The Extended Maxwell Algebra
and Two Different Ways of Constructing a Particle Model

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Abstract

The Maxwell group, an extension of the Poincaré group, which contains space-time transformations in special relativity, can itself be extended repeatedly. We examine one of these extensions, Maxwell$_3$, derive from it equations of motion obeying its symmetries using two different methods and interpret these equations. One of these methods produces equations of motion of a particle travelling in an electromagnetic field linear in space and time; the other one produces the same external field, but grants an additional term of a particle which is linearly polarized by the external field, such that the electric and magnetic polarizabilities are equal and opposite.

1 INTRODUCTION

One of the revolutionary notions of special relativity was the discovery of the relative nature of distance between two points. The space-time interval between two events in Minkowski space replaced the concept of spatial distance as an inertial frame independent quantity. The Lorentz transformations are used to interchange inertial frames of reference leaving the space-time interval invariant and (initially) keeping the origin at the same space-time coordinates of the previous frame.

Because of its property of leaving the space-time interval intact, the set of Lorentz transformations forms a symmetry group for special relativity. This group is a so-called Lie group, roughly meaning that there exists a function which parameterizes the group while obeying certain conditions of differentiability[1]. Together with spacetime translations, the Lorentz

[1] Understanding Lie groups and algebras is required for understanding this report, therefore I highly recommend reading the appendix on page 15 for an explanation of those parts of group theory which are necessary here.

group forms the Poincaré Lie group. The Poincaré group, which includes *all* of the spatial transformations leaving the spacetime interval between two events intact, can subsequently be extended into the Maxwell group. The equations of motion satisfying Maxwell symmetry[2] are those of a particle travelling in a constant external electromagnetic field, see [8]. The Maxwell group has applications in, for instance, studying the cosmological constant [2] and particles in homogeneous electromagnetic fields.
Here, in this report largely based on the 2017 paper by Gomis and Kleinschmidt [6], we investigate further extensions of the Maxwell group, dubbing the Maxwell group Maxwell$_2$ and, frankly, mostly focusing on the extension immediately succeeding Maxwell$_2$, namely Maxwell$_3$.

In [6], Gomis and Kleinschmidt outline a method for deriving the most general Lagrangian – and hence equations of motion, obeying Maxwell$_n$ symmetry. The process, one of non-linear realization [5], is to derive the most general covariant[3] quantities under Maxwell$_n$ transformation, using the Maurer-Cartan one-form $g^{-1}dg$ [4], where g is the generalized group element[5]. This one-form resides inside the group's Lie algebra, where the coefficients in front of the generators are the sought-after covariant quantities. These quantities can be combined to form the most general Lagrangian.

In an earlier paper [7], Gomis Gibbons Pope use two different methods of combining the covariant quantities into a Lagrangian for Maxwell$_2$ and show their equivalency.
Here, it is our goal to show that this equivalency does not hold for Maxwell$_3$. We will first reconstruct Maxwell$_3$ from the ground up, rederive its Maurer-Cartan one-form and then construct the Lagrangian using the two methods, which we will describe below. Consequently, we will derive the equations of motion from both Lagrangians and demonstrate that they acquire a significantly different physical interpretation.

In brief, the two methods differ in the following way: in [6], Gomis introduces new distinct variables akin to canonical momenta, f_{ab}, f_{abc}, ..., to construct the Lagrangian[6]. Let ω^{ab}, ω^{abc} be the Maxwell-covariant quantities dependent on the particle trajectory and the exter-

[2]For an EOM to satisfy a symmetry, it means that the equation of motion still holds when applying a group transformation to the coordinates the EOM depends on.

[3]A quantity is *invariant* if it stays constant under a certain transformation, *covariant* if it changes with the transformations, but retains its original information. For instance, in regular Euclidian space, the length of an object is invariant, because it does not vary under rotations and translations, but its position vector's coefficients are covariant, as those do depend on those particular transformations as applied to the basis vectors, even though the vector itself will always point to the same absolute position in space.

[4]Informally, a one-form is a mathematical object constituted by a number of differentials dx_n and corresponding coefficients. Hence, $dg = \sum_{n=1}^{N} \frac{\partial g}{\partial x_n} dx_n$.

[5]Meaning that the parameters of the function which parameterizes the Lie group are left as variables, so as to signify any possible element.

[6]These variables resemble canonical momenta, because $f_{ab} = \partial L/\partial\dot{\theta}_{ab}$, where θ^{ab} is one of the variables on which the Lagrangian depends. This is by definition the canonical momentum corresponding to θ^{ab}. A similar story holds for f_{abc}.

nal electromagnetic field. Then the Lagrangian is constructed as follows:

$$L = m\omega_1^2 + \frac{1}{2} f_{ab}\omega_2^{ab} + \frac{1}{2} f_{abc}\omega_3^{abc},$$

implying the Einstein summation convention. This would be the formulation for a Maxwell$_3$ Lagrangian, with more terms being present for higher level extensions. The alternative method described in earlier literature is to simply say that

$$L = m\omega_1^2 + \frac{a}{2}\omega_2^2 + \frac{b}{2}\omega_3^2.$$

Now, we will proceed to construct the Maxwell$_3$ algebra and run through the steps of derivation.

2 THE MAXWELL ALGEBRA

The Poincaré group is the Lie group of the isometries of Minkowski space, i.e. all transformations of the space that leave the spacetime interval intact. The Poincaré algebra consists out of two types of generators, M_{ab} for boosts and rotations and P_a for translations, where a and b both range from 0 to 3. In the Poinceré Algebra, its translation generators commute. For more information on Lie groups and algebras, see [3] or page 15.

