabla_\rho \delta\Gamma^{\rho}_{\nu\sigma} - \nabla_\sigma \delta\Gamma^{\rho}_{\nu\rho}$$

In the integral (A.1) we have $\mathrm{g}^{\mu\nu}\,\delta R_{\mu\nu}$:

$$\begin{aligned} \mathrm{g}^{\mu\nu}\,\delta R_{\mu\nu} &= \nabla_\mu\,\mathrm{g}^{\rho\nu}\,\delta\Gamma^{\mu}_{\rho\nu} - \nabla_\mu\,\mathrm{g}^{\mu\nu}\,\delta\Gamma^{\rho}_{\nu\rho} \\ \mathrm{g}^{\mu\nu}\,\delta R_{\mu\nu} &= \nabla_\mu X^\mu \end{aligned} \tag{A.6}$$

With $X^\mu \equiv \mathrm{g}^{\rho\nu}\,\delta\Gamma^{\mu}_{\rho\nu} - \mathrm{g}^{\mu\nu}\,\delta\Gamma^{\rho}_{\nu\rho}$. When we made the integration by parts we used the fact that $\nabla_\alpha\,\mathrm{g}^{\mu\nu} = 0$ since we are in a metric spacetime (this does not hold in the other formulations of GR), otherwise we would have got an extra term.

[1] As we saw in equation (2.14), there is an extra term in the transformation of the Christoffel symbol that avoids it to transform as a tensor. However, when we vary it, since we are taking the difference between $g_{\mu\nu}$ and $g_{\mu\nu} + \delta g_{\mu\nu}$, this term, which does not depend on the metric, cancels, proving that $\delta\Gamma^{\rho}_{\mu\nu}$ transforms as a tensor.

Let's input (A.3) and (A.6) in (A.1):

$$\begin{aligned}\delta S_{EH} &= \int d^4x \left[-\frac{1}{2}\sqrt{-\mathrm{g}}\,\mathrm{g}_{\mu\nu}\,\delta\,\mathrm{g}^{\mu\nu}\,R + \sqrt{-\mathrm{g}}\delta\,\mathrm{g}^{\mu\nu}\,R_{\mu\nu} + \sqrt{-\mathrm{g}}\nabla_\mu X^\mu \right] \\ &= \int d^4x\sqrt{-\mathrm{g}}\left[\left(-\frac{1}{2}R\,\mathrm{g}_{\mu\nu} + R_{\mu\nu} \right)\delta\,\mathrm{g}^{\mu\nu} + \nabla_\mu X^\mu \right]\end{aligned}$$

Since the second term is a total derivative, we can ignore it so that:

$$\delta S_{EH} = \int d^4x\sqrt{-\mathrm{g}}\left(-\frac{1}{2}R\,\mathrm{g}_{\mu\nu} + R_{\mu\nu} \right)\delta\,\mathrm{g}^{\mu\nu} \tag{A.7}$$

If we extremise the action $\delta S_{EH} = 0$, we can see:

$$G_{\mu\nu} \equiv -\frac{1}{2}R\,\mathrm{g}_{\mu\nu} + R_{\mu\nu} = 0 \tag{A.8}$$

If we now contract it, we get that $R = 0$, thus, going back to (A.8) this leads us to:

$$R_{\mu\nu} = 0$$

A.1.1 Cosmological constant

When we add the cosmological constant to S_{EH}, the field equation is derived analogously to the former case. We need to introduce now the proper factor to verify that the action has the correct dimensions (we should have included this factor before as well, but since it was one term it does not change the result):

$$\delta S = \frac{1}{16\pi G}\int d^4x\sqrt{-\mathrm{g}}\,(R - 2\Lambda) \tag{A.9}$$

As we did before, we vary the action to impose the principle of least action:

$$\begin{aligned}\delta S &= \frac{1}{16\pi G}\int d^4x\left[\delta(\sqrt{-\mathrm{g}}R) - 2\delta\sqrt{-\mathrm{g}}\Lambda\right] \\ &= \frac{1}{16\pi G}\int d^4x\left[\sqrt{-\mathrm{g}}G_{\mu\nu}\delta\,\mathrm{g}^{\mu\nu} - 2\Lambda\left(-\frac{1}{2}\sqrt{-\mathrm{g}}\,\mathrm{g}_{\mu\nu}\,\delta\,\mathrm{g}^{\mu\nu}\right)\right] \\ &= \frac{1}{16\pi G}\int d^4x\sqrt{-\mathrm{g}}\left(G_{\mu\nu} + \Lambda\,\mathrm{g}_{\mu\nu}\right)\delta\,\mathrm{g}^{\mu\nu}\end{aligned}$$

Extremising the action, $\delta S = 0$, we get:

$$G_{\mu\nu} + \Lambda\,\mathrm{g}_{\mu\nu} = 0 \tag{A.10}$$

If we contract this, we would get $R = 4\Lambda$, then, using this in (A.10):

$$R_{\mu\nu} = \Lambda\,\mathrm{g}_{\mu\nu} \tag{A.11}$$

Which are the vacuum Einstein equations in the presence of a cosmological constant.

