A GAME THEORETIC CHARACTERIZATION OF MEMBERSHIP TO SEMI-KERNELS

Bachelor Thesis

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Abstract

This paper begins by familiarizing the reader with some graph-theoretical concepts which are necessary to understand the Poison Game. Following a natural order, previous results on the aforementioned game (namely, a characterization of the existence of semi-kernels) will then be discussed. These results will, in turn, set the ground for the introduction of a new game (which will be referred to as the Local Poison Game). Thereafter, original results on the Local Poison Game (namely, a characterization of the membership to a semi-kernel) will be provided.

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1 Introduction

In mathematics, it is often the case that, once an object has been defined, and rules regarding how this object interacts with others have been provided, patterns arise. Thus, it shouldn't come as a surprise to find that different games (which boil down to different sets of rules) characterize different properties of the graphs on which they are played.

Due to the fact that there exists a great variety of different games played on graphs, it is easier to study them by grouping them into categories according to the rules that define them[1]. Some popular categories are "Meet/Avoid" games, "Sabotage" games, and "Travel" games. As seen in [4], some Meet/Avoid games such as the Cops and Robber Game, are useful for characterizing the existence of a certain type of structure in a graph called "corner", and a certain type of graph called "dismantlable". Furthermore, Meet/Avoid games have real life applications that range from the minimization of the required staff members to cover a certain geographical area to the creation of popular arcade games such as Pac-Man[4]. In a similar fashion, "Sabotage" games are used to characterize the validity of some first order logic statements[2], and have applications such as modeling the learning process between a student and a teacher[10][1]. This paper, however, will focus on games that fall in the category of Travel games. In particular, a special emphasis on a game known as the Poison Game, and a new variation of this game which will be referred to as the Local Poison Game, will be made. As will be discussed further on, such games have many different applications both within academia (in other related branches of science such as, for example, computer science) and outside academia (modelling real world scenarios[2]).

It may be important to note that Graph Theory is not the only field of study concerned with the Poison Game (and its variations). In fact, whilst "Poison Game" is the name used by authors who study the characterizations of this game from a graph-theoretical point of view (such as [6] and [8]), there are other authors (such as [11] and [13]) who prefer to use an abstract-argumentation-theoretical point of view, and thus, refer to the "Poison Game" as the "Game for Credulous Acceptance". In Abstract Argumentation Theory[11], a graph is known as an argumentation framework, and the nodes in it represent represent bits of information which are used as arguments. The directed arcs that go from one node to

[1]For more information on Meet/Avoid games, and Sabotage games, as well as their respective characterizations, the reader may refer to Appendix D.

[2]For more information on the applications of the characterizations of Travel games, the reader may refer to Appendix C.

another, represent an attack relation that indicates which arguments may be used as a counter-argument to another argument (i.e: Which arguments attack other arguments.). Under this theory, the Game for Credulous Acceptance characterizes the existence of a structure called "Credulously Admissible Argument". One of the most important properties of such a structure is that (as a consequence of the way it is constructed) it represents a set of arguments which, together, form a coherent and strong idea[13].

Under a graph-theoretical approach, credulously admissible arguments are known as semi-kernels, and it will be the main concern of this paper to show that while the Poison Game characterizes existence, the Local Poison Game characterizes membership to such structures. For convenience purposes, the graph-theoretical approach will be used throughout; as well as some fundamental aspects of notation as seen in [6] and [8].

Finally, most of the relevant concepts will be explained in the first sections of this paper; however for more information on elementary graph theory the reader may find it helpful to consult the first chapter of [3].

2 Outline

This paper will begin by providing some elementary definitions which will allow the reader to have a basic understanding of key concepts which will be frequently referred to throughout the paper. After each definition, a graphical example and a small description of it will be given.

Once the key concepts have been covered, previous results from [6] concerning the Poison game will be explained. Once again, graphical examples will be given in order to make the paper easier to follow.

Finally, original content will be presented. This will be done, first, by providing the reader with a few more definitions (and their corresponding examples) and then by introducing the new game (the Local Poison Game). Lastly, results related to this new game will be proved using the definitions that were previously introduced and following similar proof structures as those seen in [6] and [13].

For a clearer, deeper, understanding of the Local Poison Game, and the graph-theoretical concepts orbiting around it the reader may consult the Appendices.

- Appendix A contains an original result (Lemma A.1) regarding the relation between strong node congruence and player A's winning strategy.
- Appendix B Contains an important observation regarding chordless even cycles and player A's winning strategy.
- Appendix C provides the reader with an example of one of the possible applications semi-kernels have when modeling real life scenarios.
- Appendix D gives a more detailed account (compared to that of the introduction of this paper) on how other games characterize certain properties of the graphs on which they are played.

3 Elementary Definitions

The following section contains key concepts (some of which appear in [6] [8]) that will be needed to understand the role that a certain type of structure (called "semi-kernel") plays in determining the outcome of the Poison game.

The set of nodes in graph D will be referred to as $V(D)$[3], and the set of arcs as $F(D)$. Since the direction of the arcs is relevant in this paper, $e_k = (x_i, x_j)$ with $i, j, k \in \mathbb{N}$ will represent an arc that goes from node x_i to node x_j.

Definition 1 (Loop)**.** A loop is an arc (e_k) such that $e_k \in F(D) : e_k = (x_i, x_i)$ with $x_i \in V(D)$ and $i, j, k \in \mathbb{N}$. In other words, it is an arc such that the node it departs from is the same node it arrives to.

Example 1.1. Graphically, a loop looks like this:

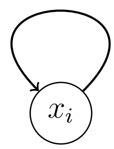

Figure 1

Definition 2 (Multiple Arcs)**.** A graph D is said to have multiple arcs if: $\exists e_k, e_l, e_m, \cdots \in F(D) : e_k = e_l = e_m = \cdots = (x_i, x_j)$ with $x_i, x_j \in V(D)$ and $i, j, k, l, m, \cdots \in \mathbb{N}$. In other words, if there exist several arcs that depart from the same node (x_i) and arrive to the same node (x_j)[4], these are called multiple arcs.

Example 2.1. Figure 2 shows how multiple arcs look like:

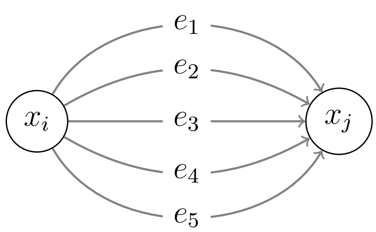

Figure 2

[3]Empty graphs will not be discussed here. Thus, it shall be assumed that $V(D) \neq \emptyset$ throughout.

[4]Assuming that the starting node and the ending one are different (i.e: $x_i \neq x_j$)

Remark. Note that although multiple arcs are not allowed in directed graphs, the existence of an arc from node x_i to node x_j and an arc from node x_j to node x_i is because the starting point and the ending one are different.

Definition 3 (Directed Graph)**.** A directed graph is a graph with no loops and no multiple arcs.

Example 3.1. The following is an example of a simple (finite) directed graph:

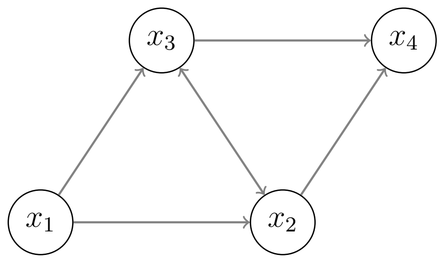

Figure 3

Definition 4 (Successor)**.** A node x_j is said to be the successor of a node x_i if $\exists e_k \in F(D) : e_k = (x_i, x_j)$ with $x_i, x_j \in V(D)$ and $i, j, k \in \mathbb{N}$. In other words, node x_j is said to be the successor of node x_i if there exists an arc going from x_i to x_j.

Example 4.1. For the graph in Figure 4, x_2 is said to be the successor of x_1

Figure 4

Definition 5 (Successors Set)**.** The successors set Γ_D^+ of a subset S_i of nodes in graph D will be denoted by $\Gamma_D^+(S_i)$, and will be defined by: $\Gamma_D^+(S_i) = \{x_m \in V(D) | \exists e_k \in F(D) : e_k = (x_j, x_m), x_j \in S_i\}$. In other words, set $\Gamma_D^+(S_i)$ is defined as the set of vertices in graph D for which there exists an arc going from any of the nodes in S_i to any other node in D.[5]

Example 5.1. Note that different subsets of graph D have different successors sets. For graph D in Figure 5 it can be said that: $S_1 = \{x_4\} \implies \Gamma_D^+(S_1) = \{x_2, x_3, x_5, x_6\}$ while $S_2 = \{x_2, x_3\} \implies \Gamma_D^+(S_2) = \{x_1, x_5, x_6\}$.

[5]Note that although Γ_D^+ is defined on subsets S_i, throughout this paper, if the subset (S_i) contains only one node (x_n), then the following will be equivalent: $\Gamma_D^+(x_n) = \Gamma_D^+(\{x_n\}) = \Gamma_D^+(S_i)$ for $S_i = \{x_n\}$.

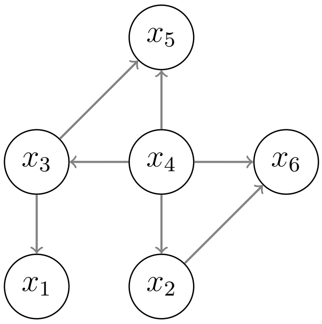

Figure 5

Definition 6 (Successors Sequence)**.** A successors sequence (denoted by $\sigma_i = x_j, x_k, x_l \dots$), is an ordered list of nodes in graph D such that any element (e.g: x_k) listed immediately to the right of any other element (e.g: x_j) is a successor of the latter in graph D (i.e: node x_k should be a successor of node x_j in graph D).

Remark. Note that successors sequences may be numbered by a sub-index; this is because a graph D may have more than one successors sequence[6]. Also note that a node may appear several times in a successors sequence.

Example 6.1. The following are successors sequences for the graph in Figure 6:

$$\sigma_1 = x_1, x_2$$

$$\sigma_2 = x_1, x_4, x_6$$

$$\sigma_3 = x_2, x_3, x_4, x_5, x_3, x_4, x_6$$

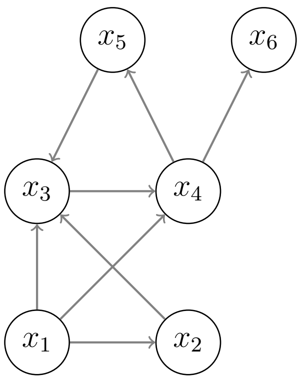

Figure 6

[6] In fact, if graph D is finite there will be a finite number of successors sequences, but if graph D is infinite it is possible for there to be uncountably many successors sequences.

Definition 7 (Position)**.** In the context of games played on graphs[7], a successors sequence may also be referred to as a position and will represent a (possibly partial) run of a game on a given graph. Since a position is, essentially, a successors sequence, it will be denoted by σ_k^j. Thus, saying that it is player i's [8] turn on position σ_k^j will mean that player i must select a successor of the last node in σ_k^j. By doing so, the state of the successors sequence σ_k^j will change. As a consequence, its upper index will shift from j to $j+1$, and the node player i selected will become the last node of position σ_k^{j+1}.

Remark. Observe that this notation emphasizes the fact that, in the context of games played on graphs, a successors sequence σ_k is not something predetermined, but rather something that is constructed as players take turns selecting nodes in the graph. It is, therefore, quite convenient to have an upper index that allows for distinctions between different states of the same successors sequence.

Example 7.1. Assume that a very simple game named G' is played on the graph D in Figure 7. This game (G') will be defined on any finite non-cyclic graph D and will consist of two players (named A and B) that take turns selecting nodes in D. Player B will begin at whichever node happens to be labeled x_1, and whoever selects a node for which there is no successor wins the game.

In this case, $\sigma_1^1 = x_1$ represents the first position for which it is player A's turn. Since there is only one option for player A to chose as a successor (namely, node x_2), by selecting it the state of the successors sequence $\sigma_1^1 = x_1$ will change to $\sigma_1^2 = x_1, x_2$. This new state represents a position in which it is player B's turn. In a similar fashion, since the only available option is node x_3 by selecting it, player B will update the successors sequence σ_1 from $\sigma_1^2 = x_1, x_2$ to $\sigma_1^3 = x_1, x_2, x_3$. So, once again, $\sigma_1^3 = x_1, x_2, x_3$ will then be a position in which it is player A's turn. The process will continue until player B wins the game by selecting node x_5. At this point, σ_1 will cease to be a partial run of the game and will become a complete one.

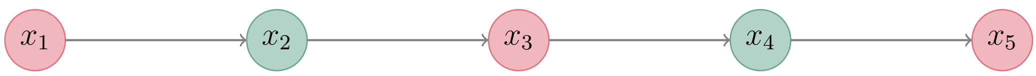

Figure 7

Definition 8 (Position Set)**.** The position set for player i will be denoted by Σ_i, and defined as: $\Sigma_i = \{\sigma_k^j |$ it is player i's turn on position $\sigma_k^j\}$. In other words, Σ_i will consist of all the positions σ_k^j for which it is player i's turn.

[7]In particular, games that involve selecting successive nodes.

[8]Note that i represents any of the players in the game. For example, in a game with two players (players A and B), it can be said that: $i \in \{A, B\}$.

