University of Groningen

Department of Physics

Bachelor Thesis

Hidden Momentum and Symmetry Constraints

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Abstract

The existence of a relativistic mechanical internal momentum in a motionless system seems to contradict basic notions of momentum conservation. In the present paper, this "hidden momentum" is analyzed in various systems and the requirements for its existence are identified. A mathematical formulation of "hidden momentum" is presented through the use of continuous symmetries and its occurrence is put into context of locally conserved currents and conserved charges.

Contents

1 Introduction

Physically realizable systems are studied and analyzed based on a series of universally accepted axioms. These axioms are first derived from daily experiences with systems that can be evaluated using classical laws of physics. Conservation of matter, energy, linear and angular momenta are the basic assumptions taken for granted in any system and they are verified by observing the motions of planets, examining thermodynamic systems such as engines or simply two balls colliding with each other. These conservation laws are now understood to come from symmetries in physical systems. As other branches of physics were being developed, such as electrodynamics, quantum mechanics and special relativity to mention some, these previously observed axioms were also found to hold in general terms. However, theoretical systems have been proposed in several areas where there would be an apparent violation of one or more of these conservation laws which would challenge the validity of our understanding of physics. In the following section, some of these paradoxes will be discussed, which challenge the notion of conservation of momentum in an isolated system.

1.1 Non-Conservation of Momentum Paradoxes

1.1.1 Jefimenko's Paradox

In terms of momentum conservation, an interesting hypothetical formulations is Jefimenko's Paradox.[1] First we will consider the system from the lab's frame Σ' as represented in Figure 1a. Particles 1 and 2 have the same charge and are moving exclusively on the x-axis. There is some distance r' between the particles at $t = 0$ and the particles' masses are assumed to be large so as to deem the acceleration as negligible. The electric field will be calculated using the Heaviside Formula (Equation (1))[2]. In this system $\theta' = 0$ because the particles motion is in 1D and $r' = (x'_2 - x'_1)$.

$$\vec{E}' = \frac{q(1 - \frac{v'^2}{c^2})}{4\pi\epsilon_0 r'^3[1 - (\frac{v'}{c}\sin\theta')^2]^{\frac{3}{2}}}\vec{r}' \tag{1}$$

As it's to be expected, the particles' coulomb forces form a Newton pair; they each feel an equal and opposite force acting upon them. Thus, there is no net force and no change in the system's momentum since $\sum_{i=1}^{N}\vec{F} = \frac{d\vec{p}_i}{dt}$ on this isolated system in this reference frame. Yet, the physics of this problem doesn't seem to agree with the previous statement when viewed from particle 1's reference frame $\sum$ shown in Figure 1b. In this frame, particle 1 is stationary ($|\vec{v}| = 0$) and thus produces a non-relativistic electric field. This creates an apparent imbalance in the net force of the system as shown in Equation (2). In this reference frame there is a net change in the system's momentum and contradicts the analysis previously performed.

$$\sum\vec{F} = \frac{q_1q_2}{4\pi\epsilon_0(x_2 - x_1)^2}\hat{x} - \frac{q_1q_2(1 - \frac{v_2^2}{c^2})}{4\pi\epsilon_0(x_2 - x_1)^2}\hat{x} = \frac{kq_1q_2v_1^2}{c^2(x_2 - x_1)^2}\hat{x} \tag{2}$$

Conservation laws are also valid in special relativity so that a system's energy or momentum should be the same regardless of the inertial frame of choice. This apparent violation of the conservation of momentum can be fixed by considering that the electromagnetic field also carries momentum within the system. Even though the charges do not feel any magnetic forces due to their positions and velocities, they still interact with the electromagnetic field created by each other. As a charge moves, the EM field it produces changes and thus there is a non-zero change of the electromagnetic momentum $\vec{p}_{field}$. We have decided to use a coulomb gauge to determine the field momentum of the configuration since one of the particles is stationary which makes the calculation straightforward. This interaction electromagnetic momentum is $\vec{p}_{field} = d\vec{A}$ where $\vec{A}_2(t) = \frac{q_2v_2}{4\pi\epsilon_0c^2(x+v_2t)}\hat{x}$ is the vector potential produced by particle 2. The change in the electromagnetic momentum evaluated at $t = 0$ becomes:

$$\frac{d\vec{p}_{field}}{dt} = -\frac{kq_1q_2v_1^2}{c^2(x_2 - x_1)^2}\hat{x} \tag{3}$$

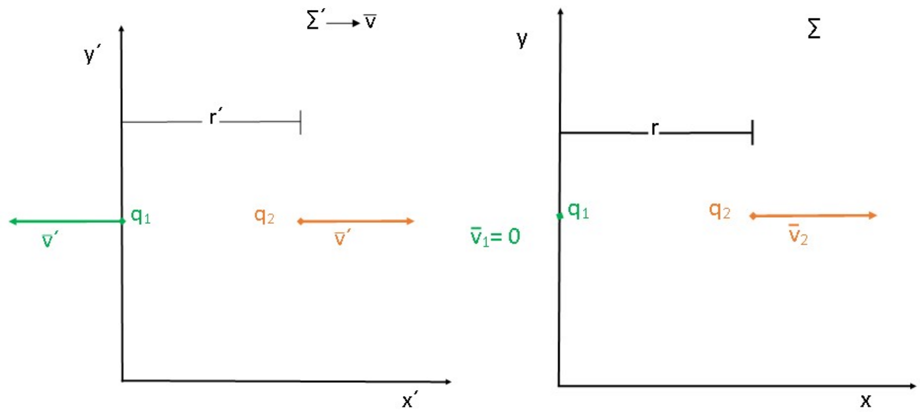

Figure 1: Jefimenko's Paradox: Lab Frame (Figure 1a on the left) q_1 frame (Figure 1b on the right)

This value is equal and opposite to the non-conserved momentum derived from Equation (3) and follows the principle that the canonical momentum, the momentum that is conserved in electromagnetic interactions, is $\frac{d\vec{P}_{\text{Canonical}}}{dt} = \frac{d(p_{\text{mech}}+p_{\text{field}})}{dt} = 0$. This paradox was solved by taking into account not only the mechanical momentum exerted on the charged particles due to the scalar potential Φ but also the momentum derived from the interactions between charges and the electromagnetic field.

1.1.2 Feynman's Paradox

An extension of Jefimenko's Paradox is Feynman's paradox. As it can be seen from Figure 2, the system is formed by two moving charged particles with charges q_1 and q_2 and velocities $\vec{v_1} = v\hat{x}$ and $\vec{v_2} = v\hat{y}$ respectively. The dilemma of Feynman's Paradox can be spotted by a classical electromagnetic (low-velocity limit) approach. In this case, particle 1 would feel the magnetic field produced by moving charge 2 but not the other way around. The Coulomb forces would form a Newton pair but the Lorentz forces would not create one as seen in Equation (85), where there is a non-zero net force in the $\hat{y}$ direction.

$$\sum \vec{F} = q_1(v_1 \cdot (v_2 \cdot \frac{kq_2}{c^2(x_2-x_1)^2})\hat{y} \tag{4}$$

Thus, there is an overall non-conserved momentum in the $\hat{y}$ direction. The same conclusion can be reached using Lagrangian Mechanics or relativistic Lorentz forces.[3]. The paradox can be solved again by considering the contribution from the electromagnetic momentum p_{field} carried by the fields. Again, the Coulomb Gauge is used to evaluate the momentum linked to the electromagnetic field. In this set-up, both particles have an initial velocity and each have a vector potential that acts on the other particle. The field momentum is calculated using Equation (5) where the vector potential is defined below.

$$\vec{p}_{field} = q_1\vec{A}_2 + q_2\vec{A}_1; \quad \vec{A}_i = \int_{-\infty}^{t} [\vec{E}_i(\tau) + \vec{\nabla}\Phi_i(\tau)]dt \tag{5}$$

$$q\frac{d\vec{A}}{dt} = [\frac{kq_1q_2v_1^2}{c^2(x_2-x_1)^2} + \frac{kq_1q_2}{(x_2-x_1)^2}(\gamma_2 - 1)]\hat{x} + \frac{-kq_2q_1v_1v_2\gamma^2}{c^2(x_2-x_1)^2}\hat{y} \tag{6}$$

The first two terms would correct the momentum in the relativistic approach to Feynman's Paradox, in fact the first term is the one derived from the Jefimenko's paradox previously discussed,

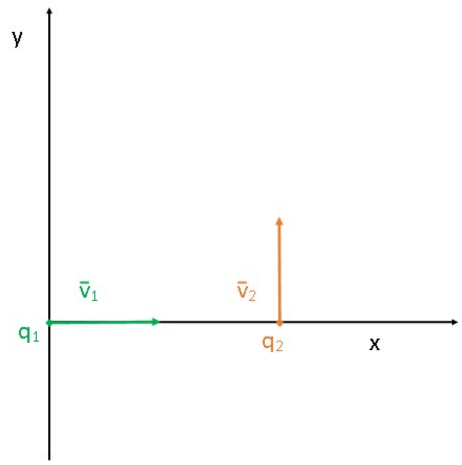

Figure 2: Feynman's Paradox

the second term would belong to the Scanio Paradox[3] where one particle remains motionless and the other moves in a perpendicular direction to the line connecting the two charges at $t = 0$ and the last term corrects the momentum imbalance expressed in Equation (85). In summary, it needs to be noted that electromagnetic field carries and exerts some momentum when we are dealing with EM fields and moving charges in the same system. In all these paradoxes, the key lies in a non-zero field momentum.

