university of groningen

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Cycle-Complete Graph Ramsey Numbers

Abstract

Given two graphs G and H, the Ramsey number $R(G, H)$ is the smallest natural number N such that every red/blue coloring of the edges of the complete graph K_N contains a red copy of G or a blue copy of H. The proofs for the following results will be given. The upper and lower bounds of the symmetric Ramsey numbers are: $2^{\frac{k}{2}} < R(k, k) < 4^{k-1}$ for $k \geq 3$. The main result is the upper bound of $R(C_m, K_n)$: For all $m \geq 3$ and $n \geq 2$, the cycle-complete graph Ramsey number $R(C_m, K_n)$ satisfies $R(C_m, K_n) \leq \lceil (m-2)(n^{\frac{1}{k}} + 2) + 1 \rceil (n-1)$, where $k = \lfloor \frac{m-1}{2} \rfloor$. Specifically for cycles of order 4 there is an upper bound: $R(C_4, K_n) < c \left(\frac{n \log(\log(n))}{\log(n)} \right)^2$, $n \to \infty$. Finally, there is an exact result: $R(C_9, K_8) = 57$.

Bachelor's Project Mathematics

July 12, 2020

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Contents

1 Introduction

1.1 Introduction to graph theory

In order to be able to understand the contents of this thesis, a basic understanding of graph theory is required.

1.1.1 Graphs

Defining the basics of graph theory will be done by following the definitions of West [20].

Definition 1 *A graph $G(V, E)$ is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its endpoints.*

Figure 1 gives an example of a graph. When there is a graph, there is also a subgraph. It is defined as follows:

Definition 2 *A subgraph of a graph G is a graph H such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$ and the assignment of endpoints to edges in H is the same as in G.*

If H contains *all* edges of G that join two vertices in $V(H)$, then H is called the subgraph "induced" by $V(H)$. If W is a subset of $V(G)$, then the subgraph of G induced by W will be denoted $\langle W \rangle_G$, or $\langle W \rangle$ in case there is no confusion about the larger graph. Figure 2 gives an example of an induced subgraph from the graph in Figure 1. Throughout this whole thesis, the number of vertices is assumed to be finite and the endpoints of an edge are assumed to be distinct. This is the basic concept of a graph. A few more definitions regarding graphs are necessary in order to elaborate on Ramsey theory.

1.1.2 Definitions

The following definition tells something about the relation between two vertices.

Definition 3 *Two vertices are called adjacent when they are endpoints of an edge.*

For example, in Figure 1, vertex 1 and vertex 2 are adjacent, but vertex 2 and vertex 4 are not adjacent. The following definitions arise from the definition of adjacency.

Definition 4 *A complete graph is a graph in which every pair of vertices is adjacent.*

Definition 5 *In a graph, a set of pairwise adjacent vertices is called a clique.*

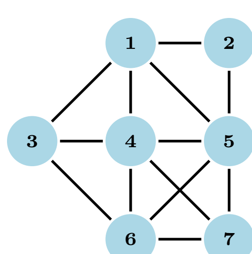

Figure 1: A graph

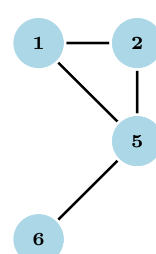

Figure 2: The subgraph induced by $\{1, 2, 5, 6\}$

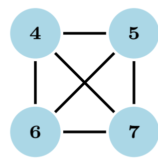

Figure 3: An induced subgraph of Figure 1 which is also a complete graph

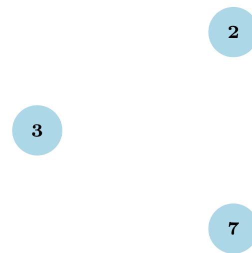

Figure 4: An induced subgraph of Figure 2 which is also an independent graph

Definition 6 *An independent set in a graph is a set of pairwise nonadjacent vertices.*

For example, in Figure 1 the vertices 4, 5, 6 and 7 form a clique and the vertices 2, 3 and 7 form an independent set. In fact, as can be seen in Figures 3 and 4, a clique or an independent set is an induced subgraph with a special property. The Figures 3 and 4 are subgraphs induced by the graph in Figure 1. Figure 3 is a clique from the graph in Figure 1 and Figure 4 is an independent set from the graph in Figure 1. The difference between a complete graph and a clique is that a clique is an induced subgraph that is complete and a complete graph is a property about the graph itself. Hence, in a complete graph, every induced subgraph is a clique. Sometimes it is necessary to know whether or how it is possible to "walk" on a graph from one vertex to the other. The next definition explains how to do that.

Definition 7 *A path in a graph is given by a sequence of distinct vertices such that two vertices are adjacent if and only if they are consecutive in the sequence.*

Definition 8 *A cycle is a path where the first vertex and the last vertex are adjacent.*

So for example, in Figure 1 there is a path from vertex 5 to vertex 3 by following the path (5,2,1,3) or (5,4,3) or (5,6,3). Moreover, there are a lot of cycles in this figure, for example, the cycle (2,1,3,6,5,2). For now, these will be the most important definitions of graph theory.

1.2 Coloring of graphs

A very important aspect of Ramsey Theory is the coloring of a graph. To be precise, the edges of a graph will be colored. This is defined as follows:

Definition 9 *A k-coloring of a graph $G(V, E)$ is a labeling $f : E(G) \to S$, where $|S| = k$.*

The labels of the function f are colors, so in words, the edges of a graph G are each given one color out of k. All edges of one color form a color class. In this thesis, only 2-colorings will be used in order to make things not too complicated. Figure 5 gives an example of a red/blue coloring of a graph with six vertices. After coloring a graph, one can look at specific properties that the color classes have. For instance, one can check whether the red color class contains any cycles or cliques.

1.3 Pigeonhole Principle

A result that will be used multiple times is the one of the pigeonhole principle. This is the pigeonhole principle as stated in [20]:

Theorem 1 *If m objects are partitioned into n classes, then some class has at least $\lceil \frac{m}{n} \rceil$ objects.*

Proof: Assume for a contradiction that each class has at most $\lceil \frac{m}{n} \rceil - 1$ objects. There are n classes, so the total number of objects is $n(\lceil \frac{m}{n} \rceil - 1) < n \cdot \frac{m}{n} = m$, but there were assumed to be m objects, so that is a contradiction. Therefore, it does not hold that each class has at most $\lceil \frac{m}{n} \rceil - 1$ objects, and so some class has at least $\lceil \frac{m}{n} \rceil$ objects. ■

The statement seems obvious, but it will be very useful in the upcoming proofs.

1.4 Probabilistic Method

A method that is considered as very powerful for solving problems in the field of discrete mathematics is the probabilistic method. Paul Erdős was the first one to use and completely master this method of proof. The method can be described in a very basic way as follows [1]: “Trying to prove that a structure with certain desired properties exists, one defines an appropriate probability space of structures and then shows that the desired properties hold in the space with positive probability.” In other words, if the probability that an element from a set satisfies a certain property is positive, then there must exist an element in the set that satisfies this property. Otherwise, if none of the elements satisfy the property, then the probability that such an element exists would be zero. In this thesis, the probabilistic method will be significant in the proofs of some theorems.

2 Theory

2.1 Notation

It is useful to have some agreement on the notation of different aspects. For the most part, the notation as used in [9] will also be used in this thesis. When X is a set, the cardinality of X is denoted by $|X|$. For example, if the graph in Figure 1 is called G, then $|V(G)| = 7$ and $|E(G)| = 13$. In a graph $G(V, E)$, $|V(G)|$ is called the order and $|E(G)|$ is called the size. A cycle of order n will be denoted C_n, a complete graph of order n will be denoted K_n and a path of order n will be denoted P_n.

The neighborhood of a set of vertices X is the set of all vertices of G that are adjacent to at least one vertex of X and this will be denoted $\Gamma_G(X)$, or $\Gamma(X)$. In case X consists of only one vertex v, the neighborhood of v will be denoted $\Gamma_G(v)$, or $\Gamma(v)$.

If x and y are two vertices of a graph G, the distance $d_G(x, y)$, or $d(x, y)$, denotes the length of the shortest path in G that connects x and y.

2.2 Ramsey Theory

In order to gain insight on Ramsey Theory, a nice example to start with is the "party problem". It helps to visualise how to approach the problems that will be dealt with.

Theorem 2 *In any party of six people either three of them mutually know each other or three of them mutually do not know each other [13].*

The proof of this theorem can be found in [13]. In fact, the initial question was:

Question 1 *How many people should attend a party in order to always have three people that know each other or three people that do not know each other?*

The answer that was found in Theorem 2 is thus: "Six". This question can be turned into a problem of graph theory by letting each person at the party represent a vertex. This gives us a graph with six vertices. Imagine a party with Alice, Bob, Carol, David, Eve and Frank. Let them represent the vertices A, B, C, D, E and F, respectively, in the graph in Figure 5. There is a connection between any pair of people; either they do know each other, indicated by a blue line, or they do not know each other, indicated by a red line. As expected, there are three people that mutually know each other or three people that mutually do not know each other. For example, in Figure 5, Alice, Bob and Eve know each other, but Alice, Carol and Frank do not know each other. Another way to put this, is to ask for a red clique of order 3 or a blue clique of order 3. Then the question would be:

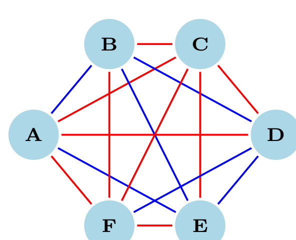

Figure 5

Question 2 *What is the minimal number of vertices n such that any 2-coloring of the edges of a complete graph of n vertices will give either a red clique of order 3 or a blue clique of order 3.*

There is yet another way to put this. Note that only one color can be assigned to one edge, so when there appears a red clique of order r in a 2-colored graph, then those same r vertices form an independent set in blue. Instead of looking for a red clique, one could also ask for a blue independent set. Now question 2 can be changed into:

Question 3 *What is the minimal number of vertices n such that any graph of n vertices has either a clique of order 3 or an independent set of order 3.*

Looking back to question 1, 2 and 3, they all have the same answer, namely: "Six". The latter formulation is the one that is used to formulate the definition of Ramsey numbers.

Definition 10 *The Ramsey number $R(n, m)$ is the smallest natural number N such that every graph of order N contains either a clique of order n or a set of m independent vertices [15].*

Definition 10 gives the definition of the classic Ramsey numbers. The example $R(3, 3) = 6$ has already been given and is supported by Figure 5. Other examples are $R(3, 4) = 9$ from [14] and $R(4, 5) = 25$ from [16]. Frank Ramsey showed in [18] that the Ramsey number $R(n, m)$ exists for every value of n and m. This was the 'go' for the search for Ramsey numbers. There is no general formula that gives the value of $R(n, m)$ or it has not been found yet. That means that each Ramsey number has to be found separately. The number of different graphs of N vertices is $2^{\frac{N(N-1)}{2}}$, so checking whether each of those graphs has a clique of order n or a set of m independent vertices is exhausting. This is one of the main reasons why the search for Ramsey numbers is so hard. For that reason, a lot of effort has been put into finding bounds for Ramsey numbers. In [7], Erdős proved one of the first upper and lower bounds.

Theorem 3 *Let $k \geq 3$. Then $2^{\frac{k}{2}} < R(k, k) < 4^{k-1}$.*

This theorem ensures that the Ramsey number being sought for is always finite. Furthermore, this lower bound shows that Ramsey numbers are increasing rapidly. Both bounds have not been improved much over the past 70 years. The best upper bound so far is due to Conlon [6] and the best lower bound is due to Spencer [19]. The results are $[1 + o(1)]\frac{\sqrt{2}}{e}k2^{\frac{k}{2}} \leq R(k, k) \leq (k-1)^{-C\frac{\log(k-1)}{\log(\log(k-1))}}\binom{2k-2}{k-1}$. From Theorem 3 it follows that the Ramsey number $R(n, m)$ is also finite, but then the upper bound is given by the maximum of 4^{n-1} and 4^{m-1}. In particular, if $m \leq n$, then $R(n, m) \leq R(n, n) < 4^{n-1}$, where the first inequality follows from [13]. Similarly, when $n \leq m$, $R(n, m)$ is finite. Now the proof of Theorem 3 will be given, however not exactly the same as in [7]. In the part on the lower bound, a modern variant of the proof is given which can be found in [12] and the part on the upper bound is inspired by a proof in [10].

Proof: First prove the first inequality: $2^{\frac{k}{2}} < R(k, k)$. Let $N \leq 2^{\frac{k}{2}}$. By letting the value of N be lower than the lower bound of $R(k, k)$, there will turn out to be a graph with N vertices that contains neither a clique of order k nor an independent set of order k. In a graph of N vertices, there are $\binom{N}{2}$ ways of choosing a pair of vertices, so N contains possibly $\binom{N}{2}$ edges. A set of $\binom{N}{2}$ elements has $2^{\binom{N}{2}}$ subsets, therefore the number of different graphs of N vertices is $2^{\binom{N}{2}}$.

The next part of the proof is following the proof as stated in [12] and it involves the probabilistic method. Let G be a graph of N vertices and assign to each edge a probability $p = \frac{1}{2}$

of being in $E(G)$. There are $\binom{N}{k}$ ways to choose k vertices out of N vertices. A graph with k vertices is complete if all possible $\binom{k}{2} = \frac{k(k-1)}{2}$ edges are in $E(G)$. Each edge has probability $\frac{1}{2}$ of being in $E(G)$, so a set of k vertices is complete with probability $2^{-\frac{k(k-1)}{2}}$. Similarly, a graph with k vertices is independent with the same probability, since in that case, all possible $\binom{k}{2} = \frac{k(k-1)}{2}$ edges are *not* in $E(G)$ with probability $2^{-\frac{k(k-1)}{2}}$. However, there were $\binom{N}{k}$ ways to choose k vertices out of N vertices, so G contains a complete graph (or independent set) of k vertices with probability $\binom{N}{k}2^{-\frac{k(k-1)}{2}}$. Note that

$$\begin{aligned}
\binom{N}{k}2^{-\frac{k(k-1)}{2}} &= \frac{N!}{k!(N-k)!}2^{-\frac{k(k-1)}{2}} \\
&= \frac{N(N-1)(N-2)\cdots(N-(k-1))}{k!}2^{-\frac{k(k-1)}{2}} \\
&< \frac{N\cdots N}{k!}2^{-\frac{k(k-1)}{2}} = \frac{N^k}{k!}2^{-\frac{k(k-1)}{2}} \\
\implies \binom{N}{k}2^{-\frac{k(k-1)}{2}} &< \frac{N^k}{k!}2^{-\frac{k(k-1)}{2}}.
\end{aligned} \tag{1}$$

Also, for $N \leq 2^{\frac{k}{2}}$ and $k \geq 3$ it holds that

$$\begin{aligned}
2N^k &\leq 2(2^{\frac{k}{2}})^k = 2^{\frac{k^2+2}{2}} \\
&= 2^{\frac{k^2-k+k+2}{2}} = 2^{\frac{k^2-k}{2}}2^{\frac{k+2}{2}} \\
&\leq 2^{\frac{k(k-1)}{2}}2^{\lceil\frac{k+2}{2}\rceil} \\
&= 2^{\frac{k(k-1)}{2}} \cdot 2\cdots 2 \\
&\leq 2^{\frac{k(k-1)}{2}} \cdot k \cdot (k-1) \cdot (k-2) \cdots \lceil\frac{k+2}{2}\rceil \\
&\leq 2^{\frac{k(k-1)}{2}} \cdot k! \\
\implies \frac{N^k}{k!} &< \frac{1}{2} \cdot 2^{\frac{k(k-1)}{2}}.
\end{aligned} \tag{2}$$

So in the end, the following is obtained by combining the results in equations (1) and (2):

$$\binom{N}{k}2^{-\frac{k(k-1)}{2}} < \frac{N^k}{k!}2^{-\frac{k(k-1)}{2}} < \frac{1}{2}.$$

This means that the probability that G contains a complete graph or independent set of order k is less than 1. Thus, the probability that G does *not* contain a complete graph or independent set of order k is positive. Therefore, when $N \leq 2^{\frac{k}{2}}$, there must exist a graph G of order N such that it contains neither a clique of order k nor a set of k independent vertices and hence $2^{\frac{k}{2}} < R(k,k)$. This proves the lower bound.

For the upper bound, a result in [8] is used. The result in [8] states that $R(k,k) \leq \binom{2k-2}{k-1}$ for $k \geq 3$. The proof that is given in [8] is very involved and covers a lot of aspects that are not covered in this thesis. Therefore, it is not suitable to present the full proof of this theorem in the same way as in [8]. However, an alternative proof will be given which was inspired by lecture notes of Fox [10]. First, show that $R(n,m) \leq R(n-1,m)+R(n,m-1)$ for all $n,m \geq 1$. To that end, let $p = R(n-1,m)+R(n,m-1)$. The goal is to show that a graph of order p always contains either a clique of order n or an independent set of m vertices. Take a vertex $v \in V(G)$. There are two cases to consider.

1. v is adjacent to at least $R(n-1, m)$ vertices. Then those $R(n-1, m)$ vertices form either a clique of order $n-1$ or an independent set of order m. In case it contains a clique of order $n-1$, vertex v can be added to form a clique of order n, since v is adjacent to all $n-1$ vertices. Hence, in either case there is a clique of order n or an independent set of m vertices.

2. v is not adjacent to at least $R(n, m-1)$ vertices. Then those $R(n, m-1)$ vertices form either a clique of order n or an independent set of $m-1$ vertices. In the latter case, the vertex v and the $m-1$ independent vertices form an independent set of m vertices, since v is not adjacent to all $m-1$ vertices. Hence, in either case there is a clique of order n or an independent set of m vertices.

It follows that $R(n, m) \leq R(n-1, m) + R(n, m-1)$ for all $n, m \geq 1$. Now induction will be used to prove that $R(n, m) \leq \binom{n+m-2}{n-1}$ for all $n, m \geq 1$. Note that $R(n, 1) = R(1, n) = 1$ for all $n \geq 1$, since a graph of one vertex always contains an independent set of one vertex or a complete graph of one vertex. Furthermore, note that $\binom{x}{0} = \binom{x}{x} = 1$ for all x [11]. For $m = 1$ and $n = 1$, it holds that $R(1, 1) = 1 \leq 1 = \binom{0}{0}$. Suppose that $R(n, 1) \leq \binom{n-1}{n-1}$ for some n and show that $R(n+1, 1) \leq \binom{n}{n}$. Then $R(n+1, 1) = 1 \leq 1 = \binom{n}{n}$. Now suppose that $R(1, m) \leq \binom{m-1}{0}$ for some m and show that $R(1, m+1) \leq \binom{m}{0}$. As before, it always hols that $R(1, m+1) = 1$, hence $R(1, m+1) = 1 \leq 1 = \binom{m}{0}$. It has been shown now that $R(n, 1) \leq \binom{n-1}{n-1}$ for all n and $R(1, m) \leq \binom{m}{0}$ for all m. It remains to show that when $R(n, m-1) \leq \binom{n+m-3}{n-1}$ and $R(n-1, m) \leq \binom{n+m-3}{n-2}$ hold for some n and m, it follows that $R(n, m) \leq \binom{n+m-2}{n-1}$. It holds that

$$R(n, m) \leq R(n-1, m) + R(n, m-1) \leq \binom{n+m-3}{n-2} + \binom{n+m-3}{n-1} = \binom{n+m-2}{n-1},$$

where the last equality is a binomial identity which can be found in [11]. This shows that $R(n, m) \leq \binom{n+m-2}{n-1}$ for all $n, m \geq 1$. Consider the case where $n = m = k$, then $R(k, k) \leq \binom{2k-2}{k-1}$ as desired. The only thing that needs to be shown is that $\binom{2k-2}{k-1} < 4^{k-1}$ for $k \geq 3$. Note that $\binom{2k-2}{k-1}$ gives the number of subsets with $k-1$ vertices and that it is less than the total number of subsets which is $2^{2k-2} = 4^{k-1}$. This shows easily that $\binom{2k-2}{k-1} < 4^{k-1}$ for $k \geq 3$.

Finally, the following inequalities are obtained for $k \geq 3$:

$$R(k, k) \leq \binom{2k-2}{k-1} < 4^{k-1},$$

hence the desired upper bound is found. ■

Looking back to the format of Question 2, the Ramsey number $R(n, m)$ can be seen as the smallest number N such that any red/blue coloring of the edges of a complete graph will either contain a red clique of order n or a blue clique of order m. The notation $R(K_n, K_m)$ would hence also be justified. The complete graph K_n contains any graph G of order $\leq n$ and the complete graph K_m contains any graph H of order $\leq m$. So instead of looking for a red K_n or a blue K_m, one could also look for a red copy of G or a blue copy of H. This is the definition of the graph Ramsey number [15]:

Definition 11 *Given two graphs G and H, the Ramsey number $R(G, H)$ is the smallest natural number N such that every red/blue coloring of the edges of the complete graph K_N contains a red copy of G or a blue copy of H.*

With this definition, the Ramsey numbers of any two graphs can be considered. This thesis focuses on the cycle-complete graph Ramsey numbers $R(C_m, K_n)$. This is the smallest integer N such that every red/blue coloring of K_N contains a red cycle of order m or a blue clique of order n. The definition of the cycle-complete graph Ramsey number will be used in the following form:

Definition 12 *The Ramsey number $R(C_m, K_n)$ is the smallest positive integer N such that every graph of order N contains either a cycle of order m or a set of n independent vertices.*

2.3 An upper bound for $R(C_m, K_n)$

The first upper bound for the cycle-complete graph Ramsey number was

$$R(C_m, K_n) \leq mn^2$$

and was found in [4]. Later, Erdős et al. [9] gave an improvement of the results by proving that for all $m \geq 3$ and $n \geq 2$, the cycle-complete graph Ramsey number $R(C_m, K_n)$ satisfies $R(C_m, K_n) \leq \lceil (m-2)(n^{\frac{1}{k}} + 2) + 1 \rceil (n-1)$, where $k = \lfloor \frac{m-1}{2} \rfloor$. In this section, an elaborate proof of this upper bound will be presented. This will be done by means of some definitions and a lemma after which the proof of the theorem will follow.

Before anything can be proven, a definition has to be introduced. It states a property that is crucial for completing the upcoming proofs.

Definition 13 *Let ℓ be a natural number. A graph G has property Π_ℓ if, for every independent set X, $|\Gamma(X)| \geq \ell|X|$.*

In other words, if a graph G has property Π_ℓ then the neighborhood of every independent set X contains more elements than ℓ times the number of elements of the set X. Erdős et al. [9] used this property in the following lemma:

Lemma 1 *Let $G(V, E)$ be a graph of order at least $(l + 1)(n - 1)$ which contains no set of n independent vertices. Then G contains an induced subgraph $\langle W \rangle$ which has property Π_ℓ.*

In other words, let $G(V, E)$ be a graph with at least $(l+1)(n-1)$ vertices such that there are no n vertices that form an independent set. In that case, there is an induced subgraph $\langle W \rangle$ which has property Π_ℓ. This lemma will be proven by using contradiction following the proof as in [9].

Proof In order to arrive at a contradiction, assume that there is no induced subgraph of $G(V, E)$ that has property Π_ℓ. If $W \subset V$ is any subset, then $\langle W \rangle$ is an induced subgraph of G. Since none of the induced subgraphs of G have property Π_ℓ, there exists an independent set $X \subset W$ such that $|\Gamma_{\langle W \rangle}(X)| < \ell|X|$.

Now, some sets will be defined. Start with the initial graph and vertex set, so define $G_1 = G$ and $W_1 = V$. The sets W_{i+1} and G_{i+1} will be defined recursively for $i = 1, 2, \ldots.$ Define $G_{i+1} = \langle W_{i+1} \rangle$ for $i = 1, 2, ...$, where W_{i+1} still has to be determined. The set W_{i+1} will be of the form $W_{i+1} = W_i \setminus B_i$ for some set of vertices B_i. Note that for every i, $\langle W_i \rangle$ is an induced subgraph of G. Hence, for every i there exists an independent set X_i such that $X_i \subset W_i$ and

$$|\Gamma_{G_i}(X_i)| < \ell|X_i|. \tag{3}$$

Define $B_i := X_i \cup \Gamma_{G_i}(X_i)$ where X_i is such an independent set. In other words, B_i consists of every vertex in X_i together with every vertex in G_i that is adjacent to a vertex in X_i.

