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Spectral almost-Riemannian geometry and the magnetic Aharonov-Bohm effect

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Abstract

In this thesis, we study the magnetostatic Aharonov-Bohm effect when charged particles are constrained to move on several spaces. To do so, we find explicit descriptions of the spectrum and eigenfunctions of a generalized Laplace-Beltrami operator which admits a vector potential A (in general, a one-form) on several two-dimensional almost-Riemannian manifolds (a generalization of Riemannian manifolds). We study three examples, namely the punctured plane and the unit cylinder both with Euclidean metric (these are in fact Riemannian manifolds) and finally the Grushin cylinder. We find in the case of the Grushin cylinder that the spectrum is extremely sensitive to changes in the magnetic flux, in contrast to the Euclidean cases.

We also discuss several modifications of this effect, including the addition of relativistic effects (from a quantum field theoretical perspective) and the addition of magnetic monopoles.

Contents

1 Introduction

The Aharonov-Bohm (AB) effect is an electromagnetic phenomenon in quantum mechanics named after Yakir Aharonov and David Bohm. In their paper, they showed that the wave function of a charged particle can be affected purely by its potentials (in the sense that an additional phase is acquired) without any electric or magnetic field being present. This was rather counter-intuitive at the time, since it was thought that only the electric and magnetic fields are physical quantities and that the scalar and vector potentials are just a mathematical construction used to simplify certain computations [1]. Their paper was thus met with skepticism by many physicists. However, the effect has been experimentally confirmed multiple times. There are two versions of this effect, one where the wave function depends on the scalar potential (the electric AB effect) and the other where the wave function depends on the vector potential (the magnetic AB effect). We will focus on the latter case, as the former case's existence is controversial (for example, see [2]). The first such experiment was based on Aharonov and Bohm's proposed experiment [3]. Their proposed setup is given in Figure 1.1. It involves splitting an electron beam at point A so that some electrons travel clockwise and others anticlockwise around the solenoid. The beams will then recombine at point F and form an interference pattern as predicted by Aharonov and Bohm in 1959 (since the phase induced by the potential is path-dependent). The experiment was first performed a year later in 1960 by Robert G. Chambers and he proved that the effect indeed exists [1].

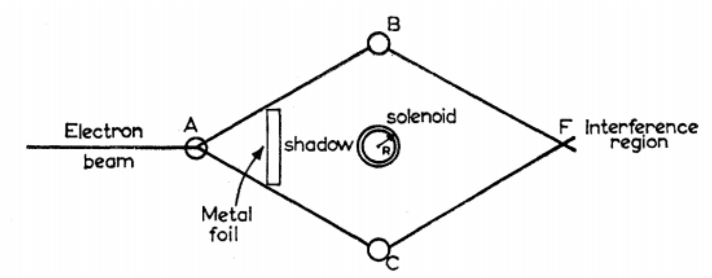

Figure 1.1: Schematic experiment to demonstrate interference with vector potential [4]

The magnetic AB effect is mathematically well-understood in the Euclidean case. However, until recently there has been little research on this effect on more abstract spaces. In this thesis, we will discuss the AB effect on several finite-dimensional almost-Riemannian manifolds. They are a generalization of Riemannian manifolds which have many well-studied applications in physics [5]. Even though the most common examples in physics are Riemannian (e.g., $\mathbb{R}^n$ with the standard dot product), almost-Riemannian manifolds have already appeared in many physical situations. For example, in classical mechanics they can be used to study orbital mechanics [6] and they play a central role in control theory [7]. Their application in quantum mechanics is rather recent: an interesting example of this is their relevance in understanding the Berry phase (also known as the geometric phase), which is especially relevant to the AB effect and the control of quantum mechanical systems [5]. We will focus on one abstract almost-Riemannian manifold called the Grushin cylinder.

We will find the spectrum of the Laplace-Beltrami (LB) operator (which is a generalization of the Laplacian) on various two-dimensional almost-Riemannian manifolds and we will analyse its relevance to the magnetic AB effect. One does this by defining a slightly more general operator, called the magnetic LB operator, which reduces to the standard LB operator in the case that the vector potential vanishes [8]. In addition to the Grushin cylinder mentioned above, we will focus on the Euclidean case and compute the spectrum on both the plane and the cylinder. This will be useful from a physical point of view, as from the magnetic LB operator, we can determine eigenvalues and eigenstates of the Hamiltonian associated with the AB effect. Finally, we will discuss the similarities and differences between the effect in the Euclidean cases and the Grushin cylinder.

This thesis is structured as follows: Section 2 contains an introduction of the AB effect in $\mathbb{R}^3$ and demonstrates that the interference pattern should indeed appear from a theoretical point of view. We also find in that section that the effect is more topological than geometrical in nature. Section 3 introduces the preliminary differential geometry concepts and defines almost-Riemannian manifolds and the Grushin cylinder. Section 4 contains a light introduction to spectral theory and also a description of the spectrum of the LB operator for each almost-Riemannian manifold. Section 5 contains some extensions and similar effects to the standard non-relativistic magnetic AB effect with an infinite solenoid (such as the Aharonov-Casher effect, which is dual to the AB effect). A comparison of the results is contained in Section 6. Finally, Section 7 contains the concluding remarks.

We use SI units throughout this thesis.

2 The Aharonov-Bohm effect

In the following, we explain the magnetic Aharonov-Bohm effect in the most commonly discussed three-dimensional Euclidean space where there are no relativistic effects. The extension to more abstract spaces will then be clear.