Example 8.1. Recalling positions σ_1^j from Example and Figure 7, it may be said that:

$$\Sigma_A = \{\sigma_1^1, \sigma_3^1\}$$

$$\Sigma_B = \{\sigma_1^2, \sigma_3^4\}$$

Definition 9 (Independent Set)**.** A subset S_i of the nodes in D is an independent set if: $\forall x_i \in S_i$ then $x_i \notin \Gamma_D^+(S_i)$ with $S_i \subset V(D)$. In other words, S_i is an independent set if no element of S_i is contained in the successors set of S_i.

Example 9.1. Sets $S_1 = \{x_1, x_3, x_5\}$ and $S_2 = \{x_2, x_4\}$ form independent sets for the graph in Figure 8.

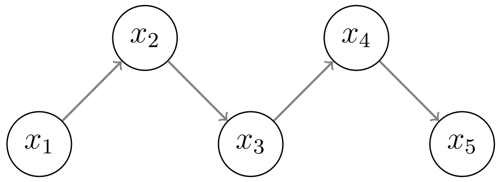

Figure 8

Definition 10 (Semi-kernel)**.** A subset S_i of the nodes in D is a semi-kernel if S_i is independent and $\forall x_m \in \Gamma_D^+(S_i) \exists e_k \in F(D) : e_k = (x_m, x_j)$ with $x_j \in S_i$. In other words, a semi-kernel is a subset S_i of the nodes in D such that S_i is an independent set and for every node in the successors set of S_i there is an arc going from this node to any node in S_i.

Example 10.1. Sets $S_1 = \{x_1, x_3, x_5\}$ and $S_2 = \{x_2, x_4, x_6\}$ are both semi-kernels in the graph shown below.

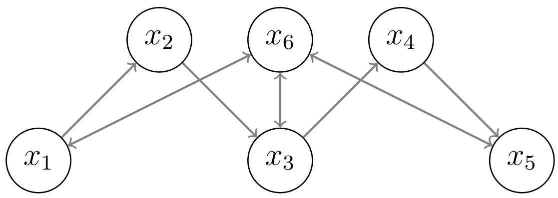

Figure 9

Definition 11 (Strategy)**.** A strategy for player i is a function such that:
$f_i : \Sigma_i \longrightarrow V(D)$. In other words, a strategy for player i is a function (f_i) that tells player i which node to select when it is his or her turn. The domain of f_i is defined on the set of all positions for which it is player i's turn, and the co-domain is defined on the set of nodes belonging to the graph on which the game is being played.

Example 11.1. Recalling game G' presented in Example 7, and playing it on graph D in Figure 10, it may be said that for position $\sigma_1^1 = x_1$ it is player$A$'s turn. Assume her strategy tells her to select node x_3 (or, in mathematical terms: $f_A(\sigma_1^1) = x_3$). Thus, by doing so, σ_1 is updated from $\sigma_1^1 = x_1$ to $\sigma_1^2 = x_1, x_3$. It will then be player B's turn again, and, assuming that his strategy tells him to select node x_4 (or, in mathematical terms: $f_B(\sigma_1^2) = x_4$), this node will then be part of σ_1. Finally, it will be player A's turn on position $\sigma_1^3 = x_1, x_3, x_4$, and assuming her strategy tells her to select node x_5 (in mathematical terms: $f_A(\sigma_1^3) = x_5$), the game will come to and end and player A will be the winner.

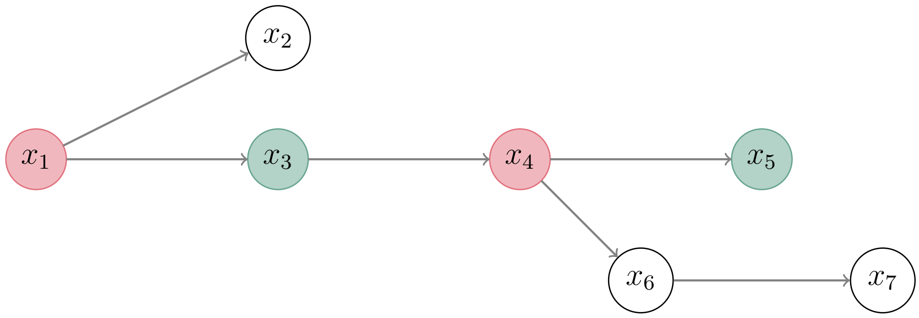

Figure 10

Definition 12 (Winning Strategy)**.** A winning strategy for player i is a strategy $f_i^W : \Sigma_i \longrightarrow V(D)$ such that it determines a successors sequence ending with a win for player i regardless of the strategy of the other player. A node x_m is said to belong to player i's winning strategy if: $f_i^W(\sigma_k^j) = x_m$ for some position σ_k^j. In other words, x_m belongs to player i's winning strategy f_i^W if it is the result of applying the winning strategy to a position σ_k^j for which it is player i's turn.

Remark. Note that the existence of a winning strategy for player i depends on the graph and the game that will be played on it. A player may have more than one winning strategy on a given graph, or may not have one at all.

Example 12.1. Assume that the simple game G' is played on graph D in Figure 11. It is clear that node x_3 doesn't belong to player A's winning strategy because if she selects it, then that would allow player B to select node x_5 and become the winner (in mathematical terms: $f_A^W(\sigma_1^1) \neq x_3$ for $\sigma_1^1 = x_1$). On the other other hand, it is clear that node x_2 must belong to player A's winning strategy, since by selecting it she will force the successors sequence σ_1 to end in a win for her (in mathematical terms: $f_A^W(\sigma_1^1) = x_2$ for $\sigma_1^1 = x_1$). By selecting node x_2, σ_1 is updated from $\sigma_1^1 = x_1$ to $\sigma_1^2 = x_1, x_2$. Then, since node x_4 is the only possible successor to node x_2 player B's strategy will consist of selecting this node (in mathematical terms: $f_B(\sigma_1^2) = x_4$ for $\sigma_1^2 = x_1, x_2$). As always, by selecting node

x_4, successors sequence σ_1 will be updated from $\sigma_1^2 = x_1, x_2$ to $\sigma_1^3 = x_1, x_2, x_4$. Finally, player A's winning strategy will tell her to select node x_5 and end the game. Thus, σ_1 becomes a complete run of the game by being updated from $\sigma_1^3 = x_1, x_2, x_4$ to $\sigma_1^4 = x_1, x_2, x_4, x_5$. Player A is the winner, and the nodes that belong to her winning strategy are x_2 and x_5.

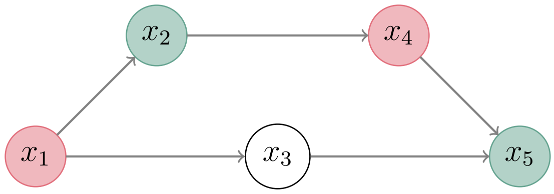

Figure 11

Example 12.2. Note, however, that if the initial setting changes, then player A may no longer have a winning strategy. Such is the case of the graph D in Figure 12. Under this new configuration of the graph previously seen in Figure 11 the only possible outcome results in player B winning the game. Since node x_1 is given as the initial setting and is not actually chosen by player B, it can only be said that node x_3 belongs to player B's winning strategy.

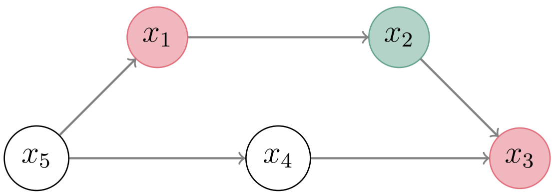

Figure 12

4 The Poison Game

Now that the ground has been set, the rules of the Poison Game will be introduced. Although the rules as seen in [6] are the same, the presentation will be slightly modified[9], and the win/lose conditions from [11] will be used with the intention of making the game easier to understand. As always, some graphical examples will be provided afterwards.

Definition 13 (Poison Game)**.** Given a directed graph D, the Poison Game on graph D is defined by the following set of rules:

I. Players A and B take turns selecting consecutive nodes in D.

II. Player A makes the first move by selecting a node of her choice.

III. By selecting a node, player B poisons it. This means that player A cannot select a node that has been selected by B in any previous turn.

IV. Player B wins if player A cannot select a node that succeeds the last node chosen by B.

V. Player A wins in any other scenario.

Example 13.1. Assume the Poison Game is played on graph D in Figure 13, and that player A begins by selecting node x_1 (in mathematical terms: $f_A^W(\sigma_1^0) = x_1$ for $\sigma_1^0 = \emptyset$). Player B's strategy will then consist of selecting node x_2 (in mathematical terms: $f_B(\sigma_1^1) = x_2$ for $\sigma_1^1 = x_1$). Finally, player A's (winning) strategy will tell her to select node x_3 and end the game (in mathematical terms: $f_A^W(\sigma_1^2) = x_2$ for $\sigma_1^2 = x_1, x_2$). So, in this run of the game, player A wins, and nodes x_1 and x_3 belong to her winning strategy.

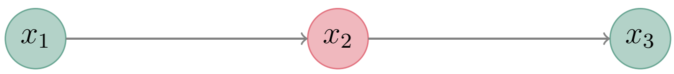

Figure 13

[9] Compared to that in [6].

Example 13.2. However, if player A selects a different node for her first move, it may be case that player B now has the winning strategy. Using the same graph D from Figure 13, if player A selects node x_2 on her first move (in mathematical terms: $f_A(\sigma_1^0) = x_2$ for $\sigma_1^0 = \emptyset$), then player B's winning strategy will tell him to select node x_3 and end the game (in mathematical terms: $f_B^W(\sigma_1^1) = x_3$ for $\sigma_1^1 = x_2$). So, in this second run of the game, player B wins and only node x_3 belongs to his winning strategy.

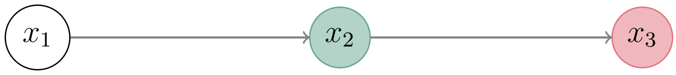

Figure 14

Example 13.3. Now consider graph D in Figure 15. Note that there are several different successors sequences σ_i that could lead to the coloring seen on graph D. However, assume that player A's winning strategy tells her to select node x_1 for her first move. Player B's strategy will then consist in selecting node x_2. In turn, player A's winning strategy will drive her to select node x_3; and, as a reply to this, player B's strategy will tell him to select node x_6. Player A can select node x_3 again (since it belongs to her winning strategy), and player B can select node x_4 this time. Then, following her winning strategy, player A will select node x_5 while player B's strategy will tell him to select node x_6 once more. Finally, player A will select node x_1. At this point it is safe to call the game to an end, since all of the nodes have been selected and it is clear that no matter which node player B selects, player A will always be able to select a successive node that belongs to her winning strategy (i.e: the run for this game σ_1 can be continued indefinitely). So, just to recap, this particular run of the game should look like: $\sigma_1 = x_1, x_2, x_3, x_6, x_3, x_4, x_5, x_6, x_1, \ldots$. Player A has a winning strategy on this graph, and the nodes belonging to it are x_1, x_3 and x_5.

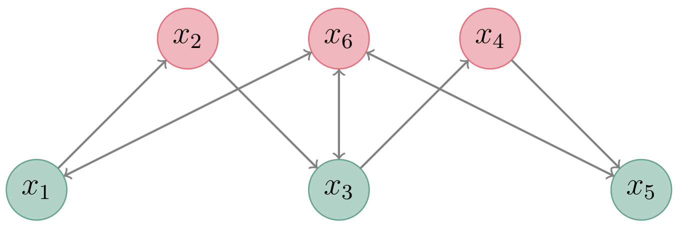

Figure 15

Remark. Note that player A has a winning strategy in this graph no matter which node she selects for her initial move. Under another run of the game, nodes x_2, x_4 and x_6 could have belonged to her winning strategy instead of nodes x_1, x_3 and x_5.

Example 13.4. Considering the graph in Figure 16, player A decides not to choose neither of the nodes x_2 (because doing so would allow player B to select node x_3 and win the game) nor x_4 (because this would invariably end with player B selecting node x_3 and thereby winning the game). So assume that player A's winning strategy tells her to select node x_1 for her first move (i.e: $f_A^W(\sigma_1^0) = x_1$ for $\sigma_1^0 = \emptyset$). Player B's strategy will then tell him to select node x_2 (i.e: $f_B(\sigma_1^1) = x_2$ for $\sigma_1^1 = x_1$). Observe that now, at position $\sigma_1^2 = x_1, x_2$ player A can either select node x_4 or x_3. Clearly, as explained earlier, node x_4 does not belong to her winning strategy; so player A decides to select node x_3 and win the game (i.e: $f_A^W(\sigma_1^2) = x_3$ for $\sigma_1^2 = x_1, x_2$).

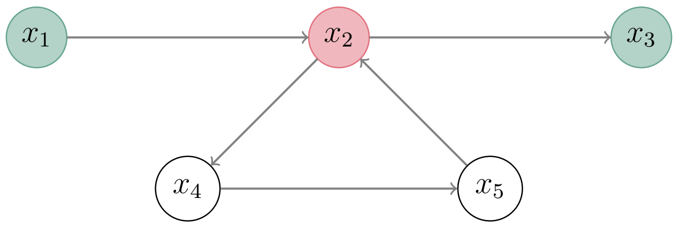

Figure 16

Example 13.5. For graph D in Figure 17 It doesn't matter which node player A selects for her first move, in the end two nodes will belong to player A's (winning) strategy, and two nodes will belong to player B's strategy. The Poison Game can have an infinite run in a graph D, thus it is considered a win for A.

