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Numerically testing the Riemann hypothesis

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Abstract

The Riemann Hypothesis states that all non-trivial zeros of the Riemann zeta function lie on the complex line $\frac{1}{2} + it$ for a real number t. In this bachelor's thesis we study a way to prove this hypothesis for $|\text{Im}(z)| \leq r$, $r \in \mathbb{R}, z \in \mathbb{C}$. The zeta function is a complex function given by an infinite sum in a part of the complex plane and analytic continued to the whole complex plane. It will be proven that the zeros of this function are all inside the complex strip with $0 < \text{Re}(z) < 1$. After that, a contour integral will be computed numerically around this strip up to $|\text{Im}(z)| < r$, to prove the Hypothesis. The zeta function is a special case of so-called L-functions. The method as described above will also be applied to an L-function of an elliptic curve to prove that its zeros are at the critical line of this L-function.

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1 Introduction

One of the Millenium Price Problems, stated by the Clay Mathematics Institute[9] is the Riemann hypothesis. This hypothesis is concerning the zeta function, an important formula in the study of prime numbers. The zeta function is defined, for complex numbers z with $\mathrm{Re}(s) > 1$ by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad z \in \mathbb{C}, \quad \mathrm{Re}(z) > 1 \tag{1}$$

This can also be written as a product over all prime numbers p:

$$\zeta(z) = \prod_{p} \frac{1}{1 - p^z}, \quad z \in \mathbb{C}, \quad \mathrm{Re}(z) > 1 \tag{2}$$

By analytic continuation, this function can be defined over the whole complex plane, except at $z = 1$.

The Riemann hypothesis, named after Bernhard Riemann, states that every nontrivial zero (the trivial zeros are those at the negative integers) of this function lies on the critical line $z = \frac{1}{2} + it$ for t real.

The main goal of this thesis is to prove that this hypothesis holds for $|\mathrm{Im}(z)| < r$ for some specific r.

To do this, an analytic continuation of the zeta function will be derived. Here we will show that there are no zeros outside the critical strip, $0 < \mathrm{Re}(z) < 1$, except the trivial zeros.

After this, we will make a contour integral containing this critical strip with $|\mathrm{Im}(z)| < 100$. The function which will be evaluated in this contour integral is the logarithmic derivative of the zeta function, $\zeta'(z)/\zeta(z)$. This will be done, since all zeros of the zeta function are simple poles of this function with residue 1. This contour integral will be evaluated using the Simpson's rule.

The second part of this paper will be about the so-called L-functions of elliptic curves. These functions are of the form

$$L(E, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where E denotes the elliptic curve. More details about this function can be found in section 6. For these functions, the Riemann hypothesis states that the zeros are on the critical line $z = 1 + it$ for t real. In this paper we will investigate the elliptic curve with the equation $y^2 + y = x^3 - x^2$. For the L-function of this curve we will show that there are no zeros outside the critical strip $0.5 < \mathrm{Re}(z) < 1.5$, except the trivial zeros and prove that the only zeros inside this critical strip with $|\mathrm{Im}(z)| < 50$ are on the critical line.

2 Preliminaries

In this text we will use many times various results from complex analysis. Therefore, we are going to define and prove some of the properties of complex functions before starting with the particular ones studied here.

First of all, we introduce the complex cosine and sine. After that, we will rewrite e^{it} in cosines and sines. After that, we will prove something about the real part of x^z and $\log(z)$.

Definition 2.1. For a complex number z, we have that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Proposition 2.1. *For a complex number z, we have that*

$$e^{iz} = \cos z + i \sin z$$

Proof. This follows directly from the definition of the sine and cosine:

$$\begin{aligned}
&\cos z + i \sin z = \\
&\frac{e^{iz} + e^{-iz}}{2} + i\frac{e^{iz} - e^{-iz}}{2i} = \\
&\frac{e^{iz} + e^{-iz} + e^{iz} - e^{-iz}}{2} = \\
&\frac{2e^{iz}}{2} = \\
&e^{iz}
\end{aligned}$$

□

Definition 2.2. For a real number $x > 0$ and a complex number z, we define

$$x^z = e^{z \ln x}$$

Proposition 2.2. *For a complex number z and a real number $x > 0$, we have that*

$$|x^z| = x^{\mathrm{Re}(z)}$$

Proof. Let x be a real and positive number, z be a complex number and a and

b be real numbers such that $z = a + bi$. Then we have

$$\begin{aligned}
|x^z| &= |e^{z \ln x}| \\
&= |e^{(a+bi) \ln x}| \\
&= |e^{a \ln x} \cdot e^{ib \ln x}| \\
&= |x^a| \cdot |e^{ib \ln x}| \\
&= x^a \cdot |\cos(b \ln x) + i \sin(b \ln x)| \\
&= x^a \cdot (\cos^2(b \ln x) + \sin^2(b \ln x))^{\frac{1}{2}} \\
&= x^a \\
&= x^{\mathrm{Re}(z)}
\end{aligned}$$

$\square$

Definition 2.3. For a complex number $z = a + bi$, we define $\log(z)$ as

$$\log(z) = \{v \in \mathbb{C} : e^v = z\}$$

Proposition 2.3. *Let z be a nonzero complex number. Then* $\mathrm{Re}(\log(z)) = \ln |z|$.

Proof. Let z be a nonzero complex number. Then z can be written as $z = re^{it}$ with $r \in \mathbb{R}_{>0}$ and $t \in \mathbb{R}$. Therefore, $z = e^{\ln(r)+it}$ and $|z| = r$.

Also note that $\log(z) = \log(e^{\ln(r)+it}) = \{\ln(r) + it + 2k\pi i \ \mid k \in \mathbb{Z}\}$. Hence we have that

$$\mathrm{Re}(\log(z)) = \ln(r) = \ln |z|.$$

$\square$

3 The Gamma Function

When analyzing the zeta function, the Gamma function will appear. Therefore, we are going to review several properties of this function here:

Definition 3.1. For a complex number $z \in \mathbb{C}$ with $\text{Re}(z) > 0$ we define

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

Now we need to analyze this function:

- For which $z \in \mathbb{C}$ does the integral in the definition of the Gamma function converge?
- How can the Gamma function be extended to a meromorphic function on the whole complex plane?
- What are the poles of the function?
- What are the zeros of the function?

This will be done in the following subsections.

3.1 Convergence for Re$(z) > 0$

In this subsection, we are going to prove that the integral defining the Gamma function converges absolutely for $z \in \mathbb{C}$ with $\text{Re}(z) > 0$. To do this, we will split the Gamma function in two parts: an integral from 0 to 1 and an integral from 1 to infinity. For both these integrals it will be proven that they converge, and therefore, the Gamma function converges.

But before we start to prove this convergence, we first need to prove something else which we will need in one of the proofs:

Lemma 3.1. *Let $a, k \in \mathbb{R}$, $a > 0$ and $n \in \mathbb{N}$. Then we have*

$$\lim_{k\to\infty} k^{a-n} \cdot e^{-k} = 0.$$

Proof. We will prove this in 3 cases: $a < n$, $a = n$ and $a > n$.

Case 1: Let $a < n$. Then $\lim_{k\to\infty} k^{a-n} = 0$ and $\lim_{k\to\infty} e^{-k} = 0$. Therefore, their product converges to 0 if $k \to \infty$.

Case 2: Let $a = n$. Then $\lim_{k\to\infty} k^{a-n} = 1$ and $\lim_{k\to\infty} e^{-k} = 0$. Therefore, their product converges to 0 if $k \to \infty$.

Case 3: Let $a > n$. Then $\lim_{k\to\infty} k^{a-n} \to \infty$. Also $\lim_{k\to\infty} e^{k} \to \infty$. Therefore, consider the following:

$$\lim_{k\to\infty} k^{a-n} \cdot e^{-k} = \lim_{k\to\infty} \frac{k^{a-n}}{e^k}$$

Take $m \in \mathbb{N}$ such that $m-1 < (a-n) \leq m$. Right now are going to apply l'Hôpital's rule m times:

$$\begin{aligned}\lim_{k\to\infty} \frac{k^{a-n}}{e^k} &\overset{H}{=} (a-n) \lim_{k\to\infty} \frac{k^{a-n-1}}{e^k} \\ &\overset{H}{=} \dots \\ &\overset{H}{=} (a-n)(a-n-1)\dots(a-n-m+1) \lim_{k\to\infty} \frac{k^{a-n-m}}{e^k}\end{aligned}$$

Since $a-n \leq m$, we have that $a-n-m \leq 0$. Therefore using Case 1 or Case 2,

$$\lim_{k\to\infty} \frac{k^{a-n}}{e^k} = (a-n)(a-n-1)\dots(a-n-m+1) \lim_{k\to\infty} \frac{k^{a-n-m}}{e^k} = 0.$$

□

Right now we are going to prove that the two integrals of the Gamma function are converging. Note that we use real numbers a instead of a complex number. This is because we will prove that the Gamma function is absolute convergent, in which case we use that $|x^{z-1}| = x^{a-1}$ (with $a = \mathrm{Re}(z)$) because of Proposition 2.2.

Lemma 3.2. *Let $a \in \mathbb{R}$, $a > 0$. Then the integral*

$$\int_0^1 x^{a-1} \cdot e^{-x} dx$$

converges.

Proof. First of all, $a-1 > -1$ since $a > 0$. Secondly, $e^{-1} \leq e^{-x} \leq 1$ for $x \in [0,1]$. Therefore, are going to write an upper and a lower bound for the integral:

$$e^{-1} \int_0^1 x^{a-1}\, dx \leq \int_0^1 x^{a-1} \cdot e^{-x} dx \leq \int_0^1 x^{a-1} dx.$$

Let us now consider the integral $\int_0^1 x^{a-1}\, dx$. This can be written as

$$\int_0^1 x^{a-1} dx = \lim_{b\to 0^+} \int_b^1 x^{a-1} dx.$$

The integral on the right hand side will be calculated:

$$\lim_{b\to 0^+} \int_b^1 x^{a-1} dx = \lim_{b\to 0^+} \left[\frac{1}{a} x^a\right]_b^1 = \lim_{b\to 0^+} \frac{1}{a}\left(1 - b^a\right) = \frac{1}{a}.$$

Note that we use here that $a \neq 0$ and in the last equality that $a > 0$.

Therefore we get

$$e^{-1}\frac{1}{a} \leq \int_0^1 x^{a-1} \cdot e^{-x} dx \leq \frac{1}{a}$$

Hence $\int_0^1 x^{a-1} \cdot e^{-x} dx$ converges. □

Lemma 3.3. *Let $a \in \mathbb{R}$, $a > 0$. Then the integral*
$$\int_1^\infty x^{a-1} \cdot e^{-x} dx$$
converges.

Proof. Let us now consider $\int_1^\infty x^{a-1} \cdot e^{-x} dx$. First note that
$$\int_1^\infty e^{-x} dx = \lim_{k\to\infty} \left[-e^{-x}\right]_1^k = \lim_{k\to\infty} \frac{1}{e} - e^{-k} = \frac{1}{e}.$$
Also note that for each $n \in \mathbb{N}$, we have, using integration by parts,
$$\begin{aligned}\int_1^\infty x^{a-n} \cdot e^{-x} dx &= \lim_{k\to\infty} \left[-x^{a-n} \cdot e^{-x}\right]_1^k + (a-n) \int_1^\infty x^{a-n-1} \cdot e^{-x} dx \\ &= \lim_{k\to\infty} \left[\frac{1}{e} - k^{a-n} \cdot e^{-k}\right] + (a-n) \int_1^\infty x^{a-n-1} \cdot e^{-x} dx.\end{aligned}$$

Because of Lemma 3.1, we have
$$\begin{aligned}\int_1^\infty x^{a-n} \cdot e^{-x} dx &= \lim_{k\to\infty} \left[\frac{1}{e} - k^{a-n} \cdot e^{-k}\right] + (a-n) \int_1^\infty x^{a-n-1} \cdot e^{-x} dx \\ &= \frac{1}{e} + (a-n) \int_1^\infty x^{a-n-1} \cdot e^{-x} dx.\end{aligned}$$
Take $m \in \mathbb{N}$ such that $m < a \leq m+1$. Then we have
$$\begin{aligned}\int_1^\infty x^{a-1} \cdot e^{-x} dx =& \frac{1}{e} + (a-1) \int_1^\infty x^{a-2} \cdot e^{-x} dx \\ =& \frac{1}{e} + (a-1) \left[\frac{1}{e} + (a-2) \int_1^\infty x^{a-3} \cdot e^{-x} dx\right] \\ =& \frac{1}{e} \left(1 + (a-1)\right) + (a-1)(a-2) \int_1^\infty x^{a-3} \cdot e^{-x} dx \\ =& \cdots \\ =& \frac{1}{e}(1 + (a-1) + \cdots + (a-(m-1))) \\ & \quad + (a-1)(a-2)\cdots(a-m) \int_1^\infty x^{a-(m+1)} \cdot e^{-x} dx \\ =& \frac{1}{e} \left[1 + \sum_{k=1}^{m-1} (a-k)\right] + \left[\prod_{l=1}^{m} (a-l)\right] \cdot \int_1^\infty x^{a-(m+1)} \cdot e^{-x} dx.\end{aligned} \tag{3}$$

Since $a \leq m+1$, we have $a-(m+1) \leq 0$. Therefore, $x^{a-(m+1)} \leq 1$ for $x > 1$. Hence
$$\int_1^\infty x^{a-(m+1)} \cdot e^{-x} dx \leq \int_1^\infty e^{-x} dx = \frac{1}{e}.$$
Hence, using equation (3)it follows that $\int_1^\infty x^{a-1} \cdot e^{-x} dx$ converges. $\square$

Combining these two Lemmas, we obtain the following Proposition:

Proposition 3.1. *Let $a > 0$ and b be real numbers and take $z = a + bi \in \mathbb{C}$. Then the integral defining $\Gamma(z)$ converges absolutely.*

Proof. To start this proof, we consider the following:

$$\begin{aligned}
|\Gamma(z)| &= |\Gamma(a+bi)| \\
&= \left| \int_0^\infty x^{a-1+bi} e^{-x} dx \right| \\
&\leq \int_0^\infty |x^{a-1+bi} e^{-x}| dx \\
&= \int_0^\infty x^{a-1} \cdot |e^{-x}| dx \\
&= \int_0^\infty x^{a-1} \cdot e^{-x} dx
\end{aligned}$$

To analyze this, we split the integral and get the following:

$$|\Gamma(z)| \leq \int_0^1 x^{a-1} \cdot e^{-x} dx + \int_1^\infty x^{a-1} \cdot e^{-x} dx$$

The Gamma function converges absolute if both integrals converges. Because of Lemma 3.2, the first integral converges and because of Lemma 3.3, the second integral converges. Therefore, the Gamma function converges absolute. □

3.2 Analytic continuation

Now we know that $\Gamma(z)$ exists for $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. Since we will use the Gamma function for the whole complex plane, we are going to make an analytic continuation of the Gamma function.

This analytic continuation will be made by introducing the fact that $\Gamma(z + 1) = z\Gamma(z)$, so $\Gamma(z) = \Gamma(z + 1)/z$. Since the right hand side is defined for $\mathrm{Re}(z) > -1, z \neq 0$, the left hand side is also defined for $\mathrm{Re}(z) > -1$, $z \neq 0$. Reusing this fact will lead to an analytic continuation to the whole complex plane.

First of all, we have to prove the following property which we already mentioned:

Lemma 3.4. *For a complex number z with $Re(z) > 0$ we have that $\Gamma(z + 1) = z\Gamma(z)$.*

Proof. Let us start with $\Gamma(z + 1)$:

$$\Gamma(z+1) = \int_0^\infty e^{-x} x^z dx.$$

Integration by parts gives us

$$\Gamma(z+1) = \int_0^\infty e^{-x} x^z dx = \left[-e^{-x} x^z\right]_0^\infty + z \int_0^\infty e^{-x} x^{z-1} dx$$

Note that $[-e^{-x} \cdot x^z]_0^\infty = 0$. Hence we get

$$\Gamma(z+1) = z \int_0^\infty e^{-x} x^{z-1} dx = z\Gamma(z).$$

□

Because of this lemma, we can write $\Gamma(z) = \frac{\Gamma(z+1)}{z}$. If we iterate this, we get

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \cdots = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}$$

Therefore, we extend the Gamma function for negative values as follows:

Definition 3.2. Let $z \in \mathbb{C}$, $\mathrm{Re}(z) \leq 0$ and $z \notin \mathbb{Z}$. Then choose $n \in \mathbb{N}$ such that $\mathrm{Re}(z) + n - 1 < 0 < \mathrm{Re}(z) + n$ and define

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}$$

The reason for $z \notin \mathbb{Z}$ is because of the fact that we then divide by 0.

3.3 Poles of the function

Lemma 3.5. *The pole of the Gamma function at* 0 *is a simple pole with residue* $Res(0) = 1$.

Proof. We have that $z\Gamma(z) = \Gamma(z+1)$. At $z = 0$, the right hand side becomes $\Gamma(1) = 1$. Therefore, the residue of the Gamma function at $z = 0$ is

$$\mathrm{Res}(0) = \lim_{z \to 0} z\Gamma(z) = 1$$

□

Theorem 3.6. *The poles of the Gamma function (i.e. for* $z \in \{0, -1, -2, \cdots\}$*) are simple poles with residue* $Res(-n) = \frac{(-1)^n}{n!}$.

Proof. For $z = 0$, we have proven this already in Lemma 3.5, since $\frac{(-1)^0}{0!} = 1$.

Now take the pole at $z = -n$. By Definition 3.2,

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)(z+2)\cdots(z+n-1)(z+n)}$$

In order to show that it is a simple pole, we note that

$$(z+n)\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)(z+2)\cdots(z+n-1)}$$

If we evaluate the right hand side in $z=-n$, we obtain $\Gamma(z+n+1) = \Gamma(1) = 1$, and $z(z+1)(z+2)\cdots(z+n-1) = n!(-1)^n$. Therefore

$$\text{Res}(-n) = \lim_{z\to -n}(z+n)\Gamma(z) = \frac{1}{n!(-1)^n} = \frac{(-1)^n}{n!}.$$

Therefore the poles at $z=-n$ are simple poles with residue $\frac{(-1)^n}{n!}$. □

3.4 Zeros of the function

In this subsection we will show that the Gamma function does not have any zero. To show this, we will prove that $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ if z is not an integer. Since the right hand side is never 0, $\Gamma(z)$ can only have a zero in $z=z_0$ not an integer, if $\Gamma(1-z)$ has a pole in z_0. Because of the fact that z is not an integer, this will never happen.

To prove this, we will prove this formula for z in the subset $V = \{\frac{1}{2n}, n \in \mathbb{N}\}$. We also will prove that it therefore is true for the whole complex plane, except for the integers.

Proving that $\Gamma(z) \neq 0$ for an integer is much easier, which will be shown at the end of this subsection.

To start this proof, we first need to prove the following:

Lemma 3.7. *For a positive integer m, define*

$$a_m = \sin\left(\pi(m+\frac{1}{2})\cdot e^{it}\right).$$

For a real number $t \in (0,2\pi)\backslash\{\pi, \pi/2, 3\pi/2\}$, we have that

$$Re(a_m) \to \pm\infty \qquad as \qquad m \to \infty. \tag{4}$$

Proof. First of all, let us define $A := \pi(m+\frac{1}{2})$. If $m\to\infty$, also $A\to\infty$ and vice versa.

In this proof, we first consider the case that $t \in \{0, \pi/2, \pi, 3\pi/2, 2\pi\}$. After that, we consider $t \in (0,2\pi)\backslash\{\pi,\pi/2,3\pi/2\}$.

If $t \in \{0, \pi/2, \pi, 3\pi/2, 2\pi\}$, we have to consider 2 cases:

- if $t=\pi/2$ or $3\pi/2$, then $e^{it} = \pm i$. Therefore, a_m becomes

  $$a_m = \sin(Ae^{it}) = \sin(\pm Ai) = \frac{1}{2i}\left(e^{i\cdot Ai} - e^{-i\cdot Ai}\right) = \frac{1}{2i}\left(e^{-A} - e^{A}\right),$$

  and hence the real part equals 0.

- If $t \in \{0, \pi, 2\pi\}$, then $e^{it} = \pm 1$. Therefore, $a_m = \sin(Ae^{it}) = \sin(\pm A) \in [-1, 1]$.

In both cases, $\mathrm{Re}(a_m) \not\to \pm\infty$. Therefore, these points need to be excluded.

Suppose now that $t \in (0, 2\pi) \backslash \{\pi, \pi/2, 3\pi/2\}$. Rewriting the function gives the following:

$$\begin{aligned}
\sin(Ae^{it}) &= \sin(A\cos t + iA\sin t) \\
&= \frac{1}{2i}\left(e^{i(A\cos t + iA\sin t)} - e^{-i(A\cos t + iA\sin t)}\right) \\
&= \frac{e^{iA\cos t}e^{-A\sin t}}{2i} - \frac{e^{-iA\cos t}e^{A\sin t}}{2i}
\end{aligned}$$

Since $e^{ix} = \cos(x) + i\sin(x)$, we are going to rewrite the last line as

$$\begin{aligned}
\sin(Ae^{it}) =& \frac{(\cos(A\cos(t)) + i\sin(A\cos(t)))e^{-A\sin t}}{2i} \\
&- \frac{(\cos(A\cos(t)) + i\sin(A\cos(t)))e^{A\sin t}}{2i} \\
=& \frac{1}{2i}(\cos(A\cos(t)))(e^{-A\sin t} - e^{A\sin t}) \\
&+ \frac{i}{2i}(\sin(A\cos(t)))(e^{-A\sin t} + e^{A\sin t})
\end{aligned}$$

Since $\frac{i}{2i} = \frac{1}{2}$, the right part is the real part and the left part is the imaginary part of the equation. $\sin(A\cos(t)) \in [-1, 1]$, and therefore

$$\frac{1}{2}\sin(A\cos(t))(e^{-A\sin t} + e^{A\sin t}) \to \infty \Leftrightarrow e^{-A\sin t} + e^{A\sin t} \to \infty, \tag{5}$$

assuming that $\sin(A\cos(t)) \neq 0$ (this will be proved later). Now we have to consider 2 cases:

1. If $t \in (0, \pi)$, $\sin t > 0 \Rightarrow \lim_{A\to\infty} e^{-A\sin t} = 0$ and $\lim_{A\to\infty} e^{A\sin t} = \infty$
2. If $t \in (\pi, 2\pi)$, $\sin t < 0 \Rightarrow \lim_{A\to\infty} e^{-A\sin t} = \infty$ and $\lim_{A\to\infty} e^{A\sin t} = 0$

In both cases we have that the right hand side of equation (5) is true and therefore, the left hand side needs to be true.

The only thing we need to check is that $\sin(A\cos(t)) \neq 0$. There is an equal sign only if $A\cos(t) = k\pi$, $k \in \mathbb{Z}$. This happens exactly at 2 points of the interval $(0, 2\pi)$: $\pi/2, 3\pi/2$. Therefore, the limit of the function is not equal to ∞ for these 2 points. Therefore, these points already were excluded from the interval. Hence we have

$$\lim_{m\to\infty} \sin(\pi(m + 1/2)e^{it}) = \pm\infty, \quad t \in (0, 2\pi) \backslash \{\pi, \pi/2, 3\pi/2\}$$

□

This lemma will be used to prove the following lemma:

Lemma 3.8. *For $n \in \mathbb{N}$, we have*

$$\frac{\pi}{\sin(\pi/2n)} = \lim_{m\to\infty} \sum_{|k|\leq m} \frac{(-1)^k}{\frac{1}{2n} - k} \tag{6}$$

Proof. Define the function

$$g_n : w \mapsto \frac{\pi}{\sin(\pi w)} \cdot \frac{1}{\frac{1}{2\pi} - w}$$

We first need to compute the poles and their residues.

First of all, at $w = \frac{1}{2n}$ we have a pole. The residue can be computed as follows:

$$\begin{aligned}
Res(\frac{1}{2n}) &= \lim_{w\to\frac{1}{2n}} (w - \frac{1}{2n}) \cdot g_n(w) \\
&= \lim_{w\to\frac{1}{2n}} \frac{w - \frac{1}{2n}}{\frac{1}{2n} - w} \cdot \frac{\pi}{\sin(\pi/2n)} \\
&= (-1) \cdot \frac{\pi}{\sin(\pi/2n)} \\
&= -\frac{\pi}{\sin(\pi/2n)}
\end{aligned}$$

Secondly, if $\pi w = \pi k$ where $k \in \mathbb{Z}$, then $\sin(\pi w) = 0$, so g_n does have a pole at $w = k$ for $k \in \mathbb{Z}$. The residue can be computed as follows:

$$\begin{aligned}
Res(k) &= \lim_{w\to k} (w - k) \cdot g_n(w) \\
&= \lim_{w\to k} (w - k) \cdot \frac{\pi}{\sin(\pi w)} \cdot \frac{1}{\frac{1}{2n} - w} \\
&= \lim_{w\to k} \frac{\pi(w-k)}{\sin(\pi w)(\frac{1}{2n} - w)} \\
&\stackrel{H}{=} \lim_{w\to k} \frac{\pi}{-\sin(\pi w) + \cos(\pi w)\pi \cdot (\frac{1}{2n} - w)} \\
&= \frac{1}{\cos(\pi k) \cdot (\frac{1}{2n} - k)} \\
&= \frac{(-1)^k}{\frac{1}{2n} - k}
\end{aligned}$$

where we used the fact that $\cos(\pi k) = (-1)^k$ for all integers k.

Right now, consider a circle C_m with radius $m + \frac{1}{2}$, centered at the origin.

$$C_m = \left\{(m + 1/2)e^{it}\right\} : 0 \leq t \leq 2\pi$$

The pole at $1/2n$ is inside this circle, and also all poles at $k \in \mathbb{Z}$ with $|k| \leq m$.

