university of groningen

Bachelor's Thesis Applied Mathematics

Improving traffic flow with the introduction of autonomous vehicles

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Abstract

Human drivers have a tendency to overreact to the behaviour of the predecessor, amplifying disturbances leading to traffic congestion whereas autonomous vehicles can be controlled to attenuate such disturbances. Here, the stability properties of a human and autonomous model were studied separately before interconnecting them, giving rise to a mixed traffic model. In particular, the notion of string stability was used extensively and numerical simulations were performed to confirm the theoretical analysis. It was found that 20% of vehicles on a roadway need to be autonomous, with the objective of maintaining a time-headway of $2s$, for string stability.

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1 Introduction

In this paper, the idea of improving traffic flow using autonomous vehicles will be explored. More specifically, this will be achieved by first studying a vehicle model consisting of only human drivers followed by a model of only autonomous drivers. From there, it will be studied how these two models can be interconnected to obtain a mixed traffic model.

The motivation for studying this problem arises from the behaviour of human drivers that have the tendency to overreact to the behaviour of the predecessor, amplifying small initial disturbances leading to “phantom” traffic jams. Traffic jams are undesirable because they are a common source of accidents, increase fuel consumption and reduce throughput.

The human vehicle model studied in this paper [1] captures this behaviour of amplifying disturbances and this phenomenon will be studied using the concept of string stability. String stable behaviour refers to the attenuation of disturbances in the upstream direction and this means that human vehicles are string unstable.

An example of string unstable behaviour is shown in figure 1. Here, the lead vehicle, vehicle 1, accelerates from 20 m/s to 22 m/s introducing a disturbance, which is then amplified by the following vehicles.

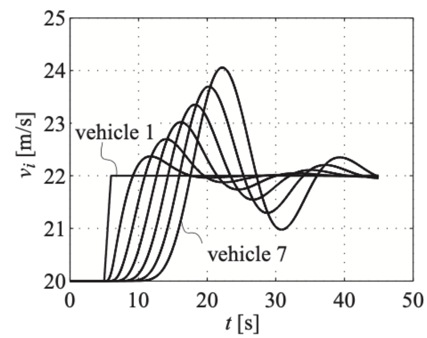

Figure 1: String unstable behaviour [2]

Automated vehicles on the other hand can be controlled to reduce the effects of such disturbances. Automated vehicles are vehicles equipped with wireless communications systems to provide real-time information of the preceding vehicle such as velocity and acceleration.

The autonomous vehicle model introduced in [3], which is string stable, will be studied and interconnected with the human vehicle model, to analyze how the evolution of disturbances is affected and whether autonomous vehicles can be used to stabilize human traffic.

2 Human vehicle model

In this section, the Optimum Velocity Model (OVM) proposed in [1] will be studied and adapted for this paper. A common approach when studying traffic models is to assume that the behaviour of a specific vehicle i is determined by the behaviour of the vehicle ahead of it, vehicle $i-1$.

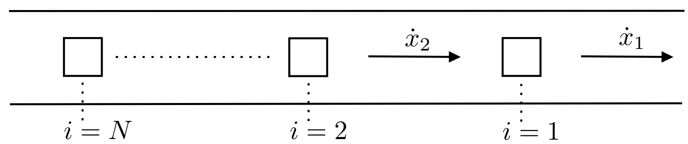

Figure 2: Notation used for the human vehicle model

The leading idea used for designing this model is that each vehicle has the objective to maintain its legal velocity $V(\Delta x_i)$, which depends on the distance between itself and the vehicle ahead, Δx_i. This means each vehicle accelerates or decelerates to maintain the legal velocity according to the motion of the previous vehicle.

The differential equation which describes this model is

$$\ddot{x}_i = a\{V(x_{i-1} - x_i) - \dot{x}_i\}, \qquad 2 \leq i \leq N, \tag{1}$$

where the following notation is used:

- x_i for the position of vehicle $i = 1, 2, ..., N$ where $i = 1$ and $i = N$ represent the first and final vehicle respectively;
- $\Delta x_i = x_{i-1} - x_i$ for headway, i.e., the distance between two successive vehicles;
- $V(\Delta x_i)$ for the legal velocity function;
- N for the total number of vehicles;
- a for the driver's sensitivity.

Furthermore, this model has several simplifying assumptions:

- Vehicles are simply points i.e. have no length;
- All vehicles are governed by the same equation.

There are two important criteria the velocity function has to satisfy to reflect realistic driver behaviour. The first one being that when headway decreases, velocity decreases to prevent collisions and the second being when headway increases, velocity increases but remains below maximum legal velocity. This can be mathematically written down as

- $V(\Delta x_i)$ is a monotonically increasing function;
- $|V(\Delta x_i)|$ has an upper bound, which will be denoted by V^{max}.

Specifically, $V^{max} = \lim_{\Delta x_i \to \infty} V(\Delta x_i)$. The velocity function suggested in [1] is

$$V(\Delta x_i) = \tanh(\Delta x_i - 2) + \tanh(2),$$

and is also shown in figure 3.

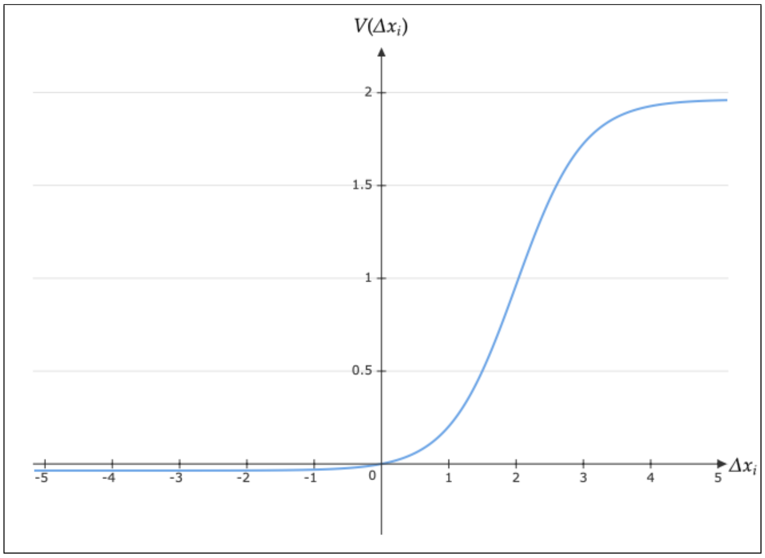

Figure 3: The velocity function

2.1 State-space model

As each vehicle is governed by a second-order differential equation, a model of N vehicles is described by a system of N second-order differential equations as

$$\begin{aligned} \ddot{x}_1 &= a\{v_{ref} - \dot{x}_1\}, \\ \ddot{x}_2 &= a\{V(x_1 - x_2) - \dot{x}_2\}, \\ &\vdots \\ \ddot{x}_N &= a\{V(x_{N-1} - x_N) - \dot{x}_N\}. \end{aligned} \tag{2}$$

where v_{ref} is the reference velocity and can also be seen as the external input into the system. To solve this system of ODEs numerically, the first step is to rewrite this system of second-order ODEs as a system of first-order ODEs, which is done by introducing the variable Δx_i such that $\Delta \dot{x}_i = v_{i-1} - v_i$ where $v_i = \dot{x}_i$. This gives the following $2N - 1$ system of first order ODEs

$$\begin{aligned} \dot{v}_1 &= a\{v_{ref} - v_1\}, \\ \dot{\Delta x}_2 &= v_1 - v_2, \\ \dot{v}_2 &= a\{V(\Delta x_2) - v_2\}, \\ &\vdots \\ \dot{\Delta x}_i &= v_{i-1} - v_i, \\ \dot{v}_i &= a\{V(\Delta x_i) - v_i\}, \\ &\vdots \\ \dot{\Delta x}_N &= v_{N-1} - v_N, \\ \dot{v}_N &= a\{V(\Delta x_N) - v_N\}, \end{aligned}$$

Note that v_{ref} should be interpreted as the objective velocity the lead vehicle attains and now the complete model is shown in figure 4. Observe that the lead vehicle, vehicle 1, has different dynamics than the rest of the vehicles because it has no vehicle ahead of it to follow.

This model (2) differs slightly from that in [1] as there a circular road was considered which meant that the $(N+1)^{th}$ vehicle was equal to the 1^{st} whereas here a straight road is considered.

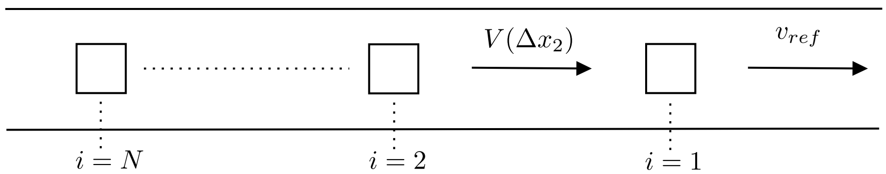

Figure 4: Straight road model

To illustrate the properties of this model, several time simulations are performed. For these simulations, the initial condition for velocity and spacing is chosen to be the equilibrium point corresponding to $v_{ref} = 1.5$. The initial spacing is denoted as Δx_{ref} and is given by

$$\Delta x_{ref} = V^{-1}(1.5) \approx 2.598$$

Note that the following condition is required to prevent "free-flow" traffic

$$v_{ref} < V^{max} \approx 1.964.$$

Free-flow traffic is the scenario where vehicle density is low enough such that all vehicles attain V^{max}. This also means perturbations introduced by the lead vehicle is not amplified by the following vehicles since the spacing between the vehicles is large enough to dissipate the disturbance.

Furthermore, a perturbation is introduced by the lead vehicle in the interval $t \in (40, 50)$ by adapting the reference velocity v_{ref} to

$$v_{ref} = 1.5 - 0.2 \sin\left(\frac{(t-40) \cdot 2\pi}{50}\right).$$

The results of these time simulations are shown in figures 5, 7 and 6. In all three figures, the perturbation introduced by the lead vehicle can be seen growing across the following vehicles.

In figure 5, the trajectory of all vehicles are plotted. Note that the slope of each line is the velocity of the corresponding vehicle, so a horizontal trajectory line implies $v = 0$. Observe that initially, all trajectories are smooth implying constant velocity whereas towards the end, the perturbation has been amplified to an extent that these vehicles have to slow down to $v = 0$.

In figure 6, the oscillation in velocity is growing across the string of vehicles. Recalling the notion of string stability explained in the introduction and the observing the similarities with figure 1, this simulation shows that the human model (2) is string unstable. This will be studied more formally in section 2.4.

In figure 7, time snapshots of the vehicle velocity for different vehicles is shown. Here, the oscillations in velocity can also be seen growing with time to the extent that the vehicles are almost stationary.

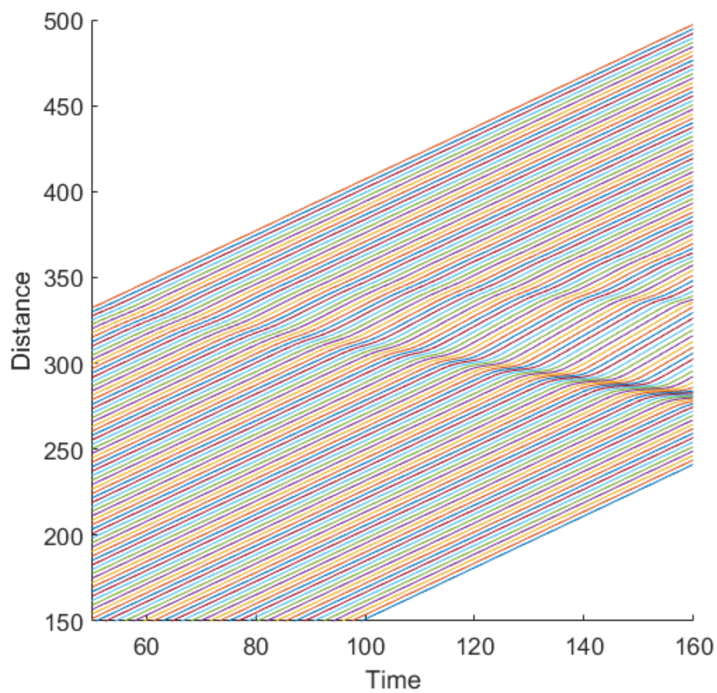

Figure 5: Trajectory of all vehicles, where each line represents the trajectory of a single vehicle. The uppermost line represents the trajectory of the first vehicle

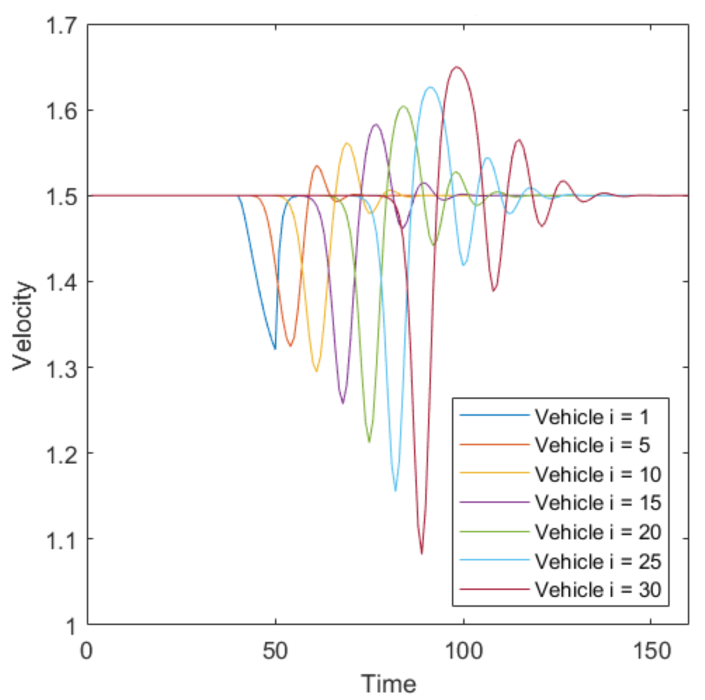

Figure 6: The time evolution of the velocities for different vehicles are shown

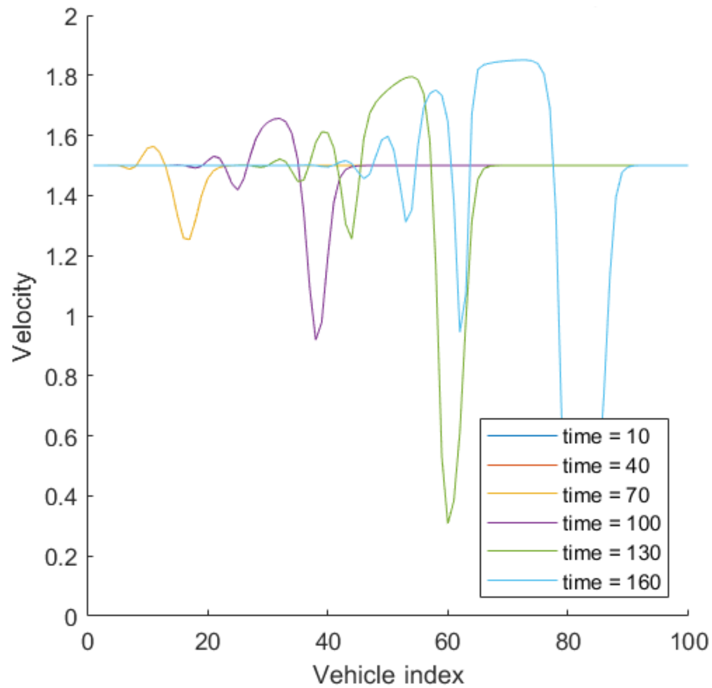

Figure 7: Time snapshots of vehicle velocity as a function of vehicle index. Recall $i = 1$ is the lead vehicle

2.2 Linearizing the model

The system (2) is not linear due to the non-linear velocity function $V(\Delta x_i)$. Once linearized, the system can be studied for its stability properties and eventually the string stability properties. From the Hartman-Grobman theorem, the behaviour of the linearized system close to the equilibrium point is a good approximation of the non-linear system. First, the state z is introduced

$$z = \begin{bmatrix} v_1 & \Delta x_2 & v_2 & \dots & \Delta x_i & v_i & \dots & \Delta x_N & v_N \end{bmatrix}$$

followed by the function $F(z, v_{ref})$

$$F(z, v_{ref}) = \begin{bmatrix} a\{v_{ref} - v_1\} \\ v_1 - v_2 \\ a\{V(\Delta x_2) - v_2\} \\ \vdots \\ v_{i-1} - v_i \\ a\{V(\Delta x_i) - v_i\} \\ \vdots \\ v_{N-1} - v_N \\ a\{V(\Delta x_N) - v_N\} \end{bmatrix}$$

where $\dot{z} = F(z, v_{ref})$.