1.2 Hidden Momentum

Having introduced some paradoxes which discuss canonical momentum conservation, we can now focus on analysing the system shown in Figure 3.[4] This system has a rectangular wire, with a width w and a length l carrying a steady current inside a homogeneous electric field $\vec{E}$ which is produced by a parallel plate capacitor. In this example, the charges will experience an increasing acceleration on the left side and a decreasing acceleration on the right side and the momenta on these segments cancel each other. This external electric field will thus increase the density of charges on the bottom horizontal segment of the wire and decrease it on the top horizontal segment. On the top segment there are N_t charges moving to the right at a velocity $\vec{v}_t$ and on the bottom segment there are N_b charges moving to the left at velocity $\vec{v}_b$. In this system, $N_b > N_t$ and $\vec{v}_b < \vec{v}_t$. Thus, the classical momentum is conserved described by Equation (7).

$$p_{class} = mN_t\vec{v}_t - mN_b\vec{v}_b = 0 \tag{7}$$

However, if we assume the particles to be moving at relativistic speeds, we would need to add a Lorentz factor and would arrive to Equation (8). The last term is derived since we know that the energy gain of a particle as it goes up the left side of the wire depends on the work done by the electric force.

$$p_{rel} = \gamma_t mN_t\vec{v}_t - \gamma_b mN_b\vec{v}_b = \frac{m\vec{I}l}{q}(\gamma_t - \gamma_b) = \frac{\vec{I}l\vec{E}w}{c^2} \tag{8}$$

The system now has some excess mechanical momentum $\frac{IlEw}{c^2}$, which didn't appear in the classical limit, that is somehow contained in the system but doesn't produce a visible manifestation in the system. This imperceptible mechanical momentum is what we will refer to as hidden momentum. Taking a look at the electromagnetic momentum, it will be carried in the system with the magnitude and direction of the Poynting Vector[5] (divided by c^2), which is calculated below.

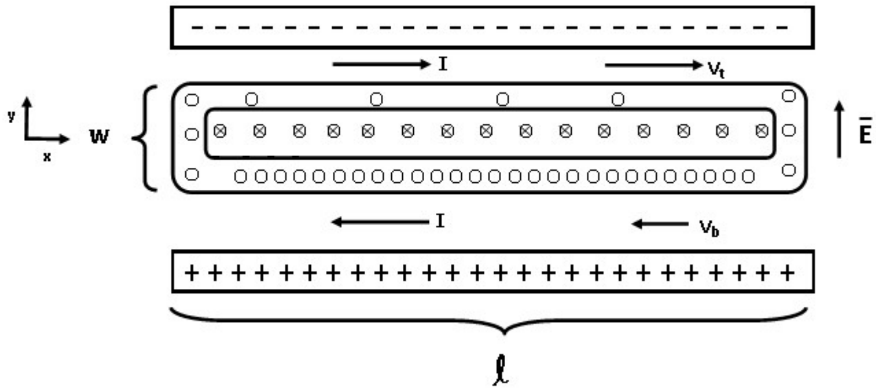

Figure 3: Rectangular wire in capacitor's electric field.

$$\vec{p}_{field} = \frac{1}{c^2}\vec{S} = -\frac{1}{c^2}\vec{I}lw\vec{E} \tag{9}$$

We can clearly see the electromagnetic momentum density in this static system is equal and opposite to the relativistic hidden momentum which would make the total momentum of the system vanish, as it is required for an isolated static system. However, the puzzling thing about this observation is that there is no a priori requirement for the electromagnetic field momentum to cancel this hidden momentum as there is no direct relationship between them.

To further understand what properties and conditions must exist for hidden momentum to be present in a system, we continue to analyze the system shown in Figure 4[6] which has a solenoid and a capacitor. First, we are going to impose some simplification on this system that will aid us with the study of this system's properties. The solenoid is made of an N number of squared cables which have edges A, B, C, D that are in positions (1,1), (-1,1), (-1,-1) and (1,-1) respectively. The square cables are one on top of the other in the z-axis to form the solenoid. Furthermore, the cables contain an incompressible charged fluid which flows frictionlessly and the cables are covered by negative charges that screen the $\vec{E}$ coming from the fluid. The fields' overall energy and momentum are related by Equation (10). The system's energy and momentum will be calculated in two different inertial frames Σ and Σ' and we will see how hidden momentum comes up in the derivations.

$$p^{\mu} = \frac{1}{c}\int T^{\mu 0} d^3x \tag{10}$$

The energy-momentum tensor of a system containing electric field and charges does not satisfy null divergence but behaves according to Equation (11)[7]. This statement means that the four-momentum will behave like a false four-vector. A false four-vector doesn't transform like a true 4-vector under Lorentz transformation[8].

$$\frac{\partial}{\partial x^{\mu}} T^{\mu\upsilon}_{\upsilon} = -\frac{1}{c} F^{\mu\upsilon} j_{\upsilon} \tag{11}$$

In the Σ frame, the solenoid is motionless and thus only produces a magnetic field $\vec{B} = B\hat{k}$. The current in the pipe depends on the charge density, fluid's velocity and cross-sectional area of the pipe as shown in Equation (12a) and since the magnetic field is that of a solenoid, it will be described by Equation (12b). The capacitor generates an electric field $\vec{E} = E\hat{j}$. It is to be noted that all self-interaction terms of the solenoid and capacitor are ignored as well as the electric state of the fluid is not altered by the $\vec{E}_{cap}$. Furthermore, the calculations on this system only consider

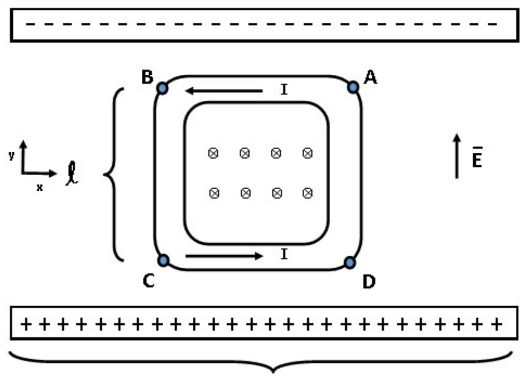

Figure 4: Square solenoid in capacitor's electric field.

the segment $Y = -1$. This is because the results are congruent for both $Y = \pm 1$ and thus to convert the results to the segment $Y = 1$, the sign of the velocity v has to be flipped $v \mapsto -v$.

$$\vec{I} = \rho \vec{v} s \tag{12a}$$

$$\vec{B} = \frac{4\pi}{c} N \rho s v \hat{k} \tag{12b}$$

The calculations are restricted to the volume inside the cube (V_0) hence $V_0 = 8$. Thus using Equation (10), the x-component of the momentum is non-zero and shown below.

$$p_{x(elec)} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{1}{4\pi c} E_y B_z dxdydz = \frac{8}{c^2} N \rho s v E_y \hat{x} \tag{13}$$

It is important to notice that the electromagnetic momentum is compensated by the mechanical pressure gradient since the pressure difference ΔP between Y = 1 and Y = -1 renders a force exerted on the fluid along X = ± 1. f_1 is the difference between the pressure exerted from $\overline{CB}$ and $\overline{AD}$. This force must be equal to the force exerted by the electric field on the charged fluid along the same segments of the wire, described by f_2. Equation (14c) is obtained by equating f_1 to f_2.

$$f_1 = -\Delta P s \hat{j} \tag{14a}$$

$$f_2 = 2 s \rho E \hat{j} \tag{14b}$$

$$\Delta P = 2 \rho E \tag{14c}$$

The fluid parallel to the x-axis on Y = -1 is represented in terms of energy and momentum in Equation (16a) and the same system after a Lorentz boost in Equation (16b). These tensor only take into account the pressure of the solenoid's charged fluid.

$$p_{x(mech)} = -4 \Delta P s N \frac{v}{c^2} \tag{15}$$

Total momentum in the x-axis is conserved. It can be check by plugging (14c) into Equation (15) which equals (13).

$$T^{\mu\upsilon} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{16a}$$

$$T^{\mu\upsilon\prime} = \begin{pmatrix} \epsilon & (\epsilon+p)\frac{v}{c} & 0 & 0 \\ (\epsilon+p)\frac{v}{c} & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{16b}$$

Now examining the system in the Σ' frame reveals an apparent non-conservation of momentum if we only regard the same sources of momentum as in frame Σ. In this frame, the volume is Lorentz contracted and the solenoid and capacitor produce both an electric and magnetic field. These values and the corresponding electromagnetic momentum are shown below.

$$V_0' = \frac{V_0}{\gamma} \tag{17a}$$

$$\vec{B}'_{sol} = \gamma B\hat{k} \quad \vec{E}'_{sol} = \gamma B\frac{u}{c}\hat{j} \tag{17b}$$

$$\vec{E}'_{cap} = \gamma E\hat{j} \quad \vec{B}'_{cap} = \gamma E\frac{u}{c}\hat{k} \tag{17c}$$

$$p'_{x(elec)} = \frac{1}{4\pi c}8\gamma B_z E_y(1+\frac{u^2}{c^2}) = \gamma(1+\frac{u^2}{c^2})p_{x(elec)} \tag{17d}$$

If we examine the fluid from this new reference frame, the four-velocity changes in the direction of the Lorentz Boost and proper time have to be taken into account where proper time (Δt) is defined as $\gamma\frac{u}{c^2}$. Below we can appreciate the difference between the fluid at the left bottom corner of the square pipe and at the right bottom corner of the pipe.

$$x_L^{\mu\prime\prime} = (0, \frac{-1}{\gamma(1+\frac{uv}{c^2})}, -1, z) \tag{18a}$$

$$x_R^{\mu\prime\prime} = (0, \frac{1}{\gamma(1+\frac{uv}{c^2})}, -1, z) \tag{18b}$$

Thus, the charged liquid undergoes a Lorentz contraction $\frac{1}{\gamma(1+u\frac{v}{c^2})}$ unlike the static charge in the insulating material, which is motionless in $\sum$, which undergoes a Lorentz contraction of $\frac{1}{\gamma}$. Since these are not equal, it means that in the $\sum'$ frame, the complete screening of electric fluid doesn't take place. The net charge per unit area is thus:

$$\rho'_{area} = \rho_{area}\gamma u\frac{v}{c^2} \tag{19}$$

The non-vanishing $\vec{E}$ emanating from the $Y = -1$ coming from the solenoid shows that the charge density is non-zero which can be proven using one of Maxwell's equations, Gauss' Law. This equation holds true because Maxwell's Equations are Lorentz covariant.

$$\vec{E}'_{sol} = \gamma\frac{u}{c^2}4\pi N\rho sv\hat{j} = 4\pi\rho'_{area}\hat{j} \tag{20}$$

This result arises in the $\sum$' frame because even though there is no charge density in $\sum$, the current j^μ is nonzero hence a nonzero charge density arises in $\sum$'. Hidden momentum, which is mechanical, arises from pressure related terms from Equation (16a). The required tensor component is shown in Equation (21a).

$$T^{10'}_{press} = \gamma\frac{v}{c}(1+\frac{u^2}{c^2})P_{Y=-1} \tag{21a}$$

$$p_{x(press)} = \frac{-4\gamma}{c^2}(1+\frac{u^2}{c^2})V\Delta Ns = \gamma(1+\frac{u^2}{c^2})p_{x(mech)} \tag{21b}$$