The Maxwell group (i.e. the first extension of Poincaré) is the group of symmetries of a particle travelling in a uniform electromagnetic field. Its algebra is given by an extension of the Poincaré algebra, where the translation generators no longer commute, but satisfy

$$[P_a, P_b] = Z_{ab}.$$

The resulting Lie algebra can be extended once more, using the commutation relation

$$[Z_{ab}, P_c] = Y_{abc}.$$

For each of the generators, it is possible to assign a level l such that for any two non-commuting generators $G_{l=l_1}$ and $G_{l=l_2}$, the generator resulting from their commutation relation satisfies

$$\left[G_{l=l_1}, G_{l=l_2}\right] = G_{l=l_1+l_2},$$

where the generators of the Lorentz algebra M_{ab} correspond to level $l = 0$ and the generators for Poincaré P_a to level $l = 1$. It can then easily be seen that Z_{ab} and Y_{abc} have levels 2 and 3 associated to them respectively, see [4].

Because every subsequent extension has a higher level generator associated to it, it is natural to call each associated group and algebra by the level of its highest level generator, such that the first extension of Poincaré with Z_{ab} becomes Maxwell$_2$ and the second extension with Y_{abc} Maxwell$_3$. The Maxwell$_2$ group is traditionally called the Maxwell group.

It can also be seen that if the highest level generator present has level $l = n$, then any commutation relation that would result in a generator of a level higher than n, should vanish.

This process of extending can be continued indefinitely, approaching Maxwell$_\infty$, see for instance [4, 7]. Here we will restrict ourselves to Maxwell$_2$ and Maxwell$_3$. I will use the term Maxwell without any subscript loosely to refer to any of the extensions beyond Poincaré itself in general.
These are thus the commutation relations for Maxwell$_3$:

$$\begin{aligned}
[M_{ab}, M_{cd}] &= g_{bc}M_{ad} - g_{bd}M_{ac} - g_{ac}M_{bd} + g_{ad}M_{bc} \\
[M_{ab}, P_c] &= g_{bc}P_a - g_{ac}P_b \\
[P_a, P_b] &= Z_{ab} \\
[M_{ab}, Z_{cd}] &= g_{bc}Z_{ad} - g_{bd}Z_{ac} - g_{ac}Z_{bd} + g_{ad}Z_{bc} \\
[Z_{ab}, Z_{cd}] &= 0 \\
[Z_{ab}, P_c] &= Y_{abc} \\
[M_{ab}, Y_{cde}] &= g_{bc}Y_{ade} - g_{ac}Y_{bde} + g_{bd}Y_{cae} - g_{ad}Y_{cbe} + g_{be}Y_{cda} - g_{ae}Y_{cdb} \\
[P_a, Y_{bcd}] &= [Z_{ab}, Y_{cde}] = \left[Y_{abc}, Y_{def}\right] = 0,
\end{aligned}$$

where $g_{\mu\nu} = (-1,1,1,1)$. It can be seen that, since Y_{abc} is the highest level generator, any commutation relation which would result in a generator of a higher level than Y_{abc}, vanishes. For greater Maxwell extensions, those commutation relations would become non-vanishing. The levels associated to each generator can be seen in table 1.

l (level)	0	1	2	3
Generator	M_{ab}	P_a	Z_{ab}	Y_{abc}

Table 1: All generators.

2.1 GOLDSTONES

Interestingly, the equations of motion that obey Maxwell symmetry, give rise to solutions that spontaneously break the given symmetry. Any particle world line in space necessarily violates translation and rotational symmetries, although any of these translated and rotated word lines still satisfy the equations of motion. It is possible to shift from symmetry breaking solution to symmetry breaking solution by applying a Maxwell transformation.
If G is a generator for a specific type of transformation, then the degree to which that type of transformation is applied in a transformation T can be modelled using the mapping from Lie algebras to Lie groups

$$T = e^{mG},$$

where m is some scalar indicating how much space has shifted under the generator G.
In the context of spontaneous symmetry breaking, m is called a Goldstone[7].

[7] In field theoretical spontaneous symmetry breaking, broken symmetries imply the existence of massless particles called Goldstone bosons.

The Goldstones associated to the generators P_a, Z_{ab} and Y_{abc} are x^a, θ^{ab} and ξ^{abc} respectively.

We can interpret the Goldstone x^a's value as the degree to which space has shifted around some reference point. Thus, if we parameterize x^a in some particle's proper time, conversely, it can be seen as that particle's position in space with respect to the origin in some rest frame. Therefore it is our goal to construct a particle Lagrangian out of these goldstons with this particular interpretation, while still preserving Maxwell symmetry.
The other Goldstones would acquire a different interpretation when parameterized in the particle's proper time. When the Lagrangian is constructed out of the given Goldstones, then θ and ξ become separate parameters which determine the value of the Lagrangian. Hence they can be seen to determine the "background" in which the particle is situated at any given moment of its proper time. This can be observed from the equations of motion as they are derived later in this report.