2.1 Constructing the Hamiltonian

We first start by deriving the electromagnetic Hamiltonian for the classical case, then we use the correspondence principle to derive the quantum mechanical Hamiltonian. Consider the motion of a particle of charge q and mass m at position $\mathbf{r}$ at time t moving through an electromagnetic field with electric scalar potential $V = V(\mathbf{r}, t)$ and magnetic vector potential $\mathbf{A} = \mathbf{A}(\mathbf{r}, t)$. In classical mechanics, the motion of the particle is given by the Lorentz force law

$$m\ddot{\mathbf{r}} = q\left(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}\right), \tag{2.1}$$

where the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ can be expressed in terms of the potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{2.2}$$

We now make a guess for a Lagrangian which will produce the Lorentz force law (2.1). We make the guess

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 - qV + q\dot{\mathbf{r}} \cdot \mathbf{A}. \tag{2.3}$$

We now show that this indeed is the correct Lagrangian. We find that

$$\frac{\partial L}{\partial \mathbf{r}} = -q\nabla V + q\nabla(\dot{\mathbf{r}} \cdot \mathbf{A}), \quad \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + q\mathbf{A}.$$

Thus, by the Euler-Lagrange equations, the equations of motion are

$$m\ddot{\mathbf{r}} = q\left(\nabla(\dot{\mathbf{r}} \cdot \mathbf{A}) - \dot{\mathbf{A}} - \nabla V\right).$$

In components, this is (we use the Einstein summation convention here)

$$\begin{aligned} m\ddot{r}^i &= q\left(\frac{\partial A_j}{\partial r^i}\dot{r}^j - \frac{\partial A_i}{\partial t} - \frac{\partial A_i}{\partial r^j}\dot{r}^j - \frac{\partial V}{\partial r^i}\right) \\ &= q\left(\left(-\frac{\partial V}{\partial r^i} - \frac{\partial A_i}{\partial t}\right) + \left(\frac{\partial A_j}{\partial r^i} - \frac{\partial A_i}{\partial r^j}\right)\dot{r}^j\right), \qquad i, j \in \{1, 2, 3\}, \end{aligned}$$

which can be simplified using the Levi-Civita symbol ϵ_{ijk} defined to be

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3), \\ -1, & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3), \\ 0, & \text{otherwise.} \end{cases}$$

This gives us the equality

$$\epsilon_{ijk}B_k = \epsilon_{ijk}\epsilon_{klm}\frac{\partial A_m}{\partial r^l} = \epsilon_{kij}\epsilon_{klm}\frac{\partial A_m}{\partial r^l} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial A_m}{\partial r^l} = \frac{\partial A_j}{\partial r^i} - \frac{\partial A_i}{\partial r^j},$$

which allows us to obtain the following equation of motion in components

$$m\ddot{r}^i = q\left(E_i + \epsilon_{ijk}\dot{r}^j B_k\right), \qquad i, j, k \in \{1, 2, 3\},$$

which agrees with equation (2.1) component-wise [9]. Thus, the Lagrangian in equation (2.3) indeed describes the motion of the particle. We derive the Hamiltonian by identifying the conjugate momentum $\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + q\mathbf{A}$, and taking a Legendre transform

$$H = \dot{\mathbf{r}} \cdot \mathbf{p} - L = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + qV.$$

Note that if one wishes to add additional interactions such as a Coulomb interaction, one just simply adds the corresponding potential to the Hamiltonian. The transition to quantum mechanics is made by using the quantum mechanical substitution $\mathbf{p} \to -i\hbar\nabla$ [10], which results in the following Hamiltonian operator on $L^2(\mathbb{R}^3)$:

$$\mathcal{H} = -\frac{\hbar^2}{2m}\left(\nabla - \frac{iq}{\hbar}\mathbf{A}\right)^2 + qV.$$

As one can see, the Hamiltonian operator has explicit dependence on the potentials and not on the fields. This motivates the Aharonov-Bohm effect. From here onwards, we discuss the magnetic version of the effect. Therefore, we assume that $\mathbf{E} = 0$ and we assume there are no additional interactions. Due to gauge freedom of the potentials ($\mathbf{A} \to \mathbf{A} + \nabla\chi$ and $V \to V - \frac{\partial\chi}{\partial t}$, where χ is a scalar function [11]), we may also assume $V = 0$. Hence in the following, the time-dependent Schrödinger equation we consider is

$$i\hbar\frac{\partial\psi}{\partial t} = \mathcal{H}\psi = -\frac{\hbar^2}{2m}\left(\nabla - \frac{iq}{\hbar}\mathbf{A}\right)^2\psi. \tag{2.4}$$

2.2 Deriving the phase shift

Consider a static electromagnetic field where $\mathbf{A} \neq 0$ and $\mathbf{B} = 0$ on a simply connected domain of $\mathbb{R}^3$ (for an explanation on simply connectedness, see Appendix A). We then claim that the solution to the Schrödinger equation in equation (2.4) is

$$\psi(\mathbf{r}, t) = e^{ig(\mathbf{r})}\psi_0(\mathbf{r}, t), \quad g(\mathbf{r}) = \frac{q}{\hbar}\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}', \tag{2.5}$$

where $\mathcal{O}$ is some arbitrarily chosen reference point in that domain (due to gauge freedom of the potentials) and ψ_0 is the solution to the Schrödinger equation where $\mathbf{A} = 0$ [12]. Note that the line integral is well-defined since $\mathbf{B} = \nabla \times \mathbf{A} = 0$ implies that $\mathbf{A}$ is irrotational, which in simply connected domains implies that $\mathbf{A}$ is conservative, so that the line integral is path independent.

Proof. Since $\nabla g = (q/\hbar)\mathbf{A}$,

$$\nabla\psi = e^{ig}(i\nabla g)\psi_0 + e^{ig}(\nabla\psi_0) = \frac{iq}{\hbar}e^{ig}\psi_0\mathbf{A} + e^{ig}(\nabla\psi_0).$$

Thus, it follows that

$$\begin{aligned}
\left(\nabla - \frac{iq}{\hbar}\mathbf{A}\right)^2\psi &= \left(\nabla - \frac{iq}{\hbar}\mathbf{A}\right) \cdot \left(e^{ig}\nabla\psi_0\right) \\
&= \nabla \cdot \left(e^{ig}\left(\nabla\psi_0\right)\right) - \frac{iq}{\hbar}e^{ig}\mathbf{A} \cdot (\nabla\psi_0) \\
&= (\nabla e^{ig}) \cdot (\nabla\psi_0) + e^{ig}\nabla^2\psi_0 - \frac{iq}{\hbar}e^{ig}\mathbf{A} \cdot (\nabla\psi_0) \\
&= e^{ig}\nabla^2\psi_0.
\end{aligned}$$

By plugging this into equation (2.5) and multiplying both sides by e^{-ig}, we obtain

$$i\hbar\frac{\partial\psi_0}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_0,$$

which is precisely equation (2.4) but with $\mathbf{A} = 0$, as required [13]. $\square$

The above result implies that in order to solve the Schrödinger equation, it suffices to determine the eigenstates in the case where $\mathbf{A} = 0$, then compute the phase factor g and multiply the eigenstates by e^{ig}. Note that we use the letter g to denote the phase factor, as it is a particular instance of geometric phase (also known as Berry's phase) [13].