Summarized, $W_{i+1} = W_i \setminus (X_i \cup \Gamma_{G_i}(X_i))$ and $G_{i+1} = \langle W_{i+1} \rangle$. In the "worst" case, X_i is chosen to consist of only one node, because a set consisting of one node is always independent. Therefore, $|X_i| \geq 1$ for $i = 1, 2, ...$, so $|W_i|$ decreases strictly in every step. Furthermore, G is a finite graph, so eventually there are no vertices left in W_i. That is, there exists a positive integer M such that $W_{M+1} = \emptyset$. With this in mind, something can be said about the relation between V and the sets B_i. Namely,

$$
\begin{aligned}
W_{M+1} &= W_M \setminus B_M \\
&= (W_{M-1} \setminus B_{M-1}) \setminus B_M \\
&= W_{M-1} \setminus (B_{M-1} \cup B_M) \\
&= ... = W_1 \setminus \cup_{i=1}^{M} B_i \\
&= V \setminus \cup_{i=1}^{M} B_i = \emptyset.
\end{aligned}
$$

From the latter and the fact that $B_i \subset V$ for all i it follows that

$$V = \bigcup_{i=1}^{M} B_i. \tag{4}$$

In every step, W_{i+1} is obtained by subtracting X_i and its neighborhood from W_i. Recall that $X_{i+1} \subset W_{i+1}$. That means that X_{i+1} and X_i are disjoint sets of vertices; they have no vertices in common. The sets X_i are independent and disjoint for all i, hence

$$X = \bigcup_{i=1}^{M} X_i \tag{5}$$

is an independent set in $G(V, E)$. The set X_i is disjoint from its neighborhood $\Gamma_{G_i}(X_i)$ because the neighborhood of X_i only contains vertices that are outside X_i. As a result, it holds that $|X_i \cup \Gamma_{G_i}(X_i)| = |X_i| + |\Gamma_{G_i}(X_i)|$. Recall equation (3), which gave a restriction to the way that X_i was chosen. Adding the term $|X_i|$ to both sides in equation 3 results in the following equations:

$$
\begin{aligned}
&|X_i| + |\Gamma_{G_i}(X_i)| < |X_i| + \ell |X_i| \\
\implies &|X_i \cup \Gamma_{G_i}(X_i)| < (\ell + 1)|X_i| \\
\implies &|B_i| < (\ell + 1)|X_i|
\end{aligned}
$$

for $i = 1, 2, ...$ Taking the union over the i's on both sides and using the fact that the sets X_i and B_i are disjoint for all i together with equations (4) and (5) gives

$$|V| < (\ell + 1)|X|. \tag{6}$$

By assumption, G contains no set of n independent vertices, so it should hold that $|X| \leq n - 1$, but then G does not have at least $(l + 1)(n - 1)$ vertices. Therefore, equation (6) contradicts the assumption that there is no induced subgraph that has property Π_ℓ. Therefore, if G is a graph of order at least $(l + 1)(n - 1)$ which contains no set of n independent vertices, then G contains an induced subgraph which has property Π_ℓ. ■

Now that this lemma has been proven, there are two more definitions that need to be given.

Definition 14 *A graph G is connected if it has a path from u to v for every $u, v \in V(G)$.*

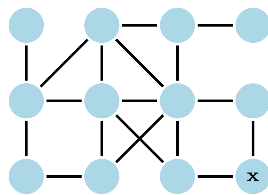

Figure 6: A graph...

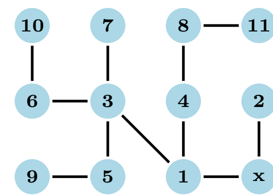

Figure 7: ...and a spanning tree

Definition 15 *A spanning tree is a subgraph of G with vertex set $V(G)$ that is connected and contains no cycles.*

A spanning tree consists thus of the same vertices as G, but perhaps not every edge in G so that the spanning tree contains no cycles. An example of a graph and a spanning tree can be found in Figures 6 and 7. These are all the requisites to give the proof of the following Theorem from [9].

Theorem 4 *For all $m \geq 3$ and $n \geq 2$, the cycle-complete graph Ramsey number $R(C_m, K_n)$ satisfies $R(C_m, K_n) \leq \lceil (m-2)(n^{\frac{1}{k}} + 2) + 1 \rceil (n-1)$, where $k = \lfloor \frac{m-1}{2} \rfloor$.*

The proof given below follows the proof given by Erdős et al [9].

Proof Assume that $G(V, E)$ is a graph of order $(\ell + 1)(n-1)$ that contains no cycle of order m and no set of n independent vertices. Then it is certain that $R(C_m, K_n) > (\ell + 1)(n-1)$. The aim of the proof is to show that if $\ell \geq \lceil (m-2)(n^{\frac{1}{k}} + 2) \rceil$, these assumptions about G lead to a contradiction. So by assuming that $R(C_m, K_n) > (\lceil (m-2)(n^{\frac{1}{k}} + 2) \rceil + 1)(n-1)$ and G containing no cycle of order m and no set of n independent vertices, a contradiction will be found and the conclusion will be that $R(C_m, K_n) \leq \lceil (m-2)(n^{\frac{1}{k}} + 2) + 1 \rceil (n-1)$.

By Lemma 1, the graph G contains an induced subgraph $H = \langle W \rangle$ which has property Π_ℓ. Then again, since H is a subgraph of G, H also contains no cycle of order m and no set of n independent vertices. From now on, graph G will not be considered anymore and only graph H and its properties will be investigated. Let x be an arbitrary vertex of H. Assume that H is connected. In case H were not connected, only the connected part that contains x would be considered.

Define the length of the shortest path in H that connects two vertices u and v to be the distance $d_H(u, v)$. Set $k = \lfloor \frac{(m-1)}{2} \rfloor$ and, for $i = 1, 2, ..., k$, define $A_i = \{v \in V \mid d_H(x, v) = i\}$. In other words, A_i represents the set of all vertices whose shortest distance to x is i. The set A_i will be referred to as the ith level.

Now a spanning tree T and a total ordering for each of the sets A_i will be constructed. This is done simultaneously by recursion. First, make an arbitrary ordering of the vertices of A_1. Then if the set A_i has already been ordered, the set A_{i+1} is ordered in the following way. Let v be an element in A_{i+1}. Create an edge in T by letting $v \in A_{i+1}$ be adjacent to the smallest vertex w in A_i such that v and w are adjacent in H. This means that each vertex in A_{i+1} is adjacent to only one vertex in A_i. After this procedure, every element in A_{i+1} is added to the spanning tree T. The vertices in A_{i+1} still need to be ordered. This is done in the following way. If vertices y and z in A_{i+1} are adjacent in T to vertices u and v in A_i, respectively, and if $u < v$, then $y < z$. Now there is a spanning tree T and an ordering for each of the sets A_i. Figures

6 and 7 give an example of a graph and a spanning tree of that graph which is obtained by following the aforementioned procedure. For the sake of clearness, in Figure 7, the levels that were obtained after labeling are $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, $A_3 = \{5, 6, 7, 8\}$ and $A_4 = \{9, 10, 11\}$.

Moving on, another definition has to be given. A sequence of vertices $v_1, v_2, ..., v_M$ in A_i that satisfies $v_1 < v_2 < \cdots < v_M$ will be called a monotonic sequence. If such a sequence forms a path $(v_1, v_2, ..., v_M)$ in $\langle A_i \rangle_H$, then it will be called a monotonic path. Since H contains no cycle of order m, there can be no monotonic path of order $m-1$. In order to show this, let $P = (v_1, ..., v_{m-1})$ be a monotonic path in $\langle A_i \rangle$ of order $m-1$ and let

$$d^* = \max_j d_T(v_j, v_{j+1}) = d_T(v_s, v_{s+1}).$$

In fact, it holds that $d_T(v_r, v_t) = d^*$ for all $r \leq s$ and $t \geq s+1$. Why is this true? The vertices in P all lie on the same level A_i so they are mutually not adjacent in T. In order to measure the distance $d_T(v_j, v_{j+1})$ between to vertices in the path P, one has to move from v_j along lower levels to reach v_{j+1}. Since v_s and v_{s+1} give the length of the maximal distance in T, it holds that $d_T(v_r, v_t) \leq d^*$ for all $r \leq s$ and $t \geq s+1$. Suppose that in order to walk from v_s to v_{s+1} in T, one has to move to a vertex in level k at the lowest, say to vertex v_k. At this point, define a left branch and a right branch. The left branch is the path from v_s to v_k and the right branch is the path from v_k to v_{s+1}. Those two branches together define the path that is taken to measure the distance $d_T(v_s, v_{s+1})$. The two branches do not have any vertices in common except for v_k, otherwise, there would have been a shorter path from v_s to v_{s+1}. Take $r \leq s$ and consider the distance from v_r to v_{s+1} in T. Since the vertices v_r and v_s in the path P are both in level i, it takes just as much steps to walk from v_r to v_k as it takes to walk from v_s to v_k. Since $r \leq s$, the vertex v_r cannot reach vertex v_{s+1} through a higher level than level k because of the ordering of the sets A_i. If the path from v_r would not have to go until the kth level to reach v_{s+1}, it had intercepted the right branch at a higher level than the kth level. However, that would mean that $r > s$ considering that each vertex in A_{i+1} is adjacent in T to the least element of A_i to which it is adjacent in H. Hence, in any case, the path from v_r has to go through v_k in order to reach v_{s+1} causing $d_T(v_r, v_{s+1}) = d^*$ for all $r \leq s$. The same argument holds when considering the distance from v_s to v_t for all $t \geq s+1$, meaning that $d_T(v_r, v_t) = d^*$ for all $r \leq s$ and $t \geq s+1$.

Moreover, whatever the value of d^* is, there exist vertices v_r and v_t for $r \leq s$ and $t \geq s+1$ such that the subpath $(v_r, v_{r+1}, ..., v_t)$ of P together with the path connecting v_r and v_t in T, forms a cycle of order m. Here is why. In total, there are k levels of vertices constructed. Recall that $k = \lfloor \frac{(m-1)}{2} \rfloor$. In case the monotonic path P is located at the kth level, the value of d^* is at most $2 \times \lfloor \frac{(m-1)}{2} \rfloor$. So in any case $d^* < m$. Now back to the arbitrary case where P is a monotonic path in $\langle A_i \rangle$. When taking the path $(v_1, ..., v_{m-1})$ together with the path connecting v_1 and v_{m-1} in T, there will be a cycle of order at least m. The order of a subpath $(v_r, v_{r+1}, ..., v_t)$ together with the path connecting v_r and v_t in T is a cycle of order $t - r + d^*$. Since the value of d^* is not changing when the value of r is being decreased or the value of t is being increased, the order of the cycle can be increased or decreased by one integer at the time. So when $t - r + d^* > m$, just decrease t and/or increase r until the cycle has order m and remove the corresponding vertices. When $t - r + d^* < m$, increase t and/or decrease r until the cycle has order m and select the corresponding vertices. However, by assumption, H contains no cycle of length m, so $\langle A_i \rangle$ contains no monotonic path of order $m-1$.

The next thing that will be proven is that $\langle A_i \rangle$ contains an independent set of at least $\lceil \frac{|A_i|}{m-2} \rceil$ vertices. For this purpose, assign to each vertex v in $\langle A_i \rangle$ as a label the order of the longest monotonic path in $\langle A_i \rangle$ which has v as its least element, that is, starting at vertex v. If two vertices have the same label, they must be independent. Otherwise, when two vertices have the same label and are adjacent, one of the two paths could be extended by the least vertex of the two, but then the labels are not the same anymore. There is no monotonic path of order $m-1$, so the labels that can be assigned are the integers 1 to $m-2$. An application of the pigeonhole principle yields that at least $\lceil \frac{|A_i|}{m-2} \rceil$ vertices have the same label and these are necessarily independent.

Define B_i to be a maximal independent subset of A_i for $i = 1, 2, ..., k$ and let $r_i = \frac{|B_i|}{|B_{i-1}|}$ with $|B_0| = 1$. Since H has property Π_ℓ and B_i is independent, it holds that $|\Gamma(B_i)| \geq \ell |B_i|$ for $i = 1, 2, ..., k$. Furthermore, B_i is adjacent in H to vertices in A_i or to vertices in A_{i+1} or A_{i-1}. That is, $\Gamma(B_i) \subseteq A_{i-1} \cup A_i \cup A_{i+1}$. Since $\lceil \frac{|A_i|}{m-2} \rceil$ was the least order of an independent subset in $\langle A_i \rangle$, it follows that $|B_i| \geq \lceil \frac{|A_i|}{m-2} \rceil$ for $i = 1, 2, ..., k$. Note that $|B_i| \geq \lceil \frac{|A_i|}{m-2} \rceil \geq \frac{|A_i|}{m-2}$ and thus $(m-2)|B_i| \geq |A_i|$. Then the following is obtained:

$$\begin{aligned} \ell |B_i| &\leq |\Gamma(B_i)| \\ &\leq |A_{i-1} \cup A_i \cup A_{i+1}| \\ &= |A_{i-1}| + |A_i| + |A_{i+1}| \\ &\leq (m-2)|B_{i-1}| + (m-2)|B_i| + (m-2)|B_{i+1}| \\ &\leq (m-2)(|B_{i-1}| + |B_i| + |B_{i+1}|). \end{aligned}$$

Rewriting this in terms of r_i gives:

$$\begin{aligned} &(m-2)(|B_{i-1}| + |B_i| + |B_{i+1}|) \geq \ell |B_i| \\ \implies &|B_{i-1}| + |B_i| + |B_{i+1}| \geq \frac{\ell}{(m-2)} |B_i| \\ \implies &\frac{|B_{i-1}|}{|B_i|} + \frac{|B_i|}{|B_i|} + \frac{|B_{i+1}|}{|B_i|} \geq \frac{\ell}{m-2} \\ \implies &\frac{1}{r_i} + 1 + r_{i+1} \geq \frac{\ell}{m-2} \\ \implies &r_{i+1} \geq \frac{\ell}{m-2} - 1 - \frac{1}{r_i}. \end{aligned} \tag{7}$$

The aim of the proof was to show that if $\ell \geq \lceil (m-2)(n^{\frac{1}{k}} + 2) \rceil$, the assumptions about G lead to a contradiction. If $\ell = \lceil (m-2)(n^{\frac{1}{k}} + 2) \rceil$, then equation (7) becomes

$$r_{i+1} \geq n^{\frac{1}{k}} + 1 - \frac{1}{r_i}, \quad i = 1, 2, ..., k-1.$$

By the following induction argument it will follow that $r_i > n^{\frac{1}{k}}$:
Base step: The vertex x can be seen as an independent set in H. Since H has property Π_ℓ, it holds that $|A_1| = |\Gamma(x)| \geq \ell |x| = \ell$. Then it follows that $r_1 = \frac{|B_1|}{|B_0|} = |B_1| \geq \lceil \frac{A_1}{m-2} \rceil \geq \lceil \frac{\ell}{m-2} \rceil = \lceil \frac{(m-2)(n^{\frac{1}{k}}+2)}{m-2} \rceil > n^{\frac{1}{k}} \implies r_1 > n^{\frac{1}{k}}$.
Induction hypothesis: Suppose that $r_i > n^{\frac{1}{k}}$ for some $i = 1, ..., k-1$.

Induction step: Show that $r_{i+1} > n^{\frac{1}{k}}$. $r_{i+1} \geq n^{\frac{1}{k}} + 1 - \frac{1}{r_i} > n^{\frac{1}{k}} + 1 - \frac{1}{n^{\frac{1}{k}}} > n^{\frac{1}{k}}$.
So by induction, $r_i > n^{\frac{1}{k}}$ for $i = 1, 2, ..., k$. Then it follows that

$$|B_k| = |\frac{B_1}{B_0}||\frac{B_2}{B_1}|\dots|\frac{B_{k-1}}{B_{k-2}}||\frac{B_k}{B_{k-1}}| = r_1 r_2 \dots r_k > n. \tag{8}$$

Equation (8) tells that $|B_k|$ is an independent subgraph of H with more than n vertices. However, this contradicts the assumption that H contains no set of n independent vertices. Therefore, if G is a graph of order $\lceil (m-2)(n^{\frac{1}{k}} + 2) + 1 \rceil (n-1)$, then it will always contain a cycle of order m or a set of n independent vertices, that is;

$$R(C_m, K_n) \leq \lceil (m-2)(n^{\frac{1}{k}} + 2) + 1 \rceil (n-1). \qquad \blacksquare$$

2.4 The special case of $R(C_4, K_n)$

At the time that [9] was written, in 1978, not much was known about the specific case of $R(C_4, K_n)$. However, Spencer and Erdős found an asymptotic bound for $R(C_4, K_n)$ which was proven in [9]. It is a stronger statement than the one in Theorem 4 for $m = 4$.

Theorem 5 $R(C_4, K_n) < c \left(\frac{n \log(\log(n))}{\log(n)} \right)^2, \qquad n \to \infty.$

The proof of this theorem is very long, so in order to keep oversight, a lemma will be proven which will be used in the proof of Theorem 5.

Lemma 2 *The function $f(k) = kn^{\frac{1}{k}}$ is decreasing as long as $k < \log\, n$.*

Proof Compute the derivative of this function:

$$\frac{d}{dk} f(k) = \frac{d}{dk} kn^{\frac{1}{k}}$$

$$= k \frac{d}{dk} n^{\frac{1}{k}} + n^{\frac{1}{k}} \frac{d}{dk} k$$

$$= k \frac{d}{dk} e^{\log(n^{\frac{1}{k}})} + n^{\frac{1}{k}}$$

$$= ke^{\log(n^{\frac{1}{k}})} \frac{d}{dk} \log(n^{\frac{1}{k}}) + n^{\frac{1}{k}}$$

$$= kn^{\frac{1}{k}} \frac{d}{dk} \frac{1}{k} \log(n) + n^{\frac{1}{k}}$$

$$= -kn^{\frac{1}{k}} \frac{1}{k^2} \log(n) + n^{\frac{1}{k}}$$

$$= -\frac{n^{\frac{1}{k}}}{k} \log(n) + n^{\frac{1}{k}}$$

$$= n^{\frac{1}{k}} (-\frac{1}{k} \log(n) + 1).$$

Summarized, $\frac{d}{dk} f(k) = n^{\frac{1}{k}} (-\frac{1}{k} \log(n) + 1)$. The function $f(k)$ is decreasing when $\frac{d}{dk} f(k) < 0$, so when $n^{\frac{1}{k}} (-\frac{1}{k} \log(n) + 1) < 0$. Compute for which values of k this holds:

$$n^{\frac{1}{k}} (-\frac{1}{k} \log(n) + 1) < 0$$

$$\implies -\frac{1}{k}\log(n) + 1 < 0$$
$$\implies 1 < \frac{1}{k}\log(n)$$
$$\implies k < \log(n).$$

Indeed, the function $f(k) = kn^{\frac{1}{k}}$ is decreasing as long as $k < \log\ n$. ■

Now the proof of Theorem 5 follows.

Proof Let $G(V, E)$ be a graph of order $R(C_4, K_{n+1}) - 1$ that contains no cycle of order 4 and no set of $n + 1$ independent vertices. Let v be a vertex that is isolated from G, that is, v is not adjacent to any vertex in G. Then $\langle V(G) \cup \{v\}\rangle_G$ forms a graph of order $R(C_4, K_{n+1})$, so it contains a set of $n + 1$ independent vertices. That means that G, which does not contain the vertex v, contains a set S of n independent vertices.

Let $T = V \setminus S$ and define $R(X) = \Gamma(X) \cap S$, for every $X \subset T$. For $k = 0, 1, ..., n$, define $T_k = \{x \mid x \in T,\ |R(x)| = k\}$ and let $N_k = |T_k|$. Since S is not part of a larger independent set, it follows that $N_0 = 0$. This becomes clear by the following argument. Take a vertex $v \in T_0$, then v is not adjacent to any vertex in S. Hence $\{v\} \cup S$ is a larger set of $n + 1$ independent vertex, but this is contradicting the assumption. Therefore, $T_0 = \emptyset$ and $N_0 = 0$. Also, $N_1 \leq 2n$ because of the following. Assume for a contradiction that $N_1 \geq 2n + 1$. Since S contains n vertices, it follows by applying the pigeonhole principle that there are three vertices in T which are adjacent to the same vertex s in S. If two of these three vertices, say a and b, are not adjacent to each other, then G has a set of $n + 1$ independent vertices by removing s from S and adding the vertices a and b to S. Otherwise, when none of the three vertices are independent of each other, they form a cycle of order 4 with the vertex s. Therefore, $N_1 \leq 2n$. Note that when both vertices a and b in T are adjacent to both s and u in S, then G would contain a cycle of order 4 by following the path (b, u, a, s, b). Hence it is not possible to have two vertices in T that are adjacent to a common pair of vertices in S.

Take a vertex $t \in T_k$, then vertex t is adjacent to k vertices in S. From those k vertices, $\binom{k}{2}$ pairs of vertices can be made. As already mentioned before, if a vertex in T is adjacent to a pair of vertices $u, s \in S$, then another vertex cannot be adjacent to both vertices of the pair u, s, for then G would contain a cycle of order 4. That means that a vertex $t \in T_k$, takes "possession" of $\binom{k}{2}$ pairs of vertices. As a consequence, every vertex in $\bigcup_{k=m}^{n} T_k$ accounts for at least $\binom{m}{2}$ pairs of vertices in S. The set S contains n vertices, so there are in total $\binom{n}{2}$ pairs of vertices in S. Putting everything together; there are $\binom{n}{2}$ pairs of vertices to possess in S and one vertex in $\bigcup_{k=m}^{n} T_k$ possesses at least $\binom{m}{2}$ pairs of vertices of S, hence the number of vertices in $\bigcup_{k=m}^{n} T_k$ is at most $\binom{n}{2}/\binom{m}{2}$. In formulas;

$$\sum_{k=m}^{n} N_k \leq \frac{\binom{n}{2}}{\binom{m}{2}} = \frac{n(n-1)}{m(m-1)}.$$

Keeping this in mind, something can be said about the order of the graph. Recall that the order of the graph G is $R(C_4, K_{n+1}) - 1$. The following is obtained:

$$R(C_4, K_{n+1}) - 1 = |V|$$

$$
\begin{aligned}
&= |S| + |T| \\
&= n + |\bigcup_{k=1}^{n} T_k| \\
&= n + |T_1| + |\bigcup_{k=2}^{m} T_k| + |\bigcup_{k=m+1}^{n} T_k| \\
&\leq n + 2n + |\bigcup_{k=2}^{m} T_k| + \frac{n(n-1)}{m(m+1)} \\
&= 3n + \sum_{k=2}^{m} N_k + \frac{n(n-1)}{m(m+1)},
\end{aligned}
$$

and hence

$$
R(C_4, K_n) < R(C_4, K_{n+1}) \leq 1 + 3n + \sum_{k=2}^{m} N_k + \frac{n(n-1)}{m(m+1)}. \tag{9}
$$

Only a bound for $\sum_{k=2}^{m} N_k$ still has to be realized. This is done by proving that if N_k is too large, then there must exist a set $A \subseteq S$ and an independent set $C \subseteq \langle T_k \rangle$ such that $R(C) \subseteq A$ and $|C| > |A|$. If this were true, then G would contain a set of at least $n+1$ independent vertices by removing A from S and adding C to it.

Let $x, y \in T$, then, again, x and y cannot be adjacent to a common pair of vertices in S. Therefore $|R(x) \cap R(y)|$ is either 0 or 1. Accordingly, each edge xy in $\langle T_k \rangle$ is classified as either type 0 or type 1. Let $M_{k,0}$ denote the number of type 0 edges in $\langle T_k \rangle$ and let $M_{k,1}$ denote the number of type 1 edges in $\langle T_k \rangle$. Moreover, let

$$
M_k = M_{k,0} + M_{k,1}. \tag{10}
$$

Let x be an arbitrary vertex in T_k and suppose that x is an endpoint of each of the following edges in $\langle T_k \rangle$: $\{xy_1\}$,...,$\{xy_\ell\}$. For $i = 1, ..., \ell$, the sets $R(y_i)$ are disjoint. Otherwise, when there is a vertex $s \in S$ such that $s \in R(y_i)$ and $s \in R(y_j)$ for some i, j, there appears a cycle of order 4 by following the path (x, y_i, s, y_j, x). Thus, there are ℓ edges that have x as an endpoint in $\langle T_k \rangle$ and each of those edges are adjacent to k vertices in S. That means that the vertex x accounts for $k\ell$ vertices in S that need to be mutually distinct, therefore $k\ell \leq n$. Suppose now that the edges $\{x, y_1\}$,...,$\{x, y_m\}$ are of type 1, that is, $|R(x) \cap R(y_i)| = 1$ for $i = 1, ..., m$. The vertices $R(x) \cap R(y_i)$ are distinct for $i = 1, ..., m$. Otherwise, when there is a vertex $s \in S$ such that $R(x) \cap R(y_i) = s = R(x) \cap R(y_j)$ for some i, j, there appears a cycle of order 4 by following the path (x, y_i, s, y_j, x). Since $x \in T_k$, x is adjacent to k vertices in S, so to make sure that no two vertices y_i and y_j are adjacent to the same vertex as x, the number of type 1 edges cannot be more than k, hence $m \leq k$. Summarized, there are two bounds; $\ell \leq \frac{n}{k}$ and $m \leq k$. From those two bounds, the following two bounds on the numbers of edges are obtained:

$$
M_k \leq N_k \frac{n}{2k} \tag{11}
$$

$$
M_{k,1} \leq N_k \frac{k}{2}, \tag{12}
$$

respectively.