Thus, we have shown that in a reference frame where the system is static, the electromagnetic momentum is equal and opposite to the mechanical momentum. The same analysis in the moving frame, reveals that there is a mechanical component that depends on pressure which is necessarily enforced to be equal and opposite to the electromagnetic momentum.
Lastly, another very similar system[9] to the two systems previously presented would be that shown in Figure 5. There is a parallel plate capacitor with the plates placed orthogonal to the x-axis. The distance between the plates is denoted by d and the thickness of each plate is denoted by w. The plates' faces which point towards each other have charges $\pm\sigma$, respectively. In this example, the solenoid is substituted by a gas which exerts a pressure P onto the capacitors' plates. For the sake of simplifying the discussion, we will assume the permittivity of the gas to be the same as of vacuum's. The attractive force f and the pressure exerted by the gas P balance each other, which makes the system as a whole motionless in the frame Σ. To aid with the discussion, the system will be divided into 3 segments as shown in Figure 6. Segments S_0 and S_2 contain each one of the plates that belong to the capacitor. They both have volume $V_0 = V_2 = Aw$ in frame Σ. The segment S_1 contains the fields and gas of the system, all inside its volume $V_1 = Ad$. We will first analyze the energy and momentum of the system by looking at the gas as a homogeneous fluid, and later on looking at the gas as made up of small particles. From electrostatics, we know that the electric field between the plates is $\vec{E} = 4\pi\sigma\hat{x}$. Assuming the electric field to be homogeneous and since the system is in equilibrium, the force between S_0 and S_2 and the corresponding pressure exerted by the gas would be:

$$\vec{f} = \pm 2\pi\sigma^2 S\hat{x} \tag{22}$$

$$P = 2\pi\sigma^2 = \frac{E^2}{8\pi} \tag{23}$$

The whole system is made up of 3 main elements: the capacitor plates, the electric field and the gas. The energy-momentum tensor $T^{\mu v}$ for each of the 3 elements is calculated below where the subscripts C, G and F indicate the capacitor plates, the gas and the fields, respectively. The energy-momentum tensors that describe electromagnetic fields and non-viscous fluids has been derived and discussed in detail in previous documents[10]. Regarding the parallel plates, the charges are located along the surface and their energy-momentum tensor only contains the energy density of the charges, ϵ_p. The energy-momentum tensor of the gas contains the energy density of the gas and the pressure exerted by the gas due to the electric field spawned by the capacitor. The energy density of the field only has the input of the electric field since there is no magnetic field in this reference frame.

$$T^{\mu v}_C = \begin{pmatrix} \epsilon_C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{24}$$

$$T^{\mu v}_G = \begin{pmatrix} \epsilon_G & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{25}$$

$$T^{\mu v}_F = \frac{E^2}{8\pi}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{26}$$

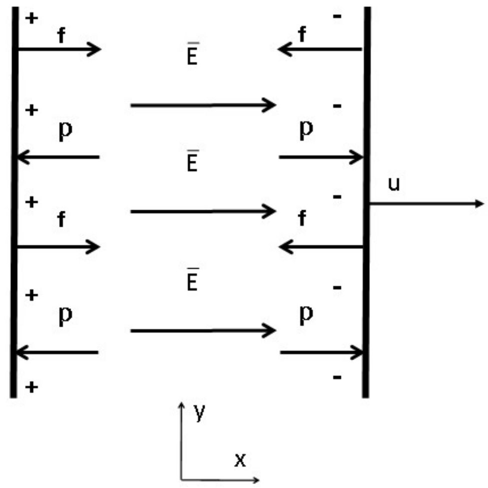

Figure 5: Parallel plate capacitor and gas system assuming the gas to be a homogeneous fluid

As shown by Equations (24), (25), (26), the momentum density for each element of the system, as described by Equation (10), vanishes. The energy of each element described by segments S_0,S_1 and, S_2 is obtained by multiplying each element's energy density by the corresponding volume. This procedure will yield the corresponding four-momenta of each segment:

$$p^{\mu}_{C} = 2V_0(\frac{\epsilon_C}{c}, 0, 0, 0) \tag{27}$$

$$p^{\mu}_{G} = V_1(\frac{\epsilon_G}{c}, 0, 0, 0) \tag{28}$$

$$p^{\mu}_{F} = V_1(\frac{E^2}{8\pi c}, 0, 0, 0) \tag{29}$$

The total four-momentum of the system is simply the sum of the four-momenta of all elements making up the system, which is

$$p^{\mu}_{total} = [2V_0\epsilon_C + V_1(\epsilon_G + P)](\frac{1}{c}, 0, 0, 0) \tag{30}$$

where Equation (23) was used. The four-vector in Equation (30) clearly shows that the system is motionless as it only contains energy, and total momentum of the system is 0. The electric field and the gas exchange some momentum through pressure to the parallel plates and both interactions cancel each other out as can been seen from Equation (23). Now the same system will be examined from a reference frame Σ' by applying a Lorentz Boost in the $\hat{x}$ direction with velocity $\vec{u} = v\hat{x}$.

Applying the Lorentz transformation to the tensors (24), (25), (26)

$$T_C'^{\mu v} = \begin{pmatrix} \gamma^2\epsilon_C & \gamma^2 v\frac{\epsilon_C}{c} & 0 & 0 \\ \gamma^2 v\frac{\epsilon_C}{c} & \gamma^2 v^2\frac{\epsilon_C}{c^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{31}$$

$$T_G'^{\mu v} = \begin{pmatrix} \gamma^2(\epsilon_G + \frac{v^2}{c^2}P) & \gamma\frac{v}{c}(\epsilon_G + P) & 0 & 0 \\ \gamma\frac{v}{c}(\epsilon_G + P) & \gamma^2(P + \frac{v^2}{c^2}\epsilon_G) & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{32}$$

$$T_F'^{\mu v} = \frac{E^2}{8\pi} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{33}$$

The tensor describing the field doesn't change due to the Lorentz invariance of electromagnetic field parallel to Lorentz boosts. In the new frame Σ', the volumes of the different segments will Lorentz contract in the direction of the boost. So multiplying tensors (31), (32) and, (33) by $\gamma\frac{2V_0}{c}$ and $\gamma\frac{V_1}{c}$ correspondingly, will yield the four-momenta of each segment in the new frame.

$$p_C'^{\mu} = \gamma 2V_0\epsilon_C(\frac{1}{c}, \frac{v}{c^2}, 0, 0) \tag{34}$$

$$p_G'^{\mu} = \gamma V_1(\frac{\epsilon_G}{c} + v^2\frac{P}{c^3}, v\frac{(\epsilon_G + P)}{c^2}, 0, 0) \tag{35}$$

$$p_F'^{\mu} = \frac{V_1}{c}(\frac{E^2}{8\pi c}, 0, 0, 0) \tag{36}$$

Adding these up will give a total four-momentum of the system:

$$p_{total}'^{\mu} = \gamma[2V_0\epsilon_C + V_1(\epsilon_G + P)](\frac{1}{c}, \frac{v}{c^2}, 0, 0) \tag{37}$$

To confirm the validity of the Equation (37) it is enough to apply the Lorentz boost to (30) which will yield the same result. Thus, p_{total} is a real four-vector. However, there is some incongruity with the momenta of the fields and the gas. If we Lorentz transform the momenta in Equations (28) and (29), we get:

$$p_G'^{\mu} = \gamma V_1\epsilon_G(\frac{1}{c}, \frac{u}{c^2}, 0, 0) \tag{38}$$

$$p_F'^{\mu} = \gamma V_1 P(\frac{1}{c}, \frac{u}{c^2}, 0, 0) \tag{39}$$

These four-vectors are not equal to the ones retrieved from the transformed energy-momentum tensor. In the same way, it can be checked that (27) is properly transformed and consistent with the values obtained from $T^{\mu v}$. We will refer to Equations (28) and (29) as false four-vectors, since they do not transform properly under Lorentz transformations. Since the total momentum does transform as a four-vector, we can conclude that it is the sum of (25) and (26) that yields a true four-momentum. A quick examination of the inconsistent four-momenta reveals that the terms by which they differ are associated with the pressure of the system.
Furthermore, let us examine the system in regards to the particles making up the gas. Assume that the gas particles only move parallel to the x-axis, collide perfectly elastically with the capacitor's plates and do not interact with each other. We will examine the motion of particle q with mass m. Assuming that the gas particle continually moves and bounces from the left and right plate, the mean energy and momentum of all the particles are related to their instantaneous values. Particle q's motion is shown in Figure 7. In the Σ frame, we assume the particle moves towards the right with velocity $v_r = v_q\hat{x}$ and four-momentum:

$$p_r^{\mu} = m\gamma(c, v_q, 0, 0) \tag{40}$$

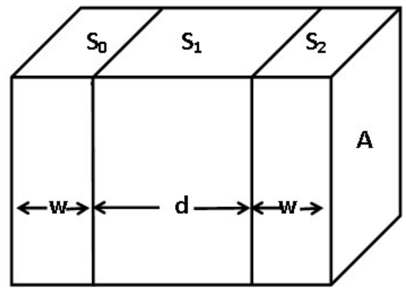

Figure 6: Three segments describing the parallel plates, gas and electric field into subsystems

The particle will remain in this motion for a time $t_r = \frac{d}{v}$. After the particle bounces from the right plate, it will have a velocity with the same magnitude in the opposite direction and a four-momentum equal to (40) but with a velocity $v_l = -v_q\hat{x}$. Thus the mean four-momentum is

$$\bar{p}^{\mu} = m\gamma(c, 0, 0, 0) \tag{41}$$

where the mean is the sum of the momenta of the particle moving to the right and moving to the left inside segment S_1. In the frame Σ' which is moving with speed u in the $\hat{x}$ direction, the four-momentum to the right would be

$$p_r'^{\mu} = m\gamma\gamma_u(c + \frac{uv_q}{c}, u + v_q, 0, 0) \tag{42}$$

The distance between capacitor plates is Lorentz contracted into

$$d' = \frac{d}{\gamma_u} \tag{43}$$

and the particle velocity across this distance is

$$v_r' = \frac{v_q + u}{1 + v_q\frac{u}{c^2}} \tag{44}$$

Thus, using these two previous equations, the new elapsed time while the particle moves to the right is

$$t_r' = \frac{\gamma_u d(1 + u\frac{v_q}{c^2})}{v} \tag{45}$$

Rewriting the equations with velocity $\vec{v} = -v\hat{x}$, will yield the same descriptions for the particle while traveling to the left after bouncing on the right plate. Using the equations derived above, the mean energy and momentum of the particle in the Σ' frame can be calculated where the relativistic energy of the particle is $E = cp^0 = \gamma mc^2$.

