university of groningen

faculty of science and engineering

mathematics and applied mathematics

COMSOL Simulation of Mixing in a Petri Dish on a Shaker

Bachelor's Project Applied Mathematics

June 2020

Student: G. Yang

First supervisor: dr. F.W. Wubs

Second assessor: dr.ir. B. Besselink

COMSOL Simulation of Mixing in a Petri Dish on a Shaker

Abstract

In this thesis, the behavior of a fluid in a Petri dish placed on a shaker is studied. In order to do this a simplification method was applied which is moving fame method where the coordinates system is binded to the Petri dish. This results in a periodic body forcing on the fluid in the Petri dish. Hereafter, this is implemented in COMSOL for the Burgers equation and a fluid with a free surface. For the Burgers equation, an analytic solution for the case where the nonlinear term is absent is given. This solution is replicated in the numerical solution of the Burgers equation for high viscosities. For the free surface fluid, we found that the moving mesh approach in COMSOL provided the best alternative. Finally, the mixing of a dilluted species in the fluid is considered which shows that shaking has a positive influence on reaching a homogeneous concentration of the species.

Keywords: Shaker; Petri dish; Computational fluid dynamics; Moving frame method; Burgers equation; Moving mesh; COMSOL Multiphysics

Constent

1 Introduction

In order to mix in a laboratory environment, one generally chooses to use a shaker. One wants to know how the liquid behaves in the Petri dish when placed on such a shaker. To understand the mixing process, building a model that can describe and solve the fluid problem is needed. After successfully constructing the model, better suggestions for the mixing process through the shaker can be provided based on the model simulation results. Based on the behaviours of the liquid on the shaker, one can clearly know how the mixing goes over time. At the same time, one can also understand how various parameters affect the mixing process, such as the volume, density and viscosity of the liquid, the frequency, amplitude and direction of the shaker. Therefore, one can choose the shaker parameters suitably for one's needs according to the simulation results. Of course, this model can also be applied to other fields by adapting the model.

Generally speaking, fluid models are usually described by systems of partial differential equations. And there are many mature numerical methods for this, such as finite difference method, finite element method and finite volume method. At present, there are many widely used numerical simulation software that can realize the solution of the partial differential equations system. COMSOL Multiphysics is the simulation software used in this project. The target of this project is the oscillating fluid in a Petri dish on a shaker. There are currently many research projects on fluid simulation, but those that are specifically aimed at the research goal of this project have not been found. Therefore, the research of this project would add a case to the field of fluid modelling and simulation.

There are many different types of laboratory shakers. Here, the reciprocating shaker is the research object of this project. For most reciprocating shakers, the oscillation can be represented by a trigonometric curve. In Chapter 2, the description of the shaker oscillation will be introduced together with the key research method of this project which is moving frame method. This method helps transform the problem from a moving boundary problem to a fixed boundary problem. In Chapter 3 and Chapter 4, the moving frame method will be implemented into the Burgers equation and COMSOL Multiphysics. As a simple case of analysis, the results of the transformed Burgers equation prove the feasibility of the method. In Chapter 5, one can see how to simulate the oscillating fluid model and fluid mixing model in COMSOL Multiphysics and how the fluid behaves. Finally, in Chapter 6, the results will be discussed and some conclusions will be drawn.

2 Moving frame method

In this chapter, a coordinate transformation named moving frame method will be introduced. During the research process of this project, the moving frame method played a vital role. And after that the basic moving frame suitable for the research object of this project will be derived in this chapter.

2.1 Moving frame attached to Petri dish

In the general partial differential equation model, partial differential equation systems, the boundary conditions of the model and the initial conditions of the model are essential. In common fluid problems, we often encounter a type of problem, which contains a set of boundary conditions that change with time, such as ice melting problems, solid structure flow problems with moving structure, and the problem of oscillating fluid similar to the research target of this project, etc. Faced with such changes in boundary conditions, it is difficult to describe and solve the model with normal means. In order to solve this problem, we can consider a basic idea, which is to bind the coordinate system to the moving boundary. This kind of method can be called the moving frame method, which is a special form of the coordinate transformation method.

In this project, the most common Petri dish is considered. The shaking mode of the shaker is reciprocating. Here we consider the general oscillation mode which can be represented by a trigonometric function. If the Petri dish placed on the shaker is regarded as a particle, its position on the x-axis is recorded as $x(t)$ in a Cartesian coordinate system, which is a function of time can be written as $x(t) = x(0) + a\ cos\ (\omega t)$, and the position of the particle on the y-axis and z-axis remains unchanged. In the general model building process, if the model only moves in one direction, one can first consider the simplified one-dimensional model in this direction.

Here, one can assume that a time-varying line segment on the x-axis $[0 + a\ cos\ (\omega t), L + a\ cos\ (\omega t)]$ represents the fluid calculation domain we are studying, where L is the diameter of the Petri dish, the parameter a is the oscillation amplitude of the shaker, ω is equal to the reciprocal of the oscillation frequency multiplied by 2π which also called angular frequency, then the endpoint of the line segment is the boundary of the model. Since the boundary and the entire calculation domain will periodically move with time, it is difficult to describe the fluid model by conventional means. It is not difficult to find that if we let $\tilde{x} = x - a\ cos\ (\omega t)$, where x is a point in the calculation domain, then $\tilde{x} \in [0, L]$ is a point on a domain that does not move with time. Therefore, I chose the following transformation to convert the original coordinate system $T = (x, t)$ to the relative coordinate system $\tilde{T} = (\tilde{x}, \tilde{t})$ which is a moving frame attached to the shaker that

$$\begin{aligned} \tilde{x} &= x - a\ cos(\omega t) \\ \tilde{t} &= t \end{aligned} \tag{2-1}$$

The coordinate transformation method above is the basis of this project. It successfully converts the moving boundary and computing domain in the original problem into a fixed boundary and computing domain.

2.2 The Burgers equation in the moving frame

In the field of computational fluid mechanics, the Burgers equation is one of the most commonly used nonlinear partial differential equations. Its form is not complicated, but it contains convection and diffusion terms. Therefore, it is widely used and is often used to simulate shock wave propagation and reflection. In order to better study the application of the moving frame method in computational fluid mechanics, here, we can assume that the fluid can be described by Burgers equation, and use this for preliminary calculation and analysis.

Here the Burgers equation in inertial Cartesian coordinate system $T = (x, t)$ is considered on moving domain $x \in [0 + a\cos(\omega t), L + a\cos(\omega t)]$ which is

$$u_t = -uu_x + \mu u_{xx} + f(x, t) \tag{2-2}$$

where x represents the position on the x-axis, t represents the time, and u is a function of x and t representing the flow velocity of the fluid at the position x and time t. The direction of the flow velocity is represented by its sign. And the function f is the source term of the velocity.