3 Maurer-Cartan One-Form

To construct the most general Lagrangian for a particle subject to Maxwell symmetries, we require quantities which are covariant under Maxwell transformations. Using these as building blocks, we can construct a Maxwell invariant Lagrangian, from which we can also derive the most general equations of motion. As mentioned in the introduction, in the method of nonlinear realizations these covariant quantities can be derived from the Maurer-Cartan one-form.
The Maurer-Cartan one-form is defined as

$$\Omega = g^{-1} dg,$$

where g is the generalised group element[8].
We are however not interested in all elements of Maxwell. We wish to observe the particle from a particular inertial reference frame and we are therefore interested in fixing a Lorentz gauge[9]. Therefore, we do not include Lorentz transformations in the generalised group element and only consider generalised group elements from the quotient group

$$\text{Maxwell}_n/\text{Lorentz}.$$

The one-form will be an element of the coset's Lie Algebra:

$$\Omega = \Omega_1^a P_a + \frac{1}{2}\Omega_2^{ab} Z_{ab} + \frac{1}{2}\Omega_3^{abc} Y_{abc}.$$

[8]That is an element of the Lie group where the parameters of the Lie group are left as variables as to encompass any possible element. d denotes the exterior derivative on g, which in our case is just using the chain rule to differentiate: $dg = \sum_{n=1}^{N} \frac{\partial g}{\partial x_n} dx_n$.
[9]This means that we do not allow variance over boosts and rotations.

The coefficients in front of the generators will be the most general combination of Goldstones covariant under Maxwell transformations.
In this section, we will derive the Maurer-Cartan one-form for Maxwell$_2$ and Maxwell$_3$ explicitly.

3.1 Computing the Maurer-Cartan one-form

An element of the coset is given by

$$g = e^{x^a P_a} e^{\frac{1}{2}\theta^{ab} Z_{ab}} e^{\frac{1}{2}\xi^{abc} Y_{abc}},$$

where every generator with its corresponding Goldstone is exponentiated exactly once through the Einstein summation convention. The $\frac{1}{2}$ in front of Z_{ab} and Y_{abc} comes from the fact that $Z_{ab} = -Z_{ba}$ and $Y_{abc} = -Y_{bac}$. Without this fraction, each generator would be counted twice. The one-form dg is given by

$$dg = \frac{\partial g}{\partial x^a} dx^a + \frac{\partial g}{\partial \theta^{ab}} d\theta^{ab} + \frac{\partial g}{\partial \xi^{abc}} d\xi^{abc}$$

We will compute each partial derivative.

We know that

$$\begin{aligned} \frac{\partial g}{\partial x^k} &= \frac{d}{dx^k} e^{x^a P_a} e^{\frac{1}{2}\theta^{ab} Z_{ab}} e^{\frac{1}{2}\xi^{abc} Y_{abc}} \\ &= \left(\frac{d}{dx^k} e^{x^a P_a} \right) e^{\frac{1}{2}\theta^{ab} Z_{ab}} e^{\frac{1}{2}\xi^{abc} Y_{abc}}. \end{aligned}$$

Remember that we cannot just write P_k in front of the exponential for the derivative, because the different Ps do not commute.

$$\frac{d}{dx^k} e^{x^a P_a} = \frac{d}{dx^k} \sum_{n=0}^{\infty} \frac{(x^a P_a)^n}{n!}$$

Where

$$\frac{d}{dx^k} (x^a P_a)^n = \sum_{m=0}^{n-1} (x^a P_a)^m P_k (x^a P_a)^{n-1-m}$$

We can keep using commutation relations to move P_k to the right. Then we get

$$\begin{aligned} \sum_{m=0}^{n-1} (x^a P_a)^m P_k (x^a P_a)^{n-1-m} &= n(x^a P_a)^{n-1} P_k - \sum_{m=1}^{n-1} (m)(x^a P_a)^{m-1} x^b Z_{bk} (x^a P_a)^{n-m-1} \\ &= n(x^a P_a)^{n-1} P_k - \frac{1}{2}(n-1)(n)(x^a P_a)^{n-2} x^b Z_{bk} - \frac{1}{6}(n-2)(n-1)(n)(x^a P_a)^{n-3} x^b x^c Y_{bkc}. \end{aligned}$$

The Z_{ab} term only appears for $n \geq 2$ and the Y_{abc} term only for $n \geq 3$. Hence,

$$\begin{aligned}\frac{d}{dx^k} g &= \left(\sum_{n=0}^{\infty} \frac{(x^a P_a)^n}{n!} P_k - \sum_{n=0}^{\infty} \frac{(x^a P_a)^n}{n!} \left(\frac{1}{2} x^b Z_{bk} \right) - \sum_{n=0}^{\infty} \frac{(x^a P_a)^n}{n!} \left(\frac{1}{6} x^b x^c Y_{bkc} \right) \right) e^{\frac{1}{2}\theta^{ab} Z_{ab}} e^{\frac{1}{2}\xi^{abc} Y_{abc}} \\ &= e^{x^a P_a} \left(P_k - \frac{1}{2} x^a Z_{ak} - \frac{1}{6} x^a x^b Y_{akb} \right) e^{\frac{1}{2}\theta^{ab} Z_{ab}} e^{\frac{1}{2}\xi^{abc} Y_{abc}}\end{aligned}$$

Because Z_{ab} and Y_{abc} commute, we can now immediately move them all the way to the right. P_k does not commute with Z_{ab}, therefore

$$\begin{aligned}P_k e^{\frac{1}{2}\theta^{ab} Z_{ab}} &= P_k \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\theta^{ab} Z_{ab}\right)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\theta^{ab} Z_{ab}\right)^n}{n!} P_k - \sum_{n=1}^{\infty} \frac{n \left(\frac{1}{2}\theta^{ab} Z_{ab}\right)^{n-1}}{n!} \frac{1}{2} \theta^{ab} Y_{abk} \\ &= e^{\frac{1}{2}\theta^{ab} Z_{ab}} \left(P_k - \frac{1}{2} \theta^{ab} Y_{abk} \right).\end{aligned}$$