Recall Cauchy's Residue Theorem: If C is a simple closed positively oriented contour and f is analytic inside and on C except at the points $z_1, z_2, \cdots, z_k$ inside C, then

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{k} Res(z_j)$$

Therefore, we have that

$$\begin{aligned}
\int_{C_m} g_n(w)dw &= \int_{C_m} \frac{\pi}{\sin(\pi w)} \cdot \frac{1}{\frac{1}{2n} - w} \\
&= 2\pi i \left(Res(\frac{1}{2n}) + \sum_{j=-m}^{m} Res(j) \right) \\
&= 2\pi i \left(-\frac{\pi}{\sin(\pi/2n)} + \sum_{|k|\leq m} \frac{(-1)^k}{\frac{1}{2n} - k} \right)
\end{aligned}$$

To integrate g_n over C_m with respect to w, we will substitute $w(t) = (m + 1/2)e^{it}$. Then $dw = (m + 1/2)ie^{it}dt$. Therefore,

$$g_n((m+1/2)e^{it}) = \frac{\pi}{\sin(\pi(m+1/2)e^{it})} \cdot \frac{1}{\frac{1}{2n} - (m+1/2)e^{it}}$$

If we take the limit of m to infinity, the second fraction, multiplied by $(m+1/2)$, becomes

$$\lim_{m\to\infty} \frac{(m+1/2)}{\frac{1}{2n} - (m+1/2)e^{it}} = \lim_{m\to\infty} \frac{(1+1/2m)}{\frac{1}{2nm} - (1+1/2m)e^{it}} = -\frac{1}{e^{it}}.$$

Evaluating the contour integral gives us

$$\begin{aligned}
\lim_{m\to\infty} \int_{C_m} g_n(w)dw &= \lim_{m\to\infty} \int_0^{2\pi} g_n((m+1/2)e^{it}) \cdot (m+1/2)ie^{it}dt \\
&= \lim_{m\to\infty} \int_0^{2\pi} \frac{\pi}{\sin(\pi(m+1/2)e^{it})} \cdot (-\frac{1}{e^{it}}) \cdot ie^{it}dt \\
&= \lim_{m\to\infty} \int_0^{2\pi} \frac{-i\pi}{\sin(\pi(m+1/2)e^{it})} dt
\end{aligned}$$

The last equation, we will rewrite into 4 integrals, namely

$$\begin{aligned}
&\lim_{m\to\infty} \int_0^{2\pi} \frac{-i\pi}{\sin(\pi(m+1/2)e^{it})} dt \\
&= \lim_{m\to\infty} \left(\sum_{n=0}^{3} \int_{\frac{n}{2}\pi}^{\frac{n+1}{2}\pi} \frac{-i\pi}{\sin(\pi(m+1/2)e^{it})} dt \right)
\end{aligned}$$

If $m \to \infty$, the fraction becomes 0 for $z \in \{(0, \frac{\pi}{2}), (\frac{\pi}{2}, \pi), (\pi, \frac{3\pi}{2}), (\frac{3\pi}{2}, 2\pi)\}$ because of Lemma 3.7. Hence we have that each integral becomes 0. Therefore, we get that

$$
\begin{aligned}
&\lim_{m\to\infty} 2\pi i \left(-\frac{\pi}{\sin(\pi/2n)} + \sum_{|k|\le m} \frac{(-1)^k}{\frac{1}{2n} - k} \right) = 0 \Rightarrow \\
&\lim_{m\to\infty} \left(-\frac{\pi}{\sin(\pi/2n)} + \sum_{|k|\le m} \frac{(-1)^k}{\frac{1}{2n} - k} \right) = 0 \Rightarrow \\
&\frac{\pi}{\sin(\pi/2n)} = \lim_{m\to\infty} \sum_{|k|\le m} \frac{(-1)^k}{\frac{1}{2n} - k}
\end{aligned}
$$

□

This result will be used later on. We only need to prove the following lemma to proceed to the actual theorem:

Lemma 3.9. *Let $U = \mathbb{C}\backslash\mathbb{Z} \cup \{0\}$, $V = \{\frac{1}{2n}, \quad n \in \mathbb{N}\}$ and let $f : U \to \mathbb{C}$ be an analytic function. Assume that $f(v) = 0$ for $v \in V$. Then $f(u) = 0$ for $u \in U$.*

Proof. First note that U is an open, connected subset of $\mathbb{C}$. Also note that $V \subset Z$.

Let Z be the set $Z = \{z \in \mathbb{U} : f(z) = 0\}$. Let S be the set $S = \{z \in U : z \text{ is a limit point of } Z\}$. This set is nonempty, since $0 \in S$ (0 is the limit point of V, which is a subset of Z). The first goal of this proof is to prove that S is closed and open.

1. S **is closed:** Since $f(z)$ is continuous, also the limit points of Z are 0. This implies that $S \subset Z$. Any limit point of S is therefore a limit point of Z. Therefore, all limit points of S are in S and hence S is closed.

2. S **is open:** Pick $z_0 \in S$. There exists a $\delta > 0$ such that inside the open disk around z_0 with radius δ, $D(z_0, \delta)$, f has a Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converging inside the disc. Suppose that the disk is not completely 0 (so at least one point z_1 exists with $f(z_1) \neq 0$). Then not all a_n's are equal to 0.

Suppose $a_m \neq 0$, but $a_n = 0$ for all $n < m$. Then we can rewrite the Taylor expansion as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = (z - z_0)^m \sum_{n=m}^{\infty} a_n (z - z_0)^{n-m} = (z - z_0)^m h(z)$$

where $h(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^{n-m}$. Right now, $h(z_0) = \sum_{n=m}^{\infty} a_n (0)^{n-m} = a_m$, since $0^0 = 1$ and $0^n = 0$ for all $n \in \mathbb{N}$.

Since h is continuous, there exists a $\delta_1 > 0$ such that $h(z) \neq 0$ for all $z \in D(z_0, \delta_1)$. Therefore, $f(z) \neq 0$ for all z such that $0 < |z - z_0| < \delta_1$. Therefore, $z_0 \notin S$, which is a contradiction. Hence $f(z) = 0$ for all $z \in D(z_0, \delta)$, so $D(z_0, \delta) \subset Z$.

It is clear that each point in $D(z_0, \delta)$ is a limit point of $D(z_0, \delta)$, so $D(z_0, \delta) \subset S$. Therefore, S is open.

Now we know that S is an open and closed subset of U. Since U is an open and connected subset of $\mathbb{C}$, we know that $S = U$. Therefore, $f(z) = 0$ for all $z \in U$. $\square$

Lemma 3.10. *Let $U = \mathbb{C} \backslash \mathbb{Z} \cup \{0\}$, $V = \{\frac{1}{2n}, \quad n \in \mathbb{N}\}$ and let $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ be analytic functions. Assume that $f(v) = g(v)$ for $v \in V$. Then $f(u) = g(u)$ for $u \in U$*

Proof. This follows directly from Lemma 3.9 with replacing $f(z)$ by $f(z) - g(z)$. $\square$

Right now we are going to start proving the theorem, stated below:

Theorem 3.11. *We have that*

$$\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)} \text{ for } z \in U := (\mathbb{C}\backslash\mathbb{Z}) \cup \{0\} \tag{7}$$

Proof. Because of Lemma 3.10, it suffies to prove that equation (7) is true for

$$z \in V := \left\{\frac{1}{2n}, \quad n \in \mathbb{N}\right\}$$

Therefore, the Theorem follows if we prove that

$$\Gamma(1+\frac{1}{2n})\Gamma(1-\frac{1}{2n}) = \frac{\pi/2n}{\sin(\pi/2n)} \tag{8}$$

First of all,

$$\begin{aligned}
\Gamma(1+\frac{1}{2n})\Gamma(1-\frac{1}{2n}) &= \int_0^\infty s^{\frac{1}{2n}} e^{-s} ds \cdot \int_0^\infty t^{-\frac{1}{2n}} e^{-t} dt \\
&= \int_0^\infty \int_0^\infty s^{\frac{1}{2n}} e^{-s} t^{-\frac{1}{2n}} e^{-t} ds dt \\
&= \int_0^\infty \int_0^\infty (s/t)^{\frac{1}{2n}} e^{-s-t} ds dt
\end{aligned}$$

To analyze this integral, we are going to change variables: $u = s + t$, $v = (s/t)$. Rewriting this, we get the following:

$$\begin{aligned}
&t = sv \Rightarrow \\
&u = s + t = s + sv = s(1+v) \Rightarrow \\
&s = u/(1+v) \Rightarrow \\
&t = uv/(1+v)
\end{aligned}$$

To put this in the integral, we need to compute the Jacobian of the substitution:

$$\frac{\partial(s,t)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{v}{v+1} & \frac{u}{(v+1)^2} \\ \frac{1}{v+1} & -\frac{u}{(v+1)^2} \end{vmatrix} = \frac{-vu-u}{(v+1)^3} = \frac{-u}{(v+1)^2}$$

Now we are going to substitute this into the integral, and get the following:

$$\begin{aligned}
\Gamma(1+\frac{1}{2n})\Gamma(1-\frac{1}{2n}) &= \int_0^\infty \int_0^\infty (s/t)^{\frac{1}{2n}} e^{-s-t} ds dt \\
&= \int_0^\infty \int_0^\infty v^{\frac{1}{2n}} e^{-u} \left| \frac{\partial(s,t)}{\partial(u,v)} \right| \cdot du dv \\
&= \int_0^\infty \int_0^\infty v^{\frac{1}{2n}} e^{-u} \frac{u}{(v+1)^2} \cdot du dv \\
&= \int_0^\infty \frac{v^{\frac{1}{2n}}}{(v+1)^2} dv \cdot \int_0^\infty e^{-u} u du
\end{aligned}$$

The first integral is a more difficult one to integrate. Therefore, we are going to compute the second integral first. This we will do using integration by parts:

$$\begin{aligned}
\int_0^\infty e^{-u} u du &= \left[-e^{-u}u\right]_0^\infty + \int_0^\infty e^{-u} du \\
&= -\left[\frac{u}{e^u}\right]_0^\infty - \left[-e^{-u}\right]_0^\infty \\
&= -\lim_{u\to\infty} \frac{u}{e^u} + 0 - (-1+0) \\
&\overset{H}{=} -\lim_{u\to\infty} \frac{1}{e^u} + 1 \\
&= 1
\end{aligned}$$

Now we are going to compute the first integral. Note that $\frac{1}{(v+1)^2} = -\frac{d}{dv}\left(\frac{1}{v+1}\right)$.

$$\begin{aligned}
\int_0^\infty \frac{v^{\frac{1}{2n}}}{(v+1)} dv &= -\left[\frac{v^{1/2n}}{v+1}\right]_0^\infty + \int_0^\infty \frac{1}{v+1} \cdot \frac{dv^{1/2n}}{dv} dv \\
&= -\lim_{v\to\infty} \left[\frac{v^{1/2n}}{v+1}\right] + 0 + \int_0^\infty \frac{1}{v+1} \cdot dv^{1/2n} \\
&= -\lim_{v\to\infty} \left[\frac{1}{v^{1-1/2n} + v^{-1/2n}}\right] + \int_0^\infty \frac{1}{v+1} \cdot dv^{1/2n} \\
&= -\left[\frac{1}{\lim_{v\to\infty} v^{1-1/2n} + 0}\right] + \int_0^\infty \frac{1}{v+1} \cdot dv^{1/2n} \\
&= 0 + \int_0^\infty \frac{1}{v+1} \cdot dv^{1/2n}
\end{aligned}$$

Let us now rewrite $w = v^{1/2n}$, so $v = w^{2n}$ and $dv^{1/2n} = dw$. Therefore we get

$$\begin{aligned}\int_0^\infty \frac{v^{\frac{1}{2n}}}{(v+1)} dv &= \int_0^\infty \frac{1}{v+1} \cdot dv^{1/2n} \\ &= \int_0^\infty \frac{1}{w^{2n}+1} dw\end{aligned}$$

Consider that $t = 1/w$ implies that $dw = -dt/t^2$. Also note that $t(1) = 1$ and $t(\infty) = 0$. This we will use on the following way:

$$\begin{aligned}\int_0^\infty \frac{1}{w^{2n}+1} dw &= \int_0^1 \frac{1}{w^{2n}+1} dw + \int_1^\infty \frac{1}{w^{2n}+1} dw \\ &= \int_0^1 \frac{1}{w^{2n}+1} dw + \int_1^0 \frac{1}{(1/t)^{2n}+1} \frac{-1}{t^2} dt \\ &= \int_0^1 \frac{1}{w^{2n}+1} dw + \int_0^1 \frac{1}{t^{-2n}+1} \cdot t^{-2} dt \\ &= \int_0^1 \frac{1}{w^{2n}+1} dw + \int_0^1 \frac{t^{2n}}{1+t^{2n}} \cdot t^{-2} dt \\ &= \int_0^1 \frac{1}{w^{2n}+1} dw + \int_0^1 \frac{t^{2n-2}}{1+t^{2n}} dt\end{aligned}$$

Right now we are going to use the property that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. This is applicable since $0 < w < 1$ and $0 < t < 1$. Therefore, we have:

$$\begin{aligned}
\int_0^\infty \frac{1}{w^{2n}+1}dw &= \int_0^1 \frac{1}{w^{2n}+1}dw + \int_0^1 \frac{t^{2n-2}}{1+t^{2n}}dt \\
&= \int_0^1 1 - w^{2n} + w^{4n} - w^{6n} + \cdots dw + \\
&\quad \int_0^1 t^{2n-2} - t^{4n-2} + t^{6n-2} - t^{8n-2} + \cdots dt \\
&= \left[1 - \frac{1}{2n+1} + \frac{1}{4n+1} - \frac{1}{6n+1} + \cdots\right] \\
&\quad + \left[\frac{1}{2n-1} - \frac{1}{4n-1} + \frac{1}{6n-1} - \frac{1}{8n-1} + \cdots\right] \\
&= \frac{1}{2n}\left(\frac{1}{0+\frac{1}{2n}} - \frac{1}{1+\frac{1}{2n}} + \frac{1}{2+\frac{1}{2n}} - \frac{1}{3+\frac{1}{2n}} + \cdots\right) \\
&\quad + \frac{1}{2n}\left(-\frac{1}{-1+\frac{1}{2n}} + \frac{1}{-2+\frac{1}{2n}} - \frac{1}{-3+\frac{1}{2n}} + \cdots\right) \\
&= \frac{1}{2n}\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{k+\frac{1}{2n}} \\
&= \frac{1}{2n}\lim_{m\to\infty}\sum_{|k|\le m} \frac{(-1)^k}{\frac{1}{2n}-k} \\
&= \frac{\pi/2n}{\sin(\pi/2n)}
\end{aligned}$$

where we applied Lemma 3.8 in the last equality. Hence we have that

$$\Gamma(1+\frac{1}{2n})\Gamma(1-\frac{1}{2n}) = \frac{\pi/2n}{\sin(\pi/2n)}$$

□

This theorem implies, together with Lemma 3.11, the next corollary:

Corollary 3.11.1. *We have that*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \textit{ for } z \in U := (\mathbb{C}\backslash\mathbb{Z}) \tag{9}$$

Proof. Because of Lemma 3.11,

$$\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)} \text{ for } z \in U := (\mathbb{C}\backslash\mathbb{Z}) \cup \{0\}.$$

Since $\Gamma(1+z) = z \cdot \Gamma(z)$, by dividing both sides by z one obtains

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \text{ for } z \in U := (\mathbb{C}\backslash\mathbb{Z}).$$

□

From this, we conclude what we want in this subsection:

Corollary 3.11.2. *The Gamma function has no zeros.*

Proof. Suppose $z \in \mathbb{Z}$.

- If $z < 0$, the Gamma function does have a pole at z, so $\Gamma(z) \neq 0$.
- If $z = 0$, the Gamma function equals 1.
- If $z > 0$, $\Gamma(z) = (z-1)! \neq 0$.

Hence $\Gamma(z) \neq 0$ for $z \in \mathbb{Z}$.

Suppose $z \in \mathbb{C} \backslash \mathbb{Z}$. Then we have

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin(\pi z)}.$$

The right hand side is not equal to 0, and therefore the left hand side needs to be unequal to 0. $\Gamma(1-z)$ is not a pole, and therefore, $\Gamma(z) \neq 0$. $\square$

3.5 Summary

In this section we have proven different theorems about the Gamma function. In the introduction of this function we were looking for the following answers:

- For which $z \in \mathbb{C}$ is the Gamma function defined?
- How can this function be extended to the whole complex plane?
- What are the poles of the function?
- What are the zeros of the function?

As seen in the previous subsections, we have answered all these questions:

- The Gamma function is defined for $z \in \mathbb{C}$ with $\text{Re}(z) > 0$.
- This can be extended to the whole complex plane by defining the function for $\text{Re}(z) \leq 0$ as in Definition 3.2.
- The function does have poles at the negative integers.
- The Gamma function does not have zero's at all.

4 The Riemann zeta function

In this section we are going to show some properties of the zeta function. The function, already defined in the introduction, is an infinite sum. Therefore, we have the following questions:

- For which $z \in \mathbb{C}$ is the zeta function defined?
- How can this function be extended to the whole complex plane?
- What are the poles of the function?
- What are the zeros of the function?

4.1 Convergence for $\mathrm{Re}(z) > 1$

Lemma 4.1. *For $\sigma \in \mathbb{R}$,*

$$\sum_{n=1}^{\infty} \frac{1}{n^\sigma} < \infty \Leftrightarrow \int_1^\infty \frac{1}{x^\sigma} dx < \infty.$$

Proof. First note that if $\sigma \leq 0$, $\frac{1}{x^\sigma}$ is monotonic increasing and if $\sigma > 0$, $\frac{1}{x^\sigma}$ is monotonic decreasing. Also note that $\int_1^2 \frac{1}{x^\sigma} dx = C$ for $C \in \mathbb{R}$, dependent on σ.

Suppose $\sigma \leq 0$. Since $\frac{1}{x^\sigma}$ is monotonic increasing, $\int_k^{k+1} \frac{1}{x^\sigma} dx \geq \int_1^2 \frac{1}{x^\sigma} dx$ for all $k \in \mathbb{N}$. Therefore,

$$\int_1^k \frac{1}{x^\sigma} dx \geq k \cdot \int_1^2 \frac{1}{x^\sigma} dx = k \cdot C$$

If we take the limit of k to infinity, the right hand side becomes infinity. Therefore, also the left hand side becomes infinity. Also $\sum_{n=1}^{k} \frac{1}{n^\sigma} \geq k \cdot \frac{1}{n^\sigma}$, and also here the right hand side goes to infinity as k goes to infinity. Therefore, for $\sigma \leq 0$, this lemma holds.

Suppose now that $\sigma > 0$. Then

$$\frac{1}{n^\sigma} > \int_n^{n+1} \frac{1}{x^\sigma} dx > \frac{1}{(n+1)^\sigma}.$$

Therefore,

$$\sum_{n=1}^{k} \frac{1}{n^\sigma} > \int_1^{k+1} \frac{1}{x^\sigma} dx > \sum_{n=1}^{k} \frac{1}{(n+1)^\sigma} = \sum_{n=2}^{k+1} \frac{1}{n^\sigma}$$

If k goes to infinity, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^\sigma} > \int_1^{\infty} \frac{1}{x^\sigma} dx > \sum_{n=2}^{\infty} \frac{1}{n^\sigma}$$

. If $\int_1^\infty \frac{1}{x^\sigma} dx < \infty$, $\sum_{n=2}^\infty \frac{1}{n^\sigma} < \infty$ and hence

$$\sum_{n=1}^{\infty} \frac{1}{n^\sigma} = \sum_{n=2}^{\infty} \frac{1}{n^\sigma} + \sum_{n=1}^{2} \frac{1}{n^\sigma} < \infty$$

If $\int_1^\infty \frac{1}{x^\sigma} dx = \infty$, then $\sum_{n=1}^\infty \frac{1}{n^\sigma} = \infty$. Hence the lemma holds for $\sigma > 0$. $\square$

Lemma 4.2. *The zeta function converges absolute for $s \in \mathbb{C}$ with $Re(s) > 1$*

Proof. Let $s = \sigma + it$, $\sigma, t \in \mathbb{R}$. Then

$$\left|\frac{1}{n^s}\right| = \frac{1}{|n^{\sigma+it}|} = \frac{1}{n^\sigma}$$

Because of Lemma 4.1,

$$\sum_{n=1}^{\infty} \frac{1}{n^\sigma} < \infty \Leftrightarrow \int_1^\infty \frac{1}{x^\sigma} dx < \infty$$

Now we calculate the integral,

$$\int_1^\infty \frac{1}{x^\sigma} dx = \lim_{t\to\infty} \left[\frac{1}{-\sigma+1}\frac{1}{x^{\sigma-1}}\right]_1^t = \lim_{t\to\infty} \left[\frac{1}{-\sigma+1}\frac{1}{t^{\sigma-1}} - \frac{1}{-\sigma+1}\right]$$

This only converges for $\sigma - 1 > 0$, or equivalent, $\sigma > 1$. Therefore, the zeta function converges absolute for $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$. $\square$

4.2 Analytic continuation

As seen in the previous subsection, the Riemann zeta function is only defined for $z \in \mathbb{C}$, $\mathrm{Re}(z) > 1$. Therefore, we are going to derive an analytic continuation of this function to evaluate it at other points. First of all, we are going to make an analytic continuation for $\mathrm{Re}(z) > 0$. After that, we will continue to the whole complex plane.

4.2.1 Analytic continuation for Re$(z) > 0$

First of all, we are going to make an analytic continuation for $\mathrm{Re}(z) > 0$. In order to do that, we need to prove first the following lemma:

Lemma 4.3. *Let $z \in \mathbb{C}$, $Re(z) > 0$ and let $n \in \mathbb{N}$. Define the function w as follows:*

$$w\colon [n, n+1] \to \mathbb{C}, \quad w(t) = \frac{1}{n^z} - \frac{1}{t^z}$$

This function satisfies the following inequality:

$$\sup_{n \le t \le n+1} |w(t)| = |w(n+1)| \le \frac{|z|}{n^{Re(z)+1}}$$

Proof. To prove this lemma, note that $w(n) = \frac{1}{n^z} - \frac{1}{n^z} = 0$. The derivative of the function is

$$w'(t) = z \cdot \frac{1}{t^{z+1}}.$$

Therefore,

$$|w'(t)| = |z| \cdot \frac{1}{|t^{z+1}|} = \frac{|z|}{t^{\mathrm{Re}(z)+1}}.$$

Since z is a fixed number and $\mathrm{Re}(z)+1 > 0$, the absolute value of the derivative is a decreasing positive function. Hence $|w(t)|$ is an increasing function with decreasing speed $|w'(t)|$.

Since we want an upper bound for $|w(t)|$, we will use the maximum speed of the function $w(t)$ (which is the maximum of $|w'(t)|$) and multiply this with the distance from t to n and add $|w(n)|$ (which equals 0). Since the speed is decreasing, $|w'(t)| \leq |w'(n)| = \frac{|z|}{n^{\mathrm{Re}(z)+1}}$. The distance from t to n is $(t-n)$ and therefore an upper bound for $|w(t)|$ is

$$|w(t)| \leq |w(n)| + (t-n) \cdot \frac{|z|}{n^{\mathrm{Re}(z)+1}} = (t-n) \cdot \frac{|z|}{n^{\mathrm{Re}(z)+1}}.$$

Since $0 \leq (t-n) \leq 1$, we have that

$$\sup_{n \leq t \leq n+1} |w(t)| \leq \frac{|z|}{n^{\mathrm{Re}(z)+1}}.$$

□

Now we are going to make an analytic continuation of the zeta function. To do this, we want to prove 2 lemmas, from which we are going to prove that a certain formula is indeed an analytic continuation of the zeta function. Lemma 4.3 is used to prove the following lemma:

Lemma 4.4. *$\zeta(z)$ can be written as $\zeta(z) = \frac{1}{z-1} + \sum_{n=1}^{\infty} \phi_n(z)$ for the functions $\phi_n(z)$ defined as*

$$\phi_n = \int_n^{n+1} \left(\frac{1}{n^z} - \frac{1}{t^z} \right) dt. \tag{10}$$

Proof. Let $z \in \mathbb{C}, \quad \mathrm{Re}(z) > 1$. Then we have

$$\int_1^{\infty} \frac{1}{t^z} dt = \left[\frac{1}{1-z} \cdot t^{1-z} \right]_{t=1}^{\infty} = \lim_{t \to \infty} \frac{1}{1-z}(t^{1-z} - 1) = \frac{1}{z-1}$$

The first integral can be rewritten as

$$\int_1^{\infty} \frac{1}{t^z} = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{t^z} dz$$

Therefore, we are going to rewrite the zeta function as

$$\begin{aligned}
\zeta(z) &= \frac{1}{z-1} - \frac{1}{z-1} + \sum_{n=1}^{\infty} \frac{1}{n^z} \\
&= \frac{1}{z-1} + \sum_{n=1}^{\infty} \left(\frac{1}{n^z} - \int_n^{n+1} \frac{1}{t^z} dt \right) \\
&= \frac{1}{z-1} + \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^z} - \frac{1}{t^z} \right) dt \\
&= \frac{1}{z-1} + \sum_{n=1}^{\infty} \phi_n(z).
\end{aligned}$$

□

Lemma 4.5. $\sum_{n=1}^{\infty} \phi_n(z)$ *converges normally on all subsets* $K \subset \mathbb{C}$ *with the following properties:*

1. *Every* $z \in K$ *has real part* $Re(z) > 0$
2. K *is a compact subset of* $\mathbb{C}$.

where $\phi_n(z) = \int_n^{n+1} \left(\frac{1}{n^z} - \frac{1}{t^z} \right) dt$

Proof. First observe that for all n, ϕ_n is defined for $\mathrm{Re}(z) > 0$ and that ϕ_n is holomorphic. What we are going to show is that for every K with properties (1) and (2) of the Lemma,

$$\sum_{n=1}^{\infty} \sup_{z \in K} |\phi_n(z)| = \sum_{n=1}^{\infty} \sup_{z \in K} \left| \int_n^{n+1} \left(\frac{1}{n^z} - \frac{1}{t^z} \right) dt \right| < \infty. \tag{11}$$

Also note that we have an upper bound for the integral,

$$\left| \int_n^{n+1} \left(\frac{1}{n^z} - \frac{1}{t^z} \right) dt \right| \leq \sup_{n \leq t \leq n+1} \left| \frac{1}{n^z} - \frac{1}{t^z} \right|. \tag{12}$$

Because of Lemma 4.3, the right hand side of this equation can be bounded by the following:

$$\left| \frac{1}{n^z} - \frac{1}{t^z} \right| \leq \frac{|z|}{n^{\mathrm{Re}(z)+1}} \quad \text{for } n \leq t \leq n+1 \tag{13}$$

Since the right hand side of this equation does not depend on t, combining equations (11), (12) and (13) gives us

$$\sum_{n=1}^{\infty} \sup_{z \in K} |\phi_n(z)| = \sum_{n=1}^{\infty} \sup_{z \in K} \left| \int_n^{n+1} \left(\frac{1}{n^z} - \frac{1}{t^z} \right) dt \right| \leq \sum_{n=1}^{\infty} \sup_{z \in K} \frac{|z|}{n^{\mathrm{Re}(z)+1}} \tag{14}$$

Right now, take an arbitrary subset K with properties (1) and (2) of the Lemma. Then we have the following:

- K is compact and therefore closed and bounded. Hence, we can take a $G \in \mathbb{R}$ such that for every $z \in K$, $|z| \leq G$
- The function mapping $z \mapsto \text{Re}(z)$ is continuous, so it has a minimum value on K, let say m. Note that $m > 0$, since $\text{Re}(z) > 0$ for all $z \in \mathbb{C}$.

Therefore, we have that for each $z \in K$ and $n \in N$,

$$\frac{|z|}{n^{\text{Re}(z)+1}} \leq \frac{G}{n^{m+1}} \quad \text{and hence } \sup_{z\in K} \frac{|z|}{n^{\text{Re}(z)+1}} \leq \frac{G}{n^{m+1}} \tag{15}$$

Now we are going to rewrite equation (14) as

$$\sum_{n=1}^{\infty} \sup_{z\in K} |\phi_n(z)| \leq \sum_{n=1}^{\infty} \sup_{z\in K} \frac{|z|}{n^{\text{Re}(z)+1}} \leq \sum_{n=1}^{\infty} \frac{G}{n^{m+1}} = G \cdot \sum_{n=1}^{\infty} \frac{1}{n^{m+1}} \tag{16}$$

The series on the right hand side can be written as $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where $p = m + 1$. This series is called a *p-series*, which converges if $p > 1$. Since $m > 0$, $p > 1$ and hence this series converges. Therefore we have $G \cdot \sum_{n=1}^{\infty} \frac{1}{n^{m+1}} < \infty$ and hence we have

$$\sum_{n=1}^{\infty} \sup_{z\in K} |\phi_n(z)| \leq G \cdot \sum_{n=1}^{\infty} \frac{1}{n^{m+1}} < \infty$$

which finishes the proof. □

Proposition 4.1. *It is possible to rewrite the zeta function as*

$$\zeta(z) = \frac{1}{z-1} + \phi(z) \tag{17}$$

where $\phi(z)$ is holomorphic for Re$(z) > 0$.