Here the transformation (2-1) is applied into this equation. Based on this coordinate transformation, one can derive the following variables in the relative coordinate system $\tilde{T} = (\tilde{x}, \tilde{t})$:

$$\begin{aligned} x &= \tilde{x} + a\,cos(\omega\tilde{t}), \\ t &= \tilde{t}. \end{aligned} \tag{2-3}$$

However, if one ignores its physical meaning and just treats the function u as a mathematical symbol, this defaults to the equation $u = \tilde{u}$, which will produce fallacy in actual situations. Because, the coordinate transformation of the moving frame used here will make the reference frame convert from the original inertial frame to a non-inertial frame. In the non-inertial system, the size and direction of the original motion physical quantity, such as velocity and acceleration, will be changed. This change on velocity can be derived from the coordinate transformation:

$$u = \frac{dx}{dt} = \frac{d[\tilde{x} + aco\,s(\omega\tilde{t})]}{d\tilde{t}} = \frac{d\tilde{x}}{d\tilde{t}} - a\omega\sin(\omega\tilde{t}) = \tilde{u} - a\omega\sin(\omega\tilde{t}). \tag{2-4}$$

According to the coordinate transformation method obtained in the previous section, it is easy to know that the relative coordinate system performs variable velocity movement relative to the original inertial reference system, and the point with zero velocity in the original coordinate system will have an additional velocity source in the relative coordinate system. The size of this extra velocity is the same as the velocity of the relative coordinate system moving relative to the original coordinate system, but the direction is opposite which is consistent with the phenomenon observed in (2-4). Here, $a\omega sin(\omega\tilde{t})$ is the inverse number of the movement velocity of the relative coordinate system relative to the original inertial system at time $t = \tilde{t}$. And according to this, one can get the expressions in relative coordinate system:

$$\begin{aligned} u_x &= \frac{\partial u}{\partial x} = \frac{\partial[\tilde{u} - a\omega\sin(\omega\tilde{t})]}{\partial\tilde{x}}\frac{\partial\tilde{x}}{\partial x} + \frac{\partial[\tilde{u} - a\omega\sin(\omega\tilde{t})]}{\partial\tilde{t}}\frac{\partial\tilde{t}}{\partial x} \\ &= \frac{\partial\tilde{u}}{\partial\tilde{x}} = \tilde{u}_{\tilde{x}}, \\ u_{xx} &= \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2[\tilde{u} - a\omega\sin(\omega\tilde{t})]}{\partial\tilde{x}^2}\frac{\partial\tilde{x}}{\partial x} + \frac{\partial^2[\tilde{u} - a\omega\sin(\omega\tilde{t})]}{\partial\tilde{x}\partial\tilde{t}}\frac{\partial\tilde{t}}{\partial x} \\ &= \frac{\partial^2\tilde{u}}{\partial\tilde{x}^2} = \tilde{u}_{\tilde{x}\tilde{x}}. \end{aligned} \tag{2-5}$$

Here, because the extra velocity term only depends on time $\tilde{t}$, this term will disappear after derivation of x. And the derivation of time will have obvious changes which is

$$\begin{aligned}
u_t = \frac{du}{dt}(x(t),t) &= \frac{\partial u}{\partial t}(x,t) + \frac{\partial u}{\partial x}(x,t)\frac{dx}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} \\
&= \frac{\partial(\tilde{u} - a\omega sin(\omega\tilde{t}))}{\partial t} \\
&+ \left(\tilde{u} - a\omega sin(\omega\tilde{t})\right)\frac{\partial(\tilde{u} - a\omega\, sin(\omega\tilde{t}))}{\partial x} \\
&= \frac{\partial\tilde{u}}{\partial\tilde{t}}\frac{\partial\tilde{t}}{\partial t} - a\omega^2\, cos(\omega\tilde{t}) + \frac{\partial\tilde{u}}{\partial x}\frac{\partial x}{\partial\tilde{t}}\frac{\partial\tilde{t}}{\partial t} \\
&+ (\tilde{u} - a\omega\, sin(\omega\tilde{t}))\frac{\partial\tilde{u}}{\partial x}\frac{\partial x}{\partial x} \\
&= \frac{\partial\tilde{u}}{\partial\tilde{t}} + u\frac{\partial\tilde{u}}{\partial x} - a\omega^2\, cos(\omega\tilde{t}) \\
&= \frac{\partial\tilde{u}}{\partial\tilde{t}}(x,\tilde{t}) + \frac{\partial\tilde{u}}{\partial x}(x,\tilde{t})\frac{dx}{d\tilde{t}} - a\omega^2\, cos(\omega\tilde{t}) \\
&= \frac{d\tilde{u}}{d\tilde{t}}(x(\tilde{t}),\tilde{t}) - a\omega^2\, cos(\omega\tilde{t}) \\
&= \tilde{u}_{\tilde{t}} - a\omega^2\, cos(\omega\tilde{t}).
\end{aligned} \tag{2-6}$$

which is consistent with the equation observed in (2-4). Therefore, the expression of the Burgers equation in the relative coordinate system should be that

$$\tilde{u}_{\tilde{t}} = -\tilde{u}\tilde{u}_{\tilde{x}} + \mu\tilde{u}_{\tilde{x}\tilde{x}} + a\omega^2\, cos(\omega\tilde{t}), \tag{2-7}$$

on a fixed domain $\tilde{x} \in [0, L]$. Here one can naturally let the source term $f = a\omega^2\, cos(\omega\tilde{t})$. And now the model has been transformed into the relative coordinate system. It is easy to represent the boundary conditions as $\tilde{u}(0,\tilde{t}) = u(L,\tilde{t}) = 0$ and the initial condition as $u(x,0) = 0$ which means that in the relative reference the Petri dish would not moving anymore and the velocity at the boundary is zero because the wall of the Petri dish and the fluid oscillation will be reflected by the velocity source term which can also be seen as a time dependent body force.

3 Analytical solutions

While experimenting with the transformed Burgers equation, one hopes to know the form of the analytical solution of the equation. But for the nonlinear Burgers equation studied here, it is too difficult to solve, and there is no good solution for the time being. So, one can chose to ignore the non-linear terms, and instead solved the main part of it to study the general form of the solution. When the influence of the nonlinear term is not significant, this solution can be regarded as the solution of the original equation.

Here, only the Burgers equation after ignoring the nonlinear term in the relative coordinate system is considered which also called the one-dimensional non-homogeneous heat conduction equation

$$u_t = \mu u_{xx} + a\omega^2 \cos(\omega t), \tag{3-1}$$

on domain $x \in [0, L]$, with boundary conditions $u(0,t) = u(L,t) = 0$ and initial condition $u(x,0) = 0$.

3.1 Separation of variables

In order to solve this non-homogeneous equation, one can first consider the homogeneous equation that

$$u_t = \mu u_{xx} \tag{3-2}$$

with boundary conditions $u(0,t) = u(L,t) = 0$. Using the separation of variables method, one can get a general solution of this homogeneous equation. The process is as follows:

Try solution $u(x,t) = f(x)g(t)$, and one gets the equation that

$$g'(t)f(x) = \mu f''(x)g(t), \tag{3-3}$$

which is equivalent to

$$\frac{g'(t)}{g(t)} = \mu \frac{f''(x)}{f(x)} = \alpha, \tag{3-4}$$

where α is a constant. However, after trying $\alpha = 0$ and $\alpha > 0$, one can find that they both lead to a contradict situation $u(x,t) \equiv 0$ which we do not want to see. Trying $\alpha < 0$, one can get that

$$g(t) = c_1 e^{\alpha t} \neq 0,$$

$$f(x) = c_2 \cos\left(\sqrt{\frac{-\alpha}{\mu}}x\right) + c_3 \sin\left(\sqrt{\frac{-\alpha}{\mu}}x\right), \tag{3-5}$$

which means that

$$u = c_1 e^{\alpha t}\left[c_2 \cos\left(\sqrt{\frac{-\alpha}{\mu}}x\right) + c_3 \sin\left(\sqrt{\frac{-\alpha}{\mu}}x\right)\right]. \tag{3-6}$$

And from the boundary condition $u(0,t) = 0$, one obtains

$$c_1 e^{\alpha t}[c_2 + 0] = 0, \tag{3-7}$$

but $c_1 = 0$ leads to contradict situation $u(x,t) \equiv 0$ which we do not want. So, trying $c_2 = 0$, one gets that

$$u = c_1 e^{\alpha t}\left[c_3 \sin\left(\sqrt{\frac{-\alpha}{\mu}}x\right)\right]. \tag{3-8}$$

From the other boundary condition $u(1,t) = 0$, one finds

$$c_1 e^{\alpha t}[c_3 sin(\sqrt{\frac{-\alpha}{\mu}}x)] = 0, \tag{3-9}$$

but $c_1 = 0$ *and* $c_3 = 0$ both lead to contradict situation $u(x,t) \equiv 0$ which we do not want. So, trying $sin(\sqrt{\frac{-\alpha}{\mu}}) = 0$ which means $\sqrt{\frac{-\alpha}{\mu}} = n\pi$, one gets the nontrivial solution

$$u = c_1 e^{-\mu(n\pi)^2 t}[sin(n\pi x)], \tag{3-10}$$

where c_1 is an arbitrary constant and n is an arbitrary integer.