So the partial derivative becomes

$$\frac{\partial g}{\partial x^k} = g \left(P_k - \frac{1}{2} x^a Z_{ak} - \frac{1}{2} \theta^{ab} Y_{abk} - \frac{1}{6} x^a x^b Y_{akb} \right).$$

Fortunately, since the Z and Y generators commute, we can take the other partial derivatives by immediately writing the generators on the right:

$$\begin{aligned}\frac{\partial g}{\partial \theta^{kl}} &= g \frac{1}{2} Z_{kl} \\ \frac{\partial g}{\partial \xi^{klm}} &= g \frac{1}{2} Y_{klm}\end{aligned}$$

Thus,

$$\begin{aligned}dg &= g \left(P_k dx^k - \frac{1}{2} x^a Z_{ak} dx^k - \frac{1}{2} \theta^{ab} Y_{abk} dx^k - \frac{1}{6} x^a x^b Y_{akb} dx^k + \frac{1}{2} Z_{kl} d\theta^{kl} + \frac{1}{2} Y_{klm} d\xi^{klm} \right) \\ &= g \left(dx^a P_a + \frac{1}{2} \left(d\theta^{ab} + dx^a x^b \right) Z_{ab} + \frac{1}{2} \left(d\xi^{abc} - \theta^{ab} dx^c + \frac{1}{3} dx^a x^b x^c \right) Y_{abc} \right).\end{aligned}$$

So, the Maurer-Cartan one-form for Maxwell$_3$ is

$$\Omega = dx^a P_a + \frac{1}{2} \left(d\theta^{ab} + dx^a x^b \right) Z_{ab} + \frac{1}{2} \left(d\xi^{abc} - \theta^{ab} dx^c + \frac{1}{3} dx^a x^b x^c \right) Y_{abc}$$

Then, if we project the same symmetry onto the coefficients as their respective generators[10]:

$$\begin{cases} \Omega_1^a &= dx^a \\ \Omega_2^{ab} &= d\theta^{ab} + \frac{1}{2} \left(dx^a x^b - dx^b x^a \right) \\ \Omega_3^{abc} &= d\xi^{abc} - \frac{1}{3} \left(2\theta^{ab} dx^c - \theta^{bc} dx^a - \theta^{ca} dx^b \right) + \frac{1}{6} \left(dx^a x^b - dx^b x^a \right) x^c \end{cases} \tag{3.1}$$

[10]This means that we use the symmetry properties of the generators and Goldstones to rearrange the terms, so that the coefficients acquire the same symmetry as the generators themselves, while keeping the value of the Maurer-Cartan one-form fixed.

For Maxwell$_2$, $Y_{abc} = 0$, effectively cancelling the last term in the MC one-form. Therefore only the first two omegas would be relevant for that level extension.

4 Equations of Motion

In this section, we will model a particle exhibiting Maxwell$_2$ and Maxwell$_3$ symmetry. First we will construct its Lagrangian using the Maurer-Cartan one-form derived in the previous section, then we will derive its equations of motion using the Euler-Lagrange equation. We will also explore the two methods of building the particle Lagrangian mentioned before.
In this section, we will purely focus on the equations of motion in their mathematical form. The next section will be dedicated to their interpretation.

To interpret x^a as a point particle's position in space, we let the Goldstones depend on the proper time τ. Then equation 3.1 becomes

$$\begin{cases} \Omega_1^a &= \dot{x}^a d\tau \\ \Omega_2^{ab} &= \dot{\theta}^{ab} d\tau + \frac{1}{2}\left(\dot{x}^a x^b - \dot{x}^b x^a\right) d\tau \\ \Omega_3^{abc} &= \dot{\xi}^{abc} d\tau - \frac{1}{3}\left(2\theta^{ab}\dot{x}^c - \theta^{bc}\dot{x}^a - \theta^{ca}\dot{x}^b\right) d\tau + \frac{1}{6}\left(\dot{x}^a x^b - \dot{x}^b x^a\right) x^c d\tau. \end{cases}$$

We can then define

$$\begin{cases} \omega_1^a &= \dot{x}^a \\ \omega_2^{ab} &= \dot{\theta}^{ab} + \frac{1}{2}\left(\dot{x}^a x^b - \dot{x}^b x^a\right). \\ \omega_3^{abc} &= \dot{\xi}^{abc} - \frac{1}{3}\left(2\theta^{ab}\dot{x}^c - \theta^{bc}\dot{x}^a - \theta^{ca}\dot{x}^b\right) + \frac{1}{6}\left(\dot{x}^a x^b - \dot{x}^b x^a\right) x^c. \end{cases} \tag{4.1}$$

Which will be the Maxwell$_3$ covariant quantities which we will construct the Lagrangian out of. For Maxwell$_2$, ω_3^{abc} can be ignored. In derivations, we will omit the subscripts $_{1\ 2\ 3}$, as the level associated to the ωs can easily be observed looking at the number of indices.
As these are the Maxwell covariant quantities, I will reference these as the *MCQ*s.