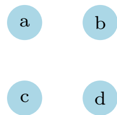

Figure 8

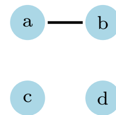

Figure 9

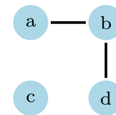

Figure 10

It still has to be shown that there exists a set $A \subseteq S$ and an independent set $C \subseteq \langle T_k \rangle$ such that $R(C) \subseteq A$ and $|C| > |A|$. This is done with the help of the probabilistic method. It will be shown that such A and C exist unless $N_k < \frac{5n^2}{kn^{1/k}}$. Let Ω denote the sample space consisting of all subsets of S. Assign the probability $P(A) = p^{|A|}(1-p)^{n-|A|}$ to each $A \subseteq S$ with the value of p to be chosen later. Equivalently, each vertex in S has independent probability p of belonging to A. Corresponding to each $A \subseteq S$, define $B = \{x \mid x \in T_k,\ R(x) \subseteq A\}$ and let C denote a maximal independent subset of B. Introduce the random variables $X_A = |A|$ and $X_C = |C|$. There are n vertices that could possibly belong to A and each vertex has probability p of belonging to A, therefore the expected value of X_A is $\mathbb{E}(X_A) = np$. The value of $|C|$ is a bit harder to find. However, a lower bound for $|C|$ can be found to narrow down the possibilities. For every edge that appears in $\langle B \rangle$, the size of the independent set C decreases with at most 1 vertex. This is explained with the help of Figures 8, 9 and 10. In Figure 8, the maximal independent subset is the set itself, so it is of order 4. When, in Figure 9, one edge is added to the graph, the maximal independent subset is decreased to order 3. When, in Figure 10, a second edge is added to the graph, the maximal independent subset is still of order 3, so it did not decrease. Therefore, for every edge in a graph, the order of the maximal independent subset is decreased with at most one. The set C is a subset of B, so a lower bound for $|C|$ is obtained by subtracting the size of $\langle B \rangle$ from $|B|$. In formulas;

$$|C| \geq |B| - |E(\langle B \rangle_G)|. \tag{13}$$

The expected value of C is then also bounded from below by the expected value of the lower bound for C.

For $x \in B$ it holds that $x \in T_k$ and $R(x) \subseteq A$. In other words, x is adjacent to k different vertices in A, say $s_1, ..., s_k \in A$. Each s_i had probability p of belonging to A, so all of the s_i together belong to A with probability p^k. Therefore, x belongs to B with probability p^k. There are N_k vertices that could possibly belong to B and each vertex has probability p^k of belonging to B, so the expected value of $|B|$ is $N_k p^k$. Moving on, the set $\langle B \rangle$ is a subset of $\langle T_k \rangle$, so it contains possibly $M_k = M_{k,0} + M_{k,1}$ edges. First consider the $M_{k,0}$ edges of $\langle T_k \rangle$ which are of type 0. An edge xy, for $x, y \in \langle T_k \rangle$, is of type 0 when $|R(x) \cap R(y)|$ is 0. Suppose that x is adjacent to $s_1, ..., s_k \in S$ and y is adjacent to $u_1, ..., u_k \in S$. Since xy is of type 0, the s_i and the u_j must be distinct for each $i, j = 1, ..., k$, that is; the s_i and u_j together account for $2k$ distinct vertices in S. The vertices x and y belong to B if each s_i and u_j belong to A. Every vertex in S has probability p of belonging to A, hence the s_i and u_j, which account for $2k$ vertices, together belong to S with probability p^{2k}. Altogether, the expected value of an edge of type 0 in B is $M_{k,0}p^{2k}$. Secondly, consider the $M_{k,1}$ edges of $\langle T_k \rangle$ that are of type 1. An edge xy, for $x, y \in \langle T_k \rangle$, is of type 1 when $|R(x) \cap R(y)| = 1$. Suppose that x is adjacent to $s_1, ..., s_k \in S$ and y is adjacent to $u_1, ..., u_k \in S$. Since xy is of type 1, the s_i and the u_j must be distinct for each $i, j = 1, ..., k$, *except* for two vertices; one of the s_i is the same as one of the u_j. That is, the s_i and u_j together account for $2k - 1$ distinct vertices in S. Again, the vertices x and y belong to B if each s_i and u_j belong to A. However, in this case, the vertices s_i and u_j together account for $2k - 1$ vertices in S, hence they all belong to A with probability p^{2k-1}. Therefore, the expected value of an edge of type 1 in B is $M_{k,1}p^{2k-1}$. The total number of edges in $\langle B\rangle$ is possibly $M_k = M_{k,0} + M_{k,1}$, so the expected value of the size of $\langle B\rangle$ is $M_{k,0}p^{2k} + M_{k,1}p^{2k-1}$. Taking the expectation on both sides of equation (13), gives the following lower bound for the expected value of $|C|$:

$$\mathbb{E}(X_C) \geq N_k p^k - \left(M_{k,0}p^{2k} + M_{k,1}p^{2k-1}\right). \tag{14}$$

Using the bounds given in equations (11) and (12), together with the fact that $p^{2k-1} - p^{2k} \geq 0$ this lower bound can be rewritten in the following way:

$$\begin{aligned}
\mathbb{E}(X_C) &\geq N_k p^k - \left(M_{k,0}p^{2k} + M_{k,1}p^{2k-1}\right) \\
&\overset{(10)}{=} N_k p^k - \left((M_k - M_{k,1})p^{2k} + M_{k,1}p^{2k-1}\right) \\
&= N_k p^k - M_k p^{2k} + M_{k,1}p^{2k} - M_{k,1}p^{2k-1} \\
&= N_k p^k - M_k p^{2k} - M_{k,1}(p^{2k-1} - p^{2k}) \\
&\overset{(11),(12)}{\geq} N_k p^k - N_k \frac{np^{2k}}{2k} - N_k \frac{k(p^{2k-1} - p^{2k})}{2} \\
&N_k \left(p^k - \frac{np^{2k}}{2k} - \frac{k(p^{2k-1} - p^{2k})}{2}\right).
\end{aligned}$$

This results in the following lower bound:

$$\mathbb{E}(X_C) \geq N_k \left(p^k - \frac{np^{2k}}{2k} - \frac{k(p^{2k-1} - p^{2k})}{2}\right). \tag{15}$$

Suppose that $p \geq \frac{k^2}{n+k^2}$, then it follows that $\frac{k(p^{2k-1}-p^{2k})}{2} \leq \frac{np^{2k}}{2k}$ by the following calculations:

$$\begin{aligned}
\frac{k}{2}(p^{2k-1} - p^{2k}) &= \frac{k}{2}p^{2k}(\frac{1}{p} - 1) \\
&\leq \frac{k}{2}p^{2k}\left(\frac{n+k^2}{k^2} - 1\right) \\
&= \frac{k}{2}p^{2k}\frac{n}{k^2} \\
&= \frac{n}{2k}p^{2k}.
\end{aligned}$$

Indeed, $\frac{k(p^{2k-1}-p^{2k})}{2} \leq \frac{np^{2k}}{2k}$. By keeping this restriction on the value of p, equation (15) will change into

$$\mathbb{E}(X_C) \geq N_k \left(p^k - \frac{np^{2k}}{k}\right).$$

Finally, set $p = (\frac{k}{2n})^{\frac{1}{k}}$. For all $k \geq 2$ and $n \geq 1$ it holds that $(\frac{k}{2n})^{\frac{1}{k}} \geq \frac{k^2}{n+k^2}$, so this choice of p respects the previously made restriction.

When filling in the value of p into the formulas for $\mathbb{E}(X_C)$ and $\mathbb{E}(X_A)$, the following comes out:

$$\mathbb{E}(X_C) \geq \frac{kN_k}{4n} \quad \text{and} \quad \mathbb{E}(X_A) = n(\frac{k}{2n})^{\frac{1}{k}}.$$

If $\mathbb{E}(X_C) > \mathbb{E}(X_A)$, then G certainly contains a set of at least $n+1$ independent vertices. This must not be the case, therefore holds that

$$\frac{kN_k}{4n} \leq n(\frac{k}{2n})^{\frac{1}{k}}$$

and hence

$$N_k \leq \frac{4n^2}{k}(\frac{k}{2n})^{\frac{1}{k}}. \tag{16}$$

This upper bound can be simplified by noting that $(\frac{k}{2})^{\frac{1}{k}} < \frac{5}{4}$ for all $k \geq 1$. This is calculated as follows: Consider $(\frac{k}{2})^{\frac{1}{k}}$ as a function of k and let $f(k) = (\frac{k}{2})^{\frac{1}{k}}$. Compute the derivative of the function $f(k)$:

$$\begin{aligned}
\frac{d}{dk}f(k) &= \frac{d}{dk}(\frac{k}{2})^{\frac{1}{k}} \\
&= \frac{d}{dk}e^{\log\left((\frac{k}{2})^{\frac{1}{k}}\right)} \\
&= e^{\log\left((\frac{k}{2})^{\frac{1}{k}}\right)} \frac{d}{dk}\log\left((\frac{k}{2})^{\frac{1}{k}}\right) \\
&= (\frac{k}{2})^{\frac{1}{k}} \frac{d}{dk}\frac{1}{k}\log(\frac{k}{2}) \\
&= (\frac{k}{2})^{\frac{1}{k}} \left(\frac{1}{k}\frac{d}{dk}\log(\frac{k}{2}) + \log(\frac{k}{2})\frac{d}{dk}\frac{1}{k}\right) \\
&= (\frac{k}{2})^{\frac{1}{k}} \left(\frac{1}{k^2} - \log(\frac{k}{2})\frac{1}{k^2}\right) \\
&= \frac{1}{k^2}(\frac{k}{2})^{\frac{1}{k}} \left(1 - \log(\frac{k}{2})\right).
\end{aligned}$$

The function attains its extrema when $\frac{d}{dk}f(k) = 0$. Note that $\frac{1}{k^2} \neq 0$ and $(\frac{k}{2})^{\frac{1}{k}} \neq 0$ for all $k \geq 1$ and compute for which values of k the extrema are attained:

$$\begin{aligned}
&\frac{d}{dk}f(k) = 0 \\
\implies &\frac{1}{k^2}(\frac{k}{2})^{\frac{1}{k}} \left(1 - \log(\frac{k}{2})\right) = 0 \\
\implies &\left(1 - \log(\frac{k}{2})\right) = 0 \\
\implies &\log(\frac{k}{2}) = 1 \\
\implies &\frac{k}{2} = e
\end{aligned}$$

$$\implies k = 2e.$$

So the extremum of $f(k)$ occurs at $k = 2e$ and its value is $f(2e) = 1.2019 < \frac{5}{4}$. When looking at the graph of $f(k)$ in Figure 20 in the Appendix, one can see that this is indeed a maximum and hence $(\frac{k}{2})^{\frac{1}{k}} < \frac{5}{4}$ for all $k \geq 1$. This changes equation (16) into

$$N_k < \frac{5n^2}{kn^{\frac{1}{k}}}. \tag{17}$$

Zoom in on the denominator of this upper bound of N_k. Consider $kn^{\frac{1}{k}}$ as a function of k with n fixed, that is; let $f(k) = kn^{\frac{1}{k}}$ be a function. As already showed in Lemma 2, the function $f(k)$ is decreasing as long as $k < \log\ n$. Let $m < k < \log\ n$, then $mn^{\frac{1}{m}} > kn^{\frac{1}{k}}$, from which it follows that $\frac{5n^2}{mn^{\frac{1}{m}}} < \frac{5n^2}{kn^{\frac{1}{k}}}$ and hence $N_m < N_k$. This means that for $m < \log\ n$ it holds that

$$\begin{aligned}\sum_{k=2}^{m} N_m &= N_1 + N_2 + N_3 + ... + N_m \\ &< N_m + N_m + N_m + ... + N_m \\ &= mN_m = \frac{5n^2}{n^{\frac{1}{m}}}.\end{aligned}$$

After a lot of computations, a bound on $\sum_{k=2}^{m} N_m$ is finally found. Filling in this bound into equation (9) which was found earlier, the following bound is obtained:

$$R(C_4, K_n) < 1 + 3n + \frac{5n^2}{n^{\frac{1}{m}}} + \frac{n^2}{m^2},$$

where also the fact that $\frac{n(n-1)}{m(m+1)} < \frac{n^2}{m^2}$ is used. Finally, take $m \sim \frac{\log\ n}{2\log(\log\ n)}$, then $\frac{n^2}{m^2} \sim (\frac{2n\log(\log\ n)}{\log\ n})^2$, therefore the bound

$$R(C_4, K_n) < c(\frac{n\log(\log\ n)}{\log\ n}), \qquad n \to \infty$$

is found, as desired. ■

2.5 An exact result

At the end of their article [9], Erdős et al. conjectured that $R(C_m, K_n) = (m-1)(n-1)+1$ for all $m \geq n \geq 3$ except $m = n = 3$. This conjecture has still not been proven so far. It has only been verified for several cases for small values of n and m. In [2], this conjecture is verified for $m = 9$ and $n = 8$. In particular, the following theorem has been proven [2]:

Theorem 6 *$R(C_9, K_8) = 57$.*

In the remaining part of this thesis, this theorem will be proven following the proof as presented in [2]. They prove Theorem 6 by means of a sequence of 8 lemmas, so that will be done here as well. In their proof however, they used the result from two theorems in [3] and it is used in a lemma in [2] to show that certain induced subgraphs have to be complete. In order to give the proof of these two theorems, another definition is needed. First, recall the definition of connectedness: A graph G is connected if it has a path from u to v for every $u, v \in V(G)$. Then this is the definition of a component:

Definition 16 *Let G be a graph and let H be a subgraph of G such that H is connected and H is not contained in any connected subgraph of G which has more vertices or edges than H has. Then H is a component of G.*

Theorem 7 *Let G be a connected graph of order $n \geq 3$ such that for any two non-adjacent vertices x and y it holds that*

$$|\Gamma(x)| + |\Gamma(y)| \geq k.$$

If $k < n$, then G contains a path of order $k+1$. [3]

Proof: Let $P = (x_1, x_2, ..., x_\ell)$ be a longest path in G. Since this is the longest path, it should not be possible to extend it by some vertex in the neighborhood of x_1 or x_ℓ, hence the neighbors of x_1 and x_ℓ are vertices of P. If $\ell = n$, then P is a path of order n, so it contains a path of order $k+1$, since $k+1 \leq n$ and then the proof is done. Suppose that $\ell < n$, then there is no cycle of order ℓ. To show this, let C_ℓ be a cycle of order ℓ. Since $\ell < n$, there is a vertex $w \in V(G)$ that is not in C_ℓ. The graph G is connected, so there is a path from w to a vertex x in C_ℓ which has order ≥ 2. The path from w to x together with the $\ell - 1$ other vertices of C_ℓ form a path of order $> \ell$, but the longest path should be of order ℓ. Therefore, there is no cycle of order ℓ in G. For that reason, the vertices x_1 and x_ℓ in P are not adjacent. Furthermore, let $A = \{x_i \in V(P) \,|\, x_1 x_{i+1} \in E(P)\}$ and $B = \{x_i \in V(P) \,|\, x_i x_\ell \in E(P)\}$. If $x_i \in A \cap B$, there is a cycle $(x_1, x_2, ..., x_i, x_\ell, x_{\ell-1}, ..., x_{i+1}, x_1)$ which is of order ℓ. Consequently, the sets A and B are disjoint subsets of $\{x_1, ..., x_{\ell-1}\}$. Therefore,

$$k \leq |\Gamma(x_1)| + |\Gamma(x_\ell)| = |A| + |B| \leq \ell - 1.$$

Since G has a path of order ℓ and $k+1 \leq \ell$, G contains a path of order $k+1$. ■

Theorem 8 *Let G be a graph of order n without a path of order $k+1$, $(k \geq 1)$. Then*

$$|E(G)| \leq \frac{k-1}{2} n.$$

Further, equality holds if and only if its components are complete graphs of order k. [3]

Proof: Theorem 8 will be proven by applying induction on n and fixing the value of k. When $n \leq k$, then there will never be a path of order $k+1$ in G because G does not have enough vertices to make a path of order $k+1$. Then G can have as much edges a possible, that is, the maximal number of edges is attained when G is a complete graph. A complete graph of order n has $\binom{n}{2}$ edges. So $|E(G)| \leq \binom{n}{2} = \frac{(n-1)}{2} n \leq \frac{(k-1)}{2} n$ as desired.

Assume now that $n \geq k+1$ and that the assertion holds for smaller values of n. If G is disconnected, write $G = \bigcup_{i=1}^{r} G_i$ where G_i are the components of G. Then the order of each G_i is smaller than n, say the order of G_i is n_i for each i. Clearly, $\sum_{i=1}^{r} n_i = n$. Since the assertion holds for smaller values of n, it holds that $|E(G_i)| \leq \frac{k-1}{2} n_i$ for all i. Then

$$|E(G)| = \sum_{i=1}^{r} |E(G_i)| \leq \sum_{i=1}^{r} \frac{k-1}{2} n_i = \frac{k-1}{2} n$$

as desired. If G is connected, then it contains no K_{k+1} since it contains no path of order $k+1$. Then by Theorem 7, for any vertex x in G, $2 \cdot |\Gamma(x)| < k \implies |\Gamma(x)| < \frac{k}{2} \leq \frac{k-1}{2}$. Then $G \setminus x$ is a graph of order $n-1$, so the assertion holds for the graph $G \setminus x$. Then

$$|E(G)| \leq |\Gamma(x)| + |E(G \setminus x)| < \frac{k-1}{2} + \frac{k-1}{2}(n-1) = \frac{k-1}{2}. ■$$

Before going into the lemmas, a lower bound of $R(C_9, K_8)$ will be given. To that end, let G be a graph of order 56. In particular, let G be a graph that consists of 7 copies of a complete graph of 8 vertices as presented in Figure 11, then G indeed has $7 \cdot 8 = 56$ vertices. The question is whether this graph contains either a cycle of order 9 or an independent set of 8 vertices or not. In Figure 11, within one complete component, the order of the largest cycle is 8, since one complete component contains only 8 vertices. Obviously, there is no cycle touching two different complete components because their vertices are not adjacent. Hence, G contains no cycles of order 9. Within one complete component, all vertices are adjacent, so in order to find an independent set in G, only one vertex can be picked from each complete component. There are 7 complete components in G so the order of the largest independent set in G is 7, thus there is no independent set of 8 vertices. Consequently, $R(C_9, K_8) > 56$ or $R(C_9, K_8) \geq 57$.

The one definition that needs to be mentioned before being able to introduce the lemmas is the following one [17]:

Definition 17 *The degree of a vertex v in a graph G, $deg(v)$, is the cardinality of the neighborhood set $\Gamma(v)$, that is, it is equal to the number of vertices that are adjacent to vertex v in G.*

Then the minimum degree of a graph, $\delta(G)$, is the minimum degree of its vertices. In the upcoming proofs, a shorthand notation for a vertex and its neighborhood will be used. Define $\Gamma[u] := \Gamma(u) \cup u$.

Lemma 3 *Let G be a graph of order ≥ 57 that contains neither a cycle of order 9 nor an independent set of order 8. Then $\delta(G) \geq 8$.*

Proof Suppose for a contradiction that $\delta(G) < 8$. Then there is a vertex u in G with degree less than 8. In that case, $|\Gamma[u]| = |\Gamma(u) \cup \{u\}| < 8 + 1 = 9$, thus $|V(G \setminus \Gamma[u])| \geq 49$. In [5], the result $R(C_9, K_7) = 49$ is proved, so $G \setminus \Gamma[u]$ contains an independent set of 7 vertices. The vertices in the graph $G \setminus \Gamma[u]$ are not adjacent to u or its neighborhood, therefore the 7 vertices in $G \setminus \Gamma[u]$ that form an independent set and the vertex u together form an independent set of 8 elements. By assumption, G contains no independent set of 8 vertices, so this is a contradiction. Therefore, $\delta(G) \geq 8$. ■

From now, G will always be a graph with minimum degree $\delta(G) \geq 8$ that contains neither a cycle of order 9 nor an independent set of 8 vertices.

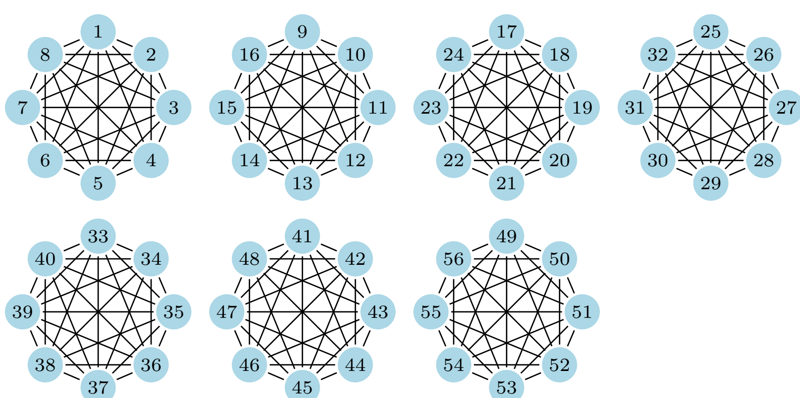

Figure 11: Graph of order 56

Lemma 4 *If G contains K_8, then $|V(G)| \geq 72$.*

Proof Let G be a graph that contains K_8 and let $U = \{u_1, ...u_8\}$ be the vertex set of the complete graph K_8 in G. Let $R = G \setminus U$ and define $U_i = \Gamma(u_i) \cap V(R)$ for each $1 \leq i \leq 8$. For all $1 \leq i \leq 8$ the vertex $u_i \in U$ is adjacent to the other 7 vertices of U and since $\delta(G) \geq 8$, u_i is also adjacent to at least one vertex outside of U, that is, u_i is adjacent to at least one vertex of R. Hence, $U_i \neq \emptyset$ for all $1 \leq i \leq 8$.

Now show that $U_i \cap U_j = \emptyset$ for all $1 \leq i < j \leq 8$. Take two vertices $u_i, u_j \in U$ for some $1 \leq i < j \leq 8$. Then there is a path of order 8 from u_i to u_j by walking along the remaining 6 vertices of U in any order. Let $w \in U_i \cap U_j$, so that a path (u_i, w, u_j) arises. By "gluing" together the path from u_i to u_j of order 8 and the path (u_i, w, u_j), there appears a cycle of order 9, which is a contradiction. Therefore, $U_i \cap U_j = \emptyset$ for all $1 \leq i < j \leq 8$.

Similarly, show that for all $1 \leq i < j \leq 8$ and for all $x \in U_i$ and $y \in U_j$, the edge xy is not in $E(G)$. Take two vertices $u_i, u_j \in U$ for some $1 \leq i < j \leq 8$. Then there is a path of order 7 from u_i to u_j by walking along 5 of the remaining 6 vertices of U in any order. Let $x \in U_i$ and $y \in U_j$ and suppose that $xy \in E(G)$. There arises a path in G joining u_i and u_j; (u_i, x, y, u_j). By "gluing" together the path from u_i to u_j of order 7 and the path (u_i, x, y, u_j), there appears a cycle of order 9, which is a contradiction. Therefore, for all $1 \leq i < j \leq 8$ and for all $x \in U_i$ and $y \in U_j$, the edge xy is not in $E(G)$.

Again, with the same reasoning, $\Gamma_R(U_i) \cap \Gamma_R(U_j) = \emptyset$ for $1 \leq i < j \leq 8$. Take two vertices $u_i, u_j \in U$ for some $1 \leq i < j \leq 8$. Then there is a path of order 6 from u_i to u_j by walking along 4 of the remaining 6 vertices of U in any order. Let $w \in \Gamma_R(U_i) \cap \Gamma_R(U_j)$ and suppose that w is adjacent to $x \in U_i$ and to $y \in U_j$, then the path (u_i, x, w, y, u_j) arises. By "gluing" together the path from u_i to u_j of order 6 and the path (u_i, x, w, y, u_j), there appears a cycle of order 9, which is a contradiction. Therefore $\Gamma_R(U_i) \cap \Gamma_R(U_j) = \emptyset$ for $1 \leq i < j \leq 8$.