3.2 Variation of coefficients

From the process in previous part, one can say that the general solution of the homogenous equation (3-8) is

$$u_h = \sum_{n=1}^{\infty} c_n e^{-\mu(n\pi)^2 t} sin(n\pi x). \tag{3-11}$$

Based on u_h, one can find the solution of the non-homogenous equation (3-7). Using variation of coefficients to get a particular solution by letting c_n depending on t, which can be noted as $c_n(t)$. Substituting $u = u_h$ into the above equation (3-7), one gets

$$\begin{aligned}\sum_{n=1}^{\infty}[\frac{dc_n}{dt} e^{-\mu(n\pi)^2 t} sin(n\pi x) + c_n(t)\frac{\partial\left(e^{-\mu(n\pi)^2 t} sin(n\pi x)\right)}{\partial t}] \\ = \sum_{n=1}^{\infty} \mu c_n(t)\frac{\partial^2(e^{-\mu(n\pi)^2 t} sin(n\pi x))}{\partial x^2} \\ + a\omega^2 cos(\omega t)\end{aligned} \tag{3-12}$$

which is equivalent to

$$\sum_{n=1}^{\infty} \frac{dc_n}{dt} e^{-\mu(n\pi)^2 t} sin(n\pi x) = a\omega^2 cos(\omega t). \tag{3-13}$$

In order to make the two sides similar so that one can continue to calculate, one needs to find the sine expansion of $f(x) = 1$, on domain $[0,1]$ which can be found by an odd extension of this function on the domain $[-1,1]$ in the way below:

$$f(x) = \begin{cases} 1, & x \in (0,1], \\ 0, & x = 0, \\ -1, & x \in [-1,0) \end{cases} \tag{3-14}$$

to satisfy the boundary conditions. According to Fourier expansion formula, one has that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x), \tag{3-15}$$

where

$$a_n = \int_{-1}^{1} f(x) cos\ (n\pi x) dx = 0, \tag{3-16}$$

$$b_n = \int_{-1}^{1} f(x) sin\,(n\pi x)dx = \int_{0}^{1} sin\,(n\pi x)dx + \int_{-1}^{0} -sin\,(n\pi x)dx$$
$$= \frac{2(\cos(0) - \cos(n\pi))}{n\pi} = \frac{2 - 2(-1)^n}{n\pi}.$$

So, the equation (3-19) can be written as

$$\sum_{n=1}^{\infty} \frac{dc_n}{dt} e^{-\mu(n\pi)^2 t} sin(n\pi x) = a\omega^2 cos(\omega t) \sum_{n=1}^{\infty} \frac{2 - 2(-1)^n}{n\pi} sin\,(n\pi x). \tag{3-17}$$

Then, for each n we have

$$\frac{dc_n}{dt} = \frac{[2 - 2(-1)^n]a\omega^2}{n\pi} e^{\mu(n\pi)^2 t} cos(\omega t). \tag{3-18}$$

Integrate on both sides and according to the partial integration method one can get that

$$\begin{aligned} c_n(t) = \int \frac{[2 - 2(-1)^n]a\omega^2}{n\pi} e^{\mu(n\pi)^2 t} cos(\omega t)\, dt \\ = \frac{[2 - 2(-1)^n]a\omega^2}{n\pi} \left(\frac{1}{\omega} e^{\mu(n\pi)^2 t} sin(\omega t) \right. \\ \left. + \frac{\mu(n\pi)^2}{\omega^2} e^{\mu(n\pi)^2 t} cos(\omega t) \right) \\ - \frac{\mu^2(n\pi)^4}{\omega^2} \int \frac{[2 - 2(-1)^n]a\omega^2}{n\pi} e^{\mu(n\pi)^2 t} cos(\omega t)\, dt \\ = \frac{[2 - 2(-1)^n]a\omega^2}{n\pi} \left(\frac{1}{\omega} e^{\mu(n\pi)^2 t} sin(\omega t) \right. \\ \left. + \frac{\mu(n\pi)^2}{\omega^2} e^{\mu(n\pi)^2 t} cos(\omega t) \right) - \frac{\mu^2(n\pi)^4}{\omega^2} c_n(t). \end{aligned} \tag{3-19}$$

So, one can derive that the coefficient $c_n(t)$ is given by

$$c_n(t) = \frac{[2 - 2(-1)^n]}{n\pi} \frac{a\omega^3 \sin(\omega t) + a\omega^2\mu(n\pi)^2 cos(\omega t)}{\omega^2 + \mu^2(n\pi)^4} e^{\mu(n\pi)^2 t}. \tag{3-20}$$

So far, one can say that the particular solution of equation (3-8) is that

$$\begin{aligned} u_p \\ = \sum_{n=1}^{\infty} \left(\frac{[2 - 2(-1)^n]}{n\pi} \right. \\ \left. \cdot \frac{a\omega^3 sin(\omega t) + a\omega^2\mu(n\pi)^2 cos(\omega t)}{\omega^2 + \mu^2(n\pi)^4} e^{\mu(n\pi)^2 t} \right) e^{-\mu(n\pi)^2 t} sin(n\pi x) \\ = \sum_{n=1}^{\infty} \frac{[2 - 2(-1)^n]}{n\pi} \cdot \frac{a\omega^3 sin(\omega t) + a\omega^2\mu(n\pi)^2 cos(\omega t)}{\omega^2 + \mu^2(n\pi)^4} \cdot sin(n\pi x) \end{aligned} \tag{3-21}$$

One can add the solution of the homogeneous equations which arbitrary but fixed coefficients to find another particular solution. Finally, the general solution of the non-homogeneous equation (3-7) is given by

$$u_g = u_p + u_h = \sum_{n=1}^{\infty} \frac{[2 - 2(-1)^n]}{n\pi} \cdot \frac{a\omega^3 \, sin(\omega t) + a\omega^2 \mu (n\pi)^2 \, cos(\omega t)}{\omega^2 + \mu^2 (n\pi)^4} \cdot sin(n\pi x) + \sum_{n=1}^{\infty} c_n e^{-\mu (n\pi)^2 t} \, sin(n\pi x). \tag{3-22}$$

3.3 Results analysis

One can see from the general solution (3-22) that when time t is sufficiently big u_h will be very small which can be ignored. The other part u_p is the main part and one can see that its period is $2\pi/\omega$, its amplitude depends on the parameters a, ω and μ. So, for large t the solution behaves as

$$u_g \sim \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \cdot \frac{a\omega^3 \, sin(\omega t) + a\omega^2 \mu ((2k-1)\pi)^2 \, cos(\omega t)}{\omega^2 + \mu^2 ((2k-1)\pi)^4} \cdot sin((2k-1)\pi x). \tag{3-23}$$

One can see that the coefficient for $k = 2$ is 27 times smaller than the one for $k = 1$. So, we will see merely the first term. Since, the coefficients decrease rapidly a good approximation for large t and $\omega \ll \mu\pi^2$ is

$$u_g \sim \frac{a\omega^2}{\mu} \cdot cos(\omega t) \cdot sin(\pi x) \tag{3-24}$$

which shows that the period of the velocity is proportional to $2\pi/\omega$ which is the same as the period of the forced oscillation and the amplitude is proportional to $a\omega^2/\mu$. This is an example of a forced oscillation, i.e. the period of the forcing function is returning in the solution. These characteristics will also be verified in numerical experiments.