Now consider a more general situation where the domain $D \subseteq \mathbb{R}^3$ such that $\mathbf{B} = 0$ is not necessarily simply connected. Furthermore, consider two charged particles taking two different paths C_1 and C_2 on D with the same start and end points such that there exists simply connected domains $D_i \subseteq D$ containing C_i for all $i \in \{1, 2\}$. The difference in phase is then given by (using Stokes' theorem)

$$\Delta g = g_1 - g_2 = \frac{q}{\hbar}\left[\int_{C_1} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}' - \int_{C_2} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'\right] = \frac{q}{\hbar}\oint \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}' = \frac{q\Phi_B}{\hbar}, \tag{2.6}$$

where Φ_B is the magnetic flux through the area between the paths [1]. Note that for the change of phase to be non-zero, it is necessary that D is not simply connected and that C_1 and C_2 are not homotopic on D (intuitively, this means that C_1 and C_2 cannot be continuously deformed into each other, without leaving D. For a more rigorous definition see Appendix A) [10]. Hence, just by changing the vector potential, one may affect a charged particle by a change of phase. This is precisely the magnetic Aharonov-Bohm effect. Note that this result is gauge invariant, as the line integral of a gradient of a scalar function over a closed loop is zero (this is a corollary of the fundamental theorem of line integrals).

A more concrete example is given in Figure 2.1, where we have an infinitely long solenoid along the $\hat{\mathbf{z}}$ axis and a charged particle moving outside this solenoid.

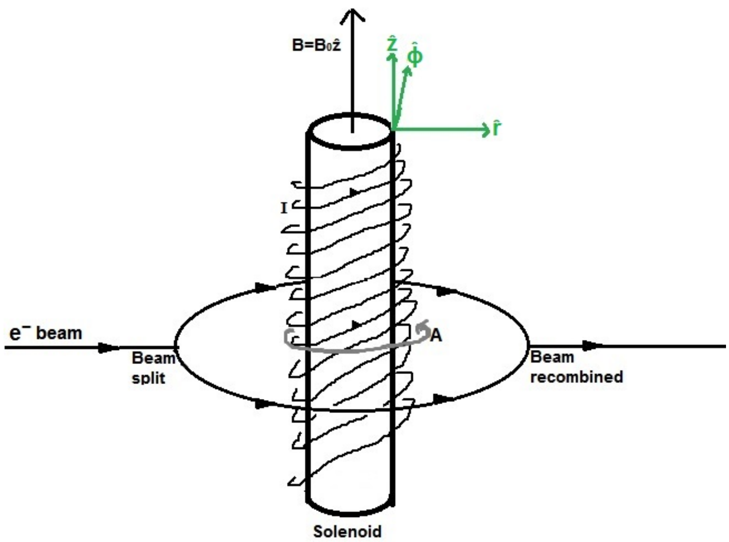

Figure 2.1: The electron beam splits, with half passing on either side of an infinitely long solenoid [13]

Using cylindrical coordinates $(r, \phi, z) \in [0, \infty) \times [0, 2\pi) \times \mathbb{R}$, the solenoid of radius $a > 0$ has constant non-zero magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$ inside the solenoid ($r \leq a$), and $\mathbf{B} = 0$ otherwise ($r > a$). Using Stokes' theorem, one obtains that the vector potential for $r > a$ is given by (where we adopt the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$)

$$\mathbf{A} = \frac{a^2 B_0}{2r}\hat{\boldsymbol{\phi}}. \tag{2.7}$$

With the notation of the previous paragraph, the region D is not simply connected. One lets C_1 and C_2 be the paths where electrons pass clockwise and anticlockwise around the solenoid respectively. The phase difference is then non-zero, namely $q\pi a^2 B_0/\hbar \neq 0$ [13]. This concrete example is the most standard way of observing the effect experimentally (the differences in phase where the beam recombines will cause an interference pattern) and it is in accordance with the experiment proposed by Aharonov and Bohm in Figure 1.1. In principle, as this phase difference is very small, the Aharonov-Bohm effect can be used to measure small differences in magnetic flux. While it is not so relevant for our analysis, one may similarly show that the vector potential inside the solenoid ($r \leq a$) with Coulomb gauge is given by [14]

$$\mathbf{A} = \frac{B_0 r}{2}\hat{\boldsymbol{\phi}}.$$

3 Differential geometry

In the following section, we introduce the preliminary differential geometry to be able to compute the spectra of the magnetic Laplace-Beltrami operator on several almost-Riemannian manifolds. In Sections 3.1 and 3.2, we introduce the preliminaries to be able to define a two-dimensional almost-Riemannian manifold, namely vector bundles and Lie brackets of vector fields. In Section 3.3, we define a Riemannian manifold, which will be particularly useful in Section 4. In Section 3.4, we define what general two-dimensional almost-Riemannian manifolds are and give two specific examples.

3.1 Vector bundles

In this section, we define a smooth vector bundle over a smooth manifold M and discuss their properties. Intuitively, one can think of them as smooth families of vector spaces parametrized by points in M. We make this more formal below. This section is based on [15].

We define the *fiber* E_p *at a point* $p \in M$ under any map $\pi : E \to M$ to be the inverse image of the singleton set $\{p\}$ under π. For any two maps $\pi : E \to M$ and $\pi' : E' \to M$, a map $\phi : E \to E'$ is called *fiber-preserving* if $\phi(E_p) \subseteq E'_p$ for all $p \in M$.

Definition 3.1. A surjective smooth map $\pi : E \to M$ of manifolds is called *locally trivial of rank* k if

(i) The fiber E_p is a k-dimensional vector space for all $p \in M$.

(ii) For all $p \in M$, there exists an open neighbourhood U of p and a fiber-preserving diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that for every $q \in U$, the restriction of ϕ to E_q, namely $\phi|_{E_q} : E_q \to \{q\} \times \mathbb{R}^k$ is a vector space isomorphism. Note that the collection $\{(U, \phi)\}$ is called a *local trivialization* for E.

Definition 3.2. A *smooth vector bundle of rank* k is a triple (E, M, π), where E and M are manifolds and $\pi : E \to M$ is a locally trivial surjective smooth map of rank k. We call E the *total space* of the vector bundle and M the *base space* of the vector bundle respectively.