$$\bar{p}'^{0} = mc\frac{\gamma_{v_r'}t_r' + \gamma_{v_l'}t_l'}{t_r' + t_l'} = mc\gamma\gamma_u(1 + \frac{v^2u^2}{c^4}) \tag{46}$$

Since the motion of the particle is restricted to the x-axis, the mean x-component of the momentum is

$$\bar{p}'^{1} = m\frac{v_r'\gamma_{v_r'}t_r' + v_l'\gamma_{v_l'}t_l'}{t_r' + t_l'} = mu\gamma\gamma_u(1 + \frac{v^2}{c^2}) \tag{47}$$

Now, we will compare the results of this macroscopic picture with the analysis of the gas as a macroscopic homogeneous fluid. The mean energy density of the particle q in Σ will be the momentum in (41) times c divided by the volume V_1 where the average energy and pressure of particle q are

$$\bar{\epsilon}_G = \frac{mc^2\gamma}{V_1} \tag{48}$$

$$\bar{P} = \frac{mv^2\gamma}{V_1} = \bar{\epsilon}_G\frac{v^2}{c^2} \tag{49}$$

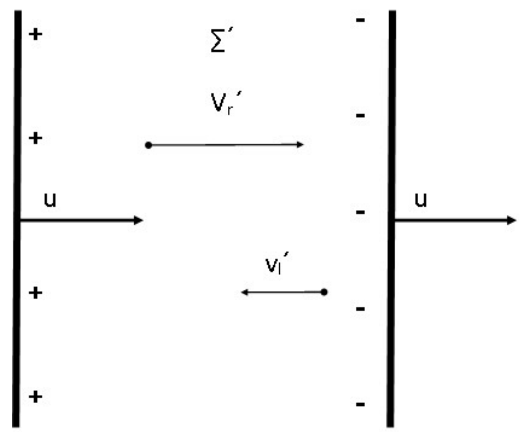

Figure 7: Particle q's motion in moving frame Σ'

Plugging in this results into the Lorentz transformed mean energy and momentum of q in Σ' (46) and (47)

$$\bar{p}'^0 = \gamma_u V_1 (\frac{\bar{\epsilon}_G}{c} + u^2 \frac{\bar{P}}{c^3}) \tag{50}$$

$$\bar{p}'^1 = \gamma_u u V_1 (\frac{\bar{\epsilon}_G + \bar{P}}{c^2}) \tag{51}$$

These results correspond with the components of the four-momentum of the gas when analyzed macroscopically as a fluid in (36). This result indicates that the apparent non-covariant Lorentz transformation of the false four-momentum of the gas are not a violation of co-variance. Equation (35) is already taking into account the properties of the individual particles making up the gas when the four-vector is Lorentz transformed. Thus the apparent false four-vectors are correct and valid covariant relativistic expressions. Equation (50) and (51) only differ with the components of the covariant vector (38) in the terms $\gamma_u V_1 u^2 \frac{P}{c^3}$ and $\gamma_u u V_1 \frac{P}{c^2}$, respectively. Both extra terms are related to the pressure in the gas and can be identified as the "hidden energy" and "hidden momentum" in this system. It is important to notice that in the system of four-vectors where hidden momentum is present in the gas((34), (35), (36)), the fields' three-momentum vanishes. However, the system of four-vectors where hidden momentum is not described((38), (39)), the fields do have a three-momentum with non-zero component in the x-direction. Pressure and the electric field are related through Equation (23) since the pressure originates by the action of the electric field on the gas. The field momentum and "hidden momentum" describe the same attribute of the system; each in different approaches which are both covariant. Additionally, it is important to realize that the system is constrained. The gas in the capacitor plate is not free to exit the system and is restricted to motion in the x-axis. This same system with an unconstrained gas would not contain hidden momentum considering that the gas' field induced pressure would be nonexistent.

Throughout the examination of the systems presented above we can derive a series of properties about hidden momentum. First and foremost, hidden momentum is an intrinsically relativistic phenomenon. It is not necessarily linked to electromagnetic systems[11] but it is easy to spot in this kind of systems thanks to our deep understanding of electromagnetic laws and their straight forward derivations in special relativity. In the last examples, it was found that matter which is moving under pressure carries additional momentum which we call hidden momentum since it is inadvertently taken into account when applying a Lorentz transformation. This hidden momentum is a purely mechanical effect that acts on the internally moving parts of a system and it arises as a way to preserve the center of energy theorem which establishes that the total momentum of a static system must vanish. The center of energy is intrinsically related to the center of mass as they are both conserved in time for a given isolated system. The formulation of the center of energy theorem is explained at the end of the following section.

2 Theory

2.1 Euler-Lagrange Equation

The Euler-Lagrange equation is useful because it's a simple and straightforward way to show the equations of motion of any system, since it only requires knowledge of the Lagrangian. The equations of motion are second order differential equations that describe how the components of a system evolve in time. In order to derive the Euler-Lagrange equation, a general Lagrangian $L(q_i, \dot{q}_i)$ where $i = 1, ..., n$ will be used. The action described by this system is shown in Equation (52). A system with q_n generalized coordinates has an infinite amount of paths for its action S.

$$S[q_i(t)] = \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t))dt \tag{52}$$

However, we are interested in finding the paths that minimize or maximize S in accordance with the Principle of Least Action. These paths are called stationary. A stationary path is one that for all nearly-equal paths, they all have virtually the same action S. This is that all small variations ΔS to the first order are 0. The Euler-Lagrange equation is derived by expanding S in terms of the tiny variations in the path where second and higher order terms are ignored because their contribution is negligible. Once the Action is expanded, ΔS will be identified and set to 0 to find the stationary paths. Rearranging ΔS will yield the Euler-Lagrange Equation.

The true path that makes the action stationary will be expressed as $\bar{q}_i(t)$. Every other path can be expressed as $\bar{q}_i(t) + \delta q_i(t)$, where $\delta q_i(t_1) = \delta q_i(t_2) = 0$. The true path variation can also be expressed by $\bar{q}_i(t) + \delta q_i(t)$ or simply by $\bar{q}_i(t) + \epsilon_i(t)$. $\epsilon_i(t)$ represents an infinitesimally small time-dependent variation in the path. The action of the slightly modified path will have a structure defined as follows: $S[\bar{q}_i(t) + \epsilon_i(t)] = S[\bar{q}_i(t)] + \Delta S + O(\epsilon^2)$. This is achieved by performing a Taylor expansion around the small time-dependent variation $\epsilon(t)$ as shown below and where $O(\epsilon^2)$ are all other higher order terms.

$$S[\bar{q}_i(t)+\epsilon_i(t)] = \int_{t_1}^{t_2} L(\bar{q}_i(t)+\epsilon_i(t), \dot{\bar{q}}_i(t)+\dot{\epsilon}_i(t))dt = \int_{t_1}^{t_2} L(\bar{q}_i(t), \dot{\bar{q}}_i(t))+\sum_{i=1}^{n}\int_{t_1}^{t_2} \left[\epsilon_i(t)\frac{\partial L}{\partial \bar{q}_1}\dot{\epsilon}_i(t)\frac{\partial L}{\partial \dot{\bar{q}}_i}\right] dt+O(\epsilon^2) \tag{53}$$

The second term of Equation (53) is ΔS. Equating ΔS to 0 and integrating by parts the last term, where the boundary term vanishes, we find:

$$\Delta S = \sum_{i=1}^{n}\int_{t_1}^{t_2} \epsilon_i(t)\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})\right] dt \tag{54}$$

Since $\Delta S = 0$ for any and all $\epsilon_i(t)$ we arrive to the Euler-Lagrange Equation:

$$\frac{\partial L}{\partial \bar{q}_i} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{\bar{q}}_i}) = 0 \tag{55}$$

There are however some special cases where the boundary term does not vanish. This occurs when the Lagrangian is subject to a predetermined constraint. We can analyze how that will affect the action of the system using calculus of variations with a similar procedure as presented above to derive the Euler-Lagrange Equation. We consider the Action described by Equation (52) and a constraint f which also goes from t_1 to t_2 and is equal to a constant C as described below. The goal is still to find a function of q which makes the action stationary with the condition that it needs to comply with the constraint imposed by function f.

$$f(q,\dot{q},t) = \int_{t_1}^{t_2} F(q,\dot{q})dt = C \tag{56}$$

Lagrange multipliers are a straightforward way of finding the required function of q. So we construct a function H, where $H = S + \lambda F$.

$$H = \int_{t_1}^{t_2} [L(q,\dot{q}) + \lambda F(q,\dot{q})]dt \tag{57}$$

Applying the Euler Lagrange Equation to this function H will yield the desired Euler-Lagrange Equation for constrained variation. Equation (58) shows the simplified Euler-Lagrange Equation for function H.

$$\frac{\partial L}{\partial q} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) + \lambda\left[\frac{\partial f}{\partial q} - \frac{d}{dt}(\frac{\partial f}{\partial \dot{q}})\right] = 0 \tag{58}$$

To follow up, the theory will be applied to a general example. Assume there is a system with two generalized coordinates, q_1 and q_2, which are dependent to each other through a holonomic constraint of the form $f(q_1, q_2, t) = 0$. Applying Hamilton's Principle to this system, which states that the motion of any dynamical system in a given time interval will aim to maximize or minimize the action integral as shown in Equation (59), and Taylor expanding the constraint we arrive to the expression shown in Equation (60).

$$\delta\int_{t_1}^{t_2} Ldt = \int_{t_1}^{t_2}([\frac{\partial L}{\partial q_1} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_1})]\delta q_1 + [\frac{\partial L}{\partial q_2} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_2})]\delta q_2)dt = 0 \tag{59}$$

$$\delta\int_{t_1}^{t_2} Ldt = \int_{t_1}^{t_2}([\frac{\partial L}{\partial q_1} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_1})]\frac{1}{\frac{\partial f}{\partial q_1}} + [\frac{\partial L}{\partial q_2} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_2})]\frac{1}{\frac{\partial f}{\partial q_2}})\delta q_1 dt = 0 \tag{60}$$

Since everything inside the brackets should be equal to zero because Equation (60) is valid for all $\delta q_1(t)$, we can extract two distinct Equations of Motion for this system, Equation (61a) and (61b). In this context, there would be two constraint forces acting on this system, $\tilde{Q}_1 = \lambda(t)\frac{\partial f}{\partial q_1}$ and $\tilde{Q}_2 = \lambda(t)\frac{\partial f}{\partial q_2}$.

