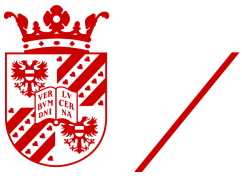
Bachelor's thesis for mathematics and physics



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Mathematical aspects of the Feynman path integral

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Abstract

Path integration methods are of crucial importance to quantum mechanics and quantum field theory. There are multiple ways the path integral can be constructed, one method uses the link between the Fokker-Planck and the Langevin equations, as is covered in this thesis. This is related to the standard derivation of the path integral in physics, in which the time is discretised. We study the mathematical problems related to the Feynman path integral, in particular the impossibility of a Lebesgue-type measure on the space of paths. It is discussed how oscillatory integrals can be used to have a well-defined 'integral' on an infinite dimensional space. This formalism is subsequently applied to both the harmonic and the anharmonic oscillator. We prove an infinite dimensional oscillatory integral exists and obtain a convergent expression for both these cases, under conditions. Examples of these conditions are the initial wavefunctions being Schwartz functions as well as conditions on the endtime, angular frequency and coupling constant. Finally, a possible approach to establish an oscillatory integral for the hydrogen atom is discussed. It is proven that the result is no longer independent of the sequence of projection operators, which is a key step towards a rigorous path integral for this system.

Contents

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1 Mathematical introduction

This bachelor's thesis is about the attempts to put the Feynman path integral on a firm mathematically rigorous footing. The Feynman path integral is a (heuristically defined) integral central to quantum field theory: to calculate a path integral one integrates over the 'space of all paths' between an initial and final state (or initial/final condition, if you will). What you obtain is called the transition density or probability amplitude, i.e. the probability that a system propagates from the given initial state to the given final state.

It is the integration domain (the space of all paths) and its associated 'measure' from which mathematical problems arise. It can be proven that a 'nice' Lebesgue-type measure does not exist on an infinite dimensional Hilbert or Banach space. The space of paths is infinite dimensional. Note that it can be made into a Banach space by using the supremum norm or into a Hilbert space (called $\mathcal{H}_T$) with the inner product

$$\langle \gamma_1, \gamma_2 \rangle = \int_0^T \dot{\gamma}_1(s) \cdot \dot{\gamma}_2(s) ds. \tag{1}$$

If one then imposes the condition that $\gamma \in \mathcal{H}_T$ if $\dot{\gamma} \in L^2([0,T])$, one obtains a Hilbert space. This space is called the Cameron-Martin space. The key theorem from which all the problems arise is the following, which is theorem 7.2.

Theorem 1.1 (Impossibility of a Lebesgue-type measure on an infinite dimensional space)**.** Let $\mathcal{H}$ be a Hilbert space. There does not exist a σ-additive measure that is both translation and rotation invariant and that assigns a finite measure to bounded open sets.

In short, this theorem ensures that one cannot construct a regular Lebesgue integral which is the mathematical representation of the Feynman path integral in physics. There are multiple ways to circumvent this issue and all of them have their advantages and drawbacks. However, the problem is that none of them works for all the cases where the path integral is used in physics. It is the purpose of this thesis to review in detail one attempt to put the path integral on a solid mathematical footing. It made sense to just consider one approach due to the vast amount of literature on (mathematical) path integrals. The conference proceedings [84] are a clear example of all the research that is being done on this fascinating topic.

Also, the aim is to review the relationship between the Feynman path integral and the Fokker-Planck equation, which is a particular class of partial differential equations (PDEs). The Fokker-Planck equation is the following PDE (where a and b are called the coefficient functions)

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}(a(x,t)p(x,t)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(b(x,t)p(x,t)) \tag{2}$$

that describes the behaviour of the so-called transition density $p(x,t)$ (which is the probability that a random variable will have value x at time t given some initial value y at time

s). For a particular choice of the coefficient functions a and b the Fokker-Planck equation can be related to a specific stochastic differential equation (SDE). Such an equation contains derivatives such as a standard PDE would, but in addition includes a noise term, which often is the Brownian motion.

The transition density also appears in quantum mechanics, where it is known as the propagator (or the probability amplitude). Every particular path integral (in particular every action) has an associated Fokker-Planck equation, which will be shown in chapter 3. This is because the path integral is a representation of the transition density for a Brownian stochastic process. The Fokker-Planck equation (which we will rewrite as the Schrödinger equation) is the PDE that the transition density (and hence the path integral) satisfies.

We will study how the path integral arises from this link, where no physics is needed to derive the path integral from the Fokker-Planck equation, except for the fact that the Schrödinger equation is the equation of motion for quantum mechanics. In this manner it becomes clear that the path integral can be viewed as a stochastic concept. Therefore many mathematical approaches towards constructing a rigorous path integral, such as [142], are justified in using stochastic methods. I would like to stress that the assumption of the noise being Brownian is crucial, the whole theory of (standard) SDEs is constructed on this basis. It is possible to extend it to different stochastic processes, although we will not do so here.

So this association between the two concepts is the first thing we will study. Subsequently, the path integral in the context of non-relativistic quantum mechanics will be introduced. It will also be shown how to solve actual problems in quantum mechanics using the path integral (the harmonic oscillator and the anharmonic oscillator). Then we will explain how the path integral formalism carriers over to quantum field theory, where special attention is given to the anharmonic (interaction) term in the Lagrangian. After that, we will examine the attempted techniques for giving a mathematical definition of the path integral. The focus will mainly be on the technique that uses oscillatory integrals, this idea originated from (linear) PDE theory, such as in [73].

In a nutshell the idea is to construct an integral in a finite dimensional Hilbert space using a so-called 'test function' ϕ, that decays fast enough such that an integral with a highly oscillating integrand still converges and can be interpreted in the Lebesgue sense (which is still possible since we are working in a finite dimensional space). These integrals generally have the form

$$\int_{\mathcal{H}} e^{\frac{i}{\hbar}\phi(x)} f(x) k(\epsilon x) dx,$$

where $\epsilon > 0$ is a parameter that will go to zero. The function ϕ is called the phase function, it will become (part of) the action in our physical interpretation. The function f is the function that is being integrated and k is the test function (that has assumed suitable decay behaviour).

Subsequently one can construct an integral on an infinite dimensional Hilbert space by using a sequence of projection operators that converges to the identity operator. That way

one obtains a sequence of (finite) oscillatory integrals each of which converges, under certain specified conditions. This approach can subsequently be used to define a mathematically rigorous path integral for the Schrödinger equation under suitable conditions for specific physical systems, successful cases being quadratic or quartic potentials for example.

Establishing a mathematically rigorous path integral for an actual physical system proceeds by establishing a Parseval-type equality. Such an equation relates the oscillatory integral on an infinite dimensional Hilbert space to a Lebesgue integral. Note that the measure in this Lebesgue integral does not satisfy all the properties in theorem 1.1 and hence is perfectly valid. This equality is derived using techniques from Fourier analysis.

By establishing functional analytic properties (such as being trace class and self-adjoint) of the operators of interest in the harmonic oscillator and anharmonic oscillator problems, we are able to obtain a formula that can be proven to solve the Schrödinger equation in a certain weak sense. The preliminaries required to understand the chapters on the oscillatory integral formalism, which include Fourier analysis, complex measure theory and functional analysis, can be found in an appendix. The scope of this appendix is rather limited, it covers only the material that is not in the undergraduate curriculum and only in a very condensed form. If the reader does not find my treatment to be enough, they are advised to consult one of the several good books that cover the area of analysis, such as the set [133–137] or [60–62]. Note that this thesis is by far not a comprehensive review of all the work that has been done on the oscillatory integral formalism itself, let alone on all the mathematical work done on the path integral.

Finally, an attempt to construct an oscillatory integral for a potential with singularities is made in this thesis. It is a big problem of the oscillatory integral formalism that there is no method to handle singularities [85, p. 614]. Since one integrates over all paths between endpoints, there will always be a path that crosses the singularity, which will make the oscillatory integral diverge. Using the Levi-Civita transformation, which is also used in celestial mechanics [1], the path integral of the hydrogen atom can be related to the one of the harmonic oscillator. This makes it possible to derive the wavefunctions and the energies of the two-dimensional hydrogen atom.

This thesis is mostly a review of existing work, although it has never been brought together in this form with both the Fokker-Planck aspect (which is from a stochastic viewpoint) as well the analytical aspect (using oscillatory integrals) of the path integral together, with a focus on the harmonic and anharmonic oscillators. A couple of things are my own work. An open problem regarding the oscillatory integral formalism is to provide a classification of functions which have an oscillatory integral. This problem is even open in finite dimension. I have made some progress myself on solving this problem, the results are covered in section 7.5.

In short, I have proven that the Fresnel space (i.e. those functions which have an oscillatory integral with quadratic phase function) includes all C^k functions such that the k-th derivative is bounded by a polynomial (i.e. belongs to the space of symbols). It also includes

all L^1 functions (therefore continuous functions with compact support have an oscillatory integral too), as well as continuous periodic functions. Most of these results were established using an independent, perhaps novel, proof. Many known results try to prove that some important class of functions (such as the Fresnel algebra) has an oscillatory integral. I have tried a different tactic, namely trying to prove properties of the space of all functions that have an oscillatory integral.

Something that follows immediately is that the set of all 'oscillatory integrable' functions is a vector space and one can also construct a seminorm on it (which is given by the absolute value of the oscillatory integral). There is a possibility that the space is a Banach space, although that is not clear yet. The classes of functions that were proven to have an oscillatory integral that were mentioned in the last paragraph were arrived at using this structural approach. The vector space structure also allows one to conclude that if a function can be decomposed in a periodic part, a rapidly growing function but with compact support (which should be L^1 but not necessarily bounded) and also a part that has (at most) polynomial growth, then the function has an oscillatory integral. There are further possibilities to extend some of these results. What should be looked at if the Fresnel space is an algebra and whether it is a Hilbert space. It seems unlikely that this will result in a complete and total classification, but it might bring further results.

Something else that is my own result is a new calculation of the index of the operator $I - L$ (which is given below and is of interest to the harmonic oscillator), given in lemma 8.5. What is also new is a calculation of the resolvent $(I - L)^{-1}$ of the following operator L

$$(L\gamma)(s) \coloneqq \int_s^T ds' \int_0^{s'} (\Omega^2\gamma)(s'')ds''.$$

These results were already in the literature, but were unfortunately incorrect. The calculation of $(I - L)^{-1}$ is necessary to obtain a rigorous expression in terms of a Lebesgue integral for the path integral of the harmonic oscillator.The new result and proof can be found in lemma 8.2. The calculation of the path integral for the harmonic oscillator (which agrees with the physical result derived in section 4.2.1) using this new operator $(I - L)^{-1}$ is done in this thesis and is given in appendix B.3.

Finally, all of chapter 9 is new work. As has been mentioned before, the construction of an oscillatory integral for a potential with a singularity remains a tough problem. Chapter 9 could be the first step on the way to a solution. In particular, the solution presented drops the condition of the independence of the result on the sequence of projection operators. To that end, a new integral is defined, known as a Duru-Kleinert integral. Such an integral has been constructed with success for the two-dimensional hydrogen atom.

There still could be major and minor issues with this construction, since some of the steps need to be formalised more than they are at present. What is interesting about this idea is the possibility of generalisation. This method, which uses a pseudo path integral

to remove singularities, could also work for different potentials with singularities. Further research will be necessary to explore the limits of this idea.

The reader is assumed to have a basic working knowledge of PDEs, probability theory, measure theory, Fourier analysis and functional analysis for the mathematical part of this thesis.

2 Physical introduction

A lot of attention in physical research is spent on trying to unify quantum field theory and general relativity. There have been great successes in making general relativity mathematically rigorous, such as in [72]. There has also been a lot of progress in making non-relativistic quantum mechanics rigorous in the Schrödinger and Heisenberg pictures, such as in [35, 69, 70].

There exists a third formulation of quantum mechanics, namely the path integral formulation. It came into being later than the other two approaches, in the 1940s. It was introduced by Richard Feynman in his thesis 'The principle of least action in quantum mechanics', which was published in 1942 [21], where it followed up on an article by Paul Dirac published in 1932 [36]. Feynman later extended the results of his thesis to quantum electrodynamics, in an article that appeared in 1948 [51] In his thesis, Feynman introduced the concept of the path integral. The path integral is given by

$$Z = \int_{q(0)=q_i, q(T)=q_f} Dq e^{\frac{i}{\hbar}S(q)}, \tag{3}$$

where q is a path with endpoints q_i and q_f and $S(q)$ is the action of the system. In the case of non-relativistic quantum mechanics, it will be the time integral of the Lagrangian. The differential Dq means that one integrates over all paths $q : [0, T] \to \mathbb{R}^3$ such that $q(0) = q_i$ and $q(T) = q_f$.

The link with the Schrödinger formalism is provided in terms of the following equation

$$\psi(y, t_2) = \int_{\mathbb{R}^3} K(t_1, t_2, x, y)\psi(x, t_1)dx, \tag{4}$$

where $K(t_1, t_2, x, y)$ signifies the path integral with as domain the space of all paths with endpoints x and y and initial t_1 and final time t_2. If the path integral of a system has been calculated, one can obtain the evolution of the wavefunction and as a result all the quantities of interest.

From equation (4) it is clear that the path integral is equal to the probability ampltitude between two states of a system. It is the probability that a system starts in an initial state and after a time $(t_2 - t_1)$ has elapsed, it will be in the final state. Note that this has a probabilistic/stochastic interpretation in terms of a transition density, which is discussed in the mathematical introduction to this thesis and also in chapters 3 and 4.

When one calculates a path integral, an integration over the space of all paths is performed. The concept turned out to be very useful, not just in physics, but also in the study of financial markets [88, 154] and even for climate dynamics [114]. It turned out that there were two ways to build up quantum field theory, one was by using canonical quantisation, the second by using the path integral [147]. The advantage of the path integral is that the transition from quantum mechanics to quantum field theory is smooth. Once one has set up

a path integral for the case of non-relativistic quantum mechanics, one can move to quantum field theory by replacing the action by a new action containing fields which is an integral of the Lagrangian density over Minkowski space.

The path integral is useful because of the simplicity of the construction, everything can be calculated from the path integral, whether it is the wavefunction, transition densities or observables (such as decay rates). The problem with the path integral is that throughout quantum field theory one will obtain many infinities as answers to calculations, which are subtracted off in most cases. That is an unsatisfactory resolution to these issues, although quantum field theory has been very successful in giving accurate predictions. Some of these issues can be resolved using effective field theory, but that does not handle all the problems with QFT. Some of these remaining issues come down to the construction of the path integral, although it must be said that the infinities also appear in the canonical quantisation formalism [147].

In order to take on this issue of a lack of mathematical rigour, a possible way to do so is to try and construct a rigorous path integral. The way to start doing so is by first studying the path integral in non-relativistic quantum mechanics, since many of the issues already appear there (as explained in the mathematical introduction). It should be remarked that Richard Feynman himself was aware of the issues with the path integral, as he once famously remarked "one must feel as Cavalieri must have felt before the invention of the calculus" [7].

Feynman calculated many path integrals by using time discretisation methods, where one uses a partition of the integration domain to calculate a finite dimensional integral and then takes the limit. This is called time slicing in both the physical and mathematical literature. Many path integrals have been calculated exactly in non-relativistic quantum mechanics by using this technique, for example for the free particle, the harmonic oscillator and the hydrogen atom [68, 88].

Of the three systems mentioned, the hydrogen atom is by far the most difficult one. This is caused by the impossibility of calculating the path integral in the standard manner for a Coulomb potential. The solution that was first published in [41, 42] instead considered an object known as the pseudopropagator. This pseudopropagator can be written as a path integral with a Hamiltonian that has a functional degree of freedom compared to the Coulomb Hamiltonian. By setting these so-called regulating functions to 1 one retrieves the original path integral.

However, if one makes a different choice for the regulating functions, the singularity $1/r$ is removed. After a Levi-Civita coordinate transformation the pseudo path integral assumes the form of the path integral for the harmonic oscillator. The latter can be solved exactly and therefore one obtains an expression for the pseudo path integral of the hydrogen atom. From this new expression one can obtain both the energies and the wavefunctions. The path integral for the hydrogen atom is covered in chapter 5.

In this thesis, I will study the Feynman path integral in the setting of non-relativistic quantum mechanics. The goal is to try to understand why the path integral has mathematical

difficulties and also why it is so useful for quantum mechanics (and quantum field theory). The reason for the focus on non-relativistic quantum mechanics is that the problem is less involved in that case. An additional advantage is that we have the opportunity to compare results of the mathematical formalism with physical calculations (which are often exact). In quantum field theory there are almost no exact calculations of path integrals, which makes it harder to compare them.

This thesis will not address many of the additional mathematical issues that quantum field theory faces, I will not attempt to provide a rigorous foundation for renormalisation nor try to say anything about Yang-Mills theory (and the assocatied Milennium prize problem). The mathematical focus lies on non-relativistic quantum mechanics, while I will discuss a bit of quantum field theory from a physical viewpoint. In general the philosophy will be that if one has certain problems with the path integral in the non-relativistic situation, then one has at least as many problems in the QFT setting.

The reader is assumed to be familiar with the standard undergraduate curriculum of physics, that includes non-relativistic quantum mechanics and classical mechanics, including an advanced course which treats Hamiltonian and Lagrangian mechanics. Some familiarity with the path integral and quantum field theory would definitely help.

3 From the Fokker-Planck equation to the path integral

In this chapter, it will be discussed how the Fokker-Planck equation and the path integral are related and why the path integral can be viewed as a stochastic concept. In order to do so, the reader has to understand the link between the Fokker-Planck equation and the theory of stochastic differential equations. In short, the transition probability density of the SDE solution satisfies the Fokker-Planck equation. Moreover, the coefficient functions of the Fokker-Planck equation can be directly related to the coefficient functions of the SDE.

This link can be used to relate the Langevin equation, an important SDE used in physics, to the Schrödinger equation, which is equal to the Fokker-Planck equation for a suitable choice of the coefficient functions. The transition density can then be written as a path integral. First, we will cover the required preliminaries in order to understand this connection, which includes measure-theoretic probability, Brownian motion, stochastic integrals and stochastic differential equations. My treatment is based on the lecture notes [96–103], the same contents can be found in [104].

3.1 Revision of probability theory

In this short section I will give a couple of definitions to ensure that it is completely clear what all the notation means and which conventions will be used. This section will not be a review of measure theoretic probability, simply because it is not necessary for the intended purposes. This section is based on [15, 24, 30].

Definition 3.1. A probability space (X, A, μ) is a set X with an associated σ-algebra A and measure μ such that $\mu(X) = 1$. The probability of an event B is given by $\mu(B)$ (where $B \in A$) [94].

Definition 3.2. Let (X, A, μ) be a probability space and $B_1, B_2 \in A$ be two events. Then the events B_1 and B_2 are called independent if

$$\mu(B_1 \cap B_2) = \mu(B_1) \cdot \mu(B_2).$$

This definition can easily be extended for n events $B_1, \ldots, B_n$ into

$$\mu(B_1 \cap \ldots \cap B_n) = \mu(B_1) \cdot \ldots \cdot \mu(B_n).$$

Definition 3.3. A map f from X to $\mathbb{R}^m$ is called a random variable, if it is A-measurable, where A is still the σ-algebra. We will take $\Omega = \mathbb{R}^n$, which will be done throughout the thesis [139]. Two random variables B and D are called independent if for all Borel sets U and V in $\mathbb{R}^n$ we have that $B^{-1}(U)$ and $D^{-1}(V)$ are independent events.

Now we can use all these concepts to introduce expectation values:

Definition 3.4. Let (X, A, P) be a probability space where P is a probability measure, Let $f(B)$ be a measurable function given in terms of a random variable B. Hence $f(B)$ maps X to $\mathbb{R}^n$. The expectation value is given by

$$\mathbb{E}[f(B)] = \int_X f(B)(x) dP(x).$$

Definition 3.5. Let (X, A, μ) be a measure space, where the measure μ is the reference measure (not necessarily a probability measure). In the relevant case here, it is the Lebesgue measure. If f is a measurable function it is called a probability density function if

$$1 = \int_X f d\mu.$$

Subsequently, the probability that an event (subset of X) 'happens' is given by

$$P(x \in B) = \int_B f d\mu.$$

Note that B must be a measurable set contained in the σ-algebra A. In the case of $\mathbb{R}^n$, we can construct the Lebesgue measure [34] and we can write the following for the probability [159]

$$P(x \in B) = \int_B f dx.$$

Finally, we introduce the Gaussian measure:

Definition 3.6. If dx is once again the standard Lebesgue measure on $\mathbb{R}^n$, then the Gaussian measure γ on $\mathbb{R}^n$ is then given by (where $B \in A$)

$$\gamma(B) \coloneqq \frac{1}{\sqrt{2\pi}^n} \int_A e^{-\frac{1}{2}\|x\|^2} dx, \tag{5}$$

where the norm is the standard norm on $\mathbb{R}^n$. This equation is the definition for the standard normal distribution. For other cases, the Gaussian measure is defined as follows

$$\gamma(B) \coloneqq \frac{1}{\sqrt{2\pi\sigma^2}^n} \int_A e^{-\frac{1}{2\sigma^2}\|x-\mu\|^2} dx, \tag{6}$$

where σ is the standard deviation and μ the average. Note that it is possible to extend the Gaussian measure to an infinite dimensional Hilbert space (unlike the Lebesgue measure), but we will not do so here [45].

We can then introduce the covariance operator:

Definition 3.7. Let P be a probability measure on a Hilbert space $\mathcal{H}$, the covariance of elements x and y is given by

$$\mathrm{Cov}(y, z) = \int_{\mathcal{H}} \langle y, x\rangle\langle z, x\rangle dP(x),$$

by the Riesz representation theorem (or the Riesz-Fréchet theorem) there exists an operator C, known as the covariance operator, such that [78]

$$\mathrm{Cov}(y, z) = \langle Cy, z\rangle. \tag{7}$$

3.2 Brownian motion

The key to introducing randomness or noise into a differential equation is a concept called Brownian motion or Wiener Process. Brownian motion is a special type of stochastic process that is also Markovian. It is defined as follows:

Definition 3.8. A Wiener process or Brownian motion W is a set of random variables indexed by a time parameter $t \geq 0$. That means that for any value $c \geq 0$, $W(c)$ is a random variable which satisfies the following properties:

1. The random process $W(t)$ is continuous in t. In particular that means that when it is visualised as a path, there are no gaps in the path.
2. At $t = 0$ we have $W(0) = 0$ with probability 1.
3. The increment of the Brownian motion $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$, for $0 \leq s \leq t$.
4. The Brownian increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent random variable for $0 \leq s \leq t \leq u \leq v$.

The Brownian motion can be visualised as being very herky-jerky, where particles are moving randomly about. We will first state some properties of the Brownian motion.

Lemma 3.9. Any Brownian motion $W(t)$ has the following moments:

1. $\mathbb{E}(W(t)) = 0$.
2. $\mathbb{E}\left((W(t))^2\right) = t$.

Proof. We first prove that the expectation value is zero. By two properties of Brownian motion we know that $W(t) = W(t) - W(0) \sim \mathcal{N}(0, t)$, which denotes a normal distribution with mean 0 and variance t. This immediately implies that $\mathrm{E}(W(t)) = 0$. To prove the second property we recall that

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2.$$

Since we just found that $\mathrm{E}(X) = 0$, we hence know that

$$t = \mathrm{Var}(W(t)) = \mathbb{E}((W(t))^2).$$

□

We now state one lemma that will be useful later (to prove a formula for the stochastic integral).

Lemma 3.10 (Quadratic Variation)**.** If for each $N \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots < t_N = T$ is a partition of $[0, T]$ and also $\Delta t = \max_{1 \le i \le N} |t_i - t_{i-1}| \to 0$ as $N \to \infty$, then we have the limit

$$\lim_{N \to \infty} \sum_{i=1}^{N} (W(t_i) - W(t_{i-1}))^2 = T, \tag{8}$$

converges in the mean square sense, i.e. $\lim_{N\to\infty} \mathbb{E}[\sum_{i=1}^{N} (W(t_i) - W(t_{i-1}))^2] = T$.

Proof. The proof can be found in [98]. □

The Brownian motion in $\mathbb{R}$ can be easily extended to $\mathbb{R}^n$ through the following statement:

Proposition 3.11. A process $\mathbf{W}(t)$ defined by

$$\mathbf{W}(t) = \begin{pmatrix} W_1(t) \\ \vdots \\ W_n(t) \end{pmatrix},$$

is a Brownian motion if and only if all the components $W_1(t), \ldots, W_n(t)$ are each a Brownian motion in $\mathbb{R}$.

Proof. The proof is rather trivial, if one imposes each of the conditions of definition 3.8 on $\mathbf{W}$ then it automatically follows that $W_1, \ldots, W_n$ satisfy that condition and vice versa. □

The Brownian motion will be the source of noise in the stochastic differential equations, it has been proven that Brownian motion is not differentiable, a proof can be found in [3, 43, 86, 112]. However, it is possible to define a stochastic integral (there a multiple ways to do so), which will be the topic of the next section. Using these stochastic integrals we will define stochastic differential equations.

3.3 Stochastic integrals

In this section we will define how one integrates over a Brownian path (i.e. $dW(t)$). The reader may recall that the Riemann integral was defined by considering Riemann sums of the form

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(a_i)(t_i - t_{i-1}),$$

where $a = t_0 < t_1 < \ldots < t_n = b$ is a partition of the interval $[a, b]$ and $a_i \in [t_{i-1}, t_i]$.We arrive at the Riemann integral by taking the supremum/infimum of the upper/lower Riemann sums (under the condition that they are equal of course) [2, 60]. We now seek to extend this definition to obtain a stochastic integral. The idea is to approximate the integral

$$I = \int_0^T W(s)dW(s),$$

similarly by taking the limit of the partial sums

$$S_N := \sum_{i=1}^{N} W(a_i)(W(t_i) - W(t_{i-1})).$$

The set $\{t_0, t_1, \ldots, t_N\}$ is a partition of $[0, T]$ and the coefficients $a_i = (1 - \alpha)t_i + \alpha t_{i-1}$, so the number α determines how we approximate the function, which depends on which representative value we take (so where on the interval). Note that we have lost the freedom of taking the a_i to be arbitrary. The following lemma will show that the choice of α determines the result of the limit of the partial sums.

Lemma 3.12. If we keep the notation from this section and we ensure that the sequence of partitions will have smaller and smaller subintervals, such that $\max_{1 \le i \le N} |t_i - t_{i-1}| \to 0$ as $N \to \infty$, then in the previously used definition of the integral the limit converges to

$$I = \lim_{N \to \infty} S_n = \frac{W(T)^2}{2} + \left(\alpha - \frac{1}{2}\right) T. \tag{9}$$

Proof. We can rewrite the partial sum S_N as

$$\begin{aligned}
S_N &= \sum_{i=1}^{N} W(a_i)(W(t_i) - W(t_{i-1})) \\
&= \frac{1}{2}\sum_{i=1}^{N}[W(t_i)^2 - W(t_{i-1})^2] - \frac{1}{2}\sum_{i=1}^{N}[W(t_i)^2 - W(t_{i-1})^2] + \sum_{i=1}^{N} W(a_i)(W(t_i) - W(t_{i-1})) \\
&= A - \frac{1}{2}\sum_{i=1}^{N}[W(t_i) - W(t_{i-1})]^2 + \sum_{i=1}^{N}[W(t_{i-1})^2 - W(t_i)W(t_{i-1})] + \sum_{i=1}^{N} W(a_i)(W(t_i) - W(t_{i-1}))
\end{aligned}$$

$$
\begin{aligned}
&= A + B + \sum_{i=1}^{N} [W(t_{i-1})^2 + W(a_i)^2 - 2W(t_{i-1})W(a_i)] \\
&+ \sum_{i=1}^{N} [-W(t_i)W(t_{i-1}) - W(a_i)^2 + W(a_i)(W(t_i) + W(t_{i-1}))] \\
&= A + B + \sum_{i=1}^{N} [W(a_i) - W(t_{i-1})]^2 + \sum_{i=1}^{N} (W(a_i) - W(t_{i-1}))(W(t_i) - W(a_i)) \\
&= A + B + C + D.
\end{aligned}
$$

Throughout the derivation we have defined that

$$A = \frac{1}{2} \sum_{i=1}^{N} [W(t_i)^2 - W(t_{i-1})^2], \tag{10}$$

$$B = -\frac{1}{2} \sum_{i=1}^{N} [W(t_i) - W(t_{i-1})]^2, \tag{11}$$

$$C = \sum_{i=1}^{N} [W(a_i) - W(t_{i-1})]^2, \tag{12}$$

$$D = \sum_{i=1}^{N} (W(a_i) - W(t_{i-1}))(W(t_i) - W(a)). \tag{13}$$

We first observe that A is in fact a telescoping sum and since $W(0) = 0$ we know that $A = \frac{1}{2}(W(T))^2$. By lemma 3.10 (quadratic variation) we know that B converges to $-\frac{T}{2}$. Now we set $a_i = (1-\alpha)t_{i-1} + \alpha t_i$, C then becomes

$$C = \sum_{i=1}^{n} [W((1-\alpha)t_{i-1} + \alpha t_i) - W(t_{i-1})]^2.$$

It follows that C converges to αT by a lemma similar to quadratic variation [50]. Finally D converges to zero because the increments are independent, which follows from the definition of Brownian motion, which completes the proof. □

We see that the choice of α is of crucial importance. There are two important choices:

- If we choose $\alpha = \frac{1}{2}$ we obtain the Stratonovich integral:

$$\int_0^T W(s) \circ dW(s) = \frac{1}{2}(W(T))^2. \tag{14}$$

The notation $\circ$ is a common notation for the Stratonovich integral. The Stratonovich calculus has the advantage that it has nice analytical properties in parallel with standard calculus. It is often used in physics [50].

- If we choose $\alpha = 0$ we obtain the Itô integral:

$$\int_0^T W(s)dW(s) = \frac{1}{2}(W(T))^2 - \frac{T}{2}. \tag{15}$$

The Itô calculus is used in finance applications since it does not depend on information in the future, so it can be easily used when one is calculating numerical approximations. It is also convenient because of the so-called Itô isometry, which will be covered below.

Both approaches have their respective advantages. I will only use the Itô integral, but the results can of course be converted to the Stratonovich integral (how to do so can be seen in equation (19)). It can always be seen from the notation which of the two integrals is being used at any given moment [90, 151]. Now we state the Itô isometry:

Lemma 3.13. If $X(s)$ is a stochastic process with certain regularity properties and be predictable, which means that

$$\left(\mathbb{E}\left[\int_0^T |X(s)|^2 ds\right]\right)^{1/2} < \infty.$$

Then we have the Itô isometry

$$\mathbb{E}\left[\left|\int_0^t X(s)dW(s)\right|^2\right] = \int_0^t \mathbb{E}[|X(s)|^2]ds, \quad t \in [0,T]. \tag{16}$$

Proof. The proof can be found in [119]. □

This lemma is very useful in practice because it allows us to transfer the expectation value from inside the integral to outside of it and vice versa. Now we are able to introduce the notion of a stochastic differential equation.

3.4 Stochastic differential equations

Definition 3.14. A stochastic differential equation (SDE), with a particular interpretation of the stochastic integral (be it Itô or Stratonovich, or something else) is often written down as

$$du = f(u)dt + g(u)dW(s). \tag{17}$$

This differential notation is just a useful shorthand for the integral notation of an SDE, which is (note that one needs to specify an initial condition $u(0)$)

$$u(t) = u(0) + \int_0^t f(u(s))ds + \int_0^t g(u(s))dW(s). \tag{18}$$

We remind the reader that the first integral is a deterministic integral and the second one is a stochastic integral.

The function $f(u)$ is called the drift term while the function $g(u)$ is called the diffusion term. When g is just a constant (i.e. independent of u), we say that the SDE has additive noise, otherwise it has multiplicative noise. We can now add a final remark on the discussion of the differences between the Itô and Stratonovich integrals. If the function $g(u)$ that is being integrated is differentiable and $u(t)$ is the solution of the Itô SDE as defined above, we have that

$$\int_0^T g(u(s)) \circ dW(s) = \int_0^T g(u(s))dW(s) + \frac{1}{2}\int_0^T g'(u(s))g(u(s))ds. \tag{19}$$

We once again stress that in order for this to be true, g must be differentiable with respect to u. If that is not the case there might be significant differences between Itô and Stratonovich SDEs.

Specifying that f and g are Lipschitz makes it possible to derive an existence and uniqueness theorem similar to deterministic ODEs [155]. Note however, that many of the SDEs that are investigated in applications such as physics and finance do not satisfy an uniqueness and existence theorem. We will now define what will be the most important SDE in the thesis, namely the Langevin equation, which is used in physics.

Example 3.15 (Langevin equation)**.** Let q denote the position of the particle, while p is its momentum (as in classical mechanics [14]). If we have a potential $V(q)$ that depends solely on the position, so the force resulting from this potential is conservative, the dynamics of the system is described by the Langevin equation (it is often written this way in physics [109])

$$\dot{p} = -\lambda p - V'(q) + \xi(t), \tag{20}$$

as a function of x however, in this form the Langevin equation is a second-order equation. Note that now we will write W instead of ξ to avoid confusion but we will replace ξ with W later. In order to write the equation as a first-order system and interpret the equation as an SDE [50, 148], we write

$$dq = pdt \tag{21}$$
$$dp = -\lambda p dt - V'(q)dt + \sigma dW(t), \tag{22}$$

where λ and σ are fixed parameters. Details about this transition from the 'ODE form' to the SDE form can be found in [29, Chapter 2] and [59, Section 4.1], I will give a short outline. In physics one imposes the following conditions on ξ

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = \delta(t - t'). \tag{23}$$

Observe that this last condition often does not hold for $t - t'$ being small in physical applications, but we use it because it simplifies the models. Moreover, it implies that the variance is infinite, which never occurs in real-life situations. Accepting this an idealisation is possible. There is even a way to circumvent this issue by defining

$$\langle \xi(t)\xi(t') \rangle = \frac{\gamma}{2} e^{-\gamma|t-t'|}. \tag{24}$$

This equation has a finite variance as a result and the limit $\gamma \to \infty$ gives a delta function. So one can first calculate the results one needs and then take the required limit, but in practical calculations this method is rather cumbersome. There is the added complication that the physical approach to the Langevin equation (as in [95]) is not generalisable to the nonlinear case [130]. More information on the stochastics of nonlinear systems can be found in [150].

There is one another big issue however, if one insists that the solution p is continuous then it follows that one is allowed to define

$$W(t) = \int_0^t \xi(t')dt',$$

where we know from the properties of the system that ξ is a continuous function. Through a derivation detailed in [59, Section 4.1] we know that W is a Wiener process. The fundamental theorem of calculus tells us that W is differentiable, which is a contradiction. By assumption, any Wiener process is continuous, but it has been proven that any such process is nowhere differentiable [3, 43, 86, 112]. Therefore we conclude that equation (20) is ill-defined. It has the merit of simplicity however and we continue to use it since our intended purposes are in physics. But the reader should know that more rigour is necessary to make what will completely mathematically sound.

3.5 Applications of the Langevin equation

We will later use the Langevin equation to derive the path integral, this section is based on [109]. Because we use the equation for such a crucial purpose, it makes sense to discuss why the equation is important and what it is used for. The Langevin equation is an equation of motion, since it describes the evolution of the momentum p over time. On the right hand side of equation (20) there are three terms. The first term, $-\lambda p$, describes a friction force. Often it is the viscosity of the fluid. The term $V'(q)$ describes a conservative force arising from a potential, while $W(t)$ is of course the noise term that introduces randomness into the

system. I will illustrate two examples where the Langevin equation, one from physics and the other one from outside physics, to illustrate the wide applicability.

Example 3.16 (Overdamped Brownian motion)**.** It might seem quite obvious, but the reason that the Langevin equation was created by Paul Langevin in 1908 was to describe the Brownian motion of particles (molecules) in a fluid [95]. Overdamped in this context means that we assume that the molecules have no net acceleration, which is the case in ordinary suspension. That means that we can ignore the acceleration term $\dot{p}$ in comparison to the friction term $-\lambda p$. Therefore we arrive at the following Langevin equation for such a system

$$p = -V'(x) + W(t). \tag{25}$$

For particular choices of V, one can obtain interesting results. If one choose a harmonic potential, i.e. $V(x) = a\frac{x^2}{2}$, one obtains

$$\dot{x} = -ax + W(t),$$

where I have redefined the factor a in the process. The equation of motion is linear in x and since W follows a Gaussian distribution, so does x. It can be shown that by multiplying by an integrating factor we get

$$\begin{aligned}
\frac{d}{dt}[xe^{at}] &= W(t)e^{at} \\
x(t) &= x_0e^{-at} + e^{-at}\int_0^t W(s)e^{as}ds, \\
\mathbb{E}[x(t)] &= x_0e^{-at} \\
\mathbb{E}[x^2(t)] &= \mathbb{E}\left[x_0^2e^{-2at} + 2x_0e^{-2at}\int_0^t W(s)e^{as}ds + e^{-2at}\int_0^t\int_0^t W(s)W(r)e^{a(s+r)}dsdr\right] \\
&= x_0^2e^{-2at} + e^{-2at}\int_0^t\int_0^t \mathbb{E}\left[W(s)W(r)\right]e^{a(s+r)}dsdr = x_0^2e^{-2at} + 2\widetilde{D}e^{-2at}\int_0^t\int_0^t \delta(s-r)e^{a(s+r)}dsdr \\
&= x_0^2e^{-2at} + 2\widetilde{D}e^{-2at}\int_0^t e^{2ar}dr = x_0^2e^{-2at} + \frac{\widetilde{D}}{a}[1-e^{-2at}].
\end{aligned}$$

Hence, $\mathrm{Var}(x(t)) = \mathbb{E}[x^2(t)] - (\mathbb{E}[x(t)])^2 = \frac{\widetilde{D}}{a}[1-e^{-2at}]$. Using the fact that x has a Gaussian distribution, we obtain the following transition density

$$p(x_0, 0, x, t) = \sqrt{\frac{a}{2\pi\widetilde{D}(1-e^{-at})}}e^{-\frac{a(x-x_0e^{-at})^2}{2\widetilde{D}(1-e^{-at})}}. \tag{26}$$

It can be shown that this transition density, which describes the evolution of this physical system, satisfies a PDE known as the Fokker-Planck equation. That will be explained in the next section. Another potential of interest is $V(x) = -a\frac{x^2}{2} + b\frac{x^4}{4}$, this is a potential with two wells. It allows one to study tunneling between the two wells.

The Langevin equation has much wider applicability. It has also found applications in chemistry, biology and electrical engineering, see [29] for more details. I will give one example from outside physics, namely biology, to illustrate the point.

Example 3.17 (Including environmental 'noise')**.** For a long time there has been a mathematical interest in population dynamics. Several different models have been created to describe the evolution of populations. Examples include the Lotka Volterra equations [155]. A challenge these models face is the question about how to include environmental effects in the differential equations that govern population dynamics. An option would be to treat these effect as random 'noise'. The competitive Lotka Volterra equations have the form

$$\begin{aligned}\frac{dN_1}{dt} &= a_1N_1 + a_2N_1^2 + a_3N_1N_2\\ \frac{dN_2}{dt} &= b_1N_2 + b_2N_2^2 + b_3N_1N_2.\end{aligned}$$

Then including a noise term gives us

$$\begin{aligned}\frac{dN_1}{dt} &= a_1N_1 + a_2N_1^2 + a_3N_1N_2 + N_1W_1\\ \frac{dN_2}{dt} &= b_1N_2 + b_2N_2^2 + b_3N_1N_2 + N_2W_2.\end{aligned}$$

This can be interpreted as a Langevin equation in the following manner. We regard N_1 and N_2 as the 'x-coordinates'. Then one can see that '$\dot{p} = 0$' and we have multiplicative noise.

Now we can establish a link with (deterministic) PDEs through the so-called Fokker-Planck equation, which is the equation that a transition density of a SDE solution satisfies.

3.6 Fokker-Planck equation and the Kramers-Moyal expansion

The Fokker-Planck equation is a (deterministic) PDE that models the evolution of the probability density of the solution of a SDE. Each SDE has its own associated Fokker-Planck equation. Before we introduce this crucial equation, we discuss the notion of a transition density.

Definition 3.18. The transition density is a type of probability density function that can be written as $p(a, t_1, b, t_2)$. It is a function that depends on a certain initial time t_1 and 'position' a and similarly a final time t_2 and 'position' b. One can think of it as the probability density that a stochastic process (such as the solution to an SDE) transitions from a certain initial value a at time t_1 to a final value b at time t_2. It satisfies the equation

$$P(u(t_1) \in A | u(t_2) = b) = \int_A p(a, t_1, b, t_2)da. \tag{27}$$

This equation says that the probability of u starting in A when it ends in b is given by the integral on the right hand side.

It can be shown that the transition density satsifies the Chapman-Kolmogorov equation

$$p(a,t_1,b,t_2) = \int_{\mathbb{R}} p(a,t_1,c,t)p(c,t,b,t_2)dc, \tag{28}$$

where it is implicit that $t_1 < t < t_2$. It is also assumed that the solution lies in $\mathbb{R}$, although the Chapman-Kolmogorov equation can of course easily be extended to solutions lying in $\mathbb{R}^n$. With the risk of getting ahead of the story too much, at this point I would like to draw the reader's attention to the crucial quantum mechanical nature of the Chapman-Kolmogorov equation. This equation can be interpreted by regarding the transition densities as probability amplitudes (in the framework of the statistical interpretation of the wavefunction), it then leads to the definition of the path integral for quantum mechanics, but this viewpoint will be discussed later on. This approach of using a Chapman-Kolmogorov type equation is known as time-slicing, because we are inserting a complete set of states at finite time intervals.

There are two different ways of looking at the Fokker-Planck equation. One way is in its relation to a specific SDE (and its solution). The other way is as an equation that the transition density of most Markov processes satisfy. We will derive the (forward) Fokker-Planck equation in the second manner. The derivation will be one-dimensional (the multi-dimensional case is analogous, but it will not be treated here to make the exposition as simple as possible). What follows will be based on [109]. We will derive the so-called Kramers-Moyal expansion. We start with a definition.

Definition 3.19. The jump moments of a system are given by

$$M_l(x,t,\Delta t) = \int dy (y-x)^l p(x,t,y,t+\Delta t). \tag{29}$$

We then use the Chapman-Kolmogorov equation in the following form

$$P(x,t+\Delta t) = \int dy\; p(y,t,x,t+\Delta t)P(y,t).$$

Let $\Delta x \coloneqq x - y$, using a Taylor series we find that

$$\begin{aligned}
&p(y,t,x,t+\Delta t)P(y,t)\\
&= p(x-\Delta x,t,(x-\Delta x)+\Delta x,t+\Delta t)P(x-\Delta x,t)\\
&= \sum_{n=1}^{\infty} \frac{(x-x-\Delta x)^n}{n!}\frac{\partial}{\partial x^n}\Big(p\big(x+\Delta x-\Delta x,t,(x+\Delta x-\Delta x)+\Delta x,t+\Delta t\big)P(x+\Delta x-\Delta x,t)\Big)\\
&= \sum_{n=1}^{\infty} \frac{(-1)^n(\Delta x)^n}{n!}\frac{\partial}{\partial x^n}\Big(p(x,t,x+\Delta x,t+\Delta t)P(x,t)\Big).
\end{aligned}$$

Integrating over y and using the Chapman-Kolmogorov equation on the left-hand side, we obtain

$$P(x, t + \Delta t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial}{\partial x^n} \Big(M_l(x, t, \Delta t) P(x, t) \Big). \tag{30}$$

We remark that $M_0(x, t, \Delta t) = 1$ and also $\lim_{\Delta t \to 0} M_l(x, t, \Delta t) = 0$, because $p(x, t, y, t) = \delta(x - y)$. Therefore, for some functions f_l, we assume that the jump moments are of the following form

$$M_l(x, t, \Delta t) = f_l(x, t)\Delta t + \mathcal{O}(\Delta t).$$

If we insert this expression into equation (30), divide by Δt and then take the limit $\Delta t \to 0$, we obtain

$$\frac{\partial P}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial}{\partial x^n} (f_l(x, t) P(x, t)). \tag{31}$$

If we can make the assumption that the jump moments are negligible for $n > 2$, we can truncate the series and obtain the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}(f_1(x, t)P(x, t)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(f_2(x, t)P(x, t)). \tag{32}$$

Fortunately, the assumption on the jump moments is often justified. Equation (32) can then be expressed in terms of transition densities by employing the Chapman-Kolmogorov equation. We have now seen that the Fokker-Planck equation arises from a stochastic process through the jump moments. The only assumption we really made on the process was that it is a Markov process. The jump moments can normally be computed by using the equation

$$M_l(x, t, \Delta t) = \int (x(t + \Delta t) - x)^l p(x, t, x(t + \Delta t), t + \Delta t). \tag{33}$$

This derivation assumes that one knows the jump moments (at least the first two) in some way in order to actually obtain the Fokker-Planck equation for a given system. In the next section we study the other perspective on the Fokker-Planck equation, namely through the link with SDEs, which will be crucial to the derivation of the path integral.

3.7 Fokker-Planck equation and SDEs

We have seen how the Fokker-Planck equation arises from the study of Markov processes. A Wiener process is Markovian too, therefore we would expect that there is a link between SDEs and the Fokker-Planck equation. Suppose we have an Itô SDE of the form

$$du = f(u)dt + G(u)dW(s).$$

Then the transition density satisfies the Fokker-Planck equation, i.e.

$$\frac{\partial}{\partial t}p(u_0,0,u,t) = -\frac{\partial}{\partial u}(f(u(t))p(u_0,0,u,t)) + \frac{1}{2}\frac{\partial^2}{\partial u^2}(G^2(u(t))p(u_0,0,u,t)). \tag{34}$$

The Fokker-Planck equation can be used to determine (probability) moments of the SDE solution. By solving the equation numerically or analytically it is possible to derive properties of the SDE solution. Now we will determine the Fokker-Planck equation of the Langevin equation:

Example 3.20 (Fokker-Planck equation of the Langevin equation)**.** The Langevin equations (21) and (22) , in vector notation (such that the equation has the form $d\mathbf{u} = \mathbf{f}(\mathbf{u})dt + \mathbf{G}(\mathbf{u})d\mathbf{W}(t)$), they are given by

$$d\begin{pmatrix} q \\ P_m \end{pmatrix} = \begin{pmatrix} P_m \\ -\lambda P_m - V'(q) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW(t). \tag{35}$$

So we have defined

$$\mathbf{u} \coloneqq \begin{pmatrix} q \\ P_m \end{pmatrix}, \mathbf{f}(\mathbf{u}) \coloneqq \begin{pmatrix} P_m \\ -\lambda P_m - V'(q) \end{pmatrix}, \mathbf{G}(\mathbf{u}) \coloneqq \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW(t). \tag{36}$$

The Fokker-Planck equation (using the multidimensional version, which we will not discuss here) is then given by

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial q}(P_m p) + \frac{\partial}{\partial P_m}((\lambda P_m + V'(q))p) + \frac{1}{2}\frac{\partial^2}{\partial P_m^2}(\sigma^2 p). \tag{37}$$

Note that here the momentum p has been renamed to P_m in equation (35)-(37), to avoid possible confusion with the transition density p.