Proof. This follows directly from Lemma 4.4 and Lemma 4.5, where

$$\phi(z) = \sum_{n=1}^{\infty} \int_{n}^{n+1} \left(\frac{1}{n^z} - \frac{1}{t^z} \right) dt$$

□

Right now we have an analytic continuation of the zeta function to the half complex plane with $\text{Re}(z) > 0$, except for $z = 1$. But we want to have an analytic continuation to the whole complex plane (maybe except for some points), which will be derived now.

4.2.2 Analytic continuation to $\mathbb{C}\backslash\{1\}$ for $\mathbf{Re}(z) > -n$

In this subsection we are going to introduce an analytic continuation to $\mathrm{Re}(z) > -n$. This will be done using the so-called Euler-MacLaurin formula. First we will prove that this formula holds. After that, we will change the formulas and constants so that it becomes an equation for the zeta function.

Before we are going to introduce the Euler-MacLaurin formula, we have to introduce the bernoulli numbers and polynomials.

Definition 4.1. Let $\mathbb{Q}[x]$ be the ring of polynomials over the rational numbers $\mathbb{Q}$, i.e. $\mathbb{Q}[x]$ consists of all elements which can be written as

$$f = \sum_{i=1}^{n} a_i x^i$$

for $a_i \in \mathbb{R}$ and $n < \infty$.

Definition 4.2. Let $\mathbb{Q}[x][[z]]$ be the ring of formal power series over the ring of polynomials $\mathbb{Q}[x]$, i.e. $\mathbb{Q}[x]$ consists of all elements which can be written as

$$g = \sum_{i=1}^{n} f_i x^i$$

for $f_i \in \mathbb{R}[x]$.

Proposition 4.2. *Let $F \in \mathbb{Q}[x][[z]]$. Then F is invertible (i.e. there exists a $G \in \mathbb{Q}[x][[z]]$ such that $F \cdot G = 1$) if and only if the constant coefficient (i.e. the coefficient of $x^0 z^0$) is a unit, or equivalent, an element of $\mathbb{Q}^{\times}$*

Proof. First of all, note that this proposition holds if it holds for $F \in \mathbb{Q}[x]$. Therefore, we will prove this proposition for $F \in \mathbb{Q}[x]$.

Suppose F is invertible. Then $F \cdot G = 1$, where G is the inverse of F. Since $F, G \in \mathbb{Q}[x]$, we can write F and G as

$$F = \sum a_n x^n, \quad G = \sum b_n x^n.$$

Therefore, $FG = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + ...$ Therefore, $a_0 b_0 = 1$, which implies that a_0 is a unit.

Suppose $F = \sum a_n x^n$ with a_0 a unit. To find $G = \sum b_n x^n$ such that G is the inverse of F, we need to solve the following system:

$$\begin{aligned} a_0 b_0 &= 1 \\ a_0 b_1 + a_1 b_0 &= 0 \\ &\vdots \\ a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0 &= 0 \\ &\vdots \end{aligned}$$

Since a_0 is a unit, $b_0 = a_0^{-1}$. Proceeding inductively, if $b_0, \ldots, b_{n-1}$ are determined in terms of a_i, then

$$b_n = a_0^{-1}(-a_1 b_{n-1} - a_2 b_{n-2} - \ldots - a_n b_0).$$

Therefore, the inverse of F exists.

Hence $F \in \mathbb{Q}[x]$ is invertible if and only if the constant term is a unit. Therefore, also $F \in \mathbb{Q}[x][[z]]$ is invertible if and only if the constant term is a unit. $\square$

We can write $e^z - 1$ as

$$e^z - 1 = z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots + \frac{z^n}{n!} + \ldots = z \cdot \left(1 + \frac{z}{2} + \ldots + \frac{z^{n-1}}{n!} + \ldots\right).$$

Therefore, the constant term of $(e^z - 1)/z$ is 1, which is a unit in $\mathbb{Q}$. Hence, $(e^z - 1)/z$ is invertible and we will denote the inverse by $z/(e^z - 1)$. Note that in this inverse all coefficients of z^n are polynomials of degree 0 (since x does not appear in the coefficients). Therefore, we can write this as

$$z/(e^z - 1) = \sum a_n z^n, \quad a_n \in \mathbb{Q}$$

The second power series which we will discuss is $e^{xz} = \sum \frac{x^n}{n!} z^n$. Here the coefficients of z^n are polynomials of degree n.

If we now multiply $z/(e^z - 1)$ and e^{xz} with each other, we get

$$z/(e^z - 1) \cdot e^{xz} = \sum a_n z^n \cdot \sum \frac{x^n}{n!} z^n = \sum c_n z^n$$

with $c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$, where $b_i = \frac{x^i}{i!} z^i$. Since b_n is a polynomial of degree n and a_n is a polynomial of degree 0, we can conclude that c_n is a polynomial of degree n. Also note that the coefficient of x^n in c^n comes from the product $a_0 b_n = x^n/n!$, and therefore, this coefficient needs to be $1/n!$.

Definition 4.3. The bernoulli polynomials are defined to be

$$B_n(x) = c_n \cdot n!$$

with c_n as described above.

Since the coefficient of x^n in c_n equals $1/n!$, the coefficient of x^n in B_n equals 1. Hence the bernoulli polynomials are monic polynomials of degree n.

Lemma 4.6. *For an integer $n \geq 0$ we have that*

$$\frac{d}{dx} B_n = n \cdot B_{n-1}$$

Proof. First of all, we have that

$$\frac{d}{dx} ze^{xz}/(e^z - 1) = z^2 e^{xz}/(e^z - 1).$$

Therefore,

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} B_n(x) z^n/n!\right) = \sum_{n=0}^{\infty} B_n(x) z^{n+1}/n!$$

The left hand side can be calculated further, using that $\frac{d}{dx} B_0(x) = 0$:

$$\begin{aligned}
\frac{d}{dx}\left(\sum_{n=0}^{\infty} B_n(x) z^n/n!\right) &= \sum_{n=0}^{\infty} \frac{d}{dx}\left(B_n(x) z^n/n!\right) \\
&= \sum_{n=0}^{\infty} \frac{d}{dx}\left(B_n(x)\right) z^n/n! \\
&= \sum_{n=1}^{\infty} \frac{d}{dx}\left(B_n(x)\right) z^n/n! \\
&= \sum_{n=0}^{\infty} \frac{d}{dx}\left(B_{n+1}(x)\right) z^{n+1}/(n+1)! \\
&= \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{B_{n+1}(x)}{n+1}\right) z^{n+1}/n!
\end{aligned}$$

Therefore, we get that $\frac{d}{dx}\frac{B_{n+1}(x)}{n+1} = B_n(x)$, and hence $\frac{d}{dx} B_{n+1}(x) = (n+1) \cdot B_n(x)$ which is equivalent to $B_n(x) = n \cdot B_{n-1}(x)$. □

Definition 4.4. The nth Bernoulli number B_n is the nth Bernoulli polynomial evaluated at 0, so $B_n = B_n(0)$.

Lemma 4.7. *For an integer $n \geq 0$, $n \neq 1$, we have that $B_n(0) = B_n(1)$, and for $n = 1$ we have $B_n(1) = B_n(0) + 1$*

Proof. We have, by definition,

$$\sum_{n=0}^{\infty} B_n(x) z^n/n! = ze^{xz}/(e^z - 1).$$

Therefore,

$$\begin{aligned}
\sum_{n=0}^{\infty} (B_n(1) - B_n(0)) z^n/n! &= \sum_{n=0}^{\infty} B_n(1) z^n/n! - \sum_{n=0}^{\infty} B_n(0) z^n/n! \\
&= ze^z/(e^z - 1) - z/(e^z - 1) \\
&= z(e^z - 1)/(e^z - 1) \\
&= z.
\end{aligned}$$

Therefore, $B_n(1) - B_n(0) = 0$ for $n \geq 0$, $n \neq 1$ and $B_1(1) - B_1(0) = 1$. Rewriting this gives us the obtained result. □

Definition 4.5. Any real number x can be uniquely written as $m+\theta$ where m is an integer and $0 \leq \theta < 1$. We call m the integer part of x and denote this by $\lfloor x \rfloor$.

Lemma 4.8. *For a function $f(r)$ which is at least n times continuously differentiable, the following formula holds:*

$$f(0) = \int_0^1 f(x)dx + \sum_{k=1}^{n} B_k/k! \cdot (f^{(k-1)}(1) - f^{(k-1)}(0)) + (-1)^{n-1}/n! \int_0^1 \overline{B_n}(x) f^{(n)}(x)dx$$

where $\overline{B_n}(x) = B_n(x - \lfloor x \rfloor)$ is the real function which is periodic with period 1, coinciding on the interval $[0,1]$ with the nth Bernoulli polynomial.

Proof. For $x \in (0,1)$, we have that $\overline{B_n}(x) = B_n(x)$. Therefore, for $n = 1$ we have

$$\begin{aligned}
\int_0^1 \overline{B_1}(x) f^{(1)}(x)dx &= \int_0^1 B_1 f^{(1)} dx \\
&= [B_1(x)f(x)]_{x=0}^{x=1} - \int_0^1 f(x) \left(\frac{d}{dx} B_1(x) \right) dx \\
&= \frac{1}{2}(f(1) + f(0)) - \int_0^1 f(x)dx \\
&= f(0) + \frac{1}{2} \cdot (f(1) - f(0)) - \int_0^1 f(x)dx \\
&= f(0) - B_1 \cdot (f(1) - f(0)) - \int_0^1 f(x)dx
\end{aligned}$$

since $B_1 = -\frac{1}{2}$ and $\frac{d}{dx} B_1(x) = \frac{d}{dx}\left(x - \frac{1}{2}\right) = 1$.

Rewriting gives us

$$f(0) = \int_0^1 f(x)dx + B_1 \cdot (f(1) - f(0)) + \int_0^1 B_1 f^{(1)} dx.$$

Therefore, the formula holds for $n = 1$.

Suppose the formula holds for $n = l$. Then

$$f(0) = \int_0^1 f(x)dx + \sum_{k=1}^{l} B_k/k! \cdot (f^{(k-1)}(1) - f^{(k-1)}(0)) + (-1)^{l-1}/l! \int_0^1 B_l(x) f^{(l)}(x)dx. \tag{18}$$

The integral in this equation can be rewritten using integration by parts, using that $\frac{d}{dx} B_{n+1}(x) = (n+1)B_n(x)$ and that $B_n = B_n(0) = B_n(1)$ for $n \leq 2$:

$$\begin{aligned}
\int_0^1 B_l(x) f^{(l)}(x)dx &= \left[\frac{1}{l+1} B_{l+1} f^{(l)}(x) \right]_{x=0}^{x=1} - \int_0^1 \frac{1}{l+1} B_{l+1}(x) f^{(l+1)}(x)dx \\
&= \frac{1}{l+1} B_{l+1} \cdot \left(f^l(1) - f^l(0) \right) - \frac{1}{l+1} \int_0^1 B_{l+1}(x) f^{(l+1)}(x)dx.
\end{aligned}$$

Therefore, we are going to rewrite equation (18) as follows:

$$
\begin{aligned}
f(0) &= \int_0^1 f(x)dx + \sum_{k=1}^{l} B_k/k! \cdot (f^{(k-1)}(1) - f^{(k-1)}(0)) + (-1)^{l-1}/l! \int_0^1 B_l(x) f^{(l)}(x)dx \\
&= \int_0^1 f(x)dx + \sum_{k=1}^{l} B_k/k! \cdot (f^{(k-1)}(1) - f^{(k-1)}(0)) + (-1)^{l-1}/l! \cdot \\
&\qquad \left(\frac{1}{l+1} B_{l+1} \cdot \left(f^l(1) - f^l(0) \right) - \frac{1}{l+1} \int_0^1 B_{l+1}(x) f^{(l+1)}(x)dx \right) \\
&= \int_0^1 f(x)dx + \sum_{k=1}^{l} B_k/k! \cdot (f^{(k-1)}(1) - f^{(k-1)}(0)) \\
&\qquad + \frac{(-1)^{l-1}}{(l+1)!} B_{l+1} \cdot \left(f^l(1) - f^l(0) \right) + \frac{(-1)^l}{(l+1)!} \int_0^1 B_{l+1}(x) f^{(l+1)}(x)dx.
\end{aligned}
$$

Since $B_{l+1} = 0$ for l even and $(-1)^{l-1} = 1$ for l odd,

$$
\frac{(-1)^{l-1}}{(l+1)!} B_{l+1} \cdot \left(f^l(1) - f^l(0) \right) = \frac{B_{l+1}}{(l+1)!} \left(f^l(1) - f^l(0) \right).
$$

Hence we get

$$
\begin{aligned}
f(0) &= \int_0^1 f(x)dx + \sum_{k=1}^{l} B_k/k! \cdot (f^{(k-1)}(1) - f^{(k-1)}(0)) \\
&\qquad + \frac{B_{l+1}}{(l+1)!} \cdot \left(f^l(1) - f^l(0) \right) + \frac{(-1)^l}{(l+1)!} B_{l+1} \cdot \int_0^1 f^{(l+1)}(x)dx \\
&= \int_0^1 f(x)dx + \sum_{k=1}^{l+1} B_k/k! \cdot (f^{(k-1)}(1) - f^{(k-1)}(0)) \\
&\qquad + \frac{(-1)^l}{(l+1)!} B_{l+1} \cdot \int_0^1 f^{(l+1)}(x)dx.
\end{aligned}
$$

By mathematical induction, this proves the formula for every integer $n \geq 1$. □

Theorem 4.9. *For a function $f(r)$ which is at least n times continuously differentiable, the Euler-MacLaurin formula holds:*

$$
\sum_{r=a}^{b-1} f(r) = \int_a^b f(x)dx + \sum_{k=1}^{n} B_k/k! \cdot (f^{(k-1)}(b) - f^{(k-1)}(a)) + (-1)^{n-1}/n! \int_a^b \overline{B_n}(x) f^{(n)}(x)dx
$$

where $\overline{B_n}(x) = B_n(x - \lfloor x \rfloor)$ is the real function which is periodic with period 1, coinciding on the interval $[0,1]$ with the nth Bernoulli polynomial.

Proof. Each integral from a to b can be rewritten as a sum of integrals from s to $s+1$ where $s = a, a+1, \cdots, b-1$. Therefore, if we prove that the equation holds for $a = s$ and $b = s+1$, then it also holds for each a and b.

Because of Lemma 4.8 we have that this holds if $s = 0$. If we now replace x by $x+a$, we obtain the required formula for $s = a$. Therefore, the Euler-MacLaurin formula holds. □

Corollary 4.9.1. *The zeta function can be written as*

$$\begin{aligned}\zeta(s) =& \frac{1}{s-1} + \frac{1}{2} + \sum_{k=2}^{n} B_k/k! \cdot (-1)^k s(s+1)\cdots(s+k-2) \\ &- s(s+1)\cdots(s+n-1)/n! \cdot \int_1^\infty \overline{B_n}(x) x^{-s-n} dx \end{aligned} \tag{19}$$

for all $s \in \mathbb{Z}$ such that the left hand side is defined. Hence the right hand side is an analytic continuation of the zeta function for a positive integer n and $Re(s) > 1-n$.

Proof. Using the Euler-MacLaurin formula with $f(r) = r^{-s}$, $a = 1$, $b = N+1$, gives a function that looks like the zeta function.

Calculating the nth derivative of $f(r)$ gives the following:

$$f^{(1)}(x) = -s \cdot r^{-s-1} = (-1)^1 s \cdot r^{-s-1}$$

$$f^{(2)}(x) = -s(-s-1) \cdot r^{-s-2} = (-1)^2 s(s+1) \cdot r^{-s-2}$$

$$f^{(n)}(x) = -s(-s-1)\cdots(-s-n+1) \cdot r^{-s-n} = (-1)^n s(s+1)\cdots(s+n-1) \cdot r^{-s-n}$$

Therefore, the equation becomes

$$\begin{aligned}\sum_{r=1}^{N} r^{-s} =& \int_1^{N+1} x^{-s} dx + B_1 \cdot (N^{-s} - 1) \\ &+ \sum_{k=2}^{n} B_k/k! \cdot ((-1)^{k-1} s(s+1)\cdots(s+k-2) \cdot ((N+1)^{-s-k+1} - 1)) \\ &+ (-1)^{n-1}/n! \int_1^{N+1} \overline{B_n}(x)(-1)^n s(s+1)\cdots(s+n-1) \cdot x^{-s-n} dx \\ =& \frac{1-N^{1-s}}{s-1} + \frac{1-N^{-s}}{2} \\ &+ \sum_{k=1}^{n} B_k/k! \cdot ((-1)^{k-1} s(s+1)\cdots(s+k-2) \cdot ((N+1)^{-s-k+1} - 1)) \\ &- s(s+1)\cdots(s+n-1)/n! \cdot \int_1^{N+1} \overline{B_n}(x) x^{-s-n} dx\end{aligned}$$

Assume that $\mathrm{Re}(s) > 1$. When N goes to infinity, we obtain the following:

$$\sum_{r=1}^{\infty} r^{-s} = \frac{1}{s-1} + \frac{1}{2} + \sum_{k=2}^{n} B_k/k! \cdot (-1)^k s(s+1)\cdots(s+k-2)$$
$$- s(s+1)\cdots(s+n-1)/n! \cdot \int_1^{\infty} \overline{B_n}(x) x^{-s-n} dx$$

Since $\overline{B_n}(x) = \overline{B_n}(x+1)$, the function is bounded by some constant. Therefore, $\int_1^{\infty} \overline{B_n}(x) x^{-s-n} dx$ converges for $\mathrm{Re}(s) > 1-n$. Therefore, this function is an analytic continuation of the zeta function for $\mathrm{Re}(s) > 1-n$. $\square$

4.2.3 Analytic continuation to $\mathbb{C}\backslash\{1\}$

Definition 4.6. Define $\xi : \mathbb{C} \to \mathbb{C}$ as

$$\xi(z) := \frac{1}{2} z(z-1)\pi^{-z/2}\Gamma(z/2)\zeta(z)$$
$$= (z-1)\pi^{-z/2}\Gamma(z/2+1)\zeta(z)$$

Theorem 4.10. *The function ξ has an analytic continuation to $\mathbb{C}$. To compute this continuation, we use that*

$$\xi(z) = \xi(1-z)$$

A proof of this theorem can be found in the Lecture notes of Evertse[4]

Corollary 4.10.1. *The zeta function has an analytic continuation to $\mathbb{C}\backslash\{1\}$ with a simple pole at $z = 1$ with residue 1. To compute the continuation, we have the following:*

$$\zeta(1-z) = 2^{1-z}\pi^{-z}\cos(\frac{1}{2}\pi z)\Gamma(z)\zeta(z) \tag{20}$$

Proof. By rewriting the definition of the ξ-function, we obtain

$$\zeta(z) = \frac{\xi(z)}{(z-1)\pi^{-z/2}\Gamma(z/2+1)}$$
$$= \frac{\xi(z)\pi^{z/2}/\Gamma(z/2+1)}{(z-1)}$$

Since the Gamma function does not have zeros, $1/\Gamma$ is analytic in the whole complex plane (since Γ is analytic). Also ξ, $\pi^{z/2}$ and $(z-1)$ are analytic on the whole complex plane. $\xi(1) \neq 0$, since $\lim_{z\to 1}(z-1)\zeta(z) \neq 0$ and is defined, and all other terms of ξ are nonzero at $z = 1$. Therefore $\zeta(z)$ is analytic on the whole complex plane, except for the simple pole at $z = 1$ (because then $z - 1$ becomes 0).

Expressing ζ in terms of ξ, one obtains

$$\begin{aligned}
\zeta(1-z) &= \frac{\xi(1-z)\pi^{(1-z)/2}/\Gamma((1-z)/2+1)}{((1-z)-1)} \\
&= \frac{\xi(z)\pi^{(1-z)/2}/(((1-z)/2)\Gamma((1-z)/2))}{-z} \\
&= \frac{\xi(z)}{-z((1-z)/2)\pi^{-(1-z)/2}\Gamma((1-z)/2)} \\
&= \frac{(z-1)\pi^{-z/2}\Gamma(z/2+1)\zeta(z)}{-z((1-z)/2)\pi^{-(1-z)/2}\Gamma((1-z)/2)} \\
&= F(z)\zeta(z)
\end{aligned}$$

where $F(z)$ is defined as

$$F(z) = \frac{(z-1)\pi^{-z/2}\Gamma(z/2+1)}{-z((1-z)/2)\pi^{-(1-z)/2}\Gamma((1-z)/2)}.$$

Before we are going to rewrite this function, first note that

$$\begin{aligned}
\Gamma(z)\Gamma(1-z) &= \frac{\pi}{\sin(\pi z)} \Rightarrow \text{(replacing } z \text{ by } \frac{1}{2}+\frac{1}{2}z) \\
\Gamma(\frac{1}{2}-\frac{1}{2}z)\Gamma(\frac{1}{2}+\frac{1}{2}z) &= \frac{\pi}{\sin(\pi(\frac{1}{2}+\frac{1}{2}z))}
\end{aligned}$$

Also note that the "duplication formula" for the Gamma function states that for $z \notin \{\frac{-n}{2} : n \in \mathbb{N} \cup \{0\}\}$,

$$\sqrt{\pi} \cdot \Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$$

A proof of this duplication formula is given in the lecture notes of Evertse [6]

Rewriting $F(z)$ gives us

$$\begin{aligned}
F(z) &= \frac{(z-1)\pi^{-z/2}\Gamma(z/2+1)}{-z((1-z)/2)\pi^{-(1-z)/2}\Gamma((1-z)/2)} \\
&= \frac{(z-1)\pi^{-z/2}\Gamma(z/2+1)}{(z/2)(z-1)\pi^{-(1-z)/2}\Gamma((1-z)/2)} \\
&= \pi^{1/2-z}\frac{\Gamma(z/2+1)/(z/2)}{\Gamma(1/2-z/2)} \\
&= \pi^{1/2-z}\frac{\Gamma(z/2)}{\Gamma(1/2-z/2)} \\
&= \pi^{1/2-z}\frac{\Gamma(\frac{1}{2}z)\Gamma(\frac{1}{2}+\frac{1}{2}z)}{\Gamma(\frac{1}{2}-\frac{1}{2}z)\Gamma(\frac{1}{2}+\frac{1}{2}z)} \\
&= \pi^{1/2-z}\frac{\Gamma(\frac{1}{2}z)\Gamma(\frac{1}{2}+\frac{1}{2}z)}{\Gamma(\frac{1}{2}-\frac{1}{2}z)\Gamma(\frac{1}{2}+\frac{1}{2}z)} \\
&= \pi^{1/2-z}\frac{\sqrt{\pi}\Gamma(z)2^{1-z}}{\pi/\sin(\frac{1}{2}\pi(1+z))} \\
&= \pi^{-z}\Gamma(z)2^{1-z}\sin(\frac{1}{2}\pi(1+z)) \\
&= 2^{1-z}\pi^{-z}\cos(\frac{1}{2}\pi z)\Gamma(z)
\end{aligned}$$

and hence we have the formula we want. ☐

4.3 Zeros of the Function

4.3.1 Zeros with real part $\mathbf{Re}(z) > 1$

Lemma 4.11. *The zeta function does not have zeros with real part $Re(z) > 1$*

Proof. Let $z = a + bi$ be a complex number with $a, \in \mathbb{R}$ and $\mathrm{Re}(z) = a > 1$. We will use the following definition of the zeta function:

$$\zeta(z) = \prod_p \frac{1}{1-p^{-z}}.$$

First of all, note that $0 \leq |p^{-z}| = p^{-a} < 1/p$, and therefore $1 - p^{-z} \neq 0$.

Let us take a positive integer $N > 1$. We will rewrite the zeta function as a product of two functions,

$$\zeta(z) = \prod_{p \leq N} \frac{1}{1-p^{-z}} \cdot \prod_{p \geq N+1} \frac{1}{1-p^{-z}}. \tag{21}$$

The first product is a finite product with terms unequal to 0, which implies that this product is unequal to zero.

The second product can be rewritten as a sum $\sum n^{-z}$, summing over $n = 1$ and all n for which the prime divisors are larger than N.

To show that this last sum is unequal to zero, we will rewrite it as $\sum n^{-z} = 1 + \sum_{n\neq 1} n^{-z}$. In this last sum, only numbers n appear with $n \geq N + 1$. Therefore, we can bound the absolute value of this sum,

$$\left|\sum_{n\neq 1} n^{-z}\right| \leq \left|\sum_{n=N+1}^{\infty} n^{-z}\right| \leq \sum_{n=N+1}^{\infty} \left|n^{-z}\right| = \sum_{n=N+1}^{\infty} n^{-a}.$$

Since $\frac{1}{n^a} < \int_{n-1}^{n}$, this last sum can be bounded by the integral $\int_N^\infty x^{-a}dx$, and therefore we have

$$\sum_{n\geq N+1}^{\infty} n^{-a} < \int_N^\infty x^{-a}dx = \frac{1}{N^{a-1}} \cdot \frac{1}{a-1}.$$

For N large enough, the right hand side is smaller than 1, since $a - 1 > 0$.

Going back to the formula $\sum n^{-z} = 1 + \sum_{n\neq 1} n^{-z}$, we want to show that this is bigger than 0. For the right hand side we have

$$1 + \sum_{n\neq 1} n^{-z} \geq 1 - \left|\sum_{n\neq 1} n^{-z}\right| \geq 1 - \sum_{n=N+1}^{\infty} n^{-a} > 1 - 1 = 0,$$

the left hand side needs to be bigger than 0.

To conclude, we will go back to equation (21). Both products on the right hand side are unequal to 0 and therefore, also the zeta function has to be unequal to 0. □

4.3.2 Zero's with Re$(z) < 0$

Lemma 4.12. *The only zero's with Re$(z) < 0$ of the zeta function are those with $z = -2n, n \in \mathbb{N}$*

Proof. Equation (20) shows that $\zeta(1-z) = 0 \Leftrightarrow 2^{1-z}\pi^{-z}\cos(\frac{1}{2}\pi z)\Gamma(z)\zeta(z){=}0$. We know that $2^{1-z} \neq 0$ and $\pi^{-z} \neq 0$ for $z \in \mathbb{C}$. Also we know that $\Gamma(z) \neq 0$ for $z \in \mathbb{C}$, proven in Corollary 3.11.2. Therefore, $\zeta(1-z) = 0$ if and only if $\cos(\frac{1}{2}\pi z) = 0$ or $\zeta(z) = 0$.

Suppose $\text{Re}(1-z) < 0$, then $\text{Re}(z) > 1$. Then, because of Lemma 4.11, $\zeta(z) \neq 0$.

Then for $\zeta(1-z)$ to be zero, one must have $\cos(\frac{1}{2}\pi z) = 0$ hence $\frac{1}{2}\pi z = (1/2+k)\pi$ for $k \in \mathbb{Z}_{\geq 0}$. Therefore, $z = \frac{(1/2+k)\pi}{\pi/2} = (1+2k)$ Therefore, $\zeta(1-(1+2k)) = \zeta(-2k)$ can be zero for $k \in \mathbb{N}$.