Since $U_i \neq \emptyset$, it holds that $|U_i| \geq 1$. Since $U_i \cap U_j = \emptyset$ for all $1 \leq i < j \leq 8$, a vertex $x \in U_i$ is adjacent to only one vertex in U, namely to u_i, so it is not adjacent to u_j for all $j \neq i$. Consequently, the vertex x must be adjacent to at least 7 vertices in R. Since x and its neighborhood are included in U_i and its neighborhood, write $x \cup \Gamma_R(x) \subset U_i \cup \Gamma_R(U_i)$ and conclude that $8 = 1 + 7 \leq |x \cup \Gamma(x)| \subset |U_i \cup \Gamma(U_i)|$. Therefore

$$|U_i \cup \Gamma_R(U_i) \cup \{u_i\}| \geq 9.$$

Since $U_i \cap U_j = \emptyset$, $\Gamma_R(U_i) \cap \Gamma_R(U_j) = \emptyset$ and $u_i \neq u_j$ for all $1 \leq i < j \leq 8$ it follows that $U_i \cup \Gamma_R(U_i) \cup \{u_i\}$ is disjoint from $U_j \cup \Gamma_R(U_j) \cup \{u_j\}$ for all for all $1 \leq i < j \leq 8$. Then the following is obtained:

$$|V(G)| \geq |\bigcup_{i=1}^{8} \left(U_i \cup \Gamma_R(U_i) \cup \{u_i\}\right)| \geq 8 \cdot 9 = 72.$$

So indeed, $|V(G)| \geq 72$. ■

Definition 18 *A star graph is a tree consisting of one vertex adjacent to all others [20].*

A star graph with n vertices will be denoted S_n.

Lemma 5 *If G contains $K_8 \setminus S_6$, then G contains K_8.*

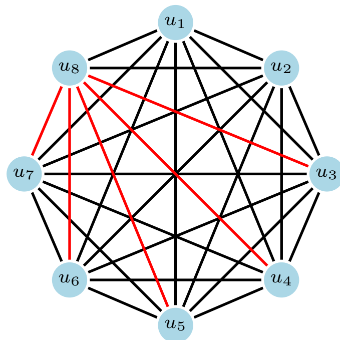

Figure 12: Star graph on a complete graph

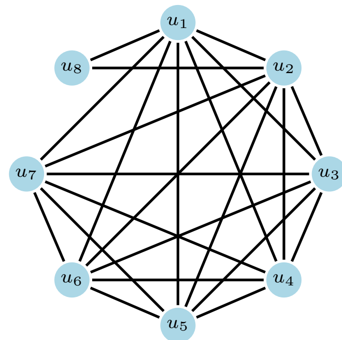

Figure 13: Star graph removed from a complete graph

Proof In Figure 12, a complete graph of order 8 is given and in particular, some of its edges are colored red. Looking closer, one can see that the red edges together with the vertices u_3, u_4, u_5, u_6, u_7 and u_8 form a star graph of order 6, i.e., S_6. The graph $K_8 \setminus S_6$ is obtained by removing the edges of S_6 from the graph K_8 as can be seen in Figure 13. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ be the vertex set of $K_8 \setminus S_6$. The induced subgraph of the vertices $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ form a complete graph of order 7. Without loss of generality, assume that the edges u_1u_8 and u_2u_8 are in $E(G)$, which is shown Figure 13. Let $R = G \setminus U$, i.e., the graph R is obtained by taking the graph G and removing the vertices of U and the edges that have a vertex of U as its endpoint. Define $U_i = \Gamma(u_i) \cap V(R)$ for each $1 \leq i \leq 8$. With the same reasoning as in Lemma 4, $U_i \cap U_j = \emptyset$ for $1 \leq i < j \leq 8$, however in this case, it does not hold for $i = 1$ and $j = 2$, since there is no path in U of order 8 from u_1 to u_2. When making a path of order 8 in U, every vertex of U has to be in the path. However, as can be seen in Figure 13, u_8 has to come after u_1 and before u_2, since u_8 is not adjacent to any other vertex in U. In this way, there is no path of order 8 that starts with u_1 and ends with u_2. Therefore, "gluing" together a path in U joining u_1 and u_2 and a path (u_1, w, u_2) for some vertex $w \in U_1 \cap U_2$ will never result in a cycle of order 9. So $U_i \cap U_j = \emptyset$ for $1 \leq i < j \leq 8$ except possibly for $i = 1$ and $j = 2$.

Since $K_8 \setminus S_6$ contains a complete graph of order 7, there is, just as in Lemma 4, a path of order 7 joining any two vertices of U. Therefore, it again holds for all $1 \leq i < j \leq 8$ and for all $x \in U_i$ and $y \in U_j$, that $xy \notin E(G)$.

Again, $\Gamma_R(U_i) \cap \Gamma_R(U_j) = \emptyset$ for all $1 \leq i < j \leq 8$, since there is a path of order 6 joining any two vertices of U.

In this lemma, it is necessary to consider another property of the sets $\Gamma_R(U_i)$ and $\Gamma_R(U_j)$. In the graph $K_8 \setminus S_6$, there is a path of order 5 between any two vertices. Take two vertices $u_i, u_j \in U$ for some $1 \leq i < j \leq 8$. A path of order 5 from u_i to u_j is made by walking along 3 of the remaining 6 vertices of U in any order. Let $x \in \Gamma_R(U_i)$ and $y \in \Gamma_R(U_j)$ and suppose that $xy \in E(G)$. Then there appears a path (u_i, v, x, y, w, u_j) where $v \in U_i$ and $w \in U_j$ are the vertices that x and y are adjacent to, respectively. By "gluing" together the path from u_i to u_j of order 5 and the path (u_i, v, x, y, w, u_j), there appears a cycle of order 9, which is a contradiction. So for all $1 \leq i < j \leq 8$ and for all $x \in \Gamma_R(U_i)$ and $y \in \Gamma_R(U_j)$, the edge $xy \notin E(G)$.

Summarized:

1. $U_i \cap U_j = \emptyset$ for $1 \leq i < j \leq 8$ except possibly for $i = 1$ and $j = 2$.
2. $xy \notin E(G)$ for all $1 \leq i < j \leq 8$ and for all $x \in U_i$ and $y \in U_j$.
3. $\Gamma_R(U_i) \cap \Gamma_R(U_j) = \emptyset$ for all $1 \leq i < j \leq 8$
4. $xy \notin E(G)$ for all $1 \leq i < j \leq 8$ and for all $x \in \Gamma_R(U_i)$ and $y \in \Gamma_R(U_j)$.

Since G is assumed to have no independent set of 8 vertices, the maximal order of an independent set is 7. As a result, at least five of the induced subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ are complete for $3 \leq i \leq 8$. In order to show this, assume that less than five of the induced subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ are complete for $3 \leq i \leq 8$. From the summation above it follows that the induced subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ for $3 \leq i \leq 8$ are mutually disjoint and not adjacent. When picking elements to make an independent set, one for sure can select one vertex from each of the subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$. This creates an independent set of 6 elements. Since less than five of the subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ for $3 \leq i \leq 8$ are complete, one can select at least two more vertices for the independent set; one from each of the subgraphs that are not complete. However, now there is an independent set of more than 7 elements, but that was assumed not to be possible. Therefore, at least five of the induced subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ are complete for $3 \leq i \leq 8$. By assumption, $\delta(G) \geq 8$, that is, the degree of each vertex is greater than 8. Assume that $k \in \{3, ..., 8\}$ is an integer such that the induced subgraph $\langle U_k \cup \Gamma_R(U_k)\rangle_G$ is complete. Take a vertex $x \in U_k$, then x is adjacent to one vertex $u_k \in U$ and at least 7 other vertices in $\langle U_k \cup \Gamma_R(U_k)\rangle_G$. Since the latter is a complete induced subgraph, the vertex x and its neighborhood $\Gamma_R(x)$ form a complete graph of order 8 in $\langle U_k \cup \Gamma_R(U_k)\rangle_G$. Consequently, the graph G contains a complete graph of order 8. ■

Lemma 6 *If G contains K_7, then G contains $K_8 \setminus S_6$ or K_8.*

Proof Let G be a graph that contains K_7 and let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ be the vertex set of the complete graph K_7. The sets R and U_i are the same as in the previous lemmas, but now they are defined for $1 \leq i \leq 7$, so define $R = G \setminus U$ and $U_i = \Gamma(u_i) \cap V(R)$ for each $1 \leq i \leq 7$. For all $1 \leq i \leq 7$ the vertex $u_i \in U$ is adjacent to the other 6 vertices of U and since $\delta(G) \geq 8$, u_i is also adjacent to at least two vertices outside of U, that is, u_i is adjacent to at least two vertices of R. Hence, $U_i \neq \emptyset$ for all $1 \leq i \leq 7$. The rest of the proof is completed by considering the following two cases:

1. $U_i \cap U_j \neq \emptyset$ for some $1 \leq i < j \leq 7$. Let $w \in U_i \cap U_j$, then look back at Figure 13. That figure represents the same situation as in this lemma, but with $u_8 = w$. That is, the vertices $u_1, ..., u_7$ in Figure 13 form a complete graph just as in this lemma and the vertex u_8 can be seen as the vertex $w \in U_i \cap U_j$. Figure 13 shows a graph $K_8 \setminus S_6$, therefore in this lemma G contains $K_8 \setminus S_6$.
2. $U_i \cap U_j = \emptyset$ for all $1 \leq i < j \leq 7$. In this second case, the same reasoning as used in lemma 4 can be applied here. Take any two vertices in U, then there is always a path of order 5, 6 or 7 joining those two vertices. Then the same results follow as in lemma 4:
   (a) $xy \notin E(G)$ for all $1 \leq i < j \leq 7$ and for all $x \in U_i$ and $y \in U_j$.
   (b) $\Gamma_R(U_i) \cap \Gamma_R(U_j) = \emptyset$ for all $1 \leq i < j \leq 7$
   (c) $xy \notin E(G)$ for all $1 \leq i < j \leq 7$ and for all $x \in \Gamma_R(U_i)$ and $y \in \Gamma_R(U_j)$.

The maximal order of an independent set is 7, since G is assumed to have no independent set of 8 vertices. By summation above, for all $1 \leq i \leq 7$ the induced subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ are mutually disjoint and not adjacent. When picking elements to make an independent set, one for sure can select one vertex from each of the subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ for $1 \leq i \leq 7$. This creates an independent set of 7 vertices. This should be the maximal order of an independent set, so it should not be possible to pick another vertex from any of the subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ that is independent to any of the 7 previous selected vertices. Therefore, the subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ are complete for $1 \leq i \leq 7$. By assumption, $\delta(G) \geq 8$, that is, the degree of each vertex is greater than 8. Take a vertex $x \in U_i$ for any $1 \leq i \leq 7$, then x is adjacent to one vertex $u_i \in U$ and at least 7 other vertices in $\langle U_i \cup \Gamma_R(U_i)\rangle_G$. Since the latter is a complete induced subgraph, the vertex x and its neighborhood $\Gamma_R(x)$ form a complete graph of order 8 in $\langle U_k \cup \Gamma_R(U_k)\rangle_G$. Consequently, the graph G contains a complete graph of order 8.

In either of the two cases described above, the set G contains $K_8 \setminus S_6$ or K_8, as desired. ■

Lemma 7 *If G contains $K_1 + P_7$, then G contains K_7.*

Before proving this fifth lemma in the sequence of eight lemmas, the meaning of $K_1 + P_7$ needs to be illustrated. The graph $K_1 + P_7$ is obtained by adding an additional vertex to the path P_7 and connecting this new vertex to each vertex of P_7. An example is given in Figure 14, where the vertices $u_2, u_3, u_4, u_5, u_6, u_7, u_8$ are the vertices of the path P_7 and the vertex u_1 is adjacent to each vertex of P_7.

Proof: Let G be a graph that contains K_1+P_7. Define $K_1 = u_1$ and $P_7 = (u_2, u_3, u_4, u_5, u_6, u_7, u_8)$. By putting those together, define $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ as the vertex set of $K_1 + P_7$. Again, let $R = G \setminus U$ and $U_i = \Gamma(u_i) \cap V(R)$ for each $1 \leq i \leq 8$. Now two cases will be distinguished: $U_4 \cap U_6 = \emptyset$ and $U_4 \cap U_6 \neq \emptyset$. From either case it will follow that G contains K_7.

1. $U_4 \cap U_6 = \emptyset$. By assumption, the degree of each vertex is greater than 8. Each vertex $u_i \in U$ is adjacent to at most 7 other vertices in U for $1 \leq i \leq 8$. For example, u_1 is adjacent to 7 vertices in U (namely to u_2, u_3, u_4, u_5, u_6, u_7, u_8) and u_3 is adjacent to 3 vertices in U (namely to u_1, u_2 and u_4). Therefore, each vertex u_i is adjacent to at least one vertex in R, hence $U_i \neq \emptyset$ for all $1 \leq i \leq 8$. By assumption, there should be no cycle of order 9 in the graph G. Because of this, it holds that $U_i \cap U_j = \emptyset$ for some

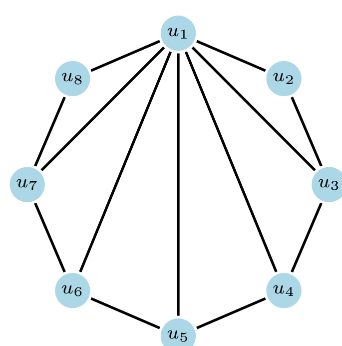

Figure 14: A graph $K_1 + P_7$

$2 \leq i < j \leq 8$. Let $u_i, u_j \in U$ for some $2 \leq i < j \leq 8$. If there is a path of order 8 in U from u_i to u_j, then $U_i \cap U_j$ must be empty. Otherwise, "gluing" together the path of order 8 and the path (u_i, w, u_j) for $w \in U_i \cap U_j$ creates a cycle of order 9. The paths in U of order 8 from u_i to u_j for $2 \leq i < j \leq 8$ that can be made are:

- from u_2 to u_3: $(u_2, u_1, u_8, u_7, u_6, u_5, u_4, u_3)$
- from u_2 to u_4: $(u_2, u_3, u_1, u_8, u_7, u_6, u_5, u_4)$
- from u_2 to u_5: $(u_2, u_3, u_4, u_1, u_8, u_7, u_6, u_5)$
- from u_2 to u_6: $(u_2, u_3, u_4, u_5, u_1, u_8, u_7, u_6)$
- from u_2 to u_7: $(u_2, u_3, u_4, u_5, u_6, u_1, u_8, u_7)$
- from u_2 to u_8: $(u_2, u_3, u_4, u_5, u_6, u_7, u_1, u_8)$
- from u_3 to u_4: $(u_3, u_2, u_1, u_8, u_7, u_6, u_5, u_4)$
- from u_3 to u_8: $(u_3, u_2, u_1, u_4, u_5, u_6, u_7, u_8)$
- from u_4 to u_5: $(u_4, u_3, u_2, u_1, u_8, u_7, u_6, u_5)$
- from u_4 to u_6: $(u_4, u_3, u_2, u_1, u_8, u_7, u_6, u_5)$
- from u_4 to u_8: $(u_4, u_3, u_2, u_1, u_5, u_6, u_7, u_8)$
- from u_5 to u_6: $(u_5, u_4, u_3, u_2, u_1, u_8, u_7, u_6)$
- from u_5 to u_8: $(u_5, u_4, u_3, u_2, u_1, u_6, u_7, u_8)$
- from u_6 to u_7: $(u_6, u_5, u_4, u_3, u_2, u_1, u_8, u_7)$
- from u_6 to u_8: $(u_6, u_5, u_4, u_3, u_2, u_1, u_7, u_8)$
- from u_7 to u_8: $(u_7, u_6, u_5, u_4, u_3, u_2, u_1, u_8)$.

For the pairs of vertices listed above, it holds that the corresponding sets $U_i \cap U_j$ should be empty, for else there is a cycle of order 9. The remaining pairs of vertices are (u_3, u_5), (u_3, u_6), (u_3, u_7), (u_4, u_7) and (u_5, u_7) and these pairs do not have a path of order 8 in U joining them. Therefore, $U_i \cap U_j$ does not have to be empty for $(i, j) \in \{(3, 5), (3, 6), (3, 7), (4, 7), (5, 7)\}$. This gives the following conclusion: $U_i \cap U_j = \emptyset$ for all $2 \leq i < j \leq 8$ except possibly for $(i, j) \in \{(3, 5), (3, 6), (3, 7), (4, 7), (5, 7)\}$. In contrary to paths of order 8, note that $K_1 + P_7$ contains a path of order 7 in U between *any* two vertices u_i and u_j for $2 \leq i < j \leq 8$. Therefore, if there is an edge $xy \in E(G)$ for $x \in U_i$ and $y \in U_j$, a cycle of order 9 appears by "gluing" together the path in U of order 7 from u_i to u_j and the path (u_i, x, y, u_j). This means that $xy \notin E(G)$ for all $2 \leq i < j \leq 8$ and for all $x \in U_i$ and $y \in U_j$. Similarly, there is a path of order 6 in U between any two vertices u_i and u_j for $2 \leq i < j \leq 8$. Therefore, if there exists $w \in \Gamma_R(U_i) \cap \Gamma_R(U_j)$, then there appears a path of order 9 by gluing together the path of order 6 in U from u_i to u_j and the path (u_i, x, w, y, u_j), where $x \in U_i$ and $y \in U_j$ are such that w is adjacent to x and y. Consequently, $\Gamma_R(U_i) \cap \Gamma_R(U_j) = \emptyset$ for all $2 \leq i < j \leq 8$. Again, with the same reasoning, there is a path of order 5 in U between any two vertices u_i and u_j for $2 \leq i < j \leq 8$, so if there is an edge $xy \in E(G)$ for $x \in \Gamma_R(U_i)$ and $y \in \Gamma_R(U_j)$, then there appears a cycle of order 9 by gluing together the path of order 5 in U and the path (u_i, v, x, y, w, u_j) where $v \in U_i$ and $w \in U_j$ are adjacent to x and y respectively. Therefore, $xy \notin E(G)$ for all $2 \leq i < j \leq 8$ and for all $x \in \Gamma_R(U_i)$ and $y \in \Gamma_R(U_j)$. Summarized, there are four properties:

(A) $U_i \cap U_j = \emptyset$ for all $2 \leq i < j \leq 8$ except possibly for $(i, j) \in \{(3, 5), (3, 6), (3, 7), (4, 7), (5, 7)\}$

(B) $xy \notin E(G)$ for all $2 \leq i < j \leq 8$ and for all $x \in U_i$ and $y \in U_j$

(C) $\Gamma_R(U_i) \cap \Gamma_R(U_j) = \emptyset$ for all $2 \leq i < j \leq 8$

(D) $xy \notin E(G)$ for all $2 \leq i < j \leq 8$ and for all $x \in \Gamma_R(U_i)$ and $y \in \Gamma_R(U_j)$.

In contrary to the previous lemmas, the induced subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ and $\langle U_j \cup \Gamma_R(U_j)\rangle_G$ for $2 \leq i < j \leq 8$ are not necessarily disjoint. As can be seen in the latter summation, $U_i \cap U_j$ does not have to be empty for $(i,j) \in \{(3,5),(3,6),(3,7),(4,7),(5,7)\}$, hence for these pairs of (i,j) the subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ and $\langle U_j \cup \Gamma_R(U_j)\rangle_G$ do not have to be disjoint. Looking closer to the pairs $(3,5),(3,6),(3,7),(4,7),(5,7)$, one can see that either 3 or 7 appears in each of the pairs. If the vertices u_3 and u_7 are taking out of consideration, then $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ and $\langle U_j \cup \Gamma_R(U_j)\rangle_G$ are disjoint and not adjacent for $i,j = 2,4,5,6,8$ and $i < j$. When picking elements to make an independent set, one for sure can select one vertex from each of the subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ for $i = 2,4,5,6,8$. This creates an independent set of five elements. The maximum order of an independent set is 7, so at most two more vertices can be included in the independent set. This is only satisfied when at least three of the five subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ for $i = 2,4,5,6,8$ are complete.

Now two claims will be proven. The first claim is: $|\Gamma_R(U_i)| \geq 7$ and so $|U_i \cup \Gamma_R(U_i)| \geq 8$ for $i = 2,8$. Proof: As stated before $U_8 \neq \emptyset$, so $|U_8| \geq 1$. Let $y \in U_8$ and y is adjacent to $x \in \{u_1,u_2,u_3,u_4,u_5,u_6,u_7\}$. As listed before, there are paths of order 8 in U between any pair of vertices except for for $(i,j) \in \{(3,5),(3,6),(3,7),(4,7),(5,7)\}$. Note that 8 does not occur in any of these pairs. Hence, there is a path of order 8 in U from any of the vertices $u_1,u_2,u_3,u_4,u_5,u_6,u_7$ to the vertex u_8. If y is adjacent to u_i for $1 \leq i \leq 7$, a cycle of order 9 arises by gluing together the path of order 8 in U from u_i to u_8 and the path (u_8,y,u_i). This is a contradiction, so y is not adjacent to u_i for each $1 \leq i \leq 7$. The vertex y is hence adjacent to only one vertex of U. Since $\delta(G) \geq 8$, y has to be adjacent to at least 7 other vertices, so $|\Gamma_R(y)| \geq 7$, thus $|\{y\} \cup \Gamma_R(y)| \geq 8$. Note that $\{y\} \cup \Gamma_R(y) \subset U_8 \cup \Gamma_R(U_8)$, which causes $|U_8 \cup \Gamma_R(U_8)| \geq 8$. By symmetry of $K_1 + P_7$, a similar argumentation using $i = 2$ gives the same result. Thus, $|\Gamma_R(U_i)| \geq 7$ and so $|U_i \cup \Gamma_R(U_i)| \geq 8$ for $i = 2,8$.

The second claim is: If there is $i \in \{4,5,6\}$ such that $|\Gamma_R(U_i)| < 6$, then $|\Gamma_R(U_j)| \geq 6$ and so $|U_j \cup \Gamma_R(U_j)| \geq 7$ for any $j \in \{4,5,6\}$ with $i \neq j$. Proof: Assume that $|\Gamma_R(U_4)| < 6$. By property (A) above, $U_4 \cap U_i$ for all $i \in \{2,3,5,6,8\}$. Let $y \in U_4$, then y is by definition adjacent to u_4 and possibly also to u_1 and u_7. In order to show that $|\Gamma_R(U_5)| \geq 6$, assume that $|\Gamma_R(U_5)| < 6$. By property (A) above, $U_5 \cap U_i = \emptyset$ for $i \in \{2,4,6,8\}$. Then for $w \in U_5$, by definition w is adjacent to u_5 and possibly also to u_1, u_3 and u_7. Since $\delta(G) \geq 8$ and $|\Gamma_R(U_5)| < 6$, the vertex w should be adjacent to at least 3 vertices in U. The vertex w is always adjacent to u_5, so it has to be adjacent to at least two of u_1, u_3 and u_7. Recall that y is adjacent to u_4 and possibly also to u_1 and u_7 Now there are three cases:

- If w is adjacent to u_1 and u_3, then $(u_2,u_3,w,u_5,u_6,u_7,y,u_4,u_1,u_2)$ is a cycle of order 9.
- If w is adjacent to u_1 and u_7, then $(u_2,u_3,u_4,y,u_7,w,u_5,u_6,u_1,u_2)$ is a cycle of order 9.
- If w is adjacent to u_3 and u_7, then $(u_2,u_3,w,u_7,u_6,u_5,u_4,y,u_1,u_2)$ is a cycle of order 9.