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_1}) - \frac{\partial L}{\partial q_1} - \lambda(t)\frac{\partial f}{\partial q_1} = 0 \tag{61a}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_2}) - \frac{\partial L}{\partial q_2} - \lambda(t)\frac{\partial f}{\partial q_2} = 0 \tag{61b}$$

In systems where hidden momentum is present, there are implicit constraints being enforced just like in Figure 3, 4 and 5. In these particular cases the moving charges are constrained to drift inside the cable or between the plates. Otherwise the charges would be displaced with or against the electric field, depending on the sign of the charges. However, it is important to clarify that hidden momentum does not arise due to the constraints of a system; constraint motion is a requisite to analyze the effect of external forces acting on the system.

2.2 Noether's Theorem

Noether's theorem states that for every differentiable symmetry in the action of a Lagrangian, there is a corresponding conserved current linked to it. In simpler words, for every continuous symmetry of the Lagrangian, there is a conserved quantity along trajectories that satisfies the equations of motion. In this case, a symmetry is a formulation of a physical law expressed as a continuous group or Lie group. The three most common symmetries of the equations of motion are: Translations in Space, Rotations in Space and Time Translations, which lead to canonical momentum, angular momentum and energy being conserved respectively. For example, a translation in physical space will have the same action as the original. Thus, all stationary paths, including small variations in the action ΔS are the same.

The Noether Procedure to obtain the Noether currents and conserved Noether charges will be discussed next. Assume there is a symmetry transformation applied to the generalized coordinate q of the form:

$$q' = q \longmapsto q + \delta_s q \tag{62}$$

where $\delta_s q$ depends on a tiny constant ϵ. The s next to the delta represents that it is indeed a symmetry variation and differentiate it from any other variation δ. First, the symmetry transformation shown in Equation (62) will transform $L \longmapsto L'$ and the requirement $S = S'$ which correspond to L and L' needs to hold. Furthermore, the tiny constant parameter ϵ will become a tiny time dependent parameter $\epsilon(t)$. A time dependent parameter will change the way the Lagrangian is transformed, as it will now behave like:

$$L \longmapsto L + \epsilon Q \tag{63}$$

Recalling that on solutions to the equations of motion, $\Delta S = 0$ if

$$\epsilon(t_1) = \epsilon(t_2) = 0 \tag{64}$$

This statement is simply the Principle of Least Action discussed in the previous section. Therefore,

$$0 = \Delta S = \int_{t_1}^{t_2} \dot{\epsilon} Q dt = -\int_{t_1}^{t_2} \epsilon \dot{Q} dt. \tag{65}$$

Since Equation (65) should be valid for any and all ϵ, then $\dot{Q} = 0$ and thus Q is the Noether Charge; the conserved quantity related to the generalized coordinate q of the example Lagrangian.
Noether Currents arise from continuous symmetries in physical systems and field theory is required to derive the proof of conserved Noether Currents. First, we define a Lagrangian $L(\phi(x), \dot{\phi}(x))$ which depends on some generalized field $\phi(x)$. The Lagrangian density $\mathcal{L}$ defines the Lagrangian when integrated by the volume as shown in Equation (66). Applying a symmetry transformation where the field is transformed by a small quantity $\delta\phi(x)$, where the infinitesimal change depends on a tiny space-time dependent variation $\epsilon(x_\mu)$.

$$L = \int \mathcal{L} d^3x \tag{66}$$

$$\phi' = \phi \longmapsto \phi(x_\mu) + \delta_s \phi \tag{67}$$

The new Lagrangian would be defined by the new transformed fields $L' = L(\phi', \dot{\phi}')$. The Lagrangians L and L' might still be equivalent to each other, $L = L'$, but their Actions will change. The difference between the Action of the transformed system and the original system is defined by Equation (68).

$$\delta S = \int \left[\left(\frac{\partial \mathcal{L}'}{\partial \epsilon(x_\mu)} - \partial_\mu \frac{\partial \mathcal{L}'}{\partial \partial_\mu \epsilon(x_\mu)} \right) \delta_s \epsilon(x_\mu) + \partial_\mu \left(\frac{\partial \mathcal{L}'}{\partial \partial_\mu \epsilon(x_\mu)} \delta_s \epsilon(x_\mu) \right) \right] dx \tag{68}$$

The first term on the right hand side of Equation (68) needs to vanish for any $\delta_s \epsilon$ similar to the procedure in section 2.1 and thus the Euler-Lagrange Equations can be retrieved:

$$\frac{\partial \mathcal{L}'}{\partial \epsilon(x_\mu)} - \partial_\mu \frac{\partial \mathcal{L}'}{\partial \partial_\mu \epsilon(x_\mu)} = 0 \tag{69}$$

The action is a surface term under x-independent transformations as that was an assumption for the derivation and as such, $\frac{\partial \mathcal{L}'}{\partial \epsilon(x_\mu)} = \frac{\partial}{\partial x_\mu} \Lambda^\mu$. Equation (69) can be reshaped to define a current with no four-divergence.

$$j^\mu = \frac{\partial \delta_s \mathcal{L}'}{\partial \partial_\mu \epsilon(x_\mu)} - \Lambda^\mu \tag{70a}$$

$$\frac{\partial j^\mu}{\partial x^\mu} = 0 \tag{70b}$$

l0.5

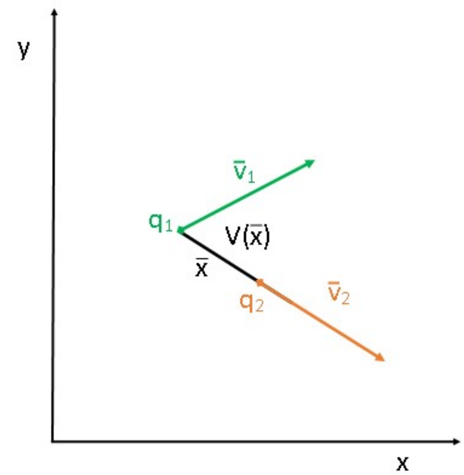

Figure 8: Two particles q_1 and q_2 with some kinetic energies proportional to their velocities $\vec{v_1}$ and $\vec{v_2}$ and a potential $V(\bar{x})$ between them.

Equation (69) represents a Noether current and it's four-divergenceless is the Noether Current Conservation Law. This conservation of current is a local conservation law. Assuming all fields to vanish at infinity, we arrive at the global conservation of charge as shown below:

$$\frac{d}{dt}Q(t) = \int \partial_0 j^0(x,t) d^3x \tag{71}$$

To follow up on the theory behind Noether's charges and currents, we are going to examine how spatial translations relate to canonical momentum conserved in a system. Let's examine as reference a system described by two particles with some potential between them as shown in Figure 8. In a space translation, a vector $\vec{a}$ would be added to the position of both particles where:

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \tag{72}$$

Notice how this space translations doesn't alter the kinetic or potential energies of the system. Thus, the Lagrangian and the momentum of the system are:

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - V(|x_1 - x_2|) \tag{73a}$$

$$p = m_1\dot{x}_1 + m_2\dot{x}_2 \tag{73b}$$

The particles paths that satisfy the equations of motion are regarded as the true paths of the system $(\bar{x}_1(t), \bar{x}_2(t))$. If we allow for tiny path variations, such as infinitesimally small, time-dependent translations $\epsilon(t)$, then the true paths will transform such as:

$$\bar{x}_1(t) \longmapsto \bar{x}_1(t) + \epsilon(t) \tag{74a}$$

$$\bar{x}_2(t) \longmapsto \bar{x}_2(t) + \epsilon(t) \tag{74b}$$

The addition of a small epsilon translation will affect the action of the Lagrangian. The variable $\bar{x}$(t) encompasses $\bar{x}_1$ and $\bar{x}_2$. As explained in section 2.1, the action will have three terms:

$$S[\bar{x}(t) + \epsilon(t)] = S[\bar{x}(t)] + \Delta S + O(\epsilon^2) \tag{75}$$

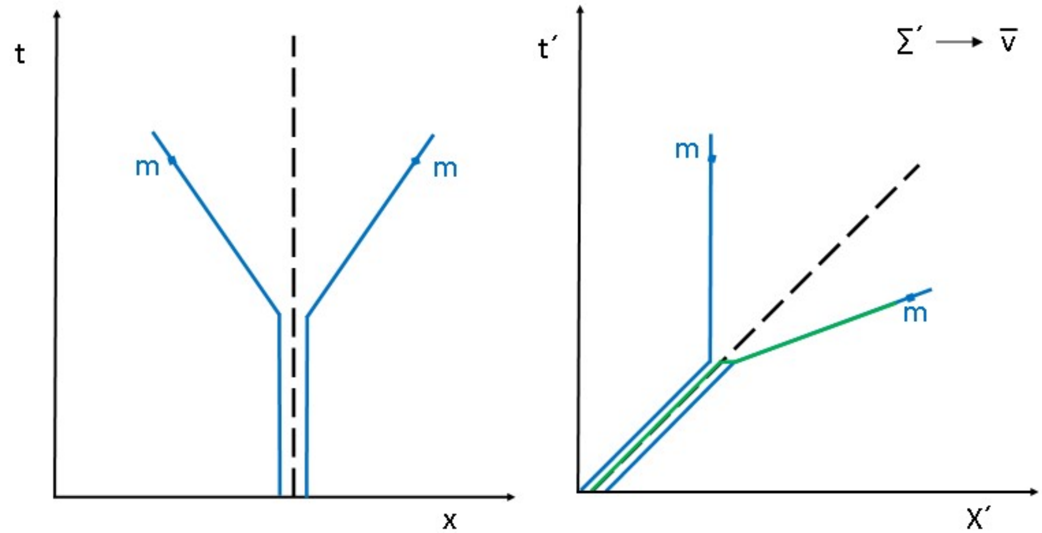

Figure 9: Center of mass-energy: 9a (to the left) and 9b (to the right)

The action of the Lagrangian for the system in Fig. 8, with the addition of the time-dependent parameter ϵ(t), becomes:

$$S[\bar{x}(t)+\epsilon(t)] = \int_{t_1}^{t_2} [\frac{1}{2}m_1(\dot{\bar{x}}_1+\dot{\epsilon(t)})^2 + \frac{1}{2}m_2(\dot{\bar{x}}_2+\dot{\epsilon(t)})^2 - V(|\bar{x}_1+\epsilon(t)-\bar{x}_2-\epsilon(t)|)]dt \tag{76a}$$

$$= \int_{t_1}^{t_2} [L = \frac{1}{2}m_1\dot{\bar{x}}_1^2 + \frac{1}{2}m_2\dot{\bar{x}}_2^2 - V(|\bar{x}_1-\bar{x}_2|)] + \int_{t_1}^{t_2} [m_1\dot{\bar{x}}_1\dot{\epsilon(t)} + m_2\dot{\bar{x}}_2\dot{\epsilon(t)}]dt + 2\dot{\epsilon}(t)^2 \tag{76b}$$

For a stationary path we require ΔS to be 0, and the condition that $\epsilon(t_1) = \epsilon(t_2) = 0$ needs to be imposed to ensure we stay on the true paths at the beginning and end of the action. Applying integration by parts to ΔS and realizing that the stationary path condition is valid for any and all ϵ(t) we arrive to the equation below:

$$0 = m_1\ddot{\bar{x}}_1 + m_2\ddot{\bar{x}}_2 = \frac{d}{dt}(m_1\dot{\bar{x}}_1 + m_2\dot{\bar{x}}_2) = \frac{d}{dt}P \tag{77}$$

So for a system with translational symmetry, total linear momentum is conserved.