We only need to check if this is not a removable zero. Note that for $\text{Re}(z) > 1$, the Gamma function does not have a pole. Also all other terms of $\zeta(1-z)$ does not have a pole at $z = 1 + 2k$ for $k \in \mathbb{N}$

The only point we have to check right now is $z = 0$. At $z = 0$, the zeta function does have a simple pole with residue $\text{Res}(0) = 1$. The residue of the cosine function will be calculated to be

$$\text{Res}(z = 1) = \lim_{z\to 1} \cos(\frac{1}{2}\pi z)/(z-1) \stackrel{H}{=} -\frac{1}{2}\pi \sin(\frac{1}{2}\pi)/1 = -\frac{1}{2}\pi.$$

Therefore,

$$\lim_{z\to 1} \zeta(1-z) = \lim_{z\to 1} 2^{1-z}\pi^{-z} \cos(\frac{1}{2}\pi z)\Gamma(z)\zeta(z) = -\frac{1}{2}$$

Hence the only negative zero's of the zeta function are those with $z = -2n$, $n \in \mathbb{N}$. $\square$

4.3.3 Zero's with Re$(z) \in \{0, 1\}$

In this part we are going to look at the case that $\zeta(z) = 0$ for $\text{Re}(z) \in \{0, 1\}$, $z \neq 0, 1$ (these points will be excluded, since we have proven that at $z = 0$, $\zeta(z) = -1/2$ and at $z = 1$ the zeta function has a pole). First we are going to look at $\text{Re}(z) = 1$. Then, by formula (20), we have that $\zeta(1-z) = 0$. Therefore, z is a zero if and only if $1 - z$ is a zero.

Definition 4.7. Let $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Then we define the principal value of the logarithm to be

$$\text{Log}(z) = \ln(|z|) + i\text{Arg}(z)$$

where $\text{Arg}(z) \in (-\pi, \pi)$.

Lemma 4.13. *We can write the principal value of the logarithm of* $\frac{1}{1-z}$, $|z| < 1$ *as an infinite series:*

$$Log(\frac{1}{1-z}) = \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

Proof. First of all, we have that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.$$

Therefore, if we integrate $\frac{1}{1-z}$, we get

$$\begin{aligned} \text{Log}(1-z) \cdot -1 &= \sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} z^n. \end{aligned}$$

Note that $\text{Log}(1-z) \cdot -1$ can be rewritten as

$$\text{Log}(1-z) \cdot -1 = \text{Log}((1-z)^{-1}) = \text{Log}(\frac{1}{1-z}).$$

Therefore, we have that

$$\text{Log}(\frac{1}{1-z}) = \sum_{n=1}^{\infty} \frac{1}{n} z^n.$$

□

Definition 4.8. Define the function $ord_{z=z_0}$ as a function mapping a meromorphic (around z_0) function f to the integers as follows:

$$ord_{z=z_0}(f) = n \quad \text{if } f \text{ does have a zero of order } n \text{ at } z_0$$

$$ord_{z=z_0}(f) = -n \quad \text{if } f \text{ does have a pole of order } n \text{ at } z_0$$

i.e., the limit from $z \to z_0$ of $(z-z_0)^{\text{ord}} \cdot f(z)$ is a complex number unequal to 0.

Lemma 4.14. *Let f and g be meromorphic functions around z_0. Then*

$$ord_{z=z_0}(f \cdot g) = ord_{z=z_0}(f) + ord_{z=z_0}(g).$$

Proof. Since f and g are meromorphic, we have that the functions

$$(z-z_0)^m f(z) \text{ and } (z-z_0)^n g(z)$$

are holomorphic. Let $m = ord(f)$ and $n = ord(g)$. Then by definition of a zero of order n (or a pole of order n), these functions are indeed holomorphic. Therefore, also

$$(z-z_0)^m f(z) \cdot (z-z_0)^n g(z) = (z-z_0)^{n+m} f(z) g(z)$$

is holomorphic and hence $ord_{z=z_0}(f(z) \cdot g(z)) = n + m$. □

Theorem 4.15. *There are no zeros of the zeta function with $Re(z) = 1$.*

Proof. Let us consider $z = 1 + it$, $t \in \mathbb{R}$. Then we have to consider 2 cases.

1. If $t = 0$, $z = 1$. At this point, the zeta function does have a pole, so there is no zero at this point.
2. The remaining case is $t \neq 0$. This we are going to discuss now.

Let us fix a real number $t \neq 0$. Suppose, for a contradiction, that $\zeta(1+it) = 0$. To analyze this, we define a new function:

$$F(z) := (\zeta(z))^3 \cdot (\zeta(z+it))^4 \cdot \zeta(z+2it)$$

$\zeta(z)$ has a simple pole at $z = 1$ and $\zeta(z + it)$ has a zero at $z = 1$. Therefore,

$$\begin{aligned} ord_{z=1}F(z) &= 3 \cdot ord_{z=1}\zeta(z) + 4 \cdot ord_{z=1}\zeta(z+it) + ord_{z=1}\zeta(z+2it) \\ &\geq 3 \cdot (-1) + 4 \cdot 1 + 0 \\ &= 1 \end{aligned}$$

Therefore, $F(z)$ has a zero at $z = 1$.

Now we are going to prove that $|F(\sigma)| > 1$ for $\sigma > 1$. Because of the continuity of the function, we need to have that

$$\lim_{\sigma \to 1^+} F(\sigma) = F(1) = 0$$

which contradicts the fact that $|F(\sigma)| > 1$ voor $\sigma > 1$. Therefore, our assumption that $\zeta(1 + it) = 0$ needs to be false.

To prove that $|F(\sigma)| > 1$, we are going to prove that $\mathrm{Log}|F(\sigma)| > 0$, which implies that $|F(\sigma)| > 1$.

For p prime, we have the following:

$$\begin{aligned} \ln|F(\sigma)| &= \ln \prod_p \left(\left| \frac{1}{1-p^{-\sigma}} \right|^3 \cdot \left| \frac{1}{1-p^{-\sigma-it}} \right|^4 \cdot \left| \frac{1}{1-p^{-\sigma-2it}} \right| \right) \\ &= \sum_p \left(3\ln \left| \frac{1}{1-p^{-\sigma}} \right| + 4\ln \left| \frac{1}{1-p^{-\sigma-it}} \right| + \ln \left| \frac{1}{1-p^{-\sigma-2it}} \right| \right) \end{aligned}$$

We also have that we can write every complex number z as $z = re^{i\theta}$ with $r \geq 0$ and $r, \theta \in \mathbb{R}$.

Because of Lemma 2.3 and Lemma 4.13 we have, for $r < 1$

$$\begin{aligned} \ln \left| \frac{1}{1-re^{i\theta}} \right| &= Re \left(\mathrm{Log} \frac{1}{1-re^{i\theta}} \right) \\ &= Re \left(\sum_{n=1}^{\infty} \frac{(re^{i\theta})^n}{n} \right) \\ &= \sum_{n=1}^{\infty} \frac{r^n}{n} Re(e^{in\theta}) \\ &= \sum_{n=1}^{\infty} \frac{r^n}{n} \cdot \cos(n\theta) \end{aligned}$$

Since we can write $p^{-\sigma-kit}$ as $p^{-\sigma-kit} = p^{-\sigma} \cdot e^{-ikt\ln(p)}$

$$\begin{aligned}
\ln|F(\sigma)| &= \sum_p \left(3\ln\left|\frac{1}{1-p^{-\sigma}}\right| + 4\ln\left|\frac{1}{1-p^{-\sigma-it}}\right| + \ln\left|\frac{1}{1-p^{-\sigma-2it}}\right|\right) \\
&= \sum_p \left(3\sum_{n=1}^{\infty}\frac{p^{-n\sigma}}{n} + 4\sum_{n=1}^{\infty}\frac{p^{-n\sigma}}{n}\cdot\cos(n\theta) + \sum_{n=1}^{\infty}\frac{p^{-n\sigma}}{n}\cdot\cos(2n\theta)\right) \\
&= \sum_p \sum_{n=1}^{\infty}\frac{p^{-n\sigma}}{n}(3 + 4\cos(n\theta) + \cos(2n\theta)) \\
&= \sum_p \sum_{n=1}^{\infty}\frac{p^{-n\sigma}}{n}\cdot 2(1+\cos(n\theta))^2 \\
&\geq 0
\end{aligned}$$

where $\theta = t\log(p)$. The last inequality comes from the fact that $\frac{p^{-n\sigma}}{n} > 0$ and that $2(1+\cos(n\theta))^2 \geq 0$. Also note that $p^{-\sigma} < 1$ for $\sigma > 1$. Hence $\log|F(\sigma)| \geq 0$ and therefore $|F(\sigma)| \geq 1$. Now we have the contradiction we wanted, and therefore we have proven that there are no zero's on the line $\mathrm{Re}(z) = 1$. □

4.3.4 Zeros in $[0, 1]$

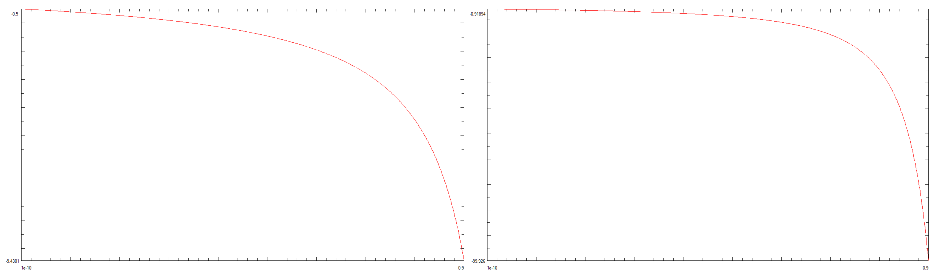
Now we are going to prove that there are no zero's at the real axis from 0 to 1. To do this, we will use the rewritten form of the zeta function introduced in Lemma 4.4, which is holomorphic for $\mathrm{Re}(z) > 0$. Here we will prove that this function is monotonically decreasing.

Theorem 4.16. *The zeta function does not have any zeros at the real line from 0 to 1.*

Proof. First of all, $\zeta(0) = -\frac{1}{2}$. We will show that the function is monotonically decreasing, and therefore there are no zeros in $(0, 1)$.

This will be done by showing two plots: one of the zeta function itself and one of the derivative of the zeta function. In the first plot, we can already see that the zeta function is monotonically decreasing, but to give still more evidence, we can see in the second plot that the derivative of the zeta function also decreases monotonically. This implies that the zeta function is not only monotonically decreasing, but also concave down.

Here are the plots of the zeta function (on the left) and its derivative (on the right) on the line $[0, 0.9]$ (between 0.9 and 1 the plots are decreasing very fast):



In these plots it can be seen that the zeta function and the derivative are maximal at 0 in this interval, where they have the values $\zeta(0) = -1/2$ and $\zeta'(0) \approx -0.92$. $\square$

4.3.5 Zeros with $0 < \mathbf{Re}(z) < 1$

According to the Riemann hypothesis, the only zeros in this region are those with real part equal to $\frac{1}{2}$. Until now, no one has proven this hypothesis. Therefore, we will not attempt to provide a proof for this. Instead, we will give numerical evidence to make the hypothesis plausible. This we will do in the next section.

4.4 Summary

In this section we have proven different theorems about the zeta function. In the introduction of this section we were looking for the following answers:

- For which $z \in \mathbb{C}$ is the zeta function defined?
- How can this function be extended to the whole complex plane?
- What are the poles of the function?
- What are the zeros of the function?

As seen in the previous subsections, we have answered all questions, except the last one:

- The zeta function can in particular be defined for $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$
- The function can be extended to $\mathrm{Re}(z) > 1 - n$ using equation (19) or, extended to the whole complex plane, using equation (20).
- The function has only one simple pole, namely at $z = 1$
- The function has zeros only at the negative integers and inside the strip with $0 < \mathrm{Re}(z) < 1$. According to the Riemann Hypothesis, these zeros should satisfy $\mathrm{Re}(z) = 1/2$.

5 Zeros of the Riemann zeta function inside the complex strip $0 < \mathrm{Re}(z) < 1$

In this section we will use a numerical method to make it plausible that the zeros of the zeta function are located at the line $1/2 + it$.

5.1 The function $\zeta'(z)/\zeta(z)$

In this subsection we are going to verify that the zeros and poles of the zeta function are poles of $\zeta'(z)/\zeta(z)$. Furthermore, since supercomputers were used to prove that the first 10 trillion zeros are simple [13], the zeros which will be used in this thesis are simple. This implies that the poles of the function $\zeta'(z)/\zeta(z)$ have residue 1 if the zeta function does have a zero.

Lemma 5.1. $\zeta(z) \neq \infty \Rightarrow \zeta'(z) \neq \infty$

Proof. This is just a consequence of the fact that the zeta function is analytic. □

Theorem 5.2. *Consider the function $\zeta'(z_0)/\zeta(z_0)$. This function has simple poles at the zeros of the zeta function with residue $ord_{z=z_0}\zeta(z)$ and it does have a simple pole at $z = 1$ with residue -1. Furthermore, no other poles appear.*

Proof. This proof will consist of three parts. First of all, we will prove that the function does have simple poles at the zeros of the zeta function with residue $\mathrm{ord}_{z=z_0}\zeta(z)$. Secondly, we will prove that the function does have a simple pole at $z = 1$ with residue -1. Thirdly, we will prove that no other poles will appear.

First of all, since the zeta function is analytic, it can be written as a Laurent series,

$$\zeta(z) = \sum_{n=-\infty}^{\infty} A_n(z - z_0)^n$$

Since the zeta function does not have a pole at $z_0 \neq 1$, we know that $A_n = 0$ for $n < 0$. If $\zeta(z_0) = 0$, also $A_0 = 0$, and therefore the zeta function around z_0 can be written as

$$\zeta(z) = \sum_{n=\mathrm{ord}_{z=z_0}\zeta(z)}^{\infty} A_n(z - z_0)^n$$

Let $\mathrm{ord}_{z=z_0}\zeta(z) = m$. Then, for $0 < k \leq m$ we have

$$\zeta^{(k)}(z) = \sum_{n=m}^{\infty} n(n-1)\cdots(n-k+1) \cdot A_n(z - z_0)^{n-k}$$

Evaluating this at z_0, this is equal to 0, except for $k = m$, where we get $\zeta^{(m)}(z_0) = m!A_m$. Therefore, $\zeta'(z)/\zeta(z)$ equals, applying l'hopitals rule $m - 1$ times, $\zeta'(z)/\zeta(z) = m!A_m/0 \to \infty$

The residue will be calculated as follows:

$$\begin{aligned}
Res(z_0) &= \lim_{z\to z_0}(z-z_0)\cdot\frac{\zeta'(z)}{\zeta(z)}\\
&\stackrel{H}{=} \lim_{z\to z_0}\frac{\zeta^{(m)}(z)}{(z-z_0)^{-1}\cdot\zeta^{(m-1)}(z)}\\
&= \frac{m!A_m}{\lim_{z\to z_0}\sum_{n=m}^{\infty}(m-1)!A_n(z-z_0)^{n-m}}\\
&= \frac{m!A_m}{(m-1)!A_m+\sum_{n=m+1}^{\infty}(m-1)!A_n(z_0-z_0)^{n-m}}\\
&= \frac{m!A_m}{(m-1)!A_m}\\
&= m = \mathrm{ord}_{z=z_0}\zeta(z).
\end{aligned}$$

Now we have proven the first part of the theorem.

For the second part, we will again use the Laurent series. Since the zeta function does have a simple pole at $z_0 = 1$, we will write it as

$$\zeta(z) = \sum_{n=-1}^{\infty} A_n(z-z_0)^n$$

Therefore, the derivative becomes

$$\zeta'(z) = \sum_{n=-1}^{\infty} nA_n(z-z_0)^{n-1} = \sum_{n=-2}^{\infty}(n+1)A_{n+1}(z-z_0)^n$$

where the last equation does have a pole of order 2. Therefore, $\zeta'(z)/\zeta(z) \to \infty$

The residue is therefore:

$$
\begin{aligned}
Res(1) &= \lim_{z\to 1}(z-1)\cdot\frac{\zeta'(z)}{\zeta(z)}\\
&= \lim_{z\to 1}(z-1)\cdot\frac{\sum_{n=-2}^{\infty}(n+1)A_{n+1}(z-1)^n}{\sum_{n=-1}^{\infty}A_n(z-1)^n}\\
&= \lim_{z\to 1}\frac{(z-1)^2}{(z-1)}\cdot\frac{\sum_{n=-2}^{\infty}(n+1)A_{n+1}(z-1)^n}{\sum_{n=-1}^{\infty}A_n(z-1)^n}\\
&= \lim_{z\to 1}\frac{\sum_{n=-2}^{\infty}(n+1)A_{n+1}(z-1)^{n+2}}{\sum_{n=-1}^{\infty}A_n(z-1)^{n+1}}\\
&= \lim_{z\to 1}\frac{\sum_{n=0}^{\infty}(n-1)A_{n-1}(z-1)^n}{\sum_{n=0}^{\infty}A_{n-1}(z-1)^n}\\
&= \frac{(-1)A_{-1}+\sum_{n=1}^{\infty}(n-1)A_{n-1}\cdot 0^n}{A_{-1}+\sum_{n=1}^{\infty}A_{n-1}\cdot 0^n}\\
&= \frac{(-1)A_{-1}}{A_{-1}}\\
&= -1.
\end{aligned}
$$

Now we have proven the second part of the theorem. For the third part, we use lemma 5.1 to note that, when the zeta function does not have a pole or a zero at z_0, then also the derivative does not have a pole there. Therefore, no other poles appear than those described above. □

As noted in the start of this subsection, the zeros which we will use in this project are simple. Therefore, the poles of the function $\zeta'(z)/\zeta(z)$ have residue 1 (if the zeta function has a zero there) and -1 (at $z_0 = 1$, where the zeta function has a pole).

5.2 The contour integral

In this subsection we are going to construct a contour integral. In order to do this, we will integrate the function $\zeta'(z)/\zeta(z)$ along the following contour: we will construct a rectangle with corners $2+Ai$, $2-Ai$, $-1-Ai$ and $-1+Ai$, where $A \in \mathbb{R}_{>0}$. This contour is chosen, since the upper part has many things in common with the lower part: the integral from 2 to $2+Ai$ is the complex conjugate of the integral from 2 to $2-Ai$. This will be proven in this section. To prove that $\overline{\zeta(s)} = \zeta(\overline{s})$, we first prove that it holds for $\mathrm{Re}(s) > 0$. After that, we will prove that it also holds for $\zeta(1-s)$ with $\mathrm{Re}(s) > 0$. To prove this, we also prove that the Gamma function has this property.

When this is proven, it will follow that $\overline{\zeta'(s)/\zeta(s)} = \zeta'(\overline{s})/\zeta(\overline{s})$. As a consequence, we only have to compute the upper part of the contour integral (with $\mathrm{Im}(s) \geq 0$) and add the conjugate of this to it, to get the value of the total contour integral.

Lemma 5.3. *$\overline{\zeta(z)} = \zeta(\overline{z})$ for $Re(z) > 0$*

Proof. To prove this lemma, we are going to use equation (10):

$$\zeta(z) = \frac{1}{z-1} + \sum_{n=1}^{\infty} \int_{n}^{n+1} \left(\frac{1}{n^z} - \frac{1}{t^z} \right) dt$$

Therefore,

$$\begin{aligned}
\overline{\zeta(z)} &= \overline{\frac{1}{z-1} + \sum_{n=1}^{\infty} \int_{n}^{n+1} \left(\frac{1}{n^z} - \frac{1}{t^z} \right) dt} \\
&= \overline{\frac{1}{z-1}} + \sum_{n=1}^{\infty} \overline{\int_{n}^{n+1} \left(\frac{1}{n^z} - \frac{1}{t^z} \right) dt} \\
&= \frac{1}{\overline{z}-1} + \sum_{n=1}^{\infty} \int_{n}^{n+1} \overline{\left(\frac{1}{n^z} - \frac{1}{t^z} \right)} dt \\
&= \frac{1}{\overline{z}-1} + \sum_{n=1}^{\infty} \int_{n}^{n+1} \left(\frac{1}{n^{\overline{z}}} - \frac{1}{t^{\overline{z}}} \right) dt \\
&= \zeta(\overline{z})
\end{aligned}$$

Here we used that $\overline{u+w} = \overline{u} + \overline{w}$ and $\overline{n^z} = n^{\overline{z}}$. ∎

Lemma 5.4. $\overline{\Gamma(z)} = \Gamma(\overline{z})$

Proof. Recall that the Gamma function is defined as

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

Let us rewrite this, using that $z = (a+1) + bi$ for $a, b \in \mathbb{R}$.

$$\begin{aligned}
\Gamma(z) &= \int_0^\infty x^{z-1} e^{-x} dx \\
&= \int_0^\infty e^{(z-1)\log(x)} e^{-x} dx \\
&= \int_0^\infty e^{(a+bi)\log(x) - x} dx \\
&= \int_0^\infty e^{a\log(x) - x} e^{ib\log(x)} dx \\
&= \int_0^\infty x^a e^{-x} (\cos(b\log(x)) + i\sin(b\log(x))) dx
\end{aligned}$$

Therefore, we get

$$
\begin{aligned}
\overline{\Gamma(z)} &= \overline{\int_0^\infty x^a e^{-x}(\cos(b\log(x)) + i\sin(b\log(x)))dx} \\
&= \int_0^\infty \overline{x^a e^{-x}(\cos(b\log(x)) + i\sin(b\log(x)))}dx \\
&= \int_0^\infty x^a e^{-x}(\cos(b\log(x)) - i\sin(b\log(x)))dx \\
&= \int_0^\infty x^a e^{-x}(\cos(b\log(x)) + i\sin(b\log(x)) - 2i\sin(b\log(x)))dx \\
&= \int_0^\infty x^a e^{-x}(e^{ib\log(x)} - 2i\cdot\frac{1}{2i}\left(e^{ib\log(x)} - e^{-ib\log(x)}\right))dx \\
&= \int_0^\infty x^a e^{-x}(e^{ib\log(x)} - e^{ib\log(x)} + e^{-ib\log(x)})dx \\
&= \int_0^\infty x^a e^{-x}e^{-ib\log(x)}dx \\
&= \int_0^\infty x^a e^{-x}x^{-ib}dx \\
&= \int_0^\infty x^{a-ib}e^{-x}dx \\
&= \int_0^\infty x^{\overline{z}-1}e^{-x}dx \\
&= \Gamma(\overline{z}).
\end{aligned}
$$

□

Theorem 5.5. $\overline{\zeta(z)} = \zeta(\overline{z})$

Proof. First of all, for $\mathrm{Re}(z) > 0$, we already know that this statement holds. If we now consider the equation $\zeta(1-z) = 2^{1-z}\pi^{-z}\cos(\frac{1}{2}\pi z)\Gamma(z)\zeta(z)$, then $\overline{\zeta(1-z)} = \zeta(1-\overline{z})$ if it holds for each term of the equation for $z \leq 1$.

For $2^{1-z}\pi^{-z}$ we know that $\overline{2^{1-z}\pi^{-z}} = \overline{2^{1-z}}\,\overline{\pi^{-z}} = 2^{1-\overline{z}}\cdot\pi^{-\overline{z}}$.

For $\cos(\frac{1}{2}\pi z)$, we know that, for $z = a + bi$,

$$\begin{aligned}
\overline{\cos(\frac{1}{2}\pi z)} &= \overline{\frac{1}{2}\left(e^{i\frac{1}{2}\pi z} + e^{-i\frac{1}{2}\pi z}\right)} \\
&= \frac{1}{2}\left(\overline{e^{i\frac{1}{2}\pi z}} + \overline{e^{-i\frac{1}{2}\pi z}}\right) \\
&= \frac{1}{2}\left(\overline{e^{(ai-b)\frac{1}{2}\pi}} + \overline{e^{-(ai-b)\frac{1}{2}\pi}}\right) \\
&= \frac{1}{2}\left(e^{\overline{(ai-b)}\frac{1}{2}\pi} + e^{\overline{-(ai-b)}\frac{1}{2}\pi}\right) \\
&= \frac{1}{2}\left(e^{-(ai+b)\frac{1}{2}\pi} + e^{(ai+b)\frac{1}{2}\pi}\right) \\
&= \frac{1}{2}\left(e^{-i(a-bi)\frac{1}{2}\pi} + e^{i(a-bi)\frac{1}{2}\pi}\right) \\
&= \frac{1}{2}\left(e^{-i\overline{z}\frac{1}{2}\pi} + e^{i\overline{z}\frac{1}{2}\pi}\right) \\
&= \cos(\frac{1}{2}\pi\overline{z})
\end{aligned}$$

For zeta and Gamma, this is proven in the previous lemma's. Therefore, we have that, for $z \leq 1$, $\overline{\zeta(1-z)} = \overline{2^{1-z}\pi^{-z}\cos(\frac{1}{2}\pi z)\Gamma(z)\zeta(z)} = \zeta(1-\overline{z})$. Therefore, we have that $\overline{\zeta(z)} = \zeta(\overline{z})$. □

Now we know that $\overline{\zeta(z)} = \zeta(\overline{z})$. Therefore, if $\zeta(z_0) = 0$, also $\zeta(\overline{z_0}) = 0$. Combining this with the fact that if $\zeta(z) = 0$, then also $\zeta(1-z) = 0$, gives a nice property: suppose that there exists a zero which is not on the critical line and which is not real. Then there need to be at least 3 additional zeros.

The contour is chosen such that if z_0 lies in the contour, then also $\overline{z_0}$, $\overline{1-z_0}$ and $1-z_0$ lie in the contour. Therefore, when evaluating the contour, we need to observe a multiple of 4 as the contribution of zeros inside the contour that are not on the critical line.

If there is a zero on the critical line with $\frac{1}{2} + it$, then also $\frac{1}{2} - it$ is a zero. Therefore, when evaluating the contour, we need to observe a multiple of 2 as the contribution of zeros inside the contour at the critical line.

Right now we know that $\overline{\zeta(z)} = \zeta(\overline{z})$. Therefore, also $\overline{\zeta'(z)} = \zeta'(\overline{z})$. Hence, if we have $Z(z) = \zeta'(z)/\zeta(z)$, it follows that $\overline{Z(z)} = \overline{\zeta'(z)/\zeta(z)} = \zeta'(\overline{z})/\zeta(\overline{z}) = Z(\overline{z})$.