Finally we can discuss our intended topic, the link between the Fokker-Planck equation, the Langevin equation and the path integral.

3.8 Relationship between the Fokker-Planck equation and the path integral

Now we can make a connection between the Fokker-Planck equation and the path integral (which we have yet to define). In a nutshell it comes down to the following: The path integral is an integral representation of a transition probability density, while the Fokker-Planck equation (as we have just seen) is a PDE that the transition density must satisfy. So the path integral is the 'solution' to the Fokker-Planck equation. Both approaches have their merits, sometimes (that means for some physical systems) it easier to solve the Fokker-Planck equation, in other cases it might be easier to calculate the path integral. In this light,

it is a little surprising that the quantum field theory literature, where the path integral is used intensively, rarely mention the stochastic aspects.

This section will be based on [28, 29, 52, 88, 109, 125, 163]. What we will do is the following, we will make a suitable choice for the coefficient functions such that the Fokker-Planck equation becomes the Schrödinger equation. Then we use the results from the previous sections to link the Schrödinger equation to a Langevin equation. Finally, we write the probability density of the Langevin equation as a path integral, where we recognise the action in the integrand.

The derivation will be one-dimensional, although it can be extended of course. We start out with the one-dimensional Langevin equation (which has been covered before), which is

$$\dot{x} = -V'(x) + W. \tag{38}$$

As was pointed out before, this equation has mathematical problems, but is commonplace in physics. The derivation that will follow will not be completely rigorous. There exists no construction of the path integral using rigorous stochastics that works for all cases and we will not discuss a rigorous stochastic construction in this thesis. The point here is simply to show the link between the Fokker-Planck equation and the path integral. A rigorous mathematical discussion will come later, in the discussion of oscillatory integrals.

We know that $W(t)$ has a Gaussian distribution, it can be shown that its density is [17, Section 7.3]

$$P(W(t)) = \exp\left(-\frac{1}{4D}\int_0^T W^2(t)dt\right), \tag{39}$$

we stress that this is a deterministic integral and note that D is half the variance. This result is obtained for example by taking the limit of the time discretisation. Recall that the Wiener process $W(t)$ is continuous, which means that the Riemann integral is well-defined [2, 60]. Now we transform the equation for the probability of W into one for the probability for x, by using the Langevin equation we have

$$P(x) = P(W)J(x) \propto \exp\left(-\frac{1}{4}\int_0^T (\dot{x}^2 + V'(x))^2 dt\right) J(x),$$

where $J(x)$ is the Jacobian for the change of variable from W to x, which is proportional to

$$J(x) \propto \exp\left(\frac{1}{2}\int_0^T V''(x)dt\right). \tag{40}$$

This can be derived by using a time discretisation, where the Jacobian is calculated for the finite dimensional case and subsequently a limit $N \to \infty$ is taken. The reader can consult [63, Sections 2.1-2.2] for further details. This leads to the following expression

$$P(x) \propto \exp\left(\int_0^T -\frac{1}{4}(\dot{x}^2 + V'(x))^2 + \frac{1}{2}V''(x)dt\right) = \exp\left(-\frac{S}{D}\right), \tag{41}$$

where we have defined what will become the action of the system, i.e.

$$S(x,\dot{x}) = \int_0^T -\frac{1}{4}(\dot{x}^2 + V'(x))^2 + \frac{1}{2}V''(x)dt. \tag{42}$$

Equation (41) is the key equation of this section, it allows us to derive our quantity of interest: the transition density, that is what the path integral represents. This is given by (note that x is a path here and no longer a point)

$$\begin{aligned} p(x_0,t_0,x_f,t_f) &= \int_{x(t_0)=x_0} Dx\delta(x-x_f)P(x) = \int_{x(t_0)=x_0,x(t_f)=x} DxP(x) \\ &= \int_{x(t_0)=x_0,x(t_f)=x_f} \exp\left(-\frac{S}{D}\right)Dx. \end{aligned}$$

We can now adapt this result so that it is applicable to the Schrödinger equation and that the path integral will become the Feynman path integral. In order to so, we first find the stationary density of the Fokker-Planck equation, i.e. the density $p(a,s,b,t)$ with the property that $\frac{\partial p}{\partial t} = 0$. Setting the time derivative equal to the zero in the Fokker-Planck equation gives us (note there is only x dependence here)

$$0 = -\frac{\partial}{\partial x}(f(x)p_{st}(x)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(g(x)p_{st}(x)),$$

we can integrate once and get

$$0 = -f(x)p_{st}(x) + \frac{1}{2}\frac{\partial}{\partial x}(g(x)p_{st}(x)).$$

Using separation of variables this gives us the solution

$$p_{st}(x) = \frac{A}{g(x)}\exp\left(2\int_0^x \frac{f(x')}{g(x')}dx'\right),$$

note that the lower limit of the integral is arbitrary, if it is changed it can be absorbed into the normalisation constant A, which we choose such that the total probability is 1. Now we use these results in the Fokker-Planck equation. We fix a and s and set the function g to be constant, moreover we assume $f(x,t) = f(x)$. We then have

$$\frac{\partial}{\partial t}p(a,s,x,t) = -\frac{\partial}{\partial x}(f(x)p(a,s,x,t)) + \frac{1}{2}B\frac{\partial^2}{\partial x^2}(p(a,s,x,t)).$$

We then introduce the change of variable $p(a,s,x,t) = (p_{st}(x))^{1/2}\psi(x,t)$. This leads to the following

$$(p_{st})^{1/2}\frac{\partial\psi}{\partial t} = -f(x)(p_{st}(x))^{1/2}\frac{\partial\psi}{\partial x} - f(x)\psi\frac{\partial(p_{st}(x))^{1/2}}{\partial x} - \psi(p_{st}(x))^{1/2}\frac{\partial f}{\partial x}$$

$$
\begin{aligned}
&+\frac{1}{2}B\frac{\partial^2\psi}{\partial x^2}(p_{st}(x))^{1/2}+B\frac{\partial\psi}{\partial x}\frac{\partial(p_{st}(x))^{1/2}}{\partial x}+\frac{1}{2}B\psi\frac{\partial^2(p_{st}(x))^{1/2}}{\partial x^2}\\
&=-f(x)(p_{st}(x))^{1/2}\frac{\partial\psi}{\partial x}-\frac{(f(x))^2}{B}\psi(p_{st}(x))^{1/2}-\psi(p_{st}(x))^{1/2}\frac{\partial f}{\partial x}\\
&+\frac{1}{2}B\frac{\partial^2\psi}{\partial x^2}(p_{st}(x))^{1/2}+f(x)\frac{\partial\psi}{\partial x}(p_{st}(x))^{1/2}+\frac{1}{2B}\psi(p_{st}(x))^{1/2}(f(x))^2\\
&+\frac{1}{2}\psi(p_{st}(x))^{1/2}\frac{\partial f}{\partial x}.
\end{aligned}
$$

Subsequently divide by p_{st}, since the stationary probability density is never zero and rearrange terms to get

$$
\begin{aligned}
&\frac{\partial\psi}{\partial t}=-\frac{1}{2}\psi\frac{\partial f}{\partial x}+\frac{1}{2}B\frac{\partial^2\psi}{\partial x^2}-\frac{1}{2B}\psi(f(x))^2 \implies B\frac{\partial\psi}{\partial t} =\frac{B^2}{2}\frac{\partial^2\psi}{\partial x^2}-\frac{1}{2}\left(B\frac{\partial f}{\partial x}+(f(x))^2\right)\psi\\
&\implies B\frac{\partial\psi}{\partial t}=\frac{B^2}{2}\frac{\partial^2\psi}{\partial x^2}+U(x)\psi
\end{aligned}
$$

The observant reader has already noticed that the choice $B=i\hbar$ yields the Schrödinger equation in the standard form, as found in many quantum mechanics textbooks, such as [65]. Note that in light of this interpretation we have defined

$$
U(x):=-\frac{1}{2}\left(B\frac{\partial f}{\partial x}+(f(x))^2\right) \tag{43}
$$

as the 'potential' of the system. The reader might object that this system needs to have a solution for each possible potential that occurs in physics in order for the link between the Schrödinger and Fokker-Planck equations to be established completely. However, we remark that for any choice of potential that is smooth in a region, we have the differential equation

$$
\frac{df}{dt}=-\frac{1}{B}(2U(x)+(f(x))^2):=G(x,f).
$$

Under quite weak assumptions, namely that U must be locally Lipschitz continuous, the differential equation will have a unique solution [155]. So for each 'standard' potential U that occurs in physics (this includes the Coulomb potential, since it is everywhere smooth except at the origin), there exists a suitable f. We relate the 'action' of the system to the action in the Feynman path integral as follows

$$
S(x)=\frac{i}{\hbar}\int_0^T\left(\frac{1}{2}\dot{x}^2-U(x)\right)dt\to\int_0^T\left(-\frac{1}{4D}(\dot{x}^2+(V'(x))^2)+\frac{1}{2}V''(x)\right)dt.
$$

Note that the cross term becomes

$$
\int_0^T\dot{x}V'(x)dt=\int_{x_0}^{x(T)}V'(x)dx=V(x_T)-V(x_0),
$$

which does not depend on the path but only on the endpoints and therefore can be included in the normalisation constant.

Finally, we can link the Fokker-Planck equation, which has rewritten in the form of a Schrödinger equation, to the path integral through the Langevin equation. Provided that the following equality holds

$$-\frac{1}{2}\left(B\frac{\partial f}{\partial x}+(f(x))^2\right)=U(x)=\frac{1}{2}(V'(x))^2-DV''(x),$$

the solutions of the Fokker-Planck equation

$$\frac{\partial}{\partial t}p(a,s,x,t)=-\frac{\partial}{\partial x}(f(x)p(a,s,x,t))+\frac{1}{2}B\frac{\partial^2}{\partial x^2}(p(a,s,x,t)).$$

correspond to the path integral

$$p(x_0,t_0,x,t)=\int_{x(t_0)=x_0,x(t)=x}\exp\left(-\frac{S}{D}\right),\quad S(x)=\frac{i}{\hbar}\int_0^T\frac{1}{2}\dot{x}^2-U(x)dt.$$

The key step in between the two formalisms was the Langevin equation

$$dx=-V'(x)dt+Wdt.$$

The relevance of this result is the following: the fact that we can construct something resembling a path integral is not inherent to physics. It merely follows from the fact that the evolution of a quantum mechanical system follows a Schrödinger equation and this specific linear PDE can be represented as a path integral. Thus the path integral is first and foremost a stochastic concept, rather than a physics concept. Of course this derivation has been rather heuristic and needs to be made rigorous mathematically, but it gives a flavour of how the path integral is a result of stochastic theory. Of course, its physical implications need to be investigated thoroughly, which will be done in the next chapters.

A final remark about generalising this result to n dimensions is in order. In that case we have n-dimensional Brownian motion $\mathbf{W}(t)$ (which was discussed in proposition 3.11), such that each of the components is a one-dimensional Brownian motion. The multi-dimensional Langevin equation is

$$dx_i=A_i(\mathbf{x})dt+W_idt,$$

or, in vector notation,

$$d\mathbf{x}=\mathbf{A}(\mathbf{x})dt+\mathbf{W}dt,$$

where we impose the following condition on the Brownian motion

$$\mathbb{E}[W_i(t)W_j(t')]=2D_{ij}\delta(t-t'),$$

for some matrices A and D. Then the 'action' (better known as Onsager-Machlup functional) is given by

$$S(x) = \int_0^T \frac{1}{4} \sum_{i,j=1}^{n} (\dot{x}_i - A_i(\mathbf{x})) D_{ij}^{-1} (\dot{x}_j - A_j(\mathbf{x})) + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial A_i}{\partial x_i}, \tag{44}$$

where of course, we impose the condition that D is invertible. Further details for the derivation of this result can be found in [64]. To obtain the form of the potential U analogous to equation (43), one uses the multi-dimensional Fokker-Planck equation and its correspondence with the Langevin equation.

The purpose of this chapter was to give a short outline of the link between the path integral and the Fokker-Planck equation (and the required preliminaries). Please note that by no means this chapter constitutes a comprehensive treatment of SDE theory and its relationship with physics. For example, there exists the monograph [29] on the Langevin equation and its applications to physics and chemistry. Another interesting application of stochastic processes lies in the white noise calculus. This is a method to define a rigorous path integral, using Gaussian measures [108, Section 6.3].

As a final remark, the reader should note that the path integral has also been extensively applied to statistical mechanics, which can be approached using quantum field theory methods to tackle the (practically) infinite number of degrees of freedom. Given the connection between the path integral and Brownian motion, and the relevance of the latter in describing molecular motion, this is not a surprise.

Moreover, the wide range of applications of the Fokker-Planck equation in stochastic analysis should not come as a surprise. After all, this equation is the bridge between PDEs and SDEs, which are very different fields of study. From the Fokker-Planck equation one can obtain a lot of information on the properties of an SDE solution. SDEs have many applications themselves, not least in quantitative finance [16]. Now that we have studied the path integral from the Fokker-Planck perspective, we turn to a quantum mechanical perspective in the next chapter.

4 The path integral in quantum mechanics

Now that we have understood the link between the Fokker-Planck equation and the path integral, we discuss how it is commonly approached by physicists. The path integral is an integral form of the probability amplitude. This is the probability a particle propagates from an initial state q_I at time $t = 0$ to a final state q_F at time $t = T$ in a given system. The probability amplitude is commonly written as $\langle q_F|e^{-iHT}|q_I\rangle$, where $H = \frac{p^2}{2m} + V(q)$ denotes the Hamiltonian. The path integral itself is given by the sum over all the paths between these two states q_I and q_F.

How does one sum over all paths? The approach that is used in physics is the so-called time-slicing approach. The time interval $[0, T]$ is divided up into small subintervals over which the particle moves on a straight line segment or a classical path (i.e. a path determined by the principle of least action [14, 21]), then one subsequently takes the limit of the length of these subintervals going to zero and obtains the path integral that way. Using this time-slicing approach mathematical attempts have been made to make the path integral rigorous, there has been significant progress, however the work is not complete yet [7, 8, 55, 57, 81, 93, 108]. We should remark that the time-slicing technique is also highly useful to compute actual path integrals [88], such as for the hydrogen atom and other atomic systems. This chapter is based on [53, 88, 121, 162].

4.1 Constructing the non-relativistic path integral

As we saw in the last chapter, the path integral is an expression for the transition density. In particular, we are interested in the transition density between the wavefunction at position q_I at an initial time $t = 0$ and another stationary state q_F at a final time $t = t_F$. We rewrite the transition amplitude by

$$\langle q_F, T|q_I, 0\rangle = \langle q_F|e^{-iHT}|q_I\rangle = \langle q_F|\underbrace{e^{-iH\delta t}\dots e^{-iH\delta t}}_{N \text{ times}}|q_I\rangle, \tag{45}$$

where $N = \frac{T}{\delta t}$. We can then rewrite the equation above by utilising the fact that $|q\rangle$ forms a complete set of states, which means that $\int dq|q\rangle\langle q| = 1$. Then using intermediate states $q_1, \dots, q_{N-1}$ we find that

$$\langle q_F|e^{-iHT}|q_I\rangle = \int dq_1 \dots dq_{N-1}\langle q_F|e^{-iH\delta t}|q_{N-1}\rangle\langle q_{N-1}|e^{-iH\delta t}|q_{N-2}\rangle \dots \langle q_2|e^{-iH\delta t}|q_1\rangle\langle q_1|e^{-iH\delta t}|q_I\rangle. \tag{46}$$

We will evaluate one of these inner products and find

$$\langle q_2|e^{-iH\delta t}|q_1\rangle = \int \frac{dp}{2\pi}\langle q_2|e^{-iH\delta t}|p\rangle\langle p|q_1\rangle = \int \frac{dp}{2\pi}\langle q_2|e^{-i\delta t\left(\frac{p^2}{2m}+V(\hat{q})\right)}|p\rangle\langle p|q_1\rangle$$

$$= e^{-i\delta t V(\hat{q})} \int \frac{dp}{2\pi} e^{-i\delta t \frac{p^2}{2m}} \langle q_2|p\rangle\langle p|q_1\rangle = e^{-i\delta t V(\hat{q})} \int \frac{dp}{2\pi} e^{-i\delta t \frac{p^2}{2m}} e^{ipq_2} e^{-ipq_1}$$
$$= e^{-i\delta t V(\hat{q})} \int \frac{dp}{2\pi} e^{-i\delta t \frac{p^2}{2m}} e^{ip(q_2-q_1)} = e^{-i\delta t V(\hat{q})} \sqrt{\frac{-im}{2\pi\delta t}} e^{[im(q_2-q_1)^2]/2\delta t}$$
$$= e^{-i\delta t V(\hat{q})} \sqrt{\frac{-im}{2\pi\delta t}} e^{(im\delta t/2)[(q_2-q_1)/\delta t]^2}.$$

Here we denote by $\hat{p}$ and $\hat{q}$ the multiplication operators with eigenvalues p and q for the states $|p\rangle$ and $|q\rangle$. This is because the states $|p\rangle$ and $|q\rangle$ are defined as follows

$$\hat{p}|p\rangle = p|p\rangle, \quad \hat{q}|q\rangle = q|q\rangle.$$

So one should distinguish between p being an operator, a scalar (an eigenvalue) and a label for a state. We can insert the obtained result into equation (46) and find

$$\langle q_F|e^{-iHT}|q_I\rangle = \left(\frac{-im}{2\pi\delta t}\right)^{N/2} \int dq_1 \dots dq_{N-1} e^{i\delta t \sum_{i=1}^{N} [\frac{m}{2}[(q_i-q_{i-1})/\delta t]^2 - V(q_i)]},$$

at this point we are going to take the limit $\delta t \to 0$. We observe that $(q_i - q_{i-1})/\delta t \to \dot{q}$, $\delta t \sum_{i=1}^{N} \to \int_0^T dt$ and finally we obtain

$$\lim_{\delta t \to 0} \left(\frac{-im}{2\pi\delta t}\right)^{N/2} \int dq_1 \dots dq_{N-1} = \int Dq, \tag{47}$$

which is the integral over all paths between q_I and q_F, to indicate that we integrate over paths we use Dq instead of dq. Taking this limit is the step that is causing all the mathematical problems with quantum field theory and quantum mechanics in Feynman's formulation. In order to do so, one has to justify that taking the limit is allowed, mathematically speaking, which means that $\left(\frac{-im}{2\pi\delta t}\right)^{N/2} dq_1 \dots dq_{N-1}$ converges to a suitable measure Dq.

It is precisely this measure Dq which is a source of mathematical troubles. The problem with it is that this 'measure' is treated as though it has 'nice properties' (σ-additive, translation and rotation invariant and assigns finite nonzero measure to bounded open sets), while that is impossible in the case of an infinite dimensional Hilbert space (or Banach space). This was discussed in the introduction and will also be covered later. Putting these mathematical concerns aside we obtain the path integral

$$\langle q_F|e^{-iHT}|q_I\rangle = \int Dq(t) e^{i\int_0^T dt\left(\frac{m}{2}\dot{q}^2 - V(q)\right)}. \tag{48}$$

A comment about what this transition amplitude means seems in order. The path integral is an expression for the propagator. The propagator, denoted by K, is related to the Green's function (or fundamental solution) of the Schrödinger equation as follows

$$G(0, T, q_I, q_F) = \Theta(T) K(0, T, q_I, q_F), \tag{49}$$

where Θ denotes the Heaviside step function. Recall that the Green's function solves the Schrödinger equation with a noise term in the following fashion

$$i\hbar\frac{\partial G(x,t_1,y,t_2)}{\partial t} - HG(x,t_1,y,t_2) = \delta(t_1-t_2)\delta(x-y). \tag{50}$$

The propagator (or rather the Green's function) describes the evolution of the wavefunction in the following manner

$$\psi(y,t_2) = \int_{\mathbb{R}^3} K(t_1,t_2,x,y)\psi(x,t_1)dx. \tag{51}$$

From this crucial equation it is apparent why the path integral formalism is equivalent to the Schrödinger formalism. In the latter, the evolution of the wavefunction is described by the Schrödinger equation. Given an initial wavefunction, the Schrödinger equation gives us the evolution of the physical system for all future times. In the path integral formalism, the path integral tells us the evolution for all future times too [67, 68]. The link between the two formalisms is hence given in equations (49) and (50). This also makes clear how one can do 'ordinary' quantum mechanics with the path integral.

Once again we observe something that was stressed in the previous chapter, namely the inherently stochastic nature of the path integral in non-relativistic quantum mechanics. If the reader compares equations (28) (the Chapman-Kolmogorov equation) and (51), one can immediately notice a connection: In the former, the probability is equal to an integral over the transition density times the probability distribution at an earlier time; In the latter, the wavefunction at a later time is equal to an integral over the wavefunction at an earlier time multiplied by the propagator, which acts as the transition density.

The derivation itself, which uses the time-slicing procedure (because we discretise the time) is a way of writing the transition density (which is what the path integral is after all) as an integral. We have discretised the time and then applied the Chapman-Kolmogorov equation (28) to split the propagator into $N-1$ propagators by utilising a complete set of states.

Then we used the properties of the solutions to the Schrödinger equation to rewrite the transition densities over small time intervals and subsequently took an ill-defined limit. This is related to the derivation of the path integral using the Fokker-Planck and Langevin equations which was covered in the chapter on the stochastic aspects of the path integral.

The relation might not be immediately obvious, but the Fokker-Planck equation can be derived from the Chapman-Kolmogorov equation and in the derivation in the previous chapter the properties of the Schrödinger equation were also used, namely that it can be considered as a Fokker-Planck equation. In addition, the time was discretised in order to find the probability distribution of the Brownian motion W in equation (39) and the Jacobian of the coordinate transformation from W to x in equation (40). So this physical derivation in this chapter has relations to the stochastic approach from the previous chapter.

A final remark about the notation, or rather the translation/change of notation, is in order to prevent confusion. In the subsequent 'mathematical' chapters the notation $Dq(t)$ will not be used since it is not a rigorously defined concept, instead the notation $d\gamma$ will be used, where $\gamma : [0, T] \to \mathbb{R}^3$ is a path in space. This path has the property that

$$\gamma(0) = q_I, \gamma(\delta t) = q_1, \gamma(2\delta t) = q_2, \ldots, \gamma(T) = q_F.$$

The space of paths (known as the Cameron-Martin space) can be made into a Hilbert space. Using techniques from linear PDE theory we can define a path integral on this space, but all of this will be covered later. The purpose of this chapter is to study the path integral from a physics standpoint, which means that it will be explained how the path integral can be used to solve physical problems. Later on we will cover a possible approach to make the concept of the path integral rigorous.

Having covered what the path integral is and does in detail, we turn towards utilising this tool to obtain actual results on physical systems in quantum mechanics.

4.2 Performing calculations with the path integral

The path integral in the setting of non-relativistic quantum mechanics can just as well be used to derive solutions of quantum mechanical problems as the Schrödinger or Heisenberg formalisms. The drawback of the path integral approach however, is that it is much more tedious in the case of quantum mechanics (but not in the case of quantum field theory, which will be explained later).

In this section, I will treat two examples of standard problems in quantum mechanics, the harmonic oscillator and the anharmonic oscillator by using the path integral formalism. The harmonic oscillator can be solved exactly, which is what we will do. The anharmonic oscillator cannot be solved exactly, which is also the case for the Schrödinger formalism. In the Schrödinger formalism one resorts to various approximation techniques, such as the Rayleigh-Schrödinger perturbation theory, as covered in [65]. In the case of the path integral, several perturbation techniques exist too, which will be explained. This section is based on [53, 58, 121].

4.2.1 Harmonic oscillator

The harmonic oscillator is one of the most important systems in physics, because it is exactly solvable and many systems can (to first order) be approximated as a harmonic oscillator. The path integral of the harmonic oscillator can be calculated exactly, we will explain how this works in this section. Note that we will set $\hbar = 1$

What we will do is the following, we will calculate the action for the classical harmonic oscillator exactly. Subsequently we consider quantum deviations from the classical path and derive that the action can be split into a quantum part and a classical part. We calculate

the quantum part of the action and then we arrive at the path integral. The Hamiltonian of the classical harmonic oscillator is

$$H = T + V = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2.$$

This leads to the equation of motion

$$\ddot{q} + \omega^2 q = 0.$$

This equation has as a solution a linear combination of a sine and a cosine with angular frequency ω. The paths $q(t)$ satisfy the following initial conditions: $q(0) = q_I$ and $q(T) = q_F$. Using trigonometric identities, the solution to the initial value problem can then be written as

$$q_c(t) = \frac{1}{\sin(\omega T)}(q_F \sin(\omega t) + q_I \sin(\omega(T-t))).$$

The subscript c has been added to signify that the path is a classical solution. The velocity is hence given by

$$\dot{q}_c(t) = \frac{\omega}{\sin(\omega T)}(q_F \cos(\omega t) - q_I \cos(\omega(T-t))).$$

The action for the classical path is then given by

$$S[q_c] = \int_0^T L dt = \frac{1}{2}m\int_0^T (\dot{q}_c^2 - \omega^2 q_c^2) dt. \tag{52}$$

We can evaluate this expression explicitly by using integration by parts on the $\dot{q}^2$ term and find (by using the classical equation of motion)

$$S[q_c] = \frac{1}{2}m[q_c\dot{q}_c]_0^T - \frac{1}{2}m\int_0^T q_c(\ddot{q}_c + \omega^2 q_c)dt = \frac{1}{2}m[q_c\dot{q}_c]_0^T = \frac{1}{2}m(q_F\dot{q}_c(T) - q_I\dot{q}_c(0)). \tag{53}$$

Now we use the results that $\dot{q}_c(T) = \frac{\omega(q_F\cos(\omega T) - q_I)}{\sin(\omega T)}$ and $\dot{q}_c(0) = \frac{\omega(q_F - q_I\cos(\omega T))}{\sin(\omega T)}$, we obtain the following expression for the action

$$S[q_c] = \frac{m\omega(-q_F q_I + q_F^2\cos(\omega T) - q_F q_I + q_I^2\cos(\omega T)}{2\sin(\omega T)} = \frac{m\omega(-2q_F q_I + (q_I^2 + q_F^2)\cos(\omega T))}{2\sin(\omega T)}. \tag{54}$$

Now moving on to the quantum case, in general we can write a path as

$$q(t) = q_c(t) + f(t), \quad \text{where } f(0) = f(T) = 0.$$

Note that the function f also satisfies the equation of motion and that we may write

$$S[q] = S[q_c] + S[f]. \tag{55}$$

This is allowed because we can write

$$\begin{aligned} S[q] = S[q_c + f] &= \frac{1}{2}m\int_0^T (\dot{q}_c + \dot{f})^2 - \omega^2(q_c + f)^2 dt \\ &= \frac{1}{2}m\int_0^T \dot{q}_c^2 + 2\dot{q}_c\dot{f} + \dot{f}^2 - \omega^2(q_c^2 + 2q_c f + f^2)dt \\ &= S[q_c] + S[f] + m\int_0^T \dot{q}_c\dot{f} - \omega^2 q_c f dt \\ &= S[q_c] + S[f] + m\dot{q}_c f|_0^T - m\int_0^T f(\ddot{q}_c - \omega^2 q_c)dt \\ &= S[q_c] + S[f]. \end{aligned}$$

We also have that $dq = df$. So now we can write the path integral as follows

$$K(0, T, q_I, q_F) = \int dq e^{iS[q]} = e^{iS[q_c]}\int df e^{iS[f]}, \tag{56}$$

where K is the propagator, which was introduced in equation (51). Since $f(0) = f(T) = 0$ and we only need an expression of f on a compact interval, in order to calculate the path integral we can use a Fourier sine series

$$f(t) = \sum_{k=1}^{\infty} a_k\sqrt{\frac{2}{T}}\sin\left(\frac{k\pi t}{T}\right). \tag{57}$$

We proceed by calculating the integral $\int df e^{iS[f]}$. We first calculate $S[f]$: using the orthonormality of the functions $\sin\left(\frac{n\pi t}{T}\right)$ we get

$$\begin{aligned} S[f] &= \frac{1}{2}m\int_0^T (\dot{f}^2 - \omega^2 f^2)dt = \frac{1}{2}m\frac{2}{T}\int_0^T \sum_{k=1}^{\infty}\left[\frac{k^2\pi^2}{T^2} - \omega^2\right] a_k^2 \sin^2\left(\frac{k\pi t}{T}\right) \\ &= \frac{1}{2}m\sum_{k=1}^{\infty}\left[\frac{k^2\pi^2}{T^2} - \omega^2\right] a_k^2. \end{aligned}$$

Assume now that the differential df is of the form

$$df = C\prod_{i=k}^{\infty} da_k, \tag{58}$$

where C is the normalisation constant. The reasons we make this assumption is that we are integrating over all possible paths f, which means that you integrate over all possible Fourier constants a_k. Now we can write down the path integral as follows

$$\int df e^{iS[f]} = C\prod_{k=1}^{\infty}\int da_k e^{iS[f]} = C\prod_{k=1}^{\infty}\int da_k e^{\frac{i}{2}m\left[\frac{k^2\pi^2}{T^2} - \omega^2\right]a_k^2}.$$

We need to calculate each of the integrals $\int da_k e^{i\left[\frac{k^2\pi^2}{T^2}-\omega^2\right]a_k^2}$, which are of the form $\int_{-\infty}^{\infty} e^{ibx^2}dx$, we will set

$$b := \frac{m}{2}\left(\frac{k^2\pi^2}{T^2}-\omega^2\right)a_k^2.$$

The integrals are equal to

$$\int_{-\infty}^{\infty} e^{ibx^2}dx = \sqrt{\frac{\pi i}{b}}.$$

This integral can be shown to be convergent by the change of variable $t = x^2$ and by using Dirichlet's test. Using this result we have

$$\int df e^{iS[f]} = C\prod_{i=1}^{\infty}\int da_k e^{\frac{i}{2}m\left[\frac{k^2\pi^2}{T^2}-\omega^2\right]a_k^2} = C\prod_{k=1}^{\infty}\sqrt{\frac{\pi i}{\frac{m}{2}\left[\frac{k^2\pi^2}{T^2}-\omega^2\right]a_k^2}} \tag{59}$$

In the free case, so when $\omega = 0$, the result must equal the result of the path integral for the free particle. That result is $\sqrt{\frac{m}{2\pi iT}}$, so the expression above must be equal to that if you set $\omega = 0$ [13]. So we have in that case

$$\int df e^{iS[f]} = C\prod_{k=1}^{\infty}\sqrt{\frac{\pi i}{\frac{m}{2}\frac{k^2\pi^2a_k^2}{T^2}}} = C\prod_{k=1}^{\infty}\sqrt{\frac{2T^2i}{m\pi k^2a_k^2}}.$$

That means that the normalisation constant assumes the form

$$C = \sqrt{\frac{m}{2\pi iT}}\prod_{k=1}^{\infty}\sqrt{\frac{m\pi k^2a_k^2}{2T^2i}}. \tag{60}$$

Note that this infinite product is divergent, since it is of the form $\prod_{k\in\mathbb{N}} k$. So if we do not put a strong assumption on the behaviour of the Fourier coefficients a_k, the expression diverges. This is not a problem, since as a whole (the constant times the other infinite product) the expression converges. If we insert the obtained expression for the normalisation constant into equation (59) we find

$$\int df e^{iS[f]} = \sqrt{\frac{m}{2\pi iT}}\prod_{k=1}^{\infty}\sqrt{\frac{\pi i}{\frac{m}{2}\left[\frac{k^2\pi^2}{T^2}-\omega^2\right]a_k^2}}\prod_{j=1}^{\infty}\sqrt{\frac{m\pi j^2a_j^2}{2T^2i}} \tag{61}$$

$$= \sqrt{\frac{m}{2\pi iT}}\prod_{k=1}^{\infty}\sqrt{\frac{\pi}{\frac{k^2\pi^2}{T^2}-\omega^2}}\prod_{j=1}^{\infty}\sqrt{\frac{\pi j^2}{T^2}} \tag{62}$$

$$= \sqrt{\frac{m}{2\pi iT}}\prod_{k=1}^{\infty}\sqrt{\frac{\pi^2}{\pi^2-\frac{\omega^2T^2}{k^2}}} = \sqrt{\frac{m}{2\pi iT}}\prod_{k=1}^{\infty}\sqrt{\frac{1}{1-\frac{\omega^2T^2}{k^2\pi^2}}}. \tag{63}$$

Subsequently we make use of the following infinite product identity

$$\prod_{k=1}^{\infty}\left(1-\frac{\omega^2T^2}{k^2\pi^2}\right)=\frac{\sin(\omega T)}{\omega T}, \tag{64}$$

this follows by looking at the zeros of both sides of the equation and the behaviour as $\omega T \to 0$. Inserting this equation into equation (63) we finally find

$$\int df e^{iS[f]}=\sqrt{\frac{m}{2\pi iT}}\sqrt{\frac{\omega T}{\sin(\omega T)}}=\sqrt{\frac{m\omega}{2\pi i\sin(\omega T)}}.$$

Combining this result with equations (54) and (56) we find

$$K(0,T,q_I,q_F)=e^{iS[q_c]}\int df e^{iS[f]}=\exp\left(\frac{im\omega(-2q_Fq_I+(q_I^2+q_F^2)\cos(\omega T))}{2\sin(\omega T)}\right)\sqrt{\frac{m\omega}{2\pi i\sin(\omega T)}}. \tag{65}$$

We can see that the propagator (as a function of q_F and T) satisfies the Schrödinger equation, we have that

$$\begin{aligned}
\frac{\partial K(0,t,q_I,q)}{\partial t}&=\sqrt{\frac{m\omega}{2\pi i\sin(\omega t)}}\frac{-im\omega^2\cos(\omega t)(-2qq_I+(q_I^2+q^2)\cos(\omega t))-im\omega^2(q_I^2+q^2)\sin^2(\omega t)}{2\sin^2(\omega t)}\\
&\quad\exp\left(\frac{im\omega(-2qq_I+(q_I^2+q^2)\cos(\omega t))}{2\sin(\omega t)}\right)-\exp\left(\frac{im\omega(-2qq_I+(q_I^2+q^2)\cos(\omega t))}{2\sin(\omega t)}\right)\\
&\quad\cdot\sqrt{\frac{m\omega^3\cos^2(\omega t)}{8\pi i\sin^3(\omega t)}}\\
\frac{\partial K(0,t,q_I,q)}{\partial q}&=\exp\left(\frac{im\omega(-2qq_I+(q_I^2+q^2)\cos(\omega t))}{2\sin(\omega t)}\right)\sqrt{\frac{m\omega}{2\pi i\sin(\omega t)}}\frac{im\omega(-2q_I+2q\cos(\omega t))}{2\sin(\omega t)}\\
\frac{\partial^2 K(0,t,q_I,q)}{\partial q^2}&=\exp\left(\frac{im\omega(-2qq_I+(q_I^2+q^2)\cos(\omega t))}{2\sin(\omega t)}\right)\sqrt{\frac{m\omega}{2\pi i\sin(\omega t)}}\frac{-m^2\omega^2(-2q_I+2q\cos(\omega t))^2}{4\sin^2(\omega t)}\\
&\quad+\exp\left(\frac{im\omega(-2qq_I+(q_I^2+q^2)\cos(\omega t))}{2\sin(\omega t)}\right)\sqrt{\frac{m\omega}{2\pi i\sin(\omega t)}}\frac{2im\omega\cos(\omega t)}{2\sin(\omega t)}
\end{aligned}$$

The propagator must satisfy the equation

$$i\frac{\partial K(0,t,q_I,q)}{\partial t}=\frac{-1}{2m}\frac{\partial^2K(0,t,q_I,q)}{\partial q^2}+\frac{1}{2}m\omega^2x^2K(0,t,q_I,q).$$

If we insert the obtained expressions (and omitting the common exponential and square root) we obtain

$$\frac{m\omega^2\cos(\omega t)(-2qq_I+(q_I^2+q^2)\cos(\omega t))+m\omega^2(q_I^2+q^2)\sin^2(\omega t)}{2\sin^2(\omega t)}-\frac{i\omega\cot(\omega)}{2}$$

$$= -\frac{\omega i \cot(\omega t)}{2} + \frac{m\omega^2(-2q_I + 2q\cos(\omega t))^2}{8\sin^2(\omega t)} + \frac{1}{2}m\omega^2 q^2$$
$$\frac{\cos(\omega t)(-2qq_I + (q_I^2 + q^2)\cos(\omega t)) + (q_I^2 + q^2)\sin^2(\omega t)}{\sin^2(\omega t)} = \frac{(-q_I + q\cos(\omega t))^2}{\sin^2(\omega t)} + q^2$$
$$-2qq_I\cos(\omega t) + (q_I^2 + q^2) = (-q_I + q\cos(\omega t))^2 + q^2\sin^2(\omega t),$$

which does indeed hold. Therefore the found propagator in equation (65) does indeed solve the Schrödinger equation and the path integral formula in (65) is correct. Later on, when we will construct a rigorous path integral for the harmonic oscillator, we will arrive at equation (65) too.

A convenient use of path integrals is to compute energy levels. The idea is to derive the partition function

$$Z = \sum_n e^{-\beta E_n}. \tag{66}$$

from the path integral. If we set $T = i\beta$, the path integral is related to the partition function in the following way

$$Z = \int_{-\infty}^{\infty} dx \langle x|e^{-\beta H}|x\rangle. \tag{67}$$

We can now calculate the partition function for the harmonic oscillator, we set $q_I = q_F = x$ in equation (65) and find

$$\begin{aligned}
Z &= \int_{-\infty}^{\infty} dx \sqrt{\frac{m\omega}{2\pi i \sin(i\omega\beta)}} \exp\left(\frac{im\omega(-2x^2 + 2x^2\cos(i\omega\beta))}{2\sin(i\omega\beta)}\right) \\
&= \int_{-\infty}^{\infty} dx \sqrt{\frac{m\omega}{2\pi i \sinh(\omega\beta)}} \exp\left(-x^2\frac{im\omega(1-\cosh(\omega\beta))}{\sinh(\omega\beta)}\right) \\
&= \sqrt{\frac{m\omega}{2\pi i\sinh(\omega\beta)}}\sqrt{\frac{\pi\sinh(\omega\beta)}{im\omega(1-\cosh(\omega\beta))}} = \sqrt{\frac{1}{2\cosh(\omega\beta)-2}} = \sqrt{\frac{1}{4\sinh^2(\omega\beta/2)}} \\
&= \frac{1}{2\sinh(\omega\beta/2)} = \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}} = \sum_{n=0}^{\infty} e^{-\beta(n+1/2)\omega}.
\end{aligned}$$

That means that we arrive at the energy levels $E_n = (n + 1/2)\omega$, recall that we had set $\hbar = 1$. If we had included the factor in the calculations, we would obtain

$$E_n = \left(\frac{1}{2} + n\right)\hbar\omega, \tag{68}$$

which is exactly the same result as obtained in the Schrödinger formalism [113]. We now move to the anharmonic oscillator.

4.2.2 Anharmonic oscillator

The anharmonic oscillator is a harmonic oscillator perturbed by a quartic term λx^4, this means that we have the following Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 - \lambda x^4. \tag{69}$$

This system is of great importance in quantum field theory because it leads to interactions, which will be discussed in the next chapter.

Before we discuss the possible options one has to calculate or approximate the path integral for such a system, it is important to mention one mathematical result for the Cauchy problem of the anharmonic oscillator. For a Schrödinger equation with a potential V that satisfies the following inequality

$$V(x) \geq C(1 + |x|)^{2+\epsilon}, \quad \text{for } |x| \to \infty$$

with $\epsilon, C > 0$, the solution of the Schrödinger equation is nowhere C^1 [160]. That means in particular that the solution to this Cauchy problem only satisfies the Schrödinger equation in a weak sense. One can find more about weak solutions in most books on PDE, such as [49].

It should be said that these results in [160] are only for one dimension, but it is reasonable to assume the theorem (or a modified version of it) extends to three dimensions. This is relevant because it puts a constraint on what kind of approximations of this path integral we are able to obtain using perturbation theory.

For example, the path integral being nowhere C^1 means it is impossible to do a Taylor expansion as an approximation. A Taylor approximation for path integrals is often used in physics and especially for quantum electrodynamics it leads to accurate results [129, 162]. However, it does not converge at all and is even ill-defined. It is unclear whether the perturbation series that results from such an approximation is in fact an asymptotic series or not [44].

Moreover, a semiclassical approximation often leads to divergences as well, as discussed in [88, Chapter 4]. Such an approximation expands the path integral as a series in $\hbar$, using the fact that path integrals with quadratic actions can be calculated exactly. The coefficients of such a series for the anharmonic oscillator grow very fast and some resummation procedure is necessary to obtain a convergent series [20].

Variational perturbation theory is such a procedure. This technique first uses a variational approach to approximate the path integral and is also way to obtain a convergent perturbation series. We will only discuss the calculation of approximations of path integrals [53]. The resulting approximations from this technique for the energies are very good, as can be seen later from the data. For readers who find the treatment here too concise or want to read about the derivation of convergent perturbation series, the definitive treatment of

variational perturbation theory remains [88, Chapter 5] (which is the main reference for this chapter).

Before we turn to variational perturbation theory, we will discuss the exact solution for the path integral. Suppose we want to calculate the path integral

$$\int Dq e^{\frac{i}{\hbar}\int_0^T \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 - \lambda x^4 dt}, \tag{70}$$

where x is understood as a function of t. This path integral does not have a simple exact solution. The problem is that the system does not have a classical solution which can be written in terms of simple functions. In the classical case, the Euler-Lagrange equation for the anharmonic oscillator would be

$$m\ddot{x} + m\omega^2 x + 3\lambda x^3 = 0,$$

which is not a differential equation that has an obvious solution. Indeed, the solution is given by elliptic sines. Therefore, using the technique of the previous section where we consider quantum deviations from the classical path is not a feasible approach. It should be noted that an exact solution for the anharmonic path integral does exist. The exact solution contains elliptic sines and elliptic integrals which makes it very complicated and hard to use in practice. It can be found in [68, paragraph 6.2.2.10]. A derivation can be found [111, 138]. The result however, is given in terms of an infinite series and cannot be used in practice.

There are several ways to still obtain approximate answers. The first (and probably most obvious) one is using a perturbation expansion in λ. What that means is that one writes for a general potential $V(x)$ (which is λx^4 in our case)

$$\begin{aligned}
&\int Dq e^{-\frac{i}{\hbar}\int_0^T V(x)dt} e^{\frac{i}{\hbar}\int_0^T \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2} \\
&= \int Dq \Bigg[1 - \frac{i}{\hbar}\int_0^T V(x(t_1))dt_1 - \frac{1}{2!\hbar^2}\int_0^T V(x(t_1))dt_1 \int_0^T V(x(t_2))dt_2 \\
&+ \frac{i}{3!\hbar^3}\int_0^T V(x(t_1))dt_1 \int_0^T V(x(t_2))dt_2 \int_0^T V(x(t_3))dt_3 + \dots \Bigg] e^{\frac{i}{\hbar}\int_0^T \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 dt}.
\end{aligned}$$

As was said before, this approach (known as the Dyson series in quantum field theory) has a strong disadvantage, namely that the perturbation series diverges. It is called an asymptotic series. An asymptotic series is a series that is formally equal to a given function, but it does not converge to that function. That means that the partial sums do not constitute a Cauchy sequence. That has strong implications for the use of perturbation series in quantum mechanics and quantum field theory, although the results yield surprising accuracy.

We will not pursue this method any further here since we will discuss it in the context of quantum field theory (and discuss the link with the Feynman diagrams). There are

physical arguments such that the perturbation series should diverge in the case of quantum electrodynamics, as argued by Dyson in [44]. However, there has been no mathematical proof that it is the case.

There is one method that works in producing a perturbation series that converges (not necessarily to the exact solution, but close to it), which uses variational perturbation theory. This was first developed in [53] (more details can be found in [88]). Note that this method always converges, regardless of the size of the coupling constant λ. Suppose we have an Euclidean path integral (i.e. the path integral has been Wick rotated, so we have let time be imaginary)

$$Z = \int Dx e^{-\int_0^\beta \frac{1}{2}\dot{x}^2 - V(x(t))dt}, \tag{71}$$

where we have set $m = 1$ for sake of notational convenience. If we decompose the paths x as follows

$$x(t) = x_0 + \sum_{n=1}^{\infty} \left[x_n e^{i\omega_n t} + \overline{x_n} e^{-i\omega_n t} \right], \tag{72}$$

where we have defined $\omega_n = \frac{2\pi n}{\beta}$. Then the path integral can be decomposed as follows

$$Z = \int \frac{dx_0}{\sqrt{2\pi\beta}} \prod_{n=1}^{\infty} \int \frac{d\,\mathrm{Re}(x_n) d\,\mathrm{Im}(x_n)}{\pi/(\beta\omega_n^2)} e^{-\beta \sum_{n=1}^{\infty} \omega_n^2 |x_n|^2 - \int_0^\beta V(x(t))dt}, \tag{73}$$

where we remark that the integration over x_n is implicit in the case of V, since x_n is contained in the variable x. If we could perform the infinite product of integrals, we would end up with an integral of the form

$$Z = \int \frac{dx_0}{\sqrt{2\pi\beta}} e^{-\beta W_0(x_0)}. \tag{74}$$

In general it is impossible to find W_0 exactly. What was found in [53] by Feynman and Kleinert however, was a very good approximation of W_0, which we call W_1. To find W_1 the following procedure is followed. First one calculates a quantity V_{a^2} that is known as the 'smeared potential', it is given by

$$V_{a^2}(x) = \int \frac{dx'}{\sqrt{2\pi a^2}} e^{-\frac{1}{2a'^2}(x-x')^2} V(x'). \tag{75}$$

At first the parameter a^2 is unknown. Then we introduce another parameter Ω and the auxiliary potential

$$\widetilde{W}_1(x_0, a^2, \Omega) = \frac{1}{\beta} \ln \left(\frac{\sinh(\beta\Omega/2)}{\beta\Omega/2} \right) - \frac{\Omega^2}{2} a^2 + V_{a^2}(x_0). \tag{76}$$

Finally, we consider a^2 and Ω to be functions of x_0 and we take W_1 to be the minimum with respect to a^2 and Ω. Calculations lead to the following relations for Ω and a^2

$$a^2 = \frac{1}{\beta\Omega^2}\left(\frac{\beta\Omega}{2}\coth\left(\frac{\beta\Omega}{2}\right) - 1\right) \tag{77}$$

$$\Omega^2(x_0) = 2\frac{\partial}{\partial a^2}V_{a^2}(x_0) = \frac{\partial^2}{\partial x_0^2}V_{a^2}(x_0). \tag{78}$$

The approximation of the Euclidean path integral that we have obtained in this manner only requires that the potential V is smooth, there are no requirements on the coupling constant λ. Using the approximation of the path integral we can also obtain approximations of the energies. If we apply these techniques to the anharmonic oscillator, we arrive at the following smeared potential (here we set $\omega^2 = 1$) [88]

$$V_{a^2}(x_0) = \frac{x_0^2}{2} + \lambda x^4 + \frac{a^2}{2} + 6\lambda x_0^2 a^2 + 3\lambda a^4. \tag{79}$$

We have the following relations for a^2 and Ω^2 in this specific case

$$a^2 = \frac{1}{\beta\Omega^2}\left(\frac{\beta\Omega}{2}\coth\left(\frac{\beta\Omega}{2}\right) - 1\right) \tag{80}$$

$$\Omega^2(x_0) = 1 + 12\lambda x_0^2 + 12\lambda a^2(x_0). \tag{81}$$

We start out with an initial approximation $\Omega(x_0) = 0$, use that to calculate a^2, using the expression for a^2 to calculate Ω^2 and so forth. The iteration process converges (for a given x_0) rapidly [53]. We find the free energy by performing the integral

$$Z_1 = e^{-F_1/k_BT} = \int_{-\infty}^{\infty}\frac{dx_0}{\sqrt{2\pi\hbar^2\beta/m}}e^{-\widetilde{W}_1(x_0)k_B/T},$$

where m is the mass. The classical limit of the approximation is given by

$$Z_{cl} = \int_{-\infty}^{\infty}\frac{dx_0}{\sqrt{2\pi\hbar^2\beta/m}}e^{-V(x)k_B/T}. \tag{82}$$

One can see the accuracy of this approximation by inspecting the following data:

λ	E_1	E_{ex}	$\Omega(0)$	$a^2(0)$
0.1	0.5603	0.559146	1.222	0.4094
0.2	0.6049	0.602405	1.370	0.3650
0.3	0.6416	0.637992	1.487	0.3363
0.4	0.6734	0.668773	1.585	0.3154
0.5	0.7017	0.696176	1.627	0.2991
1.0	0.8125	0.803771	2.000	0.2500
10	1.5313	1.50497	4.000	0.1250
50	2.5476	2.49971	6.744	0.0741
100	3.1924	3.13138	8.474	0.0590
500	5.4258	5.31989	14.446	0.0346

Table 1: Comparison between the exact energy and the approximated energies of the anharmonic oscillator. The data has been taken from [88, Table 5.1].