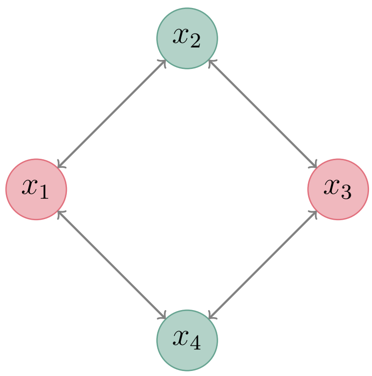

Figure 17

Example 13.6. Unlike the situation in Figure 17, player A no longer has a winning strategy for the graph in Figure 18 regardless of the node she selects for her first move. In particular, assume player A's strategy tells her to select node x_2 for her first move (i.e: $f_A(\sigma_1^0) = x_2$ for $\sigma_1^0 = \emptyset$). Then, due to a lack of other options, player B's winning strategy will tell him to poison node x_3 (i.e: $f_B^W(\sigma_1^1) = x_3$ for $\sigma_1^1 = x_2$). Both players will then continue to take turns selecting nodes until player A's strategy tells her to select node x_1 (i.e: $f_A(\sigma_1^4) = x_1$ for $\sigma_1^4 = x_2, x_3, x_4, x_5$). At this point, player B's winning strategy will tell him to poison node x_2. Note that this node had previously been selected by player A on her first turn[10] (i.e: $f_B^W(\sigma_1^5) = x_2$ for $\sigma_1^5 = x_2, x_3, x_4, x_5, x_1$). By poisoning node x_2, player A is rendered unable to select a successor, since the only successor for node x_2 is node x_3 but the latter has already been poisoned by player B. Thus, player B wins the game.

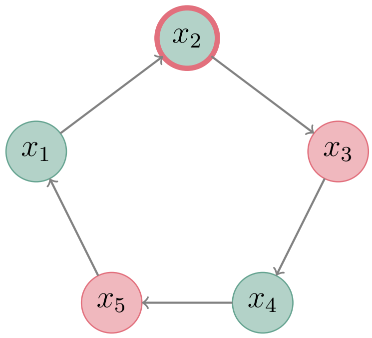

Figure 18

Example 13.7. As a final example, imagine that graph D is as seen in Figure 19. Notice that this is an infinite graph for which each node x_i has x_{i+1} as its successor. In this graph, player A may select any node for her first move and still have a winning strategy since she will always be able to select an unpoisoned successor for any node poisoned by player B. In other words, player A has a winning strategy because for any node x_i that player B poisons in his turn, player A will always be able to select node x_{i+1} due to the fact that the graph is infinite.

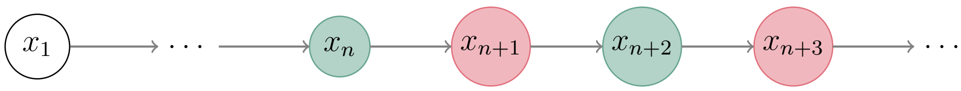

Figure 19

[10] To illustrate this, node x_2 is depicted with a green filling and a red border.

5 Results on the Poison Game

As seen in [6], this section will present results that explain the conditions under which player A has a winning strategy when the Poison Game is played on a graph (D). Before these results are presented, some more definitions shall be provided:

Definition 14 (Path)**.** If, given a successors sequence σ_i, it is the case that $\forall x_j, x_k \in V(D) : \sigma_i = \ldots, x_j, \ldots, x_k, \ldots$ and $x_j \neq x_k$, then successors sequence σ_i is referred to as a path. In other words, if all the nodes that are part of a successors sequence σ_i are different from one another, then σ_i may be referred to as a path.

Example 14.1. Considering graph D in Figure 20, it may be said that $\sigma_1 = x_1, x_2, x_4, x_5$ and $\sigma_2 = x_1, x_3, x_4$ are paths because none of the nodes in them appear more than once. However, successors sequence $\sigma_3 = x_1, x_2, x_4, x_5, x_4, x_5$ is not a path since nodes x_4 and x_5 appear more than once.

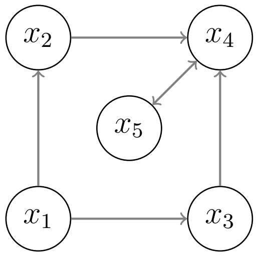

Figure 20

Definition 15 (Outwardly Finite Graph)**.** A graph is outwardly finite if there is a finite number of successors for each of the nodes of the graph.

Example 15.1. It can be seen from graph D in Figure 21 that each of the nodes has a finite number of successors. Namely: $\Gamma_D^+(x_1) = \{x_2, x_3\}$, $\Gamma_D^+(x_2) = \{x_4\}$, $\Gamma_D^+(x_3) = \{x_2\}$,and $\Gamma_D^+(x_4) = \{x_2\}$. Thus, it can be said that graph D is an outwardly finite graph.

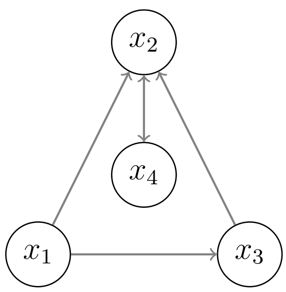

Figure 21

Definition 16 (Progressively Finite Graph)**.** In a similar fashion, a graph is called progressively finite if none of the nodes are the origin of an infinite path.

Example 16.1. Recall that a path is a successors sequence in which no node appears more than once. Then, focusing on graph D in Figure 22, it becomes evident that the longest paths that stem from node x_1 are either $\sigma_1 = x_1, x_2, x_3$ or $\sigma_2 = x_1, x_3, x_2$[11]. Similarly, the longest paths that stem from nodes x_2 and x_3 respectively are: $\sigma_3 = x_2, x_3$ and $\sigma_4 = x_3, x_2$. All of these paths are finite, hence the graph in Figure 22 is progressively finite.

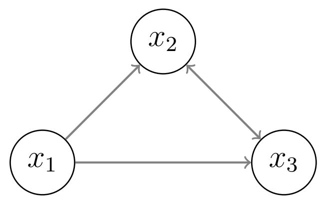

Figure 22

Now that these additional definitions have been explained, the results obtained by P.Duchet and H.Meyniel in [6] will be stated and a proof[12] will be provided.

Theorem 1 (Characterization of the Existence of Semi-kernels)**.** *Let D be a progressively and outwardly finite graph. Player A has a winning strategy if and only if D has a semi-kernel.*

Proof. Since Theorem 1 involves an if and only if statement, both directions need to be proved. The first half of this proof will show that the statement is true when going from left to right. Naturally, the second half will show that the statement is also true when going from right to left.

Left-to-Right: Assume that player A has a winning strategy. Assume also, as expected, that she makes her first move on one of the nodes that belong to her winning strategy. Now, notice that each time player A selects a node that she had previously selected, player B has the opportunity to poison a different successor. This, nonetheless does not represent an issue since it has been assumed that player A has a winning strategy, meaning that she will always be able to

[11] Note that although there could be an infinite successors sequence which repeats nodes x_2 and x_3 indefinitely, this successors sequence cannot be considered a path since it constantly repeats nodes.

[12] Following the proof in [6]

select an unpoisoned[13] successor of any node poisoned by player B during her turn. These unpoisoned successors of the nodes selected by player B must be independent because otherwise they could be poisoned by player B at some point during the run of the game and this would contradict the assumption that player A has a winning strategy. The set of independent nodes in D that are unpoisoned successors of the nodes selected by player B (in addition to the initial node selected by player A) form a set which will be denoted by S (in mathematical terms: $S = \{x_n | x_n$ is unpoisoned and $\forall x_n, x_m \in S \nexists e_l \in F(D) : e_l = (x_n, x_m)\})$. Finally, notice that another consequence of player A having a winning strategy is that all the successors of the nodes in the successors set of S must belong in S since they must be independent and unpoisoned; otherwise A wouldn't have a winning strategy[14].Therefore, the fact that A has a winning strategy implies the existence of a semi-kernel in graph D.

Right-to-Left: Now assume that graph D has a semi-kernel, and player A makes her first move on any of the nodes in this semi-kernel (which will be denoted by S). Since player A selected a node in the semi-kernel S, then by definition, any successor poisoned by B will not be in S since all of the nodes in the successors set of S do not belong in S (in mathematical terms: $\forall x_j \in \Gamma^+_D(S) \implies x_j \notin S$). Likewise, by definition, all the successors of the nodes in the successors set of S will belong in S (in mathematical terms: $\forall x_i \in \Gamma^+_D(x_j)$ with $x_j \in \Gamma^+_D(S)$ then $x_i \in S$). So no node in S can ever be poisoned by player B and A will have a way of selecting a node in S each time B makes a move. Therefore, the existence of a semi-kernel in graph D implies that player A has a winning strategy.

□

The Poison Game shows that, since player A is allowed to mark the beginning of the game by selecting any node of her preference, the existence of a semi-kernel in the directed graph on which the game is being played is enough to guarantee that player A has a winning strategy. The game, thus, characterizes the existence of semi-kernels in directed graphs. No attention is payed to the characterization of the membership to semi-kernels since player A's freedom of choice for her first node determines whether or not the nodes she selects in future turns will belong to a semi-kernel or not. In the following section, this possibility (namely, the possibility of characterizing membership to semi-kernels) will be explored.

[13]If a node falls in the category of "unpoisoned" during a run of the Poison Game, then this is because it was never selected by player B. In other words, such a node does not belong to player B's strategy. Thus it can be said that if node x_m is unpoisoned this is the same as saying that $f_B(\sigma^j_k) \neq x_m$ for all positions σ^j_k.

[14]In mathematical terms: A has a winning strategy $\implies \forall x_n \in \Gamma^+_D(S)$ then $\forall x_m \in \Gamma^+_D(x_n)$ it will be the case that $x_m \in S$

6 The Local Poison Game

In this section, a variation of the Poison Game will be proposed. This variation will have different rules to those of the Poison Game as seen in Section 4, but it will attempt to maintain some essential aspects of the player's allowed behavior. After the rules have been provided, graphical examples will follow in order to give the reader a better notion of how the new game works.

Definition 17 (Local Poison Game)**.** Given a directed graph D, the Local Poison Game on graph D is defined by the following set of rules:

I. A node, denoted by X, is given as the initial setting[15].

II. Players A and B take turns selecting consecutive nodes on D.

III. Player B makes the first move by selecting a successor of node X.

IV. By selecting a node, player B poisons it. This means that player A cannot select a node that has been selected by B in any previous turn.

V. Player B is allowed to backtrack[16]. By backtracking, player B ends a successors sequence and begins a new one. The first node of the new successors sequence is the node whose alternative successor is selected by B as a result of backtracking.

VI. Player B wins if player A cannot select a node that succeeds the last node chosen by B.

VII. Player A wins in any other scenario.

[15]Equivalently, it can be said that: $\sigma_1^1 = X$ is given as the initial position.

[16]Note that backtracking here means that player B is allowed to select a (possibly alternative) successor of any of the nodes previously selected by A or a (possibly alternative) successor of the initial node X.

Example 17.1. The following is an example of the outcome of a run of the Local Poison Game on graph D in Figure 23. Maintaining the color scheme used up until now, nodes shaded in red were selected (poisoned) by player B, while nodes shaded in green were selected by player A. The blank node (node X) is the one given as the initial setting. There are several ways in which this outcome can be reached; however, in order to illustrate the way backtracking works, this outcome will be reached by using two successors sequences.

Since the initial node X is given, and (according to the game rules) player B marks the beginning of the game by poisoning a successor of node X, it can then be said that: $f_B(\sigma_1^1) = x_1$ for $\sigma_1^1 = X$. As with previous games, by poisoning node x_1 the state of σ_1 will be updated from $\sigma_1^1 = X$ to $\sigma_1^2 = X, x_1$. Then, player A's winning strategy will tell her to select node x_2 (i.e: $f_A^W(\sigma_1^2) = x_2$ for $\sigma_1^2 = X, x_1$). Now, assume player B's strategy tells him to poison node x_7 (i.e: $f_B(\sigma_1^3) = x_7$ for $\sigma_1^3 = X, x_1, x_2$), to which player A will respond by selecting node x_4 (i.e: $f_A^W(\sigma_1^4) = x_4$ for $\sigma_1^4 = X, x_1, x_2, x_7$). Naturally players B and A will then select nodes x_5 and x_6 respectively. Thus, successors sequence σ_1 should have the following shape at this point of the (partial) run of the game: $\sigma_1 = X, x_1, x_2, x_7, x_4, x_5, x_6$. However, since node x_3 was never selected during this partial run of the game, player B might backtrack in order to attempt to change the outcome of the game. So assume, player B does backtrack, and by doing so, he creates a new successors sequence σ_2 and poisons an alternative successor of node x_2 (namely, node x_3). This could be written as: $f_B(\sigma_2^1) = x_3$ for $\sigma_2^1 = x_2$. Player A's winning strategy will then tell her to select node x_4 once again (i.e: $f_A^W(\sigma_2^2) = x_4$ for $\sigma_2^2 = x_2, x_3$), and, naturally, the game will end with players B and A selecting nodes x_5 and x_6 respectively. Thus, it can be said that this particular outcome results from successors sequences $\sigma_1 = X, x_1, x_2, x_7, x_4, x_5, x_6$ and $\sigma_2 = x_2, x_3, x_4, x_5, x_6$. This run of the game ends with player A being the winner.