Of course, if w is adjacent to all of u_1, u_3 and u_7, then any of the paths given above would give a contradiction. Therefore, w is adjacent to u_5 and at most one of u_1, u_3 and u_7. Then w is adjacent to at most 2 vertices of U, and so it should be adjacent to at least 6 vertices of R. This gives a contradiction, so $|\Gamma_R(U_5)| \geq 6$. The claim has now been proven for $i = 4$ and $j = 5$. Now continue with $j = 6$. Show that $|\Gamma_R(U_6)| \geq 6$, by assuming that $|\Gamma_R(U_6)| < 6$. By property (A) above, $U_6 \cap U_i = \emptyset$ for $i \in \{2, 4, 5, 7, 8\}$. For $w \in U_6$, w is adjacent u_1, u_3 and u_6, since $|\Gamma_R(U_6)| < 6$ causes w to be adjacent to at least 3 vertices of U. Then there is a path $(u_8, u_7, y, u_4, u_3, w, u_6, u_5, u_1, u_8)$ which is a cycle of order 9, a contradiction. This implies that w is adjacent to u_6 and at most one of u_1 and u_3. Then w should be adjacent to at least 6 vertices of R, hence $|\Gamma_R(U_6)| \geq 6$. This proves the claim for $i = 4$ and $j = 6$. By making use of the symmetry of $K_1 + P_7$ and using the same argument as above, but now for $i = 6$, it follows that if $|\Gamma_R(U_6)| < 6$, then both $|\Gamma_R(U_4)| \geq 6$ and $|\Gamma_R(U_5)| \geq 6$. It remains to show the claim for $i = 5$ and $j = 4, 6$. When $i = 5$, $|\Gamma_R(U_5)| < 6$. By property (A) above, $U_5 \cap U_i = \emptyset$ for $i \in \{2, 4, 6, 8\}$. Any $y \in U_5$ is adjacent to u_5 and it is possibly adjacent to u_1, u_3 and u_7. Assume for a contradiction that $|\Gamma_R(U_4)| < 6$. Since by property (A), $U_4 \cap U_i = \emptyset$ for all $i \in \{2, 3, 5, 6, 8\}$, a vertex $w \in U_4$ is adjacent to u_4 and possibly also to u_1 and u_7. Noting that $\delta(G) \geq 8$ and $|\Gamma_R(U_4)| < 6$, w should be adjacent to all of u_4, u_1 and u_7. The cycle $(u_8, u_7, w, u_4, u_3, u_1, y, u_5, u_6)$ of order 9 appears, a contradiction. Therefore, $|\Gamma_R(U_4)| \geq 6$. This shows the claim for $i = 5$ and $j = 4$. Finally, let $j = 6$. Then, assume again for a contradiction that $|\Gamma_R(U_6)| < 6$. Using the symmetry of $K_1 + P_7$ and the same argumentation as above, the vertex $w \in U_6$ is adjacent to u_6 and possibly to u_1, u_3. Then the path $(u_8, u_7, y, u_3, u_4, u_5, u_6, w, u_1, u_8)$ of order 9 gives a contradiction, thus $|\Gamma_R(U_6)| \geq 6$. Finally, it follows that if there is $i \in \{4, 5, 6\}$ such that $|\Gamma_R(U_i)| < 6$, then $|\Gamma_R(U_j)| \geq 6$ and so $|U_j \cup \Gamma_R(U_j)| \geq 7$ for any $j \in \{4, 5, 6\}$ with $i \neq j$.

The first claim was: $|\Gamma_R(U_i)| \geq 7$ and so $|U_i \cup \Gamma_R(U_i)| \geq 8$ for $i = 2, 8$. The second claim was: If there is $i \in \{4, 5, 6\}$ such that $|\Gamma_R(U_i)| < 6$, then $|\Gamma_R(U_j)| \geq 6$ and so $|U_j \cup \Gamma_R(U_j)| \geq 7$ for any $j \in \{4, 5, 6\}$ with $i \neq j$. Together: the induced subgraphs $\langle U_i \cup \Gamma_R(U_i) \rangle_G$ contain 7 vertices for at least for values of $i \in \{2, 4, 5, 6, 8\}$. By properties (A), (B), (C) and (D), the induced subgraphs are disjoint and not adjacent. When picking elements to make an independent set, one for sure can select one vertex from each of the subgraphs $\langle U_i \cup \Gamma_R(U_i) \rangle_G$ for $i \in \{2, 4, 5, 6, 8\}$. This creates an independent set of 5 elements. The maximal order of an independent set is 7, so it should not be possible to pick one more vertex from each of the 5 subgraphs such that all vertices form an independent set. One should be able to pick at most two more vertices from $\langle U_i \cup \Gamma_R(U_i) \rangle_G$ for $i \in \{2, 4, 5, 6, 8\}$. Therefore, at least three of the induced subgraphs $\langle U_i \cup \Gamma_R(U_i) \rangle_G$ for $i \in \{2, 4, 5, 6, 8\}$ should be complete. Then at least two of them contain 7 vertices, and so they contain a complete graph of order 7. Finally, G contains a complete graph of order 7.

2. $U_4 \cap U_6 \neq \emptyset$. Let $u_9 \in U_4 \cap U_6$. Now redefine the sets U, R and U_i. To that end, let $U' = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9\}$, $R' = G \setminus U'$ and $U'_i = \Gamma(u_i) \cap V(R')$. With the existence of u_9, some edges $u_i u_j$ cannot be in G for some $2 \leq i < j \leq 9$, since otherwise there will be a cycle of order 9.

    - When $u_2 u_9 \in E(G)$, $(u_2, u_9, u_6, u_7, u_8, u_1, u_5, u_4, u_3, u_2)$ is a cycle of order 9.
    - When $u_3 u_9 \in E(G)$, $(u_3, u_9, u_4, u_5, u_6, u_7, u_8, u_1, u_2, u_3)$ is a cycle of order 9.
    - When $u_5 u_9 \in E(G)$, $(u_2, u_3, u_4, u_9, u_5, u_6, u_7, u_8, u_1, u_2)$ is a cycle of order 9.

- When $u_7u_9 \in E(G)$, $(u_2, u_3, u_4, u_5, u_6, u_9, u_7, u_8, u_1, u_2)$ is a cycle of order 9.
- When $u_8u_9 \in E(G)$, $(u_2, u_3, u_4, u_9, u_8, u_7, u_6, u_5, u_1, u_2)$ is a cycle of order 9.

Therefore, $u_2u_9, u_3u_9, u_5u_9, u_7u_9, u_8u_9 \notin E(G)$. The vertices u_i for $i \in \{2,3,5,7,8,9\}$ are each adjacent to at most 3 vertices in U'. Since $\delta(G) \geq 8$, each of the vertices u_i for $i \in \{2,3,5,7,8,9\}$ must be adjacent to at least 5 or 6 vertices in R', thus $|\Gamma_{R'}(u_i)| \geq 1$ and so $U'_i \neq \emptyset$ for all $i \in \{2,3,5,7,8,9\}$. There are 4 properties that are similar to properties (A), (B), (C) and (D) above. The first one is: $U'_i \cap U'_j = \emptyset$ for all $i,j \in \{2,3,5,7,8,9\}$ and $i \neq j$. This will be shown pair by pair.

- When $w \in U'_2 \cap U'_3$, $(u_2, w, u_3, u_4, u_5, u_6, u_7, u_8, u_1, u_2)$ is a cycle of order 9.
- When $w \in U'_2 \cap U'_5$, $(u_2, u_3, u_4, u_1, u_8, u_7, u_6, u_5, w, u_2)$ is a cycle of order 9.
- When $w \in U'_2 \cap U'_7$, $(u_2, u_3, u_4, u_5, u_6, u_1, u_8, u_7, w, u_2)$ is a cycle of order 9.
- When $w \in U'_2 \cap U'_8$, $(u_2, u_3, u_4, u_5, u_6, u_7, u_1, u_8, w, u_2)$ is a cycle of order 9.
- When $w \in U'_2 \cap U'_9$, $(u_2, u_3, u_4, u_5, u_1, u_7, u_6, u_9, w, u_2)$ is a cycle of order 9.
- When $w \in U'_3 \cap U'_5$, $(u_3, u_4, u_9, u_6, u_7, u_8, u_1, u_5, w, u_3)$ is a cycle of order 9.
- When $w \in U'_3 \cap U'_7$, $(u_3, u_4, u_9, u_6, u_5, u_1, u_8, u_7, w, u_3)$ is a cycle of order 9.
- When $w \in U'_3 \cap U'_8$, $(u_3, u_2, u_1, u_4, u_5, u_6, u_7, u_8, w, u_3)$ is a cycle of order 9.
- When $w \in U'_3 \cap U'_9$, $(u_3, u_4, u_5, u_1, u_8, u_7, u_6, u_9, w, u_3)$ is a cycle of order 9.
- When $w \in U'_5 \cap U'_7$, $(u_5, u_1, u_2, u_3, u_4, u_9, u_6, u_7, w, u_5)$ is a cycle of order 9.
- When $w \in U'_5 \cap U'_8$, $(u_5, u_4, u_3, u_2, u_1, u_6, u_7, u_8, w, u_5)$ is a cycle of order 9.
- When $w \in U'_5 \cap U'_9$, $(u_5, u_4, u_3, u_2, u_1, u_7, u_6, u_9, w, u_5)$ is a cycle of order 9.
- When $w \in U'_7 \cap U'_8$, $(u_7, u_6, u_5, u_4, u_3, u_2, u_1, u_8, w, u_7)$ is a cycle of order 9.
- When $w \in U'_7 \cap U'_9$, $(u_7, u_6, u_5, u_1, u_2, u_3, u_4, u_9, w, u_7)$ is a cycle of order 9.
- When $w \in U'_8 \cap U'_9$, $(u_8, u_7, u_6, u_5, u_1, u_3, u_4, u_9, w, u_5)$ is a cycle of order 9.

The second property is: $xy \notin E(G)$ for all $i,j \in \{2,3,5,7,8,9\}$ with $i \neq j$ and for all $x \in U'_i$ and $y \in U'_j$. This holds by the following arguments:

- When $x \in U_2$ and $y \in U_3$, $(u_2, u_1, u_7, u_6, u_5, u_4, u_3, y, x, u_2)$ is a cycle of order 9.
- When $x \in U_2$ and $y \in U_5$, $(u_2, u_3, u_4, u_1, u_7, u_6, u_5, y, x, u_2)$ is a cycle of order 9.
- When $x \in U_2$ and $y \in U_7$, $(u_2, u_3, u_4, u_5, u_6, u_1, u_7, y, x, u_2)$ is a cycle of order 9.
- When $x \in U_2$ and $y \in U_8$, $(u_2, u_3, u_4, u_5, u_6, u_1, u_8, y, x, u_2)$ is a cycle of order 9.
- When $x \in U_2$ and $y \in U_9$, $(u_2, u_3, u_4, u_5, u_1, u_6, u_9, y, x, u_2)$ is a cycle of order 9.
- When $x \in U_3$ and $y \in U_5$, $(u_3, u_2, u_1, u_8, u_7, u_6, u_5, y, x, u_3)$ is a cycle of order 9.
- When $x \in U_3$ and $y \in U_7$, $(u_3, u_2, u_1, u_4, u_5, u_6, u_7, y, x, u_3)$ is a cycle of order 9.
- When $x \in U_3$ and $y \in U_8$, $(u_3, u_2, u_1, u_5, u_6, u_7, u_8, y, x, u_3)$ is a cycle of order 9.
- When $x \in U_3$ and $y \in U_9$, $(u_3, u_4, u_5, u_1, u_7, u_6, u_9, y, x, u_3)$ is a cycle of order 9.
- When $x \in U_5$ and $y \in U_7$, $(u_5, u_4, u_3, u_2, u_1, u_8, u_7, y, x, u_5)$ is a cycle of order 9.
- When $x \in U_5$ and $y \in U_8$, $(u_5, u_4, u_3, u_2, u_1, u_7, u_8, y, x, u_5)$ is a cycle of order 9.
- When $x \in U_5$ and $y \in U_9$, $(u_5, u_4, u_3, u_2, u_1, u_6, u_9, y, x, u_5)$ is a cycle of order 9.
- When $x \in U_7$ and $y \in U_8$, $(u_7, u_6, u_5, u_4, u_3, u_1, u_8, y, x, u_7)$ is a cycle of order 9.

- When $x \in U_7$ and $y \in U_9$, $(u_7, u_6, u_5, u_1, u_3, u_4, u_9, y, x, u_7)$ is a cycle of order 9.
- When $x \in U_8$ and $y \in U_9$, $(u_8, u_7, u_6, u_5, u_1, u_4, u_9, y, x, u_8)$ is a cycle of order 9.

The third property is: $\Gamma_{R'}(U'_i) \cap \Gamma_{R'}(U'_j) = \emptyset$ for all $i, j \in \{2, 3, 5, 7, 8, 9\}$ with $i \neq j$.

- When $w \in \Gamma_{R'}(U'_2) \cap \Gamma_{R'}(U'_3)$, $(u_2, u_1, u_6, u_5, u_4, u_3, x, w, y, u_2)$ is a cycle of order 9 for $x \in U'_3$ and $y \in U'_2$ such that x and y are adjacent to w.
- When $w \in \Gamma_{R'}(U'_2) \cap \Gamma_{R'}(U'_5)$, $(u_2, u_3, u_4, u_1, u_6, u_5, x, w, y, u_2)$ is a cycle of order 9 for $x \in U'_5$ and $y \in U'_2$ such that x and y are adjacent to w.
- When $w \in \Gamma_{R'}(U'_2) \cap \Gamma_{R'}(U'_7)$, $(u_2, u_3, u_4, u_5, u_1, u_7, x, w, y, u_2)$ is a cycle of order 9 for $x \in U'_7$ and $y \in U'_2$ such that x and y are adjacent to w.
- When $w \in \Gamma_{R'}(U'_2) \cap \Gamma_{R'}(U'_8)$, $(u_2, u_3, u_4, u_5, u_1, u_8, x, w, y, u_2)$ is a cycle of order 9 for $x \in U'_8$ and $y \in U'_2$ such that x and y are adjacent to w.
- When $w \in \Gamma_{R'}(U'_2) \cap \Gamma_{R'}(U'_9)$, $(u_2, u_3, u_4, u_1, u_6, u_9, x, w, y, u_2)$ is a cycle of order 9 for $x \in U'_9$ and $y \in U'_2$ such that x and y are adjacent to w.
- When $w \in \Gamma_{R'}(U'_3) \cap \Gamma_{R'}(U'_5)$, $(u_3, u_2, u_1, u_7, u_6, u_5, x, w, y, u_3)$ is a cycle of order 9 for $x \in U'_5$ and $y \in U'_3$ such that x and y are adjacent to w.
- When $w \in \Gamma_{R'}(U'_3) \cap \Gamma_{R'}(U'_7)$, $(u_3, u_2, u_1, u_5, u_6, u_7, x, w, y, u_3)$ is a cycle of order 9 for $x \in U'_7$ and $y \in U'_3$ such that x and y are adjacent to w.
- When $w \in \Gamma_{R'}(U'_3) \cap \Gamma_{R'}(U'_8)$, $(u_3, u_1, u_5, u_6, u_7, u_8, x, w, y, u_3)$ is a cycle of order 9 for $x \in U'_8$ and $y \in U'_3$ such that x and y are adjacent to w.
- When $w \in \Gamma_{R'}(U'_3) \cap \Gamma_{R'}(U'_9)$, $(u_3, u_2, u_1, u_7, u_6, u_9, x, w, y, u_3)$ is a cycle of order 9 for $x \in U'_9$ and $y \in U'_3$ such that x and y are adjacent to w.
- When $w \in \Gamma_{R'}(U'_5) \cap \Gamma_{R'}(U'_7)$, $(u_5, u_4, u_3, u_1, u_8, u_7, x, w, y, u_5)$ is a cycle of order 9 for $x \in U'_7$ and $y \in U'_5$ such that x and y are adjacent to w.
- When $w \in \Gamma_{R'}(U'_5) \cap \Gamma_{R'}(U'_8)$, $(u_5, u_4, u_1, u_6, u_7, u_8, x, w, y, u_5)$ is a cycle of order 9 for $x \in U'_8$ and $y \in U'_5$ such that x and y are adjacent to w.
- When $w \in \Gamma_{R'}(U'_5) \cap \Gamma_{R'}(U'_9)$, $(u_5, u_1, u_2, u_3, u_4, u_9, x, w, y, u_5)$ is a cycle of order 9 for $x \in U'_9$ and $y \in U'_5$ such that x and y are adjacent to w.
- When $w \in \Gamma_{R'}(U'_7) \cap \Gamma_{R'}(U'_8)$, $(u_7, u_6, u_5, u_4, u_1, u_8, x, w, y, u_7)$ is a cycle of order 9 for $x \in U'_8$ and $y \in U'_7$ such that x and y are adjacent to w.
- When $w \in \Gamma_{R'}(U'_7) \cap \Gamma_{R'}(U'_9)$, $(u_7, u_8, u_1, u_3, u_4, u_9, x, w, y, u_7)$ is a cycle of order 9 for $x \in U'_9$ and $y \in U'_7$ such that x and y are adjacent to w.
- When $w \in \Gamma_{R'}(U'_8) \cap \Gamma_{R'}(U'_9)$, $(u_8, u_7, u_6, u_1, u_4, u_9, x, w, y, u_8)$ is a cycle of order 9 for $x \in U'_9$ and $y \in U'_8$ such that x and y are adjacent to w.

The fourth property is: $xy \notin E(G)$ for all $i, j \in \{2, 3, 5, 7, 8, 9\}$ with $i \neq j$ and for all $x \in \Gamma_R(U_i)$ and $y \in \Gamma_R(U_j)$.

- When $x \in \Gamma_R(U_2)$ and $y \in \Gamma_R(U_3)$, then $(u_2, u_1, u_5, u_4, u_3, w, y, x, v, u_2)$ is a cycle of order 9 where $v \in U_2$ and $w \in U_3$ are adjacent to x and y respectively.
- When $x \in \Gamma_R(U_2)$ and $y \in \Gamma_R(U_5)$, then $(u_2, u_3, u_4, u_1, u_5, w, y, x, v, u_2)$ is a cycle of order 9 where $v \in U_2$ and $w \in U_5$ are adjacent to x and y respectively.
- When $x \in \Gamma_R(U_2)$ and $y \in \Gamma_R(U_7)$, then $(u_2, u_3, u_4, u_1, u_7, w, y, x, v, u_2)$ is a cycle of order 9 where $v \in U_2$ and $w \in U_7$ are adjacent to x and y respectively.

- When $x \in \Gamma_R(U_2)$ and $y \in \Gamma_R(U_8)$, then $(u_2, u_3, u_4, u_1, u_8, w, y, x, v, u_2)$ is a cycle of order 9 where $v \in U_2$ and $w \in U_8$ are adjacent to x and y respectively.
- When $x \in \Gamma_R(U_2)$ and $y \in \Gamma_R(U_9)$, then $(u_2, u_1, u_5, u_4, u_9, w, y, x, v, u_2)$ is a cycle of order 9 where $v \in U_2$ and $w \in U_9$ are adjacent to x and y respectively.
- When $x \in \Gamma_R(U_3)$ and $y \in \Gamma_R(U_5)$, then $(u_3, u_2, u_1, u_6, u_5, w, y, x, v, u_3)$ is a cycle of order 9 where $v \in U_3$ and $w \in U_5$ are adjacent to x and y respectively.
- When $x \in \Gamma_R(U_3)$ and $y \in \Gamma_R(U_7)$, then $(u_3, u_2, u_1, u_6, u_7, w, y, x, v, u_3)$ is a cycle of order 9 where $v \in U_3$ and $w \in U_7$ are adjacent to x and y respectively.
- When $x \in \Gamma_R(U_3)$ and $y \in \Gamma_R(U_8)$, then $(u_3, u_1, u_6, u_7, u_8, w, y, x, v, u_3)$ is a cycle of order 9 where $v \in U_3$ and $w \in U_8$ are adjacent to x and y respectively.
- When $x \in \Gamma_R(U_3)$ and $y \in \Gamma_R(U_9)$, then $(u_3, u_4, u_5, u_6, u_9, w, y, x, v, u_3)$ is a cycle of order 9 where $v \in U_3$ and $w \in U_9$ are adjacent to x and y respectively.
- When $x \in \Gamma_R(U_5)$ and $y \in \Gamma_R(U_7)$, then $(u_5, u_4, u_1, u_8, u_7, w, y, x, v, u_5)$ is a cycle of order 9 where $v \in U_5$ and $w \in U_7$ are adjacent to x and y respectively.
- When $x \in \Gamma_R(U_5)$ and $y \in \Gamma_R(U_8)$, then $(u_5, u_1, u_6, u_7, u_8, w, y, x, v, u_5)$ is a cycle of order 9 where $v \in U_5$ and $w \in U_8$ are adjacent to x and y respectively.
- When $x \in \Gamma_R(U_5)$ and $y \in \Gamma_R(U_9)$, then $(u_5, u_1, u_3, u_4, u_9, w, y, x, v, u_5)$ is a cycle of order 9 where $v \in U_5$ and $w \in U_9$ are adjacent to x and y respectively.
- When $x \in \Gamma_R(U_7)$ and $y \in \Gamma_R(U_8)$, then $(u_7, u_6, u_5, u_1, u_8, w, y, x, v, u_7)$ is a cycle of order 9 where $v \in U_7$ and $w \in U_8$ are adjacent to x and y respectively.
- When $x \in \Gamma_R(U_7)$ and $y \in \Gamma_R(U_9)$, then $(u_7, u_6, u_5, u_4, u_9, w, y, x, v, u_7)$ is a cycle of order 9 where $v \in U_7$ and $w \in U_9$ are adjacent to x and y respectively.
- When $x \in \Gamma_R(U_8)$ and $y \in \Gamma_R(U_9)$, then $(u_8, u_1, u_5, u_6, u_9, w, y, x, v, u_8)$ is a cycle of order 9 where $v \in U_8$ and $w \in U_9$ are adjacent to x and y respectively.

In the end there are 4 properties:

(A1) $U'_i \cap U'_j = \emptyset$ for all $i, j \in \{2, 3, 5, 7, 8, 9\}$ and $i \neq j$.

(B1) $xy \notin E(G)$ for all $i, j \in \{2, 3, 5, 7, 8, 9\}$ with $i \neq j$ and for all $x \in U'_i$ and $y \in U'_j$.

(C1) $\Gamma_{R'}(U'_i) \cap \Gamma_{R'}(U'_j) = \emptyset$ for all $i, j \in \{2, 3, 5, 7, 8, 9\}$ with $i \neq j$.

(D1) $xy \notin E(G)$ for all $i, j \in \{2, 3, 5, 7, 8, 9\}$ with $i \neq j$ and for all $x \in \Gamma_R(U_i)$ and $y \in \Gamma_R(U_j)$.

Since the maximum order of an independent set is 7, at least five of the induced subgraphs $\langle U'_i \cup \Gamma_{R'}(U'_j)\rangle_G$ for $i \in \{2, 3, 5, 7, 8, 9\}$ are complete graphs. It remains to find out which of the induced subgraphs have at least 7 vertices. Let $w \in U'_i$ for some $i \in \{2, 5, 8, 9\}$. By property (A1), w is adjacent to only one vertex in U', namely u_i. This implies that $|\Gamma_{R'}(w)| \geq 7$, by recalling that $\delta(G) \geq 8$. Since $\Gamma_{R'}(w) \subset U'_i \cup \Gamma_{R'}(U'_i)$, it follows that $|U'_i \cup \Gamma_{R'}(U'_i)| \geq 7$ for each $i \in \{2, 5, 8, 9\}$. Therefore, at least three of the induced subgraphs $\langle U'_i \cup \Gamma_{R'}(U'_i)\rangle_G$ for $i \in \{2, 3, 5, 7, 8, 9\}$ contain a complete graph of order 7 and so G contains a complete graph of order 7. ■

Lemma 8 *If G contains $K_1 + P_6$, then G contains $K_1 + P_7$ or K_7.*

Proof: Consider $K_1 + P_6$, where $K_1 = u_1$ and $P_6 = (u_2, u_3, u_4, u_5, u_6, u_7)$. The graph of $K_1 + P_6$ is presented in Figure 15. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$, $R = G \setminus U$ and $U_i = \Gamma(u_i) \cap V(R)$

for each $1 \leq i \leq 7$. Each vertex $u_i \in U$ is adjacent to at most 6 other vertices in U for $1 \leq i \leq 7$. However, since $\delta(G) \geq 8$, each vertex $u_i \in U$ must also be adjacent to at least 2 vertices that are not in U for each $1 \leq i \leq 7$. That is, $u_i \in U$ is adjacent to 2 vertices in R and therefore $|U_i| \geq 2$ for all $1 \leq i \leq 7$. There are two cases to consider: $U_i \cap U_j = \emptyset$ for all $2 \leq i \leq 7$ and $U_i \cap U_j \neq \emptyset$ for all $2 \leq i \leq 7$.