Let us now consider different paths on the boundary of our contour integral. Let $\gamma_1(t) = -1 + Ait$, $\gamma_2(t) = 2 + Ait$ and $\gamma_3(t) = (-1+3t) + Ai$. Therefore, we have the following formulas:

$$\begin{array}{llll}
\gamma_1(t) = -1 + Ait & \gamma_1'(t) = Ai & \overline{\gamma_1(t)} = -1 - Ait & \overline{\gamma_1'(t)} = -Ai \\
\gamma_2(t) = 2 + Ait & \gamma_2'(t) = Ai & \overline{\gamma_2(t)} = 2 - Ait & \overline{\gamma_2'(t)} = -Ai \\
\gamma_3(t) = (-1+3t) + Ai & \gamma_3'(t) = 3 & \overline{\gamma_3(t)} = (-1+3t) - Ai & \overline{\gamma_3'(t)} = 3
\end{array}$$

Consider the following 3 integrals:

$$C_1 = \int_{-1}^{-1+iA} Z(z)dz, \quad C_2 = \int_{2}^{2+iA} Z(z)dz, \quad C_3 = \int_{-1+Ai}^{2+iA} Z(z)dz.$$

These integrals can be rewritten on the following way:

$$C_i = \int_{\gamma_i(0)}^{\gamma_i(1)} Z(z)dz = \int_0^1 Z(\gamma_i(t))\gamma_i'(t)dt.$$

Also consider the following integrals

$$D_1 = \int_{-1}^{-1-iA} Z(z)dz, \quad D_2 = \int_{2}^{2-iA} Z(z)dz, \quad D_3 = \int_{-1-Ai}^{2-iA} Z(z)dz.$$

These integrals can be rewritten on the following way:

$$D_i = \int_{\overline{\gamma_i(0)}}^{\overline{\gamma_i(1)}} Z(z)dz = \int_0^1 Z\left(\overline{\gamma_i(t)}\right)\overline{\gamma_i'(t)}dt.$$

Here we used the fact that $\frac{d}{dt}\overline{\gamma(t)} = \overline{\gamma'(t)}$. This last integral can be rewritten, using the fact that $\overline{Z(z)} = Z(\overline{z})$, as

$$D_i = \int_0^1 \overline{Z(\gamma_i(t))} \cdot \overline{\gamma_i'(t)}dt = \int_0^1 \overline{Z(\gamma_i(t))\gamma_i'(t)}dt = \overline{\int_0^1 Z(\gamma_i(t))\gamma_i'(t)dt} = \overline{C_i}$$

Hence we have

$$\int_{-1}^{-1+iA} Z(z)dz = \overline{\int_{-1}^{-1-iA} Z(z)dz},$$

$$\int_{2}^{2+iA} Z(z)dz = \overline{\int_{2}^{2-iA} Z(z)dz},$$

$$\int_{-1+iA}^{2+iA} Z(z)dz = \overline{\int_{-1-iA}^{2-iA} Z(z)dz}.$$

5.3 The zeros on the line $1/2 + it$

There are already many zeros of the zeta function known. In 2005 already the first 10^{13} zeros were calculated by Gourdon[21]. But for this thesis we will use the zeros which are calculated in the thesis of Van der Meer[12]. In his thesis he calculated the first 29 zeros (all zeros below 100). Below you can find these zeros:

14.13472514	52.97032148	79.33737502
21.02203964	56.44624770	82.91038085
25.01085758	59.34704400	84.73549298
30.42487613	60.83177852	87.4252746
32.93506159	65.11254405	88.80911121
37.58617816	67.07981053	92.49189927
40.91871901	69.54640171	94.65134404
43.32707328	72.06715767	95.87063423
48.00515088	75.70469070	98.83119422
49.77383248	77.14484007	

5.4 Testing the Riemann Hypothesis

For testing the Riemann Hypothesis, we will make use of PARI/GP. In this subsection we first will introduce the Simpson's rule. After that, we will apply this to the logarithmic derivative of the zeta function. This will be done in 2 steps: first we will make use of the built-in functions and after that we will derive own functions instead of the built-in functions. For these tests, we take the contour integral as described in Subsection 5.2.

5.4.1 Simpson's Rule

The Simpson's rule works with lagrange interpolation. In an interval $[a, b]$, the function $f(z)$ which needs to be integrated will be approached by a polynomial of at most degree 2. This will be done using the following formula:

$$p(z) = f(a)\frac{(z-m)(z-b)}{(a-m)(a-b)} + f(m)\frac{(z-a)(z-b)}{(m-a)(m-b)} + f(b)\frac{(z-a)(z-m)}{(b-a)(b-m)} \quad (22)$$

where $m = \frac{a+b}{2}$ is the midpoint of the interval $[a, b]$.

Integrating this polynomial gives us the following formula:

$$I_s = \int_a^b p(z)dz = \frac{b-a}{6}\left[f(a) + 4f(m) + f(b)\right] \quad (23)$$

Theorem 5.6. *The error when approximating $f(z)$ by $p(z)$ is bounded by the following:*

$$E \leq \left|f^{(4)}(c)\right| \cdot \frac{h^5}{2880}$$

for some $c \in [a, b]$ and with $h = b - a$ is the width of the interval.

Proof. When approximating $f(z)$ by $p(z)$, the error E is, using that I is the exact value of the integral and I_s as described above,

$$
\begin{aligned}
E_i &= |I - I_s| \\
&= \left| \int_a^b f(z)dz - \int_a^b p(z)dz \right| \\
&= \left| \int_a^b [f(z) - p(z)]\, dz \right|.
\end{aligned}
$$

$f(z) - p(z)$ can be calculated to be

$$f(z) - p(z) = \frac{f^{(4)}(\tilde{c})}{24}(z-a)(z-m)^2(z-b).$$

for some $\tilde{c}$ in the interval $[a, b]$. Let c be such that $f^{(4)}(c)$ is the maximum of the fourth derivative in the interval $[a, b]$. Then

$$f(z) - p(z) \leq \frac{f^{(4)}(c)}{24}(z-a)(z-m)^2(z-b).$$

Substituting this in the error formula gives

$$
\begin{aligned}
E_i &= \left| \int_a^b [f(z) - p(z)]\, dz \right| \\
&\leq \left| \int_a^b \left[\frac{f^{(4)}(c)}{24}(z-a)(z-m)^2(z-b) \right] dz \right| \\
&= \left| \frac{f^{(4)}(c)}{24} \int_a^b \left[(z-a)(z-m)^2(z-b) \right] dz \right| \\
&= \left| \frac{f^{(4)}(c)}{24} \cdot \left(-\frac{(b-a)^5}{120} \right) \right| \\
&= \left| f^{(4)}(c) \right| \cdot \frac{(b-a)^5}{2880} \\
&= \left| f^{(4)}(c) \right| \cdot \frac{h^5}{2880}
\end{aligned}
$$

where h is the length of the interval $[a, b]$. □

Let us now consider an interval $[A, B]$ which is devided in n intervals with length h. When evaluating these intervals as described above, the error of the i'th interval, E_i, will be less than or equal to $\left| f^{(4)}(c) \right| \cdot \frac{h^5}{2880}$ as seen in the theorem above. The total error of this approximation will also be bounded, like the following theorem states:

Theorem 5.7. *If we consider the interval $[A, B]$, the approximation as described above does have the following error bound:*

$$E_{total} \leq \frac{h^4}{2880} \cdot (B - A)M,$$

where $\max_{z\in[A,B]} |f^{(4)}(z)| \leq M$.

Proof. Since $\max_{z\in[A,B]} |f^{(4)}(x)| \leq M$, $\max_{x\in[a,b]} |f^{(4)}(x)| \leq M$. Therefore, for each error E_i we have

$$E_i \leq \frac{h^5}{2880} \cdot M.$$

Note that the amount of subintervals n equals $(B-A)/(b-a) = (B-A)/h$. Hence,

$$E_{total} = \sum_{i=1}^{n} E_i \leq n\frac{h^5}{2880} \cdot M = \frac{h^4}{2880} \cdot (B-A)M$$

□

5.4.2 Using built-in functions

In this subsection, we will calculate the contour integral using built-in functions. After that, we will do an error analysis.

Calculation of the contour integral Before we are going to this functions, we first need to make a function which applies the Simpson's rule with the following variables:
Input variables:

- a: the starting point of the integral;
- b: the endpoint of the integral;
- N: the number of subintervals in which the function will be integrated;
- f: the function which will be integrated.

Output variable:

- Area: the value of the integral.

The code is as follows:

```
Simpson(a,b,N,f)={
    h=(b-a)/N;Area=0;ye=f(a);
    for(n=0,N-1,yb=ye;ym=f((n+1/2)*h+a);
    ye=f((n+1)*h+a);Area+=(yb+4*ym+ye)*h/6)
}
```

Now we have a function for the Simpson's rule. What needs to be done is to define the function f. This will be done using the built-in functions for the zeta function and the derivative of it. Combining them gives us the following function:

```
Z(z)=lfun(1,z,1)/zeta(z);
```

To make a function which calculates the amount of zeros in the contour integral, we first have to take a look at the value of the contour integral. The value of the contour integral is the sum of all residues multiplied by $2\pi i$,

$$\oint_C \frac{\zeta'(z)}{\zeta(z)}dz = 2\pi i \cdot \sum Res(z) \tag{24}$$

If we sum up the integrals, we need to divide by $2\pi i$ to obtain the sum of the residues. After that, we have to add 1, because of the simple pole at $z = 1$ (which has residue -1). Dividing by 2 gives the number of zeros with $\mathrm{Im}(z) > 0$ inside the contour, counted with multiplicity.

These steps are done in the following code. Here we have the following variables:
Input variables:

- A: the value as described in subsection 5.2;
- N: the amount of steps used in the Simpson's rule.

Output variable:

- Amountzero: the amount of zeros in the contour integral with $\mathrm{Im}(s) > 0$.

```
Total(A,N)={
    Simpson(2,2+A*I,N,Z); Area1=Area;
    Simpson(2+A*I,-1+A*I,N,Z); Area2=Area;
    Simpson(-1+A*I,-1,N,Z); Area3=Area;
    Totarea=Area1+Area2+Area3-conj(Area3+Area2+Area1);
    Amountzero=(Totarea/(2*Pi*I)+1)/2;
}
```

When trying Total(100,1000), we obtain approximate 29, which is the answer we had to obtain, with an accuracy of about 10^{-9}, so really accurate.

The exact output is

```
29.00000000141727236 27... - 2.3385717063745702440 E-40*I
```

Error analysis Since we are working numerically, we have to check what the error estimate is. This will be done by doing an error analysis of the Simpson's rule. An error estimate about the built-in functions is not possible.

The error function for the Simpson's rule is

$$E_{total} \leq \frac{h^4}{2880}|B - A|M.$$

In our case, we have to do three times an error analysis: one of the line from 2 to $2 + 100i$, one of the line from $2 + 100i$ to $-1 + 100i$ and one of the line from $-1 + 100i$ to -1.

The first one and the last one have the same stepwidth $h = 0.1$. The second one has stepwidth $h = 0.003$. The maximum of the fourth derivative of the logarithmic derivative of the zetafunction is smaller than 14 on the contour. A calculation showing this can be found in Appendix A. The width of the interval $|B - A|$ is 100, 3 and 100 resp. Therefore, the maximum errors of these three integrals are

$$\begin{aligned}
E_{I_1} \leq& \frac{0.1^4}{2880} \cdot 100 \cdot 14 = \frac{7}{144000} \approx 4.86 \cdot 10^{-5} \\
E_{I_2} \leq& \frac{0.003^4}{2880} \cdot 100 \cdot 14 \approx 3.94 \cdot 10^{-11} \\
E_{I_3} \leq& \frac{0.1^4}{2880} \cdot 100 \cdot 14 = \frac{7}{144000} \approx 4.86 \cdot 10^{-5}
\end{aligned}$$

The total error of the contour integral therefore becomes $2 \cdot (4.86 \cdot 10^{-5} + 3.94 \cdot 10^{-11} + 4.86 \cdot 10^{-5}) \approx 10^{-4}$. This will be divided by 4π, so that the total error is bounded by $\frac{10^{-4}}{4\pi} \leq 10^{-5}$. Therefore, the outcome is accurate enough.

There is only a problem with this method: using the built-in functions won't allow us to do an error check for the functions. Could it be possible that there are more zero's, but because of an error in one of these functions, we won't obtain them?

Therefore, we are going to create an expression for $\zeta'(z)/\zeta(z)$, which we will implement as function $Z(z)$.

5.4.3 Using self-made functions

In this subsection, we will calculate the contour integral using self-made functions. After that, we will do an error analysis.

Calculation of the contour integral To calculate the contour integral, we first discuss the zeta function and its derivative. After that, we discuss how to compute numerically $\zeta'(s)/\zeta(s)$. This will be implemented instead of the built-in functions.

When analyzing the zeta function, there will appear two integrals. For these integrals, we create a new Simpson's rule, so that variables from one calculation of the Simpson's rule will not be messed up with variables from another calculation of the Simpson's rule (since they are used at the same moment).

To analyze the zeta function, we use the analytic continuation of the zeta function as defined in Subsection 4.2.2. Since we only need the function for

$\text{Re}(s) \geq -1$, n needs to be at least 3. If $n = 3$, we obtain the following function:

$$\begin{aligned}\sum_{r=1}^{\infty} r^{-s} =& \frac{1}{s-1} + \frac{1}{2} + \sum_{k=2}^{3} B_k/k! \cdot (-1)^k s(s+1)\cdots(s+k-2) \\ & - s(s+1)(s+2)/6 \cdot \int_1^{\infty} \overline{B_3}(x) x^{-s-3} dx \\ =& \frac{1}{s-1} + \frac{1}{2} + B_2/2 \cdot s - B_3/6 \cdot s(s+1) \\ & - s(s+1)(s+2)/6 \cdot \int_1^{\infty} \overline{B_3}(x) x^{-s-3} dx\end{aligned}$$

Since we also need the derivative of the zeta function, this will be derived now:

$$\begin{aligned}\frac{d}{ds}\sum_{r=1}^{\infty} r^{-s} =& \frac{d}{ds}\Big(\frac{1}{s-1} + \frac{1}{2} + B_2/2 \cdot s - B_3/6 \cdot s(s+1) \\ & - s(s+1)(s+2)/6 \cdot \int_1^{\infty} \overline{B_3}(x) x^{-s-3} dx\Big) \\ =& \frac{d}{ds}\frac{1}{s-1} + \frac{d}{ds}\frac{1}{2} + \frac{d}{ds}B_2/2 \cdot s - \frac{d}{ds}B_3/6 \cdot s(s+1) \\ & - \frac{d}{ds}s(s+1)(s+2)/6 \cdot \int_1^{\infty} \overline{B_3}(x) x^{-s-3} dx \\ =& -\frac{1}{(s-1)^2} + B_2/2 - B_3/6 \cdot (2s+1) \\ & - (3s^2+6s+2)/6 \cdot \int_1^{\infty} \overline{B_3}(x) x^{-s-3} dx \\ & + s(s+1)(s+2)/6 \cdot \int_1^{\infty} \overline{B_3}(x) x^{-s-3} \log(x) dx\end{aligned}$$

Since we have here two integrals from which one also appears in the zeta function, we will store these values to use it twice instead of calculating it twice. These values will be stored as B1 and B2.

In these values B1 and B2, the third bernoully polynomial appears. Therefore, we will define a function with this polynomial:

```
B(x)=x^3-3/2*x^2+1/2*x
```

The values B1 and B2 now have to be computed by the Simpson's rule. Since we need here an efficient program, we will introduce a new program, specified to these integrals. Also, since the old Simpson's program also will be used, we add to each variable "new" to make sure that no variable will be messed up with a variable of the other Simpson's program. Note that there are some splits in the code lines to fit the page.

```
SimpsonNew(Nnew,endnew,snew,logpower)={
    Areanew=0;
```

```
    for(nnew=1,endnew-1,
    Areanew+=sum(mnew=1,Nnew/2,2*(B((2*mnew-1)/Nnew)/
    (nnew+(2*mnew-1)/Nnew)^(snew+3)*
    (log(nnew+(2*mnew-1)/Nnew))^logpower)
    +B((2*mnew)/Nnew)/(nnew+(2*mnew)/Nnew)^(snew+3)
    *(log(nnew+(2*mnew)/Nnew))^logpower)/(1.5*Nnew));
    Areanew=Areanew
}
```

The function $Z(s)$ becomes right now

$$\begin{aligned}Z(s) =&[-\frac{1}{(s-1)^2} + B_2/2 - B_3/6 \cdot (2s+1) - (3s^2+6s+2)/6 \cdot \int_1^\infty \overline{B_3}(x)x^{-s-3}dx \\ &+ s(s+1)(s+2)/6 \cdot \int_1^\infty \overline{B_3}(x)x^{-s-3}\log(x)dx]/ \\ &[\frac{1}{s-1} + \frac{1}{2} + B_2/2 \cdot s - B_3/6 \cdot s(s+1) - s(s+1)(s+2)/6 \cdot \int_1^\infty \overline{B_3}(x)x^{-s-3}dx]\end{aligned}$$

Using this will give us the following function for $Z(s)$ in GP/Pari. Note that there are some splits in the code lines to fit the page:

```
Z(s)={B1=SimpsonNew(300,300,s,0);B2=SimpsonNew(300,300,s,1);
    ZZ=(-1/(s-1)^2+bernreal(2)/2-bernreal(3)/6*(2*s+1)
    -(3*s^2+6*s+2)/6*B1+s*(s+1)*(s+2)/6*B2)/
    (1/(s-1)+.5+bernreal(2)/2*s-bernreal(3)/6*s*(s+1)-
    s*(s+1)*(s+2)/6*B1)
}
```

The function $Total(100, 100)$ gives the following answer (after 1 hour, 40 minutes and 19 seconds),

```
29.000386500511871920... - 1.0523572678685566098 E-39*I,
```

which also is approximate 29.

Error analysis To analyze this contour integral, we have made some approximations. The integrals B1 and B2 are computed from 1 to 300 instead of 1 to infinity, the integrals are computed with Simpson's rule, and at the end the total integral is also computed using the Simpson's rule.

The approximation of 1 to 300 instead of 1 to infinity can be made without causing a big error, since the functions are very close to zero for values greater than 300. Since the other errors will be maximums over the whole interval, we will assume that this error is approximate zero.

The error of B1 and B2 are bounded by $2.8996199 \cdot 10^{-10}$ and $1.115 \cdot 10^{-6}$ resp. A calculation showing this can be found in Appendix A. Right now we will calculate the error of the total function.

The maximum value M is 14. The stepwidth is 1, 0.03 and 1 for the intervals $[2, 2+100 \cdot i]$, $[2+100 \cdot i, -1+100 \cdot i]$ and $[-1+100 \cdot i, -1]$ resp. The interval width is 100, 3 and 100 resp. Therefore, the error bounds become

$$
\begin{aligned}
E_{I_1} \leq & \frac{1^4}{2880} \cdot 100 \cdot 14 = \frac{7}{144000} \approx 4.86 \cdot 10^{-1} \\
E_{I_2} \leq & \frac{0.03^4}{2880} \cdot 100 \cdot 14 \approx 3.94 \cdot 10^{-7} \\
E_{I_3} \leq & \frac{1^4}{2880} \cdot 100 \cdot 14 = \frac{7}{144000} \approx 4.86 \cdot 10^{-1}
\end{aligned}
$$

Summing these errors gives us a total maximum error of 0.9722. Since this is much bigger than all other errors, we will calculate further with this error. At the end, we will take this approximation in account.

Therefore, the amount of zeros has to be in the interval [28.02818,29.97259]. The only integer inside this interval is 29, but since the boundaries are approximate 28 and 30, we also will take them into account.

First of all, 28 is not possible, since there are already 29 zeros known.

Suppose there is at least one more zero in the upper half of the contour integral, $z_0 = x_0 + iy_0$. Then, as mentioned in subsection 5.2, there are four more zeros in the contour integral, from which two in the upper half. Therefore, if the amount of zeros is unequal to 29 (the amount of zeros with $\mathrm{Re}(z) = 1/2$), it has to be at least 31. Therefore, there are only 29 zeros inside this interval.

From this we can conclude that the 29 zeros on the critical line are simple. Furthermore, since there are no zeros in this interval than those on the critical line, the Riemann Hypothesis is true for $-100 \leq \mathrm{Im}(z) \leq 100$.

6 L-function of an elliptic curve

The zeta function is a special form of so-called L-functions. In this section we will take a look at the L-function of an elliptic curve. First of all, we will define what an elliptic curve is and give some other important definitions. After that, we will consider the L-function of a specific elliptic curve. For this L-function we are going to answer the following questions:

- For which $s \in \mathbb{C}$ is this L-function defined?
- How can this function be extended to the whole complex plane?
- What are the poles of the function?
- What are the zeros of the function?

6.1 Preliminaries

6.1.1 Elliptic curve

In this section we take a look at elliptic curves. An elliptic curve is defined as follows:

Definition 6.1. For $a_1, a_2, a_3, a_4, a_6 \in \mathbb{Z}$, the equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

is a Weierstrass equation. The polynomial $f \in \mathbb{Z}[x, y]$ for this equation is defined as

$$f(x, y) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6$$

For this equation we define the following equations:

$$\begin{array}{lcllcl}
b_2 & = & a_1^2 + 4a_2 & c_4 & = & b_2^2 - 24b_4 \\
b_4 & = & a_1a_3 + 2a_4 & c_6 & + & -b_2^3 + 36b_2b_4 - 216b_6 \\
b_6 & = & a_3^2 + 4a_6 & \Delta & = & -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6 \\
b_8 & = & b_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2 & j & = & c_4^3/\Delta
\end{array}$$

If $\Delta \neq 0$, the Weierstrass equation defines an elliptic curve $E/\mathbb{Q}$. Here, Δ is called the discriminant of the Weierstrass equation.

We will continue with elliptic curves over the finite fields F_p. This will be defined as the elliptic curve of definition 6.1, with coefficients a_1, a_2, a_3, a_4, a_6 modulo p. This elliptic curve will be denoted by $\tilde{E}$.

6.1.2 Minimal elliptic curve

Before going into details, we first need the following: the elliptic curve needs to be minimal, i.e. the discriminant needs to be as small as possible.

In terms of new coordinates $\xi = u^2x + r$, $\eta = u^3y + sx + t$ with $u, r, s, t \in \mathbb{Q}$ and $u \neq 0$, the Weierstrass equation changes into a new Weierstrass equation (although the new coefficients could now be rational numbers). The new equation has discriminant $u^{-12}\Delta$. This is called a change of coordinates.

In the next definition, we state what is meant by a minimal elliptic curve:

Definition 6.2. A Weierstrass equation for an elliptic curve $E/\mathbb{Q}$ is called minimal if, using a change of coordinates, the absolute value of the discriminant Δ is as small as possible, subject to the condition that $a_1, a_2, a_3, a_4, a_6 \in \mathbb{Z}$.

As shown above, a change of variables changes the discriminant with the factor u^{-12}. Since we want our a_j's to be integers, we can only apply such a change of variables if the discriminant is divisible by u^{12}.

Below there is an example from Washington[20] to make clear what this definition means:

Example 6.1. Let us start with the equation E_1, given by

$$y^2 = x^3 - 270000x + 128250000.$$

The discriminant can be calculated to be $\Delta = -2^{12}3^{12}5^{12}11$. Before we go further, we already can conclude that we can apply three times a change of variables, with $u = 5$, $u = 3$ and $u = 2$.

Changing the coordinates using $x = 25x_1$ and $y = 125y_1$ transforms the equation into

$$y_1^2 = x_1^3 - 432x_1 + 8208.$$

Right now the discriminant is $\Delta = -2^{12}3^{12}11$. Changing the coordinates using $x_1 = 9x_2 - 12$ and $y_1 = 27y_2$ gives us the equation

$$y_2^2 = x_2^3 - 4x_2^2 + 16.$$

Now the discriminant is $\Delta = -2^{12}11$. Changing once more the coordinates using $x_2 = 4x_3$ and $y_2 = 8y_3 + 4$ gives us the equation

$$y^2 + y = x^3 - x^2.$$

The discriminant is $\Delta = -11$, which is, according to Washington, the minimal Weierstrass equation for E_1.

6.1.3 Singular points

The definition about the minimal equation is needed, because of the following: if $\Delta \equiv 0 \mod p$, then there is a singular point $P = (x_0, y_0)$ at the elliptic curve modulo p. If Delta is as small as possible, it has less prime divisors.

This singular point can be found using the formulas $\frac{\partial f(P)}{\partial x} = \frac{\partial f(P)}{\partial y} = 0$. We are going to write the Taylor expansion of $f(x, y)$ at (x_0, y_0) as

$$\begin{aligned} f(x,y) - f(x_0,y_0) =& \lambda_1(x-x_0)^2 + \lambda_2(x-x_0)(y-y_0) + \lambda_3(y-y_0)^2 - (x-x_0)^3 \\ =& [(y-y_0) - \alpha(x-x_0)][(y-y_0) - \beta(x-x_0)] - (x-x_0)^3 \end{aligned}$$

Definition 6.3. A singular point P is a node if $\alpha \neq \beta$. A singular point P is a cusp if $\alpha = \beta$.

If a point P is a node, then there are 2 different tangent lines at P, namely $(y - y_0) = \alpha(x - x_0)$ and $(y - y_0) = \beta(x - x_0)$. If a point P is a cusp, then there is a unique tangent line at P.

What is meant here with a tangent line is the following: Suppose $P = (0, 0)$. Then we have $f(x, y) = 0$ for f as introduced in definition 6.1. Therefore, $a_6 = 0$.Since P is a singular point, we also know that $\frac{\partial f(P)}{\partial x} = \frac{\partial f(P)}{\partial y} = 0$, so also $a_3 = a_4 = 0$. The function $f(x, y)$ becomes therefore $f(x, y) = y^2 + a_1xy - x^3 - a_2x^2$. A tangent line through $(0, 0)$ in the direction of (a, b) has the formula $t \cdot (a, b)$. The intersections with $f(x, y)$ we get from $f(ta, tb) = 0$. We have

$$f(ta, tb) = (b^2 + a_1ab - a_2a^2)t^2 - a^3t^3.$$

We are interested in the case that the multiplicity of the zero at $t = 0$ is 3. This happens when $b^2 + a_1ab - a_2a^2 = 0$. If $a_1^2 + 4a_2 = 0$, there is a unique tangent line at P. If $a_1^2 + 4a_2 \neq 0$, there are two different tangent lines at P.

Proposition 6.1. *Let $E/\mathbb{Q}$ be given by the Weierstrass equation, being minimal. Then*

- *E has a node if and only if $\Delta = 0$ and $c_4 \neq 0$*
- *E has a cusp if and only if $\Delta = 0$ and $c_4 = 0$*

Proof. A proof of this can be found in [16]. □

Definition 6.4. Let $\tilde{E}$ be defined as earlier, being minimal. Then

- if $\tilde{E}$ is non-singular (i.e. does not contain singular points), then $\tilde{E}$ is an elliptic curve over $\mathbb{F}_p$ and E does have a good reduction at p.
- if $\tilde{E}$ has a cusp, then E has additive reduction at p.
- if $\tilde{E}$ has a node, we have 2 cases:
  - if $\alpha, \beta \in \mathbb{F}_p$, then E has split multiplicative reduction at p.
  - if $\alpha \notin \mathbb{F}_p$ or $\beta \notin \mathbb{F}_p$, then E has non-split multiplicative reduction at p.

The elliptic curve which will be evaluated in this thesis is the minimal curve of example 6.1:

$$E : y^2 + y = x^3 - x^2.$$

In this case, $\tilde{E} = E$. Therefore, $b_2 = -4$, $b_4 = 0$, $b_6 = 1$, $b_8 = -1$ and $\Delta = -11$. Hence $\Delta = 0 \mod p$ if and only if $p = 11$ when p is a prime. If $p = 11$, $c_4 = 16 \not\equiv 0 \mod 11$. Therefore, this curve will have a node.

Proposition 6.2. *The node of the curve $\tilde{E}$ is at $P = (8, 5)$ and E has split multiplicative reduction at 11.*

Proof. First of all, $f(x,y) = y^2 + y - x^3 + x^2$. $\frac{df(P)}{dx} = \frac{df(P)}{dy} = 0$ implies that $2y_0 + 1 = 0$ and $-3x_0^2 + 2x_0 = 0$. The first equation implies that $y_0 = 5$ and the second one $x_0 = 0$ or $x_0 = 8$.

Suppose $x_0 = 0$. Then $f(x_0, y_0) = 5^2 + 5 = 30 \not\equiv 0 \mod 11$. Therefore, (x_0, y_0) is not a point at $\tilde{E}$.