Now, a constant input $v_{ref}(t) = \bar{v}_{ref}$ is chosen and the corresponding equilibrium $(\bar{z}, \bar{v}_{ref})$ is determined by solving $F(\bar{z}, \bar{v}_{ref}) = 0$. This results in $v_i = V(\Delta x_i) = \bar{v}_{ref}$ for $i = 1, \dots, N$. Physically, this means that the spacing between the vehicles is such that all vehicles travel at $\bar{v}_{ref}$, which is simply $\Delta\bar{x}_{ref}{=}V^{-1}(\bar{v}_{ref})$. Hence, the equilibrium point of this system is

$$\bar{z} = [\bar{v}_{ref} \quad \Delta\bar{x}_{ref} \quad \bar{v}_{ref} \quad \bar{x}_{ref} \quad \dots \quad \Delta\bar{x}_{ref} \quad \bar{v}_{ref}]^T.$$

Now, the dynamics of the system is rewritten in coordinates that measure the deviation from this equilibrium point. The deviation from the equilibrium point $\bar{z}$ will be denoted as $\tilde{z}$. Similarly, the deviation for the nominal input $\bar{v}_{ref}$ is introduced as

$$\tilde{z}(t) = z(t) - \bar{z}, \qquad \tilde{v}_{ref}(t) = v_{ref}(t) - \bar{v}_{ref}.$$

In terms of $\tilde{z}$, the dynamics read

$$\dot{\tilde{z}}(t) = \dot{z}(t) \;-\; 0 \;= F(\bar{z} + \tilde{z}(t),\; \bar{v}_{ref} + \tilde{v}_{ref}(t)).$$

This is just a restatement of the non-linear dynamics but now, assuming $\tilde{z}$ and $\tilde{v}_{ref}$ are small, a Taylor series approximation of F around $(\tilde{z}, \tilde{v}_{ref}) = (0, 0)$ can be used. Note this is equivalent to taking a Taylor series approximation around $(z, v_{ref}) = (\bar{z}, \bar{v}_{ref})$, which gives

$$\dot{\tilde{z}} = F(\bar{z} + \tilde{z}, \bar{v}_{ref} + \tilde{v}_{ref}) \approx F(\bar{z}, \bar{v}_{ref}) + \frac{\partial F}{\partial z}(\bar{z}, \bar{v}_{ref})\tilde{z} + \frac{\partial F}{\partial v_{ref}}(\bar{z}, \bar{v}_{ref})\tilde{v}_{ref},$$

where

$$\frac{\partial F}{\partial z} = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial z_1} & \cdots & \frac{\partial f_n}{\partial z_n} \end{bmatrix}, \qquad \frac{\partial F}{\partial v_{ref}} = \begin{bmatrix} \frac{\partial f_1}{\partial v_{ref}} \\ \vdots \\ \frac{\partial f_n}{\partial v_{ref}} \end{bmatrix}$$

Note that $F(\bar{z}, \bar{v}_{ref}) = 0$ by definition of the equilibrium. Now, computing the matrix of partial derivatives $\partial F/\partial z$ and evaluating it at the equilibrium point gives

$$\frac{\partial F}{\partial z}(\bar{z}, \bar{v}_{ref}) = \begin{bmatrix} -a & & & & & & & & \\ 1 & 0 & -1 & & & & & & \\ 0 & ab & -a & & & & & & \\ & & 1 & 0 & -1 & & & & \\ & & 0 & ab & -a & & & & \\ & & & & & \ddots & & & \\ & & & & & & \ddots & & \\ & & & & & & 1 & 0 & -1 \\ & & & & & & 0 & ab & -a \end{bmatrix} \quad \text{where} \quad b = \frac{d}{dx}V(x)\big|_{x=\Delta\bar{x}_{ref}}. \tag{3}$$

Note that the matrix has a block lower triangular structure which will be useful later as it simplifies the computation of the eigenvalues and also the inverse. As for $\partial F/\partial v_{ref}$,

$$\frac{\partial F}{\partial v_{ref}}(\bar{z}, \bar{v}_{ref}) = \begin{bmatrix} a & 0 & \dots & 0 \end{bmatrix}^T.$$

Now, the output equation of the nonlinear system is chosen to be

$$y(t) = v_i.$$

This output is chosen because it will be used to study string stability of the system. The output equation of the linearized system is simply

$$\begin{aligned}\tilde{y}(t) &= y(t) - \bar{y} \\ &= v_i - v_{ref} \\ &= \tilde{v}_i(t) \\ &= e_{2i-1}^T \tilde{z}(t),\end{aligned}$$

where e_{2i-1} is the vector with a one in position $2i-1$ and zero otherwise. The position of the one corresponds to the position of $\tilde{v}_i$ in $\tilde{z}(t)$. From here on forward, the linearized human vehicle model will be denoted as

$$\begin{aligned}\dot{z}(t) &= Az(t) + B\ v_{ref}(t), \\ y(t) &= C_i z(t).\end{aligned} \tag{4}$$

where $A = \partial F/\partial z(\bar{z}, \bar{v}_{ref})$, $B = \partial F/\partial z(\bar{z}, \bar{v}_{ref})$ and $C_i = e_{2i-1}^T$. Here, the notation is abused by removing the tildes $\tilde{.}$ since only the linearized dynamics will be used unless mentioned otherwise.

In figure 8, time simulations of the linearized human vehicle model are performed with the same parameters as with the non-linear model. Observe that the string unstable behaviour is still captured by this linearized model, which means that perturbations are still amplified by the following vehicles.

Furthermore, it is important to keep in mind that the linearized human vehicle model describes the dynamics of the deviation from the equilibrium point. This is reflected in figure 8 which shows the deviation from the equilibrium velocity $v_{ref} = 1.5$.

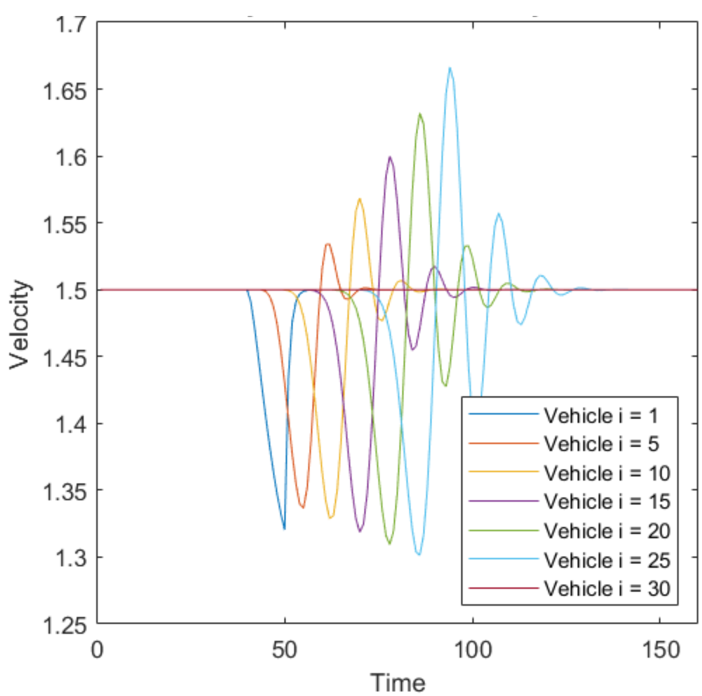

Figure 8: Time evolution of the velocities of different vehicles with the linearized dynamics

Furthermore, from the system matrix A in (3), observe that for $2 \le i \le N$, the behaviour of vehicle i is determined by v_{i-1}. Hence, the linearized human vehicle model for an individual vehicle can be written as

$$\Sigma_H^i : \begin{cases} \begin{bmatrix} \dot{\Delta x_i} \\ \dot{v_i} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ ab & -a \end{bmatrix} \begin{bmatrix} \Delta x_i \\ v_i \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_{i-1}, \\ \\ v_i = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_i \\ v_i \end{bmatrix}, \end{cases} \tag{5a}$$

for which the following notation will be used

$$\begin{cases} z_i^H = A^H \, z_i^H + B^H \, u_i^H, \\ y_i^H = C^H \, z_i^H. \end{cases}$$

The lead vehicle has slightly different dynamics

$$\Sigma_H^1 : \begin{cases} \dot{v}_1 = -av_1 + av_{ref}, \\ v_1 = v_1, \end{cases} \tag{5b}$$

and this will be notated as

$$\begin{cases} z_1^H = A_1^H \, z_1^H + B_1^H \, u_1^H, \\ y_1^H = C_1^H \, z_1^H. \end{cases}$$

Hence, interaction between the vehicles can be seen in figure 9.

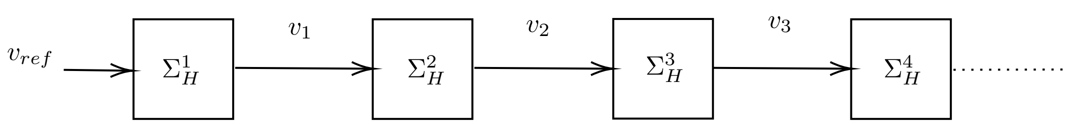

Figure 9: The connection between human vehicles

Note that the system matrix A (3) can now also be written as

$$A = \begin{bmatrix} A_1^H & & & & \\ B^H C_1^H & A^H & & & \\ & B^H C^H & \ddots & & \\ & & \ddots & \ddots & \\ & & & B^H C^H & A^H \end{bmatrix}, \tag{5c}$$

which will be useful for interconnecting different vehicles later on.

2.3 Stability

The concept of stability plays an important role in systems theory because it characterizes the asymptotic behaviour of systems. Given the homogeneous system

$$\dot{z}(t) = Az(t),$$

this system is asymptotically stable if every solution tends to zero for $t \to \infty$ for any initial condition $z_0 \in \mathbb{R}^n$

$$\lim_{t\to\infty} z(t) \ = \ 0 \quad \forall\, z_0 \in \mathbb{R}^n,$$

and this is true if and only if

$$\sigma(A) \subset \mathbb{C}_- = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq 0\}, \tag{6a}$$

where $\sigma(A)$ is the set of eigenvalues of the system matrix A (5c). Looking at the matrix A, only the block diagonal matrices have to considered for the computation of the eigenvalues. Note that the matrices on the diagonal of (5c) consists of one 1×1 matrix and $(N-1)$ identical 2×2 matrices. Hence, $\sigma(A) = \sigma(A_1^H) \cup \sigma(A^H) \cup \ldots \cup \sigma(A^H)$ where

$$\sigma(A_1^H) = \{-a\} \quad \text{and} \quad \sigma(A^H) = \left\{\frac{1}{2}\left(-a \pm \sqrt{a^2 - 4ab}\right)\right\}$$

Note that a is defined to be positive so $\sigma(A_1^H) \subset \mathbb{C}_-$. For $\sigma(A^H) \subset \mathbb{C}_-$, it is required that $a^2 - 4ab < a^2$ which implies $b > 0$. Recalling that b is the derivative of the velocity function evaluated at Δx_{ref}, this condition translates to $V'(\Delta x) > 0$ and note that this is always true since the $V(\Delta x)$ is a monotonically increasing function. Hence, the system is always asymptotically stable.

2.4 String Stability

In this section, the notion of string stability will be explained in detail and the results here can be used to explain the observations from the numerical results in figures 5, 6, 7 and 8. String stability can be used to study how the parameters of the traffic model affect the evolution of perturbations along a string of vehicles. This will also be used to study the autonomous vehicle model later.