4.1 Maxwell$_2$

Using the MCQs from equation 4.1, we construct the Lagrangian as follows[11]

$$L = m\omega_1^2 + \frac{a}{2}\omega_2^2.$$

As a tensor with upper indices is contravariant and with lower indices is covariant, a contracted tensor produces an invariant scalar. Thus we can see that this Lagrangian is indeed invariant under Maxwell transformations.
We can then use the Euler-Lagrange equation for each of the Goldstones x^k and θ^{kl}. We will also use the contracted tensor chain rule,

$$\frac{\partial \omega^2}{\partial j^k} = 2\omega_A \frac{\partial \omega^A}{\partial j^k}, \tag{4.2}$$

[11] ω^2 should be interpreted as ω contracted with itself: $\omega_A\omega^A$, summing over the set of indices A.

where A indicates the ω's indices and j^k is an arbitrary variable on which ω^A depends.
We arrive at

$$\dot{\omega}_{kl} = 0, \tag{4.3}$$

when varying with respect to θ^{kl} and,

$$m\ddot{x}^k + a\omega^{ka}\dot{x}_a = 0, \tag{4.4}$$

when varying with respect to x^k.

4.2 Maxwell$_3$: Contracted MCQs

We will use the same method for the Maxwell$_3$ equation of motion. Again using equation 4.1, we construct the Lagrangian.

$$L = m\omega_1^2 + \frac{a}{2}\omega_2^2 + \frac{b}{2}\omega_3^2.$$

When we use the Euler-Lagrange equation and equation 4.2, varying with respect to ξ^{klm}, we arrive at

$$\dot{\omega}_{klm} = 0. \tag{4.5}$$

Doing the same for θ^{kl} we get,

$$a\dot{\omega}_{kl} = -\frac{b}{3}\left(2\omega_{kla} - \omega_{akl} - \omega_{lak}\right)\dot{x}^a,$$

and then using the Jacobi identity[12]

$$a\dot{\omega}_{kl} = -b\omega_{kla}\dot{x}^a. \tag{4.6}$$

Then varying with respect to x^k,

$$2m\ddot{x}_k + 2a\omega_{ka}\dot{x}^a - b\omega_{abk}\dot{\theta}^{ab} + b\omega_{kab}\left(\dot{x}^a x^b - \dot{x}^b x^a\right) = 0,$$

which using the Jacobi identity, simplifies to

$$m\ddot{x}^k + a\omega^{ka}\dot{x}_a - \frac{b}{2}\omega^{abk}\omega_{ab} = 0. \tag{4.7}$$

4.3 Maxwell$_3$: Momentum-MCQ Coupling

In this section, we reconstruct the Lagrangian using an alternative method, resulting in different equations of motion.
We use the same MCQs from equation 4.1. Then as in [6], we introduce two new independent variables f_{ab} and f_{abc}, which we use in the Lagrangian as follows:

$$L = -m\sqrt{-\omega_a\omega^a} + \frac{1}{2}f_{ab}\omega_2^{ab} + \frac{1}{2}f_{abc}\omega_3^{abc}.$$

[12] This is explained in the section on Lie algebras inside the appendix on page 15.

These f variables resemble canonical momenta, because $f_{ab} = \partial L / \partial \dot{\theta}_{ab}$. This is by definition the canonical momentum corresponding to θ^{ab}. The same story holds for f_{abc} and ξ^{abc}. In [7], it is shown that this and the previous method are equivalent for Maxwell$_2$.

Using the Euler-Lagrange equations as before for the canonical momenta, we acquire

$$\omega_{klm} = 0,$$

and,

$$\omega_{kl} = 0.$$

Varying with respect to ξ^{klm} gives

$$\dot{f}_{klm} = 0. \tag{4.8}$$

Using the Euler-Lagrange equations for the other variables, is more interesting. Varying with respect to θ^{kl},

$$\dot{f}_{kl} = -f_{kla}\dot{x}^a. \tag{4.9}$$

Then we do the same for x^k; we get

$$m\ddot{x}_k + f_{ka}\dot{x}^a + \frac{1}{2}\dot{x}^a x^b \left(f_{bka} + f_{abk} + \frac{1}{3}\left(2f_{kab} - f_{akb} - f_{bka}\right) \right) = 0.$$

This simplifies to

$$m\ddot{x}^k + f^{ka}\dot{x}_a = 0. \tag{4.10}$$

5 The Equations of Motion's Interpretation

We are left with three sets of equations of motion.
For Maxwell 2,

$$\begin{cases} m\ddot{x}^a + a\omega^{ab}\dot{x}_b = 0, & \text{(equation 4.4)} \\ \dot{\omega}^{ab} = 0. & \text{(equation 4.3)} \end{cases}$$

For Maxwell 3, from the $f_A\omega^A$ method,

$$\begin{cases} m\ddot{x}^a + f^{ab}\dot{x}_b = 0, & \text{(equation 4.10)} \\ \dot{f}^{ab} + f^{abc}\dot{x}_c = 0, & \text{(equation 4.9)} \\ \dot{f}^{abc} = 0. & \text{(equation 4.8)} \end{cases}$$

For Maxwell 3, from the ω^2 method,

$$\begin{cases} m\ddot{x}^a + a\omega^{ab}\dot{x}_b - \frac{b}{2}\omega^{bca}\omega_{bc} = 0, & \text{(equation 4.7)} \\ a\dot{\omega}^{ab} + b\omega^{abc}\dot{x}_c = 0, & \text{(equation 4.6)} \\ \dot{\omega}^{abc} = 0. & \text{(equation 4.5)} \end{cases}$$

We can simplify these by partially solving the equations.