In figure 1 one can see the free energy F_1 obtained by this approximation plotted against the exact solution F_{ex} (which was obtained from several known exact eigenvalues for the anharmonic oscillator).

At this point the reader might wonder what the perturbative aspect of this variational approach is. The approximations to the path integral that have been explained in this section can subsequently be used to obtain convergent perturbative series for the energies in terms of the coupling constant λ. The useful aspect of these perturbative series is that they converge also for large values of the coupling constant λ. More details on this matter can be found in [88, Chapter 5].

This concludes the treatment of the path integral for the anharmonic oscillator, we have seen that it is not easy to solve it exactly, while variational perturbation theory is a technique that leads to a reasonable approximation of the path integral. Much more has been written about asymptotic series in quantum mechanics (especially in relation to perturbation expansions), the reader is referred to [58] and [107, Section 3.5] for further details. Also for perturbation theory in general, much more can be found in [88].

A final remark about operators is in order. For quantum mechanics it is of importance that an operator is self-adjoint, since it means that the eigenvalues (which have the interpretation of measurements) of the operators are real. It can however be shown that while a Hamiltonian of the form $H = p^2/2m + x^4$ is (essentially) self-adjoint, while $H = p^2/2m - x^4$ is not. This has implications for the path integral, since in the mathematical analysis that will follow we require the operators to be self-adjoint. This has nothing to do with physics per se, but it is a mathematical requirement to ensure that the construction does not have any problems.

A full analysis of all the implications and importance of self-adjointness for Schrödinger operators would warrant another thesis however and the reader is advised to consult [70, 146]

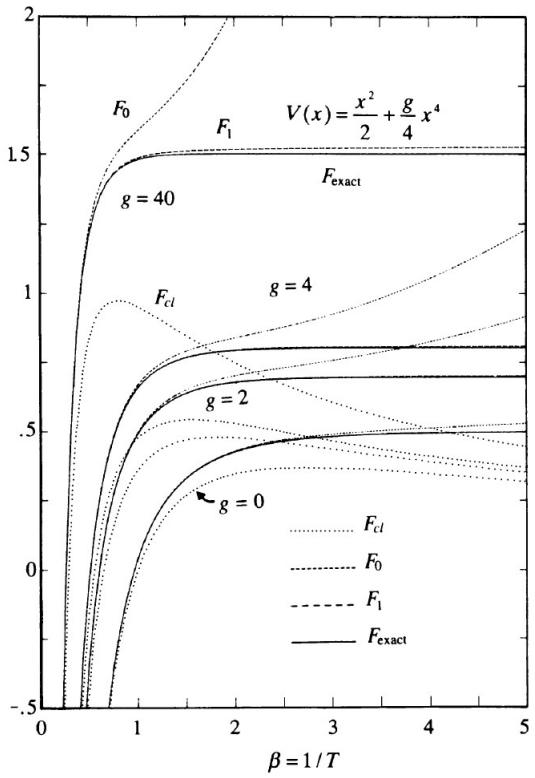


Figure 1: The free energy plotted as a function of $\beta = 1/T$. The curve F_1 is obtained by the variational perturbation theory approximation, while F_{ex} is the exact solution and F_{cl} is the classical limit given in equation (82). The curve F_0 is the approximation also obtained by variational perturbation theory by making the choice $\Omega = 0$ and $a^2 = \beta/12$ (which is not optimal). The approximations and exact solutions have been plotted for different coupling strengths, note that $g = 4\lambda$ in our case. This figure has been taken from [88, page 381].

for further details. Note that from a physical standpoint operators need not be self-adjoint. An example is the Hamiltonian $H = \frac{p^2}{2m} + ix^3$, which is not self-adjoint but does have a real spectrum, which means that it gives physically meaningful answers. The same is true for the operator $p^2/2m - x^4$. More about that can be found in [4, 18, 19].

In the next section we discuss the path integral in the setting of quantum field theory, in particular how the anharmonic (quartic) term in the potential leads to interactions and how these can be studied by using a perturbation series.

5 Solving path integrals using the Duru-Kleinert transform

A big problem surrounding Feynman's original path integral formula

$$\langle q_F, T | q_I, 0 \rangle = \int Dq e^{\frac{i}{\hbar} S(q,\dot{q})},$$

is that it is hard to make sense of it when the potential has a singularity. The approximations (using the time slicing/discretisation) often do not have finite values (due to paths crossing the singularity) which makes it impossible to handle limits.

In this chapter, we discuss a method that is known as the Duru-Kleinert transform. The idea of this transformation is that one uses the Fourier transform of the propagator, which is known as the fixed-energy amplitude. Then an object known as the pseudopropagator can be defined. The pseudopropagator is obtained by shifting the Hamiltonian with a constant (the fixed energy) and multiplying the operator by regulating functions. The pseudopropagator has a functional degree of freedom (with respect to the regulating functions) which we can use to remove the singularity by making a suitable function choice.

Using this technique, the path integral for a system with a singularity can be related to the path integral of a system without a singularity. For many interesting potentials (such as the hydrogen atom), this is the key step towards an analytical solution. This approach allows one to tackle path integrals that were previously untractable.

Finally, it should be remarked that the physics of the system is not changed in any way by the transform. The propagator related to the original system results in the same path integral as the pseudopropagator. What we are simply doing is discretising the time in a (position-dependent) different manner. This does result in a convergent expression, while the discretisation of the original path integral is divergent. The propagator itself does exist, because the singularity is never crossed due to quantum fluctuations, but path collapse does occur in the finite dimensional case. That is why the Duru-Kleinert transform is used.

This chapter is predominantly based on [88]. It should be said that this chapter together with chapter 9 form a coherent whole. This chapter explains how the path integral of the hydrogen atom is calculated in physics (in two dimensions). Chapter 9 explains how we can give this method a rigorous mathematical foundation using oscillatory integrals. That is why chapter 9 does not directly succeed this chapter, since the material on oscillatory integrals must be introduced first.

5.1 Convergence of discretised path integrals and path collapse

Suppose we were to approach the calculation of the path integral for the hydrogen atom in the straightforward way. Discretising the path integral and associated action would give us

(in two dimensions)

$$\langle q_F|U_E^N(s_F-s_I)|q_I\rangle = \frac{1}{2\pi i\epsilon_s\hbar/m}\int \frac{d^2x_1\dots d^2x_N}{\sqrt{2\pi\hbar i/m}^{2N}} e^{\frac{i}{\hbar}A_E^N},$$

where the action is given by

$$A_E^N = \sum_{k=1}^{N+1}\left(\frac{m}{2\epsilon_s}(x_k-x_{k-1})^2 - \epsilon_s\frac{1}{x_k}\right).$$

One can observe that this integral is not going to convergence, due to $\frac{1}{x_k}$ being present in the action. This is known as path collapse. It might then seem that it is impossible to define a path integral for the hydrogen atom at all. However, due to thermodynamic reasons (or quantum fluctuations) it never happens that a particle crosses the singularity. More about these reasons can be found in [88, Section 12.1].

Therefore it is possible to define a path integral, since the singularity is not a problem in the continuum. We must however, find a way to construct finite dimensional convergent path integrals, before we can take a well-defined limit. In order to do so, we will exploit a symmetry in the Hamiltonian in the next section.

5.2 Fixed-energy amplitude and its functional degree of freedom

In the previous section it became clear why for some potentials the path integral can simply not be independent of the chosen time slicing. This is why a particular discretisation must be chosen in order to give meaning to the Coulomb path integral. The first thing that will be done is to define the Fourier transform of the propagator, namely the fixed-energy amplitude

$$\langle q_F,|q_I\rangle_E \coloneqq \int_{-\infty}^{\infty} dt_F e^{iE(t_F-t_I)/\hbar}\Theta(t_F-t_I)\langle q_F,t_F|q_I,t_I\rangle = \int_{t_I}^{\infty} dt_F e^{iE(t_F-t_I)/\hbar}\langle q_F,t_F|q_I,t_I\rangle. \tag{83}$$

The fixed energy amplitude can be viewed as matrix elements of the form

$$\langle q_F|q_I\rangle_E = \langle q_F|\hat{R}(E)|q_I\rangle, \tag{84}$$

where the resolvent operator $\hat{R}$ is defined to be

$$\hat{R}(E) = \frac{i\hbar}{E-H+i\eta}. \tag{85}$$

This equation follows because the resolvent operator is the Fourier transform of the evolution operator, i.e.

$$\hat{R}(E) = \int_{-\infty}^{\infty} dt_F e^{iE(t_F-t_I)/\hbar}\Theta(t_F-t_I)U(t_F,t_I) = \int_{t_I}^{\infty} dt_F e^{iE(t_F-t_I)/\hbar}U(t_F,t_I),$$

Note that the $i\eta$ in the definition of the resolvent operator in equation (85) ensures the causality. One can recover the propagator from the fixed-energy amplitude by the inverse Fourier transform

$$\langle q_F, t_F | q_I, t_I \rangle = \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} e^{-iE(t_F - t_I)/\hbar} \langle q_F | q_I \rangle_E. \tag{86}$$

Now that the fixed-energy amplitude has been introduced, one can observe that the resolvent operator can also be written as

$$\hat{R}(E) = f_r \frac{i\hbar}{f_l(E - H + i\eta) f_r} f_l. \tag{87}$$

The two functions f_l and f_r are known as the regulating functions. If one compares this equation with (84), one sees that this leads to a new Hamiltonian

$$H_E = f_l(H - E) f_r,$$

known as the pseudohamiltonian. In principle there are no assumptions on the behaviour of the regulating functions, but one wants to pick them such that the new pseudohamiltonian is more well-behaved. Because the resolvent operator itself is not changed, a system with this new Hamiltonian H_E leads to the same fixed-energy amplitude. This means that

$$\langle q_F | q_I \rangle_E = \langle q_F | \hat{R} | q_I \rangle = \int_{s_I}^{\infty} ds_F \langle q_F | U_E(s_F - s_I) | q_I \rangle, \tag{88}$$

where we have defined the pseudotime evolution operator

$$U_E(s) = f_r(x) \exp\left(-is f_l(x)(H - E) f_r(x)\right) f_l(x).$$

The variable s is referred to as the pseudotime.

The reader might reasonably ask at this point what the advantage of this approach is. What we have gained is an additional functional degree of freedom compared to the propagator (or time-evolution amplitude). This is because we are free to choose the regulating functions, which means we choose to get rid of the singularity. For example, for the Coulomb potential, we can pick $f_l(x) f_r(x) = r$, which removes the singularity. The details will be seen in the next section. The reason that this approach using a pseudotime works is that the width of the discretisation is space-dependent, i.e.

$$dt = ds f_l(x_k) f_r(x_{k-1}). \tag{89}$$

This has the advantage that when the singularity is being approached, the slices get thinner and thinner and that prevents the discretisation from diverging.

If we succeed in calculating $\langle q_F | U_E(s_F - s_I) | q_I \rangle$, which is known as the pseudopropagator, we can use equation (88) to calculate the fixed-energy amplitude. Where possible, we can use the inverse Fourier transform to obtain the path integral.

Note that we have assumed that the regulating functions f_l and f_r depend only on position and not on momentum. This is solely for simplification purposes, but in principle the derivation can be extended to the case where the functions depend on momentum as well. It is not true in general that H_E is a self-adjoint operator.The form of its spectrum (continuous or discrete) is of interest, but more about that will be said in chapter 9. Its importance lies in the fact that eventually we would like to reconstruct the Duru-Kleinert transform rigorously using oscillatory integrals.

Now it will be discussed how the pseudopropagator is actually calculated. One can show that if the time is discretised in $N+1$ intervals of length ϵ_s, one has that

$$\langle q_F|U_E^N(s_F-s_I)|q_I\rangle = f_r(q_F)f_l(q_I)\int \frac{d^nx_1\dots d^nx_N d^np_1\dots d^np_{N+1}}{(2\pi\hbar)^{n(N+1)}}e^{\frac{i}{\hbar}A_E^N},$$

where we have defined the following discretised action

$$A_E^N = \sum_{k=1}^{N+1} p_k(x_k-x_{k-1}) - \epsilon_s f_l(x_k)[H(p_k,x_k)-E]f_r(x_k). \tag{90}$$

The fixed-energy amplitude becomes

$$\langle q_F|q_I\rangle_E \approx (N+1)\int_0^\infty d\epsilon_s \langle q_F|U_E^N(s_F-s_I)|q_I\rangle.$$

If one takes the limit $N\to\infty$ one obtains the following pseudopropagator

$$\langle q_F|U_E(s_F-s_I)|q_I\rangle = f_r(q_F)f_l(q_I)\int \frac{Dx(s)Dp(s)}{2\pi\hbar}e^{\frac{i}{\hbar}\int_0^S ds\left[px'-H_E(p,x)\right]}, \tag{91}$$

where x' is the derivative with respect to the pseudotime. If we opt for a standard Hamiltonian of the form

$$H = \frac{p^2}{2m}+V(x),$$

then the momentum variables in the discretised pseudopropagator can be integrated out. The result is

$$\langle q_F|U_E^N(s_F-s_I)|q_I\rangle = \frac{f_r(q_F)f_l(q_I)}{\sqrt{2\pi i\epsilon_s f_l(q_F)f_r(q_I)\hbar/m}^n}\int \frac{d^nx_1\dots d^nx_N}{\sqrt{2\pi\hbar i/m}^{nN}\sqrt{f(x_1)\dots f(x_N)}}e^{\frac{i}{\hbar}A_E^N}, \tag{92}$$

where the action is given by

$$A_E^N = \sum_{k=1}^{N+1}\frac{m}{2\epsilon_s f_l(x_k)f_r(x_{k-1})}(x_k-x_{k-1})^2 - \epsilon_s f_l(x_k)[V(x_k)-E]f_r(x_{k-1}). \tag{93}$$

Taking the limit $N \to \infty$ we end up with the following pseudopropagator

$$\langle q_F|U_E(s_F-s_I)|q_I\rangle = f_r(q_F)f_l(q_I)\int Dx(s)\exp\left(\frac{i}{\hbar}\int_0^S ds\left[\frac{m}{2f_l(x)f_r(x)}x'^2 - f_l(x)(V-E)f_r(x)\right]\right). \tag{94}$$

This expression holds for any choice of regulating functions. Of course we recover the original path integral if we pick $f_l(x) = f_r(x) = 1$. A final remark about the functional symmetry seems in order. The ability to choose arbitrary regulating functions holds for the fixed-energy amplitude in equation (88), but not for the discretised pseudopropagator in equation (92). So different choices of discretisation lead to different outcomes. It is equation (94) that we will use to solve the hydrogen atom.

5.3 The hydrogen atom in two dimensions

In this section, we will solve the path integral for the two-dimensional hydrogen atom. In the Schrödinger formalism, the wavefunctions of the hydrogen atom can be obtained and the corresponding Bohr energies can be calculated. The path integral formalism being equivalent to the Schrödinger picture, it should be able to calculate solution for the hydrogen atom as well.

A completely satisfactory solution was achieved in 1990 in [88] and our treatment is based on that. We will only discuss the hydrogen atom in two dimensions since in three dimensions we have to account for torsion which will complicate the discussion. This has to do with the coordinate transformation being nonholonomic which leads to flat space being mapped to a space with curvature. Recall that the hydrogen atom has the following Hamiltonian

$$H = \frac{p^2}{2m} - \frac{e}{r}.$$

We make the following choices for the regulating functions

$$f_l(x) = (f(x))^{1-\lambda}, \quad f_r(x) = (f(x))^{\lambda}, \quad f(x) = |x| = r.$$

This choice gives us the following result for the pseudoaction (given in equation (90))

$$A_E^N = \sum_{k=1}^{N+1}\left(p_k\Delta x_k - \epsilon_s r_k^{1-\lambda}r_{k-1}^{\lambda}[p_k^2/2m - E] + \epsilon_s e^2\right).$$

Note that we dropped the factor $(r_k/r_{k-1})^{\lambda}$, because it converges to 1 in the limit, we also defined $\Delta x_k = x_k - x_{k-1}$. The pseudopropagator can be written as

$$\langle q_F|U_E^N(s_F-s_I)|q_I\rangle = r_F^{\lambda}r_I^{1-\lambda}\int \frac{d^n\Delta x_2\ldots d^n\Delta x_{N-1}d^n\Delta p_1\ldots d^n\Delta p_{N+1}}{(2\pi\hbar)^{2(N+1)}}e^{\frac{i}{\hbar}A_E^N}.$$

If we integrate out the momentum variables, we obtain that

$$\langle q_F|U_E^N(s_F-s_I)|q_I\rangle = \frac{r_F^\lambda r_I^{1-\lambda}}{2\pi i\epsilon_s r_F^{1-\lambda}r_I^\lambda\hbar/m}\int\frac{d^2\Delta x_2\dots d^2\Delta x_{N+1}}{\sqrt{2\pi\hbar i/m}^{2N}\sqrt{r_1\cdot\ldots\cdot r_N}}e^{\frac{i}{\hbar}A_E^N},$$

with the following action

$$A_E^N = (N+1)\epsilon_s e^2 + \sum_{k=1}^{N+1}\frac{m}{2\epsilon_s r_k^{1-\lambda}r_{k-1}^\lambda}(\Delta x_k)^2 + \epsilon_s r_k E.$$

Note that in the last step we replaced $r_k^{1-\lambda}r_{k-1}^\lambda$ by r_k, since in the limit $N\to\infty$ it results in the same expression. Finally, in the limit we obtain the following expression for the hydrogen atom for the pseudopropagator

$$\langle q_F|U_E(s_F-s_I)|q_I\rangle = r_F^\lambda r_I^{1-\lambda}\int Dx(s)\exp\left(e^2S+\frac{i}{\hbar}\int_0^S ds\left[\frac{m}{2r}x'^2+Er\right]\right). \tag{95}$$

This expression can be calculated by performing the following coordinate transformation, known as the Levi-Civita transformation

$$\begin{aligned} x_a^2 &= u_a^2 - u_b^2\\ x_b^2 &= 2u_a u_b. \end{aligned}$$

This is precisely the point where the treatment differs from three dimensions. In that case, the coordinate transformation is only defined differentially. The coordinate transformation becomes nonholonomic (i.e. path-dependent) and the codomain is a curved space. It can be shown in the two dimension space is that the codomain is also flat, see [88, Section 13.2] for more details. There it is shown that the Riemann tensor is identically zero and hence the space is flat. The coordinate transformation gives us that

$$x'^2 = 4ru'^2.$$

This leads to the following expression for the action

$$A = e^2S + \frac{i}{\hbar}\int_0^S ds\left[\frac{4m}{2}u'^2+Eu^2\right].$$

One can recognise that this is the action of a harmonic oscillator, with effective mass $\mu=4m$ and frequency $\omega=\sqrt{-E/2m}$. The minus sign is not surprising, since we are considering bound states. If the energy is positive, we just have a free particle, which constitutes a different path integral. Alternatively, we can obtain the positive energy case by analytic

continuation. To conclude, the pseudopropagator of the hydrogen atom is given by the following formula

$$\langle u_b|U_E(s_b-s_a)|u_a\rangle = \frac{e^{ie^2(s_b-s_a)/\hbar}}{4}(K_{harmonic}(u_a,s_a,u_b,s_b)+K_{harmonic}(u_a,s_a,-u_b,s_b)).$$

Using the results from the previous chapter we obtain

$$\langle u_b|U_E(\Delta S)|u_a\rangle = \frac{e^{i\Delta S e^2/\hbar}}{2}\frac{\mu\omega}{2\pi i\sin(\omega\Delta S)}\exp\left(\frac{i\mu\omega(u_a^2+u_b^2)\cos(\omega\Delta S)}{\sin(\omega\Delta S)}\right)\cosh\left(\frac{\mu\omega u_a u_b}{\sin(\omega\Delta S)}\right).$$

We then set out to calculate the fixed-energy amplitude

$$\langle u_b|u_a\rangle_E = \frac{1}{2}\int_0^\infty dS e^{iSe^2/\hbar}\frac{\mu\omega}{2\pi i\sin(\omega S)}\exp\left(\frac{i\mu\omega(u_a^2+u_b^2)\cos(\omega S)}{\sin(\omega S)}\right)\cosh\left(\frac{i\mu\omega u_a u_b}{\sin(\omega S)}\right)$$

We introduce the notation

$$\begin{aligned} F(S) &= \sqrt{\frac{\mu\omega}{2\pi i\sin(\omega S)}} \\ \rho &= e^{-2i\omega S} \\ k &= \frac{\mu\omega}{2\hbar} \\ \nu &= \frac{e^2}{2\hbar}, \end{aligned}$$

subsequently we calculate that

$$\begin{aligned} \pi(F(S))^2 &= \frac{\mu\omega}{2\hbar i\sin(\omega S)} = k\frac{1}{i\sin(\omega S)} = k\frac{2}{1/\sqrt{\rho}-\sqrt{\rho}} = k\frac{2\sqrt{\rho}}{1-\rho} \\ e^{ie^2S/\hbar}(F(S))^2 &= k\frac{2\rho^{1/2-\nu}}{\pi(1-\rho)}. \end{aligned}$$

Then the fixed energy amplitude takes the form

$$\langle u_b|u_a\rangle_E = -i\frac{m}{\pi\hbar}\int_0^1 d\rho\frac{\rho^{-1/2-\nu}}{\pi(1-\rho)}\cos\left(2k\frac{2\sqrt{\rho}}{1-\rho}\sqrt{(x_b\cdot x_a+r_b r_a)/2}\right)\exp\left(-k\frac{1+\rho}{1-\rho}(r_b+r_a)\right).$$

Note that this integral does only converge if $\nu < \frac{1}{2}$, more details can be found in [88, Section 13.2]. Note that using a spectral decomposition we obtain the radial wave functions, in particular we find the following energy levels

$$E_n = -\frac{me^4}{\hbar^4(n-1/2)^2}.$$

These results are in agreement with the paper [161] on the two-dimensional hydrogen atom. Hence, the Duru-Kleinert transform has been successful. To conclude, in this chapter it has been explained why it is hard to calculate the path integral for a potential with a singularity. It has been covered why the problem lies with the discretisation, rather than with the path integral itself. The techniques have been used to approach the hydrogen atom in two dimensions. In chapter 9 there will be an attempt to give it a mathematical foundation, using oscillatory integrals in particular. But first, the path integral in the context of quantum field theory will be discussed.

6 The path integral in quantum field theory

In this chapter I will explain the uses of the path integral in quantum field theory. We saw that in non-relativistic quantum mechanics the path integral has few advantages over the Schrödinger formalism, except in the classical limit. The limit $\hbar \to 0$ is much easier to handle in the path integral formulation as opposed to the Schrödinger formalism. However, in quantum field theory, that is very different. In that case it is extremely difficult to do practical calculations with the Schrödinger formalism, which means that in practice is seldom to never used [122].

The reason for that is that in the transition from non-relativistic quantum mechanics to quantum field theory, the wavefunction $\psi(\mathbf{x}, t)$, which is a function, becomes a functional $\Psi(\varphi_t(\mathbf{x}), t)$. The QFT wavefunction depends on the time t as well as the field configuration φ_t at time t. The Schrödinger equation hence becomes a functional differential equation, which is hard to solve in practice. Techniques for doing so have not been adequately developed yet. Hardly any realistic QFT problems can be solved in the Schrödinger picture, which means that we have to choose a different strategy. There are two predominant methods, canonical quantisation and path integration, we will look at the latter [122].

The path integral on the other hand, is very practical for the purpose of perturbation expansions, as will soon become clear. In addition to that, these perturbation expansions can be highly accurate.

First I will explain how the path integral changes if one moves from non-relativistic quantum mechanics to quantum field theory. Then the path integral will be treated in the free field theory (no interaction) case and then we will include an interaction term. Then I will explain how this can be linked to the Feynman diagrams. The reader will soon see how hard it already is to construct a mathematically rigorous path integral in the case of non-relativistic quantum mechanics, it is even harder in the case of quantum field theory. However, the purpose of this section is to show the reader the route that lies ahead for the mathematical formalisation of the path integral and also explain why developing a path integral for the anharmonic term is of such great importance. This chapter is based on [32, 44, 58, 71, 121, 122, 128, 129, 140, 147, 162].

6.1 From quantum mechanics to quantum field theory

Recall that in quantum mechanics the path integral was given by

$$\langle q_F | e^{iHT} | q_F \rangle = \int Dq e^{\frac{i}{\hbar} S(q)}, \quad S(q) = \int_0^T dt L(q, \dot{q}, t). \tag{96}$$

In quantum field theory, the main dynamical variables are the fields themselves, as opposed to position or momentum. So, whereas in classical or quantum mechanics time was the independent variable and position or momentum were the dependent variables [14], in quantum

field theory the position and momentum/velocity become independent variables upon which the fields depend.

So from now on the fields themselves are the dynamical variables of interest. This development is really not surprising: Quantum field theory is a relativistic theory, so space and time should be on an equal footing, in particular that means that position cannot depend on time. It can be shown that the appearance of fields in quantum field theory is a result of quantisation (which is called second quantisation), see [122] for further details. Let $\varphi(x)$ be a field and x be a four-vector, the action that we need to consider is now

$$S(\varphi) = \int d^4x \mathcal{L}(\varphi),$$

where $\mathcal{L}$ is called the Lagrangian density. We will use the convention that x denotes a four-vector while $\mathbf{x}$ is a three-vector. The path integral now becomes

$$Z = \int D\varphi e^{\frac{i}{\hbar}\int d^4x \left[(\partial_\mu\varphi)(\partial^\mu\varphi) - V(\varphi)\right]}.$$

Note that a very important change has occurred here. In the non-relativistic case, the "differential" in the path integral was Dq, i.e. an integral over all paths between two fixed endpoints. Instead, in the QFT case, we have $D\varphi$: an integral over all field configurations in spacetime between two endpoints.

6.2 Considering interactions

The path integral $\int D\varphi e^{\frac{i}{\hbar}\int d^4x[(\partial_\mu\varphi)(\partial^\mu\varphi) - V(\varphi)]}$ only gives us quantum fluctuations, therefore we add a source term $J(x)\varphi(x)$ and the path integral becomes

$$Z = \int D\varphi e^{\frac{i}{\hbar}\int d^4x \left[(\partial_\mu\varphi)(\partial^\mu\varphi) - V(\varphi) + J(x)\varphi(x)\right]}. \tag{97}$$

The source term creates or annihilates particles. In general it is impossible to calculate the action analytically, except in the free field theory case, when the Lagrangian density is given by

$$\mathcal{L}(\varphi) = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - m^2\varphi^2.$$

Using the Euler-Lagrange equation for fields, which is given by

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) - \frac{\partial\mathcal{L}}{\partial\varphi} = 0,$$

we obtain the Klein-Gordon equation

$$\partial_\mu\partial^\mu\varphi + m^2\varphi = 0. \tag{98}$$

The solutions are of the from $\varphi(x) = e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}$, under the condition that $\omega^2 = k^2 + m^2$. There are no interactions in the field theory given by the path integral in equation (97), since the equation of motion is linear in φ (which means that if you have different fields $\varphi_1, \ldots, \varphi_n$, you obtain n separate equations of motion for each field). Now we add an interaction term to the action in the path integral, which is the anharmonic $-\frac{\lambda}{4!}\varphi^4$ term. This time the equations will no longer be linear and hence there will be interaction in this field theory. The path integral becomes

$$Z(J,\lambda) = \int D\varphi e^{\frac{i}{\hbar}\int d^4x\left[(\partial_\mu\varphi)(\partial^\mu\varphi) - V(\varphi) + J(x)\varphi(x) - \frac{\lambda}{4!}\varphi^4\right]} = \int D\varphi e^{-\frac{i}{\hbar}\int \frac{\lambda}{4!}\varphi^4} e^{\frac{i}{\hbar}\int d^4x\left[(\partial_\mu\varphi)(\partial^\mu\varphi) - V(\varphi) + J(x)\varphi(x)\right]},$$

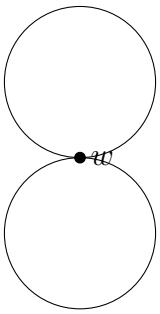
so what has changed with regards to the path integral is the appearance of the term $e^{-\frac{i}{\hbar}\int \frac{\lambda}{4!}\varphi^4}$. What you do to calculate solutions to this path integral is expand in factors of J and λ (assuming that both are small enough for the approximation to be useful). There are two ways in which we can calculate terms in the perturbation expansion. Each term will be associated with one or multiple Feynman diagrams, but more will be said about that later. These two ways either use functional derivatives (Julian Schwinger advocated this approach) or what is known in theoretical physics as the Wick contraction. I will explain both. If we proceed with using functional derivatives, We have the equation

$$Z(J) = e^{i\int d^4x \mathcal{L}_i\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} Z_0(J)$$

Note that $\mathcal{L}_i$ has become a functional different operator. Basically what you now do is say you want to know a given order in λ and J, you take the appropriate terms in the Taylor expansion, multiply them and you will get the result. An example is if you take two terms, of first order in λ and second order in J, we get

$$\begin{aligned} &i\int d^4w\lambda\left(\frac{\delta}{\delta J(w)}\right)^4 \int d^4x d^4y d^4z d^4v \left[J(x)D(x-y)J(y)J(z)D(z-v)J(v)\right] \\ &= i\lambda\int d^4w D(w-w)D(w-w). \end{aligned}$$

This is the case of a particle that propagates from the point w and back and does the same once again. Note that this expression is divergent since one integrates over all momentum, a possible resolution is to note that conservation of momentum must hold and hence fix the values of the momentum and omit the integral. Even then the expression is still divergent, because $D(0)$ is divergent. The way to handle this problem is by renormalisation, which we will not cover here. There is the following Feynman diagram associated with this:



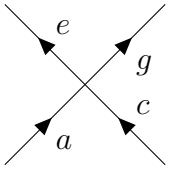
Now we suppose we consider a higher order term, of first order in λ and fourth order in J. Omitting the differentials and writing D_{xy} instead of $D(x-y)$ and J_x instead of $J(x)$, we obtain

$$i\lambda \int d^4w \lambda \left(\frac{\delta}{\delta J(w)}\right)^4 \int\int\int\int\int\int\int\int J_a D_{ab} J_b J_c D_{cd} J_d J_e D_{ef} J_f J_g D_{gh} D_h.$$

There is one connected term (we will ignore the disconnected term and also the combinatorial factors), it is

$$i\lambda \int d^4w \int\int\int\int D_{aw} D_{cw} D_{ew} D_{gw} J_a J_c J_e J_g. \tag{99}$$

What this means physically speaking is that two particles are created at a and c, they propagate to w, scatter and subsequently propagate to e and g, where they are annihilated. The associated Feynman diagram of this process is:



Now I will explain the approach that uses Wick contraction. We expand the exponent $e^{i\int d^4x \mathcal{L}_i\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)}$ first in J and then in λ, we get

$$\begin{aligned} Z(J,\lambda) &= \sum_{s=0}^{\infty} \frac{1}{s!} \int dx_1 \ \ldots \ dx_s J(x_1) \cdot \ldots \cdot J(x_s) \int D\varphi \ \varphi(x_1) \cdot \ldots \cdot \varphi(x_s) e^{\frac{i}{\hbar}\int d^4x \left[(\partial_\mu\varphi)(\partial^\mu\varphi) - V(\varphi) - \frac{\lambda}{4!}\varphi^4\right]} \\ &= Z(0,0) \sum_{s=0}^{\infty} \frac{1}{s!} \int dx_1 \ \ldots \ dx_s J(x_1) \cdot \ldots \cdot J(x_s) G^{(s)}(x_1, \ldots, x_s). \end{aligned}$$

We have defined the s-point Green's function $G^{(s)}(x_1, \ldots, x_s)$ to be

$$G^{(s)}(x_1, \ldots, x_s) = \frac{1}{Z(0,0)} \int D\varphi \ \varphi(x_1) \cdot \ldots \cdot \varphi(x_s) e^{\frac{i}{\hbar}\int d^4x \left[(\partial_\mu\varphi)(\partial^\mu\varphi) - V(\varphi) - \frac{\lambda}{4!}\varphi^4\right]}.$$

What one now does in order to evaluate the terms in the perturbation expansion, we use a result that is known in physics as Wick's theorem. It is given by

$$\frac{\delta}{\delta J(x_1)} \dots \frac{\delta}{\delta J(x_s)} Z_0(J)\bigg|_{J=0} = \sum_{\text{pairings}} D(x_{i_1} - x_{i_2}) \dots D(x_{i_{2s-1}} - x_{i_{2s}}),$$

where $i_1, \dots, i_s$ is a permutation of $1, \dots, s$. So let us consider the case of first order in λ (hence fourth order in the functional derivative of J) and also fourth order in J, using the Wick contraction we obtain the following result

$$-i\lambda \int d^4w J(x_1)J(x_2)J(x_3)J(x_4)D(x_1 - w)D(x_2 - w)D(x_3 - w)D(x_4 - w),$$

This is the same result as we obtained by using functional derivatives, because in equation (99) we have that $J(a) = \delta(x_1 - a), J(c) = \delta(x_2 - c), J(e) = \delta(x_3 - e), J(g) = \delta(x_3 - g)$. Then we obtain

$$-i\lambda \int d^4w D(a - w)D(c - w)D(e - w)D(g - w). \tag{100}$$

which is the same result.

A final note about the potentials is in order. The reader might ask why we would be more interested in $\lambda\varphi^4$-theory rather than $\lambda\varphi^3$, although the latter is commonly looked at in quantum field theory [129]. The reason is that if one looks at the potential of $\lambda\varphi^3$ it is obviously a cubic, which means that it has no global minimum and therefore the system does not have a stable ground state, since it can in that case always leave the local minimum due to quantum fluctuations.

This problem is not there in the case of $\lambda\varphi^4$, since the quartic does have a global minimum. Therefore, when adding a $\lambda\varphi^4$ term one is still justified in speaking about an oscillator (although it is anharmonic). Of course we can calculate a perturbation expansion for the cubic potential, but the problems are not apparent from that expansion itself. That is why in constructing a rigorous path integral the quartic case of such great importance.

Another reason for investigating a quartic term $\lambda\varphi^4$ has to do with renormalisation. The $\lambda\varphi^4$ theory can be renormalised. If one started out with a Lagrangian density that included a $\lambda\varphi^5$ term, one would be a obliged to include a correction $\lambda\varphi^6$ as a counterterm. That correction would necessitate another correction of the order and so on $\lambda\varphi^4$. In other words, any Lagrangian density containing a term $\lambda\varphi^k$ with $k > 4$ is not renormalisable, see [31, page 106] for more details on this matter. That is why we stick with a quartic term $\lambda\varphi^4$ instead of considering a higher order.

6.3 Feynman rules

One of the great insights by Feynman was his observation of a pattern in the Feynman diagrams. Each component of a Feynman diagram, be it a loop, vertex or internal/external

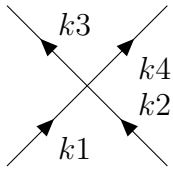
line can be associated with a certain part of the expression of a term in the perturbation expansion. We will use the example of the interaction term $\frac{\lambda}{4!}$ which we have discussed earlier on. First you draw a Feynman diagram of the process that is of interest. We have the following Feynman rules [162]:

1. Each line should be labelled with a momentum k and a propagator $\frac{i}{k^2-m^2+i\epsilon}$ (the $i\epsilon$ is there to shift the poles off the real line, it is known as the $i\epsilon$-prescription).

2. For each vertex one should write $-i\lambda(2\pi)^4\delta^{(4)}(\sum_i k_i - \sum_j k_j)$, where $\sum_i k_i$ are the momenta flowing into the vertex (recall that lines have a direction) and $\sum_j k_j$ are the momenta flowing out of the vertex. The purpose of the Dirac delta function is to enforce conservation of momentum.

3. All the momenta need to be integrated over, so we add $\int \frac{d^4k}{(2\pi)^4}$ for momenta of internal lines.

These rules can of course be derived properly, but we will not do so because they can be found in any QFT textbook and they are not needed for our purposes. There is one final rule about the combinatorial factors, but that is not of relevance here. We will cover one example here. Let us look at the picture we obtained earlier



Applying the rules, i.e. associating a propagator with each line and the term $-i\lambda(2\pi)^4\delta^{(4)}(\sum_i k_i - \sum_j k_j)$ with the vertex, we find the following expression

$$-i\lambda(2\pi)^4\delta^{(4)}(k_1+k_2-k_3-k_4)\prod_{j=1}^{4}\frac{i}{k_j^2-m^2+i\epsilon}.$$

This is in agreement with the expression we obtained by using the functional derivative and Wick approaches, which gave the result in equation (100), although it might not seem that way at a glance. The link is the following expression of the propagator

$$D(x_j-w)=\int d^4k_j\frac{e^{\pm ik_j(x_j-w)}}{k_j^2-m^2+i\epsilon}.$$

It does not matter whether we choose the plus or minus sign. Using this expression we obtain

$$-i\lambda\int d^4w D(x_1-w)D(x_2-w)D(x_3-w)D(x_4-w)$$

$$-i\lambda \int d^4w \int d^4k_1 \dots d^4k_4 e^{-i(k_1+k_2-k_3-k_4)w} \prod_{j=1}^{4} \frac{i}{k_j^2 - m^2 + i\epsilon}$$
$$-i\lambda \int d^4k_1 \dots d^4k_4 \delta(k_1 + k_2 - k_3 - k_4) \prod_{j=1}^{4} \frac{i}{k_j^2 - m^2 + i\epsilon}.$$

If we ignore the integration over $k_1, \dots, k_4$, the result is the same. The reason for that is that the moment of the four external lines is fixed, so there is no need to integrate over the possible momenta.

Now it has been covered how the path integral changes in the transition to quantum field theory and why the path integral has distinct advantages over the Schrödinger picture, which is not the case for non-relativistic quantum mechanics. It has also been covered how and why the anharmonic term leads to interactions and how the path integral is used in quantum field theory. Now we turn to the issue of a rigorous mathematical foundation of the path integral.

7 The path integral defined by oscillatory integrals

In this chapter we will discuss one approach to define the Feynman path integral rigorously, namely from the perspective of oscillatory integrals. Oscillatory integrals are a concept that originated from PDE theory. Note that we will focus on non-relativistic quantum mechanics in this chapter. The generalisation to quantum field theory using this approach is still very far away, due to all kinds of analytical problems, which will be mentioned in the discussion.

7.1 The problem

The following will all be based on [108], some of this material can also be found in [8]. We have seen in the previous two chapters that Feynman considered the path integral in his thesis [21], in the form

$$K(0,T,x,y) = \int Dq e^{\frac{i}{\hbar}S(q,\dot{q})}, \tag{101}$$

where Dq is the differential for integration over the space of continuous paths $q : [0,T] \to \mathbb{R}^n$ with fixed endpoints $q(0) = x, q(T) = y$. The symbol S is the action evaluated along the path q, which is

$$S(q) = \int_0^T L(q(t), \dot{q}(t))dt \tag{102}$$

Note that Dq is the part of equation (101) that has been giving so many problems, since it is used in physics as a Lebesgue-type measure, even though the Lebesgue measure is not defined on an infinite dimensional space such as the space of all paths [7]. We will explain what the problem exactly is. Note that we do need some kind of measure on the space to be able to consider an integral. It is reasonable to assume that the space of paths has at least some kind of structure, such as a topological structure or a metric. One possible definition would be to consider the space of paths with 'finite kinetic energy', such that $\int_0^T |\dot{\gamma}(t)|^2 dt < \infty$. This leads to the following definition:

Definition 7.1. The Cameron-Martin space is the space of all paths $\gamma : [0,T] \to \mathbb{R}^n$ such that the integral $\int_0^T |\dot{\gamma}(t)|^2 dt$ is finite. If one defines the inner product

$$\langle \gamma_1, \gamma_2 \rangle = \int_0^T \dot{\gamma}_1(t) \cdot \dot{\gamma}_2(t) dt, \tag{103}$$

the space becomes a Hilbert space, which will we denoted by $\mathcal{H}_T$. Note that we are taking the inner product of $\dot{\gamma}(t)$ instead of $\gamma(t)$ because this will become the kinetic energy part of the action in the construction.

Note the shift of notation, in earlier chapters, the symbol q was used to denote a path. This is in accordance with the physical literature. In the mathematical literature however,

the symbol γ is used for a path in this context and we will follow that convention for the mathematically oriented chapters such as this one.

Note that another option for a definition of the space of paths would be to define a Banach space on the set of all paths by defining the supremum norm on it. However, in both cases it is hopeless to define a Lebesgue measure on these spaces, since they are infinite dimensional. We have the following result

Theorem 7.2 (Impossibility of Lebesgue-type measure on an infinite dimensional Hilbert space)**.** Let $\mathcal{H}$ be a Hilbert space. There does not exist a σ-additive measure that is both translation and rotation invariant and that assigns a finite measure to bounded open sets.

Proof. Assume such a measure μ does exist. Take an orthonormal infinite set of vectors $\{e_i\}$ for $i \in \mathbb{N}$, this is always possible in an infinite dimensional Hilbert space. Now define the open balls

$$B_i := \left\{ x \in \mathcal{H} | \|x - e_i\| < \frac{1}{2} \right\}.$$

We prove that these sets are disjoint. Suppose that were not true, then there would exist $y \in B_i \cap B_j$ with $i \neq j$. That would mean that $\|x - e_i\| < \frac{1}{2}$ and $\|x - e_j\| < \frac{1}{2}$. Using the triangle inequality we obtain

$$\|e_i - e_j\| \leq \|x - e_i\| + \|x - e_j\| < 1.$$

Squaring this inequality gives that $1 > \langle e_i - e_j, e_i - e_j \rangle = \langle e_i, e_i \rangle - 2\langle e_i, e_j \rangle + \langle e_j, e_j \rangle = 2$, which is a contradiction. So the open balls are disjoint. Moreover, for any $x \in B_i$, we have $\|x\| < 2$, because

$$\|x\| \leq \|x - e_i\| + \|e_i\| < 2.$$

So the union of the open balls B_i lies in the open ball $B(2) := \{x \in \mathcal{H} |\ \|x\| < 2\}$. According to the definition of the measure μ, $B(2)$ must have finite measure, since it is open and bounded. We observe that through translation invariance, all the B_i's have the same positive measure $a > 0$ (since they are bounded open sets). However, from the σ-additivity, we have that

$$\infty > \mu(B(2)) \geq \sum_{i=1}^{\infty} \mu(B_i) = \sum_{i=1}^{\infty} a = \infty,$$

which is a contradiction. Therefore such a measure μ cannot exist. □

Note that this proof can be extended to separable infinite dimensional Banach spaces [45, Theorem 1.1]. We will work with the Cameron-Martin space since it has the advantage of a well-defined 'kinetic energy'. Now we face the issue of how to approach the problem. There exist several ways to attack it. I will briefly discuss them in the next section, before we turn to the main focus of this chapter: the oscillatory integral formalism.

7.2 Different approaches towards a rigorous construction of the Feynman path integral

There are several ways to construct a rigorous Feynman path integral. One way is to start out with a finite dimensional approximation, i.e. by performing the time-slicing rigorously, subsequently taking the limit of the interval length to zero. In many cases, one must make several mathematical assumptions for this limit to be rigorous. This is the main technique used in physics to calculate path integrals, such as done in [67, 68, 88]. More about this can be bound in [55–57, 91–93, 116, 117]. Some of these results have been extended to the Dirac equation. The construction of a rigorous path integral for the Klein-Gordon equation does not seem to be very difficult, the Klein-Gordon equation is given by

$$0 = \partial_\mu \partial^\mu \phi + m^2 \phi = \partial_t^2 - \Delta \phi + m^2 \phi.$$

This is just a wave equation with a d'Alembertian operator instead of the usual Laplacian. The solution can (most likely) be constructed in terms of an integral of the following form [153]

$$u(x,t) = \int e^{i(t\sqrt{m^2+\xi^2}+x\cdot\xi)} \psi(\xi) d\xi,$$

where $\sqrt{1+|\xi/m|^2}$ is a solution of the Klein-Gordon equation. Physically speaking, the Klein-Gordon equation corresponds to bosons. However, if one turns to the Dirac equation

$$(i\gamma^i \partial_i - m)\psi = 0,$$

which covers fermions, things become much more complicated. Then a path integral has to be constructed for a Dirac operator. A Dirac operator in general is any operator D that has the property that $D^2 = \Delta$. These operators are in general unbounded and therefore dealing with them, not to mention defining a path integral construction, is much more difficult. For a discussion of Dirac operators, one can refer to [48, 126, 145, 157, 158]. Some results using the time slicing method (and several other approaches) have been achieved, more information can be found in [80–82].

Several other approaches exist too, for example by using Wiener integrals. This means that you construct a new measure that does not have at least one of the 'Lebesgue-type' properties but that is still useful and allows you to construct a meaningful path integral. Another approach works via paths on a Riemannian manifold. More about these approaches can be found in [85, 142].