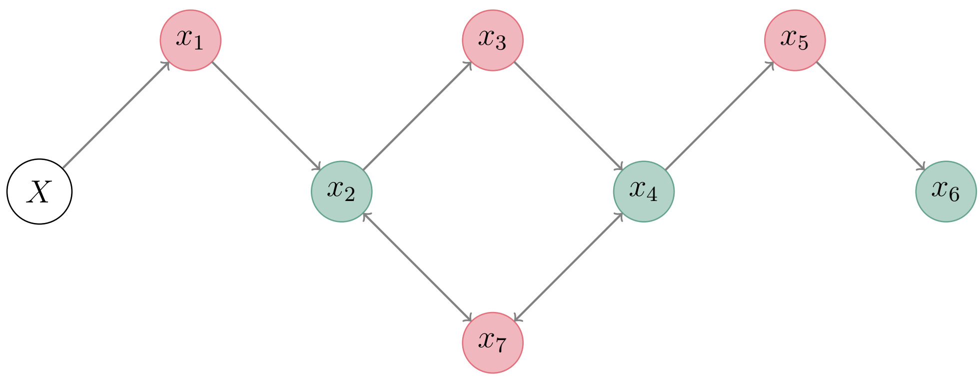

Figure 23

As was done when presenting the Poison Game, a few more examples will be discussed.

Example 17.2. Consider graph D in Figure 24 [17]. Notice that regardless, of the node that is given as the initial setting, payer A will always have a winning strategy because for each node that player B poisons, player A will always be able to select an unpoisoned successor. As is the case with many of the examples provided in this paper, there are several ways that the coloring in graph D can be attained. Assume, however that it is the result of player B's strategy telling him to select node x_3 as the successor of X (i.e: $f_B(\sigma_1^1) = x_3$ for $\sigma_1^1 = X$). Then, player A's winning strategy might tell her to select the initial node X (i.e: $f_A^W(\sigma_1^2) = X$ for $\sigma_1^2 = X, x_3$)[18]. To avoid the pointlessness of selecting node x_3 once again, assume player B's strategy tells him to poison node x_1 this time (i.e: $f_B(\sigma_1^3) = x_1$ for $\sigma_1^3 = X, x_3, X$). Finally, player A's winning strategy will tell her to select node x_2 (i.e: $f_A^W(\sigma_1^4) = x_2$ for $\sigma_1^4 = X, x_3, X, x_1$). At this point it is safe to say the game has ended with player A being the winner because there are no unselected nodes which player B might try to poison by backtracking in an attempt to change the outcome of the game, and extending successors sequence σ_1 indefinitely will not provide any new information since player A will always select either node X or x_2 and player B will always select either node x_1 or x_3.

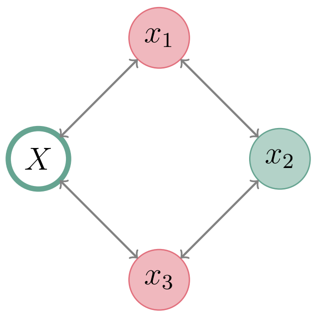

Figure 24

Example 17.3. Now consider graph D in Figure 25, and notice that regardless of the node that is given as the initial setting, it will not be player A who has a winning strategy, but rather player B. So, assume that the top node of graph D is given to be node X. Then, player B's winning strategy will tell him to poison node x_1 (i.e: $f_B^W(\sigma_1^1) = x_1$ for $\sigma_1^1 = X$), to which player A will respond by selecting node x_2 (i.e: $f_A(\sigma_1^2) = x_2$ for $\sigma_1^2 = X, x_1$). Finally, player B's winning

[17] Notice that it is, in fact, the same graph as the one in Figure 17 except that this time the Local Poison Game is being played on it.

[18] A green border will be used to represent that node X was given as the initial setting and, during the run of the game, was selected by player A.

strategy will tell him to select node X (i.e: $f_B^W(\sigma_1^3) = X$ for $\sigma_1^3 = X, x_1, x_2$)[19]. This will put an end to the game since player A will be unable to select an unpoisoned successor.

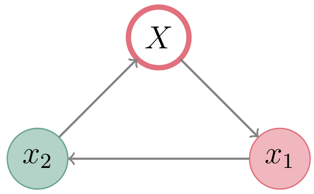

Figure 25

Example 17.4. Finally, consider graph D in Figure 26. Notice that it is an infinite graph, and, in a very similar fashion to what was seen in Figure 19, player A will always have a winning strategy no matter which node is given as the initial setting (node X). This is due to the fact that, since the graph is infinite, for any node x_i poisoned by player B in his turn, player A will always be able to reply by selecting node x_{i+1}.

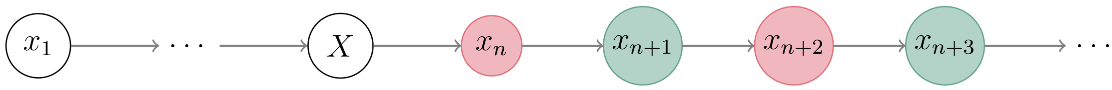

Figure 26

Now that the rules of the Local Poison Game (and an a couple examples) have been given, some related, useful definitions will be provided. These definitions will be required in order to state and prove results related to the Local Poison Game.

[19]Recalling Figure 24, a red border will be used to represent that node X was given as the initial setting and, during the run of the game, was selected by player B.

7 Further Definitions

Definition 18 (Cycle)**.** Given a graph D, a cycle (denoted by C_i) is a subset of the nodes in $V(D)$ for which $\exists \sigma_k$ that includes all the nodes in C_i, and has the following form: $\sigma_k = x_i, \ldots, x_j : x_i = x_j$ and $\forall x_m, x_n : x_m \neq x_i$ or $x_n \neq x_i$ then $x_m \neq x_n$. In other words, a cycle is a subset C_i of the nodes in the graph, that can be represented by a successors sequence (that includes all the nodes in C_i) in which the first node is equal to the last node and all other nodes are different from one another (i.e: if the last node of the successors sequence is omitted, a path is formed by all the other nodes in the successors sequence). In C_i, the i indicates the number of nodes that are part of the cycle. Observe that cycles can be even or odd depending on the number of nodes that constitute them.

Example 18.1. It may be said that the graph in Figure 17 and the one in Figure 27 are both even cycles. Successors sequence $\sigma_1 = x_1, x_2, x_3, x_4, x_1$ describes the even cycle of the form C_4 in Figure 17 [20]. Meanwhile, the graph in Figure 27 has two different even cycles. The first one, of the form C_4, consists of nodes x_1, x_2, x_3 and x_6 (This cycle can be described by the following successors sequence: $\sigma_1 = x_1, x_2, x_3, x_6, x_1$). The second cycle, of the form C_6, includes all the nodes in the graph; and can be described by the following successors sequence: $\sigma_2 = x_1, x_2, x_3, x_4, x_5, x_6, x_1$.

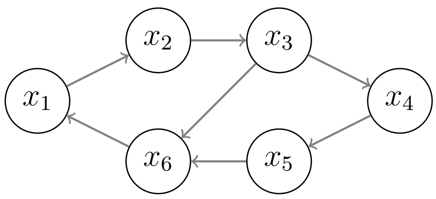

Figure 27

Example 18.2. Similarly, the graph in Figure 18 and the one in Figure 28 are both odd cycles. Successors sequence $\sigma_1 = x_1, x_2, x_3, x_4, x_5, x_1$ describes the odd cycle of the form C_5 in Figure 18. Meanwhile, the graph in Figure 28 has two different odd cycles. The first one, of the form C_5, consists of nodes x_1, x_2, x_3, x_6 and x_7 (This cycle can be described by the following successors sequence: $\sigma_1 = x_1, x_2, x_3, x_6, x_7, x_1$). The second cycle, of the form C_7, includes all the nodes in the graph and can be described by the following successors sequence: $\sigma_2 = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_1$.

[20] Note it is even because it is a cycle that consists of 4 nodes.

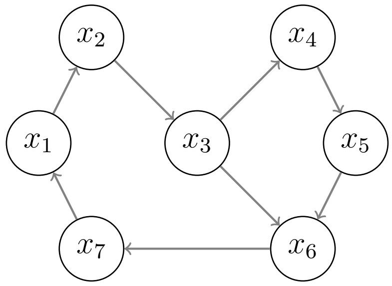

Figure 28

Definition 19 (Chordless Cycle)**.** A cycle C_i for which $\forall x_m \in C_i$ then $|\Gamma_D^+(x_m) \cap C_i| = 1$ is called a chordless cycle. In other words, if all of the nodes that form part of the cycle C_i have only one successor within the cycle then the cycle is said to be chordless. A chordless cycle may be denoted by: C_i^-.

Example 19.1. Graph D from Figure 17 is said to be a chordless even cycle because all of the nodes in it have only one successor within the cycle, and the cycle has an even number of nodes (namely 4). On the other hand, the cycle that includes all the nodes from the graph in Figure 27 cannot be considered a chordless cycle because node x_3 has more than one possible successor in the cycle.

Example 19.2. In a very similar fashion, graph D from Figure 18 is said to be a chordless odd cycle because all of the nodes in it have only one successor within the cycle, and the number of nodes is odd (namely 5). Likewise, the cycle that includes all the nodes from the graph in Figure 28 cannot be considered a chordless cycle because node x_3 has more than one possible successor in the cycle.

Definition 20 (Sequential Numbering)**.** It is possible to take node X as a reference, and number the rest of the nodes in graph D with respect to it by following a particular successors sequence σ_i. Such numbering will be referred to as sequential numbering, and will be done by assigning numbers of the form a^{σ_i} (with $a, i \in \mathbb{N}$) to each of the nodes in the graph. This assignment will be denoted by tuples of the form (x_j, a^{σ_i}) where a indicates the number that node x_i has with respect to node X according to the successors sequence σ_i. Thus, under successors sequence σ_i, number 1^{σ_i} will be assigned to the successor of node X, number 2^{σ_i} will be assigned to the successor of the successor of node X, and so on. Each time the same node (e.g: x_j) appears in a given successors sequence (σ_i) a new value of a will be associated to it, and with it a new tuple $(x_j, a_k^{\sigma_i})$ will be created.

Remark. Note that a node x_i may be assigned different values of a under different successors sequences (σ_i); and also note that it may take more than one successors sequence to number all of the nodes in graph D with respect to node X.

Example 20.1. Considering $\sigma_1 = X, x_1, x_2$ and $\sigma_2 = X, x_2$ for the graph in Figure 29 note that node x_2 is assigned different a values under σ_1 and σ_2. Namely, it can be said that: $(x_2, 2^{\sigma_1})$ and $(x_2, 1^{\sigma_2})$. Also note that for $\sigma_3 = X, x_1, x_2, x_1, x_2$ nodes x_1 and x_2 are assigned different a values under the same successors sequence σ_3. Namely it can be said that: $(x_1, 1^{\sigma_3})$ and $(x_1, 3^{\sigma_3})$ hold for node x_1 while $(x_2, 2^{\sigma_1})$ and $(x_2, 4^{\sigma_1})$ hold for node x_2.

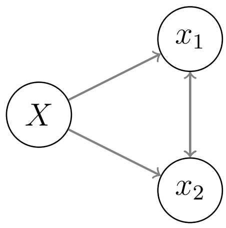

Figure 29

Definition 21 (Sequential Numbering Set)**.** The tuples generated when following the sequential numbering process for a specific successors sequence σ_i may be grouped together in a set which will be referred to as the sequential numbering set of σ_i. This set will be denoted by N_i (where the i in the sub-index of N must match that of σ_i). Note that if a node x_j appears several times in the same successors sequence σ_i, then all of the tuples generated by x_j will form part of the corresponding sequential numbering set of σ_i.

Example 21.1. Each successors sequence in graph D (pictured in Figure 30) generates a sequential numbering set.

$$\sigma_1 = X, x_3, x_4, x_5, x_6 \implies N_1 = \{(x_3, 1^{\sigma_1}), (x_4, 2^{\sigma_1}), (x_5, 3^{\sigma_1}), (x_6, 4^{\sigma_1})\}$$
$$\sigma_2 = X, x_3, x_2, x_5, x_6, x_5 \implies N_2 = \{(x_3, 1^{\sigma_2}), (x_2, 2^{\sigma_2}), (x_5, 3^{\sigma_2}), (x_6, 4^{\sigma_2}), (x_5, 5^{\sigma_2})\}$$
$$\sigma_3 = X, x_1, x_2, x_4, x_5, x_6 \implies N_3 = \{(x_1, 1^{\sigma_3}), (x_2, 2^{\sigma_3}), (x_4, 3^{\sigma_3}), (x_5, 4^{\sigma_3}), (x_6, 5^{\sigma_3})\}$$

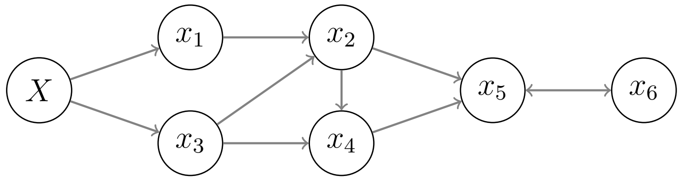

Figure 30

Definition 22 (Node Congruence)**.** If $a_m^{\sigma_i} \equiv a_n^{\sigma_i} \equiv 0 \pmod 2$ $\forall a_m^{\sigma_i}, a_n^{\sigma_i} : (x_j, a_m^{\sigma_i}),$ $(x_j, a_n^{\sigma_i}) \in N_i$ with x_j fixed $\implies$ x_j is congruent to 0 (mod 2) under successors sequence σ_i. In other words, a node x_j is said to be congruent to 0 (mod 2) under successors sequence σ_i if $a_k^{\sigma_i} \equiv 0 \pmod 2$ holds for all the values of $a_k^{\sigma_i}$ that form tuples $(x_j, a_k^{\sigma_i})$ in the sequential numbering set of σ_i with a particular (fixed) node x_j. Abusing notation, this will be denoted by $x_j^{\sigma_i} \equiv 0 \pmod 2$.