1. $U_i \cap U_j = \emptyset$ for all $2 \leq i \leq 7$. Also for this lemma, in some cases it might not be possible to have an edge $xy \in E(G)$ for $x \in U_i$ and $y \in U_j$. When there is an edge $xy \in E(G)$ for $x \in U_i$ and $y \in U_j$, then there is a path of order 4 given by (u_i, x, y, u_j). In that case, it should not be possible to form a path of order 7 in U from u_i to u_j, for else there is a cycle of order 9. Here is a list of all the paths of order 7 between vertices u_i and u_j for $2 \leq i < j \leq 7$:

   - from u_2 to u_3: $(u_2, u_1, u_7, u_6, u_5, u_4, u_3)$
   - from u_2 to u_4: $(u_2, u_3, u_1, u_7, u_6, u_5, u_4)$
   - from u_2 to u_5: $(u_2, u_3, u_4, u_1, u_7, u_6, u_5)$
   - from u_2 to u_6: $(u_2, u_3, u_4, u_5, u_1, u_7, u_6)$
   - from u_2 to u_7: $(u_2, u_3, u_4, u_5, u_6, u_1, u_7)$
   - from u_3 to u_4: $(u_3, u_2, u_1, u_7, u_6, u_5, u_4)$
   - from u_3 to u_7: $(u_3, u_2, u_1, u_4, u_5, u_6, u_7)$
   - from u_4 to u_5: $(u_4, u_3, u_2, u_1, u_7, u_6, u_5)$
   - from u_4 to u_7: $(u_4, u_3, u_2, u_1, u_5, u_6, u_7)$
   - from u_5 to u_6: $(u_5, u_4, u_3, u_2, u_1, u_7, u_6)$
   - from u_5 to u_7: $(u_5, u_4, u_3, u_2, u_1, u_6, u_7)$
   - from u_6 to u_7: $(u_6, u_5, u_4, u_3, u_2, u_1, u_7)$

   Hence, for the pairs of vertices u_i and u_j listed above the edge xy should not be in $E(G)$ for $x \in U_i$ and $y \in U_j$. In particular, $xy \notin E(G)$ for all $x \in U_i$ and $y \in U_j$ and for all $2 \leq i < j \leq 7$ except possibly for $(i,j) \in \{(3,5), (3,6), (4,6)\}$. The next argumentation shows that $\Gamma_R(U_i) \cap \Gamma_R(U_j) = \emptyset$ for all $2 \leq i < j \leq 7$. If there would exist a $w \in \Gamma_R(U_i) \cap \Gamma_R(U_j)$ for some $2 \leq i < j \leq 7$, then there is a path of order 5 given by (u_i, x, w, y, u_j), where $x \in U_i$ and $y \in U_j$ such that x and y a both adjacent to w. In that case, it should not be possible to have a path of order 6 in U from u_i to u_j, for else

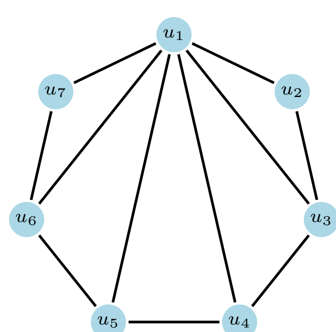

Figure 15: A graph $K_1 + P_6$

there is a cycle of order 9 in G. However, there is in fact a path of order 6 between any 2 vertices u_i and u_j for all $2 \leq i < j \leq 7$ by the following list:

- from u_2 to u_3: $(u_2, u_1, u_6, u_5, u_4, u_3)$
- from u_2 to u_4: $(u_2, u_1, u_7, u_6, u_5, u_4)$
- from u_2 to u_5: $(u_2, u_3, u_1, u_7, u_6, u_5)$
- from u_2 to u_6: $(u_2, u_3, u_4, u_1, u_7, u_6)$
- from u_2 to u_7: $(u_2, u_3, u_4, u_5, u_6, u_7)$
- from u_3 to u_4: $(u_3, u_1, u_7, u_6, u_5, u_4)$
- from u_3 to u_5: $(u_3, u_2, u_1, u_7, u_6, u_5)$
- from u_3 to u_6: $(u_3, u_2, u_1, u_4, u_5, u_6)$
- from u_3 to u_7: $(u_3, u_2, u_1, u_5, u_6, u_7)$
- from u_4 to u_5: $(u_4, u_3, u_1, u_7, u_6, u_5)$
- from u_4 to u_6: $(u_4, u_3, u_2, u_1, u_7, u_6)$
- from u_4 to u_7: $(u_4, u_3, u_2, u_1, u_6, u_7)$
- from u_5 to u_6: $(u_5, u_4, u_3, u_1, u_7, u_6)$
- from u_5 to u_7: $(u_5, u_4, u_3, u_2, u_1, u_7)$
- from u_6 to u_7: $(u_6, u_5, u_4, u_3, u_1, u_7)$

Therefore, there must not exist a vertex $w \in \Gamma_R(U_i) \cap \Gamma_R(U_j)$ for some $2 \leq i < j \leq 7$, since otherwise there would be a cycle of order 9. That means that $\Gamma_R(U_i) \cap \Gamma_R(U_j) = \emptyset$ for all $2 \leq i < j \leq 7$. Furthermore, it holds that $xy \notin E(G)$ for all $2 \leq i < j \leq 7$ and for all $x \in \Gamma_R(U_i)$ and $y \in \Gamma_R(U_j)$. This will be shown by a similar argumentation as before. When there is an edge $xy \in E(G)$ for $x \in \Gamma_R(U_i)$ and $y \in \Gamma_R(U_j)$, then there is a path of order 6 given by (u_i, v, x, y, w, u_j) where $v \in U_i$ and $w \in U_j$ are adjacent to x and y respectively. Since the graph G does not contain any cycles of order 9, there must be no paths of order 5 in U from u_i to u_j for $2 \leq i < j \leq 7$. However, it turns out that there actually is a path of order 5 between any pair of vertices u_i and u_j for $2 \leq i < j \leq 7$, by the following list:

- from u_2 to u_3: $(u_2, u_1, u_5, u_4, u_3)$
- from u_2 to u_4: $(u_2, u_1, u_6, u_5, u_4)$
- from u_2 to u_5: $(u_2, u_1, u_7, u_6, u_5)$
- from u_2 to u_6: $(u_2, u_1, u_4, u_5, u_6)$
- from u_2 to u_7: $(u_2, u_1, u_5, u_6, u_7)$
- from u_3 to u_4: $(u_3, u_1, u_6, u_5, u_4)$
- from u_3 to u_5: $(u_3, u_2, u_1, u_6, u_5)$
- from u_3 to u_6: $(u_3, u_2, u_1, u_7, u_6)$
- from u_3 to u_7: $(u_3, u_4, u_5, u_6, u_7)$
- from u_4 to u_5: $(u_4, u_1, u_7, u_6, u_5)$
- from u_4 to u_6: $(u_4, u_3, u_1, u_7, u_6)$
- from u_4 to u_7: $(u_4, u_3, u_2, u_1, u_7)$

- from u_5 to u_6: $(u_5, u_4, u_1, u_7, u_6)$
- from u_5 to u_7: $(u_5, u_4, u_3, u_1, u_7)$
- from u_6 to u_7: $(u_6, u_5, u_4, u_1, u_7)$

Therefore, there is a path of order 5 between any pair of vertices u_i and u_j for $2 \leq i < j \leq 7$. That means that if for $x \in \Gamma_R(U_i)$ and $y \in \Gamma_R(U_j)$, the edge xy would be in $E(G)$, then there is a cycle of order 9 combining the path of order 5 in U from u_i to u_j with the path (u_i, v, x, y, w, u_j) where $v \in U_i$ and $w \in U_j$ are adjacent to x and y respectively. Hence $xy \notin E(G)$ for all $2 \leq i < j \leq 7$ and for all $x \in \Gamma_R(U_i)$ and $y \in \Gamma_R(U_j)$. In the end, the following results are obtained:

(A) $xy \notin E(G)$ for all $x \in U_i$ and $y \in U_j$ and for all $2 \leq i < j \leq 7$ except possibly for $(i,j) \in \{(3,5),(3,6),(4,6)\}$.

(B) $\Gamma_R(U_i) \cap \Gamma_R(U_j) = \emptyset$ for all $2 \leq i < j \leq 7$.

(C) $xy \notin E(G)$ for all $2 \leq i < j \leq 7$ and for all $x \in \Gamma_R(U_i)$ and $y \in \Gamma_R(U_j)$.

Since G does not contain an independent set of 8 vertices, the order of the maximal independent set is 7. Now try to construct an independent set. By the results (A), (B) and (C) above, it holds for some $2 \leq i < j \leq 7$ that the induced subgraph $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ is disjoint from and not adjacent to $\langle U_j \cup \Gamma_R(U_j)\rangle_G$. Property (A) says that some of those induced subgraphs may not be disjoint or not adjacent. It says that xy can be in $E(G)$ for $x \in U_i$ and $y \in U_j$ for $(i,j) \in \{(3,5),(3,6),(4,6)\}$. Looking closer to those pairs of i and j one can see that either 3 or 6 appears in each pair. Therefore, when 3 and 6 are taken out of consideration, then for $i = 2,4,5,7$, the induced subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ are disjoint and nonadjacent from each other. Now pick one vertex from each induced subgraph $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ for $i = 2,4,5,7$. Then this forms an independent of 4 vertices since the induced subgraphs are disjoint and not adjacent to each other. It should not be possible to pick one more vertex from each of the induced subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ for $i = 2,4,5,7$ such that all vertices are adjacent to each other, because that would make a set of 8 independent vertices. When picking one more vertex from each induced subgraph $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ for $i = 2,4,5,7$, then this forms a set of 8 vertices. However, this set of 8 elements should not be independent, therefore, at least one of the induced subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ for $i = 2,4,5,7$ should be complete. Say, $\langle U_k \cup \Gamma_R(U_k)\rangle_G$ is complete for some $k = 2,4,5,7$. Let $x \in U_k$. The vertex x is adjacent to $u_k \in U$ and possibly also to u_1, and so x is adjacent to at least 6 vertices in R, since $\delta(G) \geq 8$. Then $|\Gamma_R(x)| \geq 6$, thus $|\{x\} \cup \Gamma_R(x)| \geq 7$ and so $|\langle U_k \cup \Gamma_R(U_k)\rangle_G| \geq 7$. That means that G contains a K_7.

2. $U_i \cap U_j \neq \emptyset$ for some $2 \leq i < j \leq 7$. Let u_8 be a vertex in G such that $u_8 \in U_r \cap U_s$. Now redefine the sets U, R and U_i. To that end, let $U' = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$, $R' = G \setminus U'$ and $U'_i = \Gamma(u_i) \cap V(R')$. For $2 \leq i \leq 8$, each vertex $u_i \in U'$ is adjacent to at most 7 other vertices in U'. However, since $\delta(G) \geq 8$, each vertex $u_i \in U'$ must also be adjacent to at least one vertex in R', hence $U'_i \neq \emptyset$ for all $2 \leq i \leq 8$. For each pair (r,s), that is for $u_8 \in U_r \cap U_s$, the following properties will be shown for some set J:

(A1) $U'_i \cap U'_j = \emptyset$ for all $i,j \in J$.

(B1) $xy \notin E(G)$ for all $i,j \in J$ and for all $x \in U'_i$ and $y \in U'_j$.

(C1) $\Gamma_{R'}(U'_i) \cap \Gamma_{R'}(U'_j) = \emptyset$ for all $i,j \in J$.

(D1) $xy \notin E(G)$ for all $i,j \in J$ and for all $x \in \Gamma_R(U_i)$ and $y \in \Gamma_R(U_j)$.

By now, it must be clear that when assuming that properties (A1), (B1), (C1) and (D1) do not hold, the existence of paths in U' of order 8, 7, 6 and 5, respectively, between vertices u_i and u_j causes G to contain cycles of order 9. This is due to the fact that if properties (A1), (B1), (C1) and (D1) do not hold, then there are paths of order 3, 4, 5 and 6, respectively, from u_i to u_j through R'. For each pair (r, s), where $r, s \in \{2, 3, 4, 5, 6, 7\}$ and $r \neq s$, properties (A1), (B1), (C1) and (D1) will be proven.

- Let $(r, s) = (6, 7)$ and $J = \{2, 4, 5, 8\}$. Then $u_8 \in U_6 \cap U_7$. The following table shows that for all $i, j \in J$ with $i < j$, there are paths of order 8, 7, 6 and 5 between vertices u_i and u_j. Since there cannot be cycles of order 9 in G, the properties (A1), (B1), (C1) and (D1) must hold.

(i,j)	path of order 8	path of order 7	path of order 6	path of order 5
$(2,4)$	$(u_2, u_3, u_1, u_7, u_8, u_6, u_5, u_4)$	$(u_2, u_1, u_7, u_8, u_6, u_5, u_4)$	$(u_2, u_1, u_7, u_6, u_5, u_4)$	$(u_2, u_1, u_6, u_5, u_4)$
$(2,5)$	$(u_2, u_3, u_4, u_1, u_7, u_8, u_6, u_5)$	$(u_2, u_3, u_1, u_7, u_8, u_6, u_5)$	$(u_2, u_1, u_7, u_8, u_6, u_5)$	$(u_2, u_1, u_7, u_6, u_5)$
$(2,8)$	$(u_2, u_3, u_4, u_5, u_6, u_1, u_7, u_8)$	$(u_2, u_3, u_4, u_5, u_1, u_7, u_8)$	$(u_2, u_3, u_4, u_1, u_7, u_8)$	$(u_2, u_3, u_1, u_7, u_8)$
$(4,5)$	$(u_4, u_3, u_2, u_1, u_7, u_8, u_6, u_5)$	$(u_4, u_3, u_2, u_1, u_7, u_6, u_5)$	$(u_4, u_3, u_2, u_1, u_6, u_5)$	$(u_4, u_3, u_2, u_1, u_5)$
$(4,8)$	$(u_4, u_3, u_2, u_1, u_5, u_6, u_7, u_8)$	$(u_4, u_3, u_2, u_1, u_6, u_7, u_8)$	$(u_4, u_3, u_2, u_1, u_7, u_8)$	$(u_4, u_5, u_6, u_7, u_8)$
$(5,8)$	$(u_5, u_4, u_3, u_2, u_1, u_6, u_7, u_8)$	$(u_5, u_4, u_3, u_2, u_1, u_7, u_8)$	$(u_5, u_4, u_3, u_1, u_7, u_8)$	$(u_5, u_4, u_1, u_7, u_8)$

- Let $(r, s) = (2, 3)$. The graph in Figure 15 is symmetric, so since properties (A1), (B1), (C1) and (D1) hold for $(r, s) = (6, 7)$ and $J = \{2, 4, 5, 8\}$, they also hold for $(r, s) = (2, 3)$ and $J = \{4, 5, 7, 8\}$
- Let $(r, s) = (5, 7)$ and $J = \{2, 4, 6, 8\}$. Then $u_8 \in U_5 \cap U_7$. The following table shows that for all $i, j \in J$ with $i < j$, there are paths of order 8, 7, 6 and 5 between vertices u_i and u_j. Since there cannot be cycles of order 9 in G, the properties (A1), (B1), (C1) and (D1) must hold.

(i,j)	path of order 8	path of order 7	path of order 6	path of order 5
$(2,4)$	$(u_2, u_3, u_1, u_6, u_7, u_8, u_5, u_4)$	$(u_2, u_3, u_1, u_7, u_6, u_5, u_4)$	$(u_2, u_1, u_7, u_6, u_5, u_4)$	$(u_2, u_1, u_6, u_5, u_4)$
$(2,6)$	$(u_2, u_3, u_4, u_1, u_7, u_8, u_5, u_6)$	$(u_2, u_3, u_1, u_7, u_8, u_5, u_6)$	$(u_2, u_1, u_7, u_8, u_5, u_6)$	$(u_2, u_3, u_4, u_5, u_6)$
$(2,8)$	$(u_2, u_3, u_4, u_5, u_6, u_1, u_7, u_8)$	$(u_2, u_3, u_4, u_5, u_1, u_7, u_8)$	$(u_2, u_3, u_4, u_1, u_7, u_8)$	$(u_2, u_3, u_1, u_7, u_8)$
$(4,6)$	$(u_4, u_3, u_2, u_1, u_5, u_8, u_7, u_6)$	$(u_4, u_3, u_1, u_7, u_8, u_5, u_6)$	$(u_4, u_3, u_2, u_1, u_7, u_6)$	$(u_4, u_3, u_2, u_1, u_6)$
$(4,8)$	$(u_4, u_3, u_2, u_1, u_5, u_6, u_7, u_8)$	$(u_4, u_3, u_2, u_1, u_6, u_7, u_8)$	$(u_4, u_3, u_2, u_1, u_7, u_8)$	$(u_4, u_5, u_6, u_7, u_8)$
$(6,8)$	$(u_6, u_5, u_4, u_3, u_2, u_1, u_7, u_8)$	$(u_6, u_5, u_4, u_3, u_1, u_7, u_8)$	$(u_6, u_5, u_4, u_1, u_7, u_8)$	$(u_6, u_5, u_1, u_7, u_8)$

- Let $(r, s) = (2, 4)$. The graph in Figure 15 is symmetric, so since properties (A1), (B1), (C1) and (D1) hold for $(r, s) = (5, 7)$ and $J = \{2, 4, 6, 8\}$, they also hold for $(r, s) = (2, 4)$ and $J = \{3, 5, 7, 8\}$.
- Let $(r, s) = (4, 7)$ and $J = \{2, 5, 6, 8\}$. Then $u_8 \in U_4 \cap U_7$. The following table shows that for all $i, j \in J$ with $i < j$, there are paths of order 8, 7, 6 and 5 between vertices u_i and u_j. Since there cannot be cycles of order 9 in G, the properties (A1), (B1), (C1) and (D1) must hold.
- Let $(r, s) = (2, 5)$. The graph in Figure 15 is symmetric, so since properties (A1), (B1), (C1) and (D1) hold for $(r, s) = (4, 7)$ and $J = \{2, 5, 6, 8\}$, they also hold for $(r, s) = (2, 5)$ and $J = \{3, 4, 7, 8\}$.
- Let $(r, s) = (3, 7)$ and $J = \{2, 4, 6, 8\}$. Then $u_8 \in U_3 \cap U_7$. The following table shows that for all $i, j \in J$ with $i < j$, there are paths of order 8, 7, 6 and 5 between vertices

(i,j)	path of order 8	path of order 7	path of order 6	path of order 5
$(2,5)$	$(u_2,u_3,u_1,u_6,u_7,u_8,u_4,u_5)$	$(u_2,u_3,u_4,u_1,u_7,u_6,u_5)$	$(u_2,u_3,u_1,u_7,u_6,u_5)$	(u_2,u_1,u_7,u_6,u_5)
$(2,6)$	$(u_2,u_3,u_1,u_7,u_8,u_4,u_5,u_6)$	$(u_2,u_1,u_7,u_8,u_4,u_5,u_6)$	$(u_2,u_3,u_1,u_4,u_5,u_6)$	(u_2,u_3,u_4,u_5,u_6)
$(2,8)$	$(u_2,u_3,u_1,u_7,u_6,u_5,u_4,u_8)$	$(u_2,u_3,u_4,u_5,u_1,u_7,u_8)$	$(u_2,u_3,u_4,u_1,u_7,u_8)$	(u_2,u_3,u_1,u_7,u_8)
$(5,6)$	$(u_5,u_1,u_2,u_2,u_4,u_8,u_7,u_6)$	$(u_5,u_4,u_3,u_2,u_1,u_7,u_6)$	$(u_5,u_4,u_3,u_2,u_1,u_6)$	(u_5,u_4,u_3,u_1,u_6)
$(5,8)$	$(u_5,u_4,u_3,u_2,u_1,u_6,u_7,u_8)$	$(u_5,u_4,u_3,u_1,u_6,u_7,u_8)$	$(u_5,u_4,u_1,u_6,u_7,u_8)$	(u_5,u_1,u_6,u_7,u_8)
$(6,8)$	$(u_6,u_5,u_4,u_3,u_2,u_1,u_7,u_8)$	$(u_6,u_5,u_4,u_3,u_1,u_7,u_8)$	$(u_6,u_5,u_4,u_1,u_7,u_8)$	(u_6,u_5,u_1,u_7,u_8)

u_i and u_j. Since there cannot be cycles of order 9 in G, the properties (A1), (B1), (C1) and (D1) must hold.

(i,j)	path of order 8	path of order 7	path of order 6	path of order 5
$(2,4)$	$(u_2,u_1,u_5,u_6,u_7,u_8,u_3,u_4)$	$(u_2,u_3,u_8,u_7,u_6,u_5,u_4)$	$(u_2,u_1,u_7,u_6,u_5,u_4)$	(u_2,u_1,u_6,u_5,u_4)
$(2,6)$	$(u_2,u_3,u_8,u_7,u_1,u_4,u_5,u_6)$	$(u_2,u_3,u_8,u_7,u_1,u_5,u_6)$	$(u_2,u_3,u_1,u_4,u_5,u_6)$	(u_2,u_3,u_4,u_5,u_6)
$(2,8)$	$(u_2,u_1,u_7,u_6,u_5,u_4,u_3,u_8)$	$(u_2,u_1,u_6,u_5,u_4,u_3,u_8)$	$(u_2,u_1,u_5,u_4,u_3,u_8)$	(u_2,u_1,u_4,u_3,u_8)
$(4,6)$	$(u_4,u_5,u_1,u_2,u_3,u_8,u_7,u_6)$	$(u_4,u_1,u_2,u_3,u_8,u_7,u_6)$	$(u_4,u_3,u_2,u_1,u_7,u_6)$	(u_4,u_3,u_1,u_7,u_6)
$(4,8)$	$(u_4,u_5,u_6,u_7,u_1,u_2,u_3,u_8)$	$(u_4,u_5,u_6,u_1,u_2,u_3,u_8)$	$(u_4,u_5,u_1,u_2,u_3,u_8)$	(u_4,u_1,u_2,u_3,u_8)
$(6,8)$	$(u_6,u_5,u_4,u_3,u_2,u_1,u_7,u_8)$	$(u_6,u_5,u_4,u_3,u_1,u_7,u_8)$	$(u_6,u_5,u_4,u_1,u_7,u_8)$	(u_6,u_5,u_1,u_7,u_8)

- Let $(r,s)=(2,6)$. The graph in Figure 15 is symmetric, so since properties (A1), (B1), (C1) and (D1) hold for $(r,s)=(3,7)$ and $J=\{2,4,6,8\}$, they also hold for $(r,s)=(2,6)$ and $J=\{3,5,7,8\}$.
- Let $(r,s)=(2,7)$ and $J=\{4,5,6,8\}$. Then $u_8\in U_3\cap U_7$. The following table shows that for all $i,j\in J$ with $i<j$, there are paths of order 8, 7, 6 and 5 between vertices u_i and u_j. Since there cannot be cycles of order 9 in G, the properties (A1), (B1), (C1) and (D1) must hold.

(i,j)	path of order 8	path of order 7	path of order 6	path of order 5
$(4,5)$	$(u_4,u_3,u_1,u_2,u_8,u_7,u_6,u_5)$	$(u_4,u_3,u_2,u_1,u_7,u_6,u_5)$	$(u_4,u_3,u_2,u_1,u_6,u_5)$	(u_4,u_3,u_2,u_1,u_5)
$(4,6)$	$(u_4,u_5,u_1,u_3,u_2,u_8,u_7,u_6)$	$(u_4,u_1,u_3,u_2,u_8,u_7,u_6)$	$(u_4,u_3,u_2,u_1,u_7,u_6)$	(u_4,u_3,u_1,u_7,u_6)
$(4,8)$	$(u_4,u_3,u_2,u_1,u_5,u_6,u_7,u_8)$	$(u_4,u_3,u_2,u_1,u_6,u_7,u_8)$	$(u_4,u_3,u_2,u_1,u_7,u_8)$	(u_4,u_5,u_6,u_7,u_8)
$(5,6)$	$(u_4,u_5,u_1,u_2,u_3,u_8,u_7,u_6)$	$(u_4,u_1,u_2,u_3,u_8,u_7,u_6)$	$(u_4,u_3,u_2,u_1,u_7,u_6)$	(u_4,u_3,u_1,u_7,u_6)
$(5,8)$	$(u_5,u_4,u_3,u_2,u_1,u_6,u_7,u_8)$	$(u_5,u_4,u_3,u_1,u_6,u_7,u_8)$	$(u_5,u_4,u_1,u_6,u_7,u_8)$	(u_5,u_1,u_6,u_7,u_8)
$(6,8)$	$(u_6,u_5,u_4,u_3,u_2,u_1,u_7,u_8)$	$(u_6,u_5,u_4,u_3,u_1,u_7,u_8)$	$(u_6,u_5,u_4,u_1,u_7,u_8)$	(u_6,u_5,u_1,u_7,u_8)

- Let $(r,s)=(5,6)$ and $J=\{2,4,7,8\}$. Then $u_8\in U_5\cap U_6$. The following table shows that for all $i,j\in J$ with $i<j$, there are paths of order 8, 7, 6 and 5 between vertices u_i and u_j. Since there cannot be cycles of order 9 in G, the properties (A1), (B1), (C1) and (D1) must hold.
- Let $(r,s)=(3,4)$. The graph in Figure 15 is symmetric, so since properties (A1), (B1), (C1) and (D1) hold for $(r,s)=(5,6)$ and $J=\{2,4,7,8\}$, they also hold for $(r,s)=(3,4)$ and $J=\{2,5,7,8\}$.
- Let $(r,s)=(4,6)$ and $J=\{2,5,7,8\}$. Then $u_8\in U_4\cap U_6$. The following table shows that for all $i,j\in J$ with $i<j$, there are paths of order 8, 7, 6 and 5 between vertices u_i and u_j. Since there cannot be cycles of order 9 in G, the properties (A1), (B1), (C1) and (D1) must hold.