Example 3.3. The triple (TM, M, π) with TM the tangent bundle of a smooth manifold M and $\pi : TM \to M$ the canonical projection (defined to be the map $\pi(p, v) = p$ with $p \in M$ and $v \in T_pM$) is a smooth vector bundle.

To avoid notational pedantry, we will from this point onward write the vector bundle (E, M, π) as $\pi : E \to M$.

Definition 3.4. A *section of a vector bundle* $\pi : E \to M$ is a map $\sigma : M \to E$ such that for all $p \in M$, we have $\pi(\sigma(p)) = p$ (i.e., a right-inverse of π). A section is said to be smooth if the map σ is smooth as a map of manifolds.

3.2 Vector fields and Lie brackets

In the following section, we define a smooth vector field of a smooth manifold M and the Lie bracket of two vector fields. This section is based on [5], [15].

Definition 3.5. A *vector field* is a section of the vector bundle $\pi : TM \to M$ (see Example 3.3). A vector field is called *smooth* if this section is smooth as a map of manifolds. We denote the space of all smooth vector fields on M by $\mathcal{X}(M)$.

Remark 3.6. Note that it is also possible to define a vector field as a map that is a derivation of the algebra $C^\infty(M)$. That is to say, a linear map $X : C^\infty(M) \to C^\infty(M)$ (as opposed to $M \to TM$) which satisfies the Leibniz rule

$$X(fg) = fX(g) + X(f)g,$$

for all $f, g \in C^\infty(M)$. The vector field X is called smooth if for every $f \in C^\infty(M)$, the function Xf is smooth. This definition is more convenient to use for some arguments in this thesis. Note that it is actually possible to relate the two definitions. With respect to Definition 3.5, any vector field induces a derivation $L_X : C^\infty(M) \to C^\infty(M)$ (this is called a Lie derivative, for more detail we refer to [16], [17]). However, using this distinction of notation can be pedantic, so we will use the same symbol, as the definition used is usually clear from context.

Let $x^1, \ldots, x^n$ be local coordinates on a n-dimensional smooth manifold M. In the following, we use the basis $\{\frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^n}|_p\}$ for the tangent space T_pM and the basis $\{dx^1|_p, \ldots, dx^n|_p\}$ for the cotangent space T_p^*M. In a coordinate chart $(U, x^1, \ldots, x^n)$ of M (so U is an open neighbourhood of a point $p \in M$) we can write $X = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i}$, where the ξ^i are smooth coordinate functions on U.

We now introduce a way to generate another smooth vector field from two vector fields. To do so, we first introduce Lie algebras.

Definition 3.7. A *Lie algebra* over a field $\mathbb{K}$ is a pair $(V, [\cdot,\cdot])$, where V is a vector space over $\mathbb{K}$ and $[\cdot,\cdot] : V \times V \to V$ is a map (called the Lie bracket) satisfying the following properties for all $a, b \in \mathbb{K}$ and $x, y, z \in V$:

(i) Bilinearity: $[ax + by, z] = a[x, z] + b[y, z]$ and $[z, ax + by] = a[z, x] + b[z, y]$,

(ii) Anticommutativity: $[x, y] = -[y, x]$,

(iii) The Jacobi identity: $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$.

Definition 3.8. Let M be a smooth manifold and let $X, Y \in \mathcal{X}(M)$ be two vector fields. Their Lie bracket is defined for all $f \in C^\infty(M)$ to be

$$[X, Y]f := X(Yf) - Y(Xf). \tag{3.1}$$

It can easily be verified that the set $\mathcal{X}(M)$ forms a vector space over $\mathbb{R}$ with standard addition of functions and scalar multiplication of a function.

Proposition 3.9. *Let M be a smooth manifold. The vector space $\mathcal{X}(M)$ forms a Lie algebra over $\mathbb{R}$ with the Lie bracket in equation* (3.1).

Proof. Let $X, Y \in \mathcal{X}(M)$. We first need to show that $[X, Y] \in \mathcal{X}(M)$ as well. Suppose that $f \in C^\infty(M)$ is an arbitrary smooth function over M. Since X and Y are smooth, it follows that Xf and Yf are also smooth. Thus, $X(Yf)$ and $Y(Xf)$ are smooth. A sum of smooth functions is also smooth, hence $[X, Y]f$ is smooth, as required.

We now need to check properties (i)-(iii) for a Lie bracket. Showing bilinearity (i) and anticommutativity (ii) is straightforward, so we will only prove the Jacobi identity (iii). We have for $X, Y, Z \in \mathcal{X}(M)$ and $f \in C^\infty(M)$ that

$$\begin{aligned}
[X, [Y, Z]]f &= X([Y, Z]f) - [Y, Z](Xf) \\
&= X(Y(Zf) - Z(Yf)) - (Y(Z(Xf)) - Z(Y(Xf))) \\
&= X(Y(Zf)) + Z(Y(Xf)) - X(Z(Yf)) - Y(Z(Xf)).
\end{aligned}$$

By a relabelling of the vector fields, we find that

$$[Z, [X, Y]]f = Z(X(Yf)) + Y(X(Zf)) - Z(Y(Xf)) - X(Y(Zf)),$$

$$[Y,[Z,X]]f = Y(Z(Xf)) + X(Z(Yf)) - Y(X(Zf)) - Z(X(Yf)).$$

We thus indeed find that

$$([X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]])f = 0,$$

as required. □

Proposition 3.10. *Let M be a n-dimensional smooth manifold and let $(U, x^1, \ldots, x^n)$ be a coordinate chart of M. Let X and Y be the vector fields on this coordinate chart defined by*

$$X = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}, \quad Y = \sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j},$$

with ξ^i and η^j coordinate functions defined on U. One has the following coordinate expression of the Lie bracket

$$[X,Y] = \sum_{i,j} \left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}. \tag{3.2}$$

Proof. We apply $[X,Y]$ to some function $f \in C^\infty(U)$. We then have by bilinearity of the Lie bracket that

$$\begin{aligned}
[X,Y]f &= \left[\sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}, \sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j} \right] f \\
&= \sum_{i,j} \left[\xi^i \frac{\partial}{\partial x^i}, \eta^j \frac{\partial}{\partial x^j} \right] f \\
&= \sum_{i,j} \xi^i \frac{\partial}{\partial x^i} \left(\eta^j \frac{\partial f}{\partial x^j} \right) - \eta^j \frac{\partial}{\partial x^j} \left(\xi^i \frac{\partial f}{\partial x^i} \right).
\end{aligned}$$