Analogously, If $a_m^{\sigma_i} \equiv a_n^{\sigma_i} \equiv 1 \pmod 2$ $\forall a_m^{\sigma_i}, a_n^{\sigma_i} : (x_j, a_m^{\sigma_i}), (x_j, a_n^{\sigma_i}) \in N_i$ with x_j fixed $\implies$ x_j is congruent to 1 (mod 2) under successors sequence σ_i. In other words, a node x_j is said to be congruent to 1 (mod 2) under successors sequence σ_i if $a_k^{\sigma_i} \equiv 1 \pmod 2$ holds for all the values of $a_k^{\sigma_i}$ that form tuples $(x_j, a_k^{\sigma_i})$ in the sequential numbering set of σ_i with a particular (fixed) node x_j. Similarly to the previous case, this will be denoted by $x_j^{\sigma_i} \equiv 1 \pmod 2$.

Finally, If $\exists a_n^{\sigma_i}, a_m^{\sigma_i}$ such that $(x_j, a_n^{\sigma_i}), (x_j, a_m^{\sigma_i}) \in N_i$ for a fixed x_j and $a_n^{\sigma_i} \not\equiv a_m^{\sigma_i} \equiv 1 \pmod 2$ or $a_n^{\sigma_i} \not\equiv a_m^{\sigma_i} \equiv 0 \pmod 2$ $\implies$ node x_j is incongruent under successors sequence σ_i. In other words, if a node x_j appears in several tuples $(x_j, a_k^{\sigma_i}) \in N_i$, and neither $a^{\sigma_i} \equiv 0 \pmod 2$ nor $a^{\sigma_i} \equiv 1 \pmod 2$ holds for all the a^{σ_i} values in the tuples, then x_j will be called an incongruent node under successors sequence σ_i. This will be denoted by: $x_j^{\sigma_i} \not\equiv 0 \pmod 2$ and $x_j^{\sigma_i} \not\equiv 1 \pmod 2$.

Example 22.1. According to the successors sequences σ_i provided below (which correspond to the graph in Figure 31), it can be said that node $x_1^{\sigma_1} \equiv 1 \pmod 2$ because the tuples in which x_1 appears are $(x_1, 1^{\sigma_1})$ and $(x_1, 3^{\sigma_1})$; and in both cases it can be seen that $1^{\sigma_1} \equiv 1 \pmod 2$ and $3^{\sigma_1} \equiv 1 \pmod 2$ respectively. Likewise it can be said that node $x_2^{\sigma_1} \equiv 0 \pmod 2$. However, it must be noted that node x_2 is an incongruent node under successors sequence σ_2 because it appears in two tuples (namely $(x_2, 2^{\sigma_1})$ and $(x_2, 5^{\sigma_1})$), and clearly $5^{\sigma_1} \not\equiv 2^{\sigma_1} \equiv 0 \pmod 2$.

$$\sigma_1 = X, x_1, x_2, x_1, x_2, x_3 \implies N_1 = \{(x_1, 1^{\sigma_1}), (x_2, 2^{\sigma_1}), (x_1, 3^{\sigma_1}), (x_2, 4^{\sigma_1}), (x_3, 5^{\sigma_1})\}$$
$$\sigma_2 = X, x_1, x_2, x_4, x_5, x_2 \implies N_2 = \{(x_1, 1^{\sigma_2}), (x_2, 2^{\sigma_2}), (x_4, 3^{\sigma_2}), (x_5, 4^{\sigma_2}), (x_2, 5^{\sigma_2})\}$$

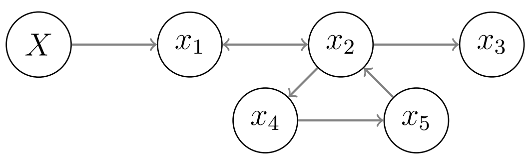

Figure 31

Definition 23 (Strong Node Congruence)**.** If for every possible σ_i it holds that $a_m^{\sigma_i} \equiv a_n^{\sigma_i} \equiv 0 \pmod 2$ $\forall a_m^{\sigma_i}, a_n^{\sigma_i} : (x_j, a_m^{\sigma_i}), (x_j, a_n^{\sigma_i}) \in N_i$ with x_j fixed $\implies x_j$ is strongly congruent to 0 (mod 2). In other words, a node x_j is said to be strongly congruent to 0 (mod 2) if $a_k^{\sigma_i} \equiv 0 \pmod 2$ holds for all values of $a_k^{\sigma_i}$ that form tuples $(x_j, a_k^{\sigma_i})$ in the sequential numbering sets of all possible successors sequences σ_i. Analogously, node x_j may be said to be strongly congruent to 1 (mod 2). These two cases will be denoted by $x_j \overset{s}{\equiv} 0 \pmod 2$ and $x_j \overset{s}{\equiv} 1 \pmod 2$ respectively.

Example 23.1. Due to the form that graph D from Figure 32 has, it is impossible for any of the nodes in it to be incongruent under any possible successors sequence σ_i. It is also impossible for any of the nodes in it to be simultaneously congruent to 0 (mod 2) under a successors sequence σ_i and congruent to 1 (mod 2) under another successors sequence σ_j. Thus, after inspecting the nodes, it can be said that $x_1 \overset{s}{\equiv} 1 \pmod 2$, $x_2 \overset{s}{\equiv} 1 \pmod 2$, and $x_4 \overset{s}{\equiv} 1 \pmod 2$; while $x_3 \overset{s}{\equiv} 0 \pmod 2$. Two examples of successors sequences (σ_1 and σ_2) as well as a graphical representation of graph D are provided below.

$$\sigma_1 = X, x_1, x_3, x_4, x_3, x_4 \implies N_1 = \{(x_1, 1^{\sigma_1}), (x_3, 2^{\sigma_1}), (x_4, 3^{\sigma_1}), (x_3, 4^{\sigma_1}), (x_4, 5^{\sigma_1})\}$$
$$\sigma_2 = X, x_2, x_3, x_4 \implies N_2 = \{(x_2, 1^{\sigma_2}), (x_3, 2^{\sigma_2}), (x_4, 3^{\sigma_2})\}$$

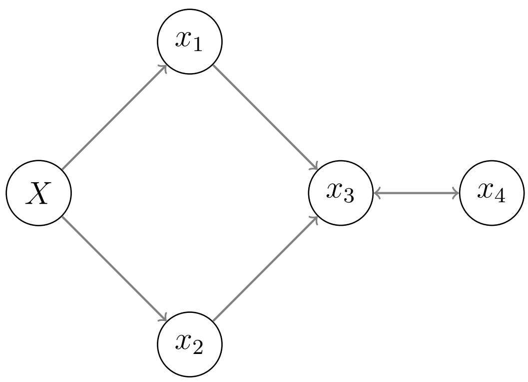

Figure 32

Definition 24 (Set S_i^*)**.** Let D be a graph. If player A has a winning strategy on graph D under the Local Poison Game, then set S_i^* will be defined as:

$$S_i^* = \left\{ x_n \mid x_n = X \text{ or } f_A^W(\sigma_k^j) = x_n \right\}$$ [21]

[21] In other words, for node to belong in the set S_i^* it is necessary for the node to be equal to node X or to be a node such that it belongs to player A's winning strategy.

Now, the construction of set S_i^* is a task which will depend on both players[22], and will be done as follows:

Given an initial node $X \in V(D)$, player B will begin by selecting (poisoning) a successor of such a node (i.e. he must select a node x_m such that $f_B(\sigma_1^1) = x_m$ for $\sigma_1^1 = X$). Player B's selection of a successor of node X will give player A a non-empty[23] successors set (namely, $\Gamma_D^+(x_m)$) from which, in turn, she shall select a node. Player A's role in the construction of set S_i^* will then consist of selecting a node x_n such that: $f_A^W(\sigma_1^2) = x_n$ for $\sigma_1^2 = X, x_m$. The node (x_n) that player A selects according to this criterion will form part of set S_i^*[24]. Thus, At this point, set S_i^* should consist of the following nodes: $S_i^* = \{X, x_n\}$. Players A and B will then continue to take turns selecting successive nodes in D, and the process that was just explained will repeat itself. That is, every time player B poisons a node x_m (meaning $f_B(\sigma_k^j) = x_m$ for some position σ_k^j) then player A will select a node $x_n \in \Gamma_D^+(x_m)$ such that $f_A^W(\sigma_k^{j+1}) = x_n$. Finally, each time player A selects a node according to this criterion, the node will become part of set S_i^*.

On a finite graph D this process will continue until player B is unable to poison a successor of the node selected by A, or until both players fall in an chordless even cycle ($C_{2n}^- : n \in \mathbb{N}$)[25]. In either case, the game is called to an end (with A being the winning player), and with it the construction process of set S_i^* also comes to an end.

Remark. Notice that player B backtracking does not affect the construction of set S_i^* in any way. It is for this reason that σ_k^j was used in the paragraph above.

Example 24.1. Notice that player A has a winning strategy on graph D in Figure 33. The construction of two different sets S_1^* and S_2^* for this graph will now be explained:

[22] Following [13].

[23] If it were empty, then A wouldn't be able to select a successor and the game would immediately come to an end with player B being the winner. This contradicts the definition's requirement of A having a winning strategy.

[24] Definition 24 requires player A to have a winning strategy. Note that having one implies that A will always be able to select a successor of any node poisoned by player B. Therefore, the existence of a node x_n such that it is a successor of node x_m, and it belongs to player A's winning strategy is guaranteed by the requirements of the definition.

[25] See Definition 19 and Examples 19.1, and 19.2 as well as Appendix B.

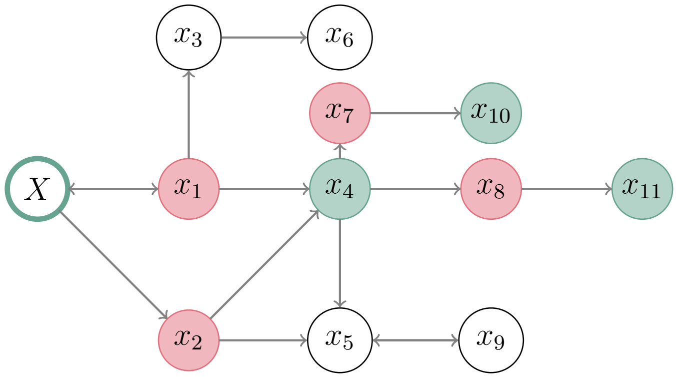

Figure 33

- **Constructing Set $\mathbf{S_1^*}$**: Begin by noticing that that, by definition, $X \in S_1^*$. Also notice that $\Gamma_D^+(X) = \{x_1, x_2\}$. Now assume that player B selects node x_1 as the successor of node X. This will give player A a set of the form $\Gamma_D^+(x_1) = \{x_3, x_4\}$ from which she must select a successor. Observe that node x_3 is strongly congruent to 0 (mod 2)[26] but it is not part of player A's winning strategy (i.e: $f_A^W(\sigma_1^2) \neq x_3$ for $\sigma_1^2 = X, x_1$) because selecting it as a successor of node x_1 would then allow player B to select node x_6 and win the game. This implies that $f_A^W(\sigma_1^2) = x_4$ for $\sigma_1^2 = X, x_1$ because by selecting node x_4 player A will force player B poison either of the nodes x_7, x_8 or x_5 during his turn, and, in any of these cases, this would determine a successors sequence ending in a win for A[27]. By selecting node x_4, according to the aforementioned criterion, this node will now form part of set S_1^* (which at this point of the construction process should consist of the following nodes: $S_1^* = \{X, x_4\}$). Since $\Gamma_D^+(x_4) = \{x_7, x_8\}$, it may be assumed that player B selects node x_7 as the successor of node x_4 leaving player A node x_{10} as the only option to choose as a successor of node x_7. Finally, notice that since node x_{10} is definitely in player A's winning strategy because it determines a win for player A (and passes the strong congruence test from Lemma A.1 in Appendix A), then is must also be part of set S_1^*. Considering the scenario in which payer B decides not to backtrack, the game comes to an end with player A being the winner and set S_1^* having the following form: $S_1^* = \{X, x_4, x_{10}\}$

[26] i.e: $x_3 \overset{s}{\equiv} 0 \pmod 2$.