Definition 1 (Vehicle string stability): Consider a string of $N \in \mathbb{N}$ interconnected vehicles. This system is string stable if and only if

$$\|v_i(t)\|_{\mathcal{L}_2} \leq \|v_{i-1}(t)\|_{\mathcal{L}_2}, \quad \forall t \geq 0, \quad 2 \leq i \leq N, \tag{6b}$$

where $v_i(t)$ is the velocity of vehicle i; $v_1(t) \in \mathcal{L}_2$ is a given input signal, and $v_i(0) = 0$ for $2 \leq i \leq N$. Also, $\|\cdot\|_{\mathcal{L}_2}$ denotes the 2$-$norm and i the vehicle index with $i = 1$ indicating the lead vehicle. This definition states $\|v_i(t)\|_{\mathcal{L}_2}$ must decrease in the upstream direction [3]

To study string stability, first consider the transfer function $T_i(s)$ from the input velocity $\hat{v}_{ref}$ to output velocity $\hat{v}_i$

$$\hat{v}_i(s) = T_i(s)\, \hat{v}_{ref}(s) \qquad i = 1, \ldots, N,$$

where $s \in \mathbb{C}$ and $\hat{v}(s)$ denotes the Laplace transform of $v(t)$. Now, dividing $\hat{v}_i$ by $\hat{v}_{i-1}$ gives $\Gamma_i(s)$ which is the transfer function from the "input" velocity $\hat{v}_{i-1}(s)$ to the "output" velocity $\hat{v}_i(s)$, i.e.,

$$\hat{v}_i(s) = \Gamma_i(s)\hat{v}_{i-1}(s), \qquad 2 \ \leq \ i \ \leq N$$

The transfer function Γ_i can now be related to the definition of string stability. Rewriting the definition (6b) as

$$\sup_{v_{i-1}\neq 0} \frac{\|v_i(t)\|_{\mathcal{L}_2}}{\|v_{i-1}(t)\|_{\mathcal{L}_2}} \leq 1,$$

note that the term on the left hand side is often referred to as the $\mathcal{L}_2$ system gain which is expressed in the time domain. Assuming asymptotic stability, this expression is equivalent to the $\mathcal{H}_\infty$ norm of the transfer function Γ_i in the frequency domain [4]:

$$\|\Gamma_i(j\omega)\|_{\mathcal{H}_\infty} = \sup_{v_{i-1}\neq 0} \frac{\|v_i(t)\|_{\mathcal{L}_2}}{\|v_{i-1}(t)\|_{\mathcal{L}_2}},$$

where $\Gamma_i(j\omega)$ is evaluated along the imaginary axis. The notation $j = \sqrt{-1}$ is used to avoid confusion with the vehicle indexing i. For scalar transfer functions, $\|\Gamma_i(j\omega)\|_{\mathcal{H}_\infty}$ is equal to the supremum of $|\Gamma_i(j\omega)|$ over the frequency ω. Hence, the string stability condition reduces to checking whether the following inequality is satisfied:

$$\sup_{\omega} |\Gamma_i(j\omega)| \leq 1, \quad 2 \leq i \leq N. \tag{6c}$$

2.4.1 Computing the transfer function Γ_i

To compute $\Gamma_i(s)$, the transfer function $T_i(s)$ from the input velocity v_{ref} to the output velocity v_i needs to be determined since $\Gamma_i = T_i/T_{i-1}$. Recalling the system of the human driven model (4), the transfer function T_i is computed as

$$T_i(s) = C_i(sI - A)^{-1}B.$$

Before computing the inverse, observe that $(sI - A)$ has a lower block triangular structure so the inverse will also be lower block triangular. Due to the form of the vectors B and C_i (4), only the element $(2i-1, 1)$ of the matrix $(sI - A)^{-1}$ needs to be computed for T_i, i.e.,

$$T_i = a\big(sI - A\big)^{-1}_{2i-1,1}. \tag{7}$$

This simplifies the computation of the transfer function. Now, the required values are computed. Let

$$(sI-A)(sI-A)^{-1} = \begin{bmatrix} J_1 & & & & \\ K_1 & J_2 & & & \\ & K_2 & J_2 & & \\ & & \ddots & \ddots & \\ & & & K_2 & J_2 \end{bmatrix} \begin{bmatrix} J_1^{-1} & & & & \\ Q_2 & J_2^{-1} & & & \\ Q_3 & \cdot & J_2^{-1} & & \\ \vdots & \cdot & \cdot & \ddots & \\ Q_N & \cdot & \cdot & \cdot & J_2^{-1} \end{bmatrix} = \mathbb{I}_{2N-1},$$

where

$$J_1 = s + a, \quad K_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} s & 1 \\ -ab & s+a \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

Note that $J_1 = sI - A_1^H$ and $J_2 = sI - A^H$ respectively. For $i = 1$, the transfer function is

$$T_1 = a\big(J_1^{-1}\big)_{1,1} = \frac{a}{s+a}.$$

For the remaining transfer functions note that $T_i = a(Q_i)_{2,1}$ for $i = 2, \ldots, N$ so the matrices Q_i are solved giving the following recurrence relation:

$$\begin{aligned} Q_2 &= -J_2^{-1}K_1J_1^{-1}, \\ Q_3 &= -J_2^{-1}K_2Q_2, \\ Q_4 &= -J_2^{-1}K_2Q_3, \\ &\vdots \\ Q_N &= -J_2^{-1}K_2Q_{N-1}. \end{aligned}$$

Generally, this can be written as

$$Q_i = (-1)^{i-1}\,(J_2^{-1}K_2)^{i-2}\,(J_2^{-1}K_1J_1^{-1}), \qquad i = 2, \dots, N.$$

To obtain a general expression for Q_i, first a general expression for $(J_2^{-1}K_2)^{i-2}$ with $3 \le i \le N$ is found as

$$(J_2^{-1}K_2)^{i-2} = \left(\frac{-ab}{s^2 + as + ab}\right)^{i-2} \begin{bmatrix} 0 & (s+a)(ab)^{-1} \\ 0 & 1 \end{bmatrix}.$$

To show that this is true, assume it holds for $i - 2 = k$. To show it holds for $i - 2 = k + 1$ as well, observe

$$\begin{aligned} (J_2^{-1}K_2)^k(J_2^{-1}K_2) &= \left(\frac{-ab}{s^2 + as + ab}\right)^{k+1} \begin{bmatrix} 0 & (s+a)(ab)^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & (s+a)(ab)^{-1} \\ 0 & 1 \end{bmatrix} \\ &= (J_2^{-1}K_2)^{i-2} \end{aligned}$$

To obtain Q_i, $(J_2^{-1}K_2)^{i-2}$ is post-multiplied by $(-1)^{i-1}(J_2^{-1}K_1J_1^{-1})$ which gives

$$\begin{aligned} Q_i &= (-1)^{2i-2}\left(\frac{ab}{s^2 + as + ab}\right)^{i-1} \begin{bmatrix} 0 & (s+a)(ab)^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (ab)^{-1} \\ (s+a)^{-1} \end{bmatrix} \\ &= \left(\frac{ab}{s^2 + as + ab}\right)^{i-1} \begin{bmatrix} 0 & (ab)^{-1} \\ 0 & (s+a)^{-1} \end{bmatrix}. \end{aligned} \tag{8}$$

Note that this expression was derived only for $3 \le i \le N$. Computing $Q_2 = J_2^{-1}K_1J_1^{-1}$, it is observed that (8) holds for $i = 2$ as well. Hence, using $T_i = a(Q_i)_{2,1}$ for $2 \le i \le N$ and observing that $T_1 = a/(s + a)$, it is obtained that

$$\begin{aligned} T_i(s) &= C_i(sI - A)^{-1}B \\ &= \frac{a}{s+a}\left(\frac{ab}{s^2 + as + ab}\right)^{i-1}, \qquad 1 \;\le\; i \;\le\; N. \end{aligned} \tag{9}$$

Now, the transfer function $\Gamma_i(s)$ can be computed and checked to see whether the human-driven model is string stable. In particular,

$$\Gamma_i(s) = \frac{T_i(s)}{T_{i-1}(s)} = \frac{ab}{s^2 + as + ab}, \qquad 2 \;\le\; i \;\le N.$$

For string stability, the condition (6c) has to be satisfied for $i = 2, \dots, N$ and since Γ_i is independent of i, it only needs to be checked once. Noting again that $j = \sqrt{-1}$, Γ_i can be written as

$$\begin{aligned} \Gamma_i(j\omega) &= \frac{ab}{ab - \omega^2 + ja\omega} \\ &= \frac{ab}{ab - \omega^2 + ja\omega}\left(\frac{ab - \omega^2 - ja\omega}{ab - \omega^2 - ja\omega}\right) \\ &= \left\{(ab - \omega^2) - ja\omega\right\}\frac{ab}{(ab - \omega^2)^2 + (a\omega)^2} \end{aligned}$$

Computing $|\Gamma(j\omega)|$,

$$\begin{aligned} |\Gamma(j\omega)| &= \left\{(ab - \omega^2)^2 + (a\omega)^2\right\}^{1/2}\frac{ab}{(ab - \omega^2)^2 + (a\omega)^2} \\ &= \frac{ab}{\left\{(ab - \omega^2)^2 + (a\omega)^2\right\}^{1/2}}, \end{aligned}$$

the supremum of $|\Gamma(j\omega)|$ over ω is obtained by minimizing the denominator with respect to ω^2

$$\sup_{\omega} |\Gamma(j\omega)| = \frac{ab}{\left\{\left(ab - \left(ab - \frac{a}{2}\right)\right)^2 + a^2\left(ab - \frac{a}{2}\right)\right\}^{1/2}}.$$

For realistic parameter values $a = 1$ and $b = 1 - \tanh(x_{ref} - 2)^2$ which were also used earlier,

$$\sup_{\omega} |\Gamma(j\omega)| = 1.0478,$$

which means the human vehicle model is string unstable, which corresponds with the results of the simulations in figures 5, 6 and 7.

A common way to study transfer functions is with a Bode plot and the bode plot for Γ_i is given in figure 10. Here, the peak of the frequency-response plot is equivalent to the peak gain of the system. For a string-stable vehicle model, this peak should be less than or equal to 1.

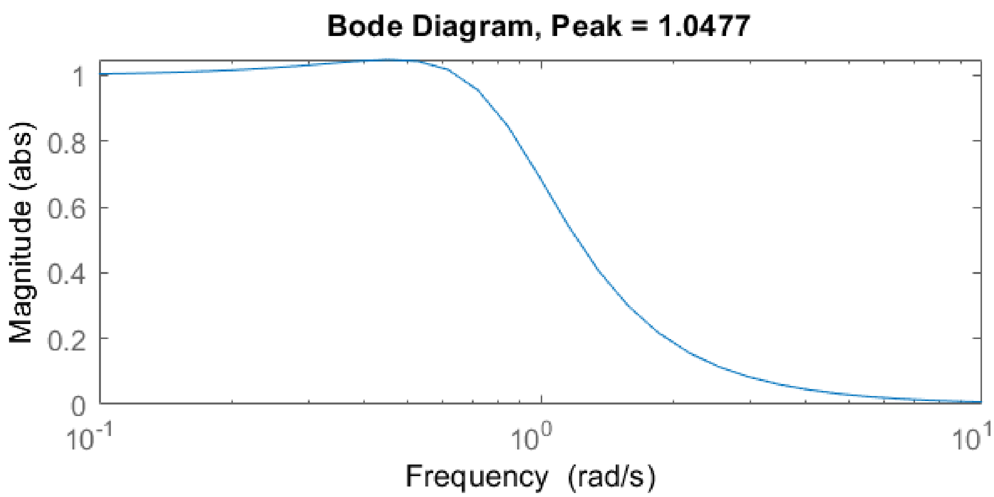

Figure 10: Bode plot of Γ_i.

3 Autonomous vehicle model

In this section, the autonomous vehicle model introduced in [3] will be studied with the goal of interconnecting it later on with the human vehicle model.

In [3], a controller is designed using wireless intervehicle communication to provide real-time information of the preceding vehicle, in addition to the information obtained by Adaptive Cruise Control (ACC) sensors such as radar. This set-up is shown in figure 11 and the resulting control system is called Cooperative ACC (CACC). This control system was implemented on a test fleet of six passenger vehicles where the results matched the theoretical analysis that the system was string stable.

The autonomous vehicles here have the objective to maintain a constant time-headway from the vehicle ahead of it. Time-headway, h, is implicitly defined as

$$\Delta x_{r,i} = h\dot{x}_i, \tag{10}$$

where $\Delta x_{r,i}$ is the desired following distance. Observe that the desired following distance increases as the velocity of the vehicle increases. Comparing this with the human vehicle model, recall that the objective of the human vehicles was to maintain the legal velocity $V(\Delta x)$.

The autonomous vehicle dynamics are described with the differential equation

$$\dot{a}_i = -\frac{1}{\tau}a_i + \frac{1}{\tau}u_i, \qquad 2 \leq i \leq N,$$

where the vehicle model can be directly written as

$$\begin{bmatrix} \dot{\Delta x}_i \\ \dot{v}_i \\ \dot{a}_i \end{bmatrix} = \begin{bmatrix} v_{i-1} - v_i \\ a_i \\ -\frac{1}{\tau}a_i + \frac{1}{\tau}u_i \end{bmatrix}, \quad 2 \leq i \leq N, \tag{11}$$

with the following notation:

- x_i for position of vehicle i;
- v_i for velocity of vehicle i;
- a_i for acceleration of vehicle i;
- u_i for external input of vehicle i;
- τ for a time constant representing engine dynamics;
- h for time headway.

The external input u_i should be interpreted as the desired acceleration and is designed in such a way that the constant time-headway is achieved.

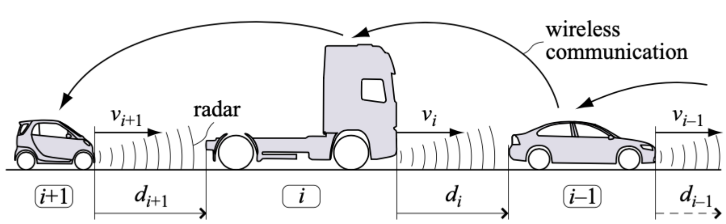

Figure 11: CACC-equipped string of vehicles. Here, $d_i = \Delta x_i$. [3]

3.1 Control design based on error dynamics

As before, consider a string of N vehicles, with Δx_i the distance between vehicle i and the vehicle ahead of it, vehicle $i-1$.

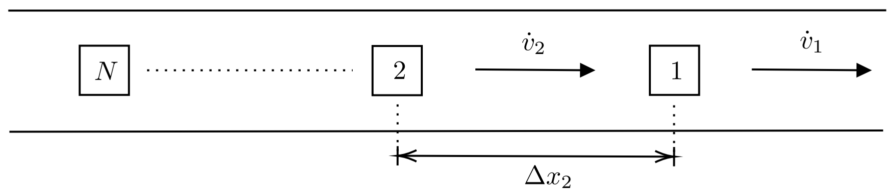

Figure 12: Overview of notation used

The objective given to each vehicle is to follow the vehicle ahead of it at a desired distance $\Delta x_{r,i}$ (10). This spacing policy is chosen because it improves string stability [3]. Now, the spacing error is defined as

$$\begin{aligned} e_i(t) &= \Delta x_i - \Delta x_{r,i} \\ &= x_{i-1} - x_i - hv_i, \end{aligned}$$

To design the controller u_i in such a way that the error dynamics are stabilized, $e_i(t)$ is differentiated twice with respect to time to introduce u_i into the error dynamics. This leads to

$$\begin{aligned} \frac{d}{dt}\dot{e}_i(t) &= \frac{d}{dt}\left(v_{i-1} - v_i - ha_i\right) \\ \ddot{e}_i(t) &= a_{i-1} - a_i - h\dot{a}_i \\ &= a_{i-1} - a_i + \frac{h}{\tau}a_i - \frac{h}{\tau}u_i. \end{aligned}$$

Now, u_i is chosen in such a way that the error dynamics is stabilized. Specifically,

$$u_i = \frac{\tau}{h}\left\{a_{i-1} - a_i\left(1 - \frac{h}{\tau}\right) + k_p e_i + k_d \dot{e}_i\right\}, \tag{12}$$

where k_p and k_d are constants with dimensions s^{-2} and s^{-1} respectively. This choice of u_i cancels terms in $\ddot{e}_i$, giving the expression

$$\ddot{e}_i = -k_d\dot{e}_i - k_p e_i.$$

Explicitly writing the error dynamics in matrix form,

$$\begin{bmatrix} \dot{e}_i \\ \ddot{e}_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix} \begin{bmatrix} e_i \\ \dot{e}_i \end{bmatrix},$$

the Routh-Hurwitz criterion can now be used to determine the stability of this system by checking the coefficients of the characteristic polynomial

$$\begin{vmatrix} s & -1 \\ k_p & s + k_d \end{vmatrix} = s^2 + k_d s + k_p.$$

For quadratic polynomials, the Routh-Hurwitz criterion states that quadratic characteristic polynomials are stable if and only if all coefficients are non-zero and have the same sign. Applying it here, the error dynamics is stable if and only if $k_p, k_d > 0$.