By the product rule, it follows that for all $i, j \in \{1, 2, \ldots, n\}$

$$\begin{aligned}
\frac{\partial}{\partial x^i} \left(\eta^j \frac{\partial f}{\partial x^j} \right) &= \frac{\partial \eta^j}{\partial x^i} \frac{\partial f}{\partial x^j} + \eta^j \frac{\partial^2 f}{\partial x^i \partial x^j}, \\
\frac{\partial}{\partial x^j} \left(\xi^i \frac{\partial f}{\partial x^i} \right) &= \frac{\partial \xi^i}{\partial x^j} \frac{\partial f}{\partial x^i} + \xi^i \frac{\partial^2 f}{\partial x^j \partial x^i}.
\end{aligned}$$

By Clairaut's theorem, we have $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$ for all i, j, thus the expression reduces to

$$\begin{aligned}
[X,Y]f &= \sum_{i,j} \left(\xi^i \frac{\partial \eta^j}{\partial x^i} \frac{\partial f}{\partial x^j} - \eta^j \frac{\partial \xi^i}{\partial x^j} \frac{\partial f}{\partial x^i} \right) \\
&= \sum_{i,j} \left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} f,
\end{aligned}$$

where in the last step, we have relabelled the indices i, j. This agrees with equation (3.2). □

3.3 Riemannian manifolds

In this section, we define a Riemannian manifold and discuss some properties.

Definition 3.11. A *Riemannian manifold* is a pair (M, g) where M is a smooth manifold and g is a map that provides each $p \in M$ an inner product $g_p : T_pM \times T_pM \to \mathbb{R}$ such that for all $X, Y \in \mathcal{X}(M)$, the map $p \mapsto g_p(X(p), Y(p))$ is smooth.

Now, suppose that $\dim(M) = n$. In local coordinates $(x^1, \ldots, x^n)$, we may write [18]

$$g = \sum_{i,j} g_{ij}\; dx^i \otimes dx^j, \quad g_{ij}|_p = g_p \left(\frac{\partial}{\partial x^i}\bigg|_p, \frac{\partial}{\partial x^j}\bigg|_p \right). \tag{3.3}$$

Also, we let g^{ij} be the entries of the inverse of the coefficient matrix $[g_{ij}]$, so that $\sum_{k=1}^{n} g^{ik} g_{kj} = \delta^i_j$. One can then show that

$$g^{ij}|_p = g_p \left(dx^i|_p, dx^j|_p \right). \tag{3.4}$$

It can be shown that any smooth manifold carries a Riemannian metric (see for instance Proposition 2.10 of [19]). In the case when the local coordinates are positively oriented, a volume form is given by

$$\omega_g = \sqrt{|\det(g_{ij})|}\; dx^1 \wedge \cdots \wedge dx^n. \tag{3.5}$$

The volume form ω_g is called the *Riemannian volume form.*

Example 3.12. Let $\mathbb{R}^n$ be endowed with the Euclidean metric. This forms a Riemannian manifold (this is called Euclidean space). In this case, we have $g_{ij} = \delta_{ij}$.

3.4 2D Almost-Riemannian manifolds

In this section, we define a generalization of two-dimensional Riemannian manifolds called two-dimensional almost-Riemannian manifolds and we study their properties. Intuitively, these are manifolds that are Riemannian outside a low codimension set (this is called the singular set) which satisfy certain conditions. In the singular set, vector fields are constrained to live on a strict subset of TM. Under appropriate conditions, one can still find geodesics (and thus measure distances between points). This section is based on [5], [20].

Definition 3.13. Let M be a smooth manifold and let $\mathcal{F} \subseteq \mathcal{X}(M)$ be a collection of smooth vector fields. We define the *Lie algebra generated by* $\mathcal{F}$ to be

$$\mathrm{Lie}(\mathcal{F}) \coloneqq \mathrm{span}\{[X_1, \ldots, [X_{j-1}, X_j]] : X_i \in \mathcal{F}, j \in \mathbb{N}\}.$$

We then say that $\mathcal{F}$ satisfies the *Hörmander condition* if for all $p \in M$, the evaluation at p of the Lie algebra generated by $\mathcal{F}$ is equal to the tangent space of M at p. That is,

$$\mathrm{Lie}_p(\mathcal{F}) = \{X(p) : X \in \mathrm{Lie}(\mathcal{F})\} = T_pM.$$

Definition 3.14. Let M be a two-dimensional connected smooth manifold. A *two-dimensional almost-Riemannian structure* on M is a pair $(\mathbf{U}, f)$ such that:

(i) $\mathbf{U}$ is a Euclidean bundle of rank 2 with base space M. A Euclidean bundle is a vector bundle whose fibers U_p are equipped with a smoothly varying inner product $\langle \cdot, \cdot \rangle_p$ with respect to p.

(ii) The map $f : \mathbf{U} \to TM$ is a smooth map such that $f(U_p) \subseteq T_pM$ and its restriction to each fiber of $\mathbf{U}$ is linear.

(iii) The collection of smooth vector fields $\mathcal{D} = \{f \circ \sigma : \sigma : M \to \mathbf{U}$ is a smooth section$\}$ satisfies the Hörmander condition.

A two-dimensional connected smooth manifold with a two-dimensional almost-Riemannian structure is called a *two-dimensional almost-Riemannian manifold* and is denoted by $(M, \mathbf{U}, f)$.

Example 3.15. One induces a Riemannian manifold by setting $\mathbf{U} = TM$ and $f : TM \to TM$ the identity map.

Now that we have defined a two-dimensional almost-Riemannian manifold, we may now discuss their properties. From now on, we will drop the term 'two-dimensional'.

Definition 3.16. Let Ω be a subset of M. An *orthonormal frame* for an almost-Riemannian structure on Ω is the pair of vector fields

$$\{X_1, X_2\} = \{f \circ \sigma_1, f \circ \sigma_2\},$$

where $\{\sigma_1, \sigma_2\}$ is an orthonormal frame on a local trivialization $\Omega \times \mathbb{R}^2$ of $\mathbf{U}$ with respect to the inner product $\langle \cdot, \cdot \rangle_p$ for every $p \in \Omega$.