A.2 Energy-Momentum tensor of a Klein-Gordon scalar

In order to study the different approaches to MOND it is convenient to properly understand how to work with the variational principle. It can be useful then to see how to derive the expression for the energy-momentum tensor. In this example, we will study the case for a Klein-Gordon scalar.

We will add to the Einstein-Hilbert action the following Lagrangian for a scalar:

$$\mathcal{L}_\phi = \frac{1}{2}\,\mathrm{g}^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 \tag{A.12}$$

The action describing the dynamics for this case would then be written as:

$$S = \int d^4x\sqrt{-\mathrm{g}}\left(R + \frac{1}{2}\,\mathrm{g}^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right) \tag{A.13}$$

Let us now vary it with respect to the metric:

$$\begin{aligned}\delta S = \int d^4x \Big[&\delta\sqrt{-\mathrm{g}}\,\mathrm{g}^{\mu\nu}\,R_{\mu\nu} + \sqrt{-\mathrm{g}}\delta\,\mathrm{g}^{\mu\nu}\,R_{\mu\nu} + \sqrt{-\mathrm{g}}g^{\mu\nu}\delta R_{\mu\nu} + \delta\sqrt{-g}\frac{1}{2}\,\mathrm{g}^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi \\ &- \delta\sqrt{-g}\frac{1}{2}m^2\phi^2 + \frac{1}{2}\sqrt{-\mathrm{g}}\delta\,\mathrm{g}^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi\Big]\end{aligned}$$

Since we already studied it in the previous section, we now know that the first two terms, corresponding to the variation $\delta(\sqrt{-\mathrm{g}}\,\mathrm{g}^{\mu\nu}\,R_{\mu\nu})$ will give $\sqrt{-\mathrm{g}}G_{\mu\nu}\delta\,\mathrm{g}^{\mu\nu}$ plus a boundary term that we can dismiss. The other terms can be calculated analogously by making use of equation (A.3). So then, we will get:

$$\begin{aligned}\delta S &= \int d^4x \Big[\sqrt{-\mathrm{g}}G_{\mu\nu}\delta\,\mathrm{g}^{\mu\nu} - \frac{1}{4}\sqrt{-g}\,\mathrm{g}_{\mu\nu}\,\delta\,\mathrm{g}^{\mu\nu}\,\mathrm{g}^{\alpha\beta}\,\partial_\alpha\phi\partial_\beta\phi + \frac{1}{4}\sqrt{-\mathrm{g}}\,\mathrm{g}_{\mu\nu}\,\delta\,\mathrm{g}^{\mu\nu}\,m^2\phi^2 \\ &\qquad + \frac{1}{2}\sqrt{-\mathrm{g}}\partial_\mu\phi\partial_\nu\phi\delta\,\mathrm{g}^{\mu\nu}\Big] \\ &= \int d^4x\sqrt{-\mathrm{g}}\left[G_{\mu\nu} - \frac{1}{4}\,\mathrm{g}_{\mu\nu}\,\mathrm{g}^{\alpha\beta}\,\partial_\alpha\phi\partial_\beta\phi + \frac{1}{4}\,\mathrm{g}_{\mu\nu}\,m^2\phi^2 + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi\right]\delta\,\mathrm{g}^{\mu\nu} \\ &= \int d^4x\sqrt{-\mathrm{g}}\left(G_{\mu\nu} + \frac{1}{2}T_{\mu\nu}\right)\delta\,\mathrm{g}^{\mu\nu} \\ &= 0\end{aligned}$$

The energy-momentum tensor is then:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\,\mathrm{g}_{\mu\nu}\left(\partial_\alpha\phi\partial^\alpha\phi - m^2\phi^2\right) \tag{A.14}$$

A.3 MOND formula: violation of conservation laws

Despite being successful in terms of phenomenology, one of the biggest flaws of the MOND formula is the violation of momentum and angular momentum conservation. This

comes from the fact that the accelerations given to a test particle by two or more attracting bodies acting together do not add linearly, [13].