3.2 State-space model

Having found a suitable controller u_i which has been shown to stabilize the error dynamics, u_i can now be substituted into the vehicle model (11) to give the autonomous vehicle model

$$\begin{aligned}\dot{\Delta x}_i &= v_{i-1} - v_i,\\ \dot{v}_i &= a_i,\\ \dot{a}_i &= k_d v_{i-1} + \frac{1}{h} a_{i-1} + k_p d_i + \alpha v_i + \beta a_i.\end{aligned}$$

where $\alpha = (-k_d - k_p h)$ and $\beta = (-1/h - k_d h)$. From this, the autonomous vehicle model for $2 \leq i \leq N$ can be written as

$$\Sigma_A^i : \begin{cases} \begin{bmatrix} \dot{\Delta x}_i \\ \dot{v}_i \\ \dot{a}_i \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ k_p & \alpha & \beta \end{bmatrix} \begin{bmatrix} \Delta x_i \\ v_i \\ a_i \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ k_d & 1/h \end{bmatrix} \begin{bmatrix} v_{i-1} \\ a_{i-1} \end{bmatrix}, \\ \begin{bmatrix} v_i \\ a_i \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_i \\ v_i \\ a_i \end{bmatrix}, \end{cases} \tag{13a}$$

for which the following shorthand notation will be used

$$\begin{cases} \dot{z}_i^A = A^A z_i^A + B^A u_i^A, \\ y_i^A = C^A z_i^A. \end{cases}$$

Considering a string of autonomous vehicles, it is clear from (13a) that the behaviour of vehicle i is determined by both v_{i-1} and a_{i-1}, for $2 \leq i \leq N$. This is different from the human driven vehicles (5a), where there the behaviour of vehicle i was determined only by v_{i-1}.

On the other hand, the lead vehicle introduces the perturbations so it has different dynamics. The dynamics for the lead vehicle is chosen as

$$\Sigma_A^1 : \begin{cases} \begin{bmatrix} \dot{v}_1 \\ \dot{a}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1/\tau \end{bmatrix} \begin{bmatrix} v_1 \\ a_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1/\tau \end{bmatrix} \begin{bmatrix} 0 \\ a_{ref} \end{bmatrix}, \\ \begin{bmatrix} v_1 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ a_1 \end{bmatrix}, \end{cases} \tag{13b}$$

which will be notated as

$$\begin{cases} \dot{z}_1^A = A_1^A z_1^A + B_1^A u_1^A, \\ y_1^A = C_1^A z_1^A. \end{cases}$$

From equations (13a) and (13b), the system matrix A for the model of N autonomous vehicles can be written as

$$A = \begin{bmatrix} A_1^A & & & \\ B^A C_1^A & A^A & & \\ & B^A C^A & \ddots & \\ & & \ddots & \\ & & B^A C^A & A^A \end{bmatrix}, \tag{13c}$$

and observe it is lower block triangular. Furthermore, the input matrix B, output matrix C_i and state vector z are simply

$$
\begin{aligned}
B &= \begin{bmatrix} 0 & 1/\tau & 0 & \dots & 0 \end{bmatrix}^T, \\
C_i &= e_{3i-2}^T, \\
z &= \begin{bmatrix} (z_1^A)^T & (z_2^A)^T & \dots & (z_N^A)^T \end{bmatrix}^T,
\end{aligned} \tag{13d}
$$

where the output is chosen to be v_i. The resulting vehicle model can be written down as

$$
\begin{aligned}
\dot{z}(t) &= Az(t) + Ba_{ref}(t), \\
y(t) &= C_i z(t).
\end{aligned} \tag{14}
$$

Note that this notation is the same as that used in section 2 for the human vehicle model. The remainder of section 3 is dedicated to studying properties of the system (14) and it should be clear that any mention of the matrices A, B or C_i correspond to these matrices and not that of the human vehicle model.

The interaction between the autonomous vehicles is illustrated in figure 13

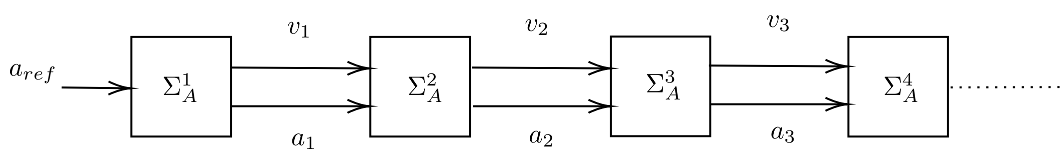

Figure 13: Connection between the autonomous vehicles

3.3 Numerical results

A time simulation of the model is performed to illustrate the properties of the autonomous vehicles. The constants h, k_d, k_p and τ are the same as those use in [3].

In figure 14, observe that the perturbation introduced by the lead vehicle is attenuated by the following vehicles i.e. it is not amplified. The reflects the string stable behaviour as expected from the autonomous vehicles. Recall that for the human vehicle model, the initial perturbation was amplified by the following vehicles as was observed in figure 8.

3.4 Linearization

Before continuing, recall that the goal is to interconnect this autonomous model with the linearized human model derived in the previous section to give a mixed traffic model. The dynamics of the linearized human model (5a) described the dynamics of the deviation from the equilibrium point.

The input of human vehicle Σ_H^i was $\tilde{v}_{i-1}$, i.e. the deviation from the equilibrium velocity of vehicle $i-1$. Supposing that vehicle $i-1$ is an autonomous vehicle, this autonomous vehicle should output $\tilde{v}_{i-1}$ so that the human vehicle can take that as an input.

Since the autonomous model is already linear, the linearized dynamics are equivalent to the current dynamics, implying in the above example that $\tilde{v}_{i-1} = v_{i-1}$ for autonomous vehicles. The same can be said for Δx_{i-1} and a_{i-1}.

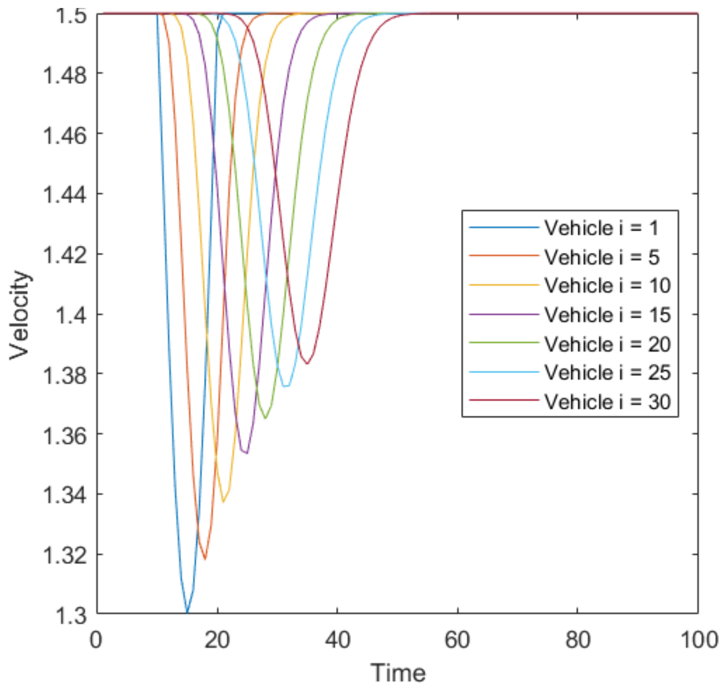

Figure 14: String stable behaviour shown by autonomous vehicles with time-headway $h = 0.7s$.

3.5 Stability

To study the stability of the system, recall the definition of stability (6a) introduced in section 2.3 Since the system matrix A of the autonomous model is also lower block triangular, the approach to computing stability is identical to that in the human model. The difference here is that the block diagonals consist of one 2×2 matrix and $(N-1)$ identical 3×3 matrices. Hence, $\sigma(A) = \sigma(A_1^A) \cup \sigma(A^A) \cup \ldots \cup \sigma(A^A)$, where

$$\sigma(A_1^A) = \{-\frac{1}{\tau},\ 0\} \quad \text{and} \quad \sigma(A_A) = \left\{-\frac{1}{h},\ -\frac{h}{2}\left(k_d \pm \sqrt{k_d^2 - \frac{4k_p}{h}}\right)\right\}.$$

First, note that the constants $k_p, k_d, \tau, h > 0$ Hence, $\sigma(A_1^A) \subset \mathbb{C}_-$. For $\sigma(A^A) \subset \mathbb{C}_-$, the condition $k_d^2 - 4k_p/h < k_d^2$ has to be satisfied and this implies $k_p/h > 0$, which is always true. Hence, $\sigma(A^A) \subset \mathbb{C}_-$ as well and therefore $\sigma(A) \subset \mathbb{C}_-$. This means that the autonomous system is also asymptotically stable.

3.6 String stability of autonomous model

To study string stability, recall the definition of string stability (6b) and how it was applied for the human driven model in section 2.4. Since the system matrix of the autonomous vehicle model also has a lower block triangular structure, the approach is identical.

For the autonomous vehicle model, $T_i(s)$ is the transfer function from the input acceleration $\hat{a}_{ref}$ to output velocity $\hat{v}_i$

$$\hat{v}_i(s) = T_i(s)\ \hat{a}_{ref}(s)$$

The transfer function is computed as

$$T_i(s) = C_i(sI - A)^{-1}B$$

where A, B and C are the system, input and output matrices of the autonomous model defined in (14). Note again that $(sI - A)$ has a lower block triangular structure so the inverse will also be lower block triangular. Furthermore, due to the structure of B and C_i, only the element $(3i-2, 2)$ of the matrix $(sI - A)^{-1}$ is required for the transfer function corresponding to vehicle i, i.e.,

$$T_i = \frac{1}{\tau}\left(sI - A\right)^{-1}_{3i-2,2}. \tag{15}$$

Following the same procedure as before, let

$$(sI - A)(sI - A)^{-1} = \begin{bmatrix} J_1 & & & & \\ K_1 & J_2 & & & \\ & K_2 & J_2 & & \\ & & \ddots & \ddots & \\ & & & K_2 & J_2 \end{bmatrix} \begin{bmatrix} J_1^{-1} & & & & \\ Q_2 & J_2^{-1} & & & \\ Q_3 & \cdot & J_2^{-1} & & \\ \vdots & \cdot & \cdot & \ddots & \\ Q_N & \cdot & \cdot & \cdot & J_2^{-1} \end{bmatrix} = \mathbb{I}_{2N+1},$$

with

$$J_1 = \begin{bmatrix} s & -1 \\ 0 & s + 1/\tau \end{bmatrix}, \quad K_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ -k_d & -1/h \end{bmatrix}, \quad J_2 = \begin{bmatrix} s & 1 & 0 \\ 0 & s & -1 \\ -k_p & -\alpha & s - \beta \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -k_d & -1/h \end{bmatrix},$$

where $\alpha = -(k_d + k_p \cdot h)$ and $\beta = -(1/h + k_d \cdot h)$. For $i = 1$, the transfer function is

$$\begin{aligned} T_1 &= \frac{1}{\tau}\left\{J_1^{-1}\right\}_{1,2} \\ &= \frac{1}{s(s\tau + 1)}. \end{aligned}$$

For $2 \leq i \leq N$, using the relation for Q_i derived in the human model, the remaining transfer functions can now be written as

$$T_i = \frac{1}{\tau}\left\{Q_i\right\}_{,2,2} \qquad i = 2, \ldots, N,$$

where

$$Q_i = (-1)^{i-1}\left(J_2^{-1}K_2\right)^{i-2}\left(J_2^{-1}K_1J_1^{-1}\right) \tag{16}$$

A general expression for $\left(J_2^{-1}K_2\right)^{i-2}$ is found which holds only for $i \geq 4$. Hence, T_2 and T_3 are worked out manually. Beginning with T_2 and T_3,

$$\begin{aligned} T_2 &= \frac{-1}{\tau}\left\{J_2^{-1}K_1J_1^{-1}\right\}_{2,2} \\ &= \frac{1}{(s)(s\tau + 1)(hs + 1)}, \end{aligned}$$

and

$$\begin{aligned} T_3 &= \frac{1}{\tau}\left\{\left(J_2^{-1}K_2\right)\left(J_2^{-1}K_1J_1^{-1}\right)\right\}_{2,2} \\ &= \frac{1}{(s)(s\tau + 1)(hs + 1)^2}, \end{aligned}$$

For $i \geq 4$, the general expression obtained for $(J_2^{-1}K_2)^{i-2}$ is

$$(J_2^{-1}K_2)^{i-2} = \frac{(-1)^{i-2}}{(hs+1)^{i-2}(s^2+hk_ds+hk_p)} \begin{bmatrix} 0 & h^2(k_p+k_ds) & hs \\ 0 & h(k_p+k_ds) & s \\ 0 & h(k_p+k_ds)s & s^2 \end{bmatrix}. \tag{17}$$

This is shown to be true with an induction argument. Assume (17) holds for $i-2=k$ where $k \in \mathbb{N}_{\geq 4}$, it is shown to hold for $i-2=k+1$ as well. This is done by post-multiplying (17) by $(J_2^{-1}K_2)$

$$\begin{aligned}
&(J_2^{-1}K_2)^k(J_2^{-1}K_2) \\
&= \frac{(-1)^k}{(hs+1)^{k+1}(s^2+hk_ds+hk_p)^2} \begin{bmatrix} 0 & h^2(k_p+k_ds) & hs \\ 0 & h(k_p+k_ds) & s \\ 0 & h(k_p+k_ds)s & s^2 \end{bmatrix} \begin{bmatrix} 0 & -(k_dh^2s+k_ph^2+hs^2+s) & 1 \\ 0 & -h(k_p+k_ds) & -s \\ 0 & -h(k_p+k_ds)s & -s^2 \end{bmatrix} \\
&= \frac{(-1)^{k+1}(s^2+hk_ds+hk_p)}{(hs+1)^{k+1}(s^2+hk_ds+hk_p)^2} \begin{bmatrix} 0 & h^2(k_p+k_ds) & hs \\ 0 & h(k_p+k_ds) & s \\ 0 & h(k_p+k_ds)s & s^2 \end{bmatrix} \\
&= (J_2^{-1}K_2)^{k+1}.
\end{aligned}$$

Observe that the matrix is independent of i. This expression can now be plugged back into Q_i in (16) giving

$$\begin{aligned}
Q_i &= (-1)^{i-1} \left(J_2^{-1}K_2\right)^{i-2} \left(J_2^{-1}K_1J_1^{-1}\right) \\
&= \frac{(-1)^{2i-2}}{(hs+1)^{i-1}(s^2+hk_ds+hk_p)} \begin{bmatrix} h^2(k_p+k_ds)/s & h\tau(s^2+hk_ds+hk_p)/(s(s+\tau)) \\ h(k_p+k_ds)/s & \tau(s^2+hk_ds+hk_p)/(s(s+\tau)) \\ h(k_p+k_ds) & \tau(s^2+hk_ds+hk_p)/(s+\tau)) \end{bmatrix}.
\end{aligned}$$

Hence, the transfer function T_i is given by

$$T_i(s) = \frac{1}{\tau}\Big\{Q_i\Big\}_{2,2} = \frac{1}{s(s+\tau)(hs+1)^{i-1}}.$$

Note that this expression for T_i also holds for $i = 1, 2, 3$ and thus this holds for $1 \leq i \leq N$. Now, since the transfer function is scalar, the condition for string stability (6c) can be checked. Note that $\Gamma_i = T_i/T_{i-1}$ is independent of i and thus the condition only needs to be checked once. Recall that $j = \sqrt{-1}$

$$\sup_\omega \left|\frac{T_i(j\omega)}{T_{i-1}(j\omega)}\right| = \sup_\omega \left|\frac{1}{hjw+1}\right| = 1.$$

Hence, this model for autonomous vehicles is string stable.