5.1 MAXWELL$_2$

Let us first examine the Maxwell$_2$ case. Since $\dot{\omega}^{ab} = 0$, we know that ω^{ab} is a constant. Let

$$qF^{ba} = a\omega^{ab}.$$

Because ω^{ab} is antisymmetric, we can interpret qF^{ba} as a particle's charge multiplied by the electromagnetic field tensor. Then the equation of motion becomes

$$m\ddot{x}^a + qF^{ba}\dot{x}_b = 0. \tag{5.1}$$

Which is identical to the equation of motion of the Lorentz force inside a constant electromagnetic field.

5.2 MAXWELL$_3$, MOMENTUM-MCQ COUPLING

The $f_A\omega^A$ approach is easier to analyse than the contracted MCQs approach. Similarly, because $\dot{f}^{abc} = 0$, we know that f^{abc} is constant. Let

$$-qS^{abc} = f^{abc}.$$

Then $\dot{f}^{ab} + f^{abc}\dot{x}_c = \dot{f}^{ab} - qS^{abc}\dot{x}_c = 0$. Then we can simply integrate $\dot{f}^{ab}$. Let

$$f^{ab} = qF^{ba} = q\int S^{abc}\dot{x}_c d\tau = q\left(S^{abc}x_c + H^{ab}\right),$$

where H^{ab} is a constant of integration (also anti-symmetric in its indices). Here too, F^{ba} can be interpreted as the electromagnetic field tensor, as it is antisymmetric, but now it is linear in space.

$$\begin{cases} m\ddot{x}^a + qF^{ba}\dot{x}_b = 0, \\ F^{ba} = S^{abc}x_c + H^{ab}, \end{cases} \tag{5.2}$$

where the tensors S^{abc} and H^{ab} are constant.

5.3 MAXWELL$_3$, CONTRACTED MCQS

The equations of motion which result from this method, are interesting, as there is an extra term present. In the same way, let

$$-qS^{abc} = b\omega^{abc},$$

where S^{abc} is a constant. Then, because $a\dot{\omega}^{ab} = -b\omega^{abc}\dot{x}_c$,

$$a\omega^{ab} = q\int S^{abc}\dot{x}_c d\tau = q\left(S^{abc}x_c + H^{ab}\right).$$

The last term in the equation of motion is $-\frac{b}{2}\omega^{bca}\omega_{bc}$. This term then becomes

$$\frac{q^2}{2a}S^{bca}\left(S_{bcd}x^d + H_{bc}\right).$$

If we choose

$$F^{ba} = S^{abc}x_c + H^{ab},$$

then the equations of motion become

$$\begin{cases} m\ddot{x}^a + qF^{ba}\dot{x}_b + \frac{q^2}{2a}\left(\partial^a F^{bc}\right)F_{bc} = 0 \\ F^{ba} = S^{abc}x_c + H^{ab}. \end{cases} \tag{5.3}$$

Interpreting F^{ba} as the electromagnetic field tensor, these equations of motion represent a particle again in an external electromagnetic field which is linear in space, but also a term of this particle's dipole moment, induced by the external field, interacting with the field itself. We will expand on this in the next section.

Here, it is briefly interesting to note that, since the electromagnetic field tensor depends on the different ωs and the ωs in turn depend on all three Goldstones, the Goldstones θ^{ab} and ξ^{abc} are indeed responsible for the particle's environment (i.e. the forces that apply to it).

Dipole Interactions

In order to interpret equation 5.3 properly, we will examine the equation of motion of a particle with an electric and magnetic dipole moment and demonstrate that this bears equivalence to equation 5.3.
A charged particle moving in an external electromagnetic field with a dipole moment, has the following Lagrangian

$$L = \frac{m}{2}\dot{x}^2 + qA_\mu\dot{x}^\mu - \frac{1}{2}F_{\mu\nu}D^{\mu\nu},$$

where A_μ is the four-potential and $D^{\mu\nu}$ is the anti-symmetric dipole tensor, see [1].

The dipole tensor can be split into the magnetic and electric part as follows

$$D^{\mu\nu} = P^{[\mu}\dot{x}^{\nu]} + \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}M_\kappa\dot{x}_\lambda,$$

where $P^{[\mu}\dot{x}^{\nu]} = \frac{1}{2}\left(P^\mu\dot{x}^\nu - P^\nu\dot{x}^\mu\right)$.

We will manipulate these quantities, to show that those specific manipulations result in the equation of motion we desire. The electric field and the magnetic field can be derived from the electromagnetic tensor as follows

$$E^\mu = F^{\mu\nu}\dot{x}_\nu \qquad B^\mu = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\nu\rho}\dot{x}_\sigma,$$

so for dipole moments proportional to their corresponding fields, we have

$$\begin{aligned} P^\mu &= \alpha E^\mu \\ &= \alpha F^{\mu\lambda}\dot{x}_\lambda \\ M_\kappa &= \beta B_\kappa \\ &= \frac{\beta}{2}\epsilon_{\mu\nu\kappa\lambda}F^{\mu\nu}\dot{x}^\lambda, \end{aligned}$$

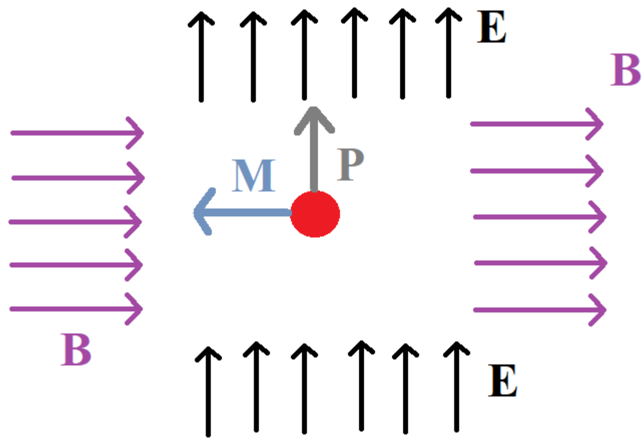

Figure 5.1: The particle described by our equations of motion. The electric polarization P points along the electric field, the magnetic polarization points antiparallel to the magnetic field. The field orientation is arbitrary.