One of the methods that has been suggested to make sense of the path integral is using the technique of oscillatory integrals, which arose from the framework of linear PDE theory. In linear PDE theory one generalises the notion of a differential operator to a pseudodifferential operator and also both oscillatory and Fourier integral operators. One defines a property of a differential operator called the symbol (which will be introduced later) and a PDE problem

is translated into a problem regarding the symbol. The whole theory was inspired by the Fourier transform, which can be used to solve all kinds of PDE problems. One can use these techniques to define a rigorous path integral as an oscillatory integral, which is not that strange after a moment of thought. The Schrödinger equation after all, is a linear PDE and there can also be approached by using these methods [83, 131].

One advantage of the oscillatory integral formalism is that it is easier to handle the classical limit $\hbar \to 0$, as opposed to other rigorous approaches. For more results on this topic, see [8, Section 10.3]. Several physical systems can be covered with this approach such as the harmonic oscillator, a uniform magnetic field and also perturbations on these systems, including an anharmonic term, which is of relevance to quantum field theory.

7.3 Finite dimensional oscillatory integrals

In this section, the first steps towards the rigorous approach to the path integral using oscillatory integrals will be explained. Suppose we want to obtain an integral of a function f that oscillates rapidly or does not have sufficiently 'nice' decay behaviour and moreover is a function on an infinite dimensional Hilbert space. In order to construct a suitable integral, we proceed in three steps:

1. First we restrict to a finite dimensional subspace and consider standard Lebesgue integrals on $\mathbb{R}^n$ by multiplying the integrand by a Schwartz function that is equal to 1 at the origin and has ϵx as the function argument. The rapid decay behaviour of the Schwartz function ensures that the integral converges.

2. Then we let the factor $\epsilon \to 0$ which means that the Schwartz function will become 1 on most of the domain. If that limit converges, we will call it the (finite dimensional) oscillatory integral.

3. Finally, in order to extend the path integral to an infinite dimensional Hilbert space, we consider a sequence of projection operators that project to finite dimensional subspaces of the Hilbert space. We require that the sequence converges to the identity operator. If that limit converges, we will call it the infinite dimensional oscillatory integral and that will become the path integral.

Oscillatory integrals have the form

$$\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\phi(x)} f(x) dx, \tag{104}$$

where $\phi(x)$ is called the phase function (in most cases it is assumed to be real). For now $\hbar$ is just an arbitrary (real) constant, of course in physics it has specific value. The function that is integrated over is f, so we speak of the oscillatory integral of the function f. These

types of integrals can be generalised to infinite dimensional Hilbert spaces, in the manner that has been described above and that will be formalised below. We will first define what a finite dimensional oscillatory integral is and then start to derive some properties of this type of integral.

Definition 7.3. An oscillatory integral for a continuous function ϕ exists if for each test function $k(x) \in \mathcal{S}(\mathbb{R}^n)$ such that $k(0) = 1$ the following limit

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\phi(x)} f(x) k(\epsilon x) dx \tag{105}$$

exists and is independent of the chosen test function. In that case the limit will be called the oscillatory integral and we will denote it by

$$\widetilde{\int_{\mathbb{R}^n}} e^{\frac{i}{\hbar}\phi(x)} f(x) dx. \tag{106}$$

Let $\Sigma \subset \mathcal{S}(\mathbb{R}^n)$, we speak of an oscillatory integral in the Σ-sense, if the limit $\epsilon \to 0$ does not depend on the choice $k \in \Sigma$. We now want to get some idea of when a certain function f has an oscillatory integral or not. In order to so, we must give a preliminary definition [66].

Definition 7.4. A map $f \in C^\infty(\mathbb{R}^n, \mathbb{C})$ belongs to the space of symbols, denoted by $S^N_\lambda(\mathbb{R}^n)$, where $N, \lambda \in \mathbb{R}$ and $0 \leq \lambda \leq 1$, if for each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n_0$ the following bound holds

$$\left| \frac{d^{\alpha_1}}{dx_1^{\alpha_1}} \cdots \frac{d^{\alpha_n}}{dx_n^{\alpha_n}} f \right| \leq C_\alpha (1 + |x|)^{N - \lambda|\alpha|}, \tag{107}$$

for some constant C_α and where we have defined $|\alpha| = \sum_{i=1}^n \alpha_i$.

We then introduce the following notation

$$S^\infty_\lambda := \bigcup_N S^N_\lambda. \tag{108}$$

We will now discuss an example to illustrate this concept.

Example 7.5. The set of polynomials of degree m, i.e. the polynomials of the form $a_m x^m + \ldots + a_1 x + a_0$, belong to the space of symbols S^m_1. That means that

$$\left| \frac{d^\alpha}{dx^\alpha} f \right| \leq C_\alpha (1 + |x|)^{N - \alpha}. \tag{109}$$

This result follows easily if one realises that the α-th derivative is of the form (for $\alpha \leq m$ of course)

$$\frac{d^\alpha}{dx^\alpha} f = b_0 + b_1 x + \ldots + b_{m-\alpha} x^{m-\alpha}$$

and it can be bounded as follows

$$\left|\frac{d^\alpha}{dx^\alpha}f\right| = |b_0 + b_1 x + \ldots + b_{m-\alpha}x^{m-\alpha}| \leq |b_0| + |b_1||x| + \ldots + |b_{m-\alpha}||x^{m-\alpha}|.$$

We then use the binomial theorem to find

$$C_\alpha(1+|x|)^{N-\alpha} = C_\alpha\left(\binom{N-\alpha}{0} + \binom{N-\alpha}{1}|x| + \ldots + \binom{N-\alpha}{N-\alpha}|x|^{N-\alpha}\right).$$

Now we can make the following choice for C_α

$$C_\alpha = \max\{b_0, b_1, \ldots, b_{N-\alpha}\},$$

now since binomial coefficients are at least equal to 1, we know that the inequality (109) is satisfied. Hence the polynomials of degree m belong to the space of symbols S^m_1.

Using the space of symbols we can prove our first result.

Proposition 7.6. If $\phi \in C^2(\mathbb{R}^n)$ with finitely many stationary points, $f \in S^N_\lambda(\mathbb{R}^n)$ (with $0 < \lambda \leq 1$) and we impose the condition that $\exists k \in \mathbb{N}$ such that the ratio $\frac{|x|^{N+1}}{|\nabla\phi|^k}$ is bounded as $|x| \to \infty$, then we have that the oscillatory integral

$$\widetilde{\int} e^{\frac{i}{\hbar}\phi(x)} f(x)dx$$

exists for any $\hbar \in \mathbb{R}\backslash\{0\}$.

Proof. This proof originally appeared in [73], our treatment is based on [11]. We first label all the critical points of the function ϕ, we name them $a_1, \ldots, a_m$ such that $\nabla\chi(a_i) = 0$. We then pick a partition of unity χ_i for $i = 0, \ldots, m$, which means that $\sum_{i=0}^m \phi_i = 1$. Note that it is always possible to choose an infinite partition of unity subordinate to some open cover in the case of a smooth manifold. Moreover, the partition of unity will be locally finite, it will then be possible to take suitable infinite sums to end up with n functions which together form a partition of unity and on top of that have the property that χ_i is equal to one in an open ball centred on a_i. We then have that

$$\chi_0 = 1 - \sum_{i=1}^m \chi_i.$$

We first observe that the integrals

$$I_i(f) := \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\phi(x)}\chi_i(x)f(x)dx$$

are well-defined since $\chi_i f \in C_0^\infty(\mathbb{R}^n)$, where the subscript 0 means that the function has compact support (note that we have chosen $\chi_1, \ldots, \chi_m$ to have compact support, while χ_0 does not). In particular the means that $\chi_i f \in L^1(\mathbb{R}^n)$ and hence the integral converges. We still need to show that

$$I_0(f) \coloneqq \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\phi(x)} \chi_0(x) f(x) dx$$

can be defined as an oscillatory integral. We first introduce the operator

$$Lf(x) = -i\hbar \frac{\chi_0(x)}{|\nabla\phi(x)|^2} \nabla\phi(x) \nabla f(x), \tag{110}$$

its adjoint is given by

$$L^* f(x) = i\hbar \frac{\chi_0(x)}{|\nabla\phi(x)|^2} \nabla\phi(x) \nabla f(x) + i\hbar \nabla \cdot \left(\frac{\chi_0(x)}{|\nabla\phi(x)|^2} \nabla\phi(x) \right) f(x).$$

In order to show that this is true, we need to show that $\langle L^* g, f \rangle = \langle g, Lf \rangle$. To shorten the equations we define

$$k(x) \coloneqq i\hbar \frac{\chi_0(x)}{|\nabla\phi(x)|^2}.$$

So we have that $Lf = -k\nabla\phi \cdot \nabla f$ and $L^* f = k\nabla\phi \cdot \nabla f + \nabla \cdot (k\nabla\phi) f$. We have the following relation between gradient and the divergence in terms of inner products, if F is a vector field and f is a scalar function, we have that

$$\langle F, \nabla f \rangle_{L^2} = \langle -\nabla \cdot F, f \rangle_{L^2}.$$

Note that this is only true for a domain with no boundary, such that the surface integral vanishes. Now we need to study the inner product $\langle h, \nabla g \cdot \nabla f \rangle_{L^2}$ for some fixed function g. Since

$$\nabla f \cdot \nabla g = \nabla \cdot (f \nabla g) - f \nabla^2 g,$$

we see that (all the inner products are the L^2 inner product)

$$\begin{aligned}
\langle g, Lf \rangle &= \langle g, -k\nabla\phi \cdot \nabla f \rangle = -\langle kg, \nabla\phi \cdot \nabla f \rangle = \langle -kg, \nabla \cdot (f\nabla\phi) - f\nabla^2\phi \rangle \\
&= \langle k\nabla^2\phi \, g, f \rangle - \langle kg, \nabla \cdot (f\nabla\phi) \rangle = \langle k\nabla^2\phi \, g, f \rangle + \langle \nabla(kg), f\nabla\phi \rangle \\
&= \langle k\nabla^2\phi \, g, f \rangle + \langle \nabla(kg) \cdot \nabla\phi, f \rangle = \langle k\nabla^2\phi + \nabla(kg) \cdot \nabla\phi, f \rangle \\
&= \langle (k\nabla^2\phi + \nabla k \cdot \nabla\phi) g + k\nabla g \cdot \nabla\phi, f \rangle = \langle (\nabla \cdot (k\nabla\phi)) g + k\nabla g \cdot \nabla\phi, f \rangle.
\end{aligned}$$

This proves that (110) is correct. The domain of L is given by

$$D(L) = \{ f \in L^2(\mathbb{R}^n) | Lf \in L^2(\mathbb{R}^n) \}$$

and similarly for $D(L^*)$. We need to show that under the hypotheses of the proposition, f belongs to $D(L^*)$. Now it becomes clear why we imposed the condition that f belongs to the space of symbols and also that the ratio $\frac{|x|^{N+1}}{|\nabla\phi|^k}$ is bounded as $|x| \to \infty$ for a natural number k. We need to bound the expression

$$i\hbar \frac{\chi_0(x)}{|\nabla\phi(x)|^2}\nabla\phi(x)\nabla f(x) + i\hbar\nabla\cdot\left(\frac{\chi_0(x)}{|\nabla\phi(x)|^2}\nabla\phi(x)\right)f(x).$$

The first term can be bounded by noting that χ_0 is part of the partition of unity and can hence be bounded by a constant. The gradient $\nabla\phi$ is nowhere zero when χ_0 is nonzero. This is the reason for assuming only a finite number of critical points: we can consider them separately using a partition of unity. Then the integral always converges since the integration domain is a compact interval and we no longer have to worry about stationary points for the remaining tails of the integral with an unbounded domain.

Moreover, the term $\frac{\nabla\phi}{|\nabla\phi|^2}$ can be bounded by a polynomial due to the assumption that was mentioned earlier. The reader might wonder why a polynomial bound is sufficiently strong given that Lebesgue integrals of polynomials with $\mathbb{R}$ as domain diverge. The resolution to the issue lies in the multiplication with the Schwartz function, which decreases superpolynomially. The reader is referred to appendix A for further details.

Now we have seen that if f belongs to the space of symbols, then Lf can be bounded by a polynomial. That means that L^*f can also be bounded by a polynomial, otherwise we would arrive at a contradiction. We can then rewrite the oscillatory integral $I_0(f)$ as follows (where $\varphi \in \mathcal{S}(\mathbb{R}^n)$)

$$\begin{aligned} I_0(f) &= \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\phi(x)}\chi_0(x)f(x)\varphi(\epsilon x)dx = \int_{\mathbb{R}^n} L\left(e^{\frac{i}{\hbar}\phi(x)}\right)f(x)\varphi(\epsilon x)dx \\ &= \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\phi(x)}L^*\left(f(x)\varphi(\epsilon x)\right)dx = \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\phi(x)}\sum_{l=0}^{m}\chi_l L^*\left(f(x)\varphi(\epsilon x)\right)dx, \end{aligned}$$

we observe again that for $\chi_1, \ldots, \chi_m$ the integral converges. For χ_0 we again apply the same trick as above. Then we see that the convergence of the integral I_0 is dependent on the convergence of the integral

$$\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\phi(x)}(L^*)^k\left(f(x)\varphi(\epsilon x)\right)dx.$$

If k is large enough the integral is absolutely convergent, because each time the operator L^* is applied the rate of decrease is faster. Hence we conclude by the dominated convergence theorem that we can move the limit $\lim_{\epsilon\to 0}$ inside the integral and obtain that I_0 converges to a well-defined oscillatory integral, since the result is independent of the choice of Schwartz function. That means that we can also conclude that the sum $I(f) = \sum_{l=1}^{m} I_m(f)$ exists as a well-defined oscillatory integral. This proves the theorem. □

Now we have seen how the concept of an oscillatory integral is related to the notion of a symbol, we can turn to a particular kind of oscillatory integral, namely the Fresnel integral. The Fresnel integral is an oscillatory integral with a quadratic phase function, we choose that kind of phase function because it translates into an inner product

7.4 Fresnel integrals

We have in particular shown in proposition 7.6 the following: if ϕ is a polynomial and $f \in S^{\infty}_{\lambda}$, with $0 < \lambda \leq 1$, then the oscillatory integral converges. This is the case because we know that for a polynomial phase function the ratio

$$\frac{|x|^{N+1}}{|\nabla\phi|^k} = \frac{|x|^{N+1}}{|Mx^{M-1}|^k}$$

is bounded (for some natural number k) as $|x| \to \infty$ under the condition that $M > 1$. If we choose $\phi(x) = x^M$ then we know that the oscillatory integral

$$\widetilde{\int_{\mathbb{R}}} e^{ix^M} f(x)dx$$

exists. Two cases have been given names, the integrals of the form

$$\widetilde{\int_{\mathbb{R}}} e^{ix^2} f(x)dx,$$

i.e. those with $M = 2$ are called Fresnel integrals. Those which have $M = 3$ are called Airy integrals. This section will focus on Fresnel integrals. Those integrals are said to have quadratic phase function (i.e. $\phi(x) = x^2$). We choose the quadratic case because it will provide us with the kinetic energy part of the action.

In this section we will establish a Parseval-type equality. This will not only have the advantage that the oscillatory integral can be computed explicitly, it will also allow us to given meaning to expressions in the infinite dimensional setting when the oscillatory integral on its own does not have an intrinsic meaning, so instead we take the limit of the dimension going to infinity (this will be made rigorous later on).

We will provide a formal definition of the Fresnel integral:

Definition 7.7. A function $g : \mathbb{R}^n \to \mathbb{C}$ is called Fresnel integrable if the limit

$$\lim_{\epsilon \to 0} \frac{1}{(2\pi i\hbar)^{n/2}} \int e^{\frac{i}{2\hbar}\langle x,x\rangle} g(x)\phi(\epsilon x)dx \tag{111}$$

exists and is independent of the test function $\phi(x) \in \mathcal{S}(\mathbb{R}^n)$ which has the property $\phi(0) = 1$. The Fresnel integral is denoted by

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle} g(x)dx. \tag{112}$$

The observant reader will notice that the only difference with the previous definition of the Fresnel integral (where we set $\phi(x) = x^2$ and $\hbar = 1$ in the definition of the oscillatory integral) is the appearance of the normalisation factor $(2\pi i\hbar)^{-n/2}$. The reason $g(x) = 1$ is because we get the following integral

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle} dx.$$

If we take the Schwartz function $\phi(x) = e^{-x^2}$ we find

$$\int e^{\frac{i}{2\hbar}x^2} e^{-\epsilon^2 x^2} dx = \sqrt{\frac{\pi}{\epsilon^2 - \frac{i}{2\hbar}}},$$

taking the limit $\epsilon \to 0$ gives us (where we ignore the normalisation constant)

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle} dx = \lim_{\epsilon\to 0} \int e^{\frac{i}{2\hbar}x^2} e^{-\epsilon^2 x^2} dx = \lim_{\epsilon\to 0} \sqrt{\frac{\pi}{\epsilon^2 - \frac{i}{2\hbar}}} = \sqrt{\frac{\pi}{-\frac{i}{2\hbar}}} = \sqrt{2\pi i\hbar}.$$

The power n of the square root appears because if we have an n-dimensional space we can split the multi dimensional integral into n one dimensional integrals of the above form. Hence if we introduce the normalisation factor $(2\pi i\hbar)^{-n/2}$ we get the following result

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle} dx = 1.$$

We will now show that the Fresnel algebra $\mathcal{F}(\mathcal{H})$ is contained in the space of function that have an oscillatory integral. We will also prove a Parseval-type equality that will be extended to the infinite dimensional case and will be applied to the harmonic oscillator in the next chapter.

Theorem 7.8 (Parseval-type equality for finite dimensional Fresnel integral)**.** Let $\mathcal{H}$ be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$. We define the function g to be

$$g(x) = e^{-\frac{i}{2\hbar}\langle x,Lx\rangle} f(x),$$

where $f(x)$ lies in the Fresnel algebra $\mathcal{F}(\mathcal{H})$ and L is a self-adjoint (linear) operator such that $I - L$ is invertible. Then g is Fresnel integrable and its Fresnel integral is given by the following Parseval-type identity

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,(I-L)x\rangle} f(x)dx = e^{-\frac{\pi i}{2}\operatorname{Ind}(I-L)} |\det(I-L)|^{-1/2} \int_{\mathbb{R}^n} e^{-\frac{i}{2\hbar}\langle x,(I-L)^{-1}x\rangle} f(x) d\hat{\mu}_f(x), \quad (113)$$

where $\operatorname{Ind}(I - L)$ is the index of the operator $I - L$, which is the number of its negative eigenvalues. Note that this notion does not have any problems at this point since we are working in a finite dimensional Hilbert space (or $\mathbb{R}^n$). Here $\hat{\mu}$ is the measure associated with the function f through the Fourier transform as given in equation (195) in appendix A.

Proof. In order to show the Parseval-type equality and the Fresnel integrability, we have to consider the following integral

$$(2\pi i\hbar)^{-n/2}\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x,(I-L)x\rangle} f(x)\phi(\epsilon x)dx, \tag{114}$$

where $\phi \in \mathcal{S}(\mathbb{R}^n)$. Note that the term $\langle x,(I-L)x\rangle$ in the exponent comes from the multiplication of the function $g(x)$ with the phase factor as can be found in equation (111), then we get $\langle x,x\rangle - \langle x,Lx\rangle = \langle x,Ix\rangle - \langle x,Lx\rangle = \langle x,(I-L)x\rangle$. We will split this proof up in several lemmas to make it more manageable. We will proceed in the following steps:

1. First we will show that the expression (114) can be transformed into

$$(2\pi i\hbar)^{-n/2}\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x,(I-L)x\rangle} f(x)h(x)dx = e^{-\frac{\pi i}{2}\operatorname{Ind}(I-L)}|\det(I-L)|^{-1/2} \tag{115}$$

$$\cdot\int_{\mathbb{R}^n\times\mathbb{R}^n} e^{-\frac{i\hbar}{2}\langle x,(I-L)^{-1}x\rangle}\tilde{h}(x-y)d\hat{\mu}(y)dx. \tag{116}$$

2. This expression will subsequently be used to show that equation (115) equals

$$e^{-\frac{\pi i}{2}\operatorname{Ind}(I-L)}|\det(I-L)|^{-1/2}\int_{\mathbb{R}^n\times\mathbb{R}^n} e^{-\frac{i}{2\hbar}\langle y+\epsilon x,(I-L)^{-1}y+\epsilon x\rangle}\tilde{\phi}(x)d\hat{\mu}(y)dx \tag{117}$$

   Here $\widetilde{\phi}$ denotes the test function after a change of variable (multiplied by a factor), the specifics will become clear in a moment.

3. Finally, we use the expression above to show that the function g is Fresnel integrable by taking the limit $\epsilon \to 0$. Taking this limit will also yield the Parseval-type equality.

□

Now we will set out to the proof of the claims:

Lemma 7.9. Claim 1 is true.

Proof. We first consider the case $f = 1$, we have to show that

$$(2\pi i\hbar)^{-n/2}\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x,(I-L)x\rangle}h(x)dx = e^{-\frac{\pi i}{2}\operatorname{Ind}(I-L)}|\det(I-L)|^{-1/2}\int_{\mathbb{R}^n} e^{-\frac{i\hbar}{2}\langle x,(I-L)^{-1}x\rangle}\tilde{h}(x)dx. \tag{118}$$

In order to prove this special case, we once again split the proof in three subcases:

1. The operator $I-L$ is positive definite.
2. The operator $I-L$ is negative definite.

3. All the other cases.

First we consider the case that $I-L$ is positive definite, hence the indicator is 0 (since those operators only have positive eigenvalues) and the term $e^{-\frac{\pi i}{2}\operatorname{Ind}(I-L)}$ equals 1. Since $I-L$ is positive definite, we can use the Cholesky decomposition. This means that we can write $I-L=A^*A$ and $(I-L)^{-1}=A^{-1}(A^{-1})^*$ for an invertible operator A. That means that we can write the inner products as

$$\langle x,(I-L)x\rangle=\langle Ax,Ax\rangle,\quad \langle x,(I-L)^{-1}x\rangle=\langle (A^{-1})^*x,(A^{-1})^*x\rangle.$$

Now we intend to rewrite the Parseval-type equality (118) for the case $f=1$ to the simplest possible case, we get that

$$\begin{aligned}
(2\pi i\hbar)^{-n/2}\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x,x\rangle}h(x)dx &= (2\pi i\hbar)^{-n/2}\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle Ax,Ax\rangle}h(x)dx\\
&= (2\pi i\hbar)^{-n/2}\det\big(A^{-1}\big)\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x,x\rangle}h(x)dx\\
(\det(I-L))^{-1/2}\int_{\mathbb{R}^n} e^{-\frac{i\hbar}{2}\langle x,(I-L)^{-1}x\rangle}\tilde{h}(x)dx &= (\det\big((I-L)^{-1}\big))^{1/2}\int_{\mathbb{R}^n} e^{-\frac{i\hbar}{2}\langle (A^{-1})^*x,(A^{-1})^*x\rangle}\tilde{h}(x)dx\\
&= (\det\big((A^{-1})^*\big))\int_{\mathbb{R}^n} e^{-\frac{i\hbar}{2}\langle (A^{-1})^*x,(A^{-1})^*x\rangle}\tilde{h}(x)dx\\
&= \int_{\mathbb{R}^n} e^{-\frac{i\hbar}{2}\langle x,x\rangle}\tilde{h}(x)dx.
\end{aligned}$$

Note that instead of Cholesky decomposition, it is also possible to use the square root of the operator, since it is positive definite. In the first equality we removed the absolute value brackets around the determinant since $I-L$ is positive definite. In the second-to-last equality we redefined the functional argument of $\tilde{h}$, where we used that A^{-1} is invertible. So by absorbing the factor $\det(A)$ into $\tilde{h}$, we obtain the equation

$$(2\pi i\hbar)^{-n/2}\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x,x\rangle}h(x)dx=\int_{\mathbb{R}^n} e^{-\frac{i\hbar}{2}\langle x,x\rangle}\tilde{h}(x)dx. \tag{119}$$

By the change of variable $x\to x'=\frac{x}{\sqrt{i\hbar}}$, one obtains the right-hand side from the left-hand side. Note that the factor $(2\pi)^{-n/2}$ is absorbed by the function $\tilde{h}$, which also has undergone a change of variable. Note that moving these factors inside $\tilde{h}$ does not affect the proof, because $\tilde{h}$ remains a Schwartz function and the precise form of the Schwartz function is irrelevant because it will disappear in the limit $\epsilon\to 0$. Alternatively, equation (119) can be proven by using the Fourier transform

$$(2\pi i\hbar)^{n/2}\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x,x\rangle}e^{i\langle x,y\rangle}dx=e^{-\frac{i\hbar}{2}\langle y,y\rangle},$$

and by using the Plancherel theorem for functions f and h

$$\int_{\mathbb{R}^n} \mathcal{F}(f)\mathcal{F}(h)dx = \int_{\mathbb{R}^n} fhdx,$$

where $\mathcal{F}$ denotes the Fourier transform. If one uses this approach, then $\tilde{h}$ is the Fourier transform of h.

The second case ($I-L$ is negative definite) is easy, one replaces $\hbar$ by $-\hbar$ and realises that now $\mathrm{Ind}(I-L) = n$. The operator $I-L$ is negative definite, the operator $-(I-L)$ is hence positive definite. For that type of operator, we have established in the previous subcase that

$$(2\pi i\hbar)^{-n/2}\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x,-(I-L)x\rangle}h(x)dx = |\det(I-L)|^{-1/2}\int_{\mathbb{R}^n} e^{-\frac{i\hbar}{2}\langle x,(-I+L)^{-1}x\rangle}\tilde{h}(x)dx.$$

Performing the replacement $\hbar \to -\hbar$ we get that

$$(2\pi i\hbar)^{-n/2}\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x,(I-L)x\rangle}h(x)dx = e^{-\frac{\pi i n}{2}}|\det(I-L)|^{-1/2}\int_{\mathbb{R}^n} e^{-\frac{i\hbar}{2}\langle x,(I-L)^{-1}x\rangle}\tilde{h}(x)dx.$$

This is exactly what we had to show, namely equation (118).

Now we have to consider all the other cases. First we remark that $I-L$ does only have nonzero eigenvalues, since it is invertible. We decompose $\mathbb{R}^n$ into the positive and negative eigenspaces as follows

$$\mathbb{R}^n = E_+ \times E_-.$$

Similarly, we decompose $I-L$ into positive definite and negative definite parts

$$I-L = (I-L)_+ \times (I-L)_-.$$

Where we define the maps $(I-L)_+ : E_+ \to E_+$ and $(I-L)_- : E_- \to E_-$. We then consider Schwartz functions h that lie in $\mathcal{S}(E_+)\times\mathcal{S}(E_-)$. We then obtain the following

$$\begin{aligned}
&(2\pi i\hbar)^{-n/2}\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x,(I-L)x\rangle}h(x)dx \\
&= (2\pi i\hbar)^{-n/2}\left(\int_{E_+} e^{\frac{i}{2\hbar}\langle x_+,(I-L)_+x_+\rangle}h_+(x_+)dx_+\right)\cdot\left(\int_{E_-} e^{\frac{i}{2\hbar}\langle x_-,(I-L)_-x_-\rangle}h_-(x_-)dx_-\right) \\
&= \left(|\det(I-L)_+|^{-1/2}\int_{\mathbb{R}^n} e^{-\frac{i\hbar}{2}\langle x_+,(I-L)_+^{-1}x_+\rangle}\tilde{h}_+(x_+)dx_+\right) \\
&\cdot e^{-\frac{i\pi\dim(E_-)}{2}}\left(|\det(I-L)_-|^{-1/2}\int_{\mathbb{R}^n} e^{-\frac{i\hbar}{2}\langle x_-,(I-L)_-^{-1}x_-\rangle}\tilde{h}_-(x_-)dx_-\right) \\
&= e^{-\frac{\pi i}{2}\mathrm{Ind}(I-L)}|\det(I-L)|^{-1/2}\int_{\mathbb{R}^n} e^{-\frac{i\hbar}{2}\langle x,(I-L)^{-1}x\rangle}\tilde{h}(x)dx.
\end{aligned}$$

We can easily generalise to general Schwartz functions $h \in \mathcal{S}(\mathbb{R}^n)$ by observing that $\mathcal{S}(E_+) \times \mathcal{S}(E_-)$ is dense in $\mathcal{S}(R_n)$. We can prove the result by using the dominated convergence theorem in taking the limit $n \to \infty$. This concludes the proof for the case $f = 1$.

For the general case where $f \in \mathcal{F}(\mathbb{R}^n)$, we first replace the operator $I - L$ with I_{rs}, which is the operator with the eigenvalue -1 with multiplicity r and the eigenvalue 1 with multiplicity s. We will first prove the statement for the operator I_{rs} and then scale the operator to $I - L$. Note that we assume that $f \in \mathcal{F}(R^n)$ and $h \in \mathcal{S}(\mathbb{R}^n)$. We have that by using equation (195) that

$$(2\pi i\hbar)^{-n/2} \int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x, I_{rs}x\rangle} f(x)h(x)dx = (2\pi i\hbar)^{-n/2} \int_{\mathbb{R}^n} d\mu_f(y) \int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x, I_{rs}x\rangle} e^{\langle y,x\rangle} h(x)dx.$$

If we then introduce the change of variable $x \to x + I_{rs}\hbar y$, then we get

$$(2\pi i\hbar)^{-n/2} \int_{\mathbb{R}^n} d\mu_f(y) \int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x, I_{rs}x\rangle} e^{\langle y,x\rangle} h(x)dx = (2\pi i\hbar)^{-n/2} \int_{\mathbb{R}^n} d\mu_f(y) \int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x, I_{rs}x\rangle} e^{\langle y,x\rangle} h(x)dx$$

□

Lemma 7.10. Claim 2 is true, namely equation (117) follows from equation (115).

Proof. We start out with equation (115) (as was proven in the previous lemma), replacing $h(x)$ by $\phi(\epsilon x)$ in this equation gives us

$$(2\pi i\hbar)^{-n/2} \int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x,(I-L)x\rangle} f(x)\phi(\epsilon x)dx = e^{-\frac{\pi i}{2}\operatorname{Ind}(I-L)} |\det(I-L)|^{-1/2}$$
$$\int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-\frac{i}{2\hbar}\langle x,(I-L)^{-1}x\rangle} \tilde{\phi}(\epsilon x - y) d\hat{\mu}(y)dx.$$

Subsequently we make the change of variable $y \to \epsilon y$, hence the measure becomes $d\hat{\mu}(y) \to \epsilon^{-n} d\hat{\mu}(y)$ (the power n arises from the fact that y is being integrated over n-dimensional space $\mathbb{R}^n$). Then we get that the function $\tilde{\phi}$ has the argument $\epsilon(x-y)$, we then redefine the function to be an argument of $\frac{x-y}{\epsilon}$ and end up with

$$(2\pi i\hbar)^{-n/2} \int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x,(I-L)x\rangle} f(x)\phi(\epsilon x)dx = e^{-\frac{\pi i}{2}\operatorname{Ind}(I-L)} |\det(I-L)|^{-1/2}$$
$$\int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-\frac{i}{2\hbar}\langle x,(I-L)^{-1}x\rangle} \tilde{\phi}\left(\frac{x-y}{\epsilon}\right) \epsilon^{-n} d\hat{\mu}(y)dx.$$

Finally we make the other change of variable $x \to x' = \epsilon^{-1}(x-y)$ (note that y is regarded as a constant with respect to integration over x), we then get that $dx' = \epsilon^n dx$. Using that we obtain

$$e^{-\frac{\pi i}{2}\operatorname{Ind}(I-L)} |\det(I-L)|^{-1/2} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-\frac{i}{2\hbar}\langle x,(I-L)^{-1}x\rangle} \tilde{\phi}\left(\frac{x-y}{\epsilon}\right) \epsilon^{-n} d\hat{\mu}(y)dx$$
$$= e^{-\frac{\pi i}{2}\operatorname{Ind}(I-L)} |\det(I-L)|^{-1/2} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-\frac{i}{2\hbar}\langle \epsilon x+y,(I-L)^{-1}(\epsilon x+y)\rangle} \tilde{\phi}(x)\, d\hat{\mu}(y)dx.$$

This equation is exactly the same as equation (117), which is what we had to show. □

Lemma 7.11. Claim 3 is true, which means that it follows from equation (117) that the function g is Fresnel integrable and on top of that we obtain the Parseval-type equality (113) from the earlier equation.

Proof. If we take the limit $\epsilon \to 0$ in equation (115) we get

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,(I-L)x\rangle} f(x)dx = e^{-\frac{\pi i}{2}\operatorname{Ind}(I-L)}|\det(I-L)|^{-1/2}\int_{\mathbb{R}^n\times\mathbb{R}^n} e^{-\frac{i}{2\hbar}\langle y,(I-L)^{-1}y\rangle}\tilde{\phi}\,(x)\,d\hat{\mu}(y)dx.$$

We are justified in taking the limit since the multiplication by the Schwartz function $\tilde{\phi}$ makes the integral bounded and we can hence utilise the dominated convergence theorem. Then we remark that $\int_{\mathbb{R}^n}\tilde{\phi}(x)dx = 1$ and using that we get the Parseval-type equality (113). This also show that the function g is Fresnel integrable, since the limit converges regardless of the test function being used. □

Remark 7.12. It might be superfluous to note that the zero operator ($\mathbf{0}(w) = 0$ for all $w \in \mathcal{H}$) is self-adjoint. When we apply this in the theorem above we have that $g(x) = f(x)$ and hence if $f(x) \in \mathcal{F}(\mathcal{H})$ it is Fresnel integrable. The theorem is a more general result but does include what we initially had set out to prove.

To conclude, in this section I have introduced the Fresnel integral, which is a specific type of finite dimensional oscillatory integral. I have shown how the Fresnel integral follows from the more general case and also proven a theorem on when a function is Fresnel integrable and we also established a Parseval-type equality for the Fresnel integral. We will now generalise the notion of the Fresnel integral, but before we do so I would like to mention a couple of things about the structure of the solution space I_F, i.e. the functions that are Fresnel integrable.

7.5 Structure of the solution space

The notion of an oscillatory integral is very appealing, but it has one big disadvantage, namely that is hard to tell whether a function is oscillatory integrable. In this section I would like to prove a few results of my own, but that could well be in the literature. After all, most of the attention in the literature on oscillatory integrals related to path integrals has been directed at proving that certain functions are Fresnel integrable (or not) and obtain Parseval-type formulae.

There are many functions which do not belong to the Fresnel algebra but that still are oscillatory integrable. Proposition 7.6 gives an indication of that. For example the polynomials belong to the space of symbols, as was proven in example 7.5, they certainly do

not belong to the Fresnel algebra. So it could be helpful for the oscillatory integral formalism to extend some of the results to functions not in the Fresnel algebra.

While the Fresnel algebra is constructed bottom-up, in this section we will try a top-down approach. Some properties of the set of functions which have an oscillatory integral will be proven. These results might be a first step towards giving a characterisation of the space of those functions, which is still an open problem (even if $\dim(\mathcal{H}) < \infty$). This section is not necessary for the rest of the thesis, so it can be skipped by the reader who is not interested.

First I will introduce notation for the set of functions that have an oscillatory integral.

Definition 7.13. The set of functions on a Hilbert space $\mathcal{H}$ that have an oscillatory integral with phase function ϕ is denoted by $I_\phi(\mathcal{H})$. In the Fresnel case, i.e. the phase function being quadratic, we write $I_F(\mathcal{H})$. We will often omit the $\mathcal{H}$ and just write I_F, if the Hilbert space is clear from the context.

The first thing that can be easily seen is that $I_\phi(\mathcal{H})$ is a vector space. We will just prove it for the Fresnel case, the general proof is completely analogous.

Proposition 7.14. The set I_F conceived as a subspace of the space of functions on $\mathcal{H}$ is a vector space, it inherits the (pointwise) addition and scalar multiplication of functions from the larger function space. The set I_F is a vector space under these operations, especially, it is algebraically closed under addition.

Proof. The fact that the axioms of the vector space are satisfied is quite straightforward to show. As was said earlier, it follows from the fact that the space of functions is a vector space. It only needs to be shown that I_F is a linear subspace of the space of functions. Obviously the zero function has an oscillatory integral. I will now show that $I_F(\mathcal{H})$ is closed under scalar multiplication and vector addition. Suppose we have that $f(x), g(x) \in I_F(\mathcal{H}$ and $a, b \in \mathbb{R}$ (the complex case is completely analogous), we then know that the limits

$$\lim_{\epsilon\to 0}(2\pi i\hbar)^{-n/2}\int e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)\phi(\epsilon x)dx$$
$$\lim_{\epsilon\to 0}(2\pi i\hbar)^{-n/2}\int e^{\frac{i}{2\hbar}\langle x,x\rangle} g(x)\phi(\epsilon x)dx$$

converge and are independent of the chosen test function $\phi \in \mathcal{S}(\mathbb{R}^n)$. But then it follows that $af(x) + bg(x) \in I_F(\mathcal{H})$, since

$$\lim_{\epsilon\to 0}(2\pi i\hbar)^{-n/2}\int e^{\frac{i}{2\hbar}\langle x,x\rangle}(af(x)+bg(x))\phi(\epsilon x)dx = a\lim_{\epsilon\to 0}(2\pi i\hbar)^{-n/2}\int e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)\phi(\epsilon x)dx$$
$$+ b\lim_{\epsilon\to 0}(2\pi i\hbar)^{-n/2}\int e^{\frac{i}{2\hbar}\langle x,x\rangle} g(x)\phi(\epsilon x)dx.$$

This can also be stated as

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle}(af(x)+bg(x))dx = a\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle}f(x)dx + b\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle}g(x)dx. \tag{120}$$

Hence we conclude that I_F is a vector space, since it is closed under vector addition and scalar multiplication. □

We first observe that several important classes of functions belongs to I_F.

Example 7.15 (Polynomials)**.** In example 7.5 it was shown that all polynomials of degree m belong to the space of symbols S_1^m. Then we can use proposition 7.6 to conclude that the polynomials are oscillatory integrable if the phase function is a polynomial. In particular, the polynomials belong to I_F. This includes all the Taylor polynomials.

Example 7.16 (L^1 space)**.** We will now show that functions of L^1 are included in I_ϕ, for any real phase function. Suppose $f \in L^1$, then

$$\int_{\mathbb{R}^n} |f| d\mu < \infty,$$

where we focus on the $\mathbb{R}^n$ case for now (since finite dimensional Hilbert spaces are isomorphic to $\mathbb{R}^n$). We can hence establish the following inequalities

$$\left|\int e^{\frac{i}{2\hbar}\varphi(x)}g(x)\phi(\epsilon x)dx\right| \leq \int \left|e^{\frac{i}{2\hbar}\varphi(x)}g(x)\phi(\epsilon x)\right| dx = \int |g(x)\phi(\epsilon x)|\, dx \leq C\int |g(x)|dx < \infty.$$

Note that it was used that a Schwartz function can always be bounded by some constant C over all of $\mathbb{R}^n$. We thus obtain the inequality

$$\lim_{\epsilon\to 0}(2\pi i\hbar)^{-n/2}\int e^{\frac{i}{2\hbar}\langle x,x\rangle}g(x)\phi(\epsilon x)dx \leq C(2\pi i\hbar)^{-n/2}\int |g(x)|dx < \infty.$$

This inequality means that by the bounded convergence theorem we can take the limit $\epsilon \to 0$ inside the integral and obtain the oscillatory integral

$$\begin{aligned}\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle}g(x)dx &= \lim_{\epsilon\to 0}(2\pi i\hbar)^{-n/2}\int e^{\frac{i}{2\hbar}\langle x,x\rangle}g(x)\phi(\epsilon x)dx = (2\pi i\hbar)^{-n/2}\int e^{\frac{i}{2\hbar}\langle x,x\rangle}g(x)\lim_{\epsilon\to 0}\phi(\epsilon x)dx \\ &= (2\pi i\hbar)^{-n/2}\int e^{\frac{i}{2\hbar}\langle x,x\rangle}g(x)dx.\end{aligned}$$

Hence the oscillatory integral exists and is bounded by $C(2\pi i\hbar)^{-n/2}\|f\|_{L^1}$ where C is any positive constant that bounds the Schwartz function $|\phi(x)|$ everywhere.

Corollary 7.17. A result from measure theory tells us that the continuous functions with compact support are dense in L^1 [34], that means that it follows from the example that any continuous function with compact support has an oscillatory integral.

We can use some of these results to prove a new theorem regarding which functions have an oscillatory integral.

Theorem 7.18. Let P be a polynomial and $f : \mathbb{R} \to \mathbb{R}$ be a C^∞ function, such that $\exists l \in \mathbb{N}_0$ for which $f^{(l)} \in S^N_\lambda(\mathbb{R})$, then $f \in I_P(\mathbb{R})$.

Proof. The function f can be written as a Taylor (Maclaurin) series of the form

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)x^i}{i!},$$

because the function is C^∞. The Taylor series is approximated by partial sums, which are known as Taylor polynomials. They are given by (we simply truncate the series)

$$f_n(x) := \sum_{i=0}^{n} \frac{f^{(i)}(0)x^i}{i!}.$$

In example 7.15 we showed that for all polynomials an oscillatory integral exists (under some conditions), in particular that means that for all Taylor polynomials an oscillatory integral exists. We are now going to prove that the Taylor series and hence the function f have a well-defined oscillatory integral under the conditions given in the theorem as follows: We will prove that the remainder is oscillatory integrable and then recall the vector space structure of I_F and hence conclude that f has an oscillatory integral. Recall that the remainder is defined by

$$R_n(x) := f(x) - f_n(x).$$

Lagrange's remainder theorem gives the following explicit form for the remainder

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1},$$

where $\xi \in [0, x]$. But now the reader should realise that actually ξ is a function of x, so we write $\xi(x)$, which means that R_n is only a function of x and not ξ. We now recall the key assumption in the theorem: a derivative of f of order l belongs to the space of symbols $S^N_\lambda(\mathbb{R})$. Now we choose $n + 1 = l$. Proposition 7.6 then tells us that R_n is oscillatory integrable. Now by the vector space structure of I_F (theorem 7.14) we see that $f(x) = f_n(x) + R_n(x)$ is oscillatory integrable, which is what we had to show. □

Remark 7.19. The condition that the function f is C^∞ is more stringent than it needs to be, to invoke the Lagrange remainder theorem we only need that the function is C^{n+1} (therefore $l \leq n+1$). In that case we are not using the space of symbols but we just bound the integral and use the dominated convergence theorem. If the derivative is bounded by some polynomial q, we obtain the inequality

$$\int e^{\frac{i}{2\hbar}P(x)} R_n(x)\phi(\epsilon x)dx \leq \int e^{\frac{i}{2\hbar}P(x)} q(x)\phi(\epsilon x)dx \leq \widetilde{\int} e^{\frac{i}{2\hbar}P(x)} q(x)dx.$$

Using the dominated convergence theorem we can move the limit $\epsilon \to 0$ inside the integral and obtain the oscillatory. The reader should realise that any function f which can be decomposed as a sum of two functions g and h where h has compact support and g has some bounded derivative hence has an oscillatory integral because of the vector space structure.

Corollary 7.20. It can easily be seen from theorem 7.18 that the sine and cosine functions do have an oscillatory integral and hence we know that partial sums of Fourier series have an oscillatory integral as well. In general any continuous periodic function has an oscillatory integral, because a continuous function maps compact intervals to compact intervals. Hence the function itself is bounded and by theorem 7.18 and remark 7.19 we know that such a function has an oscillatory integral.

Now that we know that I_F is a vector space, we can see that it has a seminorm, the natural choice for a seminorm is of course the absolute value of the Fresnel integral.

Theorem 7.21. The space I_F has a seminorm defined as follows

$$\|f(x)\|_F = \left|\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)dx\right|. \tag{121}$$

This norm will be called the Fresnel seminorm (which is denoted by the subscript F), as opposed to for example the L^p norm.

Proof. I must show that two properties are satisfied in order for $\|\cdot\|_F$ to be a seminorm. The first obvious remark is that the Fresnel norm is positive definite, i.e. $\|f(x)\|_F \geq 0$ for any $f(x) \in I_F\mathcal{H}$.
The property $\|\lambda f(x)\|_F = |\lambda|\|f(x)\|_F$ follows easily because we have that

$$\begin{aligned}\|\lambda f(x)\|_F &= \left|\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle}\lambda f(x)dx\right| = \left|\lim_{\epsilon\to 0}(2\pi i\hbar)^{-n/2}\int e^{\frac{i}{2\hbar}\langle x,x\rangle}\lambda f(x)\phi(\epsilon x)dx\right| \\ &= |\lambda|\left|\lim_{\epsilon\to 0}(2\pi i\hbar)^{-n/2}\int e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)\phi(\epsilon x)dx\right| = |\lambda|\left|\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)dx\right| = |\lambda|\|f(x)\|_F.\end{aligned}$$

The proof of the triangle inequality also follows quite straightforwardly (we can use the result that the solution space I_F is a vector space), namely by taking the absolute value inside the integral and using that the triangle inequality holds for the real numbers

$$\begin{aligned}\|f(x)+g(x)\|_F &= \left|\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle}(f(x)+g(x))dx\right| = \left|\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle}f(x)dx + \widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle}g(x)dx\right| \\ &\leq \left|\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle}f(x)dx\right| + \left|\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle}g(x)dx\right| = \|f(x)\|_F + \|g(x)\|_F.\end{aligned}$$

So we can already conclude that the map given in equation (121) is a seminorm. □

I would like to end this section with a few remarks. There is still an important property that needs to be proven, namely that $\|f\|_F = 0$ if and only if $f = 0$. If that is the case, the Fresnel seminorm becomes a norm and I_F is a normed space. The subsequent aim is to prove that the space is Banach, i.e. that all Cauchy sequences converge. Proving that sequences converge is rather trivial, since we are considering sequences in $\mathbb{R}$. What is however more difficult to show is the closedness: i.e. there is a function f that the sequence $\{f_n\}$ converges to.

Another topic that should be looked at is whether the possible norm comes from an inner product. In that case the space is a Hilbert space, but this needs to be looked at in more detail. Moreover, the results could possibly be extended to complex exponentials of functions, since that is what we are interested in from the path integral perspective. Finally, we need to look at a suitable treatment of singularities. A possibility would be to omit a small interval $(p-\epsilon, p+\epsilon)$ from the integration domain and take the limit $\epsilon \to 0+$. However, a lot of properties still need to be proven of this idea. In the next section we generalise to oscillatory integrals with an even polynomial phase function.

7.6 Generalised Fresnel integrals

Now that Fresnel integrals have been treated, we can try to extend this notion without immediately considering all finite dimensional oscillatory integrals in the following way. We replace the phase function $\varphi(x) = x^2$ by a more general polynomial $P(x)$, which is of the form

$$P(x) = A_{2M}(x,\ldots,x) + A_{2M-1}(x,\ldots,x) + A_1(x) + A_0,$$

where we impose the condition that A_{2M} is a completely symmetric positive covariant tensor. We impose the positivity condition in order to ensure that the lead coefficient is never zero, the polynomial must really be of even order. In the case the polynomial is odd there is no global minimum and that lack of a ground state leads to all kinds of physics-related problems. Of course, A_{2M} could just as well be completely negative, it is a matter of

convention. In general the A_k are k-th order covariant tensor. Note that we are using tensors here instead of multiplication because we will generalise these oscillatory integrals where we do not necessarily have the algebra structure available on the Hilbert space. Most of the results in this section were first published in [9, 11].
We recall that by using proposition 7.6 we can show that the (finite dimensional) oscillatory integral converges if f is in the space of symbols. Note that for an oscillatory integral

$$\widetilde{\int} e^{\frac{i}{\hbar}P(x)} f(x),$$

we can generalise the constant $\hbar$ to be complex under the condition that $\mathrm{Im}(\hbar) < 0$, in that case we get a negative real term in the exponent which only increases the rate of convergence. However, this generalisation is not important from a physics point of view and therefore we will not pursue it any further here.