4 Numerical results

In this section, the software COMSOL Multiphysics is used to solve the transformed Burgers equation (2-7), derived in Section 2.2, numerically. One of the methods that one can use to solve a PDE by COMSOL is entering the weak form of the PDE. The Weak Form PDE module in COMSOL is a very powerful tool with the finite element method as its core algorithm.

4.1 Model setup

The weak form of the transformed Burgers equation (2-7) can be obtained by multiplying both sides of the equation by a test function v and integrating over the domain. The expression is

$$0 = \int_{\Omega} (-u_t - uu_x + \mu u_{xx} + a\omega^2 \cos(\omega t))v dS. \tag{4-1}$$

From this, one can get the weak expression required in COMSOL software, which can be expressed in the syntax of COMSOL software as

(-ut-u*ux+mu*uxx +a*om*om*cos(om*t))*test(u), (4-2)

where test(u) represents the test function in COMSOL, the parameters a and ω are the shaker parameter data obtained according to the shaking mode of the shaker, and the parameter μ is the viscosity of the fluid. After determining the parameters of the equation and adding a Dirichlet boundary condition at both sides, one can solve it directly in COMSOL.

In the following experiments, the settings of the numerical method in COMSOL is not changed. The amount of the elements is 100 and the element shape order is quadratic. The number of degrees of freedom solved for is 201. The test space is the same as the search space which means that it is Galerkin approach. The time-dependent solver is BDF. The tolerance is set to 0.0001.

4.2 Effect of viscosity

Firstly, the influence of the viscosity is studied here. Without loss of generality, in this experiment one can set the length of the Petri dish, that is, the length of the calculation domain to $L = 1$, the shaker oscillation amplitude to $a = 0.01$, and the angular frequency of the shaker oscillation to $\omega = 1$.Here, the research target is water and related biological experiment liquids, so the viscosity coefficient should not be too big. At the same time, in order to ensure the accuracy of the calculation results and reduce the pressure of computer numerical calculation, the viscosity coefficient should not be too small. As one can see from Figure 4-1, by setting the viscosity $\mu = 10$, one obtains a number of snapshots of the velocity in the fluid domain as shown in Figure 4-1 (left). Through the output function image and the animation in the software, we can see that the velocity curve is periodic in time. In order to study its periodicity, one can intercepted the velocity data of the point at $x = 0.5$. As shown in Figure 4-1 (right), the image clearly shows the periodicity.

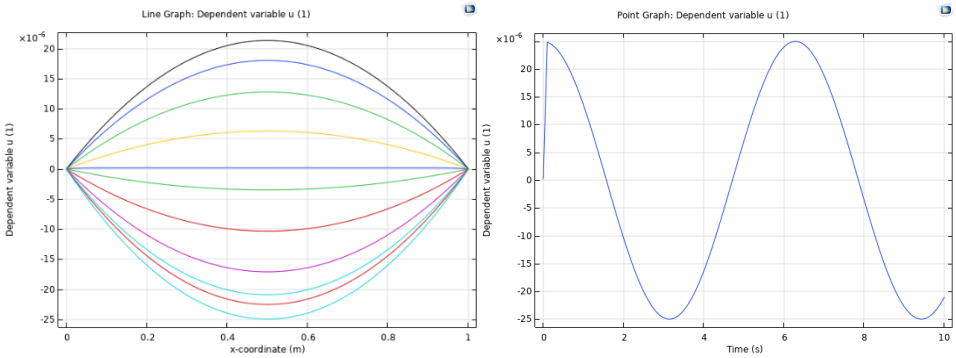

Figure 4-1: Snapshots of velocity at different moments (left) and time evolution plot of velocity at the centre point (right) when the viscosity is 10

As can be seen from Figure 4-1, the amplitude of the velocity at this point is about 0.0025, the period of the velocity curve is about 2π, and the velocity curve is smooth. From these data, one can learn that, although the Burgers equation used in the calculations contains a nonlinear term, its effect on the velocity curve is not large.

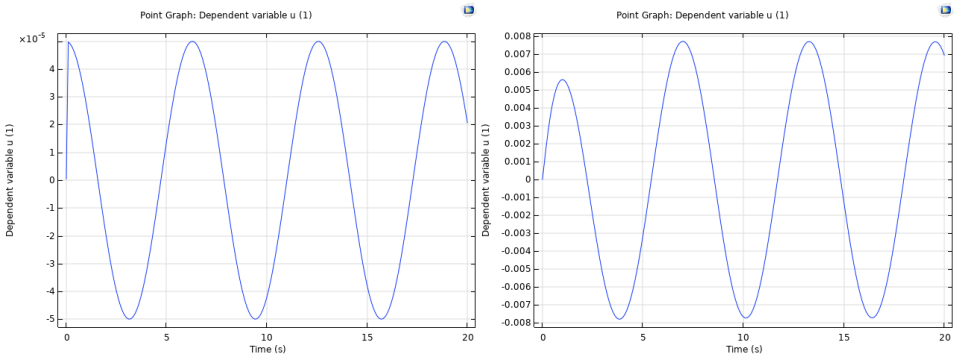

Figure 4-2: Time evolution plots of velocity with viscosities of 5 (left) and 0.025 (right)

Subsequently, the viscosity μ is gradually reduced, and the output velocity at $x = 0.5$ changes with time. The output results are shown in Figure 3-2. It can be seen from the figure that the period of the velocity curve has not changed, it is still 2π, but the time from the beginning of the calculation to the curve entering the periodical range becomes longer and longer as the viscosity decreases, and the shape of the curve also becomes more and more complicated. It is worth noting that as the viscosity value decreases, the amplitude of the velocity curve increases in proportion. The relation between viscosity and amplitude is shown in Table 4-1.

Table 4-1: Experimental data of viscosity and amplitude changes

Viscosity μ	Ratio	Reciprocal ratio	Amplitude	Ratio
10	-	-	0.000025	-
5	0.5	2	0.00005	2
1	0.2	5	0.00025	5
0.5	0.5	2	0.0005	2
0.1	0.2	5	0.0025	5
0.05	0.5	2	0.0047	1.88
0.025	0.5	2	0.0076	1.61

From the above experimental results, the lower the viscosity, the higher the amplitude of the velocity curve, and the relation between them can be approximately expressed as $A \sim 1/\mu$. When the viscosity becomes lower and lower, the velocity image will become more complicated, and the time required for the velocity curve to enter its periodic range will become longer and longer. This also shows the influence of the nonlinear term in the Burgers equation in the calculation is growing. And the time required for the speed curve to enter the periodic range is difficult to quantify, but from the image, the time required is roughly inverse proportional to the viscosity.

4.3 Effect of angular frequency

Here, the influence of the angular frequency is studied. One can follow the calculation in the previous part, fixed viscosity μ=0.5, and explore the effect on fluid velocity at x=0.5 by changing the angular frequency ω. Here, the experimental results are made for four different oscillation cycles, as shown in Figure 4-3. From top to bottom, from left to right, it is $\omega = 2, 1, 0.5, 0.1$, that is, shaker oscillation period is $T = \pi, 2\pi, 4\pi, 20\pi$ in turn.