[27] Notice that Lemma A.1 from Appendix A can be used to test whether or not a node really belongs in player A's winning strategy. In light of the fact that node x_4 determines successors sequences that end in a win for A then one might assume that it belongs to player A's winning strategy. Lemma A.1 states that if a node is part of player A's winning strategy, then that node must be strongly congruent to 0 (mod 2). Since node x_4 is strongly congruent to 0 (mod 2) it is safe to conclude that it is indeed part of player A's winning strategy.

- **Constructing Set $\mathbf{S_2^*}$**: As in the first scenario, notice that, by definition, $X \in S_2^*$. Now, if instead of selecting node x_1 as the successor of node X player B selects node x_2 this will leave player A with a set of the form: $\Gamma_D^+(x_2) = \{x_4, x_5\}$ from which she shall select a successor. Notice that node x_5 does not really belong to her winning strategy because although it may appear to force both players into a chordless even cycle of the form C_2 (which implies a win for player A) node x_5 doesn't pass the strong congruence test from Lemma A.1 in Appendix A[28]. Thus, player A will select node x_4 as the successor of node x_2. Note that, for the same reasons as those in the construction of set S_1^*, node x_4 belongs player A's winning strategy, and thus, belongs in set S_2^*. This will leave player B with a set of the form $\Gamma_D^+(x_4) = \{x_7, x_8\}$ from which to select a successor. Without loss of generality, assume he selects node x_8 and by doing so he leaves player A with the only option of selecting node x_{11} as the successor. Since node x_{11} determines a successors sequence ending in a win for player A's and it passes the strong congruence test from Lemma A.1 this node will form part of set S_2^*. At this point, set S_2 will consist of the following nodes: $S_2^* = \{X, x_4, x_{11}\}$. Now, if player B decides to backtrack and select node x_1, player A will again select node x_4 as its successor (for the same reasons mentioned in the construction of set S_1^*). Assuming player B decides to select node x_7 this time, in an attempt to have a different outcome, player A will only be left with the possibility of selecting node x_{10}. For the same reasons presented when A selected node x_{11}, node x_{10} will also be part of set S_2^*. Finally, the game comes to an end with player A being the winner, and set S_2^* consisting of the following nodes: $S_2^* = \{x_4, x_{10}, x_{11}\}$.

[28] For a concrete example that shows that it doesn't belong in player A's winning strategy consider successors sequence $\sigma_m = X, x_2, x_5, x_9, x_5$ is followed first. then, player B backtracks and forces the following successors sequence: $\sigma_m = X, x_1, x_4, x_5$. Finally, player A is rendered unable to select a successor for node x_5 since x_9 has already been poisoned by player B.

8 Results on the Local Poison Game

This section contains original results concerning the Local Poison Game. The proofs for the statements made here will use some of the definitions from Sections 3 and 7, as well as the rules for the Local Poison Game seen in Section 6.

Theorem 2 (Characterization of Membership to a Semi-kernel on a Finite Graph)**.** *Let D be a finite graph and $X \in V(D)$. Then, player A has a winning strategy under the Local Poison Game starting at node X if and only if node X is in a semi-kernel.*

Proof. Since this theorem consists of an if and only if statement, the proof will consist of two parts[29]. Firstly, the statement from left to right will be proved by contradiction. Naturally, a proof for the statement from right to left will follow.

Left-to-Right: Let $X \in S_i^*$ with S_i^* as in Definition 24[30]. Now, assume that player A has a winning strategy, and S_i^* is not a semi-kernel. Notice that if set S_i^* is not a semi-kernel, it can either be because there exists a node in the successors set of S_i^* such that there is no arc from that node to any other node within S_i^*[31], or because S_i^* is not independent[32]. Both cases will now be analyzed:

Case 1: If there does not exist an arc from a node x_m in the successors set of S_i^* to a node x_n in S_i^* this can either be because of one of the following scenarios:

I. $x_m \in \Gamma_D^+(S_i^*)$ and $\Gamma_D^+(x_m) = \emptyset$. Or, in other words, for x_m in the successors set of S_i^*, the successors set of x_m is empty. In this case, it is clear that the reason why there is no arc from x_m to a node that belongs in the set S_i^* is because x_m has no successors to start with. This implies that player A will not be able to select a successor of node x_m and will therefore lose the game.

II. By backtracking[33], or because $x_m \in C_{2n+1}^-$ (i.e: x_m belongs in a chordless odd cycle)[34] all of the possible successors of x_m have been poisoned by B.

[29] The reader should be familiar with this proof structure since it has already been used to present results on the Poison Game in Theorem 1 in Section 5.

[30] Note that it can also be the case that $S_i^* = \{X\}$ (i.e: It could be the case that S_i^* only contains one node, namely, node X).

[31] This will be referred to as **Case 1**. In mathematical terms, this case can be expressed as: $\exists x_m \in \Gamma_D^+(S_i^*) : \nexists e_k \in F(D)$ such that $e_k = (x_m, x_n)$ for some $x_n \in S_i^*$.

[32] This will be referred to as **Case 2**. Note that this case requires set S_i^* to have at least two nodes (i.e: $|S_i^*| \geq 2$). In mathematical terms, **Case 2** can be expressed as: $\exists x_m, x_n \in S_i^*$: $x_n \in \Gamma_D^+(x_m)$.

[33] For an example, see Figure B.2 in Example B.2 (Appendix B).

[34] For an example, see Figure 25 in Example 25 (Section 6).

As in the previous scenario, this implies that player A will not be able to select a successor of node x_m and will therefore lose the game.

It is clear that, in both scenarios, the game ends in a win for player B. This, however, contradicts the initial assumption that player A has a winning strategy. Therefore, it is impossible that a node x_m as described in **Case 1** exists.

Case 2: If set S_i^* is not independent, then it must be the case that some node x_n in S_i^* is a successor of another node x_m which is also in S_i^*. Now, recall that the graph D on which the game is being played is a finite graph. This implies that set $\Gamma_D^+(x_n)$ must be finite and, therefore, either of the two possible scenarios that follow apply to it:

I. $\Gamma_D^+(x_n) = \emptyset$. Or, in other words, node x_n has no successors. Notice that, by Definition 24, nodes x_m and x_n must be selected by player A during the run of the game. Then, at some point, by backtracking, player B can poison node x_n and win the game since player A will not be able to select a successor (because there are none).

II. $\Gamma_D^+(x_n) \neq \emptyset$. Or, in other words, node x_n has one or more successors. Since $\Gamma_D(x_n)$ is necessarily finite, it can be said that node x_n has k successors. Once again, notice that by Definition 24, nodes x_m and x_n must be selected by player A during the run of the game. Then, player B needs to backtrack at most $k-1$ times in order to poison all the successors of node x_n. Finally, if this were not enough to render player A unable to select a successor, player B may backtrack one more time and poison node x_n. By doing this, player A will inevitably be unable to select a successor because they have all been previously poisoned.

As in **Case 1**, it is clear that, in both scenarios, the game ends in a win for player B. Once again, this contradicts the initial assumption that player A has a winning strategy. Therefore, it is impossible that nodes x_m and x_n as described in **Case 2** exist.

Since **Case 1** and **Case 2** are incompatible with the initial assumption of player A having a winning strategy, a contradiction has been reached. Thus, it cannot be the case that set S_i^* is not a semi-kernel. Therefore, if player A has a winning strategy, this implies that node X is in a semi-kernel.

Right-to-Left: Let $X \in S_i$ for $S_i \subset V(D)$[35], and let S_i be a semi-kernel. By Definition 10 set S_i must be independent. Thus: $\forall x_m \in \Gamma^+_D(S_i) \implies x_m \notin S_i$. In particular, any node x_m that is a successor of node X is not in S_i (so whichever node x_m player B poisons in his first move will not be in S_i). Furthermore, Definition 10 states that $\forall x_m \in \Gamma^+_D(S_i) \exists x_n \in \Gamma^+_D(x_m) : x_n \in S_i$. In particular, this implies that all the successors of node X have, in turn, a successor belonging to S_i. So no matter which node x_m player B poisons in his first move, there will always a a successor (x_n) of x_m in S_i that player A will be able to select as a reply. At this point, the successors sequence of the nodes selected by players A and B should have the following form: $\sigma_1 = X, x_m, x_n$. Finally, notice that since S_i is a semi-kernel, then any successor of node x_n that is poisoned by player B will, in turn have a successor in S_i that player A will be able to select. Thus, the game will result in a win for player A. So, indeed, if node X belongs in a semi-kernel player A has a winning strategy. □

Recalling Theorem 1, it is important to note that the conditions[36] required for it are weaker when compared to those required for Theorem 2. As a consequence of this, Theorem 1 is valid for infinite graphs while Theorem 2 is only valid for finite graphs. Thus, as a next step, Theorem 2 shall be reformulated under these same weaker conditions in order to generalize its application to infinite graphs.

[35] Note that it can also be the case that $S_i = \{X\}$ (i.e: It could be the case that S_i only contains one node, namely, node X).

[36] Namely, the conditions that: 1. Graph D must be outwardly finite, and 2. Graph D must be progressively finite.

Theorem 3 (Characterization of Membership to a Semi-kernel)**.** *Let D be an outwardly and progressively finite graph, and let $X \in V(D)$. Then, player A has a winning strategy under the Local Poison Game starting at node X if and only if node X is in a semi-kernel.*

Proof. Similarly to what was done in the proof for Theorem 2, this proof will consist of two parts. The first of these will show that the statement holds from left to right, while the second will show that the statement holds from right to left.

Left-to-Right: Let $X \in S_i^*$ with S_i^* as in Definition 24[37]. Now, assume that player A has a winning strategy, and S_i^* is not a semi-kernel. Following a similar reasoning to the one in the proof for finite graphs, if set S_i^* is not a semi-kernel, it can either be because there exists a node in the successors set of S_i^* such that there is no arc from that node to any other node within S_i^*[38], or because S_i^* is not independent[39]. Both cases will now be analyzed:

Case 1: If there does not exist an arc from a node x_m in the successors set of S_i^* to a node x_n in S_i^* this can either be because of one of the following scenarios:

I. $x_m \in \Gamma_D^+(S_i^*)$ and $\Gamma_D^+(x_m) = \emptyset$. Or, in other words, for x_m in the successors set of S_i^*, the successors set of x_m is empty. In this case, it is clear that the reason why there is no arc from x_m to a node that belongs in the set S_i^* is because x_m has no successors to start with. This implies that player A will not be able to select a successor of node x_m and will therefore lose the game.

II. Since D is an outwardly finite graph, all of the nodes in $V(D)$ have a finite successors set. Thus, in particular, $\Gamma_D^+(x_m)$ must be finite. So suppose that by backtracking[40], or because $x_m \in C_{2n+1}^-$ (i.e: x_m belongs in a chordless odd cycle)[41] all of the possible successors of x_m have been poisoned by B. As in the previous scenario, this implies that player A will not be able to select a successor of node x_m and will therefore lose the game.

[37]Note that it can also be the case that $S_i^* = \{X\}$ (i.e: It could be the case that S_i^* only contains one node, namely, node X).

[38]This will be referred to as **Case 1**. In mathematical terms, this case can be expressed as: $\exists x_m \in \Gamma_D^+(S_i^*) : \nexists e_k \in F(D)$ such that $e_k = (x_m, x_n)$ for some $x_n \in S_i^*$.

[39]This will be referred to as **Case 2**. Note that this case requires set S_i^* to have at least two nodes (i.e: $|S_i^*| \geq 2$). In mathematical terms, **Case 2** can be expressed as: $\exists x_m, x_n \in S_i^*$: $x_n \in \Gamma_D^+(x_m)$.

[40]For an example, see Figure B.2 in Example B.2 (Appendix B).

[41]For an example, see Figure 25 in Example 25 (Section 6).

It is clear that, in both scenarios, the game ends in a win for player B. This, however, contradicts the initial assumption that player A has a winning strategy. Therefore, it is impossible that a node x_m as described in **Case 1** exists.

Case 2: If set S_i^* is not independent, then it must be the case that some node x_n in S_i^* is a successor of another node x_m which is also in S_i^*. Now, recall that the graph D on which the game is being played is an outwardly and progressively finite graph. In particular, this implies that the set $\Gamma_D^+(x_n)$ must be finite and, all of the nodes in $\Gamma_D^+(x_m)$ are the origin of finite paths that end in a win for player A. Therefore, either of the two possible scenarios that follow apply to node $x_n \in \Gamma_D^+(x_m)$:

I. $\Gamma_D^+(x_n) = \emptyset$. Or, in other words, node x_n has no successors. Notice that, by Definition 24, nodes x_m and x_n must be selected by player A during the run of the game. Then, at some point, by backtracking, player B can poison node x_n and win the game since player A will not be able to select a successor (because there are none).

II. $\Gamma_D^+(x_n) \neq \emptyset$. Or, in other words, node x_n has one or more successors. Since $\Gamma_D(x_n)$ is necessarily finite because D is outwardly finite, it can be said that node x_n has k successors. Once again, notice that by Definition 24, nodes x_m and x_n must be selected by player A during the run of the game. Then, all player B needs to do is backtrack at most $k-1$ times in order to poison all the successors of node x_n. Finally, if this were not enough to render player A unable to select a successor, player B may backtrack one more time and poison node x_n. By doing this, player A will inevitably be unable to select a successor because they have all been previously poisoned.