In this case, we can write the map f as $f(p, u) = u_1 X_1(p) + u_2 X_2(p)$, where $p \in M$ and $u \in U_p$. When an orthonormal frame for an almost-Riemannian structure on M exists (i.e., if $\Omega = M$), we say that the almost-Riemannian structure is *free* (in some literature, they call such an almost-Riemannian structure trivializable [8]).

Definition 3.17. The *distribution* of an almost-Riemannian structure on M is the family of subspaces $\{\mathcal{D}_p : p \in M\}$, where

$$\mathcal{D}_p \coloneqq \{X(p) : X \in \mathcal{D}\} = f(U_p) \subseteq T_p M.$$

Roughly speaking, $\mathcal{D}_p$ is a subspace per point $p \in M$ such that curves can be tangent to its vectors (in order to measure distances on almost-Riemannian manifolds). We now define a norm for an element of this subspace.

Definition 3.18. Let $v \in \mathcal{D}_p$. We define the *almost-Riemannian norm* to be

$$\|v\| \coloneqq \min\{|u| : u \in U_p \text{ s.t. } v = f(p, u)\}.$$

Definition 3.19. The *singular set* $\mathcal{Z}$ of an almost-Riemannian structure $(\mathbf{U}, f)$ of M is the set of points $p \in M$ such that $\dim(\mathcal{D}_p) < 2$. Such points $p \in \mathcal{Z}$ are called *singular points.* Otherwise, we call points $p \in M \setminus \mathcal{Z}$ *Riemannian points* (the reason why will be clear in the next theorem).

Theorem 3.20. *An almost-Riemannian structure is a Riemannian structure on $M \setminus \mathcal{Z}$.*

Proof. We provide a sketch of a proof. One can verify that the almost-Riemannian norm is indeed a norm. Furthermore, one can define an inner product at $p \in M$ via the polarization identity

$$\langle v, w \rangle_p = \frac{1}{4} \left(\|v + w\|^2 - \|v - w\|^2 \right),$$

where $v, w \in \mathcal{D}_p$. In particular, for the Riemannian points $p \in M \setminus \mathcal{Z}$, we have smoothness of the map $p \mapsto \langle X(p), Y(p) \rangle_p$, which induces a Riemannian metric. $\square$

Hence one can define a Riemannian metric and Riemannian volume form on $M \setminus \mathcal{Z}$. We now provide a useful result for a specific form of orthonormal frame.

Theorem 3.21. *Let (x^1, x^2) be a local system of coordinates on an open set $\Omega \subseteq M$. Assume that an orthonormal frame defined on Ω for a two-dimensional almost-Riemannian structure is of the form*

$$X_1(x^1, x^2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2(x^1, x^2) = \begin{pmatrix} 0 \\ f(x^1, x^2) \end{pmatrix}, \tag{3.6}$$

where $f : \Omega \to \mathbb{R}$ is a smooth function. The singular set is then given by $\mathcal{Z} = \{(x^1, x^2) : f(x^1, x^2) = 0\}$ and on $\Omega \cap (M \setminus \mathcal{Z})$, one has the following expression for the components of the Riemannian metric and volume:

$$g = (dx^1)^2 + \frac{1}{f(x^1, x^2)^2}(dx^2)^2, \quad \omega_g = \frac{dx^1 \wedge dx^2}{|f(x^1, x^2)|}.$$

Proof. In this case, the distribution $\mathcal{D}$ is given by the pair of vector fields $\{X_1, X_2\}$. To find the singular set, one must therefore find the set of points $p \in M$ such that $X_1(p)$ and $X_2(p)$ are linearly dependent. This holds if and only if p is such that $f(p) = 0$. Thus, we indeed have $\mathcal{Z} = \{(x^1, x^2) : f(x^1, x^2) = 0\}$. The metric is constructed from the identity

$$g_p(X_i(p), X_j(p)) = \delta_{ij}.$$

Subsequently, the volume form is immediately obtained from equation (3.5). □

We now discuss two examples of (free) almost-Riemannian structures, one on the unit cylinder $\mathbb{R} \times \mathbb{S}^1$ and the other on the unit sphere $\mathbb{S}^2$. Although we will not extensively discuss the almost-Riemannian structure on the sphere in this thesis, we will cover it briefly to provide further intuition.

3.4.1 The Grushin cylinder

Let $(x, \phi) \in \mathbb{R} \times [0, 2\pi)$ be the standard cylindrical coordinates on a cylinder of unit radius. The Grushin cylinder is a free almost-Riemannian structure $(\mathbf{U}, f)$ on $M = \mathbb{R} \times \mathbb{S}^1$ with Euclidean bundle $\mathbf{U} = M \times \mathbb{R}^2$ endowed with the Euclidean metric and $f(x, \phi, u_1, u_2) = (x, \phi, u_1, xu_2)$. A global orthonormal frame is then given by [5]

$$X_1(x, \phi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2(x, \phi) = \begin{pmatrix} 0 \\ x \end{pmatrix},$$

which indeed satisfies the Hörmander condition as (see Proposition 3.10)

$$[X_1, X_2](x, \phi) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and the span of X_1 and $[X_1, X_2]$ evaluated at each $p \in M$ spans T_pM. By Theorem 3.21, it follows that the singular set is given by $\mathcal{Z} = \{(x, \phi) : x = 0\}$. On that set, geodesics can only go parallel to the x-axis. On the Riemannian points $(x, \phi) \in M \setminus \mathcal{Z}$, the Riemannian metric and volume forms are given by

$$g = dx^2 + \frac{1}{x^2} d\phi^2, \quad \omega_g = \frac{dx \wedge d\phi}{|x|}.$$

To gain some intuition on why the singular set appears, one can compare the Euclidean cylinder and Grushin cylinder (see Figure 3.1). In the Euclidean case, the vector fields are constant and given by

$$X_1(x, \phi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2(x, \phi) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which span the tangent space at each point. This implies that every point on the Euclidean cylinder is a Riemannian point. However, for the Grushin cylinder, this is not the case when $p \in M$ is a point such that $x = 0$. In that case for each $\phi \in [0, 2\pi)$ we have $\dim(\operatorname{span}\{X_1(0, \phi), X_2(0, \phi)\}) = 1 < 2$.