Let's first study the dynamics of a system of two particles, assuming that the acceleration of the particles follow Milgrom's formula, $g = \sqrt{g_N a_0}$. Assume then a system consisting of two particles with masses m_1 and m_2 that interact gravitationally. For simplicity, let them be placed as follows:

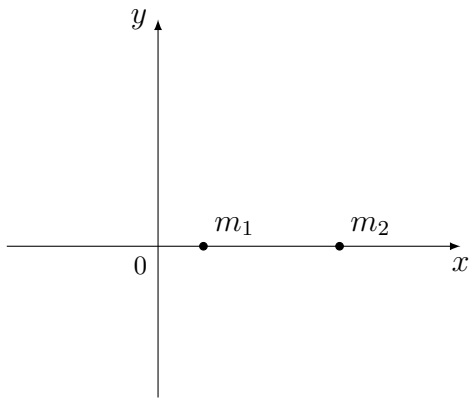

Figure A.1: System of two particles interacting gravitationally with $x_2 - x_1 \equiv r > 0$

In order to study MOND formula, these two masses must be small enough so that they follow $g = \sqrt{g_N a_0}$. Since this algorithm just applies in the non relativistic limit, we must assume that the masses are constant and the gravitational force will be the usual Newtonian one:

$$F_N = G\frac{m_1 m_2}{r^2}$$

Let us now study the conservation of the linear momentum, with the total momentum being $p \equiv p_1 + p_2$. Then, if we differentiate it:

$$\frac{dp}{dt} = \frac{dp_1}{dt} + \frac{dp_2}{dt}$$

$$= m_1\sqrt{a_0 g_{N_1}} - m_2\sqrt{a_0 g_{N_2}}$$

$$= \sqrt{a_0 F_n}\,(\sqrt{m_1} - \sqrt{m_2})$$

For the momentum to be conserved it is required that $\dot{p} = 0$. In the general case where $m_1 \neq m_2$, this does not hold (due to the fact that, unlike the Newtonian case, the two accelerations are not inversely proportional to the masses).

We can also show that the Newtonian center of mass theorem, equation A.15, does not hold in this original MOND formulation.

$$M\ddot{R}_{CM} = \sum_i m_i \ddot{r}_i \tag{A.15}$$

For this purpose, let us study a system S formed by two masses as before, but now placing this system in the gravitational field of a larger body of mass m_3:

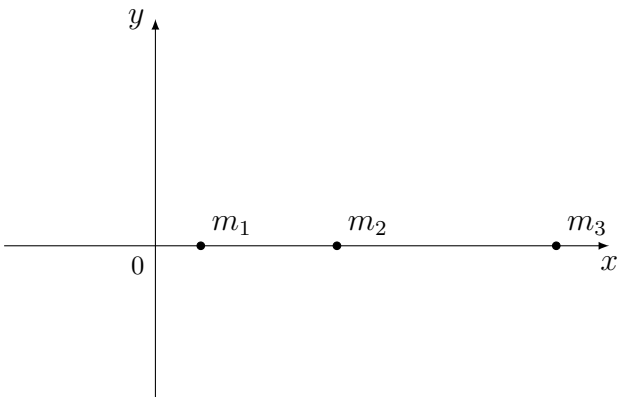

Figure A.2: System of two particles with $x_2 - x_1 \equiv r > 0$ in a gravitational field created by a particle m_3 so that $x_3 - x_2 \equiv R > 0$

We need to consider m_1 and m_2 small enough to dismiss their interaction while letting R being large enough so that the MOND acceleration can still apply to the motion of the particles 1,2 in the field of m_3. The mass m_3 produces a Newtonian acceleration g_{N_i}, $i = 1, 2$, at particle i.

By definition, the center of mass of the system S is written as:

$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{A.16}$$

By differentiating, we can see that the acceleration of the c.o.m. of the system S is, in terms of g_{N_i}:

$$\begin{aligned} \ddot{x}_{CM} &\equiv \frac{m_1 \ddot{x}_1 + m_2 \ddot{x}_2}{m_1 + m_2} \\ &= \frac{m_1 \sqrt{a_0 g_{N_1}} + m_2 \sqrt{a_0 g_{N_2}}}{m_1 + m_2} \\ &= \frac{\sqrt{a_0}}{m_1 + m_2} \left(m_1 \sqrt{g_{N_1}} + m_2 \sqrt{g_{N_2}} \right) \end{aligned} \tag{A.17}$$

Now let's see if the Newtonian theorem of the c.o.m., (A.15), holds. In this case, the external forces that are acting on the system S are F_{N_1} and F_{N_2}. If we apply the theorem to the MOND formulation with $g \approx \sqrt{a_0 g_N}$, we would get:

$$\begin{aligned} \ddot{x}_{CM} &= \sqrt{a_0 \frac{F_{N_1} + F_{N_2}}{m_1 + m_2}} \\ &= \sqrt{\frac{a_0}{m_1 + m_2}} \sqrt{m_1 g_{N_1} + m_2 g_{N_2}} \end{aligned} \tag{A.18}$$

Comparing equations (A.17) and (A.18) we see that the accelerations will only coincide if $g_{N_1} = g_{N_2}$. Thus, if indeed the individual test particles in an external field follow MOND

formula, the c.o.m. obeys the theorem only for a uniform external field. In any case, even if this external field is zero, the theorem would not be verified if one could not ignore the active gravitational mass of the particles. This shows the need of propose a complete underlying theory for this algorithm that does not describe properly the dynamics for particles.