Suppose $x_0 = 8$. Then $f(x_0, y_0) = 5^2 + 5 - 8^3 + 8^2 \equiv 0 \mod 11$. The Taylor expansion around (x_0, y_0) is

$$f(x,y) = [(y - y_0) - (x - x_0)][(y - y_0) + (x - x_0)] - (x - x_0)^3.$$

Note that $f(x_0, y_0) \equiv 0$, so we will omit it. Since $-1, 1 \in \mathbb{F}_{11}$, E has split multiplicative reduction at 11. $\square$

6.2 L-function of elliptic curves

Definition 6.5. Define N_p as the number of points in $E(\mathbb{F}_p)$, i.e. N_p is the number of elements in

$$\{O\} \cup \{(x,y) \in \mathbb{F}_p^2 : f(x,y) \equiv 0 \mod p\}$$

where O is the point at infinity. Then $a_p = p + 1 - N_p$.

The L-function of an elliptic curve is a product over primes with factors depending on the reduction at p. Therefore, we need to define the *local part at p of the L-series*:

Definition 6.6. The local part at p of the L-series is defined as

$$L_p(T) = \begin{cases} 1 - a_pT + pT^2 & \text{if } E \text{ has good reduction at } p, \\ 1 - T & \text{if } E \text{ has split multiplicative reduction at } p, \\ 1 + T & \text{if } E \text{ has non-split multiplicative reduction at } p, \\ 1 & \text{if } E \text{ has additive reduction at } p. \end{cases}$$

Now we are going to define the L-function itself:

Definition 6.7. (Hasse-Weil L-function) The L-function of an elliptic curve is defined to be

$$L(E,s) = \prod_p \frac{1}{L_p(p^{-s})}$$

for $\mathrm{Re}(s) > 3/2$

For the elliptic curve $y^2 + y = x^3 - x$, we obtain the following L-function:

$$L(s) = (1 - 11^{-s})^{-1} \prod_{p \neq 11} (1 - a_p p^{-s} + p^{1-2s})^{-1}.$$

6.3 Analytic continuation

6.3.1 Convergence for Re$(s) > \frac{3}{2}$

In this subsection we will first proof that the L-function converges for $\mathrm{Re}(s) > \frac{3}{2}$. Before we can do that, we need the following theorem

Theorem 6.1. *Hasse theorem:* $|a_p| < 2\sqrt{p}$

Proof. A proof of this theorem can be found in the master's thesis of Soeten[17]. $\square$

Corollary 6.1.1. *The L-function converges for Re*$(s) > \frac{3}{2}$

Proof. First of all, we consider the last part of the L-function,

$$\prod_{p \neq 11} (1 - a_p p^{-s} + p^{1-2s})^{-1}.$$

Because of Hasse's theorem, $|a_p| \leq 2\sqrt{p}$. Since a_p is an integer and $2\sqrt{p}$ is not, we can conclude that $|a_p| < 2\sqrt{p}$ for our elliptic curve.

Let us consider the function $1 - a_p t + pt^2$. Since $|a_p| < 2\sqrt{p}$, the zeros of this function need to be complex. Hence we have that z and $\overline{z}$ for some complex number z are the zeros of this function. Therefore, we can write the polynomial $1 - a_p t + pt^2$ as $p(t-z)(t-\overline{z})$, which implies that $z\overline{z} = 1/p$ and $z + \overline{z} = a_p/p$.

Let $c_p = \frac{1}{z}$. Then we can rewrite $p(t-z)(t-\overline{z})$ as

$$p(t-z)(t-\overline{z}) = pz\overline{z}\frac{(t-z)}{z}\frac{(t-\overline{z})}{\overline{z}} = (t/z - 1)(t/\overline{z} - 1) = (1 - c_p t)(1 - \overline{c_p} t)$$

Changing t to p^{-s} gives us

$$1 - a_p p^{-s} + p^{1-2s} = (1 - c_p p^{-s})(1 - \overline{c_p} p^{-s}).$$

Note that $\overline{c_p} c_p = p$ and hence $|c_p| = \sqrt{p}$.

Therefore, we can rewrite the last part of the L-function as

$$\begin{aligned} \prod_{p \neq 11} (1 - a_p p^{-s} + p^{1-2s}) &= \prod_{p \neq 11} (1 - c_p p^{-s})(1 - \overline{c_p} p^{-s}) \\ &= \left(\prod_{p \neq 11} (1 - c_p p^{-s}) \right) \cdot \left(\prod_{p \neq 11} (1 - \overline{c_p} p^{-s}) \right). \end{aligned}$$

Therefore, this part of the L-function converges if both products on the right hand side converges.

For $\mathrm{Re}(s) > \frac{1}{2}$, we have that $|c_p p^{-s}| = p^{\frac{1}{2} - \mathrm{Re}(s)} \leq 1$. Therefore, using the reverse triangle inequality,

$$(|1 - c_p p^{-s}|)^{-1} \leq (1 - |c_p p^{-s}|)^{-1} = (1 - p^{\frac{1}{2} - \mathrm{Re}(s)})^{-1}.$$

Therefore,

$$\prod_{p\neq 11}(|1-c_pp^{-s}|)^{-1} \leq \prod_p(|1-c_pp^{-s}|)^{-1} \leq \zeta(\mathrm{Re}(s)-\frac{1}{2}),$$

which converges for $\mathrm{Re}(s) > \frac{3}{2}$. Changing c_p to $\overline{c_p}$ gives the same result. Therefore the last part of the L-function converges for $\mathrm{Re}(s) > \frac{3}{2}$.

Since the L-function is defined as $L(s) = (1-11^{-s})^{-1}\prod_{p\neq 11}(1-a_pp^{-s}+p^{1-2s})^{-1}$, we also need to check that $(1-11^{-s})^{-1}$ is defined for this region. This is true, since the only singularities of this part are at the line $\mathrm{Re}(s) = 0$, which is outside the region with $\mathrm{Re}(s) > \frac{3}{2}$. □

6.3.2 Analytic continuation to $\mathbb{C}$

Since we want an analytic continuation of the L-function to the entire plane, we introduce a new function ξ, defined in the following definition:

Definition 6.8. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{3}{2}$. Then we define the function

$$\xi(s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(s),$$

where, for our elliptic curve, $N = 11$.

Let us now take a look at the product over the primes of the L-function. We can rewrite each term as an infinite sum over the powers of $(a_pp^{-s}-p^{1-2s})$ (except for $p = 11$),

$$(1-a_pp^{-s}+p^{1-2s})^{-1} = \sum_k(a_pp^{-s}-p^{1-2s})^k.$$

This infinite sum can be written as an infinite sum over powers of p,

$$\sum_k(a_pp^-s-p^{1-2s})^k = 1+a_pp^{-s}+(a_p^2-p)p^{-2s}+(a_p^3-2pa_p)p^{-3s}+\ldots = \sum_n a_{np}p^{-ns}.$$

For $p = 11$ we get the infinite sum

$$(1-11^{-s})^{-1} = a_{11}^n 11^{-ns}$$

If we multiply all this infinite sums with each other, we obtain the following series:

$$L(s) = (1-11^{-s})^{-1}\prod_{p\neq 11}(1-a_pp^{-s}+p^{1-2s})^{-1} = \sum_{n=1}^{\infty}a_nn^s,$$

where each a_n can be obtained by the multiplication of the infinite sums. a_{pq} will be therefore a_pa_q for p and q different primes. In the book "Elliptic curves, modular forms and their L-functions" of Lozano-Robledo [11], there are formulas to compute each a_n:

For $p \neq 11$, we have that, for $r \geq 2$,

$$a_{p^r} = a_p \cdot a_{p^{r-1}} - p \cdot a_{p^{r-2}}.$$

If the greatest common divisor of m and n equals 1, then

$$a_{mn} = a_m a_n$$

For $p \neq 11$, $a_{p^2} = a_p^2 - p$ and on this way we can calculate all a_n. The first 32 a_n's are calculated and can be found in Appendix C.

In [1], Buhler et al. proved that the Xi-function can be rewritten as an integral, which is stated in the following proposition:

Proposition 6.3. *The function $\xi(s)$ can be rewritten as*

$$\xi(s) = \int_0^\infty y^{s-1} g\left(\frac{iy}{\sqrt{N}}\right) dy$$

where the function $g(\tau)$ is defined as

$$g(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n \tau}$$

Proof. Let us start with the rewritten form.

$$\begin{aligned}\int_0^\infty y^{s-1} g\left(\frac{iy}{\sqrt{N}}\right) dy &= \int_0^\infty y^{s-1} \sum_{n=1}^{\infty} a_n e^{-2\pi n y/\sqrt{N}} dy \\ &= \int_0^\infty \sum_{n=1}^{\infty} y^{s-1} a_n e^{-2\pi n y/\sqrt{N}} dy\end{aligned}$$

Let $t = \frac{2\pi n}{\sqrt{N}} y$. Then $dy = \frac{\sqrt{N}}{2\pi n} dt$. The boundaries stay the same, since $t = y$ at the boundaries. Therefore, the formula becomes

$$\begin{aligned}\int_0^\infty \sum_{n=1}^{\infty} y^{s-1} a_n e^{-2\pi n y/\sqrt{N}} dy &= \int_0^\infty \sum_{n=1}^{\infty} \left(\frac{\sqrt{N}}{2\pi n} t\right)^{s-1} a_n e^{-t} \frac{\sqrt{N}}{2\pi n} dt \\ &= \sum_{n=1}^{\infty} \left(\frac{\sqrt{N}}{2\pi n}\right)^{s} a_n \int_0^\infty t^{s-1} e^{-t} dt \\ &= N^{s/2} (2\pi)^{-s} \sum_{n=1}^{\infty} a_n n^{-s} \int_0^\infty t^{s-1} e^{-t} dt \\ &= N^{s/2} (2\pi)^{-s} \Gamma(s) L(s) \\ &= \xi(s)\end{aligned}$$

□

Proposition 6.4. *The function $g(\tau)$ as defined in proposition 6.3, can be written as*

$$g(\tau) = q \prod (1-q^n)^2 (1-q^{11n})^2, \quad q = e^{2\pi i \tau}.$$

Proof. First of all, note that

$$\begin{aligned} g(\tau) &= q \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^{11n})^2 \\ &= q \prod_{n=1}^{\infty} (1-2q^n+q^{2n})(1-2q^{11n}+q^{22n}) \\ &= q \prod_{n=1}^{\infty} (1-2q^n+q^{2n}-2q^{11n}+4q^{12n}-2q^{13n}+q^{22n}-2q^{23n}+q^{24n}) \\ &= \sum_{n=1}^{\infty} A_n q^n \end{aligned}$$

for some A_n. This means that this proposition claims that $A_n = a_n$, where the a_n's are the coefficients of the L-function.

The proof of this result is beyond the scope of this bachelor's thesis. Therefore, we will make a table in which the coefficients A_n of the function g are compared with the coefficients a_n of the elliptic curve E:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A_n	1	-2	-1	2	1	2	-2	0	-2	-2	1	-2	4	4	-1	-4
a_n	1	-2	-1	2	1	2	-2	0	-2	-2	1	-2	4	4	-1	-4

n	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
A_n	-2	4	0	2	2	-2	-1	0	-4	-8	5	-4	0	2	7	8
a_n	-2	4	0	2	2	-2	-1	0	-4	-8	5	-4	0	2	7	8

In this table we can see that $A_n = a_n$ for all n in the table. We claim that this holds for all positive integers n. □

Proposition 6.5. *The function $g(\tau)$ as defined in proposition 6.3, satisfies the functional equation*

$$g(-1/N\tau) = -N\tau^2 g(\tau)$$

Proof. This will not be proven here, but assumed to be true. □

Proposition 6.6. *The function $\xi(s)$ can be rewritten as*

$$\xi(s) = \int_1^{\infty} g\left(\frac{iy}{\sqrt{N}}\right) \left(y^{1-s} + y^{s-1}\right) dy.$$

Proof. In this proof we will use the definition of the ξ-function given by proposition 6.3:

$$\xi(s) = \int_0^{\infty} y^{s-1} g\left(\frac{iy}{\sqrt{N}}\right) dy.$$

This integral will be split into two integrals: one from 0 to 1 and one from 1 to infinity.

Let us rewrite the first integral, using a change of coordinates $t = 1/y$, and therefore $dy = \frac{-1}{t^2}dt$:

$$\begin{aligned}
\int_0^1 y^{s-1} g\left(\frac{iy}{\sqrt{N}}\right) dy &= \int_\infty^1 t^{1-s} g\left(\frac{i}{\sqrt{N}t}\right) \frac{-1}{t^2} dt \\
&= \int_1^\infty t^{1-s} g\left(\frac{i}{\sqrt{N}t}\right) \frac{1}{t^2} dt \\
&= \int_1^\infty t^{1-s} g\left(\frac{i^2}{(N/\sqrt{N}) \cdot it}\right) \frac{1}{t^2} dt \\
&= \int_1^\infty t^{1-s} g\left(\frac{-1}{N \cdot \frac{it}{\sqrt{N}}}\right) \frac{1}{t^2} dt
\end{aligned}$$

Because of proposition 6.5, we have

$$\begin{aligned}
\int_1^\infty t^{1-s} g\left(\frac{-1}{N \cdot \frac{it}{\sqrt{N}}}\right) \frac{1}{t^2} dt &= \int_1^\infty t^{1-s} \cdot \left(-N \cdot \left(\frac{it}{\sqrt{N}}\right)^2\right) g\left(\frac{it}{\sqrt{N}}\right) \frac{1}{t^2} dt \\
&= \int_1^\infty t^{1-s} g\left(\frac{it}{\sqrt{N}}\right) dt
\end{aligned}$$

Therefore, ξ can be rewritten as

$$\begin{aligned}
\xi(s) &= \int_0^1 y^{s-1} g\left(\frac{iy}{\sqrt{N}}\right) dy + \int_1^\infty y^{s-1} g\left(\frac{iy}{\sqrt{N}}\right) dy \\
&= \int_1^\infty y^{1-s} g\left(\frac{iy}{\sqrt{N}}\right) dy + \int_1^\infty y^{s-1} g\left(\frac{iy}{\sqrt{N}}\right) dy \\
&= \int_1^\infty y^{1-s} g\left(\frac{iy}{\sqrt{N}}\right) + y^{s-1} g\left(\frac{iy}{\sqrt{N}}\right) dy \\
&= \int_1^\infty g\left(\frac{iy}{\sqrt{N}}\right) \left(y^{1-s} + y^{s-1}\right) dy
\end{aligned}$$

□

Lemma 6.2. *The function $\xi(s)$ is an entire function.*

Proof. Using the rewritten form of ξ of proposition 6.6, we will prove that $\int_1^\infty y^{s-1} g\left(\frac{iy}{\sqrt{N}}\right) dy$ converges for all $s \in \mathbb{C}$ and defines an entire function. Therefore, by replacing s by $2 - s$, also the other part of the ξ-function is entire.

Let us write $g(iy/\sqrt{N})$ as the infinite product from proposition 6.4. Then we get

$$y^{s-1} g\left(\frac{iy}{\sqrt{N}}\right) = y^{s-1} \cdot \left(e^{-s\pi y/\sqrt{N}} \prod (1-q^n)^2 (1-q^{11n})^2\right),$$

where $q = e^{-2\pi y/\sqrt{N}}$. Therefore the infinite product $\prod(1-q^n)^2(1-q^{11n})^2$ is smaller than 1. Hence we can bound the equation by

$$y^{s-1} g\left(\frac{iy}{\sqrt{N}}\right) \leq y^{s-1} \cdot \left(e^{-s\pi y/\sqrt{N}}\right).$$

Therefore, we get an upper bound for the integral,

$$\int_1^\infty y^{s-1} g\left(\frac{iy}{\sqrt{N}}\right) dy \leq \int_1^\infty y^{s-1} \cdot e^{-s\pi y/\sqrt{N}} dy.$$

The right hand side converges on every point s of the complex plane, since the last term is converging exponential to 0. Therefore, also the integral on the left hand side converges.

Because of what is said in the start of the proof, $\xi(s)$ is an entire function. □

Lemma 6.3. *For the function $\xi(s)$, we have the following functional equation:*

$$\xi(s) = \xi(2-s)$$

Proof. This follows directly from

$$\xi(s) = \int_1^\infty g\left(\frac{iy}{\sqrt{N}}\right)\left(y^{1-s} + y^{s-1}\right) dy.$$

□

Theorem 6.4. *The L-function can be rewritten as*

$$L(s) = \frac{\xi(s)}{\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)},$$

which is analytic on the whole complex plane.

Proof. From the definition of $\xi(s)$ it follows that the L-function can be written as

$$L(s) = \frac{\xi(s)}{\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)}.$$

Since $\xi(s)$ is analytic on the whole complex plane and the gamma function does not have zeros and has an analytic continuation to the whole complex plane (except at the negative integers, at which points the L-function therefore has zeros), we can conclude that the L-function is analytic on the whole complex plane. □

6.4 Zeros of the L-function

6.4.1 Zeros with $\mathbf{Re}(s) > \frac{3}{2}$

Proposition 6.7. *The L-function does not have any zero with $Re(s) > \frac{3}{2}$.*

Proof. First of all, note that

$$\sum |b_n| \text{ converges } \Rightarrow \prod(1+b_n) \text{ converges.}$$

This follows from the fact that if $\sum |b_n|$ converves then also $\sum \ln(1+|b_n|)$ converges and $\sum \ln(1+|b_n|) = \ln(\prod(1+|b_n|))$. This last expression converges if $\prod(1+|b_n|)$ converges, which implies that also $\prod(1+b_n)$ converges.

The L-function is given by

$$L(s) = (1-11^{-s})^{-1} \prod_{p \neq 11} (1 - a_p p^{-s} + p^{1-2s})^{-1}.$$

To show that this function is not zero, we need to show that the numerator of the product on the right hand side is not equal to zero.

Let us define b_p for all primes $p \neq 11$ as follows: $b_p = (-a_p p^{-s} + p^{1-2s})$. Then

$$\sum_p |b_p| \leq \sum_p \left|a_p p^{-s}\right| + \left|p^{1-2s}\right| = \left(\sum_p \left|a_p p^{-s}\right|\right) + \left(\sum_p \left|p^{1-2s}\right|\right).$$

Because of Hasse's Theorem, we have that $a_p \leq 2\sqrt{p}$. Therefore, we get

$$\begin{aligned}
\sum_p |b_p| &\leq \left(\sum_p \left|2\sqrt{p}p^{-s}\right|\right) + \left(\sum_p \left|p^{1-2s}\right|\right) \\
&= 2 \cdot \left(\sum_p \left|p^{1/2-s}\right|\right) + \left(\sum_p \left|p^{1-2s}\right|\right) \\
&\leq 2 \cdot \left(\sum_{n=1}^{\infty} \left|n^{1/2-s}\right|\right) + \left(\sum_{m=1}^{\infty} \left|m^{1-2s}\right|\right).
\end{aligned}$$

We know that $\sum_{n=1}^{\infty} n^s$ converges for $\text{Re}(s) > 1$. Therefore, the first sum on the right hand side converges for $\text{Re}(s) > \frac{3}{2}$. The second sum converges for $\text{Re}(s) > 1$. Therefore, $\sum_p |b_p|$ converges for $\text{Re}(s) > \frac{3}{2}$.

As noted in the start of this proof, this implies that $\prod(1+b_n) = \prod_{p\neq 11}(1 - a_p p^{-s} + p^{1-2s})$ converges for $\text{Re}(s) > \frac{3}{2}$.

Since the L-function is defined as $L(s) = (1-11^{-s})^{-1} \prod_{p\neq 11}(1 - a_p p^{-s} + p^{1-2s})^{-1}$, the function can be 0 only if $(1-11^{-s})^{-1} = 0$ or $\prod_{p\neq 11}(1 - a_p p^{-s} + p^{1-2s})^{-1} = 0$. For $\text{Re}(s) > \frac{3}{2}$ it is clear that the first equation never holds. The second equation holds only if $\prod_{p\neq 11}(1 - a_p p^{-s} + p^{1-2s}) \to \infty$. From what we derived above, we know that this is not the case for $\text{Re}(s) > \frac{3}{2}$. Therefore, the L-function does not have any zero with $\text{Re}(s) > \frac{3}{2}$. $\square$

6.4.2 Zeros with $\mathbf{Re}(s) < \frac{1}{2}$

Before we are going to look at the zeros of the L-function, we first will take a look at the zeros of the Xi-function.

Lemma 6.5. *The only zeros of the ξ-function are the nontrivial zeros of the L-function.*

Proof. First of all, $\xi(s)$ does not have any zero in the complex half plane defined by $\mathrm{Re}(s) > \frac{3}{2}$, since both the gamma function and the L-function do not have any zeros in this region.

Secondly, since $\xi(s)$ satisfies the functional equation $\xi(s) = \xi(2-s)$, there are also no zeros in the half complex plane with $\mathrm{Re}(s) < \frac{1}{2}$.

Right now we are left with the strip $\frac{1}{2} \leq \mathrm{Re}(s) \leq \frac{3}{2}$. Since the gamma function does not have zeros and is defined on this whole interval, the zeros in this region are the zeros of the L-function. $\square$

Theorem 6.6. *The only zeros of the L-function in the half complex plane with $Re(s) < \frac{1}{2}$ are the non-positive integers, which are simple zeros.*

Proof. First of all, we will use the rewritten form of the L-function in terms of Xi and Gamma:

$$L(s) = \frac{\xi(s)}{\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)}.$$

The function $\xi(s)$ does not have zeros and poles satisfying $\mathrm{Re}(s) < \frac{1}{2}$. Moreover $\alpha := \frac{\sqrt{N}}{2\pi}$ is just a positive constant, and $\alpha^s = e^{s \ln \alpha}$ is an analytic function of s without zeros. Hence, we only have to take a look at the Gamma function.

Suppose the Gamma function has a pole. Then the L-function will be 0 at this point. The poles of the Gamma function occur precisely at the non-positive integers. Therefore, the L-function is 0 at these points. Hence, the only zeros of the L-function in the complex half plane with $\mathrm{Re}(s) < \frac{1}{2}$ are the non-positive integers.

Since the poles of the Gamma function are simple, also the zeros of the L-function at these integers are simple. $\square$

6.5 Summary

In this section we have proven different theorems about the L-function of an elliptic curve. In the introduction of this section we were looking for the following answers:

- For which $s \in \mathbb{C}$ is the L-function function defined?
- How can this function be extended to the whole complex plane?
- What are the poles of the function?
- What are the zeros of the function?

As seen in the previous subsections, we have answered all questions, except the last one.

- The L-function (as a series and as an infinite product) is defined for $\mathrm{Re}(s) > \frac{3}{2}$.
- This function can be extended to the whole complex plane using the Xi-function
$$\xi(s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(s),$$
which is analytic on the whole complex plane.
- The L-function does not have any pole, since ξ does not have any pole.
- The trivial zeros of the L-function are the non-positive integers. Furthermore, there are no zeros outside the critical strip $\frac{1}{2} \leq \mathrm{Re}(s) \leq \frac{3}{2}$. The zeros inside this strip will be discussed in the next section.

7 Zeros of the L-function of the elliptic curve inside the complex strip $\frac{1}{2} \leq \mathrm{Re}(s) \leq \frac{3}{2}$

7.1 Zeros on the line $1 + it$

Before we are going to continue and compute the contour integral, we first need to know which zeros are on the line $\mathrm{Re}(s) = 1$. To do this, we will construct the so-called Hardy's function. More information about this type of functions can be found in the bachelor's thesis of Van der Meer [12]. To do this, we will take a look at the functional equation

$$\xi(s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s).$$

Rewriting this gives us an equation for $L(s)$:

$$L(s) = \frac{\xi(s)}{\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)}.$$

Putting $s = 1 + it$ gives us

$$\begin{aligned} L(1+it) &= \frac{\xi(1+it)}{\left(\frac{\sqrt{N}}{2\pi}\right)^{1+it} \Gamma(1+it)} \\ &= \frac{\xi(1+it)}{\left(\frac{\sqrt{N}}{2\pi}\right)\left(\frac{\sqrt{N}}{2\pi}\right)^{it} \Gamma(1+it)}. \end{aligned}$$

Multiplying both sides with $\Gamma(1+it)/|\Gamma(1+it)|$ and $\left(\frac{\sqrt{N}}{2\pi}\right)^{it}$ gives us the equation

$$\frac{\Gamma(1+it)}{|\Gamma(1+it)|}\left(\frac{\sqrt{N}}{2\pi}\right)^{it} L(1+it) = \frac{\xi(1+it)}{\left(\frac{\sqrt{N}}{2\pi}\right)|\Gamma(1+it)|}. \tag{25}$$

Let us take a look at the right hand side. The denominator is clearly a real function. The numerator is also real, which will be proven in the lemma below. Therefore, the whole right hand side, and hence the left hand side, is a real function.

Lemma 7.1. *The function $\xi(s)$ is a real function when restricted to the line $\{1+it \mid t \in \mathbb{R}\}$.*

Proof. We have $\overline{L(s)} = L(\overline{s})$ and therefore $\overline{\xi(s)} = \xi(\overline{s})$. Hence,

$$\overline{\xi(1+it)} = \xi(1-it) = \xi(2-(1-it) = \xi(1+it).$$

For a complex number s, we know that $\overline{s} = s$ if and only if s is real. Therefore, $\xi(1+it)$ is real. $\square$

Now we know that the left hand side of equation (25) is real. It will be proven that the zeros of this function are the zeros of the L-function.

Lemma 7.2. *Let $t \in \mathbb{R}$ be a real number. $L(1+it) = 0$ if and only if $\frac{\Gamma(1+it)}{|\Gamma(1+it)|}\left(\frac{\sqrt{N}}{2\pi}\right)^{it} L(1+it) = 0$.*

Proof. Before we start, we have to note the following:

1. The Gamma function only has poles at the negative integers, so the Gamma function is defined for all t on the line $1+it$;

2. The Gamma function is nowhere 0.

3. $\left(\frac{\sqrt{N}}{2\pi}\right)^{it} \neq 0$ for all t.

4. $\left(\frac{\sqrt{N}}{2\pi}\right)^{it}$ is defined for all t

The first and the second note are proven in section 3. The third and fourth note follows from the fact that $x^{it} = e^{it\ln x}$ for $x > 0$, and $|e^{it\ln x}| = 1$.

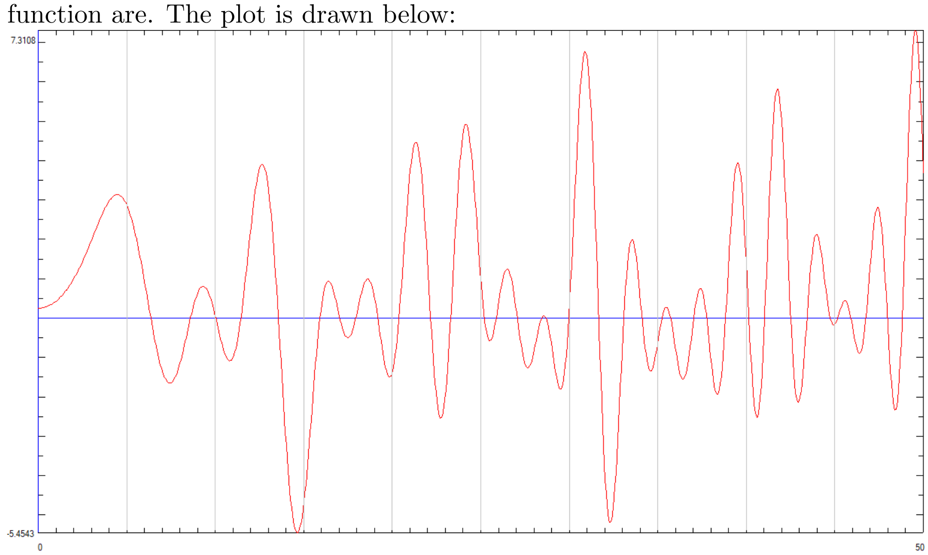
Suppose that $\frac{\Gamma(1+it)}{|\Gamma(1+it)|}\left(\frac{\sqrt{N}}{2\pi}\right)^{it} L(1+it) = 0$. From 2. and 3. it follows that $L(1+it) = 0$.