2.3 Center of Energy

In non-relativistic isolated systems, the total momentum of the system depends on the sum of all of the masses composing the system times the rate of change of the position of the center of mass with respect to time. The center of mass path of an isolated system, as shown in Figure 9a[24] by the dashed line, doesn't change in time because momentum is conserved. The center of energy, like the center of mass, follows the same path since energy in such a system is conserved. Both particles have the same mass and equal and opposite velocities at the time t they are repelled, by a compressed massless spring allowed to move at time t, from origin and thus possess the same kinetic energy at all times t. Figure 9b shows the previous system from a reference frame moving at velocity $\vec{v}$. The center of mass is still represented by the dashed line and remains constant at all times. However the center of energy doesn't follow it anymore since the only energy present in the system, kinetic energy, is not evenly distributed. After the particles suddenly repel each other in this frame, the left particle has no kinetic energy while the second one has a kinetic energy of $2mv^2$.

In this frame, the center of energy jumped abruptly from the dashed path to the path of the second particle, as shown by the green line. This approach is wrong. Mass and energy are constant in a closed isolated system and they are linearly proportional to each other as we know from $E = mc^2$. Thus, if the center of mass is constant in time for such an isolated system, the center of energy should be constant too. Just as there is a constant line (dashed line) which represents the motion of the center of mass which doesn't change through time, the same constant line should represent the motion of the center of energy which cannot change through time as the total energy of the system is conserved.

The system of a magnet and a point charge discussed by *Sidney Coleman and J. H. van Vleck*[12], is useful to understand the physical meaning of the Center of Energy. Assume there is a charged particle at rest at a long distance from a magnet whose magnetization can change. The magnet consists of 2 opposite charged disks with equal masses rotating in opposite directions along a common axis. When the disks come into contact, friction will slow them down and the magnetization of the magnet as a whole will start changing in time. Since the magnetic field of the magnet will start to change in time, it will induce an electric field which will exert a force on the particle. Since the particle is far from the magnet, the force due to the electric field will be

$$\vec{F} = \frac{e}{c}\left(\frac{\vec{r}}{r^3} \times \frac{d}{dt}\vec{M}\right) \tag{78}$$

where e is the charge of the particle, $\vec{r}$ is the vector from the magnet to the particle and $\vec{M}$ is the magnetization. The system is in an apparent non-equilibrium as there is no clear back force by the charge onto the magnet since it is electrically neutral. Now, if we place the magnet and charged particle in a closed box and let the system run we encounter a paradox: before the magnet's magnetization starts changing, the system is static. However when the magnetization starts changing, the electric charge will start moving do to the electric force. Once this particle reaches the boundary with the box, it will start pushing onto the box and the system as whole will start moving. This would be inconsistent with the law of conservation of momentum. Taking a closer look at Equation (78), it can be seen there is a clear $\frac{1}{c}$ factor in front of the parenthesis as well as an implicit $\frac{1}{c}$ in the magnetization M since this magnetization is written in terms of moving charges. This observation makes it clear that relativistic corrections such as retardation effects, variations of mass, etc exert an important effect. Moreover, the general arguments that need to be taken into account to solve the paradox are explained below. In any classical or quantum field theory which is described by a local, Lorentz-invariant Lagrangian there is a symmetric second rank tensor which is conserved.

$$T^{\mu\upsilon} = T^{\upsilon\mu} \tag{79}$$

$$\frac{\partial T^{\mu\upsilon}}{\partial x_\mu} = 0 \tag{80}$$

It will be shown in the next section that a continuous space-time translation applied to a closed system has an associated locally conserved current; the energy-momentum tensor. Since all the components of the system we are interested in can be enclosed in box, the energy and momentum of the system will be restricted to the inside of the box. Below, the energy, momentum components and center of energy components are derived:

$$E = \int T^{00} d^3x \tag{81a}$$

$$p^i = \frac{1}{c}\int T^{0i} d^3x \tag{81b}$$

$$X^i = \frac{1}{E}\int T^{00} x^i d^3x \tag{81c}$$

Based on Equations 77, 78, and 79s, the following conservation laws can be derived, where E stands for the total energy, $\vec{p}$ stands for the canonical momentum and $\vec{X}$ represents the center of energy:

$$\frac{dE}{dt} = 0 \tag{82a}$$

$$\frac{d\vec{p}}{dt} = 0 \tag{82b}$$

$$\frac{d\vec{X}}{dt} = c^2 \frac{p}{E} \tag{82c}$$

These are the law of conservation of energy, the law of conservation of momentum and the law of steady motion of the center of energy. Equation (82c) supports our previous conclusion in the system described Figure 9, where the center of energy needs to be defined by a constant line regardless of the reference frame used. The center of energy is in fact the relativistic generalization of the center of mass; after all, the mass of a particle is described by its rest-mass energy. Using these conservation laws, the magnet paradox presented above can be solved. Before the magnetization of the magnet starts to change in time, T^{00} is independent of time thus $\vec{X}$ is independent of time, which by virtue of (82c), makes $\vec{P} = 0$, so momentum is conserved and it will remain so after the magnetization starts to change. After the charged plates start to exert friction onto each other, $\frac{d\vec{X}}{dt} = 0$ and the box will not move. Even though the law of constant motion of the center of mass follows as a consequence of the conservation of momentum, it is not the same for the law of constant motion of the center of energy. In Hamiltonian mechanics, the Hamiltonian is the generator of time translations and $E\vec{X}$ is the generator of Lorentz transformations, as it will be shown in the derivation below. The generators for the Hamiltonian, the center of energy and total momentum are all part of the same group, the Poincare Group. Analyzing (82c) in a similar manner, it shows that the commutation of the generator for time translations and the generator of Lorentz transformations resulted in another generator describing space-translations. These 10 generators have to be part of continuous group. A system with invariance under transformations of this group has a conservation of mass-energy and momentum. It is shown below[13], that the law of steady motion of the center of energy is obtained as a conserved current due to Lorentz transformations. We will consider an infinitesimal Lorentz transformation on two different four-vectors of the form:

$$x^\mu \mapsto x^\mu + \omega^{\mu\upsilon} x_\upsilon d\lambda \tag{83}$$

$$y_\mu \mapsto y_\mu + \omega_{\mu\upsilon} y^\upsilon d\lambda \tag{84}$$

In a Lorentz transformation, the scalar product of these two vectors is not altered for any x and y so that

$$x^\mu y_\mu \mapsto x^\mu y_\mu + \omega^{\mu\upsilon} x_\upsilon y_\mu d\lambda + \omega_{\mu\upsilon} y^\upsilon x^\mu d\lambda \tag{85}$$

where any second order or higher terms are dropped out since they would be negligible. The second and last term can be joined by raising and lowering indices correspondingly. We will arrive to

$$x^\mu y_\mu \mapsto x^\mu y_\mu + (\omega_{\mu\upsilon} + \omega_{\upsilon\mu}) x^\mu y^\upsilon d\lambda \tag{86}$$

which is equivalent to (85). Since $x^\mu y_\mu$ needs to remain unchanged after a Lorentz transformation and x^μ and y^υ can take up any value, the omega terms inside the parenthesis need to vanish. Thus $\omega_{\mu\upsilon}$ is an anti-symmetric 4×4 matrix.

$$\omega_{\mu\upsilon} = -\omega_{\upsilon\mu} \tag{87}$$

Assume the case where $\omega^{12} = 1$,$\omega^{21} = -1$ and all other entries of the matrix are 0. Using Equation (83) we get:

$$x^1 \mapsto x^1 + \omega^{12} x_2 d\lambda = x^1 - x^2 d\lambda \tag{88}$$

$$x^2 \mapsto x^2 + \omega^{21} x_1 d\lambda = x^2 + x^1 d\lambda \tag{89}$$

These are infinitesimal rotations which can be described by the 2x2 group of rotations in 2D

$$D(\lambda) = \begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix} \tag{90}$$

such that

$$Dx^1 = -x^2 \tag{91}$$

$$Dx^2 = x^1 \tag{92}$$

Thus, it would be equal to a rotation around the z-axis. Performing the same operation with matrix entries $\epsilon^{10} = 1$ and $\epsilon^{01} = -1$ while keeping all other entries zero would be equal to using the generator D_1 of Lorentz Boosts

$$D_1(\lambda) = \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix} \tag{93}$$

where the boost would be applied in the x^1 direction. As explained in section 2.2, there is a conserved current j^μ associated with Lorentz transformations. It will be linear in ω such as

$$j^\mu = \frac{1}{2}\omega_{\lambda\rho}M^{\lambda\rho\mu} \tag{94}$$

where the $\frac{1}{2}$ coefficient prevents double counting. Since $\omega_{\lambda\rho}$ is anti-symmetric, we can define $M^{\lambda\rho\mu}$ to be anti-symmetric for the same indices. Since the current has no four-divergence as shown in (70b), the third rank tensor can be written as

$$\frac{\partial M^{\lambda\rho\mu}}{x^\mu} = 0 \tag{95}$$

Hence, there will be six global conservation laws obtained from

$$j^{\lambda\rho} = \frac{1}{2}\omega_{\lambda\rho}\int M^{\lambda\rho 0}d^3x \tag{96}$$