where α is the electric polarizability and β the magnetic susceptibility. So the dipole moment tensor becomes

$$D^{\mu\nu} = -\alpha F^{\lambda[\mu}\dot{x}^{\nu]}\dot{x}_\lambda + \frac{\beta}{4}\epsilon^{\kappa\mu\nu\lambda}\epsilon_{\kappa\alpha\beta\gamma}F^{\alpha\beta}\dot{x}^\gamma\dot{x}_\lambda.$$

Because $\epsilon^{\kappa\mu\nu\lambda}\epsilon_{\kappa\alpha\beta\gamma} = -6\delta^\mu_{[\alpha}\delta^\nu_\beta\delta^\lambda_{\gamma]}$,[13] the magnetic part

$$\begin{aligned}\frac{\beta}{4}\epsilon^{\kappa\mu\nu\lambda}\epsilon_{\kappa\alpha\beta\gamma}F^{\alpha\beta}\dot{x}^\gamma\dot{x}_\lambda &= \frac{\beta}{2}\left(F^{\mu\nu} - F^{\lambda\mu}\dot{x}^\nu\dot{x}_\lambda + F^{\lambda\nu}\dot{x}^\mu\dot{x}_\lambda\right)\\ &= -\beta\left(F^{\lambda[\mu}\dot{x}^{\nu]}\right) + \frac{\beta}{2}F^{\mu\nu}.\end{aligned}$$

Here we have used that $x_\mu x^\mu = -1$. So,

$$D^{\mu\nu} = -(\alpha+\beta)F^{\lambda[\mu}\dot{x}^{\nu]}\dot{x}_\lambda + \frac{\beta}{2}F^{\mu\nu},$$

and if we pick $\alpha = -\beta$, we acquire a neat

$$D^{\mu\nu} = \frac{\beta}{2}F^{\mu\nu}.$$

This is equivalent to picking the electric polarizability to be equal and opposite to the magnetic susceptibility. See figure 5.1.

[13]The indices surrounded by square brackets indicates summing over all the combinations of indices, adding a minus sign for odd permutations: $\delta^\mu_{[\alpha}\delta^\nu_\beta\delta^\lambda_{\gamma]} = \frac{1}{6}\left(\delta^\mu_\alpha\delta^\nu_\beta\delta^\lambda_\gamma + \delta^\mu_\beta\delta^\nu_\gamma\delta^\lambda_\alpha + \delta^\mu_\gamma\delta^\nu_\alpha\delta^\lambda_\beta - \delta^\mu_\beta\delta^\nu_\alpha\delta^\lambda_\gamma - \delta^\mu_\gamma\delta^\nu_\beta\delta^\lambda_\alpha - \delta^\mu_\alpha\delta^\nu_\gamma\delta^\lambda_\beta\right)$.

The total Lagrangian becomes

$$L = \frac{m}{2}\dot{x}^2 + qA_\mu \dot{x}^\mu - \frac{\beta}{4}F_{\mu\nu}F^{\mu\nu},$$

and using the Euler-Lagrange equation

$$\begin{aligned}\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^k} &= \frac{d}{d\tau}\left(m\dot{x}_k + qA_k\right)\\ &= m\ddot{x}_k + q\left(\partial_\mu A_k\right)\frac{dx^\mu}{d\tau}\\ &= m\ddot{x}_k + q\left(\partial_\mu A_k\right)\dot{x}^\mu,\end{aligned}$$

where we recall that the potential A_μ depends on spacetime coordinates.

$$\frac{\partial L}{\partial x^k} = q\left(\partial_k A_\mu\right)\dot{x}^\mu - \frac{\beta}{2}\left(\partial_k F_{\mu\nu}\right)F^{\mu\nu}.$$

Combining all the terms, we get

$$m\ddot{x}^k + q\left(\partial^\mu A^\kappa - \partial^\kappa A^\mu\right)\dot{x}_\mu + \frac{\beta}{2}\left(\partial^k F^{\mu\nu}\right)F_{\mu\nu} = 0.$$

Which, because of the definition of the potential, equals

$$m\ddot{x}^a + qF^{ba}\dot{x}_b + \frac{\beta}{2}\left(\partial^a F^{bc}\right)F_{bc} = 0. \tag{5.4}$$

We see that this exactly matches 5.3, when $\beta = \frac{q^2}{a}$. Hence we can see that the equations of motion obeying Maxwell$_3$ symmetry, using the ω^2 method, are identical to those of a particle moving through a linear external field, undergoing linear polarization, in such a way that the electric and magnetic polarizabilities are equal but opposite.

6 CONCLUSION

We can thus see that there indeed is a significant difference between the equations of motion and their physical interpretation depending on which method of Lagrangian construction is used. The contracted MCQs method produces the equations of motion of a particle travelling in an external electromagnetic field where the particle has a linear polarizability as an electric and magnetic dipole, where the electric polarizability is equal and opposite to the magnetic polarizability. The method of coupling the canonical momenta to the MCQs produces the same external field, but for a particle without any polarizability.