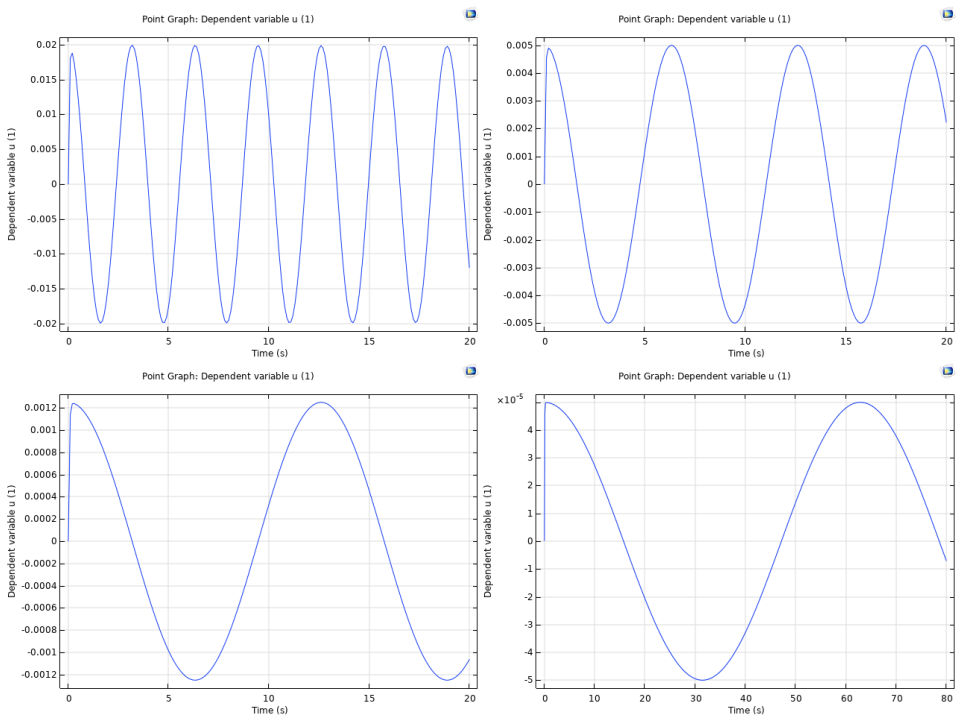

Figure 4-3: Time evolution plots of velocity with ω values in the order of 2, 1, 0.5, and 0.1

One can observe that the period of the velocity curve increases with the increase of the shaker oscillation period, while the amplitude decreases with the increase of the shaker oscillation period. Here, the data of the period and amplitude of the velocity curve changing with ω and oscillation period are shown in Table 4-2.

Table 4-2: Amplitude and period of function $u(0.5, t)$ against ω

Angular frequency ω	Square of ratio	Period of shaker movement	Ratio	Amplitude	Ratio	Period	Ratio
2	-	π	-	0.02	-	π	-
1	0.25	2π	2	0.005	0.25	2π	2
0.5	0.25	4π	2	0.0012	0.24	4π	2
0.1	0.04	20π	5	0.00005	0.042	20π	5

Combining the experimental results in the previous part, one can see that the amplitude A is also proportional to the square of the ω value to a certain extent, that is, $A \sim \omega^2/\mu$, which has an important guiding role in analytical solution to the Burgers equation. At the same time, we can also see that the longer the period, the smaller and smoother the velocity function curve amplitude. The experimental results are in line with the expectations, because in practice the smaller the angular frequency of the fluid oscillation the smaller the oscillation velocity, so the fluid oscillation will become relatively stable and slow.

5 Simulations

In Chapter 2, a basic description of the fluid model to be studied in this project has been made. In this chapter, researches and discussions on how to simulate and calculate the oscillating fluid in COMSOL will be conducted. Subsequently, the results of different experiments conducted for different purposes will be displayed and discussed.

5.1 Two-dimensional oscillating fluid modelling

In this project, the research object is the oscillating fluid. When the Petri dish is placed on the shaker, the fluid in the Petri dish will oscillate due to the interaction between the Petri dish wall and the fluid during the oscillation of the Petri dish. Here one can ignore the specific shape of the Petri dish and only consider the x-axis direction where the Petri dish is reciprocating and the y-axis direction where the height of the liquid is located, thereby forming a two-dimensional model.

5.1.1 Model setup

Here, the main parameters of the fluid model studied are that the height of the liquid surface is 0.02 meters, and the diameter of the culture dish is 0.1 meters. The fluids studied are Newtonian fluids. The model uses a multi-phase laminar flow model and does not consider its heat transfer process. Therefore, the main equations used in COMSOL is

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nabla\cdot[-p\boldsymbol{I} + \boldsymbol{K}] + \boldsymbol{F} + \rho\boldsymbol{g}, \qquad (5\text{-}1)$$
$$\rho\nabla\cdot\boldsymbol{u} = 0.$$

The sides and the bottom of the fluid are set as walls which cannot be penetrated or moved. The upper side of the fluid is set as a free surface and can move freely. In order to simulate reality, gravity will be set on the fluid. The remaining parameters of the fluid and shaker will be given in detail in the simulation experiments which will be set to different numbers for different reasons.

In COMSOL, according to the powerful meshing function of the program itself, the domain is freely divided into triangular meshes. The initial mesh of the solution domain is shown in Figure 5-1, which is physics-controlled mesh and the element size is coarse which includes 1754 solution domain elements and 140 boundary elements. The geometry shape order is also quadratic.

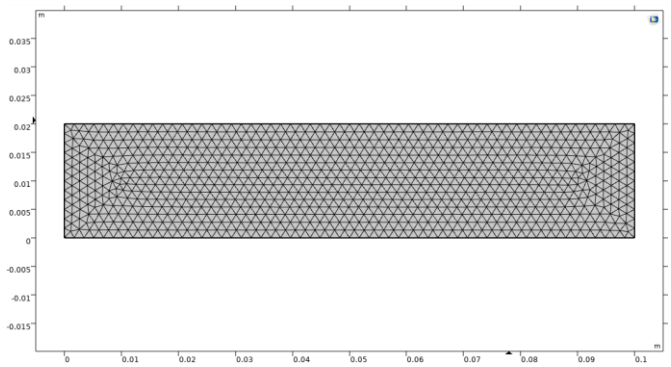

Figure 5-1: Initial meshing in COMSOL

Because the surface of the liquid is not constrained, the fluid domain is deformed greatly, and it is very irregular when the viscosity is low, so it is almost impossible to solve with the traditional fixed mesh. However, one can use the moving mesh technology to solve this problem.

5.1.2 Moving frame in COMSOL

After establishing the model in terms of geometry, one has to consider how to implement the oscillation of the fluid. In the initial model, the initial and boundary conditions of the fluid are default, and the default value is zero. Therefore, when no changes are made, the velocity field of the fluid will remain at zero. In order to achieve fluid oscillation, that is to say the reciprocating motion of the entire computing domain, we use the moving frame method introduced in Section 2.1. In order to make it possible to use the moving frame method in COMSOL software, we need to consider this method in combination with the modelling process and physical meaning of COMSOL software.

Considering the most general oscillation mode mentioned in Section 2.1, if the Petri dish placed on the shaker is regarded as a particle, its position on the x-axis is marked as $x(t)$ in a three-dimensional Cartesian coordinate system, which is a function of time. The expression can be written as $x(t) = cos\ (t)$, while the position of the particle on the other axis remains unchanged.