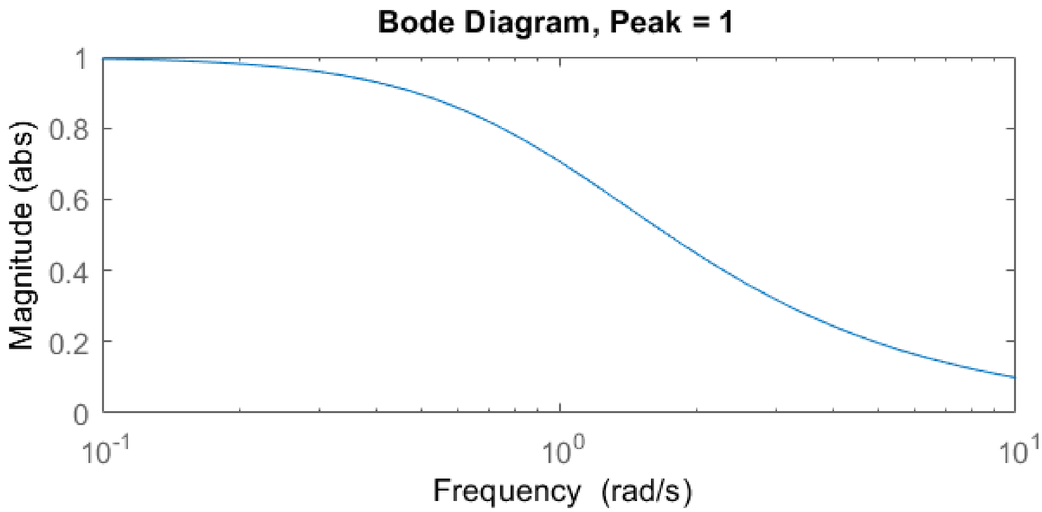

Figure 15: Bode plot of Γ_i.

The result that the autonomous model is string stable is reflected in figure 15 which has a peak of 1.

4 Mixed-traffic model

In this section, the models of human and autonomous vehicles as studied in the previous chapters will be interconnected to create a mixed-traffic model. This will be done by interconnecting the individual vehicle dynamics, as illustrated in 16.

Interconnected :

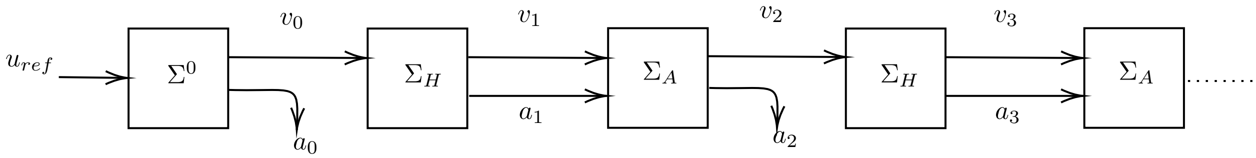

Figure 16: Interaction between different human and autonomous vehicles

Beginning with the lead vehicle, a choice has to be made between the lead vehicle in the human model (5b) and the autonomous model (13b). Recall that the lead vehicles have different dynamics than the following vehicles and therefore it will not be counted as a "human" or "autonomous" vehicle since it is a control vehicle.

Hence, the indexing of the vehicles is changed such that the lead vehicle has index $i = 0$, followed by N vehicles. This means there are now $N + 1$ vehicles, where the lead vehicle is the controlled vehicle which introduces perturbations. The dynamics of the lead vehicles are recalled as

$$\textit{Human :}$$

$$\Sigma_H^0 = \begin{cases} \dot{v}_0 & = \quad -a\, v_0 + \quad a\, v_{ref}, \\ \begin{bmatrix} v_0 \\ a_0 \end{bmatrix} & = \begin{bmatrix} 1 \\ -a \end{bmatrix} v_0 + \begin{bmatrix} 0 \\ a \end{bmatrix} v_{ref}, \end{cases}$$

$$\textit{Autonomous :}$$

$$\Sigma_A^0 = \begin{cases} \begin{bmatrix} \dot{v}_0 \\ \dot{a}_0 \end{bmatrix} & = \begin{bmatrix} 0 & 1 \\ 0 & -1/\tau \end{bmatrix} \begin{bmatrix} v_0 \\ a_0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/\tau \end{bmatrix} a_{ref}, \\ \begin{bmatrix} v_0 \\ a_0 \end{bmatrix} & = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_0 \\ a_0 \end{bmatrix}. \end{cases}$$

Note that originally, when the human model was derived, only v_i was given as an output. However for interconnection, the human vehicles Σ_H also need to output a_i since if vehicle $i+1$ is an autonomous vehicle, it requires a_i as an input. This is easily done by adding $a_0 = \dot{v}_0$ to the output equations.

This introduces a feedthrough matrix into the dynamics of the human lead vehicle which complicates computations of the transfer functions. Therefore, the lead vehicle from the autonomous model Σ_A^0 is chosen as the lead vehicle in the interconnected model.

The system, input and output matrices of the lead vehicle Σ_A^0 be denotated as A_0^A, B_0^A, C_0^A

As for the following vehicles, recall that the individual dynamics in both the human model (5a) and autonomous model (13a) are identical for $1 \le i \le N$. These dynamics are recalled as

Human :

$$\Sigma_H^i = \begin{cases} \begin{bmatrix} \dot{\Delta x}_i \\ \dot{v}_i \end{bmatrix} & = \begin{bmatrix} 0 & -1 \\ ab & -a \end{bmatrix} \begin{bmatrix} \Delta x_i \\ v_i \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_{i-1} \\ a_{i-1} \end{bmatrix}, \\ \begin{bmatrix} v_i \\ a_i \end{bmatrix} & = \begin{bmatrix} 0 & 1 \\ ab & -a \end{bmatrix} \begin{bmatrix} \Delta x_i \\ v_i \end{bmatrix}, \end{cases}$$

Autonomous :

$$\Sigma_A^i = \begin{cases} \begin{bmatrix} \dot{\Delta x}_i \\ \dot{v}_i \\ \dot{a}_i \end{bmatrix} & = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ k_p & \alpha & \beta \end{bmatrix} \begin{bmatrix} \Delta x_i \\ v_i \\ a_i \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ k_d & 1/h \end{bmatrix} \begin{bmatrix} v_{i-1} \\ a_{i-1} \end{bmatrix}, \\ \begin{bmatrix} v_i \\ a_i \end{bmatrix} & = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_i \\ v_i \\ a_i \end{bmatrix}. \end{cases}$$

Here, Σ_H^i is also adjusted to output a_i which is done by adding $\dot{v}_i = ab\Delta x_i - av_i$ to the output equations. Furthermore, the input of the human vehicle has also been adapted to include a_{i-1} even though this is not used by the human vehicle. This is clear from the input matrix. The system, input and output matrices will be denoted as A^H, B^H, C^H and A^A, B^A, C^A for the human and autonomous systems respectively. Also, recall that $\alpha = (-k_d - k_p h)$ and $\beta = (-1/h - k_d h)$.

The visual representation of these individual vehicle dynamics are given in figure 17.

Figure 17: Comparison of input-output between human and autonomous vehicles

4.1 State-space Model

Now, starting with an example, suppose vehicle $i-1$ is a human vehicle and following it, vehicle i is an autonomous vehicle. Recalling the notation used for the matrices (A^A, B^A, C^A and so forth), the dynamics of vehicle i can be written as

$$\begin{aligned} \begin{bmatrix} \dot{\Delta x}_i \\ \dot{v}_i \\ \dot{a}_i \end{bmatrix} &= A^A \begin{bmatrix} \Delta x_i \\ v_i \\ a_i \end{bmatrix} + B^A C^H \begin{bmatrix} \Delta x_{i-1} \\ v_{i-1} \end{bmatrix}, \\ &= A^A \begin{bmatrix} \Delta x_i \\ v_i \\ a_i \end{bmatrix} + B^A \begin{bmatrix} v_{i-1} \\ a_{i-1} \end{bmatrix}. \end{aligned}$$

Similarly, if vehicle $i-1$ was autonomous and vehicle i human, the dynamics of vehicle i would be

$$\begin{aligned}\begin{bmatrix}\dot{\Delta x_i} \\ \dot{v_i}\end{bmatrix} &= A^H \begin{bmatrix}\Delta x_i \\ v_i\end{bmatrix} + B^H C^A \begin{bmatrix}\Delta x_{i-1} \\ v_{i-1} \\ a_{i-1}\end{bmatrix}, \\ &= A^H \begin{bmatrix}\Delta x_i \\ v_i\end{bmatrix} + B^H \begin{bmatrix}\Delta x_{i-1} \\ v_{i-1}\end{bmatrix}.\end{aligned}$$

Hence, given an arbitrary string of vehicles such as

$$a_{ref} \to A_0 \to H_1 \to H_2 \to A_3 \to H_4 \to \dots, \tag{18}$$

the dynamics of this system can be written as

$$\dot{z}(t) = \begin{bmatrix} A_0^A & & & & & \\ B^H C_0^A & A^H & & & & \\ & B^H C^H & A^H & & & \\ & & B^A C^H & A^A & & \\ & & & B^H C^A & A^H & \\ & & & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} z_0^A \\ z_1^H \\ z_2^H \\ z_3^A \\ z_4^H \\ \vdots \end{bmatrix} + \begin{bmatrix} B_0^A \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} a_{ref},$$

where z_i refers to the state vector of the system Σ_i. Note that the dynamics are stable, $\sigma(A) \subset \mathbb{C}_-$, because the spectrum of this system matrix is simply the union of the spectra of the matrices on the diagonal. Recall that it has been shown in section 2.3 and 3.5 that the eigenvalues of these diagonal matrices are in $\mathbb{C}_-$.

4.2 Numerical Results

Here, the numerical implementation is similar as before and the code is attached in the appendix 6.7, 6.8. Note that the linearized dynamics of the human model are used instead of the non-linear dynamics. The approach used here is that given a vector such as $\begin{bmatrix}1 & 0 & 0 & 1 & 0 & \dots\end{bmatrix}^T$, the numerical model creates a system of ODEs where value 1 corresponds to an autonomous vehicle and value 0 corresponds to a human vehicle. For example, the input above corresponds to vehicle string (18).

Here, four figures are showing the numerical results. Note that the velocity of the lead vehicle is increased from 1.5 to 1.75. This input is different than in the previous sections because this allows for better interpretation of the plots in the mixed-traffic case. It is easier to distinguish between human and autonomous vehicles.

In figure 18, 20 human vehicles are modelled, without any autonomous vehicles. This is the test case to check that the string unstable behaviour of the human model is still shown. The maximum velocity attained here is 1.8917.

In figure 19, there are 4 autonomous vehicles and 16 human vehicles, with the autonomous vehicles evenly spread between the human vehicles. Observe that the maximum velocity attained is lower than in figure 18. The maximum velocity attained here is 1.7739.

In figure 20, all inputs are kept the same as in figure 19 except for time-headway, h is increased from 2 to 3. Note that this makes a noticeable difference in the behaviour of the vehicles. The maximum velocity attained here is 1.7500.

In figure 21, all inputs are kept the same as in figure 19 except the arrangement of the vehicles The maximum velocity attained here is 1.7739. Note that this is the same as in figure 19.