What we would now like to do is extend the Parseval-type equality we obtained in equation (113) to a generalised polynomial phase function. In order to do so we want to have an estimate of the Fourier transform of the function $\varphi(x) = e^{\frac{i}{\hbar}P(x)}$, this is covered in the following lemma.

Lemma 7.22. Let P be a polynomial of the form outlined above and $\hbar > 0$, in that case the Fourier transform of $e^{\frac{i}{\hbar}P(x)}$, which is given by

$$\widetilde{F}(k) = \int_{\mathbb{R}^n} e^{ik\cdot x} e^{\frac{i}{\hbar}P(x)} dx,$$

can be represented by the following entire function

$$\widetilde{F}(k) = e^{in\pi/4M} \int_{\mathbb{R}^n} e^{ie^{i\pi/4M}k\cdot x} e^{\frac{i}{\hbar}P(e^{i\pi/4M}x)} dx. \tag{122}$$

Proof. The proof can be found in [108, Lemma 2.1]. □

Remark 7.23. Note that the integral representation of $\widetilde{F}(k)$ is convergent because we can write

$$\begin{aligned}
e^{\frac{i}{\hbar}P(e^{i\pi/4M}x)} &= e^{\frac{i}{\hbar}(A_{2M}(e^{i\pi/4M}x,\dots,e^{i\pi/4M}x)+A_{2M-1}(e^{i\pi/4M}x,\dots,e^{i\pi/4M}x)+\dots+A_1(e^{i\pi/4M}x)+A_0)} \\
&= e^{\frac{i}{\hbar}A_{2M}(e^{i\pi/4M}x,\dots,e^{i\pi/4M}x)} e^{\frac{i}{\hbar}A_{2M-1}(e^{i\pi/4M}x,\dots,e^{i\pi/4M}x)} \cdot \dots \cdot e^{\frac{i}{\hbar}A_1(e^{i\pi/4M}x)} e^{\frac{i}{\hbar}A_0} \\
&= e^{\frac{i}{\hbar}e^{i\pi/4}A_{2M}(x,\dots,x)} e^{\frac{i}{\hbar}A_{2M-1}(e^{i\pi/4M}x,\dots,e^{i\pi/4M}x)} \cdot \dots \cdot e^{\frac{i}{\hbar}A_1(e^{i\pi/4M}x)} e^{\frac{i}{\hbar}A_0} \\
&= e^{-\frac{1}{\hbar}A_{2M}(x,\dots,x)} e^{\frac{i}{\hbar}A_{2M-1}(e^{i\pi/4M}x,\dots,e^{i\pi/4M}x)} \cdot \dots \cdot e^{\frac{i}{\hbar}A_1(e^{i\pi/4M}x)} e^{\frac{i}{\hbar}A_0},
\end{aligned}$$

which in particular is convergent, because $A_{2M} > 0$.

Now we will state the main theorem of this section.

Theorem 7.24. If f lies in the Fresnel algebra $\mathcal{F}(\mathbb{R}^n)$ (i.e. $f = \hat{\mu}_f$ and $\hbar \in \mathbb{R}$ the oscillatory integral (or generalised Fresnel integral) is given by the following Parseval-type equality

$$I(f) \coloneqq \widetilde{\int_{\mathbb{R}^n}} e^{\frac{i}{\hbar}P(x)} f(x)dx = \int_{\mathbb{R}^n} \widetilde{F}(k)d\mu_f(k), \tag{123}$$

where $\widetilde{F}(k)$ is the Fourier transform of $e^{\frac{i}{\hbar}P(x)}$ that was given earlier on.

Proof. Let φ be a Schwartz function, we have to consider the limit

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}P(x)} f(x)\varphi(\epsilon x)dx.$$

We can use that f belongs to the Fresnel algebra $\mathcal{F}(\mathbb{R}^n)$, which means in particular that we can write the function as

$$f(x) = \int_{\mathbb{R}^n} e^{ikx} d\mu_f(k),$$

where μ_f is a measure of bounded variation. We then first apply the Fubini-Tonelli theorem, subsequently use the dominated convergence theorem (which is allowed since f and $e^{\frac{i}{\hbar}P(x)}$ can be bounded) and finally use that the inverse Fourier transform of a product is a convolution. We obtain using these techniques

$$\begin{aligned}
\lim_{\epsilon\to 0} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}P(x)} f(x)\varphi(\epsilon x)dx &= \lim_{\epsilon\to 0} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}P(x)}\varphi(\epsilon x) \int_{\mathbb{R}^n} e^{ikx} d\mu_f(k)dx \\
&= \lim_{\epsilon\to 0} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ikx} e^{\frac{i}{\hbar}P(x)}\varphi(\epsilon x)dx d\mu_f(k) \\
&= \lim_{\epsilon\to 0} (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tilde{F}(k-\alpha\epsilon)\tilde{\varphi}(\alpha)d\alpha d\mu_f(k) \\
&= \int_{\mathbb{R}^n} \widetilde{F}(k)d\mu_f(k).
\end{aligned}$$

In the last step we used that $\int_{\mathbb{R}^n} \tilde{\varphi}(\alpha)d\alpha = (2\pi)^n$. □

Note that the existence of the oscillatory integral is ensured for the case where f belongs to the space of symbols by proposition 7.6. In this section we have extended the definition of the Fresnel integrals (i.e. those integrals with quadratic phase function, which becomes the kinetic energy in the free particle case) to more general polynomial phase functions. This includes, most interestingly, a quartic perturbation to the harmonic oscillator (so of the form $V = \lambda x^4$). As we have seen in the quantum field theory chapter, this perturbation is crucial for interactions and plays a vital role in quantum electrodynamics, therefore investigating the non-relativistic quantum mechanical case is of great physical importance for a mathematical formalism of path integrals.

7.7 Infinite dimensional oscillatory integrals

Now that we have covered finite dimensional oscillatory integrals of different kinds, we recall that the goal of this theory is to extend the 'integral' to an infinite dimensional Hilbert space, namely the Cameron-Martin space. In order to do so we will take the following approach: We start with an infinite dimensional Hilbert space $\mathcal{H}$ and considering a sequence of projection operator $\{P_i\}$ such that $P_i \leq P_{i+1}$, this means that the projections satisfy the condition $P_i(\mathcal{H}) \subset P_{i+1}(\mathcal{H})$. The operators must project to a finite dimensional subspace of the Hilbert space. Moreover, the operators converge strongly to the identity operator. We now define this type of integral

Definition 7.25. Suppose we have a measurable function $f : \mathcal{H} \to \mathbb{C}$ and a sequence of projection operators $\{P_i\}$ with the conditions imposed on them mentioned above and in addition the operator P_i projects onto an i-dimensional subspace of the Hilbert space. Morever if we have that the oscillatory integrals

$$I_j := \widetilde{\int}_{P_j\mathcal{H}} e^{\frac{i}{2\hbar}\langle P_j x, P_j x\rangle} f(P_j x) d(P_j x) \left(\widetilde{\int}_{P_j\mathcal{H}} e^{\frac{i}{2\hbar}\langle P_j x, P_j x\rangle} d(P_j x)\right)^{-1}$$

exist for each $i \in \mathbb{N}$ and that the limit $I = \lim_{i\to\infty} I_i$ is independent of the sequence of projections operators $\{P_i\}$, we call the limit the infinite dimensional oscillatory integral and denote it by

$$\widetilde{\int}_{\mathcal{H}} e^{\frac{i}{2\hbar}\langle x, x\rangle} f(x) dx. \tag{124}$$

Remark 7.26. We recall that $\left(\widetilde{\int}_{P_j\mathcal{H}} e^{\frac{i}{2\hbar}\langle P_j x, P_j x\rangle} d(P_j x)\right)^{-1} = (2\pi\hbar i)^{-n/2}$. It indeed follows that if we choose f to be the identity function, then its infinite dimensional oscillatory integral equals 1, for the simple reason that the sequence of finite dimensional oscillatory integrals is the sequence $\{1, 1, 1, 1, \ldots\}$.

Remark 7.27. The reader might think that there is contradicting notation because the integral $\widetilde{\int}$ is used for both the Fresnel and the infinite dimensional oscillatory integral. However, this is not the case since it becomes clear from which Hilbert space we are considering which integral is meant. Moreover, the definition of the infinite dimensional oscillatory integral encompasses the Fresnel integral because the phase function is the same ($\langle x, x\rangle$ in both cases) and in case $\mathcal{H}$ is finite dimensional we can just have sequence of projection operators of the form $\{P_1, P_2, \ldots, P_{n-1}, P_n, P_n, P_n, \ldots\}$ which leads to the same integral as the Fresnel integral since P_n is the identity operator on $\mathcal{H}$ for a finite dimensional Hilbert space.

Once again, the complete classification of all functions which are 'infinite dimensional oscillatory integrable' has not been achieved yet. But once again we have that this set

includes the Fresnel algebra. We will also prove another Parseval-type formula analogous to equation (113) in this section. We should note that this equation does not carry over directly, since we have to impose a couple of extra conditions to make sense of the determinant in the case of an operator on an infinite dimensional space.

We will once again consider an operator L such that $I - L$ is invertible, but now we assume that L is self-adjoint and it is trace class as well. These conditions ensure that the index of $I - L$ (the number of negative eigenvalues counted with their multiplicity) is finite. Before we are able to present the analogon to theorem 7.8 and a corresponding Parseval-type equality similar to equation (113) we must first mention an important lemma which will be useful in considering the limit of the index of $I - L$ when we are using projection operators:

Lemma 7.28. Suppose we again have a sequence of projection operators $\{P_i\}$ that satisfy all the conditions mentioned before and we also have an operator L that is both compact and self-adjoint, then the following limit is true

$$\lim_{n\to\infty} \mathrm{Ind}(I_n - P_n L P_n) = \mathrm{Ind}(I - L).$$

Proof. The proof will proceed in two steps. In the first part we show that $\lim_{n\to\infty} \mathrm{Ind}(I - P_n L P_n) \geq \mathrm{Ind}(I - L)$ and in the second part we show that $\lim_{n\to\infty} \mathrm{Ind}(I - P_n L P_n) \leq \mathrm{Ind}(I - L)$.

First we remark that $I - L$ has a finite index because it is compact, trace class and invertible. So we label the eigenvalues of L that are larger than 1 by $\lambda_1, \ldots, \lambda_m$. These are the negative eigenvalues of $I - L$. The corresponding orthonormal eigenvectors are $e_1, \ldots, e_m$. We have that

$$\langle e_i, (I - L)e_i \rangle = 1 - \lambda_i,$$

because the sequence of projection operators $\{P_i\}$ converges strongly to the identity, in particular we know that $(I - L)P_i e_j$ converges to $(I - L)e_j$ and $P_i e_j$ to e_j. That means that

$$\lim_{n\to\infty} \langle P_i e_j, (I - L)P_i e_j \rangle = 1 - \lambda_j.$$

Since projection operators are self-adjoint and also have the property that $P_i^2 = P_i$, we can write

$$\langle P_i e_j, (I - L)P_i e_j \rangle = \langle P_i e_j, (I_i - P_i L P_i)P_i e_j \rangle,$$

where I_i is the identity operator on $P_i\mathcal{H}$. We realise that for sufficiently large i we have that $\langle P_i e_j, (I - L)P_i e_j \rangle < 0$ for $j = 1, \ldots, m$. That means that $P_i e_1, \ldots, P_i e_m$ lie in a negative definite subspace of $P_i\mathcal{H}$. In the limit $P_i e_1, \ldots, P_i e_m$ are independent, that means that for sufficiently large i they are independent too. That means that there are at least m negative eigenvalues in the limit of $I_n - P_n L P_n$. So we conclude that $\lim_{n\to\infty} \mathrm{Ind}(I - P_n L P_n) \leq \mathrm{Ind}(I - L)$.

With regards to the other inequality, we observe that if $v \in P_i\mathcal{H}$ is a negative eigenvector, then $P_i v = v$. We then obtain the following inequality

$$\langle v, (I-L)v\rangle = \langle P_i v, (I-L)P_i v\rangle = \langle v, (I_i - P_i L P_i)v\rangle < 0.$$

That means that the limit of $I_i - P_i L P_i$ has at most as many negative eigenvalues as $I - L$. That means that the other inequality has been proven and the proof is complete. □

Now we are able to state the main theorem for infinite dimensional oscillatory integrals with quadratic phase function, which is analogous to theorem 7.8:

Theorem 7.29. If we have an operator $L : \mathcal{H} \to \mathcal{H}$ that satisfies all the aforementioned conditions (trace class, self-adjoint and $I-L$ must be invertible) and we consider the function

$$g(x) = e^{-\frac{i}{2\hbar}\langle x, Lx\rangle} f(x),$$

where $f(x)$ lies in the Fresnel algebra $\mathcal{F}(\mathcal{H})$. Then it follows that g is Fresnel integrable (in the infinite dimensional sense) and that the integral is given by the following Parseval-type equality

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,(I-L)x\rangle} f(x)dx = (\det(I-L))^{-1/2} \int_{\mathcal{H}} e^{-\frac{i}{2\hbar}\langle x,(I-L)^{-1}x\rangle} f(x) d\hat{\mu}_f(x), \tag{125}$$

where the determinant should be interpreted as the Fredholm determinant. This means that

$$e^{-\pi i \operatorname{Ind}(I-L)} |\det(I-L)| = \det(I-L).$$

Proof. Let $\{P_n\}$ be an arbitrary sequence of projection operators, such that they converge strongly to the identity operator I and moreover that $P_n \leq P_{n+1}$. The first thing we must show is that if f lies in $\mathcal{F}(\mathcal{H})$, then the restriction of f to $P_n\mathcal{H}$ lies in $\mathcal{F}(P_n\mathcal{H})$. We define the restriction as follows: If $x \in P_n\mathcal{H}$, then $P_n x = x$. We then define f_n by

$$f_n(x) = f(P_n x).$$

We now observe the following sequence of equalities with regards to the Fourier transform

$$f_n(x) = f(P_n x) = \int_{\mathcal{H}} e^{\langle y, P_n x\rangle} d\mu_f = \int_{\mathcal{H}} e^{\langle P_n y, x\rangle} d\mu_f = \int_{P_n\mathcal{H}} e^{\langle y, x\rangle} d\mu_{f_n}.$$

So that means that $f_n \in \mathcal{F}(P_n\mathcal{H})$. The trace class operator L can be made into a trace class operator on $P_n\mathcal{H}$ by defining

$$L_n \coloneqq P_n L P_n.$$

It can be seen that for sufficiently large n, we have that $I_n - L_n$ (where I_n is the identity operator on $P_n\mathcal{H}$) is invertible, since P_n converges strongly to the identity operator I. In

particular that means that because we know the finite dimensional Parseval-type equality to be true, we have that

$$\widetilde{\int_{P_n\mathcal{H}}} e^{\frac{i}{2\hbar}\langle P_n x,(I_n-L_n)P_n x\rangle} f(P_n x)dx$$
$$= e^{-\frac{\pi i}{2}\operatorname{Ind}(I_n-L_n)}|\det(I_n-L_n)|^{-1/2}\int_{P_n\mathcal{H}} e^{-\frac{i}{2\hbar}\langle P_n x,(I_n-L_n)^{-1}P_n x\rangle} d\hat{\mu}_f(x).$$

If one takes the limit, we have already seen that the index converges to $\operatorname{Ind}(I-L)$. The sequence L_n converges to L and we then know that $\det(I_n-L_n)$ converges to $\det(I-L)$. The operator P_n goes to I. We can then conclude that we end up with the result

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,(I-L)x\rangle} f(x)dx = (\det(I-L))^{-1/2}\int_{\mathbb{R}^n} e^{-\frac{i}{2\hbar}\langle x,(I-L)^{-1}x\rangle} f(x)d\hat{\mu}(x),$$

which is what we had to show. □

Now the reader should recall that in proving the impossibility of a Lebesgue-type measure on an infinite dimensional Hilbert space, one of the conditions that was required was translation invariance of the measure. Below we prove a result regarding how the infinite dimensional oscillatory integrals transforms under translations of the form $x \to x+a$.

Proposition 7.30. Take a function f that lies in the Fresnel algebra $\mathcal{F}(\mathcal{H})$ and define the new functions $f_a(x) := f(x+a)$ and

$$g(x) := e^{-\frac{i}{\hbar}\langle a,x\rangle} f_a(x).$$

Then g has an infinite dimensional oscillatory integral and the oscillatory integral satisfies the property

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle 2a+x,x\rangle} f_a(x)dx = e^{-\frac{i}{\hbar}\langle a,a\rangle}\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)dx. \tag{126}$$

Proof. The proof can be found in [108, Theorem 2.6]. □

As in the finite dimensional case, we have constructed infinite dimensional oscillatory integrals for a quadratic phase function. In the next section, we will extend the construction to higher order polynomial phase functions. However, in the finite dimensional case we were able to do so for polynomials of even degree for any order, in the infinite dimensional case we are only able to do so for the quartic case, which is very important nonetheless, since it concerns the anharmonic oscillator.

7.8 Generalisation of the phase function

As we did in the case of the finite dimensional Fresnel integrals, it makes sense to generalise the phase function to all kinds of polynomials in the case of infinite dimensional oscillatory integrals as well. We will prove a new Parseval-type equality, that will subsequently in the next chapter be applied to the anharmonic oscillator, of which we saw the importance in the chapters on quantum mechanics and quantum field theory. Recall that in the finite dimensional case we considered polynomials of the form

$$P(x) = A_{2M}(x, \dots, x) + A_{2M-1}(x, \dots, x) + A_1(x) + A_0.$$

We established the following Parseval-type equality

$$\widetilde{\int}_{\mathbb{R}^n} e^{\frac{i}{\hbar}P(x)} f(x)dx = \int_{\mathbb{R}^n} \widetilde{F}(k)d\mu_f(k),$$

where $\widetilde{F}(k)$ was the Fourier transform of $e^{\frac{i}{\hbar}P(x)}$, given by

$$\widetilde{F}(k) = e^{in\pi/4M} \int_{\mathbb{R}^n} e^{ie^{i\pi/4M}k\cdot x} e^{\frac{i}{\hbar}P(e^{i\pi/4M}x)}dx.$$

However, we face a problem with taking the limit $n \to \infty$. In the case of a quadratic phase function, it was perfectly possible to take the limit since it was possible to assign meaning the limit of a term $(\det(I-L))^{-1/2} \int_{\mathbb{R}^n} e^{-\frac{i}{2\hbar}\langle x,(I-L)^{-1}x\rangle} f(x)d\hat{\mu}(x)$. However assigning a limit to $e^{in\pi/4M} \int_{\mathbb{R}^n} e^{ie^{i\pi/4M}k\cdot x} e^{\frac{i}{\hbar}P(e^{i\pi/4M}x)}dx$ as $n \to \infty$ is impossible, since the first term keeps oscillating.

Nevertheless, it is still possible to extend the construction to a phase function with quartic growth. We use the construction of section 7.6, this time we consider a polynomial given by a fourth-order, positive, symmetric and covariant tensor $A : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$. If we construct an orthonormal basis on $\mathcal{H}$, we can write the tensor as

$$P(x) = A(x, x, x, x) = \sum_{j,k,l,m=1}^{n} a_{jklm} x_j x_k x_l x_m,$$

where the x_j are the components of the vectors in $\mathcal{H}$ in terms of the orthonormal basis and the a_{jklm} are the coefficients of the tensor with respect to this basis. We now proceed in a manner completely analogous to the previous sections. We first give the Fourier transform and then state the Parseval-type equality in terms of that Fourier transform. We will study generalised Fresnel integrals of the form

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,(I-B)x\rangle} e^{\frac{-i\lambda}{\hbar}P(x)} f(x)dx,$$

which is now infinite dimensional (we again choose an orthonormal basis for $\mathcal{H}$), I and B have become operators and λ is some real parameter.

Lemma 7.31. If $P(x)$ is of the form $A(x,x,x,x)$ where the tensor A has all the properties mentioned above, then the Fourier transform of the distribution

$$\frac{1}{(2\pi i\hbar)^{n/2}} e^{\frac{i}{2\hbar}x(I-B)x} e^{\frac{-i\lambda}{\hbar}P(x)},$$

which can be written as

$$\widetilde{F}(k) = \frac{1}{(2\pi i\hbar)^{n/2}} \int_{\mathbb{R}^n} e^{ik\cdot x} e^{\frac{i}{2\hbar}x(I-B)x} e^{\frac{-i\lambda}{\hbar}P(x)} dx$$

is a bounded entire function. This entire function can be represented as

$$\widetilde{F}(k) = \begin{cases} \frac{e^{in\pi/8}}{(2\pi i\hbar)^{n/2}} \int_{\mathbb{R}^n} e^{ie^{i\pi/8}k\cdot x} e^{\frac{i}{2\hbar}e^{i\pi/4}x(I-B)x} e^{\frac{\lambda}{\hbar}P(x)} dx & \lambda < 0 \\ \frac{e^{-in\pi/8}}{(2\pi i\hbar)^{n/2}} \int_{\mathbb{R}^n} e^{ie^{-i\pi/8}k\cdot x} e^{\frac{i}{2\hbar}e^{-i\pi/4}x(I-B)x} e^{-\frac{\lambda}{\hbar}P(x)} dx & \lambda > 0. \end{cases}$$

If we additionally impose the condition that $A \geq 0$, $\lambda > 0$ and $I - B$ is strictly positive, we obtain the following representation

$$\widetilde{F}(k) = \frac{1}{(2\pi i\hbar)^{n/2}} \int_{\mathbb{R}^n} e^{ie^{-i\pi/4}k\cdot x} e^{-\frac{1}{2\hbar}e^{-i\pi/4}x(I-B)x} e^{\frac{i\lambda}{\hbar}P(x)} dx = \mathbb{E}[e^{ie^{i\pi/4}k\cdot x} e^{\frac{1}{2\hbar}x\cdot Bx} e^{\frac{i\lambda}{\hbar}P(x)}],$$

where $\mathbb{E}$ denotes the expectation with respect to the Gaussian distribution with average 0.

Proof. The proof can be found in [108, Lemma 2.3]. □

Remark 7.32. If one studies in the proof in [108, Remark 2.5] one will see that the proof cannot be generalised to a higher even order polynomial. This has to with the fact that the Fourier transform of the following expression must be calculated

$$\Theta(r) r^{n-1} \frac{e^{\frac{i}{2\hbar}r^2}}{(2\pi\hbar i)^{n/2}} e^{\frac{-i\lambda}{\hbar}P(r)}, \tag{127}$$

note that Θ is the Heaviside step function. The proof relates the Fourier transform to the following integral

$$\int_0^\infty e^{ik\rho e^{i\pi/4}} \frac{e^{-\rho^2/2\hbar}}{(2\pi\hbar)^{n/2}} e^{-i\lambda P(\rho e^{i\pi/4})/\hbar} \rho^{n-1} d\rho. \tag{128}$$

In order to obtain this result, we need that $\mathrm{Re}(-i\lambda P(\rho e^{i\pi/4})) < 0$ in order to obtain convergence. For a polynomial of degree $2M$ this only holds if $0 \leq \theta \leq \frac{\pi}{2M}$. If the degree of P is larger than 4, than the angle $\pi/4$ is no longer included. This angle is crucial because we need it as part of the contour to rewrite the Fourier transform of (127) as a Gaussian integral. In equation (128) one can see the term $\frac{e^{-\rho^2/2\hbar}}{(2\pi\hbar)^{n/2}}$, which becomes part of the Gaussian measure.

The advantage of the Gaussian measure is that it does exist in an infinite dimensional Hilbert space, which allows us to take the limit $\dim(\mathcal{H}) \to \infty$. For higher order polynomials, this method can no longer be used to construct infinite dimensional oscillatory integrals.

The restriction to the quartic case limits how useful this formalism is to give the path integral, however something else should be kept in mind. If one considers Feynman diagrams with loops, in general one will encounter diverging path integrals. In order to extract a meaningful answer from quantum field theory, physicists renormalise the theory.

However, the process of renormalisation does not work for all terms in the Lagrangian density. The deep reason for that could be that the path integral cannot be rigorously constructed for such terms. At this point, this is pure speculation on the part of the author. There needs to be more comparison between the different formalisms that give a rigorous construction of the path integral to see if they all face the same problem. That could be an indication of why some terms could not be renormalisable, but this is only an outlook at this point and warrants further research.

Now we can repeat the techniques that we used earlier to move from the calculation of the Fourier transform towards establishing a Parseval-type equality in order to calculate the infinite dimensional oscillatory integral of a quartic phase function. In particular, we have the following result.

Theorem 7.33. If the function f lies in the Fresnel algebra $\mathcal{F}(\mathbb{R}^n)$ in particular that means that the generalised Fresnel integral of the function f is equal to

$$\widetilde{\int} e^{\frac{i}{2\hbar}x\cdot(I-B)x} e^{-\frac{i\lambda}{\hbar}P(x)} f(x)dx = \int_{\mathbb{R}^n} \widetilde{F}(k)d\mu_f(k).$$

Proof. The proof is analogous to the proof of theorem 7.24. □

Theorem 7.34. Let $B : \mathcal{H} \to \mathcal{H}$ be a self-adjoint trace class operator, λ a nonpositive parameter and the function f belong to the Fresnel algebra $\mathcal{F}(\mathcal{H})$. We also impose the following condition on the bounded variation measure $\hat{\mu}_f$

$$\int_{\mathcal{H}} e^{\frac{\hbar}{4}\langle k,(I-B)^{-1}} d|\mu_f|(k) < \infty.$$

Then the infinite dimensional oscillatory integral

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I-B)^{-1}x\rangle} e^{-\frac{\lambda}{\hbar}A(x,x,x,x)} f(x)dx$$

exists and is given by the following integral

$$\int_{\mathcal{H}} \mathbb{E}\left[e^{in(k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} e^{i\frac{\lambda}{\hbar}\widetilde{V}(\omega)}\right] d\mu_f(k).$$

This integral can be shown to be equal to

$$\mathbb{E}\left[e^{\frac{i}{2\hbar}\langle\omega,B\omega\rangle}e^{i\frac{\lambda}{\hbar}\widetilde{V}(\omega)}f\left(e^{i\pi/4}\omega\right)\right].$$

Both the expectation values are with respect to the standard Gaussian measure.

Proof. For a proof, see [108, Theorems 2.11 and 2.12]. □

7.9 Conclusion

Now that we have studied the approach to give the Feynman path integral a rigorous foundation by using oscillatory integrals. We first provided a construction for the finite dimensional case with a quadratic phase function and subsequently extended that to the general even polynomial case using the Fourier transform. We obtained the infinite dimensional analogues by taking limits of the dimension going to infinity. We would like to identify the advantages and drawbacks of this approach, see also [85, section 20.1].

The approach has several strong advantages. The first one is its simplicity. This formalism defines an integral by defining two limits, the first limit letting the oscillations spread over the whole interval and the second limit is letting the dimension going to infinity. Another advantage is that the formalism allows one to not only establish the existence of a path integral, but also to calculate its value. That makes it useful for checking the results against calculations done in physics, such as [67, 68].

A drawback in this approach might be in identifying where its limits are, although there has been significant progress in identifying functions which have an infinite dimensional oscillatory integral, it is very hard to give a complete classification, even if we restrict the Hilbert space to be finite dimensional and solely focus on a quadratic phase function (i.e. a Fresnel integral). Another issue is that the construction does not work at all for any potential with a singularity, such as the important case of the Coulomb potential. There exist other constructions that do work for that potential, see [85] for further details.

I would like to reiterate that there many more constructions for a rigorous path integral and they all have their advantages and drawbacks, please consult the bibliography for a more detailed overview. Now that we have developed the oscillatory integral formalism, we can apply it to both the harmonic and anharmonic oscillator systems.

8 Case studies in physics

In the last chapter we have developed the oscillatory integral formalism, in both the finite and infinite dimensional cases and for both the quadratic and quartic phase functions. We will now apply this to the physical systems that are of interest to us here, namely the harmonic and anharmonic oscillator. As was said before, anharmonic means a perturbation of the form λx^4 added to the potential energy. We will start with the harmonic oscillator. For the sake of clarity, some calculations have been moved to the appendices.

8.1 Harmonic oscillator

We have already discussed how the path integral for the harmonic oscillator can be calculated in section 4.2.1, but that does not constitute a rigorous foundation. The integration over all paths was handled by writing the path as a sine Fourier series and subsequently integrate over all the coefficients. We will not provide such a construction in this section. We will mathematically realise the path integral as an infinite dimensional oscillatory integral and subsequently utilise the Parseval equality to calculate the result, which is in the form of a Lebesgue integral. The Schrödinger equation for a harmonic oscillator is given by

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\Delta\psi + \frac{1}{2}x\Omega^2 x\psi \tag{129}$$

$$\psi(x,0) = \psi_0(x). \tag{130}$$

Note that that the wavefunction ψ lies in $L^2(\mathbb{R}^n)$ if it is a physically realisable state. Therefore Ω is an $n\times n$ matrix. From now on we will set $m=1$ to simplify the calculations. The main result will be the following existence theorem.

Theorem 8.1 (Harmonic oscillator)**.** The solution to the harmonic oscillator initial value problem given in equations (129) and (130) in the case $\psi_0 \in \mathcal{S}(\mathbb{R}^n)$ is given by the following infinite dimensional oscillatory integral over the Cameron-Martin space $\mathcal{H}_T$

$$\psi(x,T) = \widetilde{\int_{\mathcal{H}_T}} e^{\frac{i}{2\hbar}\int\dot{\gamma}^2(s)ds} e^{-\frac{i}{2\hbar}\int_0^T ds(\gamma(s)+x)\Omega^2(\gamma(s)+x)}\psi_0(\gamma(s)+x)d\gamma. \tag{131}$$

Note that we write $\gamma(s)+x$ because by definition $\gamma(T)=0$.

Before we set out to prove this key theorem, we will prove two lemmas first. These lemmas are necessary in order to justify the use of the Parseval equality

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,(I-L)x\rangle} f(x)dx = (\det(I-L))^{-1/2}\int_{\mathbb{R}^n} e^{-\frac{i}{2\hbar}\langle x,(I-L)^{-1}x\rangle} f(x)d\hat{\mu}_f(x),$$

which will be used to prove the existence of the infinite dimensional oscillatory integral (131) and give the result in terms of a Lebesgue integral. In order to do so, we define the operator $L : \mathcal{H}_T \to \mathcal{H}_T$, specifically given by

$$(L\gamma)(s) \coloneqq \int_s^T ds' \int_0^{s'} (\Omega^2\gamma)(s'')ds''. \tag{132}$$

The reason we defined this operator is that the inner product of $L\gamma_2$ with another path γ_1 is given by

$$\begin{aligned}\langle \gamma_1, L\gamma_2\rangle &= \int_0^T \dot{\gamma}_1(s) \cdot \frac{d}{ds}(L\gamma_2)(s)ds = -\int_0^T ds\dot{\gamma}_1(s) \int_0^s (\Omega^2\gamma)(s')ds' \\ &= -\gamma_1(s) \int_0^s (\Omega^2\gamma)(s')ds'\Big|_0^T + \int_0^T \gamma_1(s)\Omega^2\gamma_2(s)ds = \int_0^T \gamma_1(s)\Omega^2\gamma_2(s)ds.\end{aligned}$$

One can recognise this one term in the exponent in equation (131). To get to the second line we used the condition on the paths in the Cameron-Martin space that $\gamma(T) = 0$. From this it immediately follows that L is a self-adjoint operator, since the inner product is symmetric.

Lemma 8.2. Let the matrix Ω have eigenvalues $\Omega_1, \ldots, \Omega_n$. Now if T (which is a property of the Cameron-Martin space itself) satisfies the following condition

$$T \neq \left(n + \frac{1}{2}\right)\frac{\pi}{\Omega_j}, \quad j = 1, \ldots, n, \ n \in \mathbb{N},$$

then the inverse of the operator $I - L$ is given by

$$\begin{aligned}(I - L)^{-1}\alpha(s) = \alpha(s) - \Omega(\cos(\Omega T))^{-1}\Big[&\cos(\Omega s)\int_s^T \sin(\Omega(r - T))\alpha(r)dr \\ &+ \sin(\Omega(s - T))\int_0^s \cos(\Omega r)\alpha(r)dr\Big].\end{aligned} \tag{133}$$

Proof. Suppose that $\alpha = (I - L)\gamma$, we then know that

$$\alpha(s) = \gamma(s) - \int_s^T ds' \int_0^{s'} (\Omega^2\gamma)(s'')ds''.$$

We see that $\alpha(T) = 0$ and $\dot{\alpha}(0) = \dot{\gamma}(0)$. If one differentiates this equation twice, that leads to

$$\ddot{\alpha}(s) = \ddot{\gamma}(s) + \Omega^2\gamma(s). \tag{134}$$

What we will do is solve this differential equation for γ, the result written in terms of α will be our inverse operator. In appendix B.1 it is shown that for this boundary value problem the Green's function is given by

$$G(s,r) = \begin{cases} \Omega^{-1}(\cos(\Omega T))^{-1}\sin(\Omega(r-T))\cos(\Omega s), & \text{if } s < r \\ \Omega^{-1}(\cos(\Omega T))^{-1}\cos(\Omega r)\sin(\Omega(s-T)), & \text{if } s > r. \end{cases} \tag{135}$$

The solution to the boundary value problem is hence given by

$$\begin{aligned}\gamma(s) = \int_0^T G(s,r)\ddot{\alpha}(r)dr &= \Omega^{-1}(\cos(\Omega T))^{-1}\cos(\Omega s)\int_s^T \sin(\Omega(r-T))\ddot{\alpha}(r)dr \\ &+ \Omega^{-1}(\cos(\Omega T))^{-1}\sin(\Omega(s-T))\int_0^s \cos(\Omega r)\ddot{\alpha}(r)dr\end{aligned}$$

We rewrite this expression in terms of α (and not its derivatives) in appendix B.2. We arrive at the expression

$$\begin{aligned}(I-L)^{-1}\alpha(s) = \alpha(s) - \Omega(\cos(\Omega T))^{-1}\Bigg[&\cos(\Omega s)\int_s^T \sin(\Omega(r-T))\alpha(r)dr \\ &+ \sin(\Omega(s-T))\int_0^s \cos(\Omega r)\alpha(r)dr\Bigg].\end{aligned}$$

In the appendix it is also shown that this inverse operator satisfies the differential equation (134). One can verify for the boundary conditions that

$$\begin{aligned}\gamma(T) &= \alpha(T) = 0 \\ \dot{\gamma}(0) &= \dot{\alpha}(0),\end{aligned}$$

so there is no need to adjust to inhomogeneous boundary conditions in the case $\dot{\alpha}(0) \neq 0$, since the solution already satisfies these boundary conditions. Note that by definition γ is C^1 and it therefore also follows that α is C^1. □

Remark 8.3. A careful reader could have noticed that our formula (8.2) for the resolvent of $I - L$, does not coincide with the formula

$$(I-L)^{-1}\gamma(s) = \gamma(s) - \Omega\int_s^T \sin(\Omega(s-s'))\gamma(s')ds' + \sin(\Omega(T-s))\int_0^T (\cos\Omega T)^{-1}\Omega\cos(\Omega s')\gamma(s')ds',$$

which comes from the original paper [46]. We believe that the latter is a result of a typo: a direct computation, shows that it is not the inverse operator we were looking for.

Corollary 8.4. The case

$$T = \left(n + \frac{1}{2}\right) \frac{\pi}{\Omega_j}, \quad j = 1, \ldots, n \in \mathbb{N}, \tag{136}$$

does not have an inverse operator.

Proof. At first the calculation of the Green's function proceeds in the same way. For $s < r$ we have that

$$G(s, r) = c_1 \cos(\Omega s) + c_2 \sin(\Omega s),$$

the boundary condition $\dot{\gamma}(0) = 0$ then implies that $c_2 = 0$. If $s > r$, we have that

$$G(s, r) = c_3 \cos(\Omega s) + c_4 \sin(\Omega s),$$

the boundary condition $\gamma(T) = 0$ implies that $c_4 = 0$. Continuity if $s = r$ gives us that $c_1 = c_3$. Imposing a discontinuity becomes impossible this way. Therefore we cannot calculate an inverse operator. □

Now we state the second lemma:

Lemma 8.5. If L still satisfies the assumption $T \neq (n + 1/2) \cdot \pi/\Omega_j$, it is trace class and we have that

$$\mathrm{Ind}(I - L) = n + \sum_{j=1}^{n} \left[\left| \frac{\Omega_j T}{\pi} \right| + \frac{1}{2} \right],$$

where the square brackets denote the integer part in this case. Moreover, the Fredholm determinant is given by

$$\det(I - L) = \det(\cos(\Omega T)).$$

Proof. One can assume that the matrix Ω is diagonal without loss of generality, since it can always be brought in the from $\Omega = QDQ^{-1}$ where D is diagonal. The eigenvalues of D are the same as the ones of Ω and the determinant $\det(\cos(\Omega T))$ is unaffected because we have

$$\det(\cos(\Omega T)) = \det\left(Q \cos(DT) Q^{-1}\right) = \det(\cos(DT)).$$

The operator L is self-adjoint and positive, hence the eigenvalues are nonnegative and of the form p^2 with p real. We are looking for eigenvectors such that

$$L\gamma = p^2 \gamma,$$

this can be rewritten as follows

$$(L\gamma)(s) = \int_s^T ds' \int_0^{s'} (\Omega^2 \gamma)(s'') ds'' = p^2 \gamma(s). \tag{137}$$

If we differentiate this equation twice we obtain the following differential equation

$$p^2\ddot{\gamma}(s) = -\Omega^2\gamma(s).$$

Because of equation (137) and the definition of the Cameron-Martin space we have the initial conditions $\gamma(T) = 0$ and $\dot{\gamma}(0) = 0$. The solutions to the differential equation are of the form

$$\gamma(s) = \begin{pmatrix} \gamma_1(s) \\ \vdots \\ \gamma_n(s) \end{pmatrix} = \begin{pmatrix} A_1 \sin(\Omega_1(s-\phi_1)/p) \\ \vdots \\ A_n \sin(\Omega_n(s-\phi_n)/p) \end{pmatrix},$$

where $\Omega_1, \ldots, \Omega_n$ are the eigenvalues of the matrix Ω (they are the diagonal entries because of our earlier assumption) and $\phi_1, \ldots, \phi_n$ are phase factors. The initial condition $\dot{\gamma}(0) = 0$ tells us that

$$\frac{\Omega_j \phi_j}{p} = \left(m_j + \frac{1}{2}\right)\pi, \quad m_j \in \mathbb{Z},$$

subsequently we know from the condition $\gamma(T) = 0$ that

$$\frac{\Omega_j T}{p} = \left(m_j + \frac{1}{2}\right)\pi, \quad m_j \in \mathbb{Z}.$$

That means that the solutions for p are

$$p = \frac{\Omega_j T}{\left(m_j + \frac{1}{2}\right)\pi}, \quad m_j \in \mathbb{Z}.$$

Note that p subsequently fixes the phase ϕ_j, through the equation

$$\phi_j = \frac{p}{\Omega_j}\left(m_j + \frac{1}{2}\right)\pi, \quad m_j \in \mathbb{Z}.$$

Therefore m_j being a positive or negative integer leads to a different phase factor and hence a different eigenvector. The series

$$\prod_{j\in\mathbb{Z}} \lambda_j = \prod_{j\in\mathbb{Z}} \frac{\Omega_j^2 T^2}{\pi^2} \frac{1}{(m_j + 1/2)^2} \leq C \prod_{j=1} \frac{1}{(m_j + 1/2)^2} < \infty,$$

so since the product of the eigenvalues is finite we know that L is trace-class [124]. In order to calculate the index of $I - L$ (the index of L itself is trivially zero) we have to find the eigenvalues p^2 such that $p^2 > 1$, that comes down to

$$1 < p^2 = \frac{\Omega_j^2 T^2}{\left(m_j + \frac{1}{2}\right)^2 \pi^2}$$

$$\begin{aligned}
\frac{\pi^2}{\Omega_j^2 T^2} &< \frac{1}{\left(m_j + \frac{1}{2}\right)^2} \\
\left(m_j + \frac{1}{2}\right)^2 &< \frac{\Omega_j^2 T^2}{\pi^2} \\
-\left|\frac{\Omega_j T}{\pi}\right| < m_j + \frac{1}{2} &< \left|\frac{\Omega_j T}{\pi}\right| \\
-\left|\frac{\Omega_j T}{\pi}\right| - \frac{1}{2} < m_j &< \left|\frac{\Omega_j T}{\pi}\right| - \frac{1}{2}.
\end{aligned}$$

that means that the solution is

$$m_j = \left[-\left|\frac{\Omega_j T}{\pi}\right| - \frac{1}{2}\right] + 1, \ldots, \left[\left|\frac{\Omega_j T}{\pi}\right| - \frac{1}{2}\right].$$

The square brackets denote in this case the rounding off of the term to the biggest integer smaller than the term inside the brackets. This means that we have exactly $\left[\left|\frac{\Omega_j T}{\pi}\right| - \frac{1}{2}\right] + 1$ negative eigenvalues (counted with multiplicity). Note that we do not have to count the negative integers $m_j = \left[-\left|\frac{\Omega_j T}{\pi}\right| - \frac{1}{2}\right] + 1, \ldots, -1$, since the minus sign in p can be absorbed into the constants $A_1, \ldots, A_n$. That means that in total we have the following index (because we can repeat this procedure for each diagonal entry of Ω)

$$\mathrm{Ind}(I - L) = n + \sum_{j=1}^{n} \left[\frac{\Omega_j T}{\pi} - \frac{1}{2}\right].$$

The Fredholm determinant is given by the product of the eigenvalues, so

$$\det(I - L) = \prod_{m \in \mathbb{N}} (1 - \lambda_m),$$

where λ_n are the eigenvalues of L and equal p^2. Using an infinite product identity we get

$$\prod_{m=0}^{\infty} \left(1 - \frac{\Omega_j^2 T^2}{\left(m + \frac{1}{2}\right)^2 \pi^2}\right) = \cos(\Omega_j T).$$

Now if we take the product of the eigenvalues for each dimension we get

$$\det(I - L) = \prod_{m \in \mathbb{N}} (1 - \lambda_m) = \prod_{j=1}^{n} \cos(\Omega_j t) = \det(\cos(\Omega T)),$$

the last step follows because we assumed the matrix Ω to be diagonal. □

Remark 8.6. The result I have proven here is different from what can be found in the literature [46, Lemma 4A], which gives

$$\mathrm{Ind}(I-L)=\sum_{j=1}^{n}\left[\frac{\Omega_j T}{\pi}+\frac{1}{2}\right],$$

the difference having to do with taking into account m_j being possibly zero and the possibility that Ω_j is negative.

Now before we can set out to prove the main theorem of this section, which will be proven by applying the Parseval theorem and equality, we need to prove that the function of interest lies in the Fresnel algebra.

Lemma 8.7. If the initial wavefunction ψ_0 lies in the Schwartz algebra $\mathcal{S}(\mathbb{R}^n)$ then it follows that the function (for given x)

$$g(\gamma)=e^{-\frac{i}{2\hbar}x\Omega^2 xT}e^{-\frac{i}{\hbar}\int_0^T\gamma(s)\Omega^2 x ds}\psi_0(\gamma(0)+x).$$

lies in the Fresnel algebra $\mathcal{F}(\mathcal{H}_T)$.

Proof. The proof will be given for the one-dimensional case, although it can easily be extended to the multi-dimensional case. The function $e^{-\frac{i}{\hbar}\int_0^T ds\gamma(s)\Omega^2 x ds}$ belongs to $\mathcal{F}(\mathcal{H}_T)$ because it can be realised as follows

$$e^{-\frac{i}{\hbar}\int_0^T ds\gamma(s)\Omega^2 x ds}=\int e^{i\langle\gamma,\eta\rangle}\delta_{\eta_x}(d\eta), \tag{138}$$

where δ_{η_x} is the Dirac measure (which has bounded variation) and we have defined

$$\eta_x(s)=\frac{\Omega^2 x}{\hbar}\left(\frac{s^2}{2}-\frac{T^2}{2}\right).$$

The equation (138) then follows because we can calculate that

$$\begin{aligned}\langle\gamma,\eta_x\rangle&=\frac{1}{\hbar}\int_0^T\dot{\gamma}(s)\cdot\Omega^2 x s=\frac{1}{\hbar}\left[\gamma(s)\cdot\Omega^2 x s\right]_0^T-\frac{1}{\hbar}\int_0^T\gamma(s)\Omega^2 x ds\\&=-\frac{1}{\hbar}\int_0^T\gamma(s)\Omega^2 x ds,\end{aligned}$$

hence $e^{-\frac{i}{\hbar}\int_0^T\gamma(s)\Omega^2 x ds}$ belongs to the Fresnel algebra $\mathcal{F}(\mathcal{H}_T)$. We recall that ψ_0 is a Schwartz function in $\mathbb{R}$, this means that we can write

$$\psi_0(\gamma(0)+x)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{i(\gamma(0)+x)k}\widehat{\psi_0}(k)dk, \tag{139}$$

where $\widehat{\psi_0}$ is the Fourier transform of ψ_0. We then define the following path in the Cameron-Martin space

$$v_0(s) = T - s.$$

One can then calculate that

$$\langle \gamma, v_0 \rangle = \int_0^T \dot{\gamma} \cdot (-1) = \gamma(0) - \gamma(T) = \gamma(0).$$

This result allows to rewrite equation (139) as follows

$$\begin{aligned}\psi_0(\gamma(0) + x) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} e^{ik\gamma(0)} \widehat{\psi_0}(k) dk \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} \int_{\mathbb{R}} e^{i\langle \gamma, \eta \rangle} d\delta_{kv_0}(\eta) \widehat{\psi_0}(k) dk.\end{aligned}$$

We then introduce the measure μ_0 in the following manner

$$\int_{\mathcal{H}_T} f(\gamma) d\mu_0(\gamma) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} f(kv_0) \widehat{\psi_0}(k) dk. \tag{140}$$

We obtain the equation

$$\psi_0(\gamma(0) + x) = \int_{\mathcal{H}_T} e^{i\langle \gamma, \eta \rangle} d\mu_0(\eta).$$

Finally we can then conclude that the function g is given by

$$g(\gamma) = \int_{\mathcal{H}_T} e^{i\langle \gamma, \eta \rangle} d\mu_g(\eta), \tag{141}$$

where we have defined the measure μ_g, which is given by

$$\mu_g = e^{-\frac{i}{2\hbar} x \Omega^2 x T} \delta_{\eta_x} * \mu_0. \tag{142}$$

This is a convolution of the measures δ_{η_x} and μ_0. Hence we conclude by equation (141) that the function g belongs to the Fresnel algebra $\mathcal{F}(\mathcal{H}_T)$, which is what we had to show. □

Now that all the required lemmas have been proven, we can finally give the proof of theorem 8.1:

Proof. We need to prove that the expression

$$\psi(x, T) = \widetilde{\int_{\mathcal{H}_T}} e^{\frac{i}{2\hbar} \int \dot{\gamma}^2(s) ds} e^{-\frac{i}{2\hbar} \int_0^T ds (\gamma(s) + x) \Omega^2 (\gamma(s) + x) ds} \psi_0(\gamma(s) + x) d\gamma$$

exists. We can rewrite it as

$$\psi(x,T)=\widetilde{\int_{\mathcal{H}_T}} e^{\frac{i}{2\hbar}\langle\gamma,(I-L)\gamma\rangle}e^{-\frac{i}{2\hbar}x\Omega^2xT}e^{-\frac{i}{\hbar}\int_0^T ds\gamma(s)\Omega^2xds}\psi_0(\gamma(s)+x)d\gamma=\widetilde{\int_{\mathcal{H}_T}} e^{\frac{i}{2\hbar}\langle\gamma,(I-L)\gamma\rangle}g(\gamma)d\gamma.$$

We have proven in lemma 8.5 that the operator L is trace class and also concluded that the operator is self-adjoint. In lemma 8.7 we showed that g belongs to the Fresnel algebra $\mathcal{F}(\mathcal{H}_T)$. Now we have shown that the assumptions for theorem 7.29 are satisfied and hence we are allowed to use the Parseval-type equality for the infinite dimensional oscillatory integrals, the theorem tells us that the infinite dimensional oscillatory integral given in equation (131) exists and is given by

$$\begin{aligned}\psi(x,T)&=\widetilde{\int} e^{\frac{i}{2\hbar}\langle\gamma,(I-L)\gamma\rangle}e^{-\frac{i}{2\hbar}x\Omega^2xT}e^{-\frac{i}{\hbar}\int_0^T ds\gamma(s)\Omega^2xds}\psi_0(\gamma(s)+x)d\gamma\\&=(\det(I-L))^{-1/2}e^{-\frac{i}{2\hbar}x\Omega^2xT}\int_{\mathbb{R}^n}e^{-\frac{i}{2\hbar}\langle\gamma,(I-L)^{-1}\gamma\rangle}d\mu_g(\gamma).\end{aligned}$$

□

The only thing that is left to do is calculate the integral $\int_{\mathbb{R}^n} e^{-\frac{i}{2\hbar}\langle\gamma,(I-L)^{-1}\gamma\rangle}d\mu_g$, which will be done in the next theorem.