Here we have the coordinate transformation that

$$\begin{aligned} \tilde{x} &= x - acos(\omega t), \\ \tilde{y} &= y, \\ \tilde{t} &= t. \end{aligned} \tag{5-2}$$

And considering the physical meaning of the velocity field, which is mentioned in Section 2.2, one can get from (2-4) that

$$\tilde{u} = u + a\omega \sin(\omega \tilde{t}). \tag{5-3}$$

Furthermore, from (2-6) we can see the acceleration component on the x-axis is

$$\tilde{u}_{\tilde{t}} = u_t + a\omega^2 cos(\omega t). \tag{5-4}$$

The y component of velocity, does not change which is

$$\tilde{v} = v. \tag{5-5}$$

From the expressions above, one can see that this coordinate transformation only affects the component of each physical quantity on the x-axis that contains the term for the derivative of each order with respect to time, including displacement, velocity, and acceleration and so on. Then, to apply this coordinate transformation method, it is only necessary to change these affected terms in the differential equations used in the model. The easiest way to change all its equations is to change the acceleration component on the x-axis of all objects to the original formula plus a velocity source term which is

$$f = a\omega^2 cos(\omega \tilde{t}). \tag{5-6}$$

Here, the simplest and most direct way to apply this source term to all objects in COMSOL is to modify its gravity acceleration equation. Therefore, this velocity source term is added to the original static model and the acceleration of gravity is changed into

$$\tilde{g} = g + \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} a\omega^2 cos(\omega t) \\ -10 \end{pmatrix}. \tag{5-7}$$

In this way, one can successfully implement the original static fluid model into a model that reciprocates along the x-axis with time in COMSOL. The physical

quantities involved will be displayed as values in a relative coordinate system that is stationary relative to the Petri dish.

To consider the situation in three-dimensional space, assuming that the Petri dish is a rectangular parallelepiped, in an ideal situation, its fluid motion can be fully demonstrated by the two-dimensional model. Considering that the shape of the Petri dish is a cylinder, the behaviours of the fluid in the Petri dish will be plane symmetric about the z-axis and the x-axis of the motion axis, so the study of the three-dimensional model is not very necessary for this subject. Of course, according to the method used here, it is very easy to change the model to a three-dimensional model, but for the sake of reducing the runtime, only two-dimensional scenarios are simulated here.

Considering that the research objective of this subject is very specialized, one can expand the method. One can use this method to simulate any fluid oscillating physical scenario, that is, changing the gravity acceleration equation. For example, we often consider the behaviours of the fluid in the box during the tilting of the box, such as the tilting of the oil tank, water tank and test tube. In this situation, one finds a corresponding coordinate transformation method according to its tilting and shaking mode, and transform the original coordinate system into a relative coordinate system that shakes together with the cabinet. Then, add the corresponding difference term to the gravity acceleration equation of the model to complete the simulation of the scenario. In addition, there are many other application scenarios such as drum mixers and washing machines.

5.2 Results and analysis

In the previous section, the research method on this project and the actual simulation operation in COMSOL software have been introduced. According to the methods mentioned above, several simulation experiments, by changing the corresponding parameters of the model for several different purposes, are conducted. Here, some simulation experiments will be explained in detail, and the simulation results will be shown and analysed.

5.2.1 Velocity field

Firstly, one would like to study the velocity field of this oscillating fluid. The model building process is shown in the previous part. Here, one can set the fluid material to standard water, which is a low-viscosity fluid whose viscosity is $\mu = 0.001$ and will have unstable results when the oscillation frequency is too high. Therefore, the shaker shaking parameter is set to $\omega = 5$, which means the shaking period is $2\pi/5$ seconds, and the shaking amplitude is set to a=0.01, which means that the maximum deviation distance of the culture dish is 1 centimetre. As shown in Figure 5-2, the fluid has obvious undulations under the action of oscillation, which shows the velocity of the fluid at times $t = 0.2, 0.5, 0.8, 1.2s$ from top to bottom, from left to right. It can be seen that the undulation of the fluid is periodic to some extent. It should be noted that the velocity here is the one in the relative coordinate system.

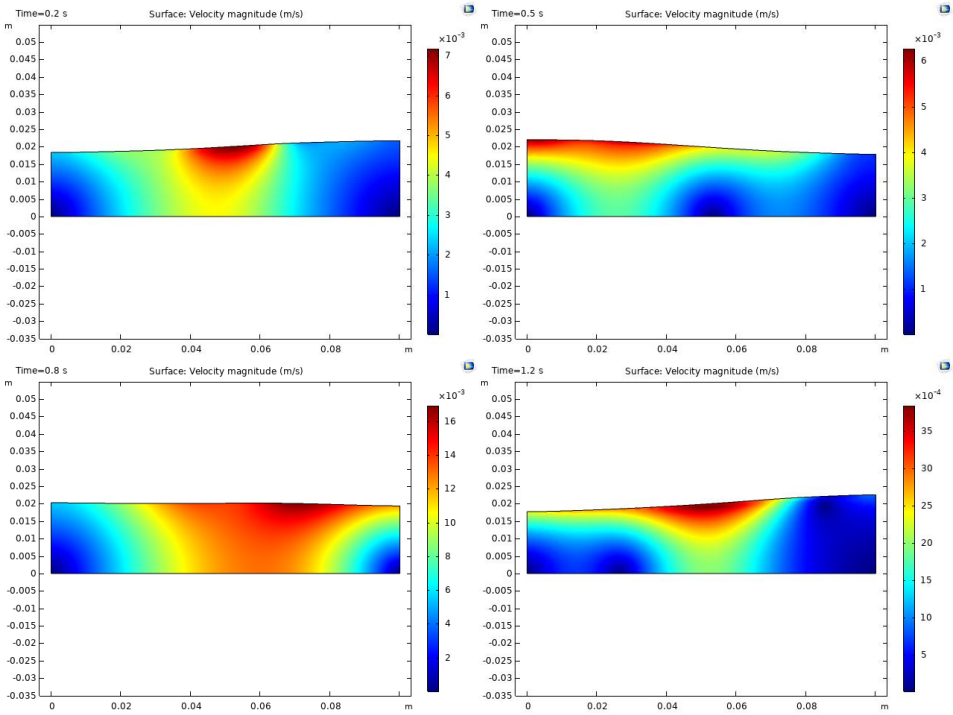

Figure 5-2: Images of fluid velocity distribution at time 0.2, 0.5, 0.8, 1.2 seconds

One finds that the place with the highest fluid surface height is the place with the fastest fluid velocity, which is a kind of water wave moving with time. One knows from the Section 5.1.2 that the accelerations due to the extra body force term is $a\omega^2 cos(\omega t) = 0.1351, -0.2003, -0.1634, 0.2400\ m/s^2$ at time $t = 0.2, 0.5, 0.8, 1.2\ s$, from top to bottom, from left to right in the Figure 5-2. The period of the acceleration due to the extra body force term is $2\pi/5 \approx 1.26s$, so one can notice that there is a certain degree of periodicity from 0.2 seconds to 1.2 seconds which is about 0.8 period. It is foreseeable that after a period of time, the fluid velocity field will enter a completely periodic state.