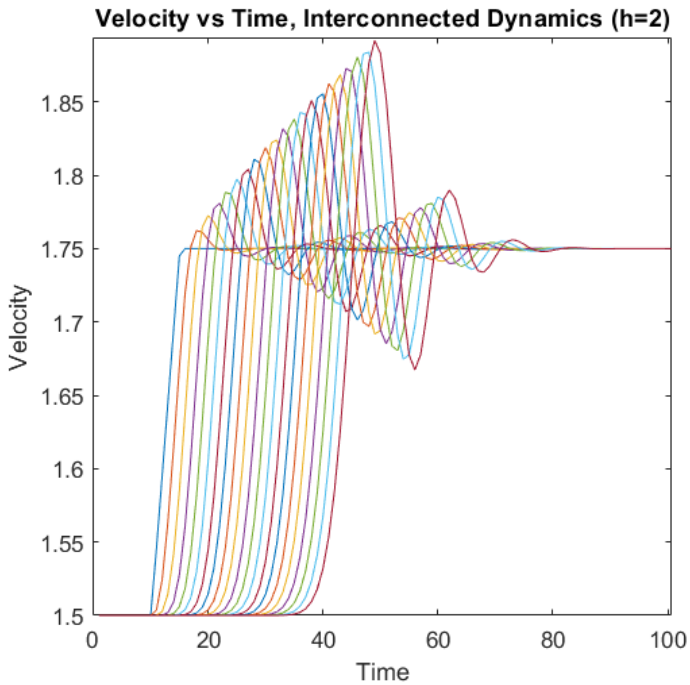

Figure 18: 20 human vehicles. Max velocity 1.8917

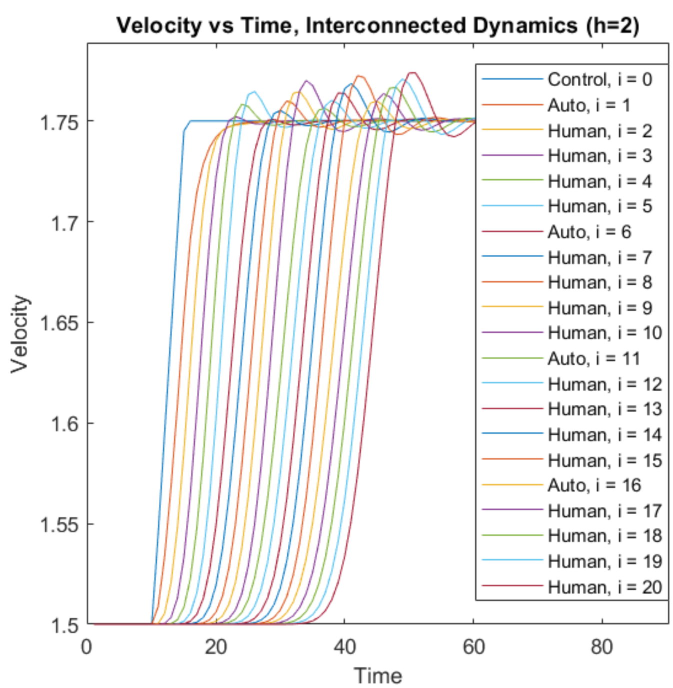

Figure 19: 16 human and 4 autonomous vehicles with $h = 2$. Max velocity 1.7739

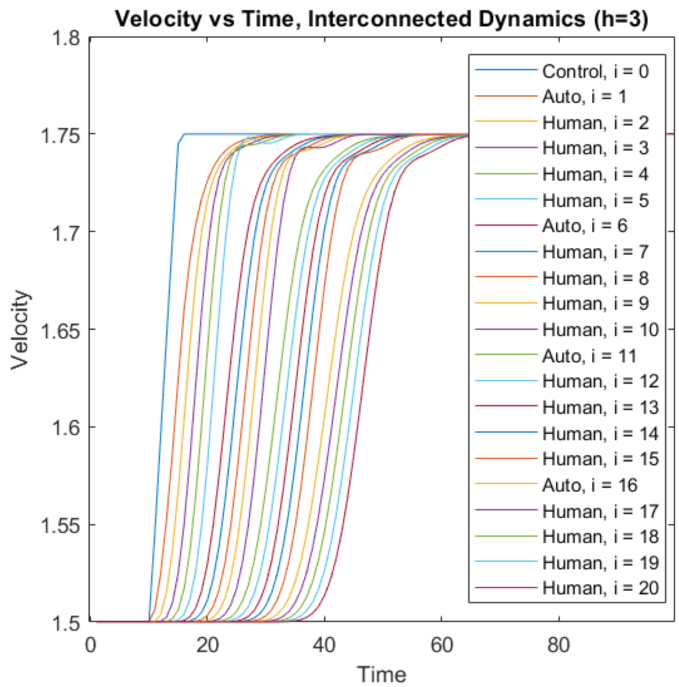

Figure 20: 16 human and 4 autonomous vehicles with $h = 3$. Max velocity 1.750

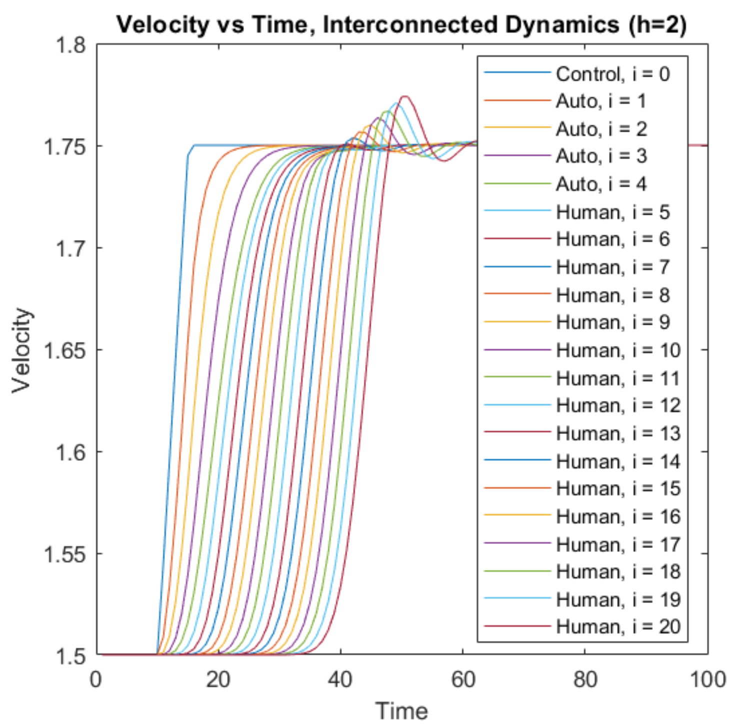

Figure 21: 16 human and 4 autonomous vehicles with $h = 2$. Max velocity 1.7739

4.3 String stability

To study string stability, transfer functions will be used as before but a slightly different approach will be used this time. Instead of computing $(sI - A)^{-1}$, the transfer functions of the individual vehicle subsystems Σ_A^0, Σ_H^i and Σ_A^i for $1 \leq i \leq N$ will be used.

From this, the transfer function matrix of an arbitrary string of vehicles can be determined. Denoting $\varphi_i = \begin{bmatrix} v_i & a_i \end{bmatrix}^T$,

$$\varphi_i(s) = \begin{cases} G_0(s)\, a_{ref} & \textit{if } i \ = \ 0, \\ G_H(s)\, \varphi_{i-1}(s) & \textit{if } i \textit{ human and } i \ \geq \ 1, \\ G_A(s)\, \varphi_{i-1}(s) & \textit{if } i \textit{ autonomous and } i \ \geq \ 1, \end{cases}$$

where $G_H(s)$ and $G_A(s)$ is the transfer function matrix of Σ_H^i and Σ_A^i respectively and note that these matrices are dimension 2×2. $G_0(s)$ is the transfer function matrix of Σ_A^0 with dimension 2×1.

$Interconnected:$

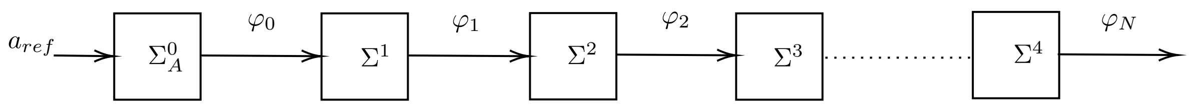

Figure 22: Σ^i is either a human or autonomous vehicle and $\varphi_i = \begin{bmatrix} v_i & a_i \end{bmatrix}^T$

Using this, the transfer function matrix of an arbitrary string of vehicles can be easily computed. This is illustrated with an example. Consider the following string of vehicles

$$a_{ref} \rightarrow \ A_0 \ \rightarrow \ H_1 \ \rightarrow \ H_2 \ \rightarrow \ A_3 \ \rightarrow \ H_4$$

The output can simply be given by

$$\varphi_4(s) \ = \ G_H\ G_A\ G_H\ G_H\ G_0\ a_{ref},$$

where $G_H\ G_A\ G_H\ G_H\ G_0$ is the transfer function matrix from a_{ref} to φ_4. Pre-multiplying this expression by $\begin{bmatrix} 1 & 0 \end{bmatrix}$ gives v_4. Using this approach, $v_i(s)$ can easily be obtained and hence string stability can be studied as before.

Here, the transfer functions matrices $G_0(s), G_H(s)$ and $G_A(s)$ are given, which read

$$\begin{aligned} G_0(s) &= C_0^A(sI - A_0^A)^{-1} B_0^A \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ 0 & s + \frac{1}{\tau} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{\tau} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{s(s\tau + 1)} \\ \frac{1}{s\tau + 1} \end{bmatrix}, \end{aligned}$$

and

$$G_H(s) = C^H(sI - A^H)^{-1}B^H$$
$$= \begin{bmatrix} 0 & 1 \\ ab & -a \end{bmatrix} \begin{bmatrix} s & 1 \\ -ab & s+a \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \frac{ab}{s^2 + as + ab} \begin{bmatrix} 1 & 0 \\ s & 0 \end{bmatrix},$$

$$G_A(s) = C^A(sI - A^A)^{-1}B^A$$
$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s & 1 & 0 \\ -ab & s & -1 \\ -k_p & -\alpha & s-\beta \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ k_d & 1/h \end{bmatrix}$$
$$= \frac{1}{(hs+1)(s^2+\zeta)} \begin{bmatrix} \zeta & s \\ \zeta s & s^2 \end{bmatrix},$$

where $\zeta = hk_p + hk_d s$. Since powers of G_H and G_A will appear when computing the transfer function matrix of arbitrary strings of vehicles, general expressions are found for $(G_H)^k$ and $(G_A)^k$, $k \in \mathbb{Z}_{\geq 1}$, as

$$(G_H)^k = \left(\frac{ab}{s^2 + as + ab}\right)^k \begin{bmatrix} 1 & 0 \\ s & 0 \end{bmatrix}^k$$
$$= \left(\frac{ab}{s^2 + as + ab}\right)^k \begin{bmatrix} 1 & 0 \\ s & 0 \end{bmatrix}$$
$$= \left(\frac{ab}{s^2 + as + ab}\right)^k M_H,$$

$$(G_A)^k = \frac{1}{(hs+1)^k(s^2+\zeta)^k} \begin{bmatrix} \zeta & s \\ \zeta s & s^2 \end{bmatrix}^k$$
$$= \frac{(s^2+\zeta)}{(hs+1)^k(s^2+\zeta)^k} \begin{bmatrix} \zeta & s \\ \zeta s & s^2 \end{bmatrix}^{k-1}$$
$$\vdots$$
$$= \frac{(s^2+\zeta)^{k-1}}{(hs+1)^k(s^2+\zeta)^k} \begin{bmatrix} \zeta & s \\ \zeta s & s^2 \end{bmatrix}$$
$$= \left(\frac{1}{hs+1}\right)^k \frac{1}{(s^2+\zeta)} \begin{bmatrix} \zeta & s \\ \zeta s & s^2 \end{bmatrix}$$
$$= \left(\frac{1}{hs+1}\right)^k M_A,$$

where M_H and M_N are terms independent of k, given by

$$M_H = \begin{bmatrix} 1 & 0 \\ s & 0 \end{bmatrix}, \qquad M_A = \frac{1}{(s^2+\zeta)} \begin{bmatrix} \zeta & s \\ \zeta s & s^2 \end{bmatrix}.$$

Furthermore, in the derivation of $(G_A)^k$, it was used that

$$\begin{bmatrix} \zeta & s \\ \zeta s & s^2 \end{bmatrix}^2 = (s^2+\zeta) \begin{bmatrix} \zeta & s \\ \zeta s & s^2 \end{bmatrix}.$$

Now, observe that these expressions can be used to compute the transfer function T_i for homogeneous strings of vehicles i.e. purely human or autonomous with a lead control vehicle Σ_A^0, defined such that

$$v_i(s) = T_i(s)\, a_{ref} \quad \textit{where} \;\; i = 0, 1, \ldots, N.$$

Determining $v_i(s)$ for both the human and autonomous vehicles, it is obtained that

$$v_i^H(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} (G_H)^i\, G_0\, a_{ref}$$
$$= \frac{a_{ref}}{s(s\tau+1)} \left(\frac{ab}{s^2 + as + ab}\right)^i,$$

$$v_i^A(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} (G_A)^i\, G_0\, a_{ref}$$
$$= \frac{a_{ref}}{s(s\tau+1)} \left(\frac{1}{hs+1}\right)^i.$$

In particular, recall that $\Gamma_i(s) = v_i/v_{i-1}$ and computing this for both the human and autonomous case gives the same expression as derived in section 2.4, 3.6.

Now, the transfer function matrices of arbitrary strings of vehicles will involve multiplying G_H by G_A and vice versa. Hence, the matrices M^H and M^A are studied and the following properties are obtained

$$M_H M_A = M_A, \qquad M_A M_H = M_H, \qquad (M_H)^i = M_H, \qquad (M_A)^i = M_A. \tag{19}$$

Furthermore,

$$M_H G_0 = G_0 = M_A G_0. \tag{20}$$

Now, consider an arbitrary string of $N+1$ vehicles ending with a human vehicle. The transfer function matrix from a_{ref} to φ_N is given by

$$G_H G_A G_A G_H G_H \dots G_0 = \left(\frac{ab}{s^2+as+ab}\right)^{n_H} \left(\frac{1}{hs+1}\right)^{n_A} M_H M_A M_A M_H M_H \dots G_0$$

where n_H and n_A denote the number of human and autonomous vehicles respectively up to and including vehicle N. Recall that the lead vehicle is neither human nor autonomous. Using the properties of M_H, M_A in (19), (20), the sequence of matrices $M_H M_A M_A M_H M_H \dots G_0$ simply reduces to G_0

To prove this, observe that any sequence of M_A and M_H ending with G_0 can be reduced to following six cases:
Case 1 :

$$\begin{aligned} M_A M_H M_A M_H \dots G_0 &= (M_A M_H)^k G_0 \\ &= (M_H)^k G_0 \\ &= M_H G_0 \\ &= G_0. \end{aligned}$$

Case 2 :

$$\begin{aligned} M_A M_H M_A M_H \dots M_A G_0 &= (M_A M_H)^k M_A G_0 \\ &= (M_H)^k M_A G_0 \\ &= M_H M_A G_0 \\ &= M_A G_0 \\ &= G_0. \end{aligned}$$

Case 3 and *Case 4* are identical to the first two, only M_H needs to be replaced with M_A and vice versa.
Case 5 and *Case 6* are the simplest cases, which are already given in (20). Hence, the transfer function from a_{ref} to v_i for a heterogeneous (human and autonomous) string of vehicles reduces to a product of scalar functions

$$\begin{aligned} T_i(s) &= \left(\frac{ab}{s^2+as+ab}\right)^{n_H} \left(\frac{1}{hs+1}\right)^{n_A} \begin{bmatrix} 1 & 0 \end{bmatrix} M_H M_A M_A M_H M_H \dots G_0 \\ &= \left(\frac{ab}{s^2+as+ab}\right)^{n_H} \left(\frac{1}{hs+1}\right)^{n_A} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{1}{s(s\tau+1)} \\ \dfrac{1}{s\tau+1} \end{bmatrix} \\ &= \left(\frac{ab}{s^2+as+ab}\right)^{n_H} \left(\frac{1}{hs+1}\right)^{n_A} \frac{1}{s(s\tau+1)} \end{aligned} \tag{21}$$

where

$$
\begin{aligned}
n_H &= \textit{number of human vehicles up to and including } i^{th} \textit{ vehicle} \\
n_A &= \textit{number of autonomous vehicles up to and including } i^{th} \textit{ vehicle} \\
a &= \textit{driver's sensitivity from human model} \\
b &= V'(x)\big|_{x=\Delta x_{ref}} \textit{ from human model} \\
h &= \textit{time-headway from autonomous model} \\
\tau &= \textit{time constant for engine dynamics from autonomous model}
\end{aligned}
$$

Observe that this means

$$
\Gamma_i(s) = \frac{v_i}{v_{i-1}} = \begin{cases} \dfrac{ab}{s^2 + as + ab} & \textit{if vehicle } i = \textit{human} \\ \\ \dfrac{1}{hs+1} & \textit{if vehicle } i = \textit{autonomous} \end{cases} \tag{22}
$$

Note that these expressions are the same as derived in the previous sections for the human and autonomous model. This implies that the transfer function from v_{i-1} to v_i (22) depends only on the type of vehicle i.

However, the overall transfer function from a_{ref} to v_i (21) depends on how many of each type of vehicle is present up to vehicle i.

To study the effects of introducing autonomous vehicles into a string of human vehicles, the transfer function

$$
\Psi_i(s) = \frac{v_i}{v_0}, \qquad 1 \ \leq \ i \ \leq N,
$$

is introduced.

5 Results of Mixed Traffic Model

When studying string stability for the purely human or autonomous models, it was sufficient to study how a particular vehicle i responded to the behaviour of its predecessor $i-1$ since all vehicles were identical. This was done by studying the transfer function Γ_i in (22).

To study how the introduction of autonomous vehicles affects the string stability properties of a human vehicle string, the definition of string stability (6b) is adapted, giving

$$
\|\Psi_i(j\omega)\|_{\mathcal{H}_\infty} = \sup_\omega \left| \frac{T_i(j\omega)}{T_0(j\omega)} \right| = \sup_\omega \left| \left(\frac{ab}{(j\omega)^2 + aj\omega + ab} \right)^{n_H} \left(\frac{1}{hj\omega + 1} \right)^{n_A} \right| = \max_{v_0 \neq 0} \frac{\|v_i(t)\|_{\mathcal{L}_2}}{\|v_0(t)\|_{\mathcal{L}_2}} \leq 1. \tag{23}
$$

From this, it can be studied whether the perturbation introduced by vehicle $i = 0$ has been attenuated or amplified by a platoon of n_H human and n_A autonomous vehicles.