In general, an anti-symmetric tensor $n \times n$ will have $\frac{n(n-1)}{2}$ independent entries. The tensor M will thus have 6 independent entries which would be 6 conservation laws. As examined previously with the ω_{12} tensor, three of these conservation laws are the familiar components of angular momentum corresponding to the rotation transformations. The associated conserved currents will be J^{12},J^{23} and J^{31}. The conservation laws associated with components J^{01},J^{02} and J^{03} are discussed below where Lorentz transformations will be applied to scalar fields. A scalar field transformed under a Lorentz transformation

$$\phi^a(x) \mapsto \phi^a(\Lambda^{-1}x) \tag{97a}$$

where the infinitesimal Lorentz transformation is described as

$$(\Lambda^{-1}x)^\rho = x^\rho - \omega^{\rho\sigma}x_\sigma d\lambda = x^\rho + \omega^{\sigma\rho}x_\sigma d\lambda \tag{98}$$

Expanding Equation (98) to the first order $d\lambda$ yields

$$\delta\phi^a = \omega_{\sigma\rho}x^\sigma\partial^\rho\phi^\sigma \tag{99}$$

Since the Lagrangian density is also a scalar and our interest lies on Lorentz invariant systems, we can substitute the general field ϕ^a by the Lagrangian density to describe the change in Lagrangian density said system described below.

$$\delta\mathcal{L} = \partial^\rho(\omega_{\sigma\rho}x^\sigma\mathcal{L}) \tag{100}$$

Equation (98) can we rewritten in such a way as to describe the change in the Lagrangian density to be caused by the divergence of a first-rank tensor F.

$$\delta\mathcal{L} = \partial^\rho(g^{\mu\rho}\omega_{\sigma\rho}x^\sigma\mathcal{L}) = \partial_\mu F^\mu \tag{101}$$

We now have all the elements needed to describe the conserved current as shown in Equation (70).

$$j^\mu = \omega_{\sigma\rho}x^\sigma[\eta^\mu_a\partial^\rho\phi^a - g^{\mu\rho}\mathcal{L}] \tag{102}$$

The expression inside the square bracket would be the conserved charge which in this case would be the energy-momentum tensor $T^{\rho\upsilon}$. Since the tensor T is not anti-symmetric for ρ and σ, we will anti-symmetrize the product $x^\sigma T^{\rho\mu}$ with respect to σ and ρ to obtain back the third-rank tensor M

$$j^\mu = \frac{1}{2}\omega_{\sigma\rho}(x^\sigma T^{\rho\mu} - x^\rho T^{\sigma\mu}) \tag{103}$$

such that

$$M^{\sigma\rho\mu} = (x^\sigma T^{\rho\mu} - x^\rho T^{\sigma\mu}) \tag{104}$$

This description for tensor M reveals that there are indeed 6 independent generators obtained by the asymmetric component pair defined by σ and ρ each of which is a four-vector defined by component μ which runs from 0 to 3. Using this new definition of M, we can define the new components J^{01},J^{02} and J^{03}.

$$J^{10} = \int[x^1T^{00} - x^0T^{10}]d^3x \tag{105}$$

$$= \int x^1T^{00}d^3x - x^0\int T^{10}d^3x \tag{106}$$

x^0 represents time t which is independent of whatever volume we are doing the integral in and t can be brought out of the integral. The second term will then be the time times the momentum in the x-direction represented by p^1. If we differentiate the current with respect to time

$$\frac{dJ^{10}}{dt} = \frac{d}{dt}\int[x^1T^{00}]d^3x - p^1 = 0 \tag{107}$$

Equation (107) is the relativistic generalization of the law of steady motion of the center of mass; the law of steady motion of the center of energy. We can compare Equation (107) to the Equation describing the change of center of mass with respect to time of non-relativistic system. $\rho(x,t)$ is the mass density and x the position.

$$\vec{v}_{CoM} = \frac{dx_{CoM}}{dt} = \frac{d}{dt}(\frac{1}{M}\int[\vec{x}\rho(x,t)]d^3x) = \frac{\vec{p}}{M} \tag{108}$$

Substituting M for $\frac{E}{c^2}$ where $c = 1$ yields the center of Energy description. The center of energy will move with a constant velocity $\frac{\vec{p}}{E}$, this guaranteeing that four-momentum is also a conserved quantity of the system. The conserved currents associated with Lorentz boosts can thus be described as:

$$j^{i0} = Ex^i - tp^i \tag{109}$$

where x^i are the components of the center of energy

$$X_i = \frac{1}{E}\int x_iT^{00}d^3x \tag{110}$$

$$E = \int T^{00}d^3x \tag{111}$$

The conservation of the center of energy is shown to be derived as a consequence of Lorentz symmetry in a system. In Figure 9, the center of kinetic energy is indeed correctly represented by the green line. Yet the center of energy is not only characterized by kinetic, potential, or other forms of exchangeable energy but also the rest mass energy of the system's constituents. Knowing this, he conserved center of energy is better described as center of mass-energy as that leads to less confusions. Thus the center of mass-energy in the two particle system discussed at the beginning of the section is located along the same dashed line as the center of mass.

3 Analysis

3.1 Continuous Symmetry and Conserved Currents

The analysis on various systems with hidden momentum presented above indicate that hidden momentum is an internal momentum linked to the motion of the particles which make up classical and well known structures such as the charges moving inside a wire or a solenoid. In section 2.2, it was shown that a system with translational symmetry has its total linear momentum conserved. Let us describe an electromagnetic system like the ones previously discussed by a Lagrangian density $\mathcal{L}(\phi)$ where we do a space-time translation, such that

$$x^\mu \mapsto x'^\mu = x^\mu + a^\mu \tag{112}$$

Where a is independent of x. The change of coordinates will make the partial derivatives transform like:

$$\begin{aligned}\frac{\partial}{\partial x^\mu} \mapsto \frac{\partial'}{\partial x^\mu} &= \frac{\partial x^\upsilon}{\partial x'^\mu}\partial_\upsilon \\ &= (\delta^\upsilon_\mu - \partial^\mu a^\upsilon)\partial_\upsilon \\ &= \partial_\mu - \partial_\mu a^\upsilon \partial_\upsilon\end{aligned} \tag{113}$$

The field will be the same quantity before and after the translation such that $\phi'(x') = \phi(x)$ while the derivative will change to:

$$\begin{aligned}\partial'_\mu \phi'(x') &= \frac{\partial x^\upsilon}{\partial x'^\mu}\partial_\upsilon \phi(x) \\ &= \partial_\mu \phi(x) - \partial_\mu a^\upsilon \partial_\upsilon \phi(x)\end{aligned} \tag{114}$$

The small change induced to the field will then be:

$$\delta\phi_s = a^\upsilon \partial_\upsilon \phi(x) \tag{115}$$

The action after the symmetry translation will be:

$$\begin{aligned}S[\phi'(x')] &= \int \mathcal{L}(\phi'(x'), \partial_\mu \phi'(x'))d^4x' \\ &= \int \left[\mathcal{L}(\phi(x), \partial_\mu\phi(x)) + \partial_\rho a^\rho \mathcal{L}(\phi(x), \partial_\mu\phi(x)) - \partial_\mu a^\upsilon \partial_\upsilon \phi(x)\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} + O(a^2)\right] d^4x \\ &= I[\phi(x)] + \int \left[\partial_\rho a^\rho \mathcal{L}(\phi(x), \partial_\mu\phi(x)) - \partial_\mu a^\upsilon \partial_\upsilon \phi(x)\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\right] d^4x + O(a^2)\end{aligned} \tag{116}$$

Since anything beyond first order approximation can be neglected, the small variation in the Action S is:

$$\begin{aligned}\delta S = S[\phi'(x')] - S[\phi(x)] &= \int \left[\partial_\rho a^\rho \mathcal{L}(\phi(x), \partial_\mu\phi(x)) - \partial_\mu a^\upsilon \partial_\upsilon \phi(x)\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\right] d^4x \\ &= \int \partial_\mu a^\upsilon \left[\delta^\mu_\upsilon \mathcal{L}(\phi(x), \partial_\mu\phi(x)) - \partial_\upsilon \phi(x)\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\right] d^4x\end{aligned} \tag{117}$$

The conserved current is the expression inside the square brackets shown on the last line of Equation (117). This expression is in fact the canonical energy-momentum tensor

$$T^{\mu\upsilon} = \eta^{\mu\upsilon}\mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\partial^\upsilon \phi(x) \tag{118}$$

The energy-stress tensor is the conserved current associated with space-time translations of electromagnetic systems. The four-momentum p^μ is the conserved charged associated to this current as it can be seen from the previously discussed Equation (10). It can further be shown[15] that the energy-momentum tensor is conserved in Lorentz transformations of an electromagnetic system which is logical since the total charge of an isolated system cannot decrease or increase from one inertial reference frame to another. In an isolated system, the energy-momentum tensor is required to have zero four-divergence since the charge cannot exit the system or else it wouldn't be conserved. However, a system formed from electromagnetic field and charges satisfies Equation (11)[7]. Using Equation (10) and (11), we arrive to the following statement regarding the four-momentum of such systems.

$$\frac{\partial p^{\mu}}{\partial x_0} = \frac{\partial}{\partial x_0}\frac{1}{c}\int T^{\mu 0} d^3x = \frac{1}{c}\int \frac{\partial}{\partial x_0} T^{\mu 0} d^3x = -\frac{1}{c^2}\int F^{\mu 0} j_0 d^3x \tag{119}$$

Equation (119) will be equal to 0 when looking at the system as a whole since the total four-momentum change is 0 in an isolated system. This statement is compatible with Equation (82c), where the center of energy will stay constant in an isolated system. In systems such as Figure 4 and 6, the center of energy of the systems as a whole remain constant and the canonical momentum is conserved. However, both of these systems can be decomposed into subsystems such as the parallel plates, the fields and charged fluid in the first system and parallel plates, fields and gas in the second. Let us take the latter system and its subsystems for this analysis as it is far simpler. The subsystem of the gas is not isolated, as it is under the influence of the electric field produced by the parallel plates. Equation 82c has to be modified to take into account the effect of this external force:

$$\sum_i (\vec{F}_{exti} \cdot v_i) r_i = \frac{d}{dt}(E\vec{X}) - c^2\vec{p} \tag{120}$$

$(\vec{F}_{extE} \cdot v_E)$ represents the power exerted by the external electric field at some position r_i in the subsystem. Since the whole system is isolated, its energy E and center of energy $\vec{X}$ do not change in time and we can arrive to the conclusion that the power exercised by the external force F_{ext} spawns a linear momentum in the subsystem which doesn't contribute to the center of energy of the system, as shown below.

$$\vec{p} = -\frac{1}{c^2}\sum_i (\vec{F}_{exti} \cdot v_i) r_i \tag{121}$$

Systems such as Jefimenko's and Feynman's Paradox are made from electromagnetic fields and charges but lack a constraint internal structure and an external force which would be required for the system to possess hidden momentum. The systems discussed in Figure 4 and Figure 5 have hidden momentum present inside the charged fluid and the gas, respectively. The first system can be regarded as an isolated solenoid with an internal structure of constrained charged particles which are under the influence of an external force produced by the electric field of the capacitor and the latter is a gas with constrained particles under the influence of a capacitor's electric field.