Theorem 8.8. The path integral in equation (131) is given by (in the one-dimensional case)

$$\psi(x,T)=\sqrt{\frac{\Omega}{2\pi i\sin(\Omega T)}}\int_{\mathbb{R}}e^{\frac{i}{2\hbar}(x\Omega\cot(\Omega T)x+y\Omega\cot(\Omega T)y-2x(\sin(\Omega T))^{-1}y}\psi_0(y)dy,\tag{143}$$

in the multi-dimensional case the path integral becomes

$$\psi(x,T)=\sqrt{\det\left(\frac{\Omega}{2\pi i\sin(\Omega T)}\right)}\int_{\mathbb{R}^n}e^{\frac{i}{2\hbar}(x\Omega\cot(\Omega T)x+y\Omega\cot(\Omega T)y-2x(\sin(\Omega T))^{-1}y}\psi_0(y)dy.\tag{144}$$

Proof. The proof will be done in the one-dimensional case, although it can easily be extended to the multi-dimensional case. What we will do is use the Fourier transform measure given in equation (142) in such a way that we can obtain a more convenient expression of the Lebesgue integral in the Parseval equality. Now we can apply the content of lemmas 8.2 and 8.5 to write that

$$\det(I-L)=\det(\cos(\Omega T)),$$

$$(I-L)^{-1}\gamma(s)=\gamma(s)-\Omega\int_s^T\sin(\Omega(s-s'))\gamma(s')ds'+\sin(\Omega(T-s))\int_0^T(\cos\Omega T)^{-1}\Omega\cos(\Omega s')\gamma(s')ds'.$$

So what we will do is evaluate the expression

$$(\det(I-L))^{-1/2}e^{-\frac{i}{2\hbar}x\Omega^2xT}\int_{\mathcal{H}_T}e^{-\frac{i}{2\hbar}\langle\gamma,(I-L)^{-1}\gamma\rangle}d\mu_g(\gamma),$$

where the function g is given by (recall that we have set $m=1$)

$$g(\gamma)=e^{-\frac{i}{2\hbar}x\Omega^2xT}e^{-\frac{i}{\hbar}\int_0^T\gamma(s)\Omega^2xds}\psi_0(\gamma(0)+x).$$

Using equation (142) we have that

$$\int_{\mathcal{H}_T\times\mathbb{R}}e^{-\frac{i\hbar}{2}\langle\gamma,(I-L)^{-1}\gamma\rangle}d\mu_g(\gamma)=e^{-\frac{i}{2\hbar}x\Omega^2Tx}\int_{\mathcal{H}_T\times\mathcal{H}_T}e^{-\frac{i\hbar}{2}\langle\gamma+\eta,(I-L)^{-1}\gamma+\eta\rangle}d\delta_{\eta_x}(\gamma)d\mu_0(\eta).$$

Then using the definition of the Dirac measure and equation (140) we find

$$\begin{aligned}&\int_{\mathcal{H}_T\times\mathbb{R}}e^{-\frac{i\hbar}{2}\langle\gamma,(I-L)^{-1}\gamma\rangle}d\mu_g(\gamma)=e^{-\frac{i}{2\hbar}x\Omega^2Tx}\int_{\mathcal{H}_T\times\mathcal{H}_T}e^{-\frac{i\hbar}{2}\langle\gamma+\eta,(I-L)^{-1}\gamma+\eta\rangle}d\delta_{\eta_x}(\gamma)d\mu_0(\eta)\\&=\frac{1}{2\pi}e^{-\frac{i}{2\hbar}x\Omega^2Tx}\int_{\mathbb{R}}e^{-\frac{i\hbar}{2}\langle kv_0+\eta_x,(I-L)^{-1}kv_0+\eta_x\rangle}e^{ikx}\widehat{\psi_0}(k)dk\\&=\frac{1}{2\pi}e^{-\frac{i}{2\hbar}x\Omega^2Tx}\int_{\mathbb{R}}e^{-\frac{i\hbar}{2}\langle\eta_x,(I-L)^{-1}\eta_x\rangle}e^{-i\hbar k\langle v_0,(I-L)^{1}\eta_x\rangle}e^{-\frac{i\hbar k^2}{2}\langle v_0,(I-L)^{-1}v_0\rangle}e^{ikx}\widehat{\psi_0}(k)dk.\end{aligned}$$

To obtain equation (143) we have to evaluate the inner products $\langle\eta_x,(I-L)^{-1}\eta_x\rangle$, $\langle v_0,(I-L)^{-1}\eta_x\rangle$ and $\langle v_0,(I-L)^{-1}v_0\rangle$ and subsequently use the Plancherel theorem. The calculations are provided in detail in appendix B.3. As a final remark, the condition that ψ_0 is a Schwartz function seems restrictive, but in fact the Schwartz functions are dense in $L^2(\mathbb{R}^n)$. It might be possible to extend the result to all L^2 functions by a limit argument, but this has not been looked at.

□

Remark 8.9 (Comparison with physics)**.** The result obtained here is in agreement with the result from physics. Note that in section 4.2.1 in equation (65) we obtained the following result for the propagator for the harmonic oscillator system

$$K(0,T,q_I,q_F)=e^{iS[q_c]}\int df e^{iS[f]}=\exp\left(\frac{im\omega(-2q_Fq_I+(q_I^2+q_F^2)\cos(\omega T)}{2\sin(\omega T)}\right)\sqrt{\frac{m\omega}{2\pi i\sin(\omega T)}}.$$

In that chapter, we set $\hbar=1$ and if we also set $m=1$ in the expression above we find it agrees with the rigorous expression we obtained here, which is

$$\sqrt{\frac{\Omega}{2\pi i\sin(\Omega T)}}e^{\frac{i}{2\hbar}[y\Omega\cot(\Omega T)y-2x\Omega(\sin(\Omega T)^{-1}y+x\Omega\cot(\Omega T)x)}$$

through the links $q_I=y, q_F=x$ and $\omega=\Omega$.

Remark 8.10 (Trigonometric special cases)**.** If one looks at equation (143), one sees that one needs to impose the condition

$$T \neq n\pi, \quad n \in \mathbb{Z},$$

for the expression to make sense. However, that is not necessary. If $T = n\pi$, then $\sin(T) = 0$ and during the derivation of the Lebesgue integral expression of the infinite dimensional oscillatory integral we get

$$\begin{aligned}\psi(x,T) &= \frac{1}{2\pi} e^{-\frac{ix^2\Omega}{2\hbar}\tan(\Omega T)} \int_{\mathbb{R}} e^{ikx(\cos(\Omega T))^{-1}} e^{-\frac{i\hbar k^2}{2}\Omega^{-1}\tan(\Omega T)} \widehat{\psi_0}(k)dk \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx(\cos(\Omega T))^{-1}}\widehat{\psi_0}(k)dk = \int_{\mathbb{R}} \delta(y - x(\cos(\Omega T))^{-1})\psi_0(y)dy = \psi_0(x(\cos(\Omega T))^{-1}).\end{aligned}$$

That means that the propagator for the case $T = n\pi$ is just a Dirac delta function. It is now also possible to make sense of the case $T = \left(n + \frac{1}{2}\right)\pi$, for which we noted in corollary 8.4 that it is impossible to calculate a Green's function. With the infinite dimensional oscillatory integral

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle\gamma,(I-L)\gamma\rangle} e^{-\frac{i}{2\hbar}x\Omega^2 xT} e^{-\frac{i}{\hbar}\int_0^T ds\gamma(s)\Omega^2 x ds}\psi_0(\gamma(s)+x)d\gamma$$

for this special case there are no problems. Therefore we can also opt to take a limit $T \to \left(n + \frac{1}{2}\right)\pi$ for the expression we found in the last theorem, in that case we get

$$\begin{aligned}&\lim_{T\to\left(n+\frac{1}{2}\right)\pi} \sqrt{\frac{\Omega}{2\pi i\sin(\Omega T)}} \int_{\mathbb{R}} e^{\frac{i}{2\hbar}[y\Omega\cot(\Omega T)y - 2x\Omega(\sin(\Omega T)^{-1}y + x\Omega\cot(\Omega T)x)}\psi_0(y)dy \\ &= \sqrt{\frac{\Omega}{\pm 2\pi i}}\int_{\mathbb{R}} e^{\mp\frac{i}{\hbar}x\Omega y}\psi_0(y)dy.\end{aligned}$$

Note that we are able to take the limit inside the Lebesgue integral because of the dominated convergence theorem, since ψ_0 is a Schwartz function.

We can thus conclude that the oscillatory integral formalism has been successful in rigorously constructing a path integral for the harmonic oscillator. We now turn to the anharmonic oscillator.

8.2 Anharmonic oscillator

The importance of the anharmonic oscillator was discussed at length in the chapter on quantum field theory, since the anharmonic term is a possible interaction term. In the previous chapter on the oscillatory integral formalism, it was discussed how to generalise the phase function from the quadratic (Fresnel) to the quartic (generalised Fresnel) case in the

context of infinite dimensional oscillatory integral. Now we would like to apply these results to construct a rigorous path integral for the anharmonic oscillator. This section is based on [10, 12, 108]. This time we consider the following Cauchy problem in $L^2(\mathbb{R}^n)$

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\Delta\psi + V(x)\psi \tag{145}$$

$$\psi(x,0) = \psi_0(x), \tag{146}$$

where the potential V is of the following form

$$V(x) = \frac{1}{2}x\Omega^2 x + \lambda|x|^4.$$

We can generalise the potential to the following form

$$V(x) = \frac{1}{2}x\Omega^2 x + \lambda C(x,x,x,x),$$

where like in the previous chapter C is a fourth-order, covariant and completely symmetric tensor. It should be mentioned that if $\lambda < 0$ the Cauchy problem loses the uniqueness of the solution. We will now construct a weak solution for the Schrödinger equation. First the final result will be given before the derivation will be discussed. Let ϕ_0 and ψ_0 be initial states in $L^2(\mathbb{R}^n)$. Then the time evolution after time t of ψ_0 is given by $e^{-\frac{i}{\hbar}Ht}\psi_0$. The path integral, which is the transition amplitude, is then given by

$$\langle\phi_0, e^{-\frac{i}{\hbar}t}\psi_0\rangle = \int_{\mathbb{R}^n}\overline{\phi(x)}\int_{\gamma(T)=x} e^{\frac{i}{2\hbar}\int_0^T(\dot\gamma(s))^2ds - \frac{i}{\hbar}\int_0^T\gamma(s)\Omega^2\gamma(s)ds - \frac{i\lambda}{\hbar}\int_0^t|\gamma(s)|^4ds}\psi_0(\gamma(t)+x)d\gamma dx. \tag{147}$$

We will take $\mathcal{H} \coloneqq \mathbb{R}^n \times \mathcal{H}_T$ as our infinite dimensional Hilbert space in this setting, where we remind the reader that $\mathcal{H}_T$ is the Cameron-Martin space. We have altered the initial conditions of the Cameron-Martin space, from now on $\gamma(0) = 0$. Just like in the case of the harmonic oscillator, we define a suitable operator that will justify using the Parseval equality

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,(I-L)x\rangle}f(x)dx = (\det(I-L))^{-1/2}\int_{\mathbb{R}^n} e^{-\frac{i}{2\hbar}\langle x,(I-L)^{-1}x\rangle}f(x)d\hat\mu_f(x).$$

Let B be an operator on $\mathcal{H}$ as follows

$$B(x,\gamma) = (z,\xi) \tag{148}$$

$$z = T\Omega^2 x + \Omega^2\int_0^T\gamma(s)ds \tag{149}$$

$$\xi(s) = \Omega^2 x\left(Ts - \frac{s^2}{2}\right) - \int_0^s\int_T^u\Omega^2\gamma(r)drdu. \tag{150}$$

It is necessary to calculate the inner product $\langle (y,\eta), B(x,\gamma)\rangle$, if we know that $B(x,\gamma) = (z,\xi)$ we get

$$\langle (y,\eta), B(x,\gamma)\rangle = \langle x, y\rangle_{\mathbb{R}^n} + \langle \eta, \xi\rangle_{\mathcal{H}_T}.$$

We evaluate each of the inner products

$$\langle x, y\rangle = y \cdot T\Omega^2 x + y \cdot \Omega^2 \int_0^T \gamma(s) ds = \int_0^T y\Omega^2 (x + \gamma(s)) ds,$$

$$\begin{aligned}
\langle \eta, \xi\rangle &= \int_0^T \dot{\eta}(s) \cdot \left[\Omega^2 x(T-s) - \int_T^s \Omega^2 \gamma(r) dr\right] ds \\
&= \left[\eta(s) \cdot \Omega^2 x(T-s) - \eta(s) \int_T^s \Omega^2 \gamma(r) dr\right]_0^T + \int_0^T \eta(s) \cdot \left[\Omega^2 x + \Omega^2 \gamma(s)\right] ds \\
&= \int_0^T \eta(s)\, \Omega^2 \left(x + \gamma(s)\right) ds.
\end{aligned}$$

Hence we obtain the final answer

$$\langle (y,\eta), B(x,\gamma)\rangle = \langle x, y\rangle_{\mathbb{R}^n} + \langle \eta, \xi\rangle_{\mathcal{H}_T} = \int_0^T (\eta(s) + y)\Omega^2 \left(x + \gamma(s)\right) ds, \tag{151}$$

which is analogous to the inner product $\langle \gamma_1, L\gamma_2\rangle$. We then introduce the following fourth-order tensor operator $A : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$

$$A((x_1,\gamma_1),(x_2,\gamma_2),(x_3,\gamma_3),(x_4,\gamma_4)) = \int_0^T (\gamma_1(s) + x_1)(\gamma_2(s) + x_2)(\gamma_3(s) + x_3)(\gamma_4(s) + x_4) ds.$$

Subsequently we introduce the following fourth order potential function (where λ is a parameter, in physical applications it is assumed to be small so that we can use a perturbation expansion, although our construction also works for the case that λ is large)

$$V_4(x,\gamma) = \lambda A((x,\gamma),(x,\gamma),(x,\gamma),(x,\gamma)) = \lambda \int_0^T |\gamma(s) + x|^4 ds.$$

We can then go ahead and define the following function

$$f(x,\gamma) = (2\pi\hbar i)^{n/2} e^{-\frac{i}{2\hbar}|x|^2} \overline{\phi(x)} \psi_0(\gamma(T) + x). \tag{152}$$

If we then impose suitable conditions on T, the endpoint of the time interval (recall that T is a fixed number when you choose a particular Cameron-Martin space). Then we can

mathematically realise the path integral for the anharmonic oscillator, given in equation (147), as the following infinite dimensional oscillatory integral

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}(|x|^2+\|\gamma\|^2)} e^{-\frac{i}{2\hbar}\langle(x,\gamma),B((x,\gamma))\rangle} e^{-\frac{i}{\hbar}V_4(x,\gamma)} f(x,\gamma)dxd\gamma. \tag{153}$$

We will derive this by using the results proven in the previous chapter on infinite dimensional oscillatory integrals with quartic phase (i.e. infinite dimensional generalised Fresnel integrals), especially theorems 7.33 and 7.34. We will first prove a couple of lemmas on the operators in question so that we are justified in using the aforementioned results. One can take these lemmas on faith and move to theorem 8.13 to read the final result.

Lemma 8.11. Let once again $\Omega_1, \ldots, \Omega_n$ be the eigenvalues of the matrix Ω^2. The operator $B : \mathcal{H} \to \mathcal{H}$ defined in equations (148)-(150) is trace class. If in addition the following conditions are satisfied

$$\Omega_i t < \frac{\pi}{2}, \quad 1 - \Omega_i \tan(\Omega_i t) > 0, \tag{154}$$

then the operator $I - B$ is strictly positive.

Proof. It is immediately apparent from equation (151) that the operator B is symmetric and self-adjoint.We will give the proof for B being trace class and strictly positive for the one-dimensional case. The proof can be easily extended to the multi-dimensional case. This means that we assume Ω^2 is a number. Suppose we have an eigenvalue c with eigenvector (v, γ), then the following equations are satisfied

$$cv = T\Omega^2 v + \Omega^2 \int_0^T \gamma(s)ds$$
$$c\gamma(s) = \Omega^2 v\left(Ts - \frac{s^2}{2}\right) - \int_0^s \int_T^u \Omega^2 \gamma(r)drdu.$$

Differentiating the second equation tells us that

$$c\ddot{\gamma}(s) + \Omega^2\gamma(s) = -\Omega^2 v.$$

One can also observe the following initial conditions to hold

$$\gamma(0) = 0, \quad \dot{\gamma}(T) = 0.$$

Subsequently one can show that the system has a unique solution if and only if the following equations holds

$$\frac{\Omega}{\sqrt{c}} \tan\left(\frac{\Omega T}{\sqrt{c}}\right) = 1. \tag{155}$$

If one plots this equation one can see that the eigenvalues are more and more distantly spaced, which means that their infinite product is finite and hence the operator B is trace class, see [10] for more details. □

Then we need another crucial lemma, which we will not prove.

Lemma 8.12. Under conditions that will be stated below, the function f, given in equation (152), belongs to the Fresnel algebra $\mathcal{F}(\mathcal{H})$, which means that it is the Fourier transform of a bounded variation measure. This measure μ_f satisfies the following condition

$$\int_{\mathcal{H}} e^{\frac{\hbar}{4}\langle (y,\eta),(I-B)^{-1}(y,\eta)\rangle} d|\mu_f|(y,\eta) < \infty,$$

where the operator B was defined in equations (148)-(150). Now we will state the required conditions for this to be true. The wavefunctions ϕ and ψ_0 lie in the set $L^2(\mathbb{R}^n)\cap\mathcal{F}(\mathbb{R}^n)$. Then we know that there exist bounded variation measures such that $\hat{\mu}_0 = \psi_0$ and $\hat{\mu}_\phi = \overline{\phi(x)}$ (the last result follows because $\mathcal{F}(\mathbb{R}^n)$ is a Banach algebra). We assume that the time T satisfies the conditions given in equation (154). Finally, we assume that the following condition holds

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{\hbar}{4}(y-(\cos(\Omega t))^{-1}x)(1-\Omega\tan(\Omega t))^{-1}(y-(\cos(\Omega t))^{-1}x)} e^{\frac{\hbar}{4}x\Omega^{-1}} e^{\frac{\hbar}{4}x\Omega^{-1}\tan(\Omega t)x} d|\mu_0|(x) d|\mu_\phi|(y) < \infty.$$

If all these conditions are satisfied, the function f belongs to the Fresnel algebra.

Proof. The first thing we observe is that the function $(2\pi\hbar i)^{n/2} e^{-\frac{i}{2\hbar}|x|^2}\overline{\phi(x)}$ is the Fourier transform of a measure of bounded variation. This is because we assumed that ϕ belongs to the Fresnel algebra $\mathcal{F}(\mathbb{R}^n)$, the factor $(2\pi\hbar i)^{n/2}$ can be absorbed in the measure μ_ϕ by a change of variable. Finally, the term $e^{-\frac{i}{2\hbar}|x|^2}$ arises from the Dirac measure δ_0, which means that

$$e^{-\frac{i}{2\hbar}|x|^2} = \int e^{-\frac{i}{2\hbar}\langle x+\gamma, x+\gamma\rangle} d\delta_0(\gamma).$$

So the required measure is the product measure $\mu_\phi \times \delta_0$, we reiterate that μ_ϕ is a measure on $\mathbb{R}^n$ and δ_0 a measure on $\mathcal{H}_T$. Because $\mathcal{F}(\mathcal{H}_T \times \mathbb{R}^n)$ is a Banach algebra, we know that the function f (given in equation (152)) belongs to $\mathcal{F}(\mathcal{H}_T \times \mathbb{R}^n)$ too. The associated measure is $(\mu_\phi \times \delta_0) * \mu_0$, which is a convolution measure. This follows from the assumption that $\psi_0 \in \mathcal{F}(\mathbb{R}^n)$.

We now have to show that the inequality

$$\int_{\mathcal{H}} e^{\frac{\hbar}{4}\langle (y,\eta),(I-B)^{-1}(y,\eta)\rangle} d|\mu_f|(y,\eta) < \infty$$

holds. By theorem A.5 from [108] we have that

$$e^{\frac{\hbar}{4}\langle (y,\eta),(I-B)^{-1}(y,\eta)\rangle} = \sqrt{\det(I-B)} F(y/\sqrt{2}, \eta/\sqrt{2}),$$

where we have defined that

$$F(y,\eta) = \int_{C_T\times\mathbb{R}^n} e^{\sqrt{\hbar}xy + \hbar n(\eta)(\omega)} e^{\frac{1}{2}\langle (x,\omega), B(x,\omega)} dN(x) dW(\omega). \tag{156}$$

Note that N and W are Gaussian measures. The space C_T is the Banach space of all continuous functions $C([0,T],\mathbb{R}^n)$ endowed with the supremum norm and the property that $\gamma(0)=0$. Note that n is a normally distributed random variable with covariance $|y|^2$ with the property that

$$\int n(x_1)n(x_2)d\mu = \langle x_1, x_2\rangle. \tag{157}$$

We subsequently use the Fubini-Tonelli theorem and also equation (151) to show that

$$\begin{aligned}
F(y,\eta) &= \int_{C_T\times\mathbb{R}^n} e^{\sqrt{\hbar}xy+\hbar n(\eta)(\omega)} e^{\frac{1}{2}\int_0^T (x+\omega(s))\Omega^2(x+\omega(s))ds} dN(x)dW(\omega) \\
&= \int_{\mathbb{R}^n} e^{\sqrt{\hbar}xy} e^{\frac{1}{2}x\Omega^2 xT} \int_{C_T} e^{\hbar n(\eta)(\omega)} e^{x\int_0^T \Omega^2\omega(s)ds} e^{\frac{1}{2}\int_0^T \omega(s)\Omega^2\omega(s)ds} dN(x)dW \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{\sqrt{\hbar}xy} e^{\frac{1}{2}x(\Omega^2T-1)x} \int_{C_T} e^{\hbar n(\eta)(\omega)} e^{\langle v_x,\omega\rangle} e^{\frac{1}{2}\int_0^T \omega(s)\Omega^2\omega(s)ds} dxdW(\omega) \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{\sqrt{\hbar}xy} e^{\frac{1}{2}x(\Omega^2T-1)x} \int_{C_T} e^{\hbar n(\eta)(\omega)} e^{n(v_x)(\omega)} e^{\frac{1}{2}\langle\omega,L\omega\rangle} dxdW(\omega).
\end{aligned}$$

The map L is the same as the one defined in the section on the harmonic oscillator, its definition can be found in equation (132). We have defined $v_x \in \mathcal{H}_T$ to be

$$v_x(s) \coloneqq \Omega^2 x(Ts - s^2/2). \tag{158}$$

Note that n is a map $n:\mathcal{H}\to\mathcal{H}^*$, where $n(x)$ is a random variable given by

$$n(x)(y) = \langle x, y\rangle.$$

Using [108, Theorem A.5] we see that the function

$$G(x) = \int_{C_T} e^{\hbar n(\eta)(\omega)} e^{n(v_x)(\omega)} e^{\frac{1}{2}\langle\omega,L\omega\rangle} dW(\omega), \tag{159}$$

can be written as

$$G(x) = \frac{1}{\sqrt{\det(\cos(\Omega T))}} e^{\frac{1}{2}\langle\sqrt{\hbar}\eta+v_x,(I-L)^{-1}(\sqrt{\hbar}\eta+v_x)\rangle}. \tag{160}$$

The inverse $(I-L)^{-1}$ was calculated in lemma 8.2. Some computation (explained in appendix B.4) gives that this is equal to

$$G(x) = \frac{1}{\sqrt{\det(\cos(\Omega T))}} e^{\frac{1}{2}\langle\sqrt{\hbar}\eta,(I-L)^{-1}\sqrt{\hbar}\eta\rangle} e^{\frac{1}{2}x(\Omega\tan(\Omega T)-T\Omega^2)x} e^{\frac{1}{2}\langle v_x,(I-L)^{-1}\sqrt{\hbar}\eta)\rangle}. \tag{161}$$

If we insert this in the expression for F (defined in equation (156)), we see that

$$F(y,\eta) = \frac{(2\pi)^{-n/2}}{\sqrt{\det(\cos(\Omega T))}} e^{\frac{1}{2}\langle\sqrt{\hbar}\eta,(I-L)^{-1}\sqrt{\hbar}\eta\rangle} \int_{\mathbb{R}^n} e^{\sqrt{\hbar}xy} e^{\frac{1}{2}x(\Omega\tan(\Omega T)-T\Omega^2)x} e^{\frac{1}{2}\langle v_x,(I-L)^{-1}\sqrt{\hbar}\eta)\rangle} dx.$$

The inequality

$$\int_{\mathcal{H}} e^{\frac{\hbar}{4}\langle(y,\eta),(I-B)^{-1}(y,\eta)\rangle} d|\mu_f|(y,\eta) < \infty$$

becomes

$$\int_{\mathcal{H}_T} F(y/\sqrt{2},\eta/\sqrt{2}) d|\mu_f|(y,\eta) < \infty.$$

The left hand side can be rewritten as

$$\begin{aligned}&\int_{\mathcal{H}_T\times\mathcal{H}_T} F((x+y)/\sqrt{2},(\gamma+\eta)/\sqrt{2}) d|\mu_\phi\times\delta_0|(y,\eta) d|\mu_0|(x,\gamma)\\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} F((x+y)/\sqrt{2},(x\gamma_T)/\sqrt{2}) d|\mu_\phi\times\delta_0|(y,\eta) d|\mu_0|(x,\gamma) < \infty,\end{aligned}$$

where $\gamma_T(s) = s$. Substituting the known results for F, we obtain the required condition. □

Now we are able to state the main theorem that proves that the infinite dimensional oscillatory integral in equation (153) exists.

Theorem 8.13. If λ is a negative parameter and the final time T satisfies all the aforementioned conditions and also ϕ and ψ_0 mention all the conditions in the previous lemma, then the function f defined in equation (152) is the Fourier transform of a bounded variation measure, the oscillatory integral equals

$$\begin{aligned}&\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}(|x|^2+\|\gamma\|^2)} e^{-\frac{i}{2\hbar}\langle(x,\gamma),B((x,\gamma))\rangle} e^{-\frac{i}{\hbar}V_4(x,\gamma)} f(x,\gamma) dx d\gamma\\ &= \int_{\mathcal{H}_T\times\mathbb{R}^n} \left(\int_{C_T\times\mathbb{R}^n} e^{ie^{i\pi/4}(x\cdot y+\sqrt{\hbar}n(\gamma)(\omega))} e^{\frac{1}{2\hbar}\int_0^T(\hbar\omega(s)+x)\Omega^2(\hbar\omega(s)+x)ds} e^{-i\frac{\lambda}{\hbar}\int_0^T|\sqrt{\hbar}\omega(s)+x|^4} dW(\omega) \right.\\ &\left. \frac{e^{-\frac{|x|^2}{2\hbar}}}{(2\pi\hbar)^{n/2}} dx \right) \mu_f(dy d\gamma).\end{aligned}$$

This can also be written as

$$i^{n/2} \int_{C_T\times\mathbb{R}^n} e^{\frac{1}{2\hbar}\int_0^T(\hbar\omega(s)+x)\Omega^2(\hbar\omega(s)+x)ds} e^{-i\frac{\lambda}{\hbar}\int_0^T|\sqrt{\hbar}\omega(s)+x|^4 ds} \overline{\phi(e^{i\pi/4}x)} \psi_0(e^{i\pi/4}\sqrt{\hbar}\omega(t)+e^{i\pi/4}x) dW(\omega) dx. \tag{162}$$

Proof. The proof is an application of theorem 7.34 and is taken from [10, Theorem 5]. What we need to check before we are allowed to use theorem 7.34, is that

$$\int_{\mathcal{H}} e^{\frac{\hbar}{4}\langle k,(I-B)^{-1}k\rangle} d|\mu_f|(k) < \infty.$$

This was verified in the last lemma (8.12). It was also shown that $I - B$ is strictly positive and B is trace class. Using theorem 7.34 we arrive at the conclusion, namely that both the infinite dimensional oscillatory integral exists and it is given in the form above. □

Note that these results have been generalised to the case that the quartic potential is time-dependent, but we will not cover that here. More details can be found in [12]. First we make a remark on the scope of these results before the properties of this solution are discussed.

Remark 8.14. The condition that ψ_0 and ϕ must lie in $\mathcal{F}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ might seem rather restrictive. However, one takes ϕ and ψ_0 to lie in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and to be of the specific form

$$|\psi_0(x)| = P(x)e^{-\frac{\alpha\hbar}{2}x^2}, \quad |\phi(x)| = Q(x)e^{-\frac{\beta\hbar}{2}x^2}.$$

Note that P and Q are polynomials, while α and β must satisfy the following conditions (we still use the notation from earlier on)

$$\begin{aligned}
0 &< \beta - \frac{1}{1 - \Omega_i \tan(\Omega_i t)} \\
0 &< \alpha - \frac{1}{\cos^2 \Omega_i t(1 - \Omega_i \tan(\Omega_i t))} + \Omega^{-1} \tan(\Omega_i t) \\
0 &< (\beta - (1 - \Omega_i \tan(\Omega_i t))^{-1}) \left(\alpha - \frac{1}{\cos^2 \Omega_i t(1 - \Omega_i \tan(\Omega_i t))} + \Omega^{-1} \tan(\Omega_i t) \right) \\
&\quad - \left(\frac{1}{\cos^2 \Omega_i t(1 - \Omega_i \tan(\Omega_i t))} \right)^2.
\end{aligned}$$

It is known that the functions ψ_0 and ϕ are dense in $L^2(\mathbb{R}^n)$, which is after all the set of all the physically realisable states/wavefunctions. So the condition is not that restrictive at all.

Now we are ready to discuss in what manner equation (162) constitutes a formal solution of the Schrödinger equation and especially the Cauchy problem (145) and (146), because it only has been proven that the path integral exists in the form of an oscillatory integral. First of all, it should be remarked that in general if the potential is time-independent and satisfies the following inequality

$$V(x) \geq C(1 + |x|)^{2+\epsilon},$$

for $\epsilon, C > 0$, then the solution of the Schrödinger equation is nowhere C^1, see [160] for further details. That means for our problem, the anharmonic oscillator (whose potential satisfies this condition) we cannot hope to have a C^2 solution. Therefore we have to opt for what is known in PDE theory as a weak solution [49]. What that precisely means, will become clear in the next theorem.

Theorem 8.15. Let λ be a real parameter (note that we have now dropped the assumption that λ is negative). Let ϕ, ψ_0 and T satisfy all the aforementioned conditions. Then the Gaussian integral

$$I_T(\phi, \psi_0) \coloneqq (i)^{n/2} \int_{C_T \times \mathbb{R}^n} e^{\frac{1}{2\hbar}\int_0^T (\hbar\omega(s)+x)\Omega^2(\hbar\omega(s)+x)ds} e^{-i\frac{\lambda}{\hbar}\int_0^T |\sqrt{\hbar}\omega(s)+x|^4 ds} \tag{163a}$$

$$\overline{\phi(e^{i\pi/4}x)}\psi_0(e^{i\pi/4}\sqrt{\hbar}\omega(t) + e^{i\pi/4}x)dW(\omega)dx \tag{163b}$$

is a quadratic form in ϕ and ψ_0. Moreover, it satisfies the Schrödinger equation in the following weak sense

$$I_0(\phi, \psi_0) = \langle \phi, \psi_0 \rangle \tag{164}$$

$$i\hbar \frac{d}{dt} I_t(\phi, \psi_0) = I_t(\phi, H\psi_0) = I_t(H\phi, \psi_0). \tag{165}$$

Proof. The first formula follows rather easily, if $t = 0$ we have that

$$\begin{aligned} I_0(\phi, \psi_0) &= (i)^{n/2} \int_{C_T \times \mathbb{R}^n} \overline{\phi(e^{i\pi/4}x)}\psi_0(e^{i\pi/4}x)dW(\omega)dx \\ &= (i)^{n/2} \int_{\mathbb{R}^n} \overline{\phi(e^{i\pi/4}x)}\psi_0(e^{i\pi/4}x)dx \\ &= \int_{\mathbb{R}^n} \overline{\phi(x)}\psi_0(x)dx = \langle \phi, \psi_0 \rangle. \end{aligned}$$

The second equality follows from the fact that W is a Gaussian measure. The third equality follows from a rotation of the coordinates. So the first equation is indeed true. Now we set out to prove the second equation, we recall that

$$H\psi = -\frac{\Delta}{2}\psi + \frac{1}{2}x\Omega^2 x + \lambda C(x, x, x, x).$$

We are looking at the inner product

$$F(z, t) = \langle \phi, e^{-\frac{ztH}{\hbar}}\psi_0 \rangle$$

for the case that $z = i$, this represents the inner product of ϕ with the solution to the Schrödinger equation with initial datum ψ_0 at time t. If however, $z \in \mathbb{R}^+$, the inner product is the solution to the heat equation

$$\frac{\partial}{\partial t}\psi = -\frac{z}{\hbar}H\psi.$$

Then we know by the Feynman-Kac formula that the inner product can be represented as

$$F(z,T) = z^{n/2} \int_{\mathbb{R}^n} \overline{\phi(\sqrt{z}x)} \int_{C_T} e^{-\frac{z^2}{2\hbar}\int_0^T (\sqrt{\hbar}\omega(s)+x)\Omega^2(\sqrt{\hbar}\omega(s)+x)ds}$$
$$e^{-\frac{z^3\lambda}{\hbar}\int_0^T C(\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x,\sqrt{\hbar}\omega(s)+x)ds}\psi_0(\sqrt{\hbar z}\omega(s)+\sqrt{z}x)dW(\omega)dx,$$

for the case that $z \in \mathbb{R}^+$. If we consider the set

$$D \coloneqq \{z \in \mathbb{C} \,|\mathrm{Re}(z) > 0\},$$

then we observe that $F(z,t)$ is well defined for $z \in \overline{D}$, because we assumed that ϕ and ψ_0 are Schwartz functions. Therefore we see that the choice $z = i \in \overline{D}$ represents the solution to the Schrödinger equation. The reader is advised to consult [10, 132] for further details. □

Thus given the circumstances, i.e. the solution is nowhere C^1, we have obtained the best we can get. I would like to close the chapter with the remark that for this system the infinite dimensional oscillatory integral approach can be linked with the analytical continuation approach that is found in [115] for example. The relevant result in this regard can be found in [108, Theorem 3.10].

8.3 Conclusion

With this, the chapter on the applications of the oscillatory integral formalism is concluded, we have successfully established infinite dimensional oscillatory integrals for both the harmonic and the anharmonic oscillator. It has been shown that the integral established in equations (143) and (144) satisfy the Schrödinger equation. Due to the result that the solution to the Schrödinger equation for the anharmonic oscillator can be nowhere C^1, we achieved a weak solution in equation (163) that satisfies the Schrödinger equation in a weak sense.

The formalism has hence been successful in this regard. There remain big problems with this approach, such as its inability to handle a Coulomb potential or indeed any potential with a singularity. Another type of potentials that still needs to be handled are even power potential of higher order than four or odd power potentials. The existing results cannot be easily extended to cover those cases, due to the convergence issues.

Another weakness of this method is that it relies heavily on the Parseval-type equality to find path integrals. For each case a Parseval-type equality is used to find an expression for the path integral formulated in terms of a convergent Lebesgue integral. As the reader knows by now, the Parseval-type equality heavily relies on the use of the Fresnel algebra. If this method really is going to provide a rigorous foundation for all potentials that appear in physics, then it needs to find different ways than by using the Fresnel algebra to construct oscillatory integrals, which is very well possible.

9 Combining the Duru-Kleinert transform with the oscillatory integral formalism

It has been studied how the Duru-Kleinert transform is useful to handle path integrals with singularities. The aim of this chapter is to give the transform a rigorous foundation using oscillatory integrals. The most important extra constraint will be that we will drop the independence of the infinite dimensional oscillatory integral on the sequence of projection operators. The reason for that is that although in the limit $N \to \infty$ the following two sliced actions

$$A_E^N = \sum_{k=1}^{N+1} \frac{m}{2\epsilon}(x_k - x_{k-1})^2 - \epsilon_s[V(x_k) - E]$$
$$A_E^N = \sum_{k=1}^{N+1} \frac{m}{2\epsilon f_l(x_k) f_r(x_{k-1})}(x_k - x_{k-1})^2 - \epsilon_s f_l(x_k)[V(x_k) - E] f_r(x_{k-1})$$

lead to the same fixed-energy amplitude. Here f_l and f_r are the regulating functions, a choice of discretisation breaks the functional symmetry and determines whether the approximations are finite or not. This of course has to do with the fact that in the discrete case, the relation between the time t and the pseudotime s is the following

$$dt = ds f_l(x_k) f_r(x_{k-1}).$$

That means that the width of the slices is space-dependent and hence path collapse might or might not be prevented. First, we will discuss how to combine the Duru-Kleinert transformation with the infinite dimensional oscillatory integrals and subsequently we will apply the results to the hydrogen atom and construct a rigorous oscillatory integral for the two-dimensional hydrogen atom.

9.1 Choosing a suitable sequence of projection operators

Once again, we will choose the Cameron-Martin space as our Hilbert space. The projection operators that are used for the standard time slicing are of the form

$$P_n : \mathcal{H}_T \to \mathbb{R}^N$$
$$P_n(\gamma) = (\gamma(0), \gamma(\Delta T), \ldots, \gamma(T - \Delta T), 0).$$

Note that in the definition of the Cameron-Martin space we included that $\gamma(T) = 0$. The width of the time slice is $\Delta T = T/(N-1)$. Note that it is not necessary for the time interval to be uniform, we just need $D_N = \{t_0, t_1, \ldots, t_N\}$ (where $t_0 = 0$ and $t_N = T$) to be a partition. Most of the results will be proven for projection operators with a fixed time

interval Δt, but they can easily be extended to arbitrary partitions. Note that for a sequence of projection operators to be used for an infinite dimensional oscillatory integral, we require that $D_N \subset D_{N+1}$. For the pseudotime slicing, we instead are using projection operators of the form

$$\begin{aligned} &P_n : \mathcal{H}_S \to \mathbb{R}^N \\ &P_n(\gamma) = (\gamma(0), \gamma(\Delta S), \ldots, \gamma(S - \Delta S), \gamma(S)), \end{aligned}$$

where the pseudotime depends both on t and the position. The precise dependence is determined by the chosen regulating functions. So the projection operator depends on the trajectory of the path in $\mathbb{R}^n$ (one should distinguish between n being the dimension of configuration space and N the dimension of the discretisation). Note that it is once again possible to generalise the results that will follow for general partitions $\{s_0, s_1, \ldots, s_N\}$ of the time interval $[0, S]$. We will now define a new type of oscillatory integral, namely one which is projection sequence dependent.

Definition 9.1. A Duru-Kleinert oscillatory integral of a measurable function $f : \mathcal{H} \to \mathbb{C}$ is a limit of a sequence of finite dimensional oscillatory integrals for a class of sequences of projection operators $\{P_j\}$, which means

$$\widehat{\int_{\mathcal{H}}} f dx \coloneqq \lim_{j\to\infty} \widetilde{\int_{P_j\mathcal{H}}} e^{\frac{i}{2\hbar}\langle P_j x, P_j x\rangle} f(P_j x) d(P_j x) \left(\widetilde{\int_{P_j\mathcal{H}}} e^{\frac{i}{2\hbar}\langle P_j x, P_j x\rangle} d(P_j x)\right)^{-1}. \tag{166}$$

One should always specify which class of projection operators one is considering if one is defining an oscillatory integral. The hat is the notation for the Duru-Kleinert oscillatory integral in order to distinguish it from the infinite dimensional oscillatory integral. Recall that for an infinite dimensional oscillatory integral the limit is independent of the chosen sequence of projection operators, under conditions mentioned in the previous chapter.

The reason why we would want to define a projection operator sequence dependent oscillatory integral is that for some choices of discretisation the path integral does not converge, while for others it does. First, we try to give the Duru-Kleinert transformation a mathematical foundation and subsequently use that to construct a Duru-Kleinert oscillatory integral for the hydrogen atom. This oscillatory integral can be related to the infinite dimensional oscillatory integral for the hydrogen atom. In this manner, we obtain a fixed-energy amplitude for the Coulomb potential.

9.2 The Duru-Kleinert transform

In this section, we first give the Duru-Kleinert transform a mathematical justification. Because of the spectral theorem, a self-adjoint operator has a basis of eigenstates. There are

of course further complications, namely that there is a discrete spectrum and a continuous spectrum. However, since we are focusing on the hydrogen atom, we will ignore these complications because we are considering bound states, they can be expanded as an infinite sum of bound eigenstates. In particular, we have the following spectral decomposition of the propagator

$$\langle x_a, t_a | x_b, t_b \rangle = \sum_n \psi_n(x_a)\overline{\psi_n}(x_b)e^{-iE_n(t_b-t_a)/\hbar}.$$

This formula can be proven by inserting a complete set of states in the expression and subsequently the Hamiltonian can be replaced by the eigenvalue. For the hydrogen atom, this is justified because the Coulomb Hamiltonian has a discrete spectrum in the negative energy range and a continuous spectrum $[0, \infty)$ in the positive energy range. Moreover, it can be proven that a basis of eigenstates exists for the space of all the physically realisable (i.e. L^2) negative energy states of the hydrogen atom. Since the positive energy states are not of relevance here [146], we can use of the complete set of eigenstates. Taking the Fourier transform of the propagator gives us that

$$\langle x_a | x_b \rangle_E = \sum_n \psi_n(x_a)\overline{\psi_n}(x_b)\frac{i\hbar}{E - E_n + i\eta}.$$

The $i\eta$-prescription ensures the causality of the time evolution operator and can be considered to be part of the energy. What we will first show is that the pseudopropagator, denoted by $\langle q_F | U_E(s_F - s_I) | q_I \rangle$ results in the same fixed-energy amplitude. We start with a definition:

Definition 9.2. The pseudo-Hamiltonian is defined to be

$$H_E = f_l(x)(H - i\hbar\partial_t)f_r(x).$$

There is the possibility that the regulating functions f_l and f_r may also depend on the momentum. Then the operator ordering needs to be handled with more care, but the derivation does not change significantly.

Theorem 9.3. The fixed-energy amplitude $\langle q_F | q_I \rangle_E$, which is the Fourier transform of the path integral $\langle q_F, t_F | q_I, t_I \rangle$, can be obtained from the pseudopropagator as follows

$$\langle q_F | q_I \rangle_E = \int_0^\infty dS \langle q_F | U_E(S) | q_I \rangle. \tag{167}$$

Recall that U_E is the time-evolution operator. The pseudopropagator is the propagator for a system with Hamiltonian $f_l(x)(H - E)f_r(x)$.

Proof. Assume that we have a solution $\phi(x, t, s)$ that solves the pseudotime Schrödinger equation

$$H_E\phi = i\hbar\partial_s\phi, \quad \text{i.e. } f_l(x)(H - i\hbar\partial_t)f_r(x)\phi = i\hbar\partial_s\phi.$$

The left-hand side of this equation is independent of the pseudotime s, which means that we can factor its dependence out in the following way $\phi(x,t,s) = \phi_{\mathcal{E}}(x,t)e^{-i\mathcal{E}s/\hbar}$. We repeat the same trick and remove the time dependence and we write

$$\phi_{\mathcal{E}}(x,t) = \phi_{\mathcal{E},E}(x)e^{-iEt/\hbar}.$$

That results in the equation

$$f_l(x)(H-E)f_r(x)\phi_{\mathcal{E},E} = \mathcal{E}\phi_{\mathcal{E},E}. \tag{168}$$

So if one makes a choice for E one gets a set of eigenvalues $\mathcal{E}_n$, hence we write for the pseudoenergies $\mathcal{E}_n(E)$. We can then obtain a spectral representation

$$\langle q_F|U_E(S)|q_I\rangle = f_r(q_F)f_l(q_I)\sum_n \phi_{\mathcal{E}_n(E)}(q_F)\overline{\phi_{\mathcal{E}_n(E)}(q_I)}e^{-iS\mathcal{E}_n(E)/\hbar}.$$

Once again such a spectral representation needs to be justified. The Hamiltonian that will be our pseudohamiltonian later is the harmonic oscillator, which does have a discrete spectrum and for which such a spectral representation can be obtained. Hence such a step is justified. Subsequently we need to evaluate the integral

$$\langle q_F|q_I\rangle_E = \int_0^\infty dS\langle q_F|U_E(S)|q_I\rangle.$$

This gives us the result

$$\langle q_F|U_E(S)|q_I\rangle = -f_r(q_F)f_l(q_I)\sum_n \phi_{\mathcal{E}_n(E)}(q_F)\overline{\phi_{\mathcal{E}_n(E)}(q_I)}\frac{i\hbar}{\mathcal{E}_n(E)}.$$

For the standard Schrödinger equation with the normal time t, we have the following representation for the fixed-energy amplitude

$$\langle q_F|q_I\rangle_E = \sum_n \psi_n(q_F)\overline{\psi_n}(q_I)\frac{i\hbar}{E-E_n+i\eta}.$$

A Taylor series expansion around $E = E_m$ then gives us

$$\frac{1}{\mathcal{E}_n(E)} \approx \frac{1}{\mathcal{E}_n'(E_m)\cdot(E-E_m+i\eta)}.$$

This follows because $\mathcal{E}_n(E_m) = 0$ since equation (168) becomes zero on the left-hand side. We conclude that the two fixed-energy amplitudes are the same, since they concern the same energies. □

Remark 9.4. This 'theorem' is not mathematically rigorous at this point, because several assumptions have not been formalised and key steps have not been made rigorous. In particular, the domains of the Hamiltonian and the pseudohamiltonian have not been defined, there are also no assumptions on the spectrum. The same goes for the regulating functions, one can very well imagine that certain choices of regulating functions will not work. It seems at present that this 'theorem' can be established rigorously, but it will require further work to do so.