(i,j)	path of order 8	path of order 7	path of order 6	path of order 5
$(2,4)$	$(u_2,u_3,u_1,u_7,u_6,u_8,u_5,u_4)$	$(u_2,u_3,u_1,u_7,u_6,u_5,u_4)$	$(u_2,u_1,u_7,u_6,u_5,u_4)$	(u_2,u_1,u_6,u_5,u_4)
$(2,7)$	$(u_2,u_3,u_4,u_5,u_8,u_6,u_1,u_7)$	$(u_2,u_3,u_4,u_5,u_8,u_6,u_7)$	$(u_2,u_3,u_4,u_5,u_6,u_7)$	(u_2,u_1,u_5,u_6,u_7)
$(2,8)$	$(u_2,u_3,u_4,u_5,u_1,u_7,u_6,u_8)$	$(u_2,u_3,u_4,u_5,u_1,u_6,u_8)$	$(u_2,u_3,u_4,u_5,u_6,u_8)$	(u_2,u_3,u_4,u_5,u_8)
$(4,7)$	$(u_4,u_3,u_2,u_1,u_5,u_8,u_6,u_7)$	$(u_4,u_3,u_2,u_1,u_5,u_6,u_7)$	$(u_4,u_3,u_2,u_1,u_6,u_7)$	(u_4,u_3,u_2,u_1,u_7)
$(4,8)$	$(u_4,u_3,u_2,u_1,u_7,u_6,u_5,u_8)$	$(u_4,u_3,u_2,u_1,u_7,u_6,u_8)$	$(u_4,u_3,u_2,u_1,u_7,u_8)$	(u_4,u_1,u_7,u_6,u_8)
$(7,8)$	$(u_7,u_1,u_2,u_3,u_4,u_5,u_6,u_8)$	$(u_7,u_1,u_2,u_3,u_4,u_5,u_8)$	$(u_7,u_1,u_3,u_4,u_5,u_8)$	(u_7,u_1,u_4,u_5,u_8)

(i,j)	path of order 8	path of order 7	path of order 6	path of order 5
$(2,5)$	$(u_2,u_3,u_4,u_8,u_6,u_7,u_1,u_5)$	$(u_2,u_3,u_4,u_1,u_7,u_6,u_5)$	$(u_2,u_3,u_1,u_7,u_6,u_5)$	(u_2,u_1,u_7,u_6,u_5)
$(2,7)$	$(u_2,u_3,u_4,u_8,u_6,u_5,u_1,u_7)$	$(u_2,u_1,u_3,u_4,u_5,u_6,u_7)$	$(u_2,u_3,u_4,u_5,u_6,u_7)$	(u_2,u_1,u_5,u_6,u_7)
$(2,8)$	$(u_2,u_3,u_4,u_5,u_1,u_7,u_6,u_8)$	$(u_2,u_3,u_4,u_5,u_1,u_6,u_8)$	$(u_2,u_3,u_4,u_5,u_6,u_8)$	(u_2,u_1,u_3,u_4,u_8)
$(5,7)$	$(u_5,u_1,u_2,u_3,u_4,u_8,u_6,u_7)$	$(u_5,u_4,u_3,u_2,u_1,u_6,u_7)$	$(u_5,u_4,u_3,u_2,u_1,u_7)$	(u_5,u_4,u_3,u_1,u_7)
$(5,8)$	$(u_5,u_4,u_3,u_2,u_1,u_7,u_6,u_8)$	$(u_5,u_4,u_3,u_1,u_7,u_6,u_8)$	$(u_5,u_1,u_2,u_3,u_4,u_8)$	(u_5,u_1,u_3,u_4,u_8)
$(7,8)$	$(u_7,u_1,u_2,u_3,u_4,u_5,u_6,u_8)$	$(u_7,u_1,u_3,u_4,u_5,u_6,u_8)$	$(u_7,u_1,u_2,u_3,u_4,u_8)$	(u_7,u_1,u_3,u_4,u_8)

- Let $(r,s)=(3,5)$. The graph in Figure 15 is symmetric, so since properties (A1), (B1), (C1) and (D1) hold for $(r,s)=(4,6)$ and $J=\{2,5,7,8\}$, they also hold for $(r,s)=(3,5)$ and $J=\{2,4,7,8\}$.
- Let $(r,s)=(3,6)$ and $J=\{2,5,7,8\}$. Then $u_8\in U_3\cap U_6$. The following table shows that for all $i,j\in J$ with $i<j$, there are paths of order 8, 7, 6 and 5 between vertices u_i and u_j. Since there cannot be cycles of order 9 in G, the properties (A1), (B1), (C1) and (D1) must hold.

(i,j)	path of order 8	path of order 7	path of order 6	path of order 5
$(2,5)$	$(u_2,u_3,u_8,u_6,u_7,u_1,u_4,u_5)$	$(u_2,u_3,u_4,u_1,u_7,u_6,u_5)$	$(u_2,u_3,u_1,u_7,u_6,u_5)$	(u_2,u_1,u_7,u_6,u_5)
$(2,7)$	$(u_2,u_3,u_8,u_6,u_5,u_4,u_1,u_7)$	$(u_2,u_1,u_3,u_4,u_5,u_6,u_7)$	$(u_2,u_3,u_4,u_5,u_6,u_7)$	(u_2,u_1,u_5,u_6,u_7)
$(2,8)$	$(u_2,u_3,u_4,u_5,u_1,u_7,u_6,u_8)$	$(u_2,u_3,u_4,u_5,u_1,u_6,u_8)$	$(u_2,u_3,u_4,u_5,u_6,u_8)$	(u_2,u_1,u_7,u_6,u_8)
$(5,7)$	$(u_5,u_4,u_1,u_2,u_3,u_8,u_6,u_7)$	$(u_5,u_4,u_3,u_2,u_1,u_6,u_7)$	$(u_5,u_4,u_3,u_2,u_1,u_7)$	(u_5,u_4,u_3,u_1,u_7)
$(5,8)$	$(u_5,u_4,u_3,u_2,u_1,u_7,u_6,u_8)$	$(u_5,u_4,u_3,u_1,u_7,u_6,u_8)$	$(u_5,u_6,u_1,u_2,u_3,u_8)$	(u_5,u_1,u_2,u_3,u_8)
$(7,8)$	$(u_7,u_6,u_5,u_4,u_1,u_2,u_3,u_8)$	$(u_7,u_1,u_3,u_4,u_5,u_6,u_8)$	$(u_7,u_6,u_1,u_2,u_3,u_8)$	(u_7,u_1,u_2,u_3,u_8)

- Let $(r,s)=(4,5)$ and $J=\{2,6,7,8\}$. Then $u_8\in U_4\cap U_5$. The following table shows that for all $i,j\in J$ with $i<j$, there are paths of order 8, 7, 6 and 5 between vertices u_i and u_j. Since there cannot be cycles of order 9 in G, the properties (A1), (B1), (C1) and (D1) must hold.

It has finally been shown that properties (A1), (B1), (C1) and (D1) hold for each pair (r,s) and some $J\subset\{2,3,4,5,6,7,8\}$ with $|J|=4$.

In case the vertex $u_8\in U_r\cap U_s$ for $(r,s)\in\{(2,3),(3,4),(4,5),(5,6),(6,7)\}$ and u_8 is adjacent to u_1, then the vertices of U' form a K_1+P_7. For example, when $u_8\in U_4\cap U_5$, then there is a path of order 7, namely, $(u_2,u_3,u_4,u_8,u_5,u_6,u_7)$ and every vertex u_i for $i\in\{2,3,4,5,6,7,8\}$ is adjacent to u_1, so that gives a K_1+P_7, where $K_1=u_1$ and $P_7=(u_2,u_3,u_4,u_8,u_5,u_6,u_7)$. Hence, the proof is done because there is a K_1+P_7 in G. When this is not the case, the following shows that there is a K_7 in G. Try to construct an independent set. Whatever the value of (r,s), properties (A1), (B1), (C1) and (D1) show that the induced subgraph $\langle U_i'\cup\Gamma_{R'}(U_i')\rangle_G$ is disjoint from and not adjacent to

(i,j)	path of order 8	path of order 7	path of order 6	path of order 5
$(2,6)$	$(u_2,u_3,u_4,u_8,u_5,u_1,u_7,u_6)$	$(u_2,u_3,u_4,u_5,u_1,u_7,u_6)$	$(u_2,u_3,u_4,u_5,u_1,u_6)$	(u_2,u_3,u_4,u_5,u_6)
$(2,7)$	$(u_2,u_3,u_4,u_8,u_5,u_6,u_1,u_7)$	$(u_2,u_3,u_4,u_8,u_5,u_6,u_7)$	$(u_2,u_3,u_4,u_5,u_6,u_7)$	(u_2,u_1,u_5,u_6,u_7)
$(2,8)$	$(u_2,u_3,u_4,u_1,u_7,u_6,u_5,u_8)$	$(u_2,u_3,u_4,u_1,u_6,u_5,u_8)$	$(u_2,u_3,u_4,u_1,u_5,u_8)$	(u_2,u_3,u_4,u_5,u_8)
$(6,7)$	$(u_6,u_5,u_8,u_4,u_3,u_2,u_1,u_7)$	$(u_6,u_5,u_4,u_3,u_2,u_1,u_7)$	$(u_6,u_5,u_4,u_3,u_1,u_7)$	(u_6,u_5,u_4,u_1,u_7)
$(6,8)$	$(u_6,u_7,u_1,u_2,u_3,u_4,u_5,u_8)$	$(u_6,u_7,u_1,u_2,u_3,u_4,u_8)$	$(u_6,u_1,u_2,u_3,u_4,u_8)$	(u_6,u_1,u_3,u_4,u_8)
$(7,8)$	$(u_7,u_6,u_5,u_1,u_2,u_3,u_4,u_8)$	$(u_7,u_6,u_1,u_2,u_3,u_4,u_8)$	$(u_7,u_1,u_2,u_3,u_4,u_8)$	(u_7,u_1,u_3,u_4,u_8)

$\langle U'_j \cup \Gamma_{R'}(U'_j)\rangle_G$ for $i,j \in J$ with $i \neq j$, where, again, $|J| = 4$. Pick one vertex from each induced subgraph $\langle U'_i \cup \Gamma_{R'}(U'_i)\rangle_G$ for $i \in J$. Then this forms an independent of 4 vertices since the induced subgraphs are disjoint from and not adjacent to each other. It should not be possible to pick one more vertex from each of the induced subgraphs $\langle U'_i \cup \Gamma_{R'}(U'_i)\rangle_G$ for $i \in J$ such that all vertices are adjacent to each other, because that would make a set of 8 independent vertices. When picking one more vertex from each induced subgraph $\langle U'_i \cup \Gamma_{R'}(U'_i)\rangle_G$ for $i \in J$, then this forms a set of 8 vertices, therefore, at least one of the induced subgraphs $\langle U'_i \cup \Gamma_{R'}(U'_i)\rangle_G$ for $i \in J$ should be complete. Since $\delta(G) \geq 8$, this complete graph contains K_7.

Therefore, in both cases, G ends up having either a K_7 or a $K_1 + P_6$. ■

Lemma 9 *If G contains K_6, then G contains $K_1 + P_6$ or K_7.*

Proof: Let G be a graph that contains K_6 and let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be the vertex set of the complete graph K_6. Define $R = G \setminus U$ and $U_i = \Gamma(u_i) \cap V(R)$ for each $1 \leq i \leq 6$. Take any vertex $u_i \in U$, then u_i is adjacent to the 5 other vertices in U, so since $\delta(G) \geq 8$, u_i is also adjacent to at least 3 vertices of R, that is $|U_i| \geq 3$ for $1 \leq i \leq 6$. The rest of the proof is completed by considering the following two cases:

1. $U_i \cap U_j \neq \emptyset$ for some $1 \leq i < j \leq 6$. Let $w \in U_i \cap U_j$ for some $1 \leq i < j \leq 6$. The graph K_6 is a complete graph, so the vertex u_i is adjacent to all the other vertices of U and u_i is also adjacent to w. Together, u_i is adjacent to 6 vertices. Those 6 vertices form a path of order 6 in the following way: start the path with vertex w, let the second vertex be u_j and let vertices 3 through 6 be the remaining vertices of U in any order. This creates the following path: $(w, u_j, u_{j_1}, u_{j_2}, u_{j_3}, u_{j_4})$ where $u_j, u_{j_1}, u_{j_2}, u_{j_3}, u_{j_4} \in U$. Since u_1 is adjacent to every vertex in the path, this forms a $K_1 + P_6$ where $K_1 = u_1$ and $P_6 = (w, u_j, u_{j_1}, u_{j_2}, u_{j_3}, u_{j_4})$. Then the proof is finished for this case.

2. $U_i \cap U_j = \emptyset$ for all $1 \leq i < j \leq 6$. This case is continued by considering two subcases.

    2.1. $xy \notin E(G)$ for all $1 \leq i < j \leq 6$ and for all $x \in U_i$ and $y \in U_j$. The graph K_6 is complete so for every two vertices $u_i, u_j \in U$ with $1 \leq i < j \leq 6$, there is a path of order 6 in U from u_i to u_j by walking along the 4 remaining vertices of U. Then the path looks like this: $(u_i, u_{i_1}, u_{i_2}, u_{i_3}, u_{i_4}, u_j)$ where all vertices in the path are distinct vertices in U. If there would be a vertex $w \in \Gamma_R(U_i) \cap \Gamma_R(U_j)$, then there is the following path from u_i to u_j: (u_i, x, w, y, u_j) where w is adjacent to $x \in U_i$ and to $y \in U_j$. The two paths $(u_i, u_{i_1}, u_{i_2}, u_{i_3}, u_{i_4}, u_j)$ and (u_i, x, w, y, u_j) together form a cycle of order 9 in G. This is a contradiction, hence $\Gamma_R(U_i) \cap \Gamma_R(U_j) = \emptyset$. Similarly, between any two vertices $u_i, u_j \in U$ with $1 \leq i < j \leq 6$, there is a path of order 5 in U from u_i to u_j by walking along 3 of the 4 remaining vertices of U. Then the path looks like this: $(u_i, u_{i_1}, u_{i_2}, u_{i_3}, u_j)$ where all vertices in the path are distinct vertices in U. Let $x \in \Gamma_R(U_i)$ and $y \in \Gamma_R(U_j)$ and suppose that $xy \in E(G)$.

(i,j)	path of order 8	path of order 7	path of order 6	path of order 5
$(3,4)$	$(u_3,u_2,u_8,u_7,u_1,u_6,u_5,u_4)$	$(u_3,u_2,u_8,u_7,u_1,u_5,u_4)$	$(u_3,u_2,u_1,u_6,u_5,u_4)$	(u_3,u_1,u_6,u_5,u_4)
$(3,5)$	$(u_3,u_2,u_8,u_7,u_1,u_4,u_6,u_5)$	$(u_3,u_2,u_8,u_7,u_1,u_4,u_5)$	$(u_3,u_2,u_1,u_6,u_4,u_5)$	(u_3,u_2,u_1,u_6,u_5)
$(3,6)$	$(u_3,u_2,u_8,u_7,u_1,u_4,u_5,u_6)$	$(u_3,u_2,u_8,u_7,u_1,u_5,u_6)$	$(u_3,u_2,u_1,u_4,u_5,u_6)$	(u_3,u_2,u_4,u_5,u_6)
$(3,7)$	$(u_3,u_4,u_5,u_6,u_1,u_2,u_8,u_7)$	$(u_3,u_4,u_5,u_6,u_2,u_8,u_7)$	$(u_3,u_5,u_6,u_2,u_8,u_7)$	(u_3,u_4,u_5,u_8,u_7)
$(3,8)$	$(u_3,u_4,u_5,u_6,u_2,u_1,u_7,u_8)$	$(u_3,u_4,u_5,u_6,u_1,u_7,u_8)$	$(u_3,u_4,u_5,u_1,u_7,u_8)$	(u_3,u_4,u_5,u_7,u_8)
$(4,5)$	$(u_4,u_3,u_2,u_8,u_7,u_1,u_6,u_5)$	$(u_4,u_2,u_8,u_7,u_1,u_6,u_5)$	$(u_4,u_3,u_2,u_1,u_6,u_5)$	(u_4,u_2,u_1,u_6,u_5)
$(4,6)$	$(u_4,u_3,u_2,u_8,u_7,u_1,u_5,u_6)$	$(u_4,u_2,u_8,u_7,u_1,u_5,u_6)$	$(u_4,u_3,u_2,u_1,u_5,u_6)$	(u_4,u_3,u_2,u_1,u_6)
$(4,7)$	$(u_4,u_5,u_6,u_1,u_3,u_2,u_8,u_7)$	$(u_4,u_5,u_6,u_1,u_2,u_8,u_7)$	$(u_4,u_5,u_6,u_2,u_8,u_7)$	(u_4,u_5,u_2,u_8,u_7)
$(4,8)$	$(u_4,u_5,u_6,u_3,u_2,u_1,u_7,u_8)$	$(u_4,u_5,u_6,u_3,u_1,u_7,u_8)$	$(u_4,u_5,u_6,u_1,u_7,u_8)$	(u_4,u_5,u_6,u_2,u_8)
$(5,6)$	$(u_5,u_4,u_3,u_2,u_8,u_7,u_1,u_6)$	$(u_5,u_3,u_2,u_8,u_7,u_1,u_6)$	$(u_5,u_4,u_3,u_2,u_1,u_6)$	(u_5,u_3,u_2,u_1,u_6)
$(5,7)$	$(u_5,u_4,u_3,u_6,u_1,u_2,u_8,u_7)$	$(u_5,u_4,u_3,u_1,u_2,u_8,u_7)$	$(u_5,u_4,u_3,u_2,u_8,u_7)$	(u_5,u_4,u_2,u_8,u_7)
$(5,8)$	$(u_5,u_4,u_3,u_6,u_2,u_1,u_7,u_8)$	$(u_5,u_4,u_3,u_6,u_1,u_7,u_8)$	$(u_5,u_4,u_3,u_1,u_7,u_8)$	(u_5,u_4,u_3,u_2,u_8)
$(6,7)$	$(u_6,u_5,u_4,u_3,u_1,u_2,u_8,u_7)$	$(u_6,u_5,u_4,u_3,u_2,u_8,u_7)$	$(u_6,u_4,u_3,u_2,u_8,u_7)$	(u_6,u_3,u_2,u_8,u_7)
$(6,8)$	$(u_6,u_5,u_4,u_3,u_2,u_1,u_7,u_8)$	$(u_6,u_5,u_4,u_3,u_1,u_7,u_8)$	$(u_6,u_5,u_4,u_3,u_2,u_8)$	(u_6,u_5,u_4,u_2,u_8)
$(7,8)$	$(u_7,u_1,u_6,u_5,u_4,u_3,u_2,u_8)$	$(u_7,u_1,u_5,u_4,u_3,u_2,u_8)$	$(u_7,u_1,u_4,u_3,u_2,u_8)$	(u_7,u_1,u_3,u_2,u_8)

Figure 16

Then there is path of order 6 from u_i to u_j, namely (u_i, v, x, y, w, u_j) where $v \in U_i$ and $w \in U_j$ are adjacent to x and y respectively. The two paths $(u_i, u_{i_1}, u_{i_2}, u_{i_3}, u_j)$ and (u_i, v, x, y, w, u_j) together form a cycle of order 9 in G. This is a contradiction, hence $xy \notin E(G)$ for all $2 \leq i < j \leq 6$ and for all $x \in \Gamma_R(U_i)$ and $y \in \Gamma_R(U_j)$. It follows that each of the induced subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ is disjoint and not adjacent for $1 \leq i \leq 6$. Since the order of an independent set in G is maximal 7, at least five of the induced subgraphs $\langle U_i \cup \Gamma_R(U_i)\rangle_G$ for $1 \leq i \leq 6$ are complete. It remains to show that one of those complete induced subgraphs contains at least 7 vertices. To that end, let $x \in U_i$ for some $1 \leq i \leq 6$. Since $U_i \cap U_j = \emptyset$ for all $1 \leq i < j \leq 6$, x is adjacent to $u_i \in U$ and to no other vertices in U. Then x is adjacent to at least 7 vertices of R, since $\delta(G) \geq 8$. That means that $|\Gamma_R(U_i)| \geq 7$ for all $1 \leq i \leq 6$ and so $|\langle U_i \cup \Gamma_R(U_i)\rangle_G| \geq 7$ for all $1 \leq i \leq 6$. And so, G contains K_7.

2.2. $xy \in E(G)$ for some $1 \leq i < j \leq 6$ and for some $x \in U_i$ and $y \in U_j$. Without loss of generality, assume that $u_7u_8 \in E(G)$ for $u_7 \in U_1$ and $u_8 \in U_2$. Then (u_1, u_7, u_8, u_2) is a path in G. Redefine the sets U, R and U_i in the following way. Define $U' = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$, $R' = G \setminus U'$ and $U'_i = \Gamma(u_i) \cap V(R')$. The following 4 properties will be shown:

(A) $U'_i \cap U'_j = \emptyset$ for all $3 \leq i < j \leq 8$.

(B) $xy \notin E(G)$ for all $3 \leq i < j \leq 8$ and for all $x \in U'_i$ and $y \in U'_j$.

(C) $\Gamma_{R'}(U'_i) \cap \Gamma_{R'}(U'_j)$ for all $3 \leq i < j \leq 8$.

(D) $xy \notin E(G)$ for all $3 \leq i < j \leq 8$ and for all $x \in \Gamma_{R'}(U'_i)$ and $y \in \Gamma_{R'}(U'_j)$.

These properties hold because there are paths of order 8, 7, 6 and 5 between any two vertices u_i and u_j for $3 \leq i < j \leq 8$ which is shown in the table in Figure 16. If the properties (A), (B), (C) and (D) were assumed not to be true, then there would be paths of order 3, 4, 5 and 6, respectively, between vertices u_i and u_j for $3 \leq i < j \leq 8$ and then in all four cases, there would be a cycle of order 9 in G. The induced subgraphs $\langle U'_i \cup \Gamma_{R'}(U'_i)\rangle_G$ are disjoint and not adjacent for $3 \leq i \leq 8$. Since the order of an independent set in G is maximal 7, at least five of the induced subgraphs $\langle U_i' \cup \Gamma_{R'}(U_i')\rangle_G$ are complete. It remains to show that one of those complete induced subgraphs contains at least 7 vertices. To that end, let $x \in U_i'$ for some $3 \leq i \leq 8$. Since $U_i' \cap U_j' = \emptyset$ for all $3 \leq i < j \leq 8$, x is adjacent to $u_i \in U'$ and to no other vertices in U'. Then x is adjacent to at least 7 vertices of R', since $\delta(G) \geq 8$. That means that $|\Gamma_{R'}(U_i')| \geq 7$ for all $3 \leq i \leq 8$ and so $|\langle U_i' \cup \Gamma_{R'}(U_i')\rangle_G| \geq 7$ for all $3 \leq i \leq 8$. And so, G contains K_7.

In both subcases, G contains K_7. Hence, $U_i \cap U_j = \emptyset$ for all $1 \leq i \leq 6$ implies that G contains K_7.

Finally, from both cases it follows that G contains either $K_1 + P_6$ or K_7. ■

Lemma 10 *If G is a graph of order ≥ 57, then G contains $K_1 + P_6$ or K_6.*

Proof: Let G be a graph of order ≥ 57. Assume for a contradiction that G contains neither $K_1 + P_6$ nor K_6. This lemma will be proven by means of three claims.

The first claim is: $|\Gamma(u)| \leq 28$ for any $u \in V(G)$. Proof: In order to arrive at a contradiction, suppose that $|\langle \Gamma_G(u)\rangle_G| \geq 29$ for some vertex u. Recall the following set: $\Gamma[u] := \Gamma(u) \cup u$. Figures 17 and 18 show an example of the induced subgraphs of the neighborhood of u, including u and excluding u respectively. Figure 18 shows that the neighborhood of u consists of 4 components: $G_1 = \{u_1, u_8\}$, $G_2 = \{u_2\}$, $G_3 = \{u_3, u_4, u_5, u_6\}$ and $G_4 = \{u_7, u_9, u_{10}\}$.