Suppose now that $L(1+it) = 0$. From 1. and 4. it follows that

$$\frac{\Gamma(1+it)}{|\Gamma(1+it)|}\left(\frac{\sqrt{N}}{2\pi}\right)^{it} L(1+it) = 0$$

. $\square$

Since we have now a real function with the same zeros as the L-function, we are going to plot a graph of this function and look at where the zeros of the function are. The plot is drawn below:



In this plot we see where the zeros approximately are. Using the solve-function of GP/Pari, we compute the following 36 zeros (the zeros below $t = 50$):

range of t	zero	range of t	zero	range of t	zero
(5,7)	6.36261389	(24,25.5)	25.20986842	(37.5,38.5)	37.76195717
(7,9)	8.60353962	(25.5,26.5)	25.87640308	(38.5,40)	38.83348637
(9,11)	10.03550910	(26.5,28)	27.06763523	(40,40.5)	40.14665618
(11,13)	11.45125861	(28,28.5)	28.44964970	(40.5,41.5)	41.03779402
(13,15)	13.56863906	(28.5,29)	28.68390988	(41.5,43)	42.51173279
(15,16.5)	15.91407260	(29,31)	29.97485995	(43,44)	43.42909861
(16.5,17.5)	17.03361032	(31,32)	31.66357557	(44,45)	44.74645453
(17.5,18.5)	17.94143357	(32,33.5)	33.08284281	(45,45.5)	45.15871414
(18.5,20)	19.18572497	(33.5,35)	34.11285248	(45.5,46.5)	45.91026196
(20,21)	20.37926046	(35,35.5)	35.23648779	(46.5,47.5)	46.75759897
(21,22.5)	22.17249029	(35.5,36)	35.72287782	(47.5,48.5)	47.97205062
(22.5,24)	23.30141550	(36,37.5)	37.03640515	(48.5,50)	48.80751812

7.2 The contour integral

In this subsection we will test if the Riemann Hypothesis is true at least in some bounded region, for the elliptic curve defined by the Weierstrass equation $y^2+y=x^3-x^2$. As seen in Lemma 6.5, the nontrivial zeros of the L-function of the elliptic curve are the only zeros of the function $\xi(s)$. Since this function does have some nice properties, we will use this function instead of the L-function for the contour integral.

The contour which we are going to use here, is the rectangle with corners $2-Ai$, $2+Ai$, Ai and $-Ai$. All zeros with $|\text{Im}(z)|<A$ are inside this rectangle.

The nice properties as mentioned are the following: as already earlier seen, $\xi(s)=\xi(2-s)$. Therefore, we also have $\xi'(s)=-\xi'(2-s)$. Before stating more properties, let us first define the paths on the border of the contour:

$$\begin{array}{ll|ll}
\gamma_1(t)=2+Ait & \gamma_1'(t)=Ai & \gamma_5(t)=-Ait & \gamma_5'(t)=-Ai \\
\gamma_2(t)=2+Ai-t & \gamma_2'(t)=-1 & \gamma_6(t)=-Ai+t & \gamma_6'(t)=1 \\
\gamma_3(t)=1+Ai-t & \gamma_3'(t)=-1 & \gamma_7(t)=1-Ai+t & \gamma_7'(t)=1 \\
\gamma_4(t)=Ai-Ait & \gamma_4'(t)=-Ai & \gamma_8(t)=2-Ai+Ait & \gamma_8'(t)=Ai
\end{array}$$

We are going to evaluate $X(s)=\xi'(s)/\xi(s)$ on these paths. We will use $X(s)$ here, because it has the property $X(s)=-X(2-s)$. Also, since $\overline{\xi(s)}=\xi(\overline{s})$, $\overline{X(s)}=X(\overline{s})$.

There are more properties of this function, stated and proved in the following theorem:

Theorem 7.3. *The function $X(s)$ has poles at the zeros of $\xi(s)$ with residue $ord_{z=z_0}\xi(s)$. No other poles appear.*

Proof. First of all, note that $\xi(s)$ does not have any pole. Therefore, also the derivative does not have a pole, which is a consequence of the fact that the Xi-function is analytic. Therefore, the only poles of $X(s)$ are at the zeros of the Xi-function.

What is left to prove, is the fact that $X(s)$ has a pole at each of the zeros of the Xi-function and that the residue is equal to $ord_{z=z_0}\xi(s)$. This proof is the same as the proof of the first part of Theorem 5.2, by changing ζ to ξ. □

Let us now again consider the paths on the border of the contour, γ_i. Combining the paths gives us the total contour integral. Let us now take a look at the different integrals. Then the following theorem arises:

Theorem 7.4. *Calculating the integral along γ_1, γ_5, γ_4 and γ_8 gives us the following relation:*

$$\int_2^{2+Ai} X(s)ds \overset{(1)}{=} \int_0^{-Ai} X(s)ds \overset{(5)}{=} \overline{\int_0^{Ai} X(s)ds} \overset{(2)}{=} \overline{\int_2^{2-Ai} X(s)ds}.$$

Calculating the integral along γ_2, γ_6, γ_3 and γ_7 gives us the following relation:

$$\int_{2+Ai}^{1+Ai} X(s)ds \overset{(3)}{=} \int_{-Ai}^{1-Ai} X(s)ds \overset{(6)}{=} \overline{\int_{Ai}^{1+Ai} X(s)ds} \overset{(4)}{=} \overline{\int_{2-Ai}^{1-Ai} X(s)ds}.$$

Remark. *Note that, because of this theorem, we only have to calculate two integrals to obtain the total contour integral by some computations.*

Proof. (1):

$$\begin{aligned}\int_2^{2+Ai} X(s)ds &= \int_0^1 X(\gamma_1(t))\gamma_1'(t)\ dt = \int_0^1 X(2+Ait)\cdot Ai\ dt\\ &= \int_0^1 -X(-Ait)\cdot Ai\ dt = \int_0^1 X(\gamma_5(t))\cdot\gamma_5'(t)\ dt\\ &= \int_0^{-Ai} X(s)ds.\end{aligned}$$

(2):

$$\begin{aligned}\int_2^{2-Ai} X(s)ds &= \int_1^0 X(\gamma_8(t))\gamma_8'(t)\ dt = \int_1^0 X(2-Ai+Ait)\cdot Ai\ dt\\ &= \int_1^0 -X(Ai-Ait)\cdot Ai\ dt = \int_1^0 X(\gamma_4(t))\cdot\gamma_4'(t)\ dt\\ &= \int_0^{Ai} X(s)ds.\end{aligned}$$

(3):

$$\begin{aligned}\int_{2+Ai}^{1+Ai} X(s)ds &= \int_0^1 X(\gamma_2(t))\gamma_2'(t)\ dt = \int_0^1 X(2+Ai-t)\cdot -1\ dt\\ &= \int_0^1 -X(-Ai+t)\cdot -1\ dt = \int_0^1 X(\gamma_6(t))\cdot\gamma_6'(t)\ dt\\ &= \int_{-Ai}^{1-Ai} X(s)ds.\end{aligned}$$

(4):

$$\begin{aligned}\int_{Ai}^{1+Ai} X(s)ds &= \int_1^0 X(\gamma_3(t))\gamma_3'(t)\ dt = \int_1^0 X(1+Ai-t)\cdot -1\ dt\\ &= \int_1^0 -X(1-Ai+t)\cdot -1\ dt = \int_1^0 X(\gamma_7(t))\cdot\gamma_7'(t)\ dt\\ &= \int_{2-Ai}^{1-Ai} X(s)ds.\end{aligned}$$

(5):

$$\begin{aligned}
\int_0^{-Ai} X(s)ds &= \int_0^1 X(\gamma_5(t))\gamma_5'(t)\ dt = \int_1^0 X(\overline{\gamma_4(t)}) \cdot \overline{-\gamma_4'(t)}\ dt \\
&= \int_0^1 \overline{X(\gamma_4(t))} \cdot \overline{\gamma_4'(t)}\ dt = \int_0^1 \overline{X(\gamma_4(t)) \cdot \gamma_4'(t)}\ dt \\
&= \overline{\int_0^1 X(\gamma_4(t)) \cdot \gamma_4'(t)\ dt} = \overline{\int_0^{Ai} X(s)ds}
\end{aligned}$$

(6):

$$\begin{aligned}
\int_{-Ai}^{1-Ai} X(s)ds &= \int_0^1 X(\gamma_6(t))\gamma_6'(t)\ dt = \int_1^0 X(\overline{\gamma_3(t)}) \cdot \overline{-\gamma_3'(t)}\ dt \\
&= \int_0^1 \overline{X(\gamma_3(t))} \cdot \overline{\gamma_3'(t)}\ dt = \int_0^1 \overline{X(\gamma_3(t)) \cdot \gamma_3'(t)}\ dt \\
&= \overline{\int_0^1 X(\gamma_3(t)) \cdot \gamma_3'(t)\ dt} = \overline{\int_{Ai}^{1+Ai} X(s)ds}
\end{aligned}$$

□

7.3 Testing the Riemann Hypothesis for the L-function of an elliptic curve

7.3.1 Using built-in functions

In GP/Pari, many built-in functions can be used to compute the contour integral. There is a function which can calculate the L-function at a point s of the elliptic curve $e1$, $elllseries(e1, s)$. Using this function and the built-in gamma function, we can construct the following function for the Xi-function:

```
e1=ellinit([0,-1,1,0,0]);
Xi(s)=elllseries(e1,s)*gamma(s)*(sqrt(11)/(2*Pi))^s;
```

Since $\xi'(s)/\xi(s)$ is the logarithmic derivative of the Xi-function, we will take the logarithm of the Xi-function and approximate the derivative with the built-in function:

```
ZZ(s)=log(Xi(s));
Z(s)=ZZ'(s);
```

Now we have a function for $\xi'(s)/\xi(s)$ and therefore, we can calculate the contour integral of this function. This will be done using the Simpson's rule, which we already used earlier:

```
Simpson(a,b,N,f)={
    h=(b-a)/N;Area=0;ye=f(a);
    for(n=0,N-1,yb=ye;ym=f((n+1/2)*h+a);
    ye=f((n+1)*h+a);Area+=(yb+4*ym+ye)*h/6);
```

```
    thearea=Area;
};
```

Since the gamma function has a pole at $s_0 = 0$, we will take the contour integral over the contour with corners $2-Ai$, $2+Ai$, $0.5+Ai$ and $0.5-Ai$. This contour will be calculated using the following function:

```
Total(A,N,f)={
    Simpson(2,2+A*I,N,f);Area1=Area;Simpson(2+A*I,.5+A*I,N,f);
    Area2=Area;Simpson(.5+A*I,.5,N,f);Area3=Area;
    Totarea=Area1+Area2+Area3-conj(Area1+Area2+Area3);
    Amountzero=(Totarea/(4*Pi*I));
};
```

When calculating $Total(50, 100, Z)$, we got, after 17 minutes, the following answer: $35.99879765805316655697\ldots + 0.E - 40 * I$

When calculating $Total(50, 1000, Z)$ we got, after approximate 3 hours, the answer $35.99999997715190717001\ldots + 0.E - 40 * I$.

Error analysis In Appendix B, the error analysis can be found. There, it is shown that the error bound for this last calculation is, assuming that the built-in functions are working correctly, approximate 0.26. Therefore, the amount of zeros, multiplied with their multiplicity, needs to be 36. Since we know that there are 36 zeros on the critical line inside the contour with $\mathrm{Im}(s) > 0$ and if there is an extra zero not on the critical line, there needs to be at least 2, this implies that there are no zeros in this area which are not on the critical line.

There is only a problem with this approximation. We do not know the accuracy of the built-in functions. Therefore, we also will compute this contour integral using self made functions, discussed in the next subsection.

7.3.2 Using self-made functions

Since we need to create self-made functions, we need to make a function for Xi and one for the derivative of Xi. This will be done using the integral from proposition 6.6.

In this integral, also the function g appears. Therefore, we will also create a self made function for this infinite product. Instead of taking the infinite product, we will approximate this by taking the product from $n = 1$ to 20. This will be done using the following code:

```
q(t)=exp(-2*Pi*t/sqrt(11));
g(t)=q(t)*prod(n=1,20,(1-q(n*t))^2*(1-q(11*n*t))^2);
```

What we need to do now is creating a function for the product inside the integrals of $\xi(s)$ and $\xi'(s)$. Note that

$$\xi(s) = \int_1^\infty g\left(\frac{iy}{\sqrt{11}}\right)(y^{1-s} + y^{s-1})dy.$$

From this formula, we will take the derivative with respect to s. This will be done using the fact that, for this function, differentiating this integral with respect to s can be done by differentiating the function under the integral with respect to s:

$$\begin{aligned}
\xi'(s) &= \frac{\partial}{\partial s}\int_1^\infty g\left(\frac{iy}{\sqrt{11}}\right)(y^{1-s}+y^{s-1})dy \\
&= \int_1^\infty \frac{\partial}{\partial s}\left(g\left(\frac{iy}{\sqrt{11}}\right)(y^{1-s}+y^{s-1})\right)dy \\
&= \int_1^\infty g\left(\frac{iy}{\sqrt{11}}\right)\ln(y)(-y^{1-s}+y^{s-1})dy
\end{aligned}$$

Combining these functions together, gives us the following function, where $derivative = 0$ for $\xi(s)$ and $derivative = 1$ for $\xi'(s)$:

```
E0(y2,s2,derivative)=g(y2)*log(y2)^derivative*
     ((-1)^derivative*y2^(1-s2)+y2^(s2-1));
```

Note that there is a split in this code.

To calculate the integrals, we approximate the integrals by taking the integral from 1 to 10 instead of ∞. This will be done using the Simpson's rule, which we adopt specific for this function:

```
Simpson2(aN2,bN2,NN2,derivative,sN2)={
     hN2=(bN2-aN2)/NN2;AreaN5=0;yeN2=E0(aN2,sN2,derivative);
     for(nN2=0,NN2-1,ybN2=yeN2;
     ymN2=E0((nN2+1/2)*hN2+aN2,sN2,derivative);
     yeN2=E0((nN2+1)*hN2+aN2,sN2,derivative);
     AreaN5+=(ybN2+4*ymN2+yeN2)*hN2/6);
     AreaTotalN=AreaN5;
};
```

Since we need to compute $\xi'(s)/\xi(s)$, we define the following functions, where $E1$ is $\xi'(s)$ and $E2$ is $\xi(s)$:

```
E1(s2)=Simpson2(1,10,1000,1,s2);
E2(s2)=Simpson2(1,10,1000,0,s2);
E(s)=E1(s)/E2(s);
```

Right now we have a function which computes $\xi'(s)/\xi(s)$. Therefore, we need to integrate this function along the contour. This will be done again using the Simpson's rule as introduced earlier:

```
Simpson(a,b,N,f)={
     h=(b-a)/N;Area=0;ye=f(a);
     for(n=0,N-1,yb=ye;ym=f((n+1/2)*h+a);
     ye=f((n+1)*h+a);Area+=(yb+4*ym+ye)*h/6);
     thearea=Area;
};
```

```
Total(A,N,f)={
    Simpson(2,2+A*I,N,f);Area1=Area;Simpson(2+A*I,1+A*I,N,f);
    Area2=Area;
    Totarea=2*Area1+2*Area2-2*conj(Area2)-2*conj(Area1);
    Amountzero=(Totarea/(4*Pi*I));
};
```

Total(50,100,E) gave after 23 minutes the answer

$$32.984488059045727\ldots + 0.E - 37 * I$$

Total(50,1000,E) gave after approximate 3 hours the answer

$$33.000001691102359\ldots + 0.E - 37 * I$$

According to our error analysis, which can be found in Appendix B, this solution is not accurate enough. The problem, however, is the time which the program needs to compute the solution of this contour integral. Right now it already needed three hours to get the answer. Because of this, we have decided not to try to get a better approximation using our own Simpson's rule. Instead of that, we have tried to use the built-in integration rule of GP/Pari itself, which will be discussed in the subsection after the following.

But there is one big issue which we have to discuss earlier: Why was the outcome 33 instead of the expected 36? This will be discussed in the next subsection.

7.3.3 The error in the calculations

First of all, note that using the built-in functions for the elliptic curve, we already calculated that the answer needs to be 36. But to try to give an answer to this question, we will show different tables with the outcome of different used functions, using different approximations (better and worse than the used approximations).

First of all, we will show the outcome of the built-in functions for $\xi(s)$ and $\xi'(s)$ with $s = 2$, $s = 2 + 25i$, $s = 2 + 50i$ and $s = 1 + 50i$. This is displayed in the following table:

s	$\xi(s)$		$\xi'(s)$	
2	0.152147		0.0385082	
$2 + 25i$	$-7.09882 \cdot 10^{-16}$	$-3.09659 \cdot 10^{-16} \cdot i$	$-1.21010 \cdot 10^{-15}$	$-2.27816 \cdot 10^{-15} \cdot i$
$2 + 50i$	$-3.07346 \cdot 10^{-32}$	$+1.26887 \cdot 10^{-32} \cdot i$	$-1.09428 \cdot 10^{-31}$	$-1.62347 \cdot 10^{-32} \cdot i$
$1 + 50i$	$2.66848 \cdot 10^{-33}$	$+5.62516 \cdot 10^{-42} \cdot i$	$2.84495 \cdot 10^{-61}$	$+1.46007 \cdot 10^{-32} \cdot i$

Now we will show some results using different amount of steps in the Simpson's rule and compare them to the built-in functions. In the following tables, the first row shows the outcome of the built-in functions, the other rows show the outcome of the use of different N's.

For $s = 2$, we have:

	$\xi(s)$	$\xi'(s)$
built-in	0.152147	0.0385082
$N = 500$	0.152147	0.0385082
$N = 1000$	0.152147	0.0385082
$N = 2000$	0.152147	0.0385082
$N = 4000$	0.152147	0.0385082

For $s = 2 + 25i$, we have:

	$\xi(s)$		$\xi'(s)$	
built-in	$-7.09882 \cdot 10^{-16}$	$-3.09659 \cdot 10^{-16} \cdot i$	$-1.21010 \cdot 10^{-15}$	$-2.27816 \cdot 10^{-15} \cdot i$
$N = 500$	$-2.08584 \cdot 10^{-8}$	$-1.57003 \cdot 10^{-8} \cdot i$	$1.83071 \cdot 10^{-8}$	$-4.01515 \cdot 10^{-8} \cdot i$
$N = 1000$	$5.99120 \cdot 10^{-9}$	$-1.78775 \cdot 10^{-8} \cdot i$	$1.82145 \cdot 10^{-8}$	$-4.23285 \cdot 10^{-8} \cdot i$
$N = 2000$	$7.64952 \cdot 10^{-9}$	$-1.80104 \cdot 10^{-8} \cdot i$	$1.82091 \cdot 10^{-8}$	$-4.24614 \cdot 10^{-8} \cdot i$
$N = 4000$	$7.75286 \cdot 10^{-9}$	$-1.80187 \cdot 10^{-8} \cdot i$	$1.82087 \cdot 10^{-8}$	$-4.24696 \cdot 10^{-8} \cdot i$

For $s = 2 + 50i$, we have:

	$\xi(s)$		$\xi'(s)$	
built-in	$-3.07346 \cdot 10^{-32}$	$+1.26887 \cdot 10^{-32} \cdot i$	$-1.09428 \cdot 10^{-31}$	$-1.62347 \cdot 10^{-32} \cdot i$
$N = 500$	$-1.08321 \cdot 10^{-7}$	$+6.53034 \cdot 10^{-9} \cdot i$	$2.56687 \cdot 10^{-8}$	$+8.68591 \cdot 10^{-9} \cdot i$
$N = 1000$	$4.15370 \cdot 10^{-9}$	$+1.74398 \cdot 10^{-9} \cdot i$	$2.55489 \cdot 10^{-8}$	$+3.90007 \cdot 10^{-9} \cdot i$
$N = 2000$	$1.08545 \cdot 10^{-8}$	$+1.47184 \cdot 10^{-9} \cdot i$	$2.55431 \cdot 10^{-8}$	$+3.62794 \cdot 10^{-9} \cdot i$
$N = 4000$	$1.12684 \cdot 10^{-8}$	$+1.45523 \cdot 10^{-9} \cdot i$	$2.55427 \cdot 10^{-8}$	$+3.61133 \cdot 10^{-9} \cdot i$

For $s = 1 + 50i$, we have:

	$\xi(s)$		$\xi'(s)$	
built-in	$2.66848 \cdot 10^{-33}$	$+5.62516 \cdot 10^{-42} \cdot i$	$2.84495 \cdot 10^{-61}$	$+1.46007 \cdot 10^{-32} \cdot i$
$N = 500$	$-1.17452 \cdot 10^{-7}$			$5.69597 \cdot 10^{-9} \cdot i$
$N = 1000$	$-4.91773 \cdot 10^{-9}$			$9.09352 \cdot 10^{-10} \cdot i$
$N = 2000$	$1.785974 \cdot 10^{-9}$			$6.37209 \cdot 10^{-10} \cdot i$
$N = 4000$	$2.19999 \cdot 10^{-9}$			$6.20599 \cdot 10^{-10} \cdot i$

In the tables above, we saw that the different N's did not change the outcome too much. Therefore, there are two possibilities left which could change the outcome too much: The approximation of $g(y)$ and the approximation of integrating $\xi(s)$ and $\xi'(s)$ from 1 to 10 instead of infinity. Therefore, we will show a table in which $g(iy/\sqrt{11})$ is computed for different y. Note that g is approximated by

$$g(\tau) \approx q \prod_{n=1}^{N} (1-q^n)^2(1-q^{11n})^2, \quad q = e^{2\pi i \tau},$$

and therefore we will also compute g for different N.

$g(iy/\sqrt{11})$	$N = 5$	$N = 10$	$N = 20$
$y = 1$	0.1028805	0.1028778	0.1028778
$y = 2$	0.0215860	0.0215860	0.0215860
$y = 5$	$7.6945577 \cdot 10^{-5}$	$7.6945577 \cdot 10^{-5}$	$7.6945577 \cdot 10^{-5}$
$y = 10$	$5.9224447 \cdot 10^{-9}$	$5.9224447 \cdot 10^{-9}$	$5.9224447 \cdot 10^{-9}$
$y = 30$	$2.0773184 \cdot 10^{-25}$	$2.0773184 \cdot 10^{-25}$	$2.0773184 \cdot 10^{-25}$

As can be seen in this table, only for $y = 1$ it matters very much how big N needs to be. If $N = 20$, $g(iy/\sqrt{11})$ does not change a lot with respect to $N = 10$. Therefore, the approximation of using $N = 20$ would not change the outcome too much.

Therefore, the only option left is the approximation of integrating $\xi(s)$ and $\xi'(s)$ from 1 to 10 instead of infinity. This is probably the reason why we got 33 instead of 36 from the contour integration. This conclusion will be used when constructing the integrals for the next subsection.

7.3.4 Using built-in integration

The integration rule which will be used in this subsection is *intnum*. This integration rule has to be implemented in the three cases where the Simpson's rule is used first. This we do by replacing the following code:

```
E1(s2)=intnum(X=1,10,E0(X,s2,1));
E2(s2)=intnum(X=1,10,E0(X,s2,0));
Total(A,N,f)={
    Area1=intnum(X=2,2+A*I,E(X)); Area2=intnum(X=2+A*I,1+A*I,E(X));
    Totarea=2*Area1+2*Area2-2*conj(Area2)-2*conj(Area1);
    Amountzero=(Totarea/(4*Pi*I));
};
```

Total(50,1000,E) gave after approximate 2 minutes the answer

$$31.95000680988360777302\ldots + 0.E - 37 * I$$

Therefore, we tried to get more accuracy by expanding the interval at which E1 and E2 are evaluating, from 1-10 to 1-20, using the following code:

```
E1(s2)=intnum(X=1,10,E0(X,s2,1))+intnum(X=10,20,E0(X,s2,1));
E2(s2)=intnum(X=10,20,E0(X,s2,0))+intnum(X=1,10,E0(X,s2,0));
```

Here we did not just change 10 in 20 but calculated the integral in two parts. This because otherwise it gave a less accurate answer. Total(50,1000,E) gave after approximate 6 minutes the answer

$$32.26236780770444717495\ldots + 0.E - 37 * I$$

Since we want to get a more accurate answer, we constructed the following functions:

```
E1(s2)=intnum(X=1,2,E0(X,s2,1))+intnum(X=2,5,E0(X,s2,1))+
        intnum(X=5,10,E0(X,s2,1))+intnum(X=10,15,E0(X,s2,1))+
        intnum(X=15,20,E0(X,s2,1))+intnum(X=20,30,E0(X,s2,1))+
        intnum(X=30,40,E0(X,s2,1))+intnum(X=40,50,E0(X,s2,1));
E2(s2)=intnum(X=1,2,E0(X,s2,0))+intnum(X=2,5,E0(X,s2,0))+
        intnum(X=5,10,E0(X,s2,0))+intnum(X=10,15,E0(X,s2,0))+
        intnum(X=15,20,E0(X,s2,0))+intnum(X=20,30,E0(X,s2,0))+
        intnum(X=30,40,E0(X,s2,0))+intnum(X=40,50,E0(X,s2,0));
g(t)=q(t)*prod(n=1,50,(1-q(n*t))^2*(1-q(11*n*t))^2)
```

Total(50,1000,E) gave after approximate 1 hour and 43 minutes the answer

$$36.06887446677726991178\ldots + 0.E - 37 * I$$

.

We cannot say too much about accuracy of this calculation. Therefore, we will show some values in the tables below which show how accurate different calculations are with respect to the "real" solutions (which come from the built-in functions of the elliptic curve).

Let us recall the definition of $g(\tau)$:

$$g(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2 \approx q\prod_{n=1}^{N}(1-q^n)^2(1-q^{11n})^2.$$

The last approximation is made, since we cannot multiply infinitely many elements numerically, so we need this approximation. With the built-in functions, we used that $N = 20$. In the last calculation, we used $N = 50$.

Right now we will look at the functions which we used for this last calculation, so with $g(\tau)$ with $N = 50$. After that, we will show what happens when $g(\tau)$ is calculated with $N = 20$. This will be done for the value $s = 2 + 50i$, where our self-made function was very inaccurate.