It can be shown, using a time slicing argument [88, Section 12.2], that the resulting path integral from the Hamiltonian

$$f_l(x)(H - i\hbar\partial_t)f_r(x),$$

is given by

$$\langle q_F|U_E(s_F-s_I)|q_I\rangle = f_r(q_F)f_l(q_I)\int Dx(s)\exp\left(\frac{i}{\hbar}\int_0^S ds\frac{m}{2f_l(x)f_r(x)}x'^2 - f_l(x)(V-E)f_r(x)\right). \tag{169}$$

In the next section, we will show that this integral can be constructed as a Duru-Kleinert oscillatory integral. This will be the path integral for the hydrogen atom.

9.3 A Duru-Kleinert oscillatory integral for the hydrogen atom in two dimensions

What we found earlier on was the following expression for the pseudopropagator of the hydrogen atom

$$\langle q_F|U_E(s_F - s_I)|q_I\rangle = r_F^\lambda r_I^{1-\lambda}\int Dx(s)\exp\left(e^2 S + \frac{i}{\hbar}\int_0^S ds\left[\frac{m}{2r}x'^2 + Er\right]\right). \tag{170}$$

Moreover, we obtained the following expression for the time sliced pseudopropagator of the hydrogen atom

$$\langle q_F|U_E^N(s_F - s_I)|q_I\rangle = \frac{r_F^\lambda r_I^{1-\lambda}}{\sqrt{2\pi i\epsilon_s r_F^{1-\lambda}r_I^\lambda\hbar/m}^n}\int\frac{d^n\Delta x_2\ldots d^n\Delta x_{N+1}}{\sqrt{2\pi\hbar i/m}^{nN}\sqrt{r_1\cdot\ldots\cdot r_N}}e^{\frac{i}{\hbar}A_E^N},$$

with the following action

$$A_E^N = (N+1)\epsilon_s e^2 + \sum_{k=1}^{N+1}\left[\frac{m}{2\epsilon_s r_k^{1-\lambda}r_{k-1}^\lambda}(\Delta x_k)^2 + \epsilon_s r_k E\right].$$

The choice $\lambda = 0$ gives us that

$$A_E^N = (N+1)\epsilon_s e^2 + \sum_{k=1}^{N+1}\left[\frac{m}{2\epsilon_s r_k}(\Delta x_k)^2 + \epsilon_s r_k E\right].$$

As was discussed earlier on, we can transform the path integral by using the Levi-Civita transformation

$$x_a = u_a^2 - u_b^2$$
$$x_b = 2u_a u_b.$$

Note that if we writing the two-dimensional vectors in the complex form $x = x_a + ix_b$ and $u = u_a + iu_b$, then we have the property that $x = u^2$. In the continuum limit, the action can then be written as

$$A = e^2 S + \frac{i}{\hbar}\int_0^S ds\left(\frac{4m}{2}u'^2 + Eu^2\right).$$

Slicing this expression results in

$$A_E^N = (N+1)\epsilon_s e^2 + \sum_{k=1}^{N+1}\left[\frac{4m}{2\epsilon_s}(\Delta u_k)^2 + \epsilon_s u_k^2 E\right].$$

For the sliced pseudopropagator, we obtain

$$\langle q_F|U_E^N(s_F - s_I)|q_I\rangle = \frac{r_I}{2\pi i\epsilon_s r_F\hbar/m}\int\frac{d^2u_1\ldots d^2u_N}{(2\pi\hbar i/4m)^N}e^{\frac{i}{\hbar}A_E^N} \tag{171}$$

$$= \frac{r_I}{2\pi i\epsilon_s r_F\hbar/m}\int\frac{d^2u_1\ldots d^2u_N}{(2\pi\hbar i/\mu)^N}e^{\frac{i}{\hbar}A_E^N} \tag{172}$$

$$A_E^N = Se^2 + \sum_{k=1}^{N+1}\left[\frac{\mu}{2\epsilon_s}(\Delta u_k)^2 + \frac{1}{2}\omega^2\epsilon_s u_k^2\right]. \tag{173}$$

where we have defined $\mu = 4m$ and $\omega = \sqrt{-2E/m}$. Note that we cannot transform the coordinates in the continuum case. There are multiple reasons for this. First, we have not constructed a rigorous path integral at all so we would need to assume that there exists one and then perform the coordinate transformation, which does not make sense. Secondly, even if one assumes the existence of the path integral, performing a coordinate transformation in the continuum case can give an incorrect answer (see [88, Section 8.3] for an example with polar coordinates).

We will now attempt to make the discussion above rigorous by using oscillatory integrals. In the next lemma, it will be shown that for the same projection operator, the finite dimensional/projected path integral in Levi-Civita coordinates is the same as the path integral in Cartesian coordinates. That allows us to conclude that the limit must also be the same. The path integral in Levi-Civita coordinates is the same as the path integral of the harmonic oscillator. An infinite dimensional oscillatory integral, is independent of the chosen sequence of projection operators. That means that the projection operators we choose in the lemma, can also be used to obtain an expression for the pseudopropagator of the hydrogen atom and hence obtain the wavefunctions and energies.

Lemma 9.5. Let $\Delta S = S/j$ and $\gamma_k = \gamma(k\Delta S)$ and define the following projection operator $P_j : \mathcal{H}_S \to P_j(\mathcal{H}_S)$

$$P_j(\gamma) = \begin{cases} \gamma_0 + \frac{s}{\Delta S}(\gamma_1 - \gamma_0), & \text{if } 0 \le s \le \Delta S \\ \gamma_1 + \frac{(s-\Delta S)}{\Delta S}(\gamma_2 - \gamma_1), & \text{if } \Delta S \le s \le 2\Delta S \\ \vdots & \\ \gamma_{j-1} + \frac{(s-(j-1)\Delta S)}{\Delta S}(\gamma_j - \gamma_{j-1}), & \text{if } (j-1)\Delta S \le s \le S \end{cases}$$

the action

$$e^2 S + \int_0^S ds \left(\frac{\mu}{2} u'^2 + \frac{1}{2} m\omega^2 u^2 \right)$$

takes the discretised form

$$A_j = e^2 S + \sum_{k=1}^{j} \left[\frac{\mu}{2\Delta S}(\gamma_k - \gamma_{k-1})^2 + \frac{\Delta S}{3}(\gamma_k^2 + \gamma_k \gamma_{k-1} + \gamma_{k-1}^2) \right].$$

The action

$$\int_0^S ds \left(\frac{m}{2r} x'^2 + Er \right),$$

takes the same discretised form.

Proof. We first look at the action. From now on we will regard u as a path in the Cameron-Martin space $\mathcal{H}_S$. The projection operator P_j hence maps $\mathcal{H}_S$ to $P_j(\mathcal{H}_S)$, which is a j-dimensional subspace of $\mathcal{H}_S$. Under the projection operator, the different components of the action can be written as

$$\begin{aligned} A &= e^2 S + \int_0^S \frac{\mu}{2} (P_j u)'^2 ds \\ &= e^2 S + \sum_{k=1}^{j} \int_{(k-1)\Delta S}^{k\Delta S} \frac{\mu}{2(\Delta S)^2} (\gamma_k - \gamma_{k-1})^2 ds \\ &= e^2 S + \sum_{k=1}^{j} \frac{\mu}{2\Delta S} (\gamma_k - \gamma_{k-1})^2 \end{aligned}$$

We calculate that

$$\int_0^S (P_j(\gamma))^2 ds = \sum_{k=1}^{j} \int_{(k-1)\Delta S}^{k\Delta S} \left(\gamma_{k-1} + \frac{s - (k-1)\Delta S}{\Delta S} (\gamma_k - \gamma_{k-1}) \right)^2 ds$$

$$
\begin{aligned}
&= \sum_{k=1}^{j} \left[\frac{\Delta S}{3(\gamma_k - \gamma_{k-1})} \left(\gamma_{k-1} + \frac{s-(k-1)\Delta S}{\Delta S}(\gamma_k - \gamma_{k-1}) \right)^3 \right]_{(k-1)\Delta S}^{k\Delta S} \\
&= \sum_{k=1}^{j} \left[\frac{\Delta S}{3(\gamma_k - \gamma_{k-1})} \left(\gamma_{k-1} + (\gamma_k - \gamma_{k-1}) \right)^3 - \frac{\Delta S}{3(\gamma_k - \gamma_{k-1})} (\gamma_{k-1})^3 \right] \\
&= \sum_{k=1}^{j} \left[\frac{\Delta S}{3} (\gamma_k^2 + \gamma_k \gamma_{k-1} + \gamma_{k-1}^2) \right]
\end{aligned}
$$

If we consider for the action

$$
\int_0^S ds \left(\frac{m}{2r} x'^2 + Er \right),
$$

we first remark that we have the following projection operator

$$
P_j(\gamma) = \begin{cases}
\left[\gamma_0 + \frac{s}{\Delta S}(\gamma_1 - \gamma_0) \right]^2, & \text{if } 0 \le s \le \Delta S \\
\left[\gamma_1 + \frac{(s-\Delta S)}{\Delta S}(\gamma_2 - \gamma_1) \right]^2, & \text{if } \Delta S \le s \le 2\Delta S \\
\vdots & \\
\left[\gamma_{j-1} + \frac{(s-(j-1)\Delta S)}{\Delta S}(\gamma_j - \gamma_{j-1}) \right]^2, & \text{if } (j-1)\Delta S \le s \le S
\end{cases}
$$

The integral over Er being the same, but for $\frac{m}{2r}x'^2$ we have that

$$
\int_0^S \frac{m}{2r} x'^2 ds = \sum_{k=1}^{j} \int_{(k-1)\Delta S}^{k\Delta S} \frac{m}{(\Delta S)^2} \frac{(\gamma_{k-1}(\gamma_k - \gamma_{k-1}) + (s/\Delta S - (k-1))(\gamma_k - \gamma_{k-1})^2)^2}{\left[\gamma_{k-1} + \frac{(s-(k-1)\Delta S)}{\Delta S}(\gamma_k - \gamma_{k-1}) \right]^2} ds
$$

If we define

$$
\begin{aligned}
a &= \frac{1}{\Delta S}(\gamma_k - \gamma_{k-1}) \\
b &= \gamma_{k-1} - (k-1)(\gamma_k - \gamma_{k-1}) \\
c &= \frac{1}{\Delta S}(\gamma_k - \gamma_{k-1})^2 \\
d &= (\gamma_k - \gamma_{k-1})(\gamma_{k-1} - (k-1)(\gamma_k - \gamma_{k-1}))
\end{aligned}
$$

$$\begin{aligned}\int_0^S \frac{m}{2r}x'^2 ds &= \sum_{k=1}^{j}\int_{(k-1)\Delta S}^{k\Delta S} \frac{m}{(\Delta S)^2}\frac{\gamma_{k-1}(\gamma_k-\gamma_{k-1})+(s-(k-1)\Delta S)(\gamma_k-\gamma_{k-1})^2}{\left[\gamma_{k-1}+\frac{(s-(k-1)\Delta S)}{\Delta S}(\gamma_k-\gamma_{k-1})\right]^2}ds \\ &= \sum_{k=1}^{j}\int_{(k-1)\Delta S}^{k\Delta S}\frac{m}{(\Delta S)^2}\frac{(cs+d)^2}{(as+b)^2}ds \\ &= \sum_{k=1}^{j}\frac{m}{(\Delta S)^2a^3}\left[-\frac{(bc-ad)^2}{as+b}-2c(bc-ad)\ln(as+b)+ac^2s\right]_{(k-1)\Delta S}^{k\Delta S}\end{aligned}$$

We first evaluate

$$\Delta S(bc-ad)=\gamma_{k-1}(\gamma_k-\gamma_{k-1})^2-(k-1)(\gamma_k-\gamma_{k-1})^3+(k-1)(\gamma_k-\gamma_{k-1})^3-\gamma_{k-1}(\gamma_k-\gamma_{k-1})^2=0.$$

That leads to the following integral

$$\int_0^S \frac{m}{2r}x'^2 ds=\sum_{k=1}^{j}\frac{m}{(\Delta S)^2a^2}c^2\Delta=\sum_{k=1}^{j}\frac{m}{\Delta S}(\gamma_k-\gamma_{k-1})^2.S$$

Thus we include that in the limit, for the given sequence of projection operators, we have that

$$\int_0^S ds\left[\frac{m}{2r}x'^2+Er\right]=\int_0^S ds\left[\frac{\mu}{2}u'^2+\frac{1}{2}m\omega^2u^2\right].$$

□

Corollary 9.6. In the end we obtained the following form for the discretised action

$$A_j=e^2S+\sum_{k=1}^{j}\left[\frac{\mu}{2\Delta S}(\gamma_k-\gamma_{k-1})^2+\frac{\Delta S}{3}(\gamma_k^2+\gamma_k\gamma_{k-1}+\gamma_{k-1}^2)\right]. \tag{174}$$

If one looks at this equation and its derivation closely, one observes that the assumption of a fixed time interval ΔS is not necessary. That means in particular that the previous lemma can be extended to projection operators of the form

$$P_j(\gamma)=\begin{cases}\gamma_0+\frac{s}{s_1}(\gamma_1-\gamma_0), & \text{if } 0\le s\le s_1\\ \gamma_1+\frac{(s-s_1)}{s_2-s_1}(\gamma_2-\gamma_1), & \text{if } s_1\le s\le s_2\\ \vdots & \\ \gamma_{j-1}+\frac{(s-s_{j-1})}{S-s_{j-1}}(\gamma_j-\gamma_{j-1}), & \text{if } s_{j-1}\le s\le S,\end{cases} \tag{175}$$

where $\gamma_k \coloneqq \gamma(s_k)$. Note that $\{s_0,s_1,\dots,s_j\}$ is a partition of $[0,S]$, with $s_0=0$ and $s_j=S$. This means that the result holds for general partitions, this will also be the case for proposition 9.7. This is of great importance because later on the sequence of projection operators must satisfy $P_l(\mathcal{H})\subset P_{l+1}(\mathcal{H})$.

Proposition 9.7. The Duru-Kleinert integrals

$$\widehat{\int}_{\mathcal{H}_T} d\gamma \exp\left(\int_0^S ds \frac{m}{2r} x'^2 + Er\right)$$

and

$$\widehat{\int}_{\mathcal{H}_T} d\gamma \exp\left(\int_0^S ds \frac{\mu}{2} u'^2 + \frac{1}{2} m\omega^2 u^2\right)$$

for the given sequence of projection operators are equal.

Proof. In the previous lemma we have already shown that for the projection operators, the integrals

$$\int_0^S ds \left[\frac{m}{2r} x'^2 + Er\right] = \int_0^S ds \left[\frac{\mu}{2} u'^2 + \frac{1}{2} m\omega^2 u^2\right]$$

are equal. Moreover we have that the differential becomes

$$\int \frac{d\gamma_2 \dots d\gamma_{j-1}}{(2\pi\hbar i)^{j-2}} = \int \frac{d\gamma_2^2 \dots d\gamma_{j-1}^2}{(2\pi\hbar i)^{j-2}\sqrt{\gamma_2^2 \dots \gamma_{j-1}^2}}.$$

□

Using this result, we can state the main theorem regarding the hydrogen atom:

Theorem 9.8. The pseudopropagator $\langle q_F | U_E(s_F - s_I) | q_I \rangle$ of the hydrogen atom, which in physics is given by [88, Equation 12.36]

$$\langle q_F | U_E(s_F - s_I) | q_I \rangle = r_F^{\lambda} r_I^{1-\lambda} \int Dx(s) \exp\left(e^2 S + \frac{i}{\hbar} \int_0^S ds \left[\frac{m}{2r} x'^2 + Er\right]\right).$$

can be constructed as an infinite dimensional Duru-Kleinert oscillatory integral of the form

$$\widehat{\int}_{\mathcal{H}_T} d\gamma \exp\left(\int_0^S ds \left[\frac{\mu}{2} u'^2 + \frac{1}{2} m\omega^2 u^2\right]\right) = \widetilde{\int}_{\mathcal{H}_T} e^{\frac{i}{2\hbar}\int \mu \dot{\gamma}^2(s) ds} e^{-\frac{i}{2\hbar}\int_0^T ds \mu(\gamma(s)+x)\omega^2(\gamma(s)+x)} d\gamma \tag{176}$$

for any sequence of projection operators $\{P_l\}$ of the form (175), with the condition that for all $j \in \mathbb{N}$

$$\{s_0, s_1, \dots, s_j\} \subset \{s_0, s_1, \dots, s_{j+1}\}.$$

Proof. In the previous proposition we have shown that the pseudopropagator can be viewed as a Duru-Kleinert oscillatory integral and it is equal to

$$r_F^{\lambda} r_I^{1-\lambda} \widehat{\int}_{\mathcal{H}_T} d\gamma \exp\left(\int_0^S ds \left[\frac{\mu}{2} u'^2 + \frac{1}{2} m\omega^2 u^2\right]\right).$$

Up till now we have not imposed boundary conditions on $\mathcal{H}_T$. We have seen in the previous chapter that the harmonic oscillator propagator can be realised as

$$r_F^{\lambda} r_I^{1-\lambda} \widetilde{\int}_{\mathcal{H}_T} e^{\frac{i}{2\hbar}\int \mu\dot{\gamma}^2(s)ds} e^{-\frac{i}{2\hbar}\int_0^T ds\mu(\gamma(s)+x)\omega^2(\gamma(s)+x)} d\gamma.$$

Suppose $u(S) = x$. By defining $v(s) = u(s) - x$ and repeating the derivation (note that the differentials do not change), we observe that we have

$$r_F^{\lambda} r_I^{1-\lambda} \widehat{\int}_{\mathcal{H}_T} d\gamma \exp\left(\int_0^S ds\left[\frac{\mu}{2}v'^2 + \frac{1}{2}m\omega^2(v+x)^2\right]\right).$$

Note that v is a complex number, while γ will be a path. This is going to be equivalent to

$$r_F^{\lambda} r_I^{1-\lambda} \widetilde{\int}_{\mathcal{H}_T} e^{\frac{i}{2\hbar}\int \mu\dot{\gamma}^2(s)ds} e^{-\frac{i}{2\hbar}\int_0^T ds\mu(\gamma(s)+x)\omega^2(\gamma(s)+x)} d\gamma.$$

This infinite dimensional oscillatory integral converges under the given sequence of projection operators since that is a defining feature of the infinite dimensional oscillatory integrals that the result is independent of the sequence of projection operators. Note that for a sequence of projection operators $\{P_l\}$ $(l \in \mathbb{N})$, the projections must satisfy $P_l(\mathcal{H}) \subset P_{l+1}(\mathcal{H})$ as a subset. Therefore we must impose the following condition on the projection times

$$\{s_0, s_1, \ldots, s_j\} \subset \{s_0, s_1, \ldots, s_{j+1}\}.$$

Recall that the projection operators were defined in equation (175). By this assumption, the next projection operator in the sequence is also a piecewise linear path with the same fixed times s_k with just one additional intermediate point of projection (inserted at a fixed time s). Assume that the additional projection time s is inserted between s_k and s_{k+1}, then it is always possible to pick the intermediate point $\gamma(s)$ such that the path is still a straight line segment between $\gamma(s_k)$ and $\gamma(s_{k+1})$. Therefore the condition $P_l(\mathcal{H}) \subset P_{l+1}(\mathcal{H})$ is satisfied.

The sequence of projection operators converges strongly to the path because a piecewise linear combination converges to the path. This allows us to justify the use theorem 8.8 to establish the existence of the Duru-Kleinert integral in equation (176). □

Theorem 9.9. The pseudopropagator is then given by

$$\langle q_F|U_E(s_F - s_I)|q_I\rangle = \frac{e^{iSe^2/\hbar}}{4}(K_{harmonic}(u_a, t_a, u_b, t_b) + K_{harmonic}(u_a, t_a, -u_b, t_b)) \quad (177)$$

Proof. The symmetry follows from [88, figure 13.1] in u_b. We then arrive at the following

expression in terms of Duru-Kleinert and infinite dimensional oscillatory integrals

$$\begin{aligned}\langle q_F|U_E(s_F-s_I)|q_I\rangle &= r_F^{\lambda} r_I^{1-\lambda}\int Dx(s)\exp\left(e^2S+\frac{i}{\hbar}\int_0^S ds\frac{m}{2r}x'^2+Er\right)\\ &= \frac{e^{iSe^2/\hbar}}{4}\widetilde{\int}_{\mathcal{H}_{[s_I,s_F]}} e^{\frac{i}{2\hbar}\int\mu\dot{\gamma}^2(s)ds}e^{-\frac{i}{2\hbar}\int_{s_I}^{s_F}ds\mu(\gamma(s)+u_b)\omega^2(\gamma(s)+u_b)}d\gamma\\ &+ \frac{e^{iSe^2/\hbar}}{4}\widetilde{\int}_{\mathcal{H}_{[s_I,s_F]}} e^{\frac{i}{2\hbar}\int\mu\dot{\gamma}^2(s)ds}e^{-\frac{i}{2\hbar}\int_{s_I}^{s_F}ds\mu(\gamma(s)-u_b)\omega^2(\gamma(s)-u_b)}d\gamma.\end{aligned}$$

□

Finally, we are able to evaluate the expression. Recall that in equation (144) we found that [143] (for the two-dimensional case)

$$K_{harmonic}(u_a,t_a,u_b,t_b)=\frac{m\omega}{2\pi i\sin(\omega\Delta S)}\exp\left(\frac{im\omega(-2u_au_b+(u_a^2+u_b^2)\cos(\omega\Delta S)}{2\sin(\omega\Delta S)}\right).$$

That results in the following expression for the pseudopropagator

$$\langle q_F|U_E(s_F-s_I)|q_I\rangle=\frac{e^{iSe^2/\hbar}}{2}\frac{m\omega}{2\pi i\sin(\omega\Delta S)}\exp\left(\frac{im\omega(u_a^2+u_b^2)\cos(\omega\Delta S)}{\sin(\omega\Delta S)}\right)\cosh\left(\frac{im\omega u_au_b}{\sin(\omega\Delta S)}\right). \tag{178}$$

Earlier on it was shown how to obtain the energies and wavefunctions from the pseudopropagator (and fixed-energy amplitude, which must be calculated first). Note that from here on, we can continue with the existing derivation of the wavefunctions and energies, as outlined in [88] and briefly summarised in chapter 5.

9.4 Limitations on the projection operators

The final question that remains is: For which sequences of projection operators can one construct a convergent Duru-Kleinert oscillatory integral? To start to answer that question, we would need to show that not all projection operators can work to construct a rigorous Duru-Kleinert path integral. In [88, Section 12.2] there is the following formula on the relation between the time t and the pseudotime s

$$t=sf_l(x)f_r(x)=sr(s). \tag{179}$$

This can be seen by comparing the time-evolution operators. That leads to the equation

$$s=\frac{t}{r(s)}.$$

If we were to define a projection operator of the form (where again $\gamma_k = \gamma(k\Delta t)$)

$$P_j(\gamma) = \begin{cases} \gamma_0 + \frac{t}{\Delta t}(\gamma_1 - \gamma_0), & \text{if } 0 \leq t \leq \Delta t \\ \gamma_1 + \frac{(t-\Delta t)}{\Delta t}(\gamma_2 - \gamma_1), & \text{if } \Delta t \leq t \leq 2\Delta t \\ \vdots & \\ \gamma_{j-1} + \frac{(t-(j-1)\Delta t)}{\Delta t}(\gamma_j - \gamma_{j-1}), & \text{if } (j-1)\Delta t \leq t \leq T. \end{cases}$$

Once again we opt for time intervals of equal length, although there is no necessity to do so, the discussion can once again be generalised to projection operators of the form (175). Using the projection operator from above, the action becomes

$$A = \int_0^T \left(\frac{m}{2} \left(\frac{d}{dt} P_j(\gamma) \right)^2 + \frac{e^2}{r} dt \right) \tag{180}$$

$$+ \sum_{k=1}^{j} \left[\frac{m}{2} (\Delta\gamma_k)^2 + \int_{(k-1)\Delta t}^{k\Delta} \frac{e^2}{\gamma_{k-1} + \frac{(t-(k-1)\Delta t)}{\Delta t}(\gamma_k - \gamma_{k-1})} dt \right]. \tag{181}$$

But if one where to integrate over all possibilities γ_k and γ_{k-1}, then one observes that some integrals are ill-defined. That is because the denominator can be zero, or rather shift from being positive to negative. In order to rewrite this operator as a function of s, we need to rewrite the equation

$$s = \frac{t}{r(t)}$$

as an equation of the form $t = f(s)$ for some function f. The relation between s and t, for given $0 \leq k \leq j$ has the form

$$\begin{aligned} s &= \frac{t}{\gamma_k + \frac{(t-t_k)}{t_{k+1}-t_k}(\gamma_{k+1} - \gamma_k)} = \frac{1}{\frac{1}{t}\left(\gamma_k - \frac{t_k}{t_{k+1}-t_k}(\gamma_{k+1} - \gamma_k)\right) + \frac{1}{t_{k+1}-t_k}(\gamma_{k+1} - \gamma_k)} \\ \frac{1}{s} &= \frac{1}{t}\left(\gamma_k - \frac{t_k}{t_{k+1}-t_k}(\gamma_{k+1} - \gamma_k)\right) + \frac{1}{t_{k+1}-t_k}(\gamma_{k+1} - \gamma_k) \\ \frac{1}{t} &= \frac{\frac{1}{s} - \frac{1}{t_{k+1}-t_k}(\gamma_{k+1} - \gamma_k)}{\gamma_k - \frac{t_k}{t_{k+1}-t_k}(\gamma_{k+1} - \gamma_k)} \\ t &= s\frac{\gamma_k - \frac{t_k}{t_{k+1}-t_k}(\gamma_{k+1} - \gamma_k)}{1 - \frac{s}{t_{k+1}-t_k}(\gamma_{k+1} - \gamma_k)} := s f_k(s). \end{aligned}$$

Rewriting the projection operator as a function of s gives us

$$P_j(\gamma)(s) = \begin{cases} \gamma_0 + \frac{s f_1(s)}{\Delta t}(\gamma_1 - \gamma_0), & \text{if } 0 \leq s \leq \Delta t / \|P_j(\gamma)(\Delta t)\| \\ \gamma_1 + \frac{(s f_2(s)-\Delta t)}{\Delta t}(\gamma_2 - \gamma_1), & \text{if } \Delta t/\|P_j(\gamma)(\Delta t)\| \leq s \leq 2\Delta t/\|P_j(\gamma)(2\Delta t)\| \\ \vdots & \\ \gamma_{j-1} + \frac{(s f_j(s)-(j-1)\Delta t)}{\Delta t}(\gamma_j - \gamma_{j-1}), & \text{if } (j-1)\Delta t/\|P_j(\gamma)((j-1)\Delta t)\| \leq s \leq T/\|P_j(\gamma)(T)\|. \end{cases}$$

That means that such an operator results in a divergent expression for the path integral of the hydrogen atom. That is because equation (181) does not converge for all choices of paths. This holds true even if the partition $\{\gamma_0, \gamma_1, \ldots, \gamma_j\}$ would not be equally distributed. Therefore we have concluded that there exists a class of projection operators for which the path integral converges and there is also a class for which the path integral does not converge.

9.5 Conclusion

In this chapter, a new type of oscillatory integral has been constructed, namely a Duru-Kleinert oscillatory integral. This type differs from the infinite dimensional oscillatory integral in the sense that the limit is only independent for a chosen class of sequences of projection operators. It has been shown in lemma 9.5 that for position-independent piecewise linear projection operators in the pseudotime, the path integral for the hydrogen atom can be related to the harmonic oscillator. This has been concluded in proposition 9.7 and theorems 9.8 and 9.9.

Finally, in the last section it has been concluded that it is impossible to construct an infinite dimensional oscillatory integral. In that section, sequences of projection operators have been investigated that clearly do not have converging oscillatory integrals. So we know that using space-independent piecewise linear projection operators with respect to the pseudotime s results in convergence of the path integral, while space-independent piecewise projection operators with respect to the time t do not have converging path integrals.

There is one question that remains to be answered: Under what conditions does a sequence of projection operators result in a converging Duru-Kleinert integral for the hydrogen atom? There is one conjecture that would seem to be true, but cannot be proven at this point.

Conjecture 9.10. For any sequence of projection operators $\{P_m\}$ with respect to the pseudotime s, such that the projections do not depend on the trajectory of the path in space (i.e. are space-independent), the Duru-Kleinert oscillatory integral converges. This is regardless of the choice of interpolation between the projection points on the condition that such a sequence must satisfy the requirements $P_m(\mathcal{H}) \subset P_{m+1}(\mathcal{H})$, they must also converge strongly to the identity operator.

A possible start for a proof would be to try to approximate such a function using piecewise linear functions and take some kind of limit, but that is all unclear at this point. It seems unlikely that there exists a theorem that provides an easily testable condition for convergence of a sequence of projection operators.

For a long time, it was open whether there could exist an oscillatory integral for the Coulomb potential, as discussed in [85, Chapter 20]. With this chapter, a possible way to construct such an oscillatory integral has been covered. At this stage, there is still a distinct

possibility that there are major or minor mistakes in this chapter, but at least the author hopes it might provide a first step towards a rigorous path integral for the hydrogen atom.

10 Discussion and outlook

Anyone who enters the field of mathematical path integrals faces one big issue, the sheer diversity of approaches to tackle the problem. These approaches often use very different mathematical perspectives and use completely different ideas. This makes it much harder to get an overview of the progress that has been made.

The interplay of different domains of mathematics and physics ensures that the literature is very split out. Each construction is worked out in different parts of the literature, which makes it hard to get the big picture. There are significant efforts to bridge the gap however, such as the books [8, 23, 55, 85, 108]. These books discuss several approaches and prove results that link different approaches, an example is [108, Theorem 3.10]. This is a good and encouraging development.

Unfortunately, there are few papers that actually compare different constructions on actual physical systems. I do not know of a paper that compares the results on the harmonic oscillator for example, while it is very important to see whether the different formalisms give the same results (or are equivalent in some way). There should be just one path integral after all.

One connection that could be promising is between the oscillatory integral formalism and the rigorous time-slicing technique, particularly since the latter also uses oscillatory integrals [55]. The oscillatory integral formalism can also be regarded as a time-slicing approach if one uses a suitable sequence of projection operators that project onto discrete points on the time interval of the Cameron-Martin space.

Something that has received little attention until now is the numerical aspect of these mathematical approaches to path integrals. It is very important to be able to numerically compute path integrals if we want to use them in the areas where there are applications, such as quantum field theory or finance. This is not to say that this area has received no attention. For example the numerical computation of oscillatory integrals has been studied extensively, see [120] for example. But the link with the mathematical approaches has not received much study. In the case of non-relativistic quantum mechanics, one faces three separate problems in the computation of infinite dimensional oscillatory integrals, these are:

1. First one tries to calculate a finite dimensional oscillatory integral on a bounded interval, that on its own is hard because the integrand is highly oscillating. That means that any of the standard integral approximation methods, such as Simpson's method (or adapted versions of it, such as Romberg's method) do not have very good convergence rates.

2. Secondly, one faces the task of calculating a finite dimensional oscillatory integral over an unbounded domain, namely $\mathbb{R}^n$ (or a Hilbert space isomorphic to it). To achieve a numerical approximation of that kind of integral one has to calculate a limit of bounded

domain approximations (which each have their own errors of course) and that makes an achievement of good convergence even more difficult.

3. Finally, one has to let the dimension of the domain go off to infinity. Thus one has to take another limit (of finite dimensional integrals with unbounded domains this time), which makes the numerical approximation even more involved.

Note that these issues are not unique to the oscillatory integral formalism. They are common to all numerical approximations of path integrals. This is because the issues, namely an oscillating integrand, unbounded domains and infinite dimension, are not unique to the formalism. Some work on numerical aspects can be found in [22, 37, 47, 105, 118, 120]. Numerical work related to the path integral is discussed in [152, 156].

Something else that should be looked at is the extension of the mathematical approaches in this thesis to other applications than quantum field theory or quantum mechanics. As was said in the introduction, the path integral has been applied to statistical mechanics [163], climate dynamics [114], financial markets [16, 154] and even to statistics [27]. It would be very interesting to see whether the rigorous constructions of the path integral that exist can be carried over to these applications or not.

Another topic that could be of interest is the extension of the oscillatory integral formalism to quantum field theory, first it needs to be extended to the Klein-Gordon equation and subsequently to the Dirac equation. It was noted earlier on that the solution to the Klein-Gordon equation can be written as an oscillatory integral of the form [153]

$$u(x,t) = \int e^{i(t\sqrt{m^2+\xi^2}+x\cdot\xi)}\psi(\xi)d\xi. \tag{182}$$

It should therefore be possible to extend the oscillatory integral approach to quantum field theory in this way, maybe first for the two-dimensional spacetime case (i.e. one spatial dimension and one temporal dimension). The Dirac equation is more difficult, as Dirac operators are nonlinear (they mix up components) and are totally unbounded, which makes constructing an oscillatory integral much more involved. It is unclear how to proceed at that point.

Even if one succeeds at finding a completely rigorous description of the path integral for the Dirac equation that works for all physical systems of interest, then we will have to deal with renormalisation in some rigorous manner. Undoubtedly, that will bring a whole new set of mathematical challenges that most likely that will be hard to deal with, but that is a matter of future concern.

11 Conclusion

In this thesis, the Feynman path integral in the context of physics has been discussed, with a focus on non-relativistic quantum mechanics. We have seen how the path integral arises from the framework of SDE theory, especially from the link between the Fokker-Planck equation and the Langevin equation. After a suitable choice of coefficient functions and factoring out the stationary density, we can rewrite the Fokker-Planck equation as

$$B\frac{\partial \psi}{\partial t} = \frac{B^2}{2}\frac{\partial^2 \psi}{\partial t^2} + U(x)\psi,$$

where the function U was defined in equation (43). The choice $B = -i\hbar$ gives the Schrödinger equation. Using the connection between SDEs and the Fokker-Planck equation, the Schrödinger equation can be related to the Langevin equation. This specific Langevin equation is of the form

$$\dot{x} = -V'(x) + W,$$

although this equation does have mathematical issues (which is not strange after a moment of thought, after all if the stochastic derivation of the path integral was completely rigorous, this thesis would not have been written). Using the properties of Brownian motion (on which this whole approach relies), one can derive that the transition density is given by the following path integral

$$p(x_0, t_0, x_f, t_f) = \int_{x(t_0)=x_0, x(t_f)=x_f} \exp\left(-\frac{S}{D}\right) Dx. \tag{183}$$

The 'action' is given by

$$S(x, \dot{x}) = \int_0^T \left(-\frac{1}{4}(\dot{x}^2 + V'(x))^2 + \frac{1}{2}V''(x)\right) dt. \tag{184}$$

This action can be related to the 'physical action' as follows

$$S(x) = \frac{i}{\hbar}\int_0^T \left(\frac{1}{2}\dot{x}^2 - U(x)\right) dt \to \int_0^T \left(-\frac{1}{4D}(\dot{x}^2 + (V'(x))^2) + \frac{1}{2}V''(x)\right) dt. \tag{185}$$

So in a nutshell first the path integral for the Langevin equation was derived, subsequently the correspondence between SDEs and the Fokker-Planck equation was used to relate it to the Schrödinger equation and link it to the path integral from physics. This whole derivation relied on the Brownian nature of SDEs. If one prefers to use Lévy processes for example, one needs to formulate SDEs (and especially stochastic integrals) using a non-Brownian process. Then one can redo the derivation outlined above, but it is doubtful whether it will work out nicely.

After that, the uses of the path integral were discussed, first in the setting of non-relativistic quantum mechanics, where we looked at the case of the harmonic oscillator and how the path integral can be calculated exactly in that case, then we studied the path integral of the anharmonic oscillator. It is not straightforward to calculate the path integral for the anharmonic oscillator, since the standard technique used in the case of the harmonic oscillator does not extend to the anharmonic case. An exact result does exist, but its form is very complicated and includes several special functions.

We concluded it could not be calculated exactly and that moreover, traditional methods of perturbation theory in order to approximate the energies do not work. Instead, variational perturbation theory was used, which does provide a convergent perturbation series. This method relies on approximating the path integral by a variational approach and use that to arrive at a perturbation series for the energies. It leads to reasonable accurate answers, as can be seen in table 1 and figure 1, especially if the coupling constant λ is small.

Then, in the case of quantum field theory, it was described how the addition of an anharmonic term leads to interactions between particles/fields and also how the perturbation expansion leads to Feynman diagrams.

Then we studied why the path integral is no ordinary Lebesgue integral, we proved that there can be no Lebesgue measure on an infinite dimensional Hilbert space such as the space of paths. Faced with that problem, one particular construction which gives a rigorous path integral was studied in this thesis, namely one which uses oscillatory integrals.

The concept of an oscillatory integral (which is commonly used in linear PDE theory) was extended to infinite dimensional spaces. The advantage of oscillatory integrals is that the presence of the Schwartz function ensures the convergence of many integrals. After having established the integrability the Fresnel algebra we turned to actual physical systems, where we succeeded in achieving a rigorous construction of the path integral for both the harmonic and the anharmonic oscillator. The key equation that was established are the two infinite dimensional Parseval-type equalities, for the quadratic case

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,(I-L)x\rangle} f(x)dx = (\det(I-L))^{-1/2} \int_{\mathbb{R}^n} e^{-\frac{i}{2\hbar}\langle x,(I-L)^{-1}x\rangle} f(x)d\hat{\mu}_f(x),$$

and for the quartic case

$$\widetilde{\int}_{\mathcal{H}} e^{\frac{i}{2\hbar}\langle x,(I-B)^{-1}x\rangle} e^{-\frac{\lambda}{\hbar}A(x,x,x,x)} f(x)dx \int_{\mathcal{H}} \mathbb{E}\left[e^{in(k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} e^{i\frac{\lambda}{\hbar}\widetilde{V}(\omega)}\right] d\mu_f(k).$$

Applying the oscillatory integral techniques to the harmonic and anharmonic oscillators is then just a matter of defining suitable operators L and B, proving that they have the required properties for the Parseval-type equalities to hold and subsequently calculating the right-hand side of those equalities.

First the harmonic oscillator was discussed, we defined the operator

$$(L\gamma)(s) \coloneqq \int_s^T ds' \int_0^{s'} (\Omega^2\gamma)(s'')ds''.$$

We chose this operator because its inner product is of the following form

$$\langle \gamma_1, L\gamma_2 \rangle = \int_0^T \gamma_1(s)\Omega^2\gamma_2(s)ds,$$

which is exactly the potential energy part of the action for the harmonic oscillator. For the inverse to be well-defined, we require the final time T to satisfy the following property

$$T \neq \left(n + \frac{1}{2}\right)\pi.$$

The reason we require this to be true is that otherwise $\cos(\Omega T) = 0$ and that means we cannot properly calculate a Green's function of the problem. Subsequently calculating the inverse $(I - L)^{-1}$ and inserting that expression in the Parseval-type equality gives us the following formula for the wavefunction

$$\psi(x,t) = \sqrt{\det\left(\frac{\Omega}{2\pi i \sin(\Omega t)}\right)} \int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}(x\Omega\cot(\Omega t)x + y\Omega\cot(\Omega t)y - 2x(\sin(\Omega t))^{-1}y}\psi_0(y)dy.$$

For the quartic case, the following operator was defined

$$\begin{aligned} &B(x,\gamma) = (y,\eta) \\ &y = T\Omega^2 x + \Omega^2 \int_0^T \gamma(s)ds \\ &\eta(s) = \Omega^2 x\left(Ts - \frac{s^2}{2}\right) - \int_0^s \int_T^u \Omega^2\gamma(r)drdu. \end{aligned}$$

Once again, we chose this operator because under the inner product of the Cameron-Martin space the following equation can be obtained

$$\langle (y,\eta), B(x,\gamma)\rangle = \langle x, y\rangle_{\mathbb{R}^n} + \langle \eta, \xi\rangle_{\mathcal{H}_T} = \int_0^T (\eta(s) + y)\Omega^2\left(x + \gamma(s)\right)ds.$$

The quartic potential is given by

$$V_4(x,\gamma) = \lambda A((x,\gamma),(x,\gamma),(x,\gamma),(x,\gamma)) = \lambda \int_0^T |\gamma(s) + x|^4 ds.$$

Under the conditions

$$\Omega_i t < \frac{\pi}{2}, \quad 1 - \Omega_i \tan(\Omega_i t) > 0,$$

the operator $I - B$ is trace class and also strictly positive. Under the additional condition that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{\hbar}{4}(y-(\cos(\Omega t))^{-1}x)(1-\Omega\tan(\Omega t))^{-1}(y-(\cos(\Omega t))^{-1}x)} e^{\frac{\hbar}{4}x\Omega^{-1}} e^{\frac{\hbar}{4}x\Omega^{-1}\tan(\Omega t)x} d|\mu_0|(x) d|\mu_\phi|(y) < \infty.$$

the Parseval-type equality for the quartic phase function holds. We have that the final infinite dimensional oscillatory integral is given by

$$\begin{aligned}
&\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}(|x|^2+\|\gamma\|^2)} e^{-\frac{i}{2\hbar}\langle(x,\gamma),B((x,\gamma))\rangle} e^{-\frac{i}{\hbar}V_4(x,\gamma)} f(x,\gamma) dx d\gamma \\
&= \int_{\mathcal{H}_T\times\mathbb{R}^n} \left(\int_{C_T\times\mathbb{R}^n} e^{ie^{i\pi/4}(x\cdot y+\sqrt{\hbar}n(\gamma)(\omega))} e^{\frac{1}{2\hbar}\int_0^T(\hbar\omega(s)+x)\Omega^2(\hbar\omega(s)+x)ds} e^{-i\frac{\lambda}{\hbar}\int_0^T|\sqrt{\hbar}\omega(s)+x|^4} dW(\omega) \right. \\
&\left. \frac{e^{-\frac{|x|^2}{2\hbar}}}{(2\pi\hbar)^{n/2}} dx \right) \mu_f(dy d\gamma).
\end{aligned}$$

This can also be written as

$$i^{n/2} \int_{C_T\times\mathbb{R}^n} e^{\frac{1}{2\hbar}\int_0^T(\hbar\omega(s)+x)\Omega^2(\hbar\omega(s)+x)ds} e^{-i\frac{\lambda}{\hbar}\int_0^T|\sqrt{\hbar}\omega(s)+x|^4 ds} \overline{\phi(e^{i\pi/4}x)} \psi_0(e^{i\pi/4}\sqrt{\hbar}\omega(t)+e^{i\pi/4}x) dW(\omega) dx.$$

It can be shown that this equation satisfies the Schrödinger equation in the weak sense using the Feynman-Kac formula.

The infinite dimensional oscillatory integral for the harmonic oscillator agrees with the calculation from physics, as was noted in remark 8.9. For the anharmonic oscillator, it is more complicated. The construction of an infinite dimensional oscillatory integral is merely an existence result in that case. The exact solution that is discussed in [111] relies on a semiclassical approach, which is nonperturbative, since there is no expansion in terms of the coupling constant λ. The result of this approach is an asymptotic expansion. The method relies heavily on the properties of the anharmonic oscillator, as explained in [23, Section 4.3]. Its classical action can be calculated exactly for example, which is not the case for many other systems.

Variational perturbation theory however, applies to a wide class of path integrals, not just the anharmonic oscillator. It can be made arbitrarily accurate [88] and can be used for analytically unsolvable path integrals as well. What would be interesting is to compare variational perturbation theory with the exact solution and also the oscillatory integral result. For example, can variational perturbation theory also be applied to oscillatory integrals? Can the exact solution be linked to the oscillatory integral? In other words, can a semiclassical approach be applied to the oscillatory integral too? In principle the answers to the questions

should be yes, if the oscillatory integral is really a mathematical representation of the path integral. Showing these things are true, is another matter and could be investigated in the future.

The oscillatory integral approach also has its disadvantages. It is unclear how to properly define infinite dimensional oscillatory integrals for potentials with a singularity. The problem is that it is difficult to make sense of the value of the integrand for paths that cross the singularity. In chapter 9 there seems to be some progress on this problem, by making use of the Duru-Kleinert transform.

This transform relates a path integral of the form

$$\int Dx(s) \exp\left(\frac{i}{\hbar}\int_0^S ds\left[\frac{m}{2}x'^2 - V(x)\right]\right)$$

to a pseudo path integral (or pseudopropagator) of the form

$$f_r(q_F)f_l(q_I)\int Dx(s) \exp\left(\frac{i}{\hbar}\int_0^S ds\left[\frac{m}{2f_l(x)f_r(x)}x'^2 - f_l(x)(V-E)f_r(x)\right]\right).$$

One can pick the regulating functions such that the singularity is removed. For the two-dimensional hydrogen atom, one chooses $f_l(x)f_r(x) = r$. This results in the $1/r$ Coulomb singularity being removed. In chapter 5 this technique was used to calculate the path integral for the two-dimensional hydrogen atom and obtains its energies.

In chapter 9, it has been attempted to combine the Duru-Kleinert transform with the oscillatory integral formalism. To that end, a new type of oscillatory integral was defined, namely a Duru-Kleinert integral. Such an integral differs from the standard infinite dimensional oscillatory integral in the sense that the limit of the finite dimensional oscillatory integrals is only independent of a given set of projection operator sequences, instead of all sequences of projection operators.

In chapter 9, a Duru-Kleinert integral of the following form was derived

$$r_F^{\lambda} r_I^{1-\lambda} \widehat{\int_{\mathcal{H}_T}} d\gamma \exp\left(\int_0^S ds\left[\frac{\mu}{2}u'^2 + \frac{1}{2}m\omega^2 u^2\right]\right).$$

It was proven that this equals the infinite dimensional oscillatory integral for the harmonic oscillator for projection operators of the form (175). This allowed us to rigorously construct the path integral for the hydrogen atom and obtain the pseudopropagator

$$\langle q_F|U_E(s_F - s_I)|q_I\rangle = \frac{e^{iSe^2/\hbar}}{2}\frac{m\omega}{2\pi i\sin(\omega\Delta S)}\exp\left(\frac{im\omega(u_a^2+u_b^2)\cos(\omega\Delta S)}{\sin(\omega\Delta S)}\right)\cosh\left(\frac{im\omega u_a u_b}{\sin(\omega\Delta S)}\right),$$

which is in agreement with results from physics, see [88, Chapter 13] for example.