A.4 Newtonian limit in GR

It might be handy to understand which approximations must be performed to get the Newtonian limit from General Relativity, since in the relativistic formulation of AQUAL we make use of some of the following results.

By Newtonian limit, we mean that we work with a weak field limit around flat space. Then, the gravity must be weak so that the metric is almost flat except for a small perturbation:

$$\mathrm{g}_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \tag{A.19}$$

Where $|h_{\mu\nu}(x)| \ll 1$ is the metric perturbation and $\eta_{\mu\nu}$ is the Minkowski metric. We also need to consider a stationary gravitational field, so that $\frac{\partial \mathrm{g}_{\mu\nu}}{\partial t} = 0$. When we calculate the covariant components of the metric, we get at the leading order:

$$\mathrm{g}^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{A.20}$$

We can now recalculate the field equations, starting from the connection $\Gamma^{\sigma}_{\nu\rho}$:

$$\Gamma^{\sigma}{}_{\nu\rho} = \frac{1}{2}\eta^{\sigma\lambda}\left(\partial_\nu h_{\lambda\rho} + \partial_\rho h_{\nu\lambda} - \partial_\lambda h_{\nu\rho}\right) \tag{A.21}$$

Thus, since the Riemann curvature (and so, the Ricci tensor and scalar) can be derived from the connection, we get to linear order:

$$R^{\sigma}{}_{\rho\mu\nu} = \frac{1}{2}\eta^{\sigma\lambda}\left(\partial_\mu\partial_\rho h_{\nu\lambda} - \partial_\mu\partial_\lambda h_{\nu\rho} - \partial_\nu\partial_\rho h_{\mu\lambda} + \partial_\nu\partial_\lambda h_{\mu\rho}\right) \tag{A.22}$$

$$R_{\mu\nu} = \frac{1}{2}\left(\partial^\rho\partial_\mu h_{\nu\rho} + \partial^\rho\partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right) \tag{A.23}$$

$$R = \partial^\mu\partial^\nu h_{\mu\nu} - \Box h \tag{A.24}$$

Where $\Box \equiv \partial_\mu\partial^\mu$ and $h \equiv h^\mu_\mu$. Thus, the Einstein tensor, (2.30), would be:

$$G_{\mu\nu} = \frac{1}{2}\left[\partial^\rho\partial_\mu h_{\nu\rho} + \partial^\rho\partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \left(\partial^\mu\partial^\nu h_{\mu\nu} - \Box h\right)\eta_{\mu\nu}\right] \tag{A.25}$$

Now that we can write the Einstein equations, (2.34), for the case where the gravity is weak and stationary:

$$\left[\partial^\rho\partial_\mu h_{\nu\rho} + \partial^\rho\partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \left(\partial^\mu\partial^\nu h_{\mu\nu} - \Box h\right)\eta_{\mu\nu}\right] = 16\pi G T_{\mu\nu} \tag{A.26}$$

Under the de Donder gauge ($\Gamma^{\alpha}_{\beta\gamma} g^{\beta\gamma} = 0$)[2] and with the definition $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$ the linearised equations (A.26) simplify:

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \tag{A.27}$$

[2]This is not relevant for the topic, so we will not deepen into it.

So now, we can implement the Newtonian limit, where $\Box = \nabla^2$ since, as we mentioned, the gravitational field is stationary. In such a limit, the energy-momentum tensor has the components:

$$T_{00} = \rho(x) \tag{A.28}$$
$$T_{ij} = 0 \qquad \forall i, j \tag{A.29}$$

Thus, we would get the components for $h_{\mu\nu}$:

$$\nabla^2 \bar{h}_{00} = -16\pi G\rho(x) \tag{A.30}$$
$$\nabla^2 \bar{h}_{0i} = \nabla^2 \bar{h}_{ij} = 0 \tag{A.31}$$

From the Poisson's equation (3.7) and choosing the suitable boundary conditions, we finally get:

$$h_{00} = -2\Phi_N(x) \tag{A.32}$$
$$h_{ij} = -2\Phi_N \delta_{ij} \tag{A.33}$$
$$h_{0i} = 0 \tag{A.34}$$

Thus, knowing that the metric is approximated by (A.19), we get the components:

$$\mathrm{g}_{00} = -1 - 2\Phi_N(x) \tag{A.35}$$
$$\mathrm{g}_{ij} = (1 - 2\Phi_N(x))\delta_{ij} \tag{A.36}$$
$$\mathrm{g}_{0i} = 0 \tag{A.37}$$

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