3.2 Unconstrained and Constrained Systems

When a system is free to move without restrictions it is considered to be unconstrained motion. An example of an unconstrained system is that of Feynman's Paradox. There is an apparent force unbalance that leads to non-conservation of momentum. However, once the electromagnetic momentum is included, this missing force is found and the net force is 0. In the case of unconstrained motion, each particle in the system feels a force, linked to the change of the vector potential with respect to time, that will in turn conserves canonical momentum. In the case of constrained systems, this force can be found in the analysis on the constrained Lagrangian in section 2.1. The constrained force depends on what the constraint applied on the system is and can be observed as a force external to the unconstrained Euler-Lagrange equation that will result in the constrained Action being stationary. In the systems discussed in section 1, the pressure gradient of the charged fluid in the solenoid or the pressure exerted by the gas particles on the plates originated from the constraints applied to the system. If the fluid were free to move or the gas wasn't constrained to move parallel to the x-axis between the plates, there would be no pressure in this systems. This constraints have corresponding forces which alter the Lagrangian density $\mathcal{L}$ of these systems. Since the Lagrangian densities change,as it can be seen from Equation (57), the energy-momentum tensor derived from them, (118), will also have to be adjusted accordingly. When a Lorentz Boost is applied to a system, the total momentum will transform covariantly when looked at the system as a whole. However when the subsystems or elements of the system are transformed separately, as discussed in the system of Figure 5, there seems to be an apparent inconsistency since the transformed four-vector of some of those subsystems appears to be non-covariant. This has been found to be false when looked at the effect of the Lorentz Boost on the individual particles of the system which carry some inherent momentum due to the pressure. Thus, when examined as a whole the total momentum seems to transform correctly as this hidden momentum is transformed implicitly. The momenta of the field and gas subsystems, (28) and (29), obtained from their respective energy-momentum tensors, (25) and (26), do not behave like four-vectors because these subsystems have forces such as the electric field induced force and pressure that are not four-vectors. These vectors transform differently under a Lorentz Boost and explain why hidden momentum is not obtained from them even though they still are covariant formulations of Equation (36) and (35). The energy-momentum tensor of the capacitor plates doesn't contain pressure or electric field terms which results in the four-momentum of the plates showing no difference between the two ways it was transformed. The energy and momentum from the energy-momentum tensor transform as a four-vector as long as the tensor is divergenceless. The tensor describing the isolated system as whole is divergenceless but this is not the case for the energy-momentum tensor describing the gas and field. Thus, both tensors describing the subsystems of the gas and fields violate the divergenceless condition and spawn local hidden energy and hidden momentum as it can be seen from (50) and (51).

4 Conclusion

Hidden momentum is observed in isolated systems that contained internally moving parts in constrained motion and require a relativistic approach. This elusive momentum is not present all throughout a system but in specific subsystems or elements whose volumes contain both charges and fields constrained. Hidden momentum is always described in terms of pressure revealing that the subsystem needs to be mechanical in nature and the energy and momentum of this subsystem doesn't transform as a four-vector under Lorentz transformations. In this type of subsystems, the energy-momentum tensor is not divergenceless (11), which causes energy and momentum to change locally. This local momentum is identified as "hidden momentum" and it can be obtained from the law of constant motion of the center of energy under an external force (120). The constant motion of the center of energy is a conserved law derived as a consequence of Lorentz symmetry. This hidden momentum does not alter the center of energy of the system and as such only exists inside the volume boundaries of a subsystem and vanishes when the whole system's momentum is analyzed. It arises as a direct result from the energy deposited by an external force, the most common of which is an electric field force, and balances the effects of this force on the subsystem. This balancing occurs because the subsystems are constrained and motionless which, due to conservation of momentum, need to remain motionless in time. This analysis also answers the unknown cause behind the equality between hidden momentum and electromagnetic momentum.

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A Lorentz Transformations

In this appendix, we will review how Euclidean vectors, four-vectors and second rank tensors transform under a Lorentz transformation. A Lorentz transformation, which is represented below as a second rank tensor for a boost in the x-direction, is a set of linear equations which relates three space and one time coordinate of two frames moving at a constant velocity with respect to each other. Lorentz transformations become relevant when we are dealing with frames moving at relativistic speeds ($v < c$). In low-velocity limits ($v << c$), we retrieve the conventional Galilean transformations which are valid in the more daily Newtonian physics.

$$L^{\mu}_{\upsilon} = \begin{pmatrix} \gamma & \frac{\gamma u}{c} & 0 & 0 \\ \frac{\gamma u}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{122}$$

Here $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$ and μ and ϵ range from 0 to 3.

A.1 Four-vectors

Four-vectors have four components, one related to time and three related to space. They behave in a singular manner under Lorentz transformations. The product of any two four-vectors is a Lorentz invariant; it is not altered under a Lorentz transformation. They are transformed such that:

$$a^{\mu} = L^{\mu}_{\upsilon} a^{\upsilon} \tag{123}$$

To see how a four-vector transforms under a Lorentz transformation, we will examine the position four-vector

$$a^{\upsilon} = (ct, \vec{a}) \tag{124}$$

where ct is the magnitude of the vector is the rest frame at time t and $\vec{a}$ represents the spatial coordinates of the vector. Applying a Lorentz transformation to this vector yields

$$a^{\mu} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & \frac{\gamma u}{c} & 0 & 0 \\ \frac{\gamma u}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{t - x\frac{v}{c^2}}{\sqrt{1-\frac{u^2}{c^2}}} \\ \frac{x-ut}{\sqrt{1-\frac{u^2}{c^2}}} \\ y \\ z \end{pmatrix} \tag{125}$$

A.2 Second-rank Tensors

Tensors are mathematical structures that are made up of vectors or other tensors. Tensors such as the energy-momentum tensor ($T^{\mu\upsilon}$) do not transform trivially under a Lorentz transformation. A Lorentz boost as described in (122) is also a tensor. Tensor products are not carried on like a matrix multiplication would since tensors transform as products of vectors. Matrix multiplication combines 2 arrays into a new third array while tensor product combines 2 tensors and yields a third new tensor. Rank 2 tensors transform like a product of two Lorentz transformations and thus pick up two Lorentz factors. One Lorentz factor will be applied due to the change of frame and the other comes from the Lorentz contraction of the space which this tensor represents. Thus tensors will transform like a product of vectors:

$$T'_{n'_1,\ldots,n'_M} = R_{n'_1 n_1} \ldots R_{n'_M n_M} T_{n_1,\ldots,n_M} \tag{126}$$

where $R_{n'_1 n_1} \ldots R_{n'_M n_M}$ are the components of the transformation of the tensor T. This second-rank tensor Lorentz transformation is shown below for a simple $T^{\mu\upsilon}$:

$$T^{\mu\upsilon\prime} = \begin{pmatrix} \gamma & \frac{\gamma u}{c} & 0 & 0 \\ \frac{\gamma u}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \gamma^2 \rho & \gamma^2 u\rho & 0 & 0 \\ \gamma^2 u\rho & \gamma^2 u^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{127}$$

If we apply the Lorentz transformation tensor as a matrix multiplication as it was done to a four-vector, it would yield:

$$T^{\mu\upsilon\prime} = \begin{pmatrix} \gamma & \frac{\gamma u}{c} & 0 & 0 \\ \frac{\gamma u}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \gamma\rho & 0 & 0 & 0 \\ \gamma\frac{u}{v}\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{128}$$

which is wrong since it can clearly be seen that it is not symmetric anymore.

A.3 Euclidean Vectors

A transformation applied to a Euclidean vector or three-vector will yield a transformed three-vector as shown in (129) where the indices n and m go from 1 to 3.

$$a'_n = R_{nm} a_m \tag{129}$$

Euclidean vectors only contain the three spatial components and cannot be transformed by applying the tensor (122) directly on them. The vectors of the Lorentz tensor that relate the spatial components to their transformed counterparts need to be applied to the vector a_m individually.
To find the velocity of a particle in a frame which is Lorentz boosted in the x-direction with velocity $\vec{u} = (u, 0, 0)$, the space and time components are defined to transform like:

$$x' = \frac{x - ut}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{130}$$

$$y' = y \tag{131}$$

$$z' = z \tag{132}$$

$$t' = \frac{t - x\frac{v}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{133}$$

Knowing this, we can define a small increment in time dt' along which a particle will move a small distance dx':

$$dt' = \gamma(dt - dx\frac{u}{c^2}) \tag{134}$$

$$dx' = \gamma(dx - udt) \tag{135}$$

$$dy' = dy \tag{136}$$

$$dz' = dz \tag{137}$$

Thus the transformed velocities will be:

$$v'_x = \frac{dx'}{dt'} = \frac{\gamma(dx - udt)}{\gamma(dt - dx\frac{u}{c^2})} = \frac{v_x - u}{1 - v_x\frac{u}{c^2}} \tag{138}$$

$$v'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma(dx - udt)} = \frac{v_y}{\gamma(v_x - u)} \tag{139}$$

$$v'_z = \frac{dz'}{dt'} = \frac{dz}{\gamma(dx - udt)} = \frac{v_z}{\gamma(v_x - u)} \tag{140}$$

where v_x, v_y and v_z are the velocities of the particle in the rest frame. This same procedure where space and time are individually transformed need to be carried on for any standard Euclidean vector such as force, acceleration, pressure,etc.