There are still mathematical difficulties with this new construction. For example it is still unclear what the correct domain of definition of the pseudohamiltonian is and what conditions need to be imposed on the regulating functions to ensure the derivation is rigorous. The general question which projection operators result in a converging oscillatory integral for the hydrogen atom and which ones in a diverging one remains partially answered. Some operators have been identified to work, while others have been identified to result in divergences. It would be interesting to see if conditions can be established on the sequences of projection operators that ensure convergence or divergence of the path integral.

If these mathematical issues get resolved, one can also ask more general questions that are of interest. The first and most obvious one is whether a rigorous path integral can be derived for the three-dimensional hydrogen atom. One could subsequently try to extend the result for molecular potentials, although it will undoubtedly become much more involved. More generally, there is the question whether the Duru-Kleinert transform can be rigorously established generally, not just for the hydrogen atom. This is related to the question about the conditions on the regulating functions and the pseudohamiltonian, but then in a more general setting.

As for now, one can conclude that the oscillatory integral formalism has had its successes. It has the advantage of clarity, although up till now there have been strong limitations on the kinds of systems for which a rigorous path integral can be constructed. Time will tell whether the Duru-Kleinert transform can expand the number of systems the formalism can be applied to.

12 Guide to the literature

Throughout this thesis, I have included references to literature on various topics relating to the path integral. I think it is helpful to provide some pointers in one place for the reader about where to start when studying the mathematical aspects of the path integral. The reason for providing this short bibliography is that the literature has many different subareas which use their own techniques and notation. Therefore it makes sense to provide some kind of overview on what has been done on mathematical path integrals up to this point.

I have split the discussion into the various approaches that exist since there are no books to my knowledge that cover several approaches with the same attention. Books that mainly cover one approach but also discuss others are [8, 23, 55, 85, 108]. The proceedings [84] and the festschrift [110] provide an overview of many different topics related to the path integral, although from a physical perspective. It is important to note that the purpose of this annotated bibliography is not to be comprehensive, its purpose is merely to give an overview that is helpful to the reader.

12.1 Oscillatory integral approach

There exist two monographs on this construction, which has received a lot of attention in this thesis. These are [108] and [8]. The book [108] has the advantage that it starts out with the infinite dimensional oscillatory integral approach right away, it also includes a discussion of open quantum systems. The book [8] includes several topics not discussed in [108], such as path integrals of relativistic bosonic fields. It starts out with a different construction, which relies more on the Fresnel algebra. In most of [8], the Parseval equality is taken as a definition of the Fresnel integral, as opposed to the construction in [108], where it is a theorem. The infinite dimensional oscillatory integrals are covered in the last chapter of the book.

With regards to papers, the cases of the harmonic and anharmonic oscillator were covered in [46]. This paper was crucial in that it provided a connection between the Fresnel integral approach of Albeverio and Hoegh-Krohn in [8] and the definition of Feynman and Hibbs in [52]. It was the first paper that introduced the notion of an infinite dimensional oscillatory integral, which became the basis for the modern oscillatory integral formalism for the path integral. The papers [9, 11] discussed an extension of the Fresnel integral to a polynomial phase function in the finite dimensional case. Potentials which are Laplace transforms of measures were covered in [6].

The paper [10] extended the Fresnel integral to a quartic phase function. A Parseval-type equality was proved in this paper for such an oscillatory integral. These results were used to establish an infinite dimensional oscillatory integral that represented the path integral for the anharmonic oscillator (that serves as a model for the $\lambda\phi^4$-theory, as was noted earlier

on). Moreover, the authors prove that the perturbative series in λ (i.e. the Dyson series) that results from the infinite dimensional oscillatory integral is Borel summable. Further results in this regard can be found in [12]. Results with regards to path integrals with electromagnetic fields can be found in [5, 7].

12.2 Related PDE aspects

The oscillatory integral formalism has close links with linear PDE theory, the definitions used in this formalism came from [73]. There are many books on PDEs from all kinds of viewpoints, I will just mention a couple of books that discuss linear PDE from a viewpoint that is related to this approach to path integrals.
The theory of oscillatory and Fourier integral operators came into its present form in the 1970s in the papers [38, 40, 73], as well as several others.

A good starting point for the theory of Fourier integral operators is [39]. The book is densely written, so it takes time to digest the concepts, but the book is remarkable for its clarity of discussion and well thought-out presentation. This book includes an introduction to symplectic geometry, which is very useful.

The definite treatise on linear PDE remains [74–77], but it is hard to use these books as a first look on topics such as Fourier integral operators. The book [66] is an alternative, which covers many of the ideas in a very concise form.

12.3 Related SDE aspects and stochastic physics

There exist many textbooks on SDE theory, examples are [106, 119]. Books that cover numerical aspects are [90, 104]. The monograph on the Fokker-Planck equation is [125]. The main monograph on the Langevin equation is [29]. An overview of stochastic physics is [151], [59] is also a useful overview of stochastic methods. The link between the Fokker-Planck equation and the path integral is discussed in [17].

12.4 Time-slicing technique (sequential approach)

The book [55] is the main monograph on the time-slicing approach. In the book, the potential is assumed to satisfy the inequality $|V(x)| \leq v_0(1 + |x|)^2$ for some constant v_0. Using that assumption results are proven the convergence of the time-slicing of the path integral (which turns out to be an oscillatory integral). This assumption is rather narrow however, since even a quartic potential is not covered by it, let alone a Coulomb potential.

The scope of this approach is much wider however. There has even been successes in applying it to the Dirac equation. Results in that regard can be found in [80, 82].

12.5 Mathematical quantum mechanics

The reader might also find it helpful to refer to a couple of books that cover quantum mechanics (and also quantum field theory) from a more general mathematical perspective. Books that are good to start with are [35, 69]. They are very readable and assume relatively little in terms of prerequisites, a basic knowledge of functional analysis and physics will do. At a higher level there are the books [70, 144]. Especially the last one spends of a lot of attention on the functional analytic aspects of quantum theory. A book that discusses quantum field theory from a mathematical perspective is [54]. None of these books spends a lot of attention on the path integral, but they might be helpful for a mathematician trying to get a more general picture.

12.6 Path integrals in physics

There are several books that discuss the path integral from a physics perspective. A very important book is [88]. The book is very comprehensive, it covers many physical systems as well as applications of the path integral outside physics. The author (along with Ismail Duru) was the first to calculate the path integral for the hydrogen atom, which is described in the papers [41, 42, 87]. Other monographs on path integrals include [25, 26]. Quantum field theory textbooks that cover the path integral are [129, 162].

A Analytical and measure theoretic preliminaries

In this section we will review the key concepts from analysis that were not in the undergraduate curriculum in Groningen. The reader who is familiar with these topics does not need to read this appendix.

A.1 Functional analysis

First we will cover a bit of graduate-level functional analysis which will be needed for our purposes. Using the concept of a Banach space it is possible to construct a so-called Banach algebra [33, 94]:

Definition A.1. A Banach algebra $(X, \|\cdot\|, \cdot)$ is a Banach space $(X, \|\cdot\|)$ with a multiplicative operation $\cdot$ such that X (together with the $+$-operation arising from the vector space structure) is an algebra. The multiplication together with the norm satisfy the following condition

$$\|a \cdot b\| \leq \|a\| \cdot \|b\|. \tag{186}$$

If the algebra has an identity e such that $e \cdot a = a \cdot e = a$ for all $a \in X$, in that case we use the term unital algebra, it is custom to assume that $\|e\| = 1$. What can be established is that $\|a\| \leq \|a\| \cdot \|e\| \implies 1 \leq \|e\|$. This makes it possible to define a new norm

$$\|a\|_2 \coloneqq \frac{\|a\|}{\|e\|},$$

which has the property that $\|e\|_2 = 1$.

Note that one does not have to assume the condition given in equation (186). If the set X has a norm under which the multiplication is continuous, then it is possible to find a norm equivalent to the former under which equation (186) holds [33]. We will introduce a couple of concepts from operator theory that will later be required:

Definition A.2. Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$ and let $\{e_i\}$ be an orthonormal basis for the space. The trace of the operator T is then defined by

$$\mathrm{Tr}(T) := \sum_{n=1}^{\infty} \langle e_i | T e_i \rangle. \tag{187}$$

Note that the trace is independent under orthonormal bases transformations.

Lemma A.3. Let T be a bounded linear operator on $\mathcal{H}$ such that $\mathrm{Tr}(|T|^p) < \infty$ for some positive p, then T is compact.

Proof. The proof can be found in [123, Lemma 3.4.5]. □

Definition A.4. A compact operator T is called trace class if we have that $\mathrm{Tr}(T) < \infty$. We denote the set of trace class operators by $B^1(\mathcal{H})$.

Definition A.5. A compact operator T is called Hilbert-Schmidt if we have that $\mathrm{Tr}(T^*T) < \infty$. We denote the set of Hilbert-Schmidt operators by $B^2(\mathcal{H})$.

The last definition might seem a bit arbitrary, why do we take the product T^*T instead of TT^*? However, one can prove the following result [123, Proposition 3.4.3]

$$\mathrm{Tr}(T^*T) = \mathrm{Tr}(TT^*). \tag{188}$$

Note that all bounded operators T of finite rank (i.e. satisfying the requirement that $\dim(T(\mathcal{H})) < \infty$) are trace class. We also have the following inclusion relation

$$B^1(\mathcal{H}) \subset B^2(\mathcal{H}).$$

So every trace class operator is a Hilbert-Schmidt operator. The sets $B^1(\mathcal{H})$ and $B^2(\mathcal{H})$ have additional structure on the space, which will be given in the next theorems.

Theorem A.6. The set of Hilbert-Schmidt operators $B^2(\mathcal{H})$ is a Hilbert space under the following inner product

$$\langle S, T\rangle \coloneqq \mathrm{Tr}(T^*S).$$

Proof. The proof can be found in [123, Theorem 3.4.9]. □

Theorem A.7. The set of trace class operators $B^1(\mathcal{H})$ is a Banach space under the following norm

$$\|T\| = \mathrm{Tr}(|T|).$$

Note that we define $|T|$ to be the operator with the property that $|T|^2 = TT^*$.

Proof. The proof can be found in [123, Theorem 3.4.12]. □

There is an important relation between the compact operators and the trace class operators:

Proposition A.8. If we introduce the following bilinear form

$$[S, T] = \mathrm{Tr}(ST).$$

Note the difference with the inner product on $B^2(\mathcal{H})$. We then have the following relations between the space of compact operators, denoted by $B^0(\mathcal{H})$, and the space of trace class operators $B^1(\mathcal{H})$ (recall that the $*$ signifies the dual space)

$$\left(B^0(\mathcal{H})\right)^* = B^1(\mathcal{H}), \quad \left(B^1(\mathcal{H})\right)^* = B(\mathcal{H}).$$

Proof. The proof can be found in [123, Theorem 3.4.13]. □

The concept of the spaces $B^1(\mathcal{H})$ and $B^2(\mathcal{H})$ can be extended as follows:

Definition A.9. The space of p-th Schatten class operators is given by all compact operators T that satisfy

$$\mathrm{Tr}(|T|^p) < \infty.$$

This space can be made into a Banach space as follows:

Theorem A.10. The space of p-th Schatten class operators is a Banach space under the following norm

$$\|T\|_p = \left(\mathrm{Tr}(|T|^p)\right)^{1/p}. \tag{189}$$

Note the analogy with the norm on the L^p spaces.

A.2 Complex measure theory

The reader is assumed to be familiar with standard (real) measure theory, this section is based on [30]. The complex measure theory covered is necessary for extending Fourier analysis to Hilbert spaces. Now the concept of a complex measure will be introduced.

Definition A.11. Suppose we have a measurable space (X, A). A complex measure μ is a map from A to $\mathbb{C}$ that satisfies the following properties:

1. $\mu(\emptyset) = 0$.
2. the measure is countably additive, so if we have a sequence of disjoint sets $\{A_i\}$ where $i \in \mathbb{N}$ and $A_i \in A$ for each i and also all the sets are disjoint, so $A_i \cap A_j = \emptyset$ if $i \neq j$. Then the measure satisfies
$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$
3. The measure cannot have infinite values (unlike in the case of a standard measure), so $\pm\infty$ is not in the range. Obviously, by definition it is possible for μ to take negative or complex values.

We now introduce the concept of variation:

Definition A.12. If μ is a complex measure on the measurable space (X, A), the variation (denoted by $|\mu|$) for a set B in the σ-algebra A is given by

$$|\mu|(A) = \sup_{\{B_i\}} \left(\sum_{i=1}^{n} |\mu(B_i)|\right), \tag{190}$$

where the supremum runs over all finite partitions of the set B into n subsets $B_i \in A$, these sets are disjoint from one another, which means that $B_i \cap B_j = \emptyset$ if $i \neq j$. Note that n is not fixed, it only has to be finite.

It can be proven that the variation $|\mu|$ is a finite measure (if μ itself is a complex measure) on the measurable space (X, A), the details can be found in [30]. So it is possible to construct a real and finite measure out of a complex measure. Note that it is not possible to define a variation in the case of a standard measure since $\mu(A_i)$ might be infinite and hence it will not be an interesting construction.

From here we can move to the idea of the total variation of a complex measure, which simply is

$$\|\mu\| = |\mu|(X). \tag{191}$$

Note that the total variation is a norm on the space of measures, which can be shown easily by using the properties of the real numbers ($|\mu(B_i)|$ is a real number and you can derive all the properties of the norm by using that the real numbers are a Banach space under the absolute value). The set of complex measures on a given measurable space, denoted by $M(X, A)$ is complete under the total variation norm, the details can again be found in [30, Proposition 4.1.8].

The set of complex measures $M(X, A)$ (with final total variation) is a Banach algebra under the following product (known as the convolution) [108]

$$\mu * \nu(B) = \int_{X \times X} \mathbb{1}_B(x + y) d\mu(x) d\nu(y). \tag{192}$$

Note that the product is commutative by the Fubini-Tonelli theorem. Finally, we want to introduce a specific measure that will appear later on (and will be the unit element of the Banach algebra $M(X, A)$):

Definition A.13. The Dirac measure δ_x for an element $x \in X$ is defined as follows

$$\delta_x(B) = \begin{cases} 1, & \text{if } x \in B \\ 0, & \text{if } x \notin B. \end{cases} \tag{193}$$

Alternatively, the Dirac measure can be written as an indicator function as follows

$$\delta_x(A) = \mathbb{1}_A(x).$$

It can be shown quite easily that the Dirac measure is actually a measure, by observing that $x \notin \emptyset$ and hence $\delta_x(\emptyset) = 0$ and moreover that if we have a sequence of pairwise disjoint sets $\{B_i\}$ then x can only be an element of one of these sets. Hence the countably additivity property is satisfied. Finally we observe that the Dirac measure is a finite measure [89, Example 1.30].

It is obvious the the Dirac measure δ_0 (note that the 0 here has to be interpreted as the zero element from the vector space structure) has unit norm, because

$$\|\delta_0\| = |\delta_0|(X) = \sup \sum_{i=1}^{n} |\delta_0(B_i)| = 1.$$

The last equality follows from the fact that for any finite partition of A into a finite number of sets B_i, the element 0 will be in only of these sets, hence the measure of that set will be 1 and the measure of the other sets will be zero. We will now show that the measure δ_0 act as a unit element under multiplication, we have

$$\mu * \delta_0(B) = \int_{X \times X} \mathbb{1}_B(x+y) d\mu(x) d\delta_0(y) = \int_X \mathbb{1}_B(x) d\mu(x) = \mu(B).$$

We can conclude that $M(X, A)$ is a Banach algebra (we omit the proofs of the other properties).

A.3 Fourier analysis

We will need some Fourier analysis to describe some of the analytical results on the Feynman path integral. This section is based on [141]. One of the key concepts from Fourier analysis is the Schwartz space [70]. It is defined to be

Definition A.14. The Schwartz space on the real numbers, denoted by $\mathcal{S}(\mathbb{R})$ is the set of all C^∞ functions, such that the function itself and all the derivatives (of any order) decrease superpolynomially as $|x| \to \infty$, this means that

$$\sup_{x \in \mathbb{R}} |x|^k |f^{(l)}(x)| < \infty,$$

this must hold for any natural numbers l and k (zero included).

The fact that the Schwartz space is a vector space, follows almost immediately. It can also be seen that the derivative of any function in the Schwartz space is still in the Schwartz space. Finally, observe that if $f(x) \in \mathcal{S}(\mathbb{R})$ then also $x \cdot f(x) \in \mathcal{S}(\mathbb{R})$. Hence we can say that the Schwartz space is closed on multiplication by polynomials and also under differentiation. We will give an easy example of a function that is in the Schwartz space

Example A.15. The function e^{-x^2}, which is the probability density of the standard normal distribution, lies in the Schwartz space. We first observe that the derivatives of e^{-x^2} are of the form $P(x)e^{-x^2}$, where $P(x)$ is an n-th order polynomial for the case of the n-th derivative. Note that for the condition $\sup_{x \in \mathbb{R}} |x|^k |f^{(l)}(x)| < \infty$ to be satisfied, one only needs to check the

behaviour of $|x|^k|f^{(l)}(x)|$ as $x \to \pm\infty$. This is because continuous functions map compact sets to compact sets [2]. Now since $x^k f^{(l)}(x)$ is continuous, the supremum on any compact interval will remain finite.

Now we need to find a suitable bound for $P(x)e^{-x^2}$, if $|x| > 1$, it is always possible to bound an n-th order polynomial $P(x)$ by $a|x|^n$ for some suitable constant a. Secondly, because e^{x^2} has a Taylor series

$$e^{x^2} = 1 + \frac{x^2}{1!} + \frac{(x^2)^2}{2!} + \dots,$$

we know that in particular that (for m even)

$$e^{x^2} \geq 1 + \frac{x^m}{(m/2)!}. \tag{194}$$

Now without loss of generality we assume that the positive integer $k + l$ (in the bound $x^k f^{(l)}(x)$) is even. This step is allowed because if $k + l$ were odd, we could redefine k by replacing it by $k + 1$ and using that

$$|x^k f^{(l)}(x)| < |x^{k+1} f^{(l)}(x)|, \text{ for } |x| > 1.$$

Since $k + l$ is even, we set $m = k + l$ and rewrite the inequality (194) for e^{x^2} as

$$e^{-x^2} \leq \frac{1}{1 + x^m/(m/2)!}.$$

Now we use the obtained bounds to get (where $P(x)$ is an l-th order polynomial since we are considering the l-th derivative of f)

$$\begin{aligned}\sup_{x\in\mathbb{R}}|x|^k|f^{(l)}(x)| &= \sup_{x\in\mathbb{R}}|x|^k|P(x)e^{-x^2}| \leq \sup_{x\in\mathbb{R}} a|x^{k+l}e^{-x^2}| \\ &\leq \sup_{x\in\mathbb{R}} a\left|\frac{x^m}{1 + x^m/(m/2)!}\right| \leq \left|\frac{x^m}{x^m/(m/2)!}\right| = \left(\frac{m}{2}\right)! < \infty.\end{aligned}$$

Hence the function e^{-x^2} belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$.

The idea of the Schwartz space can be extended to $\mathbb{R}^n$ in the following way [141]:

Definition A.16. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consists of the C^∞ functions f from $\mathbb{R}^n$ to $\mathbb{R}$ (the codomain could also be $\mathbb{C}$, the definition is analogous in that case) such that the following holds

$$\sup_{x\in\mathbb{R}^n} |x|^k|\partial^\alpha f(x)| < \infty,$$

for any k and any multi-index $\alpha \in \mathbb{N}^n$. Hence ∂^α is a shorthand for

$$\partial^\alpha = \frac{\partial}{\partial x^{\alpha_1}} \cdots \frac{\partial}{\partial x^{\alpha_n}}.$$

Remark A.17. The function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x_1, \ldots, x_n) = e^{-(x_1^2+\ldots+x_n^2)}$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. The proof is analogous to example A.15 and is left to the reader.

If we denote the space of smooth functions with compact support by $\mathcal{D}(\mathbb{R}^n)$, we have the following inclusion

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n).$$

In particular this means that the C^∞ bump functions, which exists on any smooth manifold [149], are contained in the Schwartz space, since they have compact support [79].
Before finishing this section, we will introduce one final crucial concept, which the reader most likely has seen in the context of $\mathbb{R}^n$ [108].

Definition A.18. Let μ be a complex measure with bounded variation on a Hilbert space $\mathcal{H}$. The Fourier transform of the measure is given by

$$f(x) := \int_{\mathcal{H}} e^{i\langle y,x\rangle} d\mu(x). \tag{195}$$

In order to indicate that the function f is the Fourier transform of μ it is common to use the notation $f = \hat{\mu}$.

The set of all functions f which are the Fourier transform of a bounded variation complex measure is known as the Fresnel algebra $\mathcal{F}(\mathcal{H})$. This name is proper, since $\mathcal{F}(\mathcal{H})$ is actually a Banach algebra. The unit element is given by the identity function (which is the transform of δ_0). The norm on $\mathcal{F}(\mathcal{H})$ is induced by the variation norm on $M(\mathcal{H})$ as follows

$$\|f\| \coloneqq \|\hat{\mu}\|. \tag{196}$$

Finally, the multiplication is given by pointwise multiplication of the functions. It is hard to give a complete characterisation of the Fresnel algebra. However, I will state a couple of results that are known [108].

Proposition A.19. Let $f \in \mathcal{F}(\mathcal{H})$, we know that

- The function f is uniformly continuous and bounded.
- If $\|\cdot\|$ denotes the norm given in equation (196) (i.e. through the variation norm) and $\|\cdot\|_\infty$ is the supremum norm [127]. The following inequality holds

$$\|f\|_\infty \leq \|f\|.$$

- If the Hilbert space $\mathcal{H}$ is finite dimensional, all norms on the space are equivalent and we can identify the space with $\mathbb{R}^n$. In this case we have the following very important inclusion relation

$$\mathcal{S}(\mathbb{R}^n) \subset \mathcal{F}(\mathbb{R}^n).$$

B Calculations

B.1 Calculation of Green's function

We need to find the Green's function of the following boundary value problem

$$\ddot{\gamma} + \Omega^2\gamma = \ddot{\alpha}, \quad \dot{\gamma}(0) = \dot{\alpha}(0), \gamma(T) = 0.$$

At first we will set $\dot{\alpha}(0) = 0$, which means that we consider a homogeneous boundary value problem, which will simplify the calculations. The homogeneous solution of the differential equation is

$$\gamma_{hom}(s) = A\sin(\Omega s) + B\cos(\Omega s),$$

the Green's function needs to satisfy

$$\ddot{G}(s,r) + \Omega^2 G(s,r) = \delta(s-r).$$

For $s < r$, we have

$$G(s,r) = c_1\cos(\Omega s) + c_2\sin(\Omega s). \tag{197}$$

The boundary condition $\dot{\gamma}(0) = 0$ tells us that $c_2 = 0$. For $s > r$ we have

$$G(s,r) = c_3\cos(\Omega s) + c_4\sin(\Omega s). \tag{198}$$

The boundary condition $\gamma(T) = 0$ gives us that $c_3 = -c_4\tan(\Omega T)$. That leads to the solution

$$G(s,r) = c_4(-\tan(\Omega T)\cos(\Omega s) + \sin(\Omega s)).$$

Note that is why we imposed $t \neq \left(n + \frac{1}{2}\right)\frac{\pi}{\Omega_j}$, because otherwise we would have to set $c_3 = 0$. Now we need to make sure continuity is assured when $s = r$. We must have that

$$c_1\cos(\Omega r) = -c_4\tan(\Omega T)\cos(\Omega r) + c_4\sin(\Omega r) \implies c_1 = -c_4\tan(\Omega T) + c_4\tan(\Omega r).$$

Ensuring discontinuity in the first derivative gives us that

$$\begin{aligned}
\Omega^{-1} &= c_4\tan(\Omega T)\sin(\Omega r) + c_4\cos(\Omega r) + c_1\sin(\Omega r) \\
\Omega^{-1} &= c_4\big(\tan(\Omega r)\sin(\Omega r) + \cos(\Omega r)\big) \\
c_4 &= \frac{\Omega^{-1}}{\tan(\Omega r)\sin(\Omega r) + \cos(\Omega r)} = \frac{\Omega^{-1}\cos(\Omega r)}{\sin^2(\Omega r) + \cos^2(\Omega r)} = \Omega^{-1}\cos(\Omega r).
\end{aligned}$$

Using this result we can calculate that the other constants are equal to

$$\begin{aligned}
c_1 &= -c_4\tan(\Omega T) + c_4\tan(\Omega r) = \Omega^{-1}\sin(\Omega r) - \Omega^{-1}\tan(\Omega T)\cos(\Omega r) \\
c_3 &= -\Omega^{-1}\tan(\Omega T)\cos(\Omega r).
\end{aligned}$$

Note that when $s < r$, we can write that

$$\Omega^{-1}\sin(\Omega r)\cos(\Omega s) - \Omega^{-1}\tan(\Omega T)\cos(\Omega r)\cos(\Omega s) = \Omega^{-1}(\cos(\Omega T))^{-1}\sin(\Omega(r-T))\cos(\Omega s),$$

when $s > r$ we have that

$$-\Omega^{-1}\tan(\Omega T)\cos(\Omega r)\cos(\Omega s) + \Omega^{-1}\cos(\Omega r)\sin(\Omega s) = \Omega^{-1}(\cos(\Omega T))^{-1}\cos(\Omega r)\sin(\Omega(s-T)).$$

This leads to the following Green's function

$$G(s,r) = \begin{cases} \Omega^{-1}(\cos(\Omega T))^{-1}\sin(\Omega(r-T))\cos(\Omega s), & \text{if } s < r \\ \Omega^{-1}(\cos(\Omega T))^{-1}\cos(\Omega r)\sin(\Omega(s-T)), & \text{if } s > r. \end{cases} \tag{199}$$

B.2 Calculation of $(I-L)^{-1}$

The Green's function gives us the following solution for γ

$$\begin{aligned}\gamma(s) = \int_0^T G(s,r)\ddot{\alpha}(r)dr &= \Omega^{-1}(\cos(\Omega T))^{-1}\cos(\Omega s)\int_s^T \sin(\Omega(r-T))\ddot{\alpha}(r)dr \\ &+ \Omega^{-1}(\cos(\Omega T))^{-1}\sin(\Omega(s-T))\int_0^s \cos(\Omega r)\ddot{\alpha}(r)dr.\end{aligned}$$

We first verify that this indeed solves the differential equation, we calculate that

$$\begin{aligned}\dot{\gamma}(s) &= (\cos(\Omega T))^{-1}\left[-\sin(\Omega s)\int_s^T \sin(\Omega(r-T))\ddot{\alpha}(r)dr + \cos(\Omega(s-T))\int_0^s \cos(\Omega r)\ddot{\alpha}(r)dr\right] \\ \ddot{\gamma}(s) &= -\Omega^2\gamma + (\cos(\Omega T))^{-1}\left[\sin(\Omega s)\sin(\Omega(s-T))\ddot{\alpha}(s) + \cos(\Omega(s-T))\cos(\Omega s)\ddot{\alpha}(s)\right] \\ &= -\Omega^2\gamma + (\cos(\Omega T))^{-1}\Big[\cos(\Omega T)(\sin^2(\Omega s) + \cos^2(\Omega s))\ddot{\alpha}(s) + \sin(\Omega T)(\sin(\Omega s)\cos(\Omega s) \\ &\quad - \sin(\Omega s)\cos(\Omega s))\ddot{\alpha}(s)\Big] \\ &= -\Omega^2\gamma + \ddot{\alpha}(s).\end{aligned}$$

So this is indeed the solution of the differential equation. We rewrite the given integrals into an integral over α using integration by parts

$$\begin{aligned}\int_0^s \cos(\Omega r)\ddot{\alpha}(r)dr &= \cos(\Omega r)\dot{\alpha}(r)\Big|_0^s + \Omega\int_0^s \sin(\Omega r)\dot{\alpha}(r)dr \\ &= \cos(\Omega s)\dot{\alpha}(s) + \Omega\sin(\Omega r)\alpha(r)\Big|_0^s - \Omega^2\int_0^s \cos(\Omega r)\alpha(r)dr\end{aligned}$$

$$= \cos(\Omega s)\dot{\alpha}(s) + \Omega\sin(\Omega s)\alpha(s) - \Omega^2\int_0^s \cos(\Omega r)\alpha(r)dr$$

$$\int_s^T \sin(\Omega(r-T))\ddot{\alpha}(r)dr = \sin(\Omega(r-T))\dot{\alpha}(r)\Big|_s^T - \Omega\int_s^T \cos(\Omega(r-T))\dot{\alpha}(r)dr$$

$$= -\sin(\Omega(s-T))\dot{\alpha}(s) + \Omega\cos(\Omega(s-T))\alpha(s) - \Omega^2\int_s^T \sin(\Omega(r-T))\alpha(r)dr.$$

We then obtain the final answer

$$\gamma(s) = \Omega^{-1}(\cos(\Omega T))^{-1}\left[\cos(\Omega s)\int_s^T \sin(\Omega(r-T))\ddot{\alpha}(r)dr + \sin(\Omega(s-T))\int_0^s \cos(\Omega r)\ddot{\alpha}(r)dr\right]$$

$$= (\cos(\Omega T))^{-1}\Bigg[\cos(\Omega s)\cos(\Omega(s-T))\alpha(s) - \Omega\cos(\Omega s)\int_s^T \sin(\Omega(r-T))\alpha(r)dr$$

$$+ \sin(\Omega(s-T))\sin(\Omega s)\alpha(s) - \Omega\sin(\Omega(s-T))\int_0^s \cos(\Omega r)\alpha(r)dr\Bigg]$$

$$= (\cos(\Omega T))^{-1}\Bigg[[\cos(\Omega s)\cos(\Omega T) + \sin(\Omega s)\sin(\Omega T)]\cos(\Omega s)\alpha(s)$$

$$- \Omega\cos(\Omega s)\int_s^T \sin(\Omega(r-T))\alpha(r)dr + [\sin(\Omega s)\cos(\Omega T) - \sin(\Omega T)\cos(\Omega s)]\sin(\Omega s)\alpha(s)$$

$$- \Omega\sin(\Omega(s-T))\int_0^s \cos(\Omega r)\alpha(r)dr\Bigg]$$

$$= (\cos(\Omega T))^{-1}\Bigg[\cos(\Omega T)[\cos^2(\Omega s) + \sin^2(\Omega s)]\alpha(s)$$

$$- \Omega\cos(\Omega s)\int_s^T \sin(\Omega(r-T))\alpha(r)dr - \Omega\sin(\Omega(s-T))\int_0^s \cos(\Omega r)\alpha(r)\Bigg]$$

$$= \alpha(s) - \Omega(\cos(\Omega T))^{-1}\left[\cos(\Omega s)\int_s^T \sin(\Omega(r-T))\alpha(r)dr + \sin(\Omega(s-T))\int_0^s \cos(\Omega r)\alpha(r)dr\right].$$

The calculation does not differ substantially in the multidimensional case. First we verify that this does indeed solve the differential equation, we calculate

$$\dot{\gamma}(s) = \dot{\alpha}(s) - \Omega^2(\cos(\Omega T))^{-1}\left[\cos(\Omega(s-T))\int_0^s \cos(\Omega r)\alpha(r)dr - \sin(\Omega s)\int_s^T \sin(\Omega(r-T))\alpha(r)dr\right]$$

$$\ddot{\gamma}(s) = \ddot{\alpha}(s) + \Omega^2\alpha(s) - \Omega^2\gamma(s)$$

$$- \Omega^2(\cos(\Omega T))^{-1}\left[\cos(\Omega(s-T))\cos(\Omega s)\alpha(s) + \sin(\Omega s)\sin(\Omega(s-T))\alpha(s)\right]$$

$$= \ddot{\alpha}(s) + \Omega^2\alpha(s) - \Omega^2\gamma(s) - \Omega^2\alpha(s) = \ddot{\alpha}(s) - \Omega^2\gamma(s),$$

hence the differential equation is satisfied.

B.3 Calculation of the path integral for the harmonic oscillator

In order to calculate the path integral for the harmonic oscillator, we have to evaluate the expression

$$\psi(x,T) = \sqrt{\frac{1}{\det(I-L)}}\frac{1}{2\pi}e^{-\frac{i}{2\hbar}x\Omega^2 Tx}\int_{\mathbb{R}} e^{-\frac{i\hbar}{2}\langle\eta_x,(I-L)^1\eta_x\rangle}e^{-i\hbar k\langle v_0,(I-L)^1\eta_x\rangle}e^{-\frac{i\hbar k^2}{2}\langle v_0,(I-L)^1 v_0\rangle}e^{ikx}\widehat{\psi_0}(k)dk.$$

In order to do so, we need to evaluate the inner products $\langle\eta_x,(I-L)^1\eta_x\rangle, \langle v_0,(I-L)^1\eta_x\rangle$ and $\langle v_0,(I-L)^1 v_0\rangle$. Subsequently we will use the Parseval/Plancherel theorem to rewrite the expression into an integral over coordinate space again. We start out with the formula for the inverse operator $(I-L)^{-1}$

$$(I-L)^{-1}\alpha(s) = \alpha(s) - \Omega(\cos(\Omega T))^{-1}\left[\cos(\Omega s)\int_s^T \sin(\Omega(r-T))\alpha(r)dr + \sin(\Omega(s-T))\int_0^s \cos(\Omega r)\alpha(r)dr\right]. \tag{200}$$

In order to simplify the calculations and reduce the number of steps, we will use the formula for the derivative instead (since that is what is needed to evaluate the inner product in the Cameron-Martin space)

$$\dot{\gamma}(s) = \dot{\alpha}(s) - \Omega^2(\cos(\Omega T))^{-1}\left[\cos(\Omega(s-T))\int_0^s \cos(\Omega r)\alpha(r)dr - \sin(\Omega s)\int_s^T \sin(\Omega(r-T))\alpha(r)dr\right].$$

We recall some of the earlier results

$$\begin{aligned}\int_0^s \cos(\Omega s')\eta_x(s')ds' &= \frac{x}{2\hbar\Omega}\left[(\Omega^2 s'^2-2)\sin(\Omega s') + 2\Omega s'\cos(\Omega s') - T^2\Omega^2\sin(\Omega s')\right]_0^s \\ &= \frac{x}{2\hbar\Omega}\left[(\Omega^2 s^2-2)\sin(\Omega s) + 2\Omega s\cos(\Omega s) - T^2\Omega^2\sin(\Omega s)\right]\end{aligned}$$

$$\begin{aligned}&\int_s^T \sin(\Omega(s'-T))\eta_x(s')ds' \\ &= \frac{x}{2\hbar\Omega}\left[(2-\Omega^2 s'^2)\cos(\Omega(s'-T)) + 2\Omega s'\sin(\Omega(s'-T)) + T^2\Omega^2\cos(\Omega(s'-T))\right]_s^T \\ &= \frac{x}{2\hbar\Omega}\left[(2-\Omega^2T^2) - (2-\Omega^2 s^2)\cos(\Omega(s-T)) - 2\Omega s\sin(\Omega(s-T)) + T^2\Omega^2 - T^2\Omega^2\cos(\Omega(s-T))\right]\end{aligned}$$

$$\left[\cos(\Omega(s-T))\int_0^s \cos(\Omega r)\alpha(r)dr - \sin(\Omega s)\int_s^T \sin(\Omega(r-T))\alpha(r)dr\right]$$

$$
\begin{aligned}
&= \frac{x}{2\hbar\Omega}\Big[-2\sin(\Omega s) + (\Omega^2 s^2 - 2)(\sin(\Omega s)\cos(\Omega(s-T)) - \sin(\Omega s)\cos(\Omega(s-T))) \\
&+ 2\Omega s[\cos(\Omega s)\cos(\Omega(s-T)) + \sin(\Omega(s-T))\sin(\Omega s)] \\
&- T^2\Omega^2[\sin(\Omega s)\cos(\Omega(s-T)) - \sin(\Omega s)\cos(\Omega(s-T))]\Big] \\
&= \frac{x}{2\hbar\Omega}\Big[-2\sin(\Omega s) + 2\Omega s\cos(\Omega T)\Big] \\
\dot{\gamma}(s) &= \frac{\Omega^2 x}{\hbar}s - \frac{x\Omega}{\hbar}(\cos(\Omega T))^{-1}\Big[-\sin(\Omega s) + \Omega s\cos(\Omega T)\Big] = \frac{x\Omega}{\hbar}(\cos(\Omega T))^{-1}\sin(\Omega s).
\end{aligned}
$$

We subsequently are able to calculate inner products

$$
\begin{aligned}
\langle \eta_x, (I-L)^{-1}\eta_x\rangle &= \frac{x^2\Omega^3}{\hbar^2}(\cos(\Omega T))^{-1}\int_0^T s\sin(\Omega s)ds \\
&= \frac{x^2\Omega}{\hbar^2}(\cos(\Omega T))^{-1}\Big[\sin(\Omega s) - \Omega s\cos(\Omega s)\Big]_0^T \\
&= \frac{x^2\Omega}{\hbar^2}(\cos(\Omega T))^{-1}\Big[\sin(\Omega T) - \Omega T\cos(\Omega T)\Big] = \frac{x^2\Omega}{\hbar^2}\Big[\tan(\Omega T) - \Omega T\Big].
\end{aligned}
$$

$$
\langle v_0, (I-L)^{-1}\eta_x\rangle = -\frac{x\Omega}{\hbar}(\cos(\Omega T))^{-1}\int_0^T \sin(\Omega s)ds = \frac{x}{\hbar}[1 - (\cos(\Omega T))^{-1}].
$$

For the last inner product we have that

$$
\begin{aligned}
\dot{\gamma}(s) = -1 - \Omega^2(\cos(\Omega T))^{-1}\Big[&\cos(\Omega(s-T))\int_0^s \cos(\Omega s')(T-s')ds' \\
&- \sin(\Omega s)\int_s^T \sin(\Omega(s'-T))(T-s')ds'\Big].
\end{aligned}
$$

We evaluate each of the integrals separately

$$
\begin{aligned}
\int_s^T \sin(\Omega(s'-T))v_0(s')ds' &= \Omega^{-2}\Big[(s'-T)\Omega\cos(\Omega(s'-T)) - \sin(\Omega(s'-T))\Big]_s^T \\
&= \Omega^{-2}\Big[(T-s)\Omega\cos(\Omega(s-T)) + \sin(\Omega(s-T))\Big]
\end{aligned}
$$

$$
\begin{aligned}
\int_0^s \cos(\Omega s')v_0(s')ds' &= \Omega^{-2}\Big[\Omega(T-s')\sin(\Omega s') - \cos(\Omega s')\Big]_0^s \\
&= \Omega^{-2}\Big[\Omega(T-s)\sin(\Omega s) - \cos(\Omega s) + 1\Big].
\end{aligned}
$$

$$\begin{aligned}
&\Big[\cos(\Omega(s-T))\int_0^s \cos(\Omega s')(T-s')ds' - \sin(\Omega s)\int_s^T \sin(\Omega(s'-T))(T-s')ds'\Big] \\
&= \Omega^{-2}\Big[-[\cos(\Omega(s-T))\cos(\Omega s)+\sin(\Omega s)\sin(\Omega(s-T))]+\cos(\Omega(s-T))\Big] \\
&= \Omega^{-2}\Big[-\cos(\Omega T)+\cos(\Omega(s-T))\Big]
\end{aligned}$$

$$\begin{aligned}
\dot\gamma(s) &= -(\cos(\Omega T))^{-1}\cos(\Omega(s-T)) \\
\langle v_0,(I-L)^{-1}v_0\rangle &= (\cos(\Omega T))^{-1}\int_0^T \cos(\Omega(s-T))ds = \Omega^{-1}(\cos(\Omega T))^{-1}\Big[\sin(\Omega(s-T))\Big]_0^T \\
&= \Omega^{-1}\tan(\Omega T).
\end{aligned}$$

Finally we can calculate the wavefunction to be

$$\begin{aligned}
\psi(x,T) &= \frac{1}{2\pi}e^{-\frac{i}{2\hbar}x\Omega^2 Tx}\int_{\mathbb{R}} e^{-\frac{i\hbar}{2}\langle\eta_x,(I-L)^1\eta_x\rangle}e^{-i\hbar k\langle v_0,(I-L)^1\eta_x\rangle}e^{-\frac{i\hbar k^2}{2}\langle v_0,(I-L)^1v_0\rangle}e^{ikx}\widehat{\psi_0}(k)dk \\
&= \frac{1}{2\pi}e^{-\frac{i}{2\hbar}x\Omega^2 Tx}\int_{\mathbb{R}} e^{-\frac{i\hbar}{2}\frac{x^2\Omega}{\hbar^2}\Big[\tan(\Omega T)-\Omega T\Big]}e^{-i\hbar k\frac{x}{\hbar}[1-(\cos(\Omega T))^{-1}]}e^{-\frac{i\hbar k^2}{2}\Omega^{-1}\tan(\Omega T)}e^{ikx}\widehat{\psi_0}(k)dk \\
&= \frac{1}{2\pi}e^{-\frac{ix^2\Omega}{2\hbar}\tan(\Omega T)}\int_{\mathbb{R}} e^{ikx(\cos(\Omega T))^{-1}}e^{-\frac{i\hbar k^2}{2}\Omega^{-1}\tan(\Omega T)}\widehat{\psi_0}(k)dk
\end{aligned}$$

We subsequently use the following Fourier transform formula

$$\mathcal{F}^{-1}[e^{iak^2}e^{ibk}] = \sqrt{\frac{i}{4\pi a}}e^{-i(y-b)^2/4a},$$

in this case we have that

$$\begin{aligned}
a &= -\frac{\hbar}{2}\Omega^{-1}\tan(\Omega T) \\
b &= x(\cos(\Omega T))^{-1} \\
\mathcal{F}^{-1}[e^{iak^2}e^{ibk}] &= \sqrt{\frac{\Omega}{2\pi i\tan(\Omega T)}}e^{\frac{i}{2\hbar}\Omega\cot(\Omega T)(y-x(\cos(\Omega T)^{-1})^2} \\
&= \sqrt{\frac{\Omega}{2\pi i\tan(\Omega T)}}e^{\frac{i}{2\hbar}[y\Omega\cot(\Omega T)y-2x\Omega(\sin(\Omega T)^{-1}y+x\Omega(\cos(\Omega T)\sin(\Omega T))^{-1}x)}
\end{aligned}$$

Subsequently we can use Plancherel's theorem (or Parseval's theorem if one uses the other name) to obtain

$$\begin{aligned}
\psi(x,T) &= \widetilde{\int} e^{\frac{i}{2\hbar}\langle\gamma,(I-L)\gamma\rangle} e^{-\frac{i}{2\hbar}x\Omega^2 xT} e^{-\frac{i}{\hbar}\int_0^T ds\gamma(s)\Omega^2 x ds}\psi_0(\gamma(s)+x)d\gamma \\
&= (\det(I-L))^{-1/2} e^{-\frac{i}{2\hbar}x\Omega^2 xT}\int_{\mathcal{H}_t} e^{-\frac{i}{2\hbar}\langle\gamma,(I-L)^{-1}\gamma\rangle} d\hat{\mu}(\gamma) \\
&= \sqrt{\frac{1}{\cos(\Omega T)}}\frac{1}{2\pi} e^{-\frac{i}{2\hbar}x\Omega^2 Tx}\int_{\mathbb{R}} e^{-\frac{i\hbar}{2}\langle\eta_x,(I-L)^1\eta_x\rangle} e^{-i\hbar k\langle v_0,(I-L)^1\eta_x\rangle} e^{-\frac{i\hbar k^2}{2}\langle v_0,(I-L)^1 v_0\rangle} e^{ikx}\widehat{\psi_0}(k)dk \\
&= \sqrt{\frac{\Omega}{2\pi i\sin(\Omega T)}} e^{-\frac{ix^2\Omega}{2\hbar}\tan(\Omega T)}\int_{\mathbb{R}} e^{\frac{i}{2\hbar}[y\Omega\cot(\Omega T)y - 2x\Omega(\sin(\Omega T)^{-1}y + x\Omega(\cos(\Omega T)\sin(\Omega T))^{-1}x)}\psi_0(y)dy \\
&= \sqrt{\frac{\Omega}{2\pi i\sin(\Omega T)}}\int_{\mathbb{R}} e^{\frac{i}{2\hbar}[y\Omega\cot(\Omega T)y - 2x\Omega(\sin(\Omega T)^{-1}y + x\Omega\cot(\Omega T)x)}\psi_0(y)dy,
\end{aligned}$$

which is what we had to show.

B.4 Calculation of $\langle v_x, (I-L)^{-1}v_x\rangle$

Recall that we defined

$$v_x(s) \coloneqq \Omega^2 x(Ts - s^2/2). \tag{201}$$

We now proceed with calculating the inverse

$$\begin{aligned}
(I-L)^{-1}v_x &= -\Omega x\int_0^s \sin(\Omega(s-s'))ds' + \Omega x\sin(\Omega s)T + x\Omega^2\frac{\sin(\Omega s)}{\cos(\Omega T)}\int_0^T \sin(\Omega(T-s'))(T-s')ds' \\
&= x(\cos(\Omega s)-1) + \Omega x\sin(\Omega s)T - x\frac{\sin(\Omega s)}{\cos(\Omega T)}\Big[\sin(\Omega(T-s')) + \Omega(s'-T)\cos(\Omega(T-s'))\Big]_0^T \\
&= x(\cos(\Omega s)-1) + \Omega x\sin(\Omega s)T + x\frac{\sin(\Omega s)}{\cos(\Omega T)}\Big[\sin(\Omega T) - T\Omega\cos(\Omega T)\Big] \\
&= x(\cos(\Omega s)-1) + x\sin(\Omega s)\tan(\Omega T) \\
\frac{d}{ds}(I-L)^{-1}v_x &= x\Omega\cos(\Omega s)\tan(\Omega T) - x\Omega\sin(\Omega s).
\end{aligned}$$

Subsequently we are able to calculate the integral

$$\begin{aligned}
\langle v_x, (I-L)^{-1}v_x\rangle &= x^2\Omega^3\int_0^T (T-s)\Big[\cos(\Omega s)\tan(\Omega T) - \sin(\Omega s)\Big]ds \\
&= x^2\Omega\Big[\Omega(T-s)\sin(\Omega s)\tan(\Omega T) - \cos(\Omega s)\tan(\Omega T) + \Omega(T-s)\cos(\Omega s) + \sin(\Omega s)\Big]_0^T
\end{aligned}$$

$$= x^2\Omega\Big[-\sin(\Omega T) + \tan(\Omega T) - \Omega T + \sin(\Omega T)\Big]$$
$$= x\Big[\Omega\tan(\Omega T) - \Omega^2 T\Big]x.$$

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