In other words, this condition says that the velocity attained by vehicle i is not amplified beyond the velocity of the lead vehicle. To study this, $\|\Psi_i\|$ is computed for the time-headways $h \ = \ 1, \ 1.5, \ 2$. In tables 1 to 3, a string of 10 vehicles is considered. Beginning with only human vehicles, a human vehicle is replaced with an autonomous vehicle until $\|\Psi_{10}\| = 1.0$. As per definition, this means that the string of 10 vehicles is string stable.

n_A	$\|\Psi_1\|$	$\|\Psi_2\|$	$\|\Psi_3\|$	$\|\Psi_4\|$	$\|\Psi_5\|$	$\|\Psi_6\|$	$\|\Psi_7\|$	$\|\Psi_8\|$	$\|\Psi_9\|$	$\|\Psi_{10}\|$
0	1.048	1.098	1.15	1.205	1.263	1.323	1.386	1.452	1.522	1.594
1	1.0	1.0	1.019	1.059	1.106	1.156	1.209	1.266	1.326	1.388
2		1.0	1.0	1.0	1.008	1.038	1.077	1.122	1.17	1.222
3			1.0	1.0	1.0	1.0	1.002	1.025	1.057	1.097
4				1.0	1.0	1.0	1.0	1.0	1.0	1.016
5					1.0	1.0	1.0	1.0	1.0	1.0

Table 1: $h = 1$

n_A	$\|\Psi_1\|$	$\|\Psi_2\|$	$\|\Psi_3\|$	$\|\Psi_4\|$	$\|\Psi_5\|$	$\|\Psi_6\|$	$\|\Psi_7\|$	$\|\Psi_8\|$	$\|\Psi_9\|$	$\|\Psi_{10}\|$
0	1.048	1.098	1.15	1.205	1.263	1.323	1.386	1.452	1.522	1.594
1	1.0	1.0	1.0	1.003	1.031	1.071	1.115	1.164	1.216	1.272
2		1.0	1.0	1.0	1.0	1.0	1.0	1.006	1.031	1.064
3			1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Table 2: $h = 1.5$

n_A	$\|\Psi_1\|$	$\|\Psi_2\|$	$\|\Psi_3\|$	$\|\Psi_4\|$	$\|\Psi_5\|$	$\|\Psi_6\|$	$\|\Psi_7\|$	$\|\Psi_8\|$	$\|\Psi_9\|$	$\|\Psi_{10}\|$
0	1.048	1.098	1.15	1.205	1.263	1.323	1.386	1.452	1.522	1.594
1	1.0	1.0	1.0	1.0	1.0	1.002	1.031	1.068	1.111	1.158
2		1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Table 3: $h = 2$. For $A = 2$, $\|\Psi_{12}\| > 1$

5.1 Discussion

From the "Highway Design and Traffic Safety Engineering Handbook [5]", the time headway that drivers should obey is based on the reaction time of drivers, and this varies between $1s$ and $2s$. This implies that $h = 2$ is a reasonable time-headway for the autonomous vehicles to follow and does not introduce large following distances between the vehicles. For example, driving at 100 km/h with a headway of $2s$ means the following distance should be 55.56 m. Hence, several numerical simulations are performed for $h = 2$ to compare with the results of table 3. Note that the linearized human vehicle model was used for interconnection.

In figure 23, the control vehicle is followed by 10 human vehicles and the initial perturbation is amplified by the following vehicles. This corresponds with the results obtained in the chapter 2, where the human vehicle model was shown to be string unstable.

In figure 24, an autonomous vehicle is introduced after the control vehicle in position $i = 1$. Observe that the amplification of the perturbation is clearly reduced compared to figure 23 but it is still string unstable. From table 3, $\|\Psi_{10}\| = 1.158$ for $n_A = 1$.

In figure 25, another autonomous vehicle is introduced in position $i = 2$. Here, the initial perturbation is clearly attenuated and this reflects the result in table 3 where $\|\Psi_{10}\| = 1.0$ for $n_A = 2$.

In figure 26 and 27, three of the ten vehicles are autonomous, but are arranged differently. In figure 26, the first three vehicles after the control vehicle are autonomous whereas in figure 27, the autonomous vehicles are spread evenly though the human vehicles. Note that both these figures represent string stable behaviour.

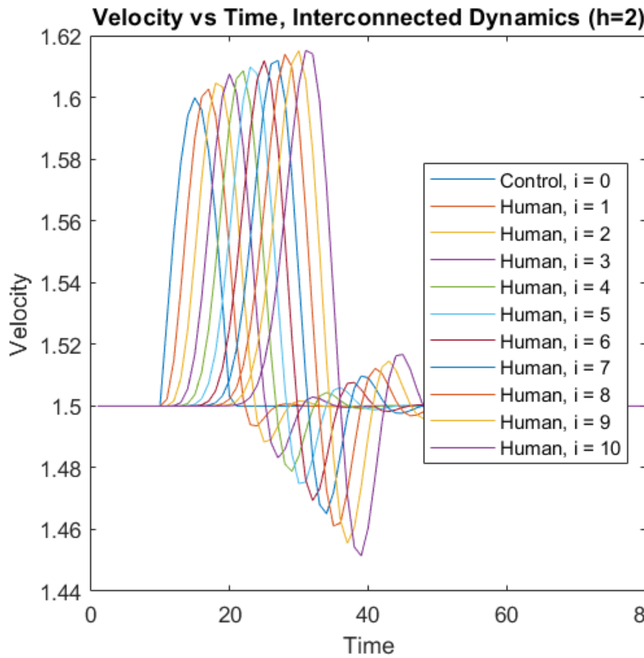

Figure 23: String unstable

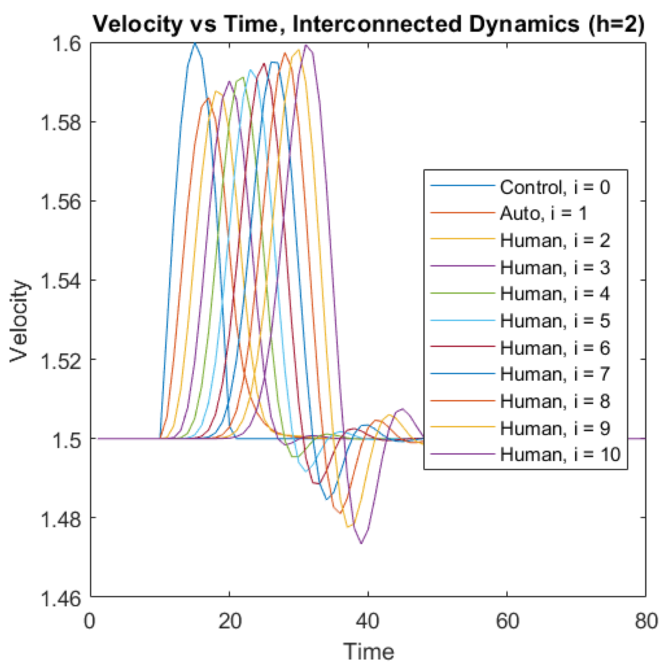

Figure 24: String unstable

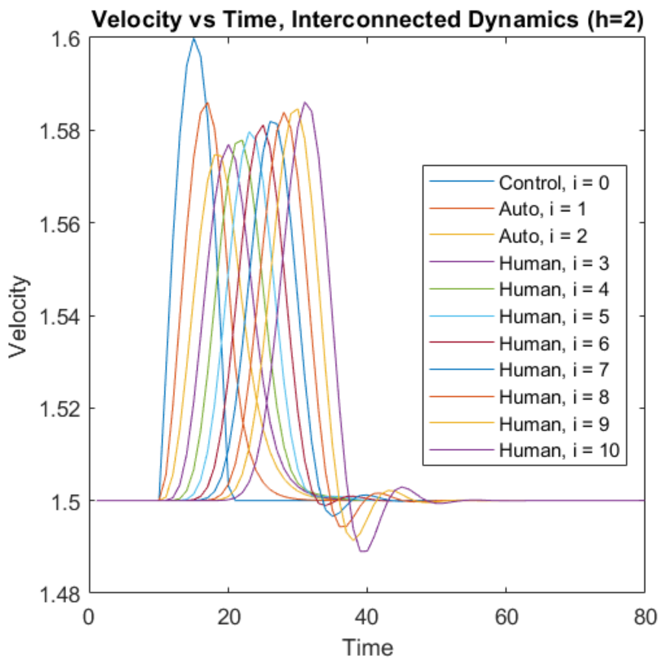

Figure 25: String stable

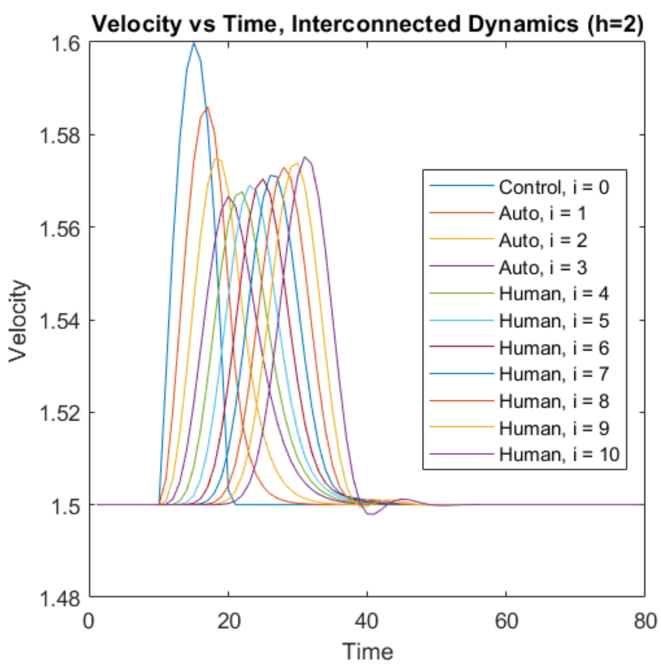

Figure 26: String stable

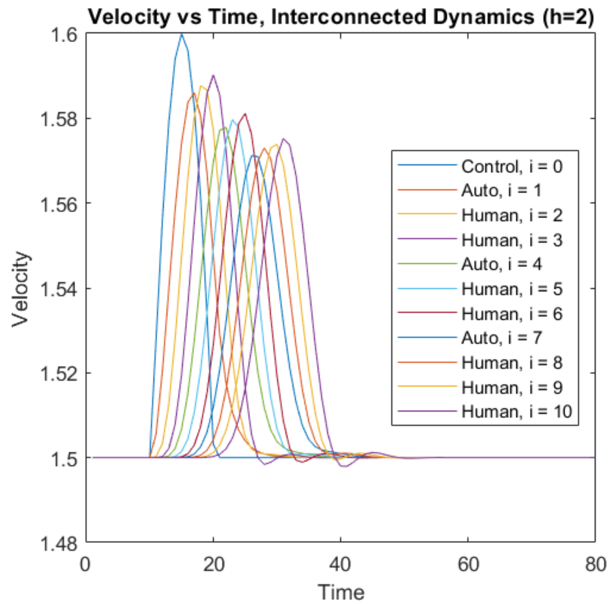

Figure 27: String stable

Furthermore, observe that the transfer function Ψ_i used in definition (23) is simply a product of scalar functions. This implies that the positions of the autonomous vehicles in the string of mixed vehicles is not important for string stability, only the ratio between human and autonomous vehicles.

However, consider a string of 100 vehicles where the first 80 vehicles are human vehicles followed by 20 autonomous vehicles. This scenario would allow the perturbations to grow through the human vehicles before being attenuated by the autonomous vehicles, which is clearly undesirable since the disturbance is first amplified by 80 vehicles. Instead, it would be better to have the autonomous vehicles evenly spread out through the human vehicles.

Another alternative would be to restrict the number of consecutive human vehicles in a string to 10 for example. This would prevent the perturbations from growing out of hand. This could also be done by enforcing a maximum deviation from the equilibrium velocity for a string to be considered string stable.

From the values computed in table 3, 1 autonomous vehicles should be able to attenuate perturbations in a string of 5 vehicles. This implies that 20% of vehicles on a roadway need to be autonomous for perturbations to be reduced.

In figures 28 and 29, the numerical simulations are performed for a string of 600 vehicles, with the autonomous vehicles evenly spread throughout the human vehicles. Observed that for 20% autonomous vehicles, string stable behaviour is observed whereas for 14.3% autonomous vehicles, string unstable behaviour is observed, as expected from the values in table 3.

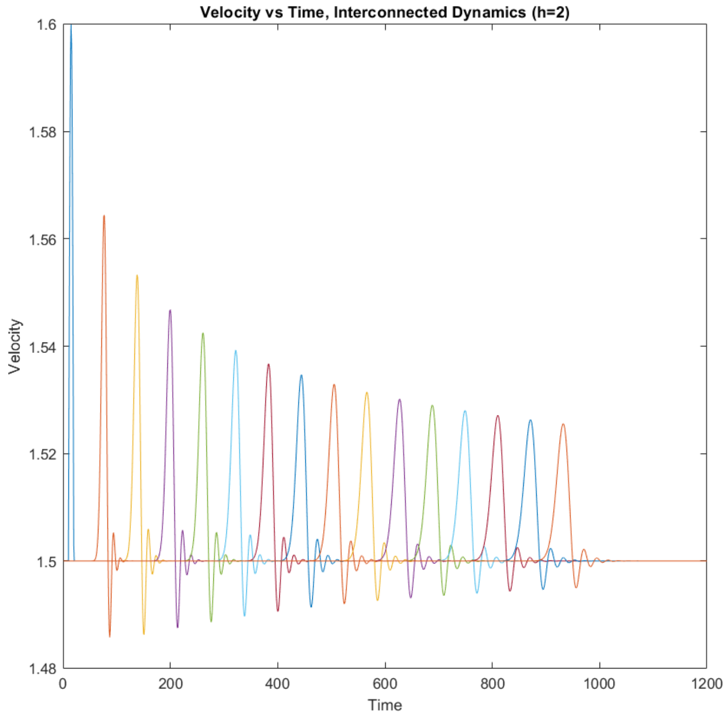

Figure 28: 600 vehicles, where 1 in 5 are autonomous i.e. 20.0%. Only a selection of vehicles are plotted

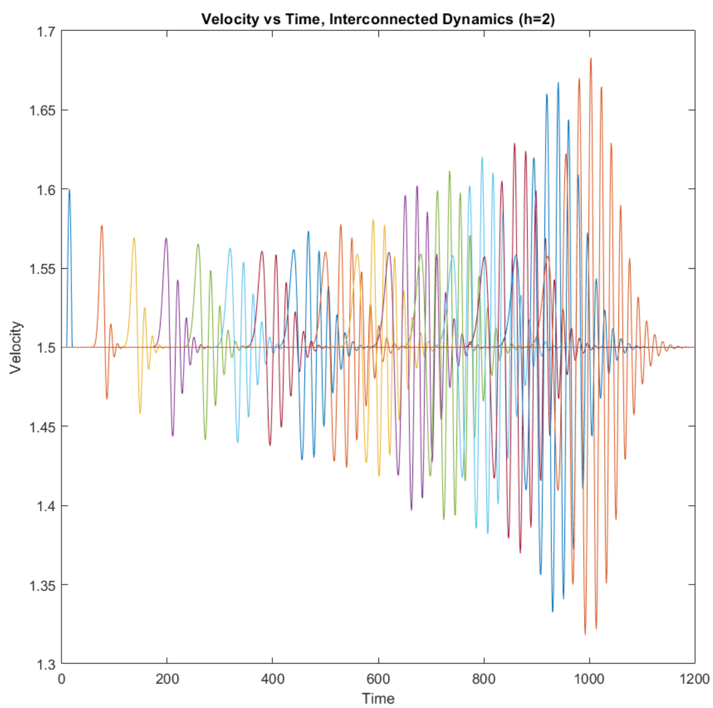

Figure 29: 600 vehicles, where 1 in 7 are autonomous i.e. 14.3%. Only a selection of vehicles are plotted

5.2 Limitations

First of all, the road segment considered has a fixed amount of vehicles and there are no vehicles entering or leaving the road. Furthermore, this model is based on a one-lane road and no consideration is made for vehicles overtaking one another.