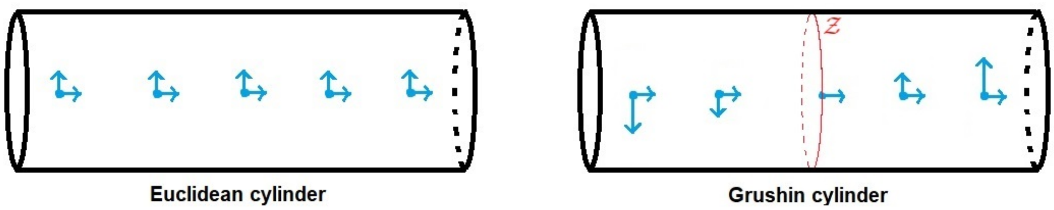

Figure 3.1: Comparison between the Euclidean cylinder and Grushin cylinder

3.4.2 The Grushin sphere

The Grushin sphere is a free almost-Riemannian structure $(\mathbf{U}, f)$ on the unit sphere $\mathbb{S}^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ with Euclidean bundle $\mathbf{U} = \mathbb{S}^2 \times \mathbb{R}^2$ and $f(\theta, \varphi, u_1, u_2) = (\theta, \varphi, u_1, \tan(\theta)u_2)$, where we have used spherical coordinates $(\theta, \varphi) \in [-\pi/2, \pi/2] \times [0, 2\pi)$ related to Cartesian coordinates by

$$(x, y, z) = (\cos(\theta)\cos(\varphi), \cos(\theta)\sin(\varphi), \sin(\theta)). \tag{3.7}$$

A global orthonormal frame is then given by [8]

$$X_1(\theta, \varphi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2(\theta, \varphi) = \begin{pmatrix} 0 \\ \tan(\theta) \end{pmatrix},$$

which similarly satisfies the Hörmander condition. By Theorem 3.21, it follows that the singular set $\mathcal{Z}$ is the set of points such that $\theta = 0$ in the spherical coordinate system (3.7). On that set, geodesics can only go parallel to the θ-axis. Note that in this case, the singular set does not include the set of points such that $\theta = \pm\pi/2$. This is only due to the natural coordinate singularity which arises when converting to spherical coordinates. On Riemannian points, the Riemannian metric and volume form are

$$g = d\theta^2 + \frac{1}{\tan^2(\theta)} d\varphi^2, \quad \omega_g = \frac{d\theta \wedge d\varphi}{|\tan(\theta)|}.$$

4 Spectra of the magnetic Laplace-Beltrami operator

In the following section, we evaluate the spectrum of an operator called the magnetic Laplace-Beltrami operator (cf. Definition 4.6) on several spaces, which is related to the Hamiltonian operator $\mathcal{H}$. We find that due to the generality of the underlying space, it is more convenient to consider electromagnetic quantities in terms of differential forms. Namely, in Cartesian coordinates we use the following one-to-one correspondence [21]:

$$\mathbf{E} = E_1\hat{\mathbf{x}} + E_2\hat{\mathbf{y}} + E_3\hat{\mathbf{z}} \longleftrightarrow E = E_1\ dx + E_2\ dy + E_3\ dz,$$

$$\mathbf{B} = B_1\hat{\mathbf{x}} + B_2\hat{\mathbf{y}} + B_3\hat{\mathbf{z}} \longleftrightarrow B = B_1\ dy \wedge dz + B_2\ dz \wedge dx + B_3\ dx \wedge dy,$$

$$\mathbf{A} = A_1\hat{\mathbf{x}} + A_2\hat{\mathbf{y}} + A_3\hat{\mathbf{z}} \longleftrightarrow A = A_1\ dx + A_2\ dy + A_3\ dz.$$

Note that strictly speaking, the magnetic field is not a vector (it is an axial vector). Nevertheless, this abuse of notation is used quite frequently in physics. With this correspondence, we have $B = dA$, where d is the exterior derivative (i.e., B is an exact two-form).

4.1 The Laplace-Beltrami operator

In the following, we define the Laplace-Beltrami operator when acted on a smooth function on a Riemannian manifold (M, g). It is a generalization of the standard Laplacian in Euclidean space. Let $\Omega^k(M)$ be the vector space of smooth k-forms on M.

Definition 4.1. Let (M, g) be a smooth n-dimensional Riemannian manifold. We define the *Hodge star operator* to be the unique isomorphism $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ with $\alpha \mapsto \star\alpha$ such that for all $\alpha, \beta \in \Omega^k(M)$, we have

$$\alpha \wedge (\star\beta) = g(\alpha, \beta)\omega_g,$$

where we define the inner product of decomposable k-forms $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k$ and $\beta = \beta_1 \wedge \cdots \wedge \beta_k$ to be

$$g(\alpha, \beta) \coloneqq \det(g(\alpha_i, \beta_j))_{i,j=1}^k,$$

which is extended to general k-forms by linearity. Note that ω_g here is the Riemannian volume form, as in equation (3.5).

Uniqueness of $\star\beta$ can be proven using the Riesz representation theorem, but since the proof is not so relevant in our analysis, we omit it. For a proof, we refer to [22]. Note that an explicit formula for the operator exists, as seen for instance in [23]. However, computing with this expression is rather tedious and it is easier to simply use the definition. We may now define the Laplace-Beltrami operator.

Definition 4.2. Let (M, g) be a smooth Riemannian manifold. The *Laplace-Beltrami operator* of a smooth function on M is defined to be

$$\Delta \coloneqq d^* d,$$

where $d^* \coloneqq \star d \star$ [8].

The notation d^* is due to the fact that with the definition above it is formally adjoint to the exterior derivative with respect to the inner product $\langle \omega, \eta \rangle = \int_M \omega \wedge \star\eta$ (up to a difference in sign). This can be proven using the anti-derivation property of the exterior derivative and Stokes' theorem (the particular case where $\partial M = \varnothing$).

Remark 4.3. Note that it is also possible to define the Laplace-Beltrami operator without use of the Hodge star operator. This approach allows for an easier derivation for an explicit expression on a coordinate chart. For more detail, see Appendix B.