$N = 50$	$\xi(s)$		$\xi'(s)$	
built-in	$-3.07346 \cdot 10^{-32}$	$+1.26887 \cdot 10^{-32} \cdot i$	$-1.21010 \cdot 10^{-15}$	$-2.27816 \cdot 10^{-15} \cdot i$
intnum	$-3.07082 \cdot 10^{-32}$	$+1.26837 \cdot 10^{-32} \cdot i$	$-1.09405 \cdot 10^{-31}$	$-1.62578 \cdot 10^{-32} \cdot i$

$N = 20$	$\xi(s)$		$\xi'(s)$	
built-in	$-3.07346 \cdot 10^{-32}$	$+1.26887 \cdot 10^{-32} \cdot i$	$-1.21010 \cdot 10^{-15}$	$-2.27816 \cdot 10^{-15} \cdot i$
intnum	$2.53109 \cdot 10^{-20}$	$+6.14679 \cdot 10^{-22} \cdot i$	$-1.70350 \cdot 10^{-23}$	$+6.14579 \cdot 10^{-22} \cdot i$

These tables indeed confirm that we need to calculate the integrals of $\xi(s)$ and $\xi'(s)$ up to at least 50 to get this answer. But this also induces that we need to calculate $g(\tau)$ up to $N = 50$ instead of the 20 which we were using.

8 Conclusion and discussion

For the zeta function and the L-function of the elliptic curve with Weierstrass equation $y^2 + y = x^3 - x^2$, we have found a way to test the Riemann hypothesis for these functions numerically. The Riemann hypothesis states that the nontrivial zeros (i.e. not the zeros on the non-positive integers) lie on a complex line, $\mathrm{Re}(z) = \frac{1}{2}$ for the zeta function and $\mathrm{Re}(z) = 1$ for the L-function of the elliptic curve. In this thesis we sketched a proof of the well-known fact that there are no nontrivial zeros outside the critical strip, $0 < \mathrm{Re}(z) < 1$ for the zeta function and $\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{3}{2}$ for the L-function.

Let us now first consider the zeta function. Since the goal of this thesis was to give a method of proving the Riemann hypothesis up to some bound $|\mathrm{Im}(z)| = r$, $r \in \mathbb{R}$, we considered the function $\zeta'(z)/\zeta(z)$, i.e., the logarithmic derivative of the zeta function. This logarithmic derivative has the property that each zero of the zeta function is a pole of this function with residue equal to the order of the zero. When looking at the bound $|\mathrm{Im}(z)| = 100$, we obtain 29 zeros up to this bound on the complex line with $\mathrm{Re}(z) = \frac{1}{2}$, $\mathrm{Im}(z) > 0$, which are all simple. We have proved that whenever z_0 is a zero, also $\overline{z_0}$ is a zero. Therefore, according to the Riemann Hypothesis, there are exactly 58 nontrivial zeros with $|\mathrm{Im}(z)| < 100$.

To test this, we have made a contour with corners $2-100i$, $2+100i$, $-1+100i$ and $-1-100i$. Using the Simpson's rule, we have calculated the contour integral of the logarithmic derivative over this contour. Dividing the outcome by $2\pi i$ (and adding 1, to get rid of the pole of the zeta function at $z_0 = 1$), gives the amount of zeros inside this contour. The outcome of this contour integral was accurate enough to conclude that there are indeed no other zeros inside this contour and that indeed the given zeros are simple.

Let us now consider the L-function of the elliptic curve. This L-function is only defined for $\mathrm{Re}(z) > \frac{3}{2}$. Multiplying this function with another function gave us the following equation:

$$\xi(z) = \left(\frac{\sqrt{11}}{2\pi}\right)^z \Gamma(z)L(z) = \int_1^\infty g\left(\frac{iy}{\sqrt{N}}\right)\left(y^{1-z} + y^{z-1}\right) dy,$$

where $\gamma(z)$ is the Gamma function and $g(\tau)$ is the infinite product $q\prod(1 - q^n)^2(1 - q^{11n})^2$ with $q = e^{2\pi i\tau}$. It is known that $\xi(z)$ is analytic on the whole complex plane and that the zeros of $\xi(s)$ correspond to the nontrivial zeros of the L-function. Therefore, we have used this function to test the Riemann Hypothesis for the L-function.

When looking at the bound $|\mathrm{Im}(z)| = 50$, we obtain 36 zeros up to this bound on the line $\mathrm{Re}(z) = 1$ with $\mathrm{Im}(z) > 0$. Therefore, according to the Riemann Hypothesis, there are 72 nontrivial zeros up to this bound.

To test this, we have made a contour with corners $2 - 50i$, $2 + 50i$, $50i$ and $-50i$. Integrating the logarithmic derivative of the Xi-function over this contour, gives us the amount of zeros inside this region. As for the zeta function, also here it holds that each zero of the L-function is a pole of this function with residue equal to the order of the zero. Therefore, dividing the outcome of the contour integral by $2\pi i$ gives us the amount of zeros of this function multiplied by their multiplicity.

In this thesis, however, we saw that it was very time-consuming to do this integration numerically with sufficient precision. We have done an integration with insufficient accuracy that already took more than 3 hours. Therefore, we have decided to do not compute this integral in this thesis. Therefore, it is shown how to prove numerically that the Riemann Hypothesis holds, but for this L-function it is not done accurately enough.

To conclude this, we have shown how to prove the Riemann Hypothesis up to a certain imaginary bound. This we also did for a specific bound for the zeta function. For the L-function of the elliptic curve, we also did do this, but with insufficient accuracy to get a conclusion from that calculation.

References

[1] Joe P. Buhler, Benedict H. Gross, and Don B. Zagier. On the conjecture of Birch and Swinnerton-Dyer for an elliptic curve of rank 3. *Mathematics of Computation*, 44(170):473–481, 1985.

[2] Jasbir S. Chahal and Brian Osserman. The Riemann Hypothesis for Elliptic Curves. *The American Mathematical Monthly*, 115(5):431–442, 2008.

[3] Dr. S.R. Dahmen. *Elliptic curves, modularity, and Fermat's Last Theorem.* VU Amsterdam, 2016.
Source: `http://www.few.vu.nl/~sdn249/modularforms16/FLTetc.pdf`.

[4] Dr. J.H. Evertse. *Analytic number theory*, chapter 9: The functional equation for the Riemann zeta function. University of Leiden, 2013.
Source: http://www.math.leidenuniv.nl/ evertse/ant13-9.pdf.

[5] Dr. J.H. Evertse. *Analytic number theory*, chapter 5: The Riemann zeta function and L-functions. University of Leiden, 2013.
Source: http://www.math.leidenuniv.nl/ evertse/ant13-5.pdf.

[6] Dr. J.H. Evertse. *Analytic number theory*, chapter 8: Euler's Gamma function. University of Leiden, 2014.
Source: http://www.math.leidenuniv.nl/ evertse/ant14-8.pdf.

[7] Dr. J.H. Evertse. *Analytic number theory*, chapter 2: Complex analysis. University of Leiden, 2014.
Source: http://www.math.leidenuniv.nl/ evertse/ant14-2.pdf.

[8] K.P. Hart. Complex power series: The complex logarithm. Source: `https://fa.its.tudelft.nl/~hart/37/onderwijs/weblecture/handout.pdf`.

[9] Clay Mathematics Institute. Millennium Problems. Source: `https://www.claymath.org/millennium-problems`, 2020. Accessed: 2020-05-18.

[10] Anthony W Knapp. *Elliptic Curves*, volume 40 of *Mathematical Notes*. Princeton University Press, 2018.

[11] Álvaro Lozano-Robledo and Álvaro Lozano-Robledo. *Elliptic curves, modular forms, and their L-functions*, chapter 5.1. The L-function of an elliptic curve. American Mathematical Society Providence, RI, 2011.

[12] Robert van der Meer. "Zeros of the Zeta Function". Bachelor's thesis, University of Groningen, 2020. `http://fse.studenttheses.ub.rug.nl/21726/`.

[13] Melissa N. Miller. "The simple zeros of the Riemann zeta-function". Honors thesis, University of Tennessee, Chattanooga, 2016.

[14] Prof. A. Ron. Lecture 19: Numerical Integration II. Scource: `http://pages.cs.wisc.edu/~amos/412/lecture-notes/lecture19.pdf`, 2010.

[15] Jean Pierre Serre. *A Course in Arithmetic*, chapter 6.3. Springer-Verlag, 1973.

[16] Joseph H. Silverman. *The Arithmetic of Elliptic Curves*, chapter 3: The geometry of elliptic curves. Springer-Verlag, New York, 2009.

[17] Mirjam Soeten. "Hasse's Theorem on Elliptic Curves". Master's thesis, University of Groningen, 2013. `http://fse.studenttheses.ub.rug.nl/10999/`.

[18] J. Veisdal. 'Prime numbers and the Riemann zeta function'. Source: `https://www.dropbox.com/s/iychwnb6qge1aze/thesis13.pdf?dl=0`, 2013.

[19] M. Waldschmidt et al. *From Number Theory to Physics*, chapter 1: An Introduction to Zeta Functions. Springer-Verlag Berlin Heidelberg, 1995.

[20] Lawrence C. Washington. *Elliptic curves: number theory and cryptography.* CRC press, 2008.

[21] Erik W. Weisstein. "Riemann Zeta Function Zeros". From MathWorld–A Wolfram Web Resource. `https://mathworld.wolfram.com/RiemannZetaFunctionZeros.html`, Accessed: 2020-06-04.

[22] T. Weston. The modular curves $X_0(11)$ and $X_1(11)$, 2001. `http://swc.math.arizona.edu/aws/2001/01Weston1.pdf`, Accessed: 2020-05-27.

A Error analysis of functions from section 5

The error analysis of different functions

Here we will first show that the maximum of the fourth derivative of the logarithmic derivative is 14. after that, the errors of the functions B1 and B2, used in the self-made functions, will be analyzed.

Logarithmic derivative of zeta function

The logarithmic derivative is given by

$$\frac{\zeta'(z)}{\zeta(z)}$$

From now on, we will denote $\zeta(z)$ as Z. The first derivative of this function is given by

$$\frac{Z \cdot Z'' - (Z')^2}{Z^2} = \frac{Z''}{Z} - \left(\frac{Z'}{Z}\right)^2$$

The second derivative of this function is given by

$$\frac{Z \cdot Z''' - Z'' \cdot Z'}{Z^2} - 2 \cdot \frac{Z'}{Z} \cdot \left(\frac{Z''}{Z} - \left(\frac{Z'}{Z}\right)^2\right) = \frac{Z'''}{Z} - 3\frac{Z'' \cdot Z'}{Z^2} + 2\left(\frac{Z'}{Z}\right)^3$$

The third derivative of this function is given by

$$\frac{Z''''}{Z} - \frac{Z''' \cdot Z'}{Z^2} - 3\frac{Z^2(Z''' \cdot Z' + (Z'')^2) - 2Z \cdot Z''(Z')^2}{Z^4} + 6\left(\frac{Z'}{Z}\right)^2\left(\frac{Z''}{Z} - \left(\frac{Z'}{Z}\right)^2\right) =$$

$$\frac{Z''''}{Z} - 4\frac{Z''' \cdot Z'}{Z^2} - 3\frac{(Z'')^2}{Z^2} + 12\frac{Z''(Z')^2}{Z^3} - 6\left(\frac{Z'}{Z}\right)^4$$

The fourth derivative of this function is given by

$$\frac{Z^{(5)}}{Z} - 5\frac{Z''''Z'}{Z^2} - 10\frac{Z'''Z''}{Z^2} + 20\frac{Z'''(Z')^2}{Z^3} + 30\frac{Z'(Z'')^2}{Z^3} - 60\frac{Z''(Z')^3}{Z^4} + 24\left(\frac{Z'}{Z}\right)^5$$

The maximum of this function is less than M=14 for the whole contour.

The error of B1

The third Bernoulli polynomial is given by

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$$

Therefore,

$$\int_1^\infty \overline{B_3}(x)x^{-s-3}dx = \sum_{n=1}^\infty \int_0^1 B_3(x)(x+n)^{-s-3} = \sum_{n=1}^\infty \int_0^1 (x^3 - \frac{3}{2}x^2 + \frac{1}{2}x)(x+n)^{-s-3}$$

We will analyze this last integral to say something about the first integral. First of all, let $P(x) = (x^3 - \frac{3}{2}x^2 + \frac{1}{2}x)$ and $Q(x) = (x+n)^{-s-3}$. Let us consider $P(x)Q(x)$

The first derivative is given by

$$P'Q + PQ'$$

The second derivative is given by

$$P''Q + P'Q' + PQ'' + P'Q' = P''Q + 2P'Q' + PQ''$$

The third derivative is given by

$$P'''Q+P''Q'+2P''Q'+2P'Q''+P'Q''+PQ''' = P'''Q+3P''Q'+3P'Q''+PQ'''$$

The fourth derivative is given by

$$\begin{aligned}&P''''Q + P'''Q' + 3P'''Q' + 3P''Q'' + 3P''Q'' + 3P'Q''' + P'Q''' + PQ'''' =\\ &P''''Q + 4P'''Q' + 6P''Q'' + 4P'Q''' + PQ''''\end{aligned}$$

Rewriting P and Q and their derivatives to the actual functions gives us the following result: the fourth derivative is given by

$$\begin{aligned}&-24(s+3)(x+n)^{-s-4} + 6(6x-3)(s+3)(s+4)(x+n)^{-s-5}\\ &\quad - 4(3x^2 - 3x + 1/2)(s+3)(s+4)(s+5)(x+n)^{-s-6}\\ &\quad + (x^3 - \frac{3}{2}x^2 + \frac{1}{2}x)(s+3)(s+4)(s+5)(s+6)(x+n)^{-s-7}\\ =&(s+3)(x+n)^{-s-4}[-24 + 6(6x-3)(s+4)(x+n)^{-1}\\ &\quad - 4(3x^2 - 3x + 1/2)(s+4)(s+5)(x+n)^{-2}\\ &\quad + (x^3 - \frac{3}{2}x^2 + \frac{1}{2}x)(s+4)(s+5)(s+6)(x+n)^{-3}]\\ =&(s+3)(x+n)^{-s-4}[-24 + (s+4)(x+n)^{-1}[6(6x-3) + (s+5)(x+n)^{-1}\cdot\\ &\quad [-4(3x^2 - 3x + 1/2) + (s+6)(x+n)^{-1}(x^3 - \frac{3}{2}x^2 + \frac{1}{2}x)]]]\end{aligned}$$

The maximum value of this derivative in the interval $x \in [0,1]$ is at $x = 0$. Putting this into the formula gives us

$$(s+3)(n)^{-s-4}[-24 + (s+4)(n)^{-1}[-18 - 2(s+5)(n)^{-1}]]$$

For n=1, the maximum value is at $s = 2 + 100i$, where M=2029270. Putting this into the error formula gives us

$$\frac{(1/300)^5}{2880} \cdot 1 \cdot 2029270 \approx 2.8996199 \cdot 10^{-10}$$

Since the maximum is decreasing very fast as n increases, this also is approximate an error bound for the whole integral.

The error of B2

Recall from the previous paragraph that the fourth derivative of $P(x)Q(x)$ is

$$P''''Q + 4P'''Q' + 6P''Q'' + 4P'Q''' + PQ''''$$

. Therefore, the fourth derivative of $P(x)Q(x)R(x)$ is given by

$$(PQ)''''R + 4(PQ)'''R' + 6(PQ)''R'' + 4(PQ)'R''' + (PQ)R''''$$

Implementing the derivatives of PQ gives us the following derivative:

$$\begin{aligned}
&(PQ)''''R + 4(PQ)'''R' + 6(PQ)''R'' + 4(PQ)'R''' + (PQ)R'''' = \\
&(P''''Q + 4P'''Q' + 6P''Q'' + 4P'Q''' + PQ'''')R + \\
&4(P'''Q + 3P''Q' + 3P'Q'' + PQ''')R' + 6(P''Q + 2P'Q' + PQ'')R'' + \\
&4(P'Q + PQ')R''' + PQR''''
\end{aligned}$$

Let $P(x) = (x^3 - \frac{3}{2}x^2 + \frac{1}{2}x)$, $Q(x) = (x+n)^{-s-3}$ and $R(x) = log(x)$. Therefore, the derivatives of these functions are

$$\begin{aligned}
&P'(x) = 3x^2 - 3x + 1/2 \\
&P''(x) = 6x - 3 \\
&P'''(x) = 6 \\
&P''''(x) = 0 \\
&Q'(x) = -(s+3)(x+n)^{-s-4} \\
&Q''(x) = (s+3)(s+4)(x+n)^{-s-5} \\
&Q'''(x) = -(s+3)(s+4)(s+5)(x+n)^{-s-6} \\
&Q''''(x) = (s+3)(s+4)(s+5)(s+6)(x+n)^{-s-7} \\
&R'(x) = x^{-1} \\
&R''(x) = -x^{-2} \\
&R'''(x) = 2x^{-3} \\
&R''''(x) = -6x^{-4}
\end{aligned}$$

Implementing in GP/Pari gives the following:

```
P0(x)=x^3-3/2*x^2+x/2;
P1(x)=3*x^2-3*x+1/2;
P2(x)=6*x-3;
P3(x)=6;
P4(x)=0;
Q0(x)=(x+n)^{-s-3};
Q1(x)=-(s+3)*(x+n)^{-s-4};
Q2(x)=(s+3)*(s+4)*(x+n)^{-s-5};
```

```
Q3(x)=-(s+3)*(s+4)*(s+5)*(x+n)^{-s-6};
Q4(x)=(s+3)*(s+4)*(s+5)*(s+6)*(x+n)^{-s-7};
R0(x)=log(x);
R1(x)=1/x;
R2(x)=-1/x^2;
R3(x)=2/x^3;
R4(x)=-6/x^4;
```

Using these functions to create the fourth derivative, gives us the following equation for this derivative. Note that there are some splits in the code to fit the page:

```
DB2(x)=(P4(x)*Q0(x)+4*P3(x)*Q1(x)+6*P2(x)*Q2(x)
+4*P1(x)*Q3(x)+P0(x)*Q4(x))*R0(x)+4*(P3(x)*Q0(x)
+3*P2(x)*Q1(x)+3*P1(x)*Q2(x)+P0(x)*Q3(x))*R1(x)
+6*(P2(x)*Q0(x)+2*P1(x)*Q1(x)+P0(x)*Q2(x))*R2(x)
+4*(P1(x)*Q0(x)+P0(x)*Q1(x))*R3(x)+P0(x)*Q0(x)*R4(x)
```

The maximum of this derivative is approximate 26.000.000. Implementing this into the error formula, gives us

$$\frac{(1/300)^5}{2880} \cdot 300 \cdot 26000000 \approx 1.115 \cdot 10^{-6}$$

B Error analysis of functions from section 7

Error of Simpsons rule on the contour integral Changing Z to X in the formula for the logarithmic derivative of the zeta function gives us

$$\frac{X^{(5)}}{X}-5\frac{X''''X'}{X^2}-10\frac{X'''X''}{X^2}+20\frac{X'''(X')^2}{X^3}+30\frac{X'(X'')^2}{X^3}-60\frac{X''(X')^3}{X^4}+24\left(\frac{X'}{X}\right)^5$$

If we denote the n'th prime of X by Xn(s), we get the following functions:

```
X0(s)=intnum(y=1,100,eta(I*y/sqrt(11),1)^2*
    eta(11*I*y/sqrt(11),1)^2*(y^(s-1)+y^(1-s)));
X1(s)=intnum(y=1,100,eta(I*y/sqrt(11),1)^2*
    eta(11*I*y/sqrt(11),1)^2*log(y)*(y^(s-1)-y^(1-s)));
X2(s)=intnum(y=1,100,eta(I*y/sqrt(11),1)^2*
    eta(11*I*y/sqrt(11),1)^2*log(y)^2*(y^(s-1)+y^(1-s)));
X3(s)=intnum(y=1,100,eta(I*y/sqrt(11),1)^2*
    eta(11*I*y/sqrt(11),1)^2*log(y)^3*(y^(s-1)-y^(1-s)));
X4(s)=intnum(y=1,100,eta(I*y/sqrt(11),1)^2*
    eta(11*I*y/sqrt(11),1)^2*log(y)^4*(y^(s-1)+y^(1-s)));
X5(s)=intnum(y=1,100,eta(I*y/sqrt(11),1)^2*
    eta(11*I*y/sqrt(11),1)^2*log(y)^5*(y^(s-1)-y^(1-s)));
Test(s)={
    XX0=X0(s);XX1=X1(s);XX2=X2(s);XX3=X3(s);XX4=X4(s);XX5=X5(s);
    Total=XX5/XX0-5*XX4*XX1/XX0^2-10*XX3*XX2/XX0^2+20*XX3*XX1^2/
        XX0^3+30*XX1*XX2^2/XX0^3-60*XX2*XX1^3/XX0^4+24*(XX1/XX0)^5
};
```

Note that there are several splits in this code.

Using this code, we obtain that M=23300000. This is an upper bound for the maximum value of the fourth derivative, which is around 2+23i. Putting this into the error formula gives us, for $N = 1000$ and hence $h = 0.05$ for the first and $h = 0.001$ for the second part,

$$\frac{0.05^5}{2880}\cdot 50\cdot 23300000 \approx 0.13$$

$$\frac{0.001^5}{2880}\cdot 1\cdot 23300000 \approx 8.09\cdot 10^{-12}$$

Therefore, the total error of the Simpson's rule on the contour integral is $0.13\cdot 2 + 8.09\cdot 10^{-12} \approx 0.26$

Error of the infinite product $g(\tau)$ There are also other approximations. $g(\tau)$ is approximated by taking the product of $n = 1$ up to $n = 50$ instead of infinity. This approximation is accurate up to 41 digits for $y = 1$. For larger y, this will be much more accurate (for example at $y = 2$, we already have an accuracy of 80 digits).

Errors of the integrals of $\xi(s)$ and $\xi'(s)$ Also the integrals of $\xi(s)$ and $\xi'(s)$ are approximated. First of all, they are computed with the Simpson's rule. Secondly, they are computed from $y = 1$ up to $y = 20$ instead of infinity.

The error of $\xi'(s)$, made by the Simpson's rule, can be calculated on the following way:

From what we did for the error for B2 in Appendix A, we have the formula for the fourth derivative of a function $P(y)Q(y)R(y)$:

$$\begin{aligned}
&(PQR)'''' = \\
&(P''''Q + 4P'''Q' + 6P''Q'' + 4P'Q''' + PQ'''')R+ \\
&4(P'''Q + 3P''Q' + 3P'Q'' + PQ''')R' + 6(P''Q + 2P'Q' + PQ'')R''+ \\
&4(P'Q + PQ')R''' + PQR''''
\end{aligned}$$

When $P(y) = y^{s-1} + y^{1-s}$, $Q(y) = g(iy/\sqrt{11})$ and $R(y) = \ln(y)$, we get the fourth derivative of the terms inside the integral of $\xi'(s)$. Therefore, calculating the derivatives of these functions and putting this in GP/Pari gives us the following code:

```
g1(t)=g'(t)
g2(t)=g''(t)
g3(t)=g'''(t)
g4(t)=g''''(t)
P0(y,s)=y^(s-1)+y^(1-s)
P1(y,s)=(s-1)*y^(s-2)+(s-1)*y^(-s)
P2(y,s)=(s-2)*(s-1)*y^(s-3)+(s-1)*s*y^(-s-1)
P3(y,s)=(s-3)*(s-2)*(s-1)*y^(s-4)+(s-1)*s*(s+1)*y^(-s-2)
P4(y,s)=(s-4)*(s-3)*(s-2)*(s-1)*y^(s-5)+(s-1)*s*
    (s+1)*(s+2)*y^(-s-3)
R0(y)=log(y);
R1(y)=1/y;
R2(y)=-1/y^2;
R3(y)=2/y^3;
R4(y)=-6/y^4;

DL0(y,s)={
    gg0=g(y);gg1=g1(y);gg2=g2(y);gg3=g3(y);gg4=g4(y);
    PP0=P0(y,s);PP1=P1(y,s);PP2=P2(y,s);PP3=P3(y,s);
    PP4=P4(y,s);RR0=R0(y);RR1=R1(y);RR2=R2(y);RR3=R3(y);
    RR4=R4(y);
    Total=(PP4*gg0+4*PP3*gg1+6*PP2*gg2+4*PP1*gg3+PP0*gg4)*RR0
    +4*(PP3*gg0+3*PP2*gg1+3*PP1*gg2+PP0*gg3)*RR1
    +6*(PP2*gg0+2*PP1*gg1+PP0*gg2)*RR2
    +4*(PP1*gg0+PP0*gg1)*RR3+PP0*gg0*RR4
}
```

Note that there are several splits in this code.

The maximum value is at $y = 1$ and $s = 50 * I$, which can be bounded by $M = 103500$. Putting this into the error formula gives us the following equation:

$$\frac{0,009^5}{2880} \cdot 19 \cdot 103500 \approx 6.83 \cdot 10^{-8}.$$

Since for larger y the solution decreases exponential, the approximation of not using $y > 10$ seems to do not affect too much the solution.

This same method can be done for the integral of $\xi(s)$, but then changing $R(y)$ to $R(y) = 1$.

The maximum value of this function is at $y = 1$ and $s = 50 * I$, which can be bounded by $M = 1293000$. Putting this into the error formula gives us the following equation:

$$\frac{0,009^5}{2880} \cdot 19 \cdot 1293000 \approx 8.53 \cdot 10^{-7}$$

Also for this equation, it holds that for larger y the solution decreases exponential. Therefore, also here the approximation of not using the part of the integral with $y > 10$ seems to do not affect too much the solution.

Hence it seems that we have a very accurate solution, but $|\xi(2+50i)| \approx 10^{-32}$ and $|\xi'(2 + 50i)| \approx 10^{-31}$. Therefore, the solution is too inaccurate to use this for our purpose.

To get this accurate enough (up to at least 10^{-34}, h needs to be at least $2.6 \cdot 10^{-8}$. Also the values of the integrals for $y > 10$ do affect the solution, since up to approximate $y = 40$, the values of the integrals are still above 10^{-32}. This implies that there needs to be a huge computer effort to get so accurate. Therefore, this will not be done in this thesis.

C The first 32 a_n's of the elliptic curve

In the table below the first 32 a_n's of the elliptic curve with weierstrass equation $y^2 + y = x^3 - x^2$ are shown. For $n = 1$ or $n = p$ with p prime, the a_n's are just given. For the other a_n's, also the calculation is given.

a_1	$=$	1	a_{17}	$=$	-2
a_2	$=$	-2	a_{18}	$=$	$a_2a_9 = 4$
a_3	$=$	-1	a_{19}	$=$	0
a_4	$=$	$a_2^2 - 2 = 2$	a_{20}	$=$	$a_4a_5 = 2$
a_5	$=$	1	a_{21}	$=$	$a_3a_7 = 2$
a_6	$=$	$a_2a_3 = 2$	a_{22}	$=$	$a_2a_{11} = -2$
a_7	$=$	-2	a_{23}	$=$	-1
a_8	$=$	$a_2a_4 - 2 \cdot a_2 = 0$	a_{24}	$=$	$a_3a_8 = 0$
a_9	$=$	$a_3^2 - 3 = -2$	a_{25}	$=$	$a_5^2 - 5 = -4$
a_{10}	$=$	$a_2a_5 = -2$	a_{26}	$=$	$a_2a + 13 = -8$
a_{11}	$=$	1	a_{27}	$=$	$a_3a_9 - 3 \cdot a_3 = 5$
a_{12}	$=$	$a_4a_3 = -2$	a_{28}	$=$	$a_4a_7 = -4$
a_{13}	$=$	4	a_{29}	$=$	0
a_{14}	$=$	$a_2a_7 = 4$	a_{30}	$=$	$a_2a_3a_5 = 2$
a_{15}	$=$	$a_3a_5 = -1$	a_{31}	$=$	7
a_{16}	$=$	$a_2a_8 - 2 \cdot a_4 = -4$	a_{32}	$=$	$a_2a_{16} - 2 \cdot a_8 = 8$