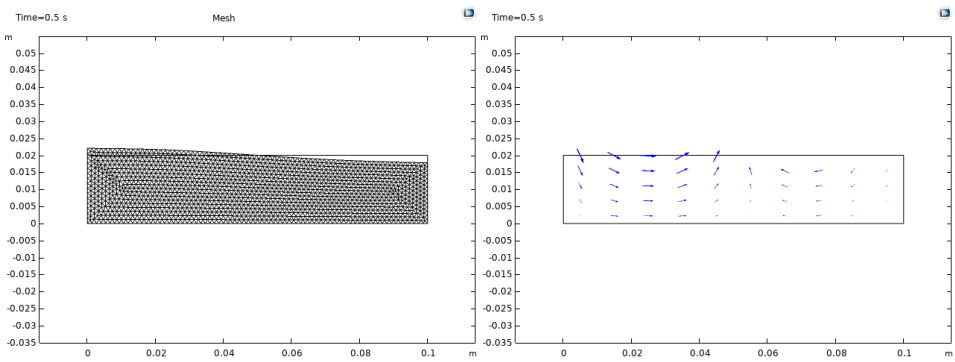

Figure 5-3: Moving grid at 0.5 seconds (left) and fluid velocity field image (right)

As shown in Figure 5-3, the left picture shows the instantaneous mesh division at a time of 0.5 seconds. The moving mesh technology provides a powerful help for simulating the free liquid surface motion of fluids. The figure on the right shows us the direction and the size of the velocity vector of the fluid at time 0.5 seconds. This

vector diagram combined with the velocity size distribution diagram of Figure 5-2 gives a good impression of the velocity field distribution of the fluid over time. One knows from Section 5.1.2 that the extra acceleration due to the extra body force term will reach the maximum value at time $t = 0.63\ s$. Therefore, before this, when the time $t = 0.5\ s$, the fluid will still tend to the left side of the Petri dish, and due to inertia and gravity, the fluid on the left side of the Petri dish will return after reaching its highest point. At $t = 0.5\ s$, a convective water wave appeared in the centre of the Petri dish to the left.

In order to be able to more clearly see the image of the relationship between the speed of the fluid and the time, one can intercept two data points in the centre of the fluid calculation domain and the centre of the right boundary of the calculation domain, and output the data for different viscosities and angular frequencies which shown in Figure 5-4 and Figure 5-5 images of velocity varying with time.

Here the upper blue curve is the cut point data of the centre point of the calculation domain which is $(0.05, 0.01)$, and the lower green curve is the cut point data of the centre point of the right boundary of the calculation domain which is $(0.1, 0.01)$.

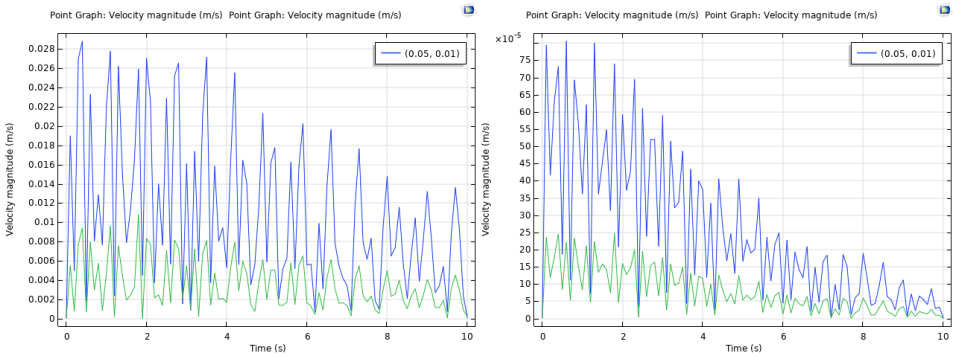

Figure 5-4: Time evolution plots of velocity with the viscosity μ=0.001 and the angular frequency ω=5 (left) and ω=1 (right)

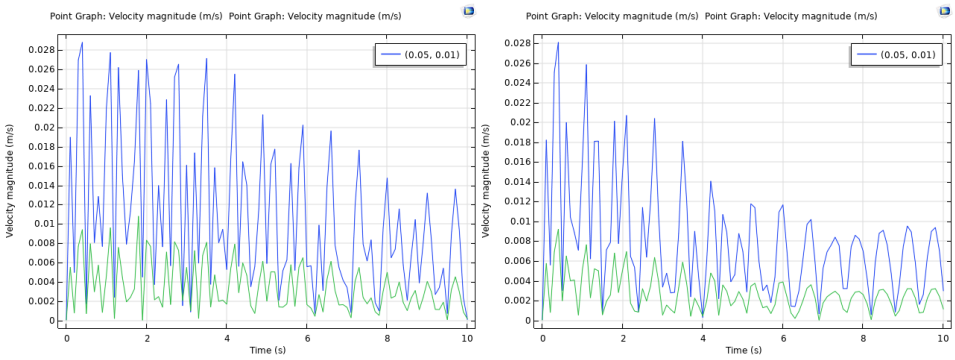

Figure 5-5: Time evolution plots of velocity with the angular frequency ω=5 and the viscosity μ=0.001 (left) and μ=0.1 (right)

One can see that the velocity of the data cut point on the right boundary is significantly smaller than the velocity of the data cut point at the centre of the calculation domain. This is because the fluid at the boundary cannot move along the x-axis direction because it is close to the inner wall of the culture dish. Therefore, its speed is only determined by the speed component in the y-axis direction, so the

overall speed is lower than the speed at the centre. One can see that clearly from Figure 5-6.

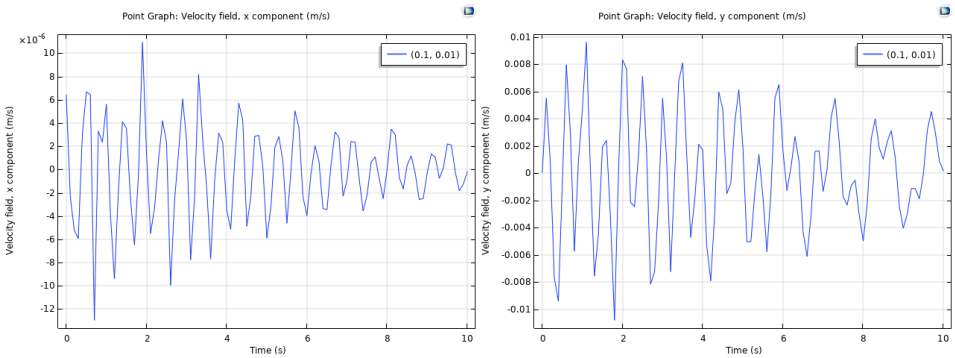

Figure 5-6: Time evolution plots of velocity in x-axis direction (left) and y-axis direction (right) at the centre cut point of the right boundary (0.1, 0.01) with ω=5 and μ=0.001

From Figures 5-4 and 5-5, one can see many patterns. First of all, if the frequency of shaking of the shaker is reduced by a factor 5, the velocity of the fluid will be greatly reduced. It can be seen that the highest peak is reduced from 0.028 to 0.0008, which is realistic. When the flow is relatively small, the movement of the fluid is very peaceful. However, due to the low viscosity of the fluid, considering the reflection of the water wave in the Petri dish, the velocity field change will be relatively irregular. Secondly, if the viscosity of the fluid is increased, it will make the entire oscillating process gentler and more regular. Its peak does not change, but after a period of time, the fluid oscillating movement becomes relatively regular, and its periodicity becomes more and more apparent.

5.2.2 Concentration distribution

Since the shaker is mainly used to mix, one would also like to see the concentration distribution in this oscillating fluid. Based on the previous model, the Transport of Diluted Species module is used here. In order to study the concentration distribution, a new geometry is built here. The small 0.01*0.01 square area in the upper middle is the initial position of the dilute solution, and its concentration is set here to 10 mol/m^3 as shown in Figure 5-7. In theory, the final concentration in the entire area will reach 0.5 mol/m^3.