Define the induced subgraph of the neighborhood of u as follows: let u be any vertex, then $\langle \Gamma_G(u)\rangle_G = \bigcup_{i=1}^{r} G_i$ where G_i is a component for each i. The goal is now to consider the case where $\langle \Gamma_G(u)\rangle_G$ has minimum number of independent vertices and then try to show that the order of the largest independent set is still greater than or equal to 8 (while the maximum number of independent vertices was assumed to be smaller than 8). To that end, suppose that $\langle \Gamma_G(u)\rangle_G$ has minimum number of independent vertices. This hold when $\langle \Gamma_G(u)\rangle_G$ has a maximum number of edges. At this point, Theorem 8 will be used. The number of edges in $\langle \Gamma_G(u)\rangle_G$ is maximal if the equality holds in the theorem. Then Theorem 8 implies that the components of $\langle \Gamma_G(u)\rangle_G$ must be complete graphs, so G_i is complete for each i. Assume for a contradiction that $\langle \Gamma_G(u)\rangle_G$ contains a path of order 6: P_6. Since each vertex in the path is in the neighborhood of u, each vertex in the path is adjacent to u. However, this creates a $K_1 + P_6$ in G where $K_1 = u$ which was assumed not to be possible. Therefore, $\langle \Gamma_G(u)\rangle_G$ contains no P_6. The components G_i contain hence no P_6. Since the G_i are complete for all i and contain

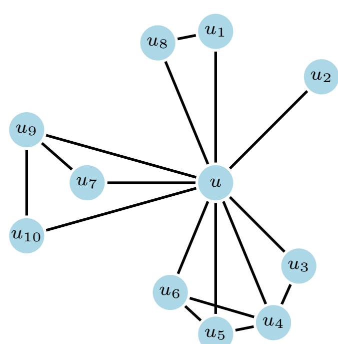

Figure 17: $\langle \Gamma[u]\rangle_G$

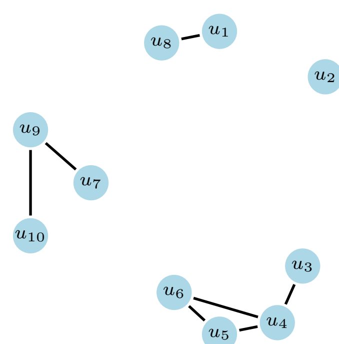

Figure 18: $\langle \Gamma(u)\rangle_G$

no P_6, G_i must be a complete graph of order at most 5 for all i. Suppose G_i is a complete of order 5. The vertex u is adjacent to every vertex in G_i, since G_i is in the neighborhood of u. Therefore, $\langle G_i \cup u \rangle_G$ form a complete graph of order 6, a K_6, in G which was assumed not to be possible. Thus, G_i must be a complete graph of order at most 4. It was assumed that $|\langle \Gamma_G(u) \rangle_G| \geq 29$. This is only possible if $\langle \Gamma_G(u) \rangle_G$ contains:

- 7 tetrahedrons and an isolated vertex; $7 \times 4 + 1 = 29$ or
- 6 tetrahedrons, a triangle and a K_2; $6 \times 4 + 3 + 2 = 29$ or
- 6 tetrahedrons and 2 triangles; $6 \times 4 + 2 \times 3 = 30 \geq 29$.

In any of these cases, there are at least 8 components in G, so there is an independent set of at least 8 vertices. This is a contradiction and therefore, $|\Gamma(u)| \leq 28$ for any $u \in V(G)$.

Let $\alpha(G)$ denote the order of the largest independent set in G. The second claim is: $\alpha(G) = 7$. Proof: By assumption, $|V(G)| \geq 57$ and G contains no C_9. In [5], the result $R(C_9, K_7) = 49$ is proved, so when the order of a graph is ≥ 49, there will always be a cycle of order 9 or an independent set of 7 vertices. In this lemma, $|V(G)| \geq 57$ and G contains no C_9, hence $\alpha(G) \geq 7$. However, G has no set of 8 independent vertices, therefore $\alpha(G) \leq 7$. The results $\alpha(G) \geq 7$ and $\alpha(G) \leq 7$ together give the result $\alpha(G) = 7$.

The second claim assures that G has an independent set of 7 vertices. Let $u_1, u_2, u_3, u_4, u_5, u_6, u_7$ be 7 independent vertices in $V(G)$. Recall that $\Gamma[u] := \Gamma(u) \cup u$. Set $\Gamma_i[u_{i+1}] = \Gamma[u_{i+1}] \setminus \left(\bigcup_{j=1}^{i} \Gamma[u_j] \right)$ for $1 \leq i \leq 6$. In the same way, set $\Gamma_i(u_{i+1}) = \Gamma(u_{i+1}) \setminus \left(\bigcup_{j=1}^{i} \Gamma(u_j) \right)$ for $1 \leq i \leq 6$. Now let $A = \bigcup_{i=1}^{6} \Gamma_i[u_{i+1}]$, $B = \bigcup_{i=1}^{6} \Gamma_i(u_{i+1})$ and $\beta = \alpha(\langle B \rangle_G)$.

The third claim is: $|\Gamma(u_1) \cup B| \geq 50$. Proof: Suppose for a contradiction that $|\Gamma(u_1) \cup B| \leq 49$. Then

$$\begin{aligned} &|\Gamma[u_1] \cup A| \\ &= |\Gamma(u_1) \cup B \cup \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}| \\ &= |\Gamma(u_1) \cup B| + |\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}| \leq 49 + 7 = 56, \end{aligned}$$

so $|\Gamma[u_1] \cup A| \leq 56$. It follows that

$$|G \setminus (\Gamma[u_1] \cup A)| \geq 57 - 56 = 1.$$

Note that $R(C_9, K_1) = 1$, then consider the graph $G \setminus (\Gamma[u_1] \cup A)$ which is obtained by removing the vertices $\Gamma[u_1] \cup A$ from the graph G. The graph $G \setminus (\Gamma[u_1] \cup A)$ contains none of the vertices u_i for all $1 \leq i \leq 7$ and also no vertices of the neighborhoods of u_i for all $1 \leq i \leq 7$. Therefore, $G \setminus (\Gamma[u_1] \cup A)$ contains a vertex, say u_8, which is not adjacent to u_i for all $1 \leq i \leq 7$. Then $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ forms an independent set in G. This means that $\alpha(G) \geq 8$, but that is a contradiction. Therefore, $|\Gamma(u_1) \cup B| \geq 50$.

Up to now, the following three claims have been proved:

1. $|\Gamma(u)| \leq 28$ for any $u \in V(G)$.
2. $\alpha(G) = 7$.
3. $|\Gamma(u_1) \cup B| \geq 50$.

Since $|V(G)| \geq 57$, by Lemma 3, $\delta(G) \geq 8$. It follows from Claim 1 that $8 \leq |\Gamma(u_1)| \leq 28$. In Claim 3 it has been shown that $|\Gamma(u_1) \cup B| \geq 50$. If $|\Gamma(u_1)| = r$, then $|B| \geq 50 - r$. Now a similar argumentation as in Claim 1 will be used to prove that $\alpha(\langle \Gamma(u_1) \rangle_G \geq \lceil \frac{r}{4} \rceil$.

Assume that $|\Gamma(u_1)| = r$. Let $\langle \Gamma(u_1) \rangle_G = \bigcup_{i=1}^{r} H_i$ where H_i is a component for each i. Consider the case where $\langle \Gamma(u_1) \rangle_G$ has minimum number of independent vertices and look whether the order of the largest independent set is still greater than or equal to 8. The graph $\langle \Gamma(u_1) \rangle_G$ has minimum number of independent vertices when $\langle \Gamma(u_1) \rangle_G$ has a maximum number of edges. By Theorem 8 it holds that the number of edges in $\langle \Gamma(u_1) \rangle_G$ is maximal if the equality holds in the theorem. Then Theorem 8 implies that the components of $\langle \Gamma(u_1) \rangle_G$ must be complete graphs, so H_i is complete for each i. With the same reasoning as before, G_i cannot contain a P_6, so G_i must have at most 5 vertices. However, G_i cannot be a complete graph of order 5, so G_i is a complete graph of order at most 4. The number of components H_i is minimal when the H_i are all tetrahedrons for all i. In that case, the number of components is $\lceil \frac{r}{4} \rceil$ and so $\alpha(\langle \Gamma(u_1) \rangle_G) = \lceil \frac{r}{4} \rceil$. The latter holds when the number of components H_i is minimal, so in general $\alpha(\langle \Gamma(u_1) \rangle_G) \geq \lceil \frac{r}{4} \rceil$. Since $|\Gamma(u_1)| = r$ and $|B| \geq 50 - r$, $\alpha(\langle B \rangle_G) \geq \lceil \frac{50-r}{4} \rceil$. Recall that $8 \leq r \leq 28$. When $8 \leq r \leq 21$, $\lceil \frac{50-r}{4} \rceil \geq 8$, thus $\alpha(G) \geq 8$, a contradiction. This means that r cannot take values from 8 to 21 and therefore $22 \leq |\Gamma(u_1)| \leq 28$. This will be separated into three cases.

1. $22 \leq |\Gamma(u_1)| \leq 25$. As shown above, when $|\Gamma(u_1)| = r$, then $\alpha(\langle \Gamma(u_1) \rangle_G) \geq \lceil \frac{r}{4} \rceil$ and $\beta = \lceil \frac{50-r}{4} \rceil$. Then $22 \leq r \leq 25$ implies that $\lceil \frac{r}{4} \rceil \geq 6$, so $\alpha(\langle \Gamma(u_1) \rangle_G) \geq 6$. Also, for those values of r, $\lceil \frac{50-r}{4} \rceil \geq 7$, so $\beta \geq 7$. Then, $\langle B \rangle_G$ has an independent set of 7 vertices. All those 7 vertices are not adjacent to vertex u_1, so they together form a set of 8 independent vertices of G and that is a contradiction.

2. $|\Gamma(u_1)| = 26$. From Claim 3, $|\Gamma(u_1) \cup B| \geq 50$, it follows that $|B| \geq 24$. In this case, $r = 26$, so with the same reasoning as above, $\alpha(\langle \Gamma(u_1) \rangle_G) \geq \lceil \frac{26}{4} \rceil = 7$ and $\beta \geq \lceil \frac{50-26}{4} \rceil = 6$. This value of β will be split into two cases.

   2.1. $\beta \geq 7$. Then, $\langle B \rangle_G$ has an independent set of 7 vertices. All those 7 vertices are not adjacent to vertex u_1, so they together form a set of 8 independent vertices of G and that is a contradiction.

   2.2. $\beta = 6$. Write the set B as $B = \Gamma_1(u_2) \cup \Gamma_2(u_3) \cup \Gamma_3(u_4) \cup \Gamma_4(u_5) \cup \Gamma_5(u_6) \cup \Gamma_6(u_7)$. The graphs $\langle \Gamma_i(u_{i+1}) \rangle_G$ for $1 \leq i \leq 6$ have minimum number of independent vertices when each $\langle \Gamma_i(u_{i+1}) \rangle_G$ has a maximum number of edges. By Theorem 8 it holds that the number of edges in $\langle \Gamma_i(u_{i+1}) \rangle_G$ is maximal if the equality holds in the theorem. Then Theorem 8 implies that the components of $\langle \Gamma_i(u_{i+1}) \rangle_G$ must be complete graphs. Each $\langle \Gamma_i(u_{i+1}) \rangle_G$ for $1 \leq i \leq 6$ cannot contain a P_6 because otherwise, that P_6 would form a $K_1 + P_6$ with $K_1 = u_{i+1}$, so the order of each component of $\langle \Gamma_i(u_{i+1}) \rangle_G$ is at most 5. Also, $\langle \Gamma_i(u_{i+1}) \rangle_G$ cannot contain a K_5 because otherwise, K_5 and u_{i+1} together form a K_6. Therefore, the components of $\langle \Gamma_i(u_{i+1}) \rangle_G$ are complete graphs of order at most 4. Since $|B| \geq 24$ and $\alpha(\langle B \rangle_G) = 6$, each subgraph $\langle \Gamma_i(u_{i+1}) \rangle_G$ must consist of one tetrahedron. Also, $|\Gamma(u_1)| = 26$, so $\Gamma(u_1)$ has minimal number of components when it has 6 tetrahedrons and one K_2. This situation is given in Figure 19. Define $\langle \Gamma(u_1) \rangle_G = \bigcup_{i=1}^{r} J_i$ where J_i is a component for each i. Define J_i for $1 \leq i \leq 6$ to be the tetrahedrons in $\langle \Gamma(u_1) \rangle_G$ and define J_7 to be the K_2 in $\langle \Gamma(u_1) \rangle_G$. Now there are two subcases.

   2.2.1. There exists a vertex $a_1 \in \bigcup_{i=1}^{6} \Gamma_i(u_{i+1})$ that is not adjacent to at least one vertex of each J_i for $1 \leq i \leq 7$. Say, a_1 is not adjacent to $x_i \in J_i$ for each

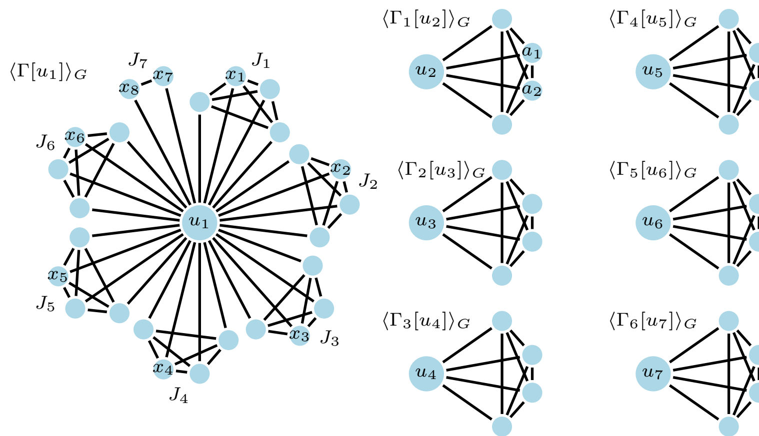

Figure 19

$1 \leq i \leq 7$. All components are independent, therefore the x_i are independent for all $1 \leq i \leq 7$. Also, a_1 is not adjacent to x_i for all $1 \leq i \leq 7$. Then $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, a_1\}$ is an independent set in G of 8 elements, which means that $\alpha(G) \geq 8$. And that is a contradiction.

2.2.2. For each vertex of $\bigcup_{i=1}^{6} \Gamma_i(u_{i+1})$ there is $1 \leq i \leq 7$ such that this vertex is adjacent to all vertices of J_i. Take two vertices from the same subgraph $\langle\Gamma_i(u_{i+1})\rangle_G$ for some $1 \leq i \leq 6$. Say, $a_1, a_2 \in \langle\Gamma_1(u_2)\rangle_G$ as in Figure 19.

- When both a_1 and a_2 are adjacent to all vertices of the same component J_i for $1 \leq i \leq 6$, then there is a cycle of order 9 by starting at a_1, walking along the vertices of J_i and u_1, then going to a_2, walking along the remaining vertices of $\langle\Gamma_1(u_2)\rangle_G$ and going back to a_1.
- When both a_1 and a_2 are adjacent to the vertices of the component $J_7 = \langle\{x_7, x_8\}\rangle_G$, then there is a $K_1 + P_6$ by letting $K_1 = a_1$ and $P_6 = (x_7, x_8, a_2, a_{i_1}, a_{i_2}, u_2)$, where a_{i_1} and a_{i_2} are the remaining two vertices of $\langle\Gamma_1(u_2)\rangle_G$.
- When a_1 is adjacent to all vertices of J_i and a_2 is adjacent to all vertices of J_k for some $1 \leq i < k \leq 6$, then there is a cycle of order 9 by starting at a_1, walking along two vertices of J_i, then going to u_1, then walking along two vertices of J_k, then walking to a_2, then to u_2 and then back to a_1.
- When a_1 is adjacent to all vertices of J_i for some $1 \leq i \leq 6$ and a_2 is adjacent to the vertices of $J_7 = \langle\{x_7, x_8\}\rangle_G$, then there is a cycle of order 9 by starting at a_1, then walking along all 4 vertices of J_i, then going to u_1, then walking along x_7 and x_8, then going to a_2 and finally back to a_1.

So in all possible ways of letting vertices of $\bigcup_{i=1}^{6} \Gamma_i(u_{i+1})$ be adjacent to all vertices of some component J_i, there will be a $K_1 + P_6$ or a cycle of order 9 and that is a contradiction. That means that when $\beta = 6$, there is a contradiction.

For all possible values of β, there is a contradiction, so $|\Gamma(u_1)| \neq 26$

3. $|\Gamma(u_1)| = 27$. From Claim 3 it follows that $|B| \geq 23$. Here $r = 27$, so $\alpha(\langle\Gamma(u_1)\rangle_G) \geq 7$ and $\beta \geq 6$. Then this can again be split into two cases.

3.1 $\beta \geq 7$. With the same reasoning as before, $\langle B\rangle_G$ has an independent set of 7 vertices.

All those 7 vertices are not adjacent to vertex u_1, so they together form a set of 8 independent vertices of G and that is a contradiction.

3.2 $\beta = 6$. This situation is the same as in Figure 19 but then both $\langle \Gamma_6(u_7) \rangle_G$ and J_7 are triangles. Then the same reasoning as in subcases 2.2.1 and 2.2.2 will give the same result for this case. Namely, there will always be a $K_1 + P_6$ or a cycle of order 9 and that is a contradiction.

All possible values of β give a contradiction, so $|\Gamma(u_1)| \neq 27$.

4. $|\Gamma(u_1)| = 28$. From Claim 3 it follows that $|B| \geq 22$. Here $r = 28$, so $\alpha(\langle \Gamma(u_1) \rangle_G) \geq 7$ and $\beta \geq 6$. Then this can again be split into two cases.

4.1 $\beta \geq 7$. With the same reasoning as before, $\langle B \rangle_G$ has an independent set of 7 vertices. All those 7 vertices are not adjacent to vertex u_1, so they together form a set of 8 independent vertices of G and that is a contradiction.

4.2 $\beta = 6$. This situation is the same as in Figure 19 but then J_7 is a tetrahedron, just like the other components of $\langle \Gamma(u_1) \rangle_G$ and $\langle \Gamma_6(u_7) \rangle_G$ is a K_2. Then the same reasoning as in subcases 2.2.1 and 2.2.2 will give the same result for this case. Namely, there will always be a $K_1 + P_6$ or a cycle of order 9 and that is a contradiction.

All possible values of β give a contradiction, so $|\Gamma(u_1)| \neq 28$.

From Claim 1 is follows that $|\Gamma(u_1)| \leq 28$. However, for all possible values of $|\Gamma(u_1)|$, there follows a contradiction by using the assumption that G contains no $K_1 + P_6$ nor K_6. Therefore, if G is a graph of order ≥ 57, then G contains $K_1 + P_6$ or K_6. ■

Finally, all lemmas have been proven, so all requisites are available to prove Theorem 6:

Theorem 6 $R(C_9, K_8) = 57$.

For the sake of clarity, here are the 8 lemmas stated together:
Lemma 3 *Let G be a graph of order ≥ 57 that contains neither a cycle of order 9 nor an independent set of order 8. Then $\delta(G) \geq 8$.*
Lemma 4 *If G contains K_8, then $|V(G)| \geq 72$.*
Lemma 5 *If G contains $K_8 \setminus S_6$, then G contains K_8.*
Lemma 6 *If G contains K_7, then G contains $K_8 \setminus S_6$ or K_8.*
Lemma 7 *If G contains $K_1 + P_7$, then G contains K_7.*
Lemma 8 *If G contains $K_1 + P_6$, then G contains $K_1 + P_7$ or K_7.*
Lemma 9 *If G contains K_6, then G contains $K_1 + P_6$ or K_7.*
Lemma 10 *If G is a graph of order ≥ 57, then G contains $K_1 + P_6$ or K_6.*

Next, the proof of Theorem 6 is given:

Proof It was already established that $R(C_9, K_8) \geq 57$. Now show that $R(C_9, K_8) \leq 57$. To that end, suppose that G is a graph of order 57 that contains neither a cycle of order 9 nor an independent set of 8 vertices. Then by Lemma 3, $\delta(G) \geq 8$, which was assumed to hold in Lemmas 6, 7, 8, 9 and 10. By Lemma 10, G contains $K_1 + P_6$ or K_6. This gives two cases:

- In case G contains $K_1 + P_6$, Lemma 8 implies that G contains $K_1 + P_7$ or K_7. When G contains $K_1 + P_7$, Lemma 7 tells that G contains K_7. So G will always contain K_7.

- In case G contains K_6, Lemma 9 implies that G contains K_1+P_6 or K_7. When G contains $K_1 + P_6$, Lemma 8 tells that G contains $K_1 + P_7$ or K_7. If G contains then $K_1 + P_7$, Lemma 7 implies that G contains K_7. So G will always contain K_7.

In either case, G will contain a complete graph K_7. Moving on, G having a K_7 means that G contains $K_8 \setminus S_6$ or K_8 by Lemma 6. If G contains $K_8 \setminus S_6$, then by Lemma 5, G contains K_8. Hence, in any case, G contains K_8. Finally, by Lemma 4, $|V(G)| \geq 72$. This is a contradiction, since by assumption, $|V(G)| = 57$. Therefore, if G is a graph of order 57, then G contains either a cycle of order 9 or an independent set of 8 vertices. This means that $R(C_9, K_8) \leq 57$. Together with the result $R(C_9, K_8) \geq 57$, the conclusion is that $R(C_9, K_8) = 57$. ■

3 Conclusion and discussion

A lot of different aspects of Ramsey Theory have been covered in this thesis. After giving all the necessary information about Graph Theory, the basics of Ramsey Theory were introduced. Then, Theorem 3 was introduced [7]: *Let* $k \geq 3$. *Then* $2^{\frac{k}{2}} < R(k,k) < 4^{k-1}$. This theorem shows that the symmetric Ramsey number is bounded from below and from above. All Ramsey numbers are hence always finite and increase rapidly when increasing k.

After that, Theorem 4 was proven [9]: *For all* $m \geq 3$ *and* $n \geq 2$*, the cycle-complete graph Ramsey number* $R(C_m, K_n)$ *satisfies* $R(C_m, K_n) \leq \lceil (m-2)(n^{\frac{1}{k}} + 2) + 1 \rceil (n-1)$*, where* $k = \lfloor \frac{m-1}{2} \rfloor$. That theorem can be considered as the main theorem in this thesis. It gives one of the first upper bounds for cycle-complete graph Ramsey numbers and it has not been improved significantly since [19], [6].

Thereafter, an upper bound regarding the special case of cycles of order 4 is proven in Theorem 5 [9]: $R(C_4, K_n) < c\left(\frac{n\log(\log(n))}{\log(n)}\right)^2$, $n \to \infty$. It shows how the cycle-complete graph Ramsey numbers behave asymptotically when considering cycles of order 4 and letting n go to infinity.

Finally, the exact calculation of a specific Ramsey number was given in Theorem 6: $R(C_9, K_8) = 57$. This theorem gives a very nice picture of what a derivation of a Ramsey number looks like when an exact answer is found instead of only bounds from above and below.

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5 Appendix

The graph in Figure 20 shows that the extreme value that was computed in Section 2.4 is indeed a maximum.

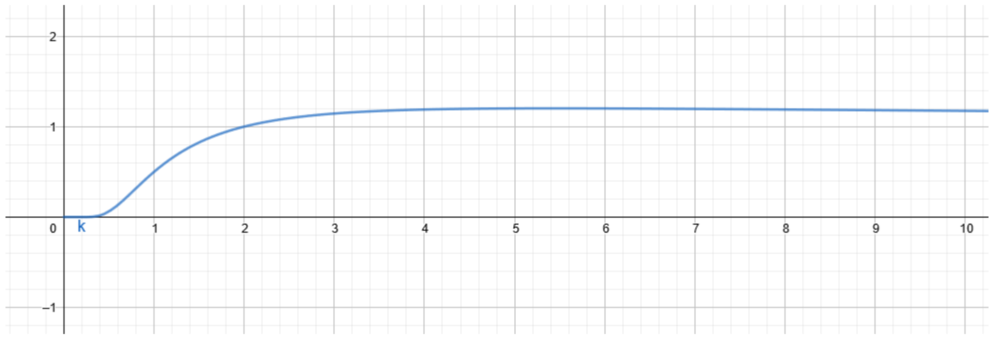

Figure 20