As in **Case 1**, it is clear that, in both scenarios, the game ends in a win for player B. Once again, this contradicts the initial assumption that player A has a winning strategy. Therefore, it is impossible that nodes x_m and x_n as described in **Case 2** exist.

Since **Case 1** and **Case 2** are incompatible with the initial assumption of player A having a winning strategy, a contradiction has been reached. Thus, it cannot be the case that set S_i^* is not a semi-kernel. Therefore, if player A has a winning strategy, this implies that node X is in a semi-kernel.

Right-to-Left: Let $X \in S_i^+$ for $S_i \subset V(D)$[42], and let S_i be a semi-kernel. Then, following a similar reasoning to the one seen in the proof for Theorem 2, since S_i is a semi-kernel this implies that $\forall x_m \in \Gamma_D^+(S_i) \implies x_m \notin S_i$ and $\forall x_m \in \Gamma_D^+(S_i) \exists x_n \in \Gamma_D^+(x_m) : x_n \in S_i$. In particular, this implies that no matter which node player B selects as a successor of node X on his first turn, the node will not belong in set S_i. Furthermore, it will necessarily be the case that whichever node player A selects as a reply to player B's first move, this node will be in the set S_i. Note now that this scheme can be repeated infinitely many times, since set S_i being a semi-kernel implies that player B can never poison nodes within the semi-kernel and it also implies that whenever player B selects a node in the successors set of S_i player A will always be able to select a successor of this node such that it belongs to set S_i. Therefore, if node X belong in a semi-kernel S_i this implies that player A has a winning strategy.

□

[42] Note that it can also be the case that $S_i = \{X\}$ (i.e: It could be the case that S_i only contains one node, namely, node X.)

9 Conclusion

The Local Poison Game receives its name from the fact that a node must be given in order for the game to be played. Although this detail may not appear to change the player experience drastically when comparing the Local Poison Game to its predecessor[43], the fact is that this variation, in conjunction with player B's possibility to backtrack, are enough to provide a completely different set of information concerning a given graph.

Unlike in The Poison Game, the existence of a semi-kernel in a graph on which the Local Poison Game will be played is no longer enough to guarantee that player A has a winning strategy. This is due to the fact that player A no longer has control over which node will mark the beginning of the game, and, therefore, has no control over the type of set to which the nodes in her strategy will belong to. Thus, when it is said that the Local Poison Game is a characterization of membership to semi-kernels, what is meant is that it provides information on the existence of a winning strategy for player A as a function of the initial node belonging to a semi-kernel.

In a broader context, this change of settings in the (initial) Poison Game might have several applications in fields related to abstract argumentation theory. It is not hard to imagine a scenario (e.g: a debate or a legal trial) in which two players (e.g: two teams in a debate club or two groups of lawyers in a court) start a point that was not chosen by either of them and try to provide arguments [13] that render the other unable to provide a valid reply. In such cases, the win/lose nature of the possible scenarios demands that the game end at some point. However, under more abstract scenarios related to theoretical aspects of Computer Science or Artificial Intelligence, the possibility of infinite runs of the Local Poison Game may have more relevance, as it may shed light on e.g: the possible states of a computer program when undergoing a certain process that complies with a specific set of conditions.

[43]i.e: The Poison Game

A Winning Strategies and Strong Congruence

This appendix contains an original result which derives from Definition 12, Definition 23, and Definition 24. The small result, presented in the form of a lemma, states that if a node belongs to player A's winning strategy under the Local Poison Game, it must then be the case that such a node is strongly congruent to 0 (mod 2).

Lemma A.1. *Let D be a directed graph, and $X \in V(D)$. If player A has a winning strategy on graph D under the Local Poison Game starting at $X \in V(D)$, then:*

$$\forall x_n : f_A^W(\sigma_k^j) = x_n \implies x_n \overset{s}{\equiv} 0 \ (mod\ 2)$$

Proof. According to rules of the Local Poison Game[1], $\sigma_1^1 = X$ is given as the initial setting. Then, according to Definition 20, number 1^{σ_1} will be assigned to the node x_m that player B selects as a successor of X because $f_B(\sigma_1^1) = x_m$ for $\sigma_1^1 = X \implies \sigma_1^2 = X, x_m \implies (x_m, 1^{\sigma_1})$ [2]. As a consequence of this, whichever node x_n player A selects as a successor of the x_m poisoned by player B, it will necessarily be the case that number 2^{σ_1} will be assigned to it because $f_A^W(\sigma_1^2) = x_n$ for $\sigma_1^2 = X, x_m \implies \sigma_1^3 = X, x_m, x_n \implies (x_n, 2^{\sigma_1})$ [3]. Now, since it has been assumed that payer A has a winning strategy, this means that she will always be able to select an unpoisoned successor for any node poisoned by player B during his turn. In particular, this implies that, under successors sequence σ_1, any node x_m poisoned by player B will be such that: $x_m^{\sigma_1} \equiv 1$ (mod 2), and any successive node x_n selected by player A as a reply will be such that $x_n^{\sigma_1} \equiv 0$ (mod 2) [4].

Notice that, according to the rules of the Local Poison Game[5], if player B decides to backtrack at some point of the game and start a new successors sequence σ_2, then σ_2 will be of the form: $\sigma_2 = x_n, x_m, \ldots$ where, by definition, x_n is either node X or a node which was previously selected by player A, and x_m is the (possibly) alternative successor of node x_n that will be poisoned by player B as a result of backtracking. Observe that, under this new successors sequence, it is the case that $x_n^{\sigma_2} \equiv 0$ (mod 2) and $x_m^{\sigma_2} \equiv 1$ (mod 2) for the first two nodes in σ_2. Then, since player B's decision to backtrack doesn't affect player A's winning strategy[6] player A will still be able to always select an unpoisoned successor for

[1]In particular, Rule I. in Definition 17 contained in Section 6.

[2]Notice that $x_m \equiv 1$ (mod 2).

[3]Likewise, notice that $x_n \equiv 0$ (mod 2).

[4]This follows from the fact that the first node x_m selected a successor of X is congruent to 1 (mod 2), the fact that both players take turns selecting successive nodes, and the fact that player A has a winning strategy.

[5]In particular, Rule V. in Definition 17 contained in Section 6.

[6]This follows from Definition 12.

any node that player B poisined during his turn. In particular, this implies that under any successors sequence σ_k started as a consequence of player B backtracking (e.g: σ_2), it will be the case that any node x_m poisoned by player B will be such that: $x_m^{\sigma_k} \equiv 1 \pmod 2$, and any successive node x_n selected by player A as a reply will be such that $x_n^{\sigma_k} \equiv 0 \pmod 2$.

So, in summary, it can be said that $\forall x_n : f_A^W(\sigma_k^j) = x_n \implies x_n^{\sigma_k} \equiv 0 \ (mod\ 2)$. Or, in other words, if given a position σ_k^j a node x_n belongs to player A's winning strategy it must then be the case that x_n is congruent to 0 (mod 2) under successors sequence σ_k.

Now suppose that $\exists x_n : f_A^W(\sigma_k^j) = x_n \implies x_n^{\sigma_k} \equiv 0 \pmod 2$ but $x_n^{\sigma_l} \equiv 1 \pmod 2$. Or, in other words, suppose that there exists a node x_n that belongs to player A's winning strategy under successors sequence σ_k such that x_n is congruent to 0 (mod 2) under σ_k but, at the same time, node x_n is congruent to 1 (mod 2) under another successors sequence σ_l. Notice then that if players A and B run successors sequence σ_k first, node x_n will be selected by player A (by hypothesis). This means that player B is allowed to backtrack and poison all of the successors of node x_n, after which player B may backtrack once again to force both players into following successors sequence σ_l. Recall that (by hypothesis): $x_n^{\sigma_l} \equiv 1 \pmod 2$. This implies that under σ_l player B poisons node x_n, and since all the successors of node x_n have already been poisoned, player A will lose the game. This clearly contradicts the initial assumption that player A has a winning strategy. Thus, it is impossible for such a node x_n to exist.

Therefore: $\forall x_n : f_A^W(\sigma_k^j) = x_n \implies x_n \overset{s}{\equiv} 0 \pmod 2$. Or, in other words, if a node x_n is such that it belongs to player A's winning strategy under successors sequence σ_k, then it must be the case that $x_n^{\sigma_i} \equiv 0 \pmod 2$ under all possible successors sequences σ_i [7]. Following Definition 23 this is equivalent to saying that node node x_n is strongly congruent to 0 (mod 2).

□

Example A.1. Consider graph D in Figure A.1. Note that player A has a winning strategy under the Local Poison Game for this graph. Now, assume that: $f_B(\sigma_1^1) = x_2$ for $\sigma_1^1 = X$. Player A can then select either node x_3 or x_4. Notice that, node x_4 may appear to be a good option for player A, since it would lead both players to fall in a chordless even cycle of the form C_2 and this would imply a win for player A. Nonetheless, player A must follow her winning strategy,

[7] This includes successors sequence σ_k

and the fact that node x_4 is congruent to 0 (mod 2) under successors sequence $\sigma_1 = X, x_2, x_4, x_6, x_8$ but congruent to 1 (mod 2) under successors sequence $\sigma_2 = X, x_1, x_3, x_4, x_6$ means that x_4 is not strongly congruent to 0 (mod 2) and therefore $f_B(\sigma_1^1) \neq x_2$ [8]. Thus, the option of selecting node x_4 is discarded by player A. This implies that $f_A^W(\sigma_1^2) = x_3$ for $\sigma_1^2 = X, x_2$ must be the case because by selecting it player A forces a successors sequence that ends in win for her (namely $\sigma_1 = X, x_2, x_3, x_5, x_7$). Finally, note that nodes x_3 and x_7 both belong in player A's winning strategy, and as a consequence, $x_3 \overset{s}{\equiv} 0 \pmod 2$ and $x_7 \overset{s}{\equiv} 0 \pmod 2$.

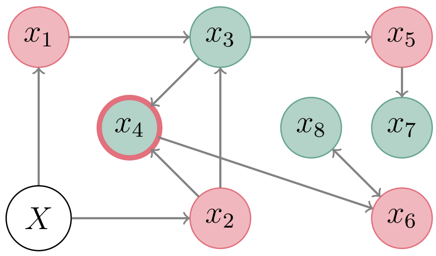

Figure A.1

[8]This is due to the fact that if successors sequence σ_1 is followed first, player B may backtrack (as soon as player A selects node x_8) and poison node x_1. This would force player A to select node x_3, and this, in turn would allow player B to poison node x_4 rendering player A unable to select any successor since node x_6 has already been poisoned.

B Chordless Even Cycles

Assuming that the Local Poison Game is being played, the sentence "falling in a chordless even cycle" refers to a scenario in which players A and B take turns selecting nodes that form part of a chordless even cycle ($C^{-}_{2n} : n \in \mathbb{N}$) an indefinite number of times. Examples of such scenarios will now be provided:

Example B.1. Graph D in Figure B.1 depicts a case in which players A and B take turns selecting successive nodes until player B poisons node x_k. At this point, player A's winning strategy tells her to select node x_{k+1} because not doing so will determine a successors sequence ending in a win a for player B [1]. By selecting this node, player A has entered a chordless even cycle (C^{-}_{4}) that stems from node x_k. Player B will then be forced to poison node x_{k+2} to which player A will reply by selecting node x_{k+3}. Finally, player B's strategy will tell him to select node x_k once again. Nonetheless, player A's winning strategy will not have changed, thus in order avoid the successors sequence that guarantees a win for B she decides to select node x_{k+1} again. In this way, players A and B will find themselves taking turns selecting nodes from the chordless even cycle C^{-}_{4} that stems from node x_k. Note that player B cannot change the situation by backtracking, since by doing so he can only select alternative successors to nodes previously selected by player A, but due to the graph's shape, there are no alternative successors to nodes previously selected by player A. Thus, since A will continue to force both players into C^{-}_{4} every time B selects node x_k it can be said that both players have fallen in an even chordless cycle, and, therefore, the game ends in a win for player A[2].

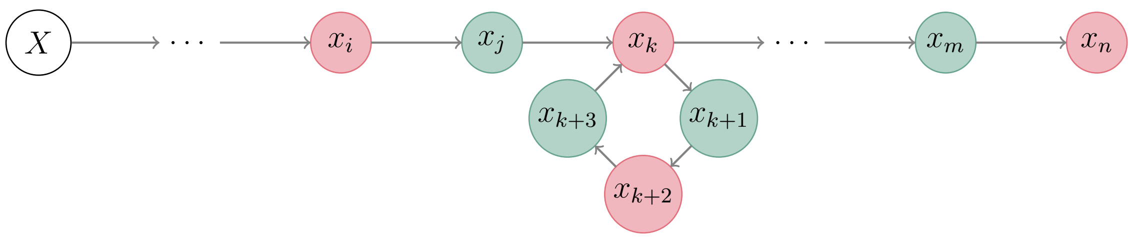

Figure B.1

[1]The graph in Figure B.1 shows the case in which node x_k has another successor besides node x_{k+1}. However, it could very well be the case that x_k doesn't have any successors at all other than x_{k+1} or indeed has multiple successors other than x_{k+1} but they all determine successors sequences that guarantee a win for player B. Other possibilities are not considered here because this appendix is dedicated to scenarios in which both players fall in a chordless even cycle.