Hence, in this particular case, one can see that the method where the Lagrangian is formed through the contraction of the omegas is the more expansive.[14]

[14] We can attempt to recover the equations of motion of the momentum coupling case from the MCQ contracted case, by sending $a \to \infty$ in equation 5.3. Trying to keep the external field fixed, this requires $\omega^{ab} \to 0$ in equation 4.7. We can see that then the last term drops out. We however also lose the antisymmetric property of the electromagnetic field tensor, since ω^{ab} becomes uniform. This means that we cannot keep an external field and we regain a particle travelling in a straight line (of infinite charge, but without any field to influence it).

Appendix: Lie Groups and Their Algebras

The goal of this section is to informally explain Lie groups and algebras in addition to some preliminaries of group theory. For more information, proofs and a text that is less dense, see [3].

Preliminaries

A group G is a set of elements defined together with an operation $\cdot$, such that $\forall g, g_1, g_2, g_3 \in G$

1.$g_1 \cdot g_2 \in G$ (Closure)

2.$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ (Associativity)

3.$\exists e \in G$ s.t. $e \cdot g = g \cdot e = g$ (Existence of identity)

4.$\exists g^{-1} \in G$ s.t. $g \cdot g^{-1} = g^{-1} \cdot g = e$ (Existence of inverse)

Any group $H \subset G$ is called a subgroup of G. H is an invariant subgroup of G if $\forall h \in H$ and $g \in G$, $ghg^{-1} \in H$.
The set $gH = \{gh | h \in H\}$ is defined as a left coset of H. One can see this as multiplying a subgroup by a group element, creating a set where all elements of the subgroup have been multiplied by g. A right coset of H, Hg, is defined as $\{hg | h \in H\}$. It is important to note that cosets are not (sub-)groups generally. Also, if $g_2 \in g_1 H$, then $g_2 H = g_1 H$.
Cosets partition the group in subsets of equal magnitude. This means that $\forall g \in G$ there exists exactly one gH given H which includes g.

It is possible to define a multiplication law between cosets themselves, such that the set of all cosets forms a group, only if H is an invariant subgroup. This multiplication law is defined as follows $g_1 H \cdot g_2 H = (g_1 \cdot g_2) H$. Hence then if H is an invariant subgroup, then $G/H = \{gH | g \in G\}$ is the quotient group pronounced *G over H*. When constructing a quotient group, one effectively divides out the subgroup, for instance let our group be

$$G = \mathbb{C} \setminus \{0\}$$

with the operation of regular complex multiplication and

$$H = \left\{ e^{i\theta} | \theta \in [0, 2\pi) \right\}.$$

Then a general coset becomes

$$\begin{aligned} gH &= \left(re^{i\phi} \right) H \\ &= \left\{ re^{i(\phi+\theta)} | \theta \in [0, 2\pi) \right\} \\ &= \left\{ re^{i\theta} | \theta \in [0, 2\pi) \right\} \\ &= rH. \end{aligned}$$

So the quotient group equals
$$G/H = \{rH | r > 0\}$$
and we recover (a group isomorphic to) the real positive number line. In other words, dividing out H entails removing all elements from the group that were obtainable only by transformation through H. The real positive number line remains, because solely those elements do not require a phase shift to obtain when starting from identity.

Lie Groups

Taken from [3], a Lie group G is a differentiable manifold G which is also a group, such that the group multiplication $\cdot : G \times G \to G$ and the map $f(g) = g^{-1}$ are differentiable maps.
Informally and for the purpose of this report, this can be seen as a group whose elements are given by a differentiable function. Let us say that $g = g(x) = g(x_1, x_2, \ldots, x_n)$, then g is everywhere differentiable. The number of parameters necessary is called the group's dimension. Techincally, this description is in general only valid locally, as manifolds resemble Euclidian space locally only.

Lie Algebras

It is possible to construct a map from a so-called Lie algebra to any component of a Lie group that is connected to the identity element. A Lie algebra is a vector space which is constructed as follows: Let G be a Lie group with elements $g(x) = g(x_1, x_2, \ldots, x_n)$. Let $D(g(x))$ or simply $D(x)$ be a matrix representation of element $g(x)$. The *generators* of the group are then given by
$$X^a = \frac{\partial}{\partial x_a} D(x)|_{x=0}$$
where x_a is the ath group coordinate and X^a its corresponding generator. The generators form a basis for the vector space.

It is now possible to express elements from the group using the elements from the Lie algebra. If all generators commute, we get
$$g(x) = e^{x^a X_a},$$
using Einstein's summation convention. If the generators do not commute, it is necessary to have multiple exponentials. Let $\{X_1, X_2, \ldots, X_k\}$ and $\{Y_1, Y_2, \ldots, Y_l\}$ be two sets of commuting generators, then
$$g(x, y) = e^{x_a X^a} e^{y_a Y^a}.$$
It is possible in general to retrieve any element of the connected subgroup, the component of the Lie group connected to the identity element, with a finite number of exponentials. Using a vector space instead of a Lie group can greatly expedite the process of doing computations.

A Lie algebra L is a vector space endowed with the product $[a, b]$, the commutator between two elements, with the properties that $\forall a, b, c \in L$

1. $[a,b] \in L$

2. $\left[\alpha a + \beta b, c\right] = \alpha\,[a,c] + \beta\,[b,c]\ \forall \alpha,\beta \in \mathbb{R}$

3. $[a,b] = -\,[b,a]$

4. $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$

The last property is called the Jacobi identity.

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