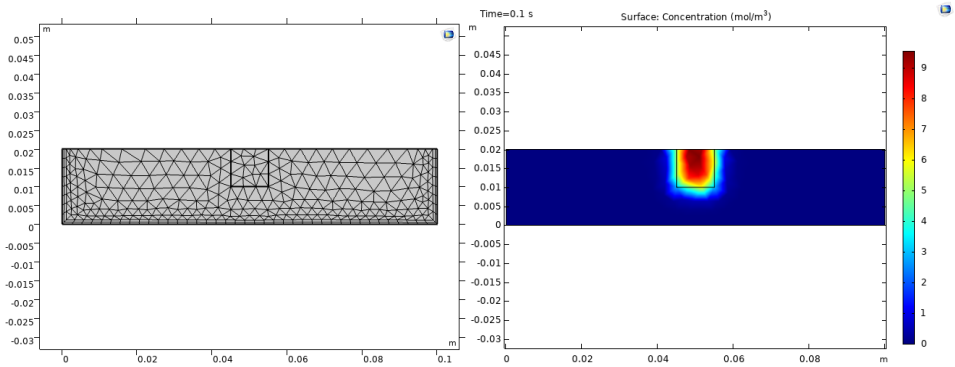

Figure 5-7: Initial mesh (left) and initial concentration distribution (right)

In order to see more clearly that the change of the concentration, one can set a cut point and output here the concentration. Here, the point $(0.02, 0.01)$ is chosen. As shown in Figure 5-8, the fluid is set as water whose density ρ is 1000 and viscosity μ is 0.001, the oscillation amplitude a is set to 0.01 and the diffusion coefficient D is set to 1×10^{-5}. For angular frequency is zero and five, one can get those two images.

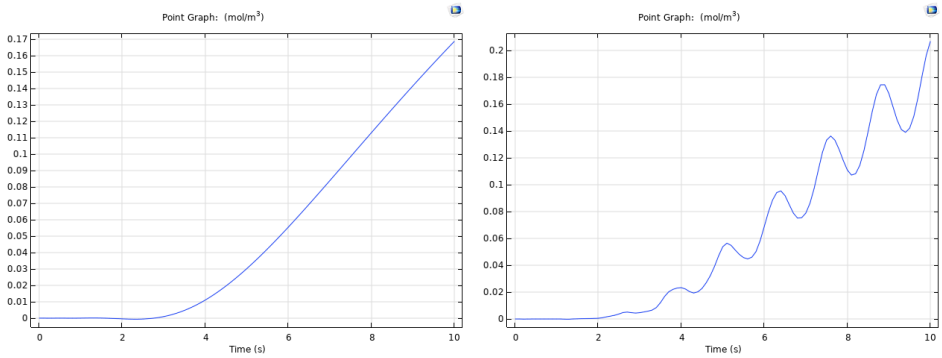

Figure 5-8: Concentration change at point (0.02,0.01) in the case where the shaker does not oscillate and only diffuses (left) and in the case where the shaker is oscillating with angular frequency ω=5 (right)

The diffusion process is affected by oscillation which is faster when the shaker is oscillating, but here the impact is not very obvious. In order to study how will different angular frequency affect the concentration contribution the angular frequency is set to 5 and 10 here with the smaller diffusion coefficient 1×10^{-9} which will reduce the impact of diffusion, as shown in Figure 5-9.

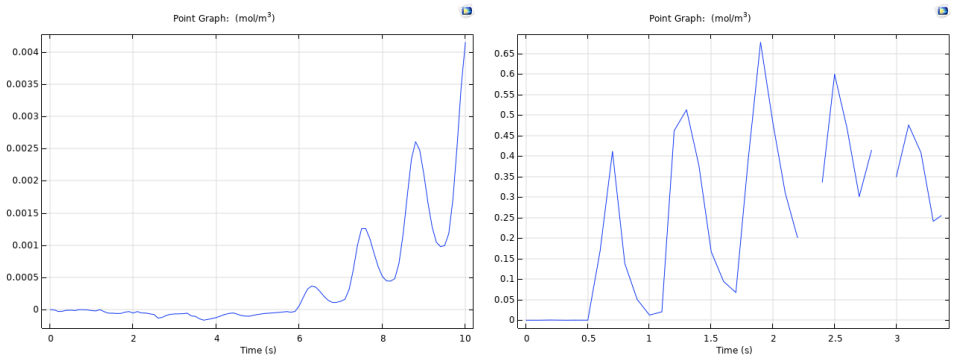

Figure 5-9: Concentration change at point (0.02,0.01) where the shaker is oscillating with angular frequency ω=5 (left) and ω=10 (right)

One can see from the Figure 5-9 that the concentration goes way faster with higher angular frequency. The point where the concentration curve disappears is that the fluid has left this point due to oscillation.

6 Discussion and Conclusions

At the beginning of the research, the moving frame method was studied, discussed and combined with the actual situation of this project. The moving frame method has been applied to the Burgers equation and the oscillating fluid model. In the subsequent COMSOL model establishment and analysis of experimental results, the numerical solution of the transformed Burgers equation was analysed in various aspects, and one of the analytical solutions was given for the linear case where the nonlinear term is omitted from the equation which fits well with the numerical results for relatively high viscosity. The simulation of two-dimensional oscillating fluid is also practical for studying the oscillating fluid behaviour.

The research of this project shows that the moving frame method of coordinate transformation is very effective for the simplification of a differential model whose boundary conditions change regularly. It is worth noting that the physical meaning of each symbol must be fully considered while performing moving frame coordinate transformation.

The experimental results show that the characteristics of the solution of the transformed Burgers equation. When the viscosity is not very low, the amplitude of the velocity of the oscillating fluid is proportional to the reciprocal of the viscosity and also the square of the ω value, which can be expressed as $A \sim \omega^2/\mu$. And its period is proportional to ω can be expressed as $T \sim 2\pi/\omega$. The experimental results also show that the fluid in the Petri dish on a periodically oscillating shaker is relatively disordered at the initial stage, and the degree of disorder is proportional to the reciprocal of viscosity. After a period of time, the fluid will enter a periodic movement.

Finally, the research of this project provides direction guidance and method innovation for the research of related topics, and there is still a lot of work to be done to quantify the more specific characteristics.

References

[1] Gledhill I. M. A., Roohani H, Forsberg K, Eliasson P, Skews B. W., Nordström J. Theoretical treatment of fluid flow for accelerating bodies[J]. Theoretical and Computational Fluid Dynamics, 2016, 30: 449–467.

[2] Li L, Sherwin S J, Bearman P W. A moving frame of reference algorithm for fluid/structure interaction of rotating and translating bodies[J]. International Journal for Numerical Methods in Fluids, 2002, 38(2):187-206.

[3] Bertini L, Cancrini N, Jona-Lasinio G. The stochastic Burgers equation[J]. Communications in Mathematical Physics, 1994, 165(2): 211-232.

[4] Wang W, Roberts A J. Diffusion approximation for self-similarity of stochastic advection in Burgers' equation[J]. Communications in Mathematical Physics, 2015, 333(3): 1287-1316.

[5] Haberman, Richard. Elementary Applied Partial Differential Equations[M]. Prentice-Hall, 1987.

[6] Wan D, Turek S. Fictitious boundary and moving mesh methods for the numerical simulation of rigid particulate flows[J]. Journal of Computational Physics, 2007, 222(1):28-56.

[7] Quarteroni A., Sacco R., Saleri F. Numerical Mathematics (Texts in Applied Mathematics)[M]. Springer-Verlag New York, Inc. 2007.