[2]Note that any chordless even cycle $C^{-}_{2n} : n \in \mathbb{N}$ works (e.g: C^{-}_{2}, C^{-}_{10} or even C^{-}_{1762} all work equally well).

Example B.2. Note that it is of great importance that the cycle be chordless. Graph D in Figure B.2 depicts a case (very similar to the one seen in Figure B.1) in which players A and B take turns selecting successive nodes until player B poisons node x_k. As explained before, player A's strategy will tell her to enter the even cycle (C_4) that stems from node x_k because not doing so would determine a successors sequence that guarantees a win for player B. However, things are very different this time because the cycle is not chordless. Once player A has selected node x_{k+1}, player B's winning strategy will tell him to poison node x_{k+2} to which player A will respond by selecting node x_{k+3}. At this point, player B's winning strategy will tell him to backtrack and poison node x_{k+3} since it is an alternative successor of x_{k+1}. By doing so, player A will be impeded to select an unpoisoned successor, and the game will immediately come to and end with player B being the winner. Thus, if player A is to have a winning strategy as a consequence of falling in an even cycle (C_{2n}) stemming from a node in the graph, it must be the case that such a cycle is chordless.

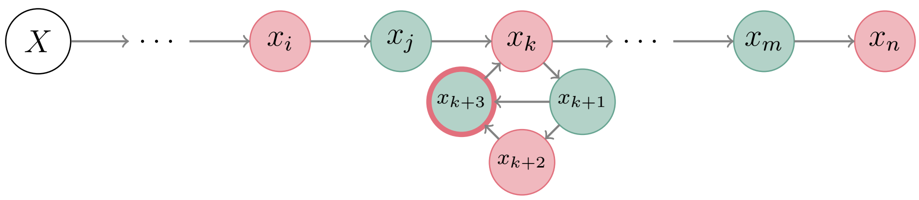

Figure B.2

Example B.3. Note, also, that player A's decision to enter the even cycle is crucial. Without it, both players might only enter it a finite number of times, or might not even enter it at all. Graph D from Figure B.3 depicts a case in which players A and B take turns selecting successive nodes until player A selects node x_j. At this point, player B's winning strategy will tell him to avoid entering the chordless even cycle C_4^- stemming from node x_j and instead poison node x_k because by doing so he will determine a successor sequence that will end in a win for him. Thus, if player A is to have a winning strategy as a consequence of falling in a chordless even cycle (C_{2n}^-) stemming from a node in the graph, it must be the case that such a cycle stems from a node poisoned by player B, and not from a node selected by player A.

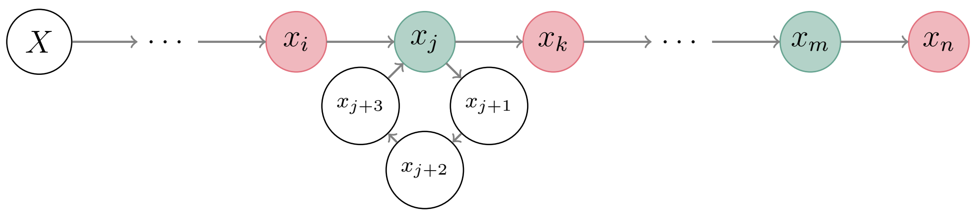

Figure B.3

C Some Applications of Semi-kernels

As mentioned in [5] and [1], semi-kernels have a wide variety of applications. Some of the most theoretical range from game theory (positional, coopertaive and combinaorial games [5]), to mathematical logic[11] and graph colorings [9]. However, "real world" applications can also be found [1], some of these range from argumentation, computation (search problems), social networks, and artificial intelligence to warfare. The following is a brief description of such sort of application.

C.1 Service Locations

Following [7] it is possible to think of a map as a directed graph in which the nodes represent cities , and the (directed) arcs represent highways connecting any two cities. Note that this reasoning can be applied on different scales. That is, instead of having cities on the map of a country or a map containing several countries, one could have locations within a certain city, and the arcs between such locations could be streets instead of highways.

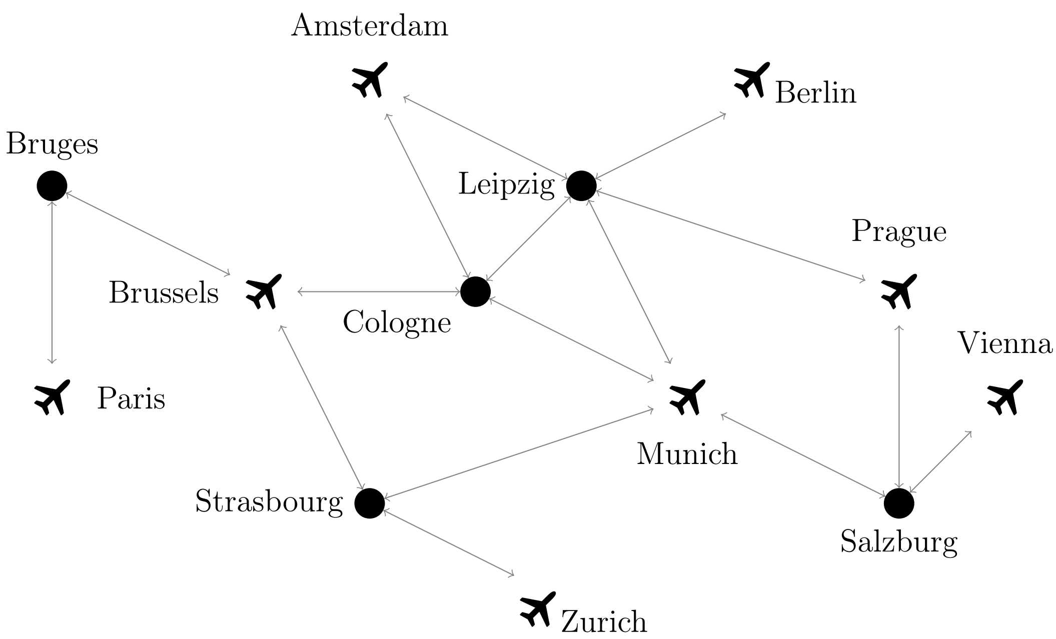

Figure C.1

Example C.1. For this particular example, imagine a car rental company desires to know where would it be convenient to have service points given the map model (graph) seen in Figure C.1. One possible solution would be to find a semi-kernel in the given graph. This happens to be an efficient solution due to the fact that having service centers on the nodes belonging to a semi-kernel guarantees that it will be possible for potential customers to have access to the company's service because they either live in a city with a service point or they live in a city connected to one with a service point. It also ensures that the company will not waste money in more service points than necessary because the nodes in a semi-kernel are independent (or, in other words, having service points in the nodes that belong to a semi-kernel guarantees that the company will not waste money on having two service points in cities which are connected to one another). Thus, for the very simple graph seen in Figure C.1, it is quite clear that it would be convenient to have service points in the cities which have airports in them[1], or possibly even at the airports themselves. This would allow potential customers to rent a car once their plane has landed and to return it at the same airport (or at any other airport) once their trip has come to an end.

Although the idea of finding efficient locations for a car rental company to have service points may be profitable for some, it is, nonetheless, a rather frivolous activity. However, the same sort of reasoning may be applied with more noble intentions such as finding efficient locations to build hospitals, schools, or public transport stations so that the inhabitants of a city always have access to these basic needs. In fact, according to [7], it may even be possible [2] to fix restrictions on the maximum and minimum distance between any two service points belonging to a semi-kernel.

[1]Cities with airports are symbolized by airplanes in the graph. Also note this is a simplified version of a real map which has been adapted to fit this example.

[2]With the aid of some additional concepts to the ones that have been discussed in this paper.

D Other Games on Directed Graphs

Although the Poison Game, and the Local Poison Game, are the main focus of this paper, it is important to mention the role that other games play in the characterization of some game theoretic notions. Although there is a wide range of different types of games, and variations on them, the following are brief explanations on some of the most known examples.

D.1 The Cops and Robbers Game

Following [4], the Cops and Robbers Game is played on reflexive graphs[1] with no multiple arcs[2]. There are two players: the cops and the robber (Note that although there may be more than one cop, they are all controlled by one player; thus they count as a single set accounted for by such a player). The game begins with the cops (denoted by C) selecting a node of their preference in the first move. The cops and the robber (denoted by R) then take turns selecting successive nodes in the graph[3]. The cops win, if at least one of them is able to *capture* the robber (i.e: occupy the same vertex as the robber) in a finite number of rounds. Intuitively, the robber wins if he can avoid being captured indefinitely.

This game falls within the category of the Meet/Avoid games mentioned in [1], and, in particular, it is considered a *vertex-pursuit game*. According to Nowakowski, Winkler and Quilliot [4], this game characterizes the existence of a certain structure (a corner) on finite graphs. A brief outline of this characterization will now be provided[4].

Definition D.1 (Corner)**.** Vertex $x_i \in V(D)$ is said to be a corner If $\exists x_j$: $\Gamma^+_D(x_i) \cup \{x_i\} \subseteq \Gamma^+_D(x_j) \cup \{x_j\}$.

Example D.1. For the graph seen in Figure D.1, it can be said that node x_2 is a corner because $\Gamma^+_D(x_2) \cup \{x_2\} \subseteq \Gamma^+_D(x_3) \cup \{x_3\}$

[1] This means that every vertex in the graph has at least one loop.

[2] Although they are allowed, they make no difference when the game is played. This is is assumed, for simplicity purposes, that the graph on which the game will be played contains no multiple arcs.

[3] Note that since the graph is reflexive, it is possible for either of the player to "pass" or "stay still" during his or her turn.

[4] The reader might find it useful to notice that instead of using the notation introduced in [4], the characterization will be explained using the same notation that has been appeared throughout this paper.

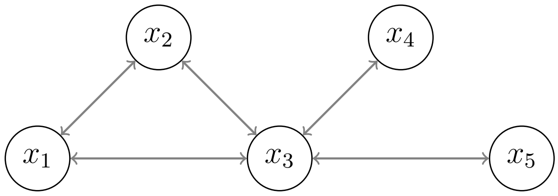

Figure D.1

Definition D.2 (Dismantlable)**.** A graph is dismantlable if there exists a sequence of deleting corners that reduces the graph to a single vertex. Or, in other words, if by deleting the corners of the graph one at a time, the graph can be reduced to a single vertex, then it is said that the graph is dismantlable.

Example D.1. Any tree is a dismantlable graph since all leafs[5] are corners. For the graph seen in Figure D.2 is is possible to begin by deleting node x_5 (since it is a corner), and then proceed by deleting nodes x_1, x_4 and x_6 since they are all corners. Finally, it could be possible to delete either node x_2 or x_3 since the have become corners after deleting the other nodes. Thus the graph has been reduced to a single node and, as a consequence, it can be called a dismantlabele graph.

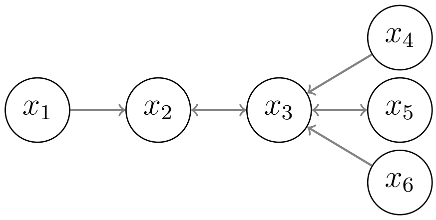

Figure D.2

Theorem D.1 (Characterization of Dismantlable Graphs)**.** *Given a finite graph D, Cops have a winning strategy if and only if graph D is dismantlable.*

Although a proof for this theorem escapes the scope of this appendix[6], special attention should be payed to the fact that the Cops and Robbers Game is, actually, a way of distinguishing graphs that can be reduced to a single node from those that can't.

[5]A leaf is a node in a tree with degree 1. The reader may refer to the first chapter of [3] for more information on these concepts.

[6]For a complete proof of this theorem and other related lemmas the reader may refer to the second chapter in [4].

D.2 Sabotage Games

Typically, sabotage games consist of two players out of which (following [12]) one is known as the Runner (denoted by R) and the other as the Blocker (denoted by B). The runner's goal is to travel between two given nodes in the graph, while the blocker's goal is to prevent this from happening. Normally, the way in which B prevents R from arriving to the given node is by deleting arcs in the graph.

According to [2], since sabotage games satisfy the three conditions in Zermelo's Theorem[7], it can be said that sabotage games are determined (i.e: It must be the case that one of the two players has a winning strategy given an initial setting an a graph). However, unlike with other games played on graphs, it is particularly difficult to compute the final outcome; this is due to the fact that every time blocker makes a move (i.e: every time blocker deletes an arc from the graph), the graph's structure changes. As a consequence of the graph structure modifying nature of these games, they have been found to be useful in the evaluation of First Order Logic statements on structures that change under evaluation[8]. The use of graphs in the evaluation of logic statements is known as logical model checking, and the way it is done is by conceiving the graph as a set (domain) of arcs and allowing existential quantifiers to run over them. The existence of an edge may be cancelled if such an edge is selected by B. This leads to a characterization of the following form: *Player R has a winning strategy if and only if the first order logic formula obtained from the graph modified by both players evaluates to true*[9].

Besides logical model checking, sabotage games may also be used to model learning processes [10](where a Learner and a Teacher interact), as well as traffic networks and border control [12].

[7]Zemerlo's Theorem says that every finite two player zero-sum game is determined.

[8]An example of a fact that might change when inspected is the classical quantum mechanics problem that arises when trying to determine the position of an atom.

[9]See [2] (page 270) for more information.

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