It is also assumed that a "human" vehicle communicates with an "autonomous" vehicle to transmit information regarding its velocity and acceleration. However, the human vehicle does not necessarily need to have wireless communication hardware. Instead, this information could also be determined using the sensors of the autonomous vehicle. A time-lag for determining and processing this information would have to be taken into consideration when designing the controller of the autonomous vehicle.

Besides that, the behaviour of all the vehicles on the highway are described by two vehicle models. In reality, this is of course different since an older person may have different reaction times to a younger person, implying a different constant a in the human model. Similarly, different autonomous vehicles would also have different time constant τ for the engine dynamics.

5.3 Conclusion

String stability is an essential requirement to prevent perturbations from being amplified in a string of vehicles. It has been shown that, theoretically and with numerical simulations, that the human vehicle model [3] is indeed string unstable whereas the autonomous vehicle model [3] is string stable.

Interconnecting both these models, it was found that for a time-headway $h = 2$, 20% of vehicles in a mixed traffic model need to be autonomous for the traffic to be stabilized i.e. for perturbations to be attenuated. Note that this will improve traffic flow on roadways leading to decreased fuel consumption and traffic accidents.

However, more restrictions should be imposed on the current definition of string stability for mixed traffic (23) to bound how large a disturbance can be amplified and still be considered string stable. Furthermore, the current limitations mentioned in 5.2 can be looked into and improved upon.

6 Appendix

6.1 ODE3_straight_road.m

```
function dwdt = ODE3_straight_road(t,w,a, N, vref)
    dwdt = zeros(2*N,1);
    for i = 1:N
        %first half of system
        dwdt(i) = w(i+N);

        %second half of system
        deltaX    = w(i+1)-w(i);
        if i == N
             dwdt(i+N) = a*( vref - w(i+N) );
             %%% the if-statement below enters the perturbation.
             Tstart = 40;
             Tend   = 50; %consider tgrid
             if t>Tstart && t< Tend
                dwdt(i+N) = a*( vref - 0.2*sin( (t-Tstart)*2*pi/Tend ) - w(i+N) );
             end
        else
            dwdt(i+N) = a*(tanh(deltaX-2) + tanh(2) - w(i+N) );
        end

    end
end
```

6.2 runscript_Straight_Road.m

```
%%%%%%%%%%%%%%
%runscript for ODE3_straight_road
%%%%%%%%%%%%%%
clear all
close all

vref = 1.5;
xref = atanh(vref-tanh(2))+2;

N = 100;               %number of vehicles
a = 1;                 %driver sensitivity
tgrid = [1 : 1 : 160]; %check time when perturbation entered

%%% initial condition %%%
initialPosition = zeros(N,1);
initialVelocity = vref*ones(N,1);
for i = 1:N
    initialPosition(i) = xref*i;
end
initialCond = [initialPosition; initialVelocity];

options= odeset('AbsTol',1e-9,'RelTol',1e-9); %reduce numerical error
```

```
[t, w] = ode45(@(t,w) ODE3_straight_road(t,w,a,N, vref), tgrid, ...
                initialCond, options);
%each row in w corresponds to solution at time t in corresponding row

figure
for i = 1:1:N
    hold on
    plot(tgrid, w(:,i))
end
axis([50 160 150 500])
pbaspect([1 1 1])
title(['Trajectory of all vehicles (\Delta x_{int}=',num2str(xref),', N=',num2str(N),') '])
xlabel('Time')
ylabel('Distance')

figure
for i = 10:30:160
   hold on
   txt = ['time = ',num2str(i)];
   plot([N:-1:1], w(i, (N+1):2*N), 'DisplayName', txt )
end
pbaspect([1 1 1])
title(['Velocity vs Vehicle Index snapshots (\Delta x_{int}=',num2str(xref),') '])
xlabel('Vehicle index')
ylabel('Velocity')
legend('Location','southeast')
legend show

figure
txt = ['Vehicle i = ',num2str(1)];
plot(tgrid, w(:,2*N), 'DisplayName', txt )
for i = 96:-5:70
   hold on
   txt = ['Vehicle i = ',num2str(N+1-(i))];
   plot(tgrid, w(:,N+i), 'DisplayName', txt )
end
pbaspect([1 1 1])
title(['Velocity vs Time (\Delta x_{int}=',num2str(xref),') '])
xlabel('Time')
ylabel('Velocity')
legend('Location','southeast')
legend show
```

6.3 ODE_Bando_Linearized.m

```
function dwdt = ODE_Bando_Linearized(t,w,N,xref)
    vinput = 0;
    if t>40 && t <50
      vinput =  -0.2*sin( (t-40)*2*pi/50 );
    end

```

```
    %%% Linearized BANDO %%%
    a  = 1;
    b  = 1-tanh(xref-2)^2;
    Ab = [0 -1; a*b -a];
    Bb = [1 0 ; 0 0];
    Cb = [0 1; a*b -a];

    %%% intialization %%%
    dwdt = zeros(1+2*(N-1),1);

    %%% vehicle 1 %%%
    dwdt(1) = -a*w(1) + a*vinput;

    %%% vehicle 2 %%%
    vehicle_2 = Ab * [w(2) ; w(3)] + Bb*Cb*[0;w(1)] ;
    dwdt(2)  = vehicle_2(1);
    dwdt(3)  = vehicle_2(2);

    %%% vehicle 3 to N %%%
    for i = 4:2:2*(N-2)
       vehicle_i = Ab * [w(i) ; w(i+1)] + Bb*Cb*[w(i-2);w(i-1)];
       dwdt(i)   = vehicle_i(1);   %%% \dot{\Delta x}
       dwdt(i+1) = vehicle_i(2);   %%% \dot{v}
    end

end
```

6.4 runscript_Bando_Linearized

```
clear all
close all

N = 30;                % number of vehicles
tgrid = [1 : 1 : 160];
initialCond=zeros(1+(N-1)*2,1);
% note all initialCondition is zero bc this is dynamics of the
% deviation from equilibrium.

vref = 1.5;
xref = atanh(vref-tanh(2))+2;

options= odeset('AbsTol',1e-9,'RelTol',1e-9);
[t, w] = ode45(@(t,w) ODE_Bando_Linearized(t,w,N, xref), tgrid, ...
             initialCond, options);

txt = ['Vehicle i = ',num2str(1)];
plot(tgrid, 1*1.5+w(:,1), 'DisplayName', txt )
counter = 2;
for i = 3 : 2 : 2*N+1
   hold on
   if mod(counter,5)== 0
```

```
        txt = ['Vehicle i = ',num2str(counter)];
        plot(tgrid, 1*1.5+w(:,i), 'DisplayName', txt )
    end
    counter = counter + 1;
end
axis([0 160 1.25 1.7])
pbaspect([1 1 1])
title(['Velocity vs Time, Linearized Dynamics'])
xlabel('Time')
ylabel('Velocity')
legend('Location','southeast')
legend show
```

6.5 ODE_Ploeg_Only.m

```
function dwdt = ODE_Ploeg_Only(t,w,N,h)
    aref = 0;
    if t>=10 && t <= 20
        aref = -0.2*(2*pi/20)*cos(2*pi*(t-10)/20);
    end

    %%% PLOEG %%% constants from ploeg paper
    h     = h;
    tau   = 0.1;
    kp    = 0.2;
    kd    = 0.7;
    alpha = -kd - kp*h;
    beta  = -1/h -kd*h;
    Ap    = [0 (-1) 0; 0 0 1; kp alpha beta];
    Bp    = [1 0; 0 0; kd 1/h];
    Cp    = [0 1 0; 0 0 1];

    dwdt = zeros(2+3*N,1);

    %%% lead vehicle      %%%
    dwdt(1) = w(2);
    dwdt(2) = (-1/tau)*w(2) + (1/tau)*aref;

    %%% vehicle 2 to N    %%%
    j = 3;
    for i = 2:N
        Cw = Cp*[0; w(j-2); w(j-1)]; %note first can be zero since unused
        vehicle_i = Ap*[w(j) ; w(j+1); w(j+2)] + Bp*Cw;
        dwdt(j)   = vehicle_i(1);
        dwdt(j+1) = vehicle_i(2);
        dwdt(j+2) = vehicle_i(3);
        j = j + 3;
    end
end
```

6.6 runscript_Ploeg_Only

```
%%% autonomous only model %%%
clear all
close all

h = 0.7;               % time headway
N = 30;                % number of autonomous vehicles
a = 1;                 % driver sensitivity
tgrid = [1 : 1 : 100]; % check time when perturbation entered
initialCond=zeros(2 + N*3 , 1);
% note all initialCondition is zero bc this is dynamics of the
% deviation from equilibrium.

options= odeset('AbsTol',1e-9,'RelTol',1e-9);
[t, w] = ode45(@(t,w) ODE_Ploeg_Only(t,w,N,h), tgrid,initialCond, options);

%%% PLOT %%%
txt = ['Vehicle i = ',num2str(1)];
plot(tgrid,1.5 + w(:,1), 'DisplayName', txt )
j = 3; %position of Delta_x_i in state vector
counter = 1;
for i = 2:N
   hold on
   if mod(i,5) == 0
     txt = ['Vehicle i = ',num2str(i)];
     plot(tgrid, 1.5 + w(:,j+1), 'DisplayName', txt )
   end
   j = j + 3;
   counter = counter + 1;
end
pbaspect([1 1 1])
title(['Velocity vs Time, Autonomous model (h=',num2str(h),')'])
xlabel('Time')
ylabel('Velocity')
legend('Location','east')
legend show
```

6.7 ODE_interconnected_new.m

```
function dwdt = ODE_interconnected_new(t,w,N, ploeg_location, N_ploeg,h)
    aref = 0;
    if t>=10 && t <=15
        aref = 0.05;
    end

    %%% BANDO %%%
    N_bando = N - 1 - N_ploeg;
    xref = atanh(1.5-tanh(2))+2;
    a  = 1;
    b  = 1-tanh(xref-2)^2;
```

```
    Ab = [0 -1; a*b -a];
    Bb = [1 0 ; 0 0];
    Cb = [0 1; a*b -a];

    %%% PLOEG %%% constants from ploeg paper
    h     = h;
    tau   = 0.1;
    kp    = 0.2;
    kd    = 0.7;
    alpha = -kd - kp*h;
    beta  = -1/h -kd*h;
    Ap    = [0 (-1) 0; 0 0 1; kp alpha beta];
    Bp    = [1 0; 0 0; kd 1/h];
    Cp    = [0 1 0; 0 0 1];

    dwdt = zeros(2+N_bando+3*N_ploeg,1);

    %%% lead vehicle  i=0 in paper    %%%
    dwdt(1) = w(2) ;
    dwdt(2) = (-1/tau)*w(2) + (1/tau)*aref;

    %%% vehicle 1 to N_ploeg + N_bando   %%%
    j = 3; %% j = 3
    for i = 2:N
        previous_vehicle_type = ploeg_location(i-1);
        current_vehicle_type  = ploeg_location(i)  ;

        if previous_vehicle_type == 0
            Cw = Cb*[w(j-2); w(j-1)];
        else
            Cw = Cp*[0 ; w(j-2); w(j-1)];
        end

        if current_vehicle_type == 0
            vehicle_i = Ab*[w(j) ; w(j+1)] + Bb*Cw;
            dwdt(j)   = vehicle_i(1);
            dwdt(j+1) = vehicle_i(2);
            j = j + 2;
        else
            vehicle_i = Ap*[w(j) ; w(j+1); w(j+2)] + ...
                        Bp*Cw;
            dwdt(j)   = vehicle_i(1);
            dwdt(j+1) = vehicle_i(2);
            dwdt(j+2) = vehicle_i(3);
            j = j + 3;
        end
    end
end
```

6.8 runscript_interconnected_new

```
%%% this is the interconnected model with autonomous first vehicle
clear all
close all

h = 2; %ploeg spacing

% number of vehicles of each type
N_ploeg = 4;
N_bando = 16;
N       = N_ploeg+ N_bando +1 ;

% location of the autonomous vehicles, 1 = present
ploeg_location    = zeros(N,1);

% This must always be one since "lead" vehicle from ploeg model
% This is important for the dynamics of the second vehicle in
% ODE_interconnected_new since that depends on what vehicle i-1 is.
ploeg_location(1) = 1;

% Now, free to decide from 2 ... N if autonomous or not
% Consider N_ploeg
 ploeg_location([2 7 12 17]) = 1;

a = 1;                 % driver sensitivity
tgrid = [1 : 1 : 150]; % check time when perturbation entered
initialCond=zeros(2 + (N_bando)*2 + N_ploeg*3 , 1);
% note all initialCondition is zero bc this is dynamics of the
% deviation from equilibrium.

options= odeset('AbsTol',1e-9,'RelTol',1e-9);
[t, w] = ode45(@(t,w) ODE_interconnected_new(t,w,N,ploeg_location, N_ploeg,h)...
                , tgrid,initialCond, options);

%%% PLOT %%%
txt = ['Control, i = ',num2str(0)];
plot(tgrid, 1.5+w(:,1), 'DisplayName', txt )
j = 3; %position of Delta_x_i in state vector
for i = 2:N
   hold on

   if ploeg_location(i) == 0
      txt = ['Human, i = ',num2str(i-1)];
   else
      txt = ['Auto, i = ',num2str(i-1)];
   end
   plot(tgrid, 1.5 +  w(:,j+1), 'DisplayName', txt )

   %update j to position of Delta_x_{i+1} in state vector
   if ploeg_location(i) == 0
       j = j + 2;
```

```
    else
        j = j + 3;
    end
end
pbaspect([1 1 1])
axis([0 100 1.5 1.8])
title(['Velocity vs Time, Interconnected Dynamics (h=',num2str(h),')'])
xlabel('Time')
ylabel('Velocity')
legend('Location','east')
%legend show
```

Preliminary references

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