Example 4.4. Consider the case where $M = \mathbb{R}^3$ is endowed with the Euclidean metric and let $f \in C^\infty(\mathbb{R}^3)$. In this case the Laplace-Beltrami operator coincides with the usual Laplacian. This can be seen by the following computation

$$\begin{aligned}
\Delta f &= d^* df \\
&= d^* \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right) \\
&= \star d \left(\frac{\partial f}{\partial x} dy \wedge dz + \frac{\partial f}{\partial y} dz \wedge dx + \frac{\partial f}{\partial z} dx \wedge dy \right) \\
&= \star \left(\frac{\partial^2 f}{\partial x^2} dx \wedge dy \wedge dz + \frac{\partial^2 f}{\partial y^2} dy \wedge dz \wedge dx + \frac{\partial^2 f}{\partial z^2} dz \wedge dx \wedge dy \right) \\
&= \star \left(\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) dx \wedge dy \wedge dz \right) \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2},
\end{aligned}$$

where we have used $\star(dx) = dy \wedge dz$, $\star(dy) = dz \wedge dx$, $\star(dz) = dx \wedge dy$ and $\star(dx \wedge dy \wedge dz) = 1$.

We now give a coordinate expression for the Laplace-Beltrami operator on two-dimensional free almost-Riemannian manifolds with the general form in Theorem 3.21.

Lemma 4.5. *Let $\{X_1, X_2\}$ be a global orthonormal frame as in equation (3.6) for a two-dimensional almost-Riemannian structure on a smooth two-dimensional connected manifold M. The Laplace-Beltrami operator on $M \setminus \mathcal{Z}$ is given by*

$$\Delta = \frac{\partial^2}{\partial (x^1)^2} + f^2 \frac{\partial^2}{\partial (x^2)^2} - \frac{1}{f} \frac{\partial f}{\partial x^1} \frac{\partial}{\partial x^1} + f \frac{\partial f}{\partial x^2} \frac{\partial}{\partial x^2}. \tag{4.1}$$

Proof. Let $\psi \in C_c^\infty(M \setminus \mathcal{Z})$ be a smooth function with compact support in $M \setminus \mathcal{Z}$. We have $g(dx^1, dx^1) = 1$ and $g(dx^2, dx^2) = f^2$ (see equation (3.4)). It then follows that $\star(dx^1) = \frac{1}{|f|}\, dx^2$, $\star(dx^2) = -|f|\, dx^1$ and $\star(dx^1 \wedge dx^2) = |f|$. Hence

$$\begin{aligned}
d^* d\psi &= d^* \left(\frac{\partial \psi}{\partial x^1} dx^1 + \frac{\partial \psi}{\partial x^2} dx^2 \right) \\
&= \star d \left(\frac{\partial \psi}{\partial x^1} \frac{1}{|f|}\, dx^2 - \frac{\partial \psi}{\partial x^2} |f|\, dx^1 \right) \\
&= \star \left(\frac{\partial}{\partial x^1} \left(\frac{\partial \psi}{\partial x^1} \frac{1}{|f|} \right) dx^1 \wedge dx^2 - \frac{\partial}{\partial x^2} \left(\frac{\partial \psi}{\partial x^2} |f| \right) dx^2 \wedge dx^1 \right) \\
&= |f| \left(\frac{\partial}{\partial x^1} \left(\frac{\partial \psi}{\partial x^1} \frac{1}{|f|} \right) + \frac{\partial}{\partial x^2} \left(\frac{\partial \psi}{\partial x^2} |f| \right) \right) \\
&= \frac{\partial^2 \psi}{\partial (x^1)^2} + |f| \frac{\partial \psi}{\partial x^1} \frac{\partial (|f|^{-1})}{\partial x^1} + f^2 \frac{\partial^2 \psi}{\partial (x^2)^2} + |f| \frac{\partial \psi}{\partial x^2} \frac{\partial (|f|)}{\partial x^2} \\
&= \frac{\partial^2 \psi}{\partial (x^1)^2} - |f| \frac{\partial \psi}{\partial x^1} \frac{1}{f \cdot |f|} \frac{\partial f}{\partial x^1} + f^2 \frac{\partial^2 \psi}{\partial (x^2)^2} + |f| \frac{\partial \psi}{\partial x^2} \frac{f}{|f|} \frac{\partial f}{\partial x^2} \\
&= \frac{\partial^2 \psi}{\partial (x^1)^2} + f^2 \frac{\partial^2 \psi}{\partial (x^2)^2} - \frac{1}{f} \frac{\partial f}{\partial x^1} \frac{\partial \psi}{\partial x^1} + f \frac{\partial f}{\partial x^2} \frac{\partial \psi}{\partial x^2},
\end{aligned}$$

which indeed agrees with equation (4.1). □

We now generalize the Laplace-Beltrami operator in order to compute the spectrum of the Aharonov-Bohm Hamiltonian on almost-Riemannian manifolds.

Definition 4.6. Let (M, g) be a smooth Riemannian manifold, $A \in \Omega^1(M)$ a differential one-form and $b \in \mathbb{R}$ a constant. We define the *magnetic Laplace-Beltrami operator* to be [8]

$$\Delta_A^b := (d - ibA)^*(d - ibA).$$

Note that one can find the following decomposition

$$\Delta_A^b \psi = \Delta\psi - ibd^*(A\psi) - ibA^*(d\psi) - b^2 A^*(A\psi), \tag{4.2}$$

so that we indeed find that Δ_A^b reduces to the standard Laplace-Beltrami operator if either $A = 0$ or $b = 0$. In the Euclidean case, one finds that the Aharonov-Bohm Hamiltonian operator and the magnetic Laplace-Beltrami operator coincide up to a scale factor, namely

$$\mathcal{H} = -\frac{\hbar^2}{2m}\Delta_A^{q/\hbar}.$$

We now state a similar result to Lemma 4.5 for the generalized operator.

Theorem 4.7. *Let $\{X_1, X_2\}$ be a global orthonormal frame as in equation* (3.6) *for a two-dimensional almost-Riemannian structure on a smooth two-dimensional connected manifold M. Let $A = A_1\, dx^1 + A_2\, dx^2$, where both A_1 and A_2 may be functions of both x^1 and x^2. The Laplace-Beltrami operator on $M \setminus \mathcal{Z}$ is given by*

$$\begin{aligned}\Delta_A^b = &\frac{\partial^2}{\partial(x^1)^2} + f^2\frac{\partial^2}{\partial(x^2)^2} - \frac{1}{f}\frac{\partial f}{\partial x^1}\frac{\partial}{\partial x^1} + f\frac{\partial f}{\partial x^2}\frac{\partial}{\partial x^2}\\ &- ib\left(2A_1\frac{\partial}{\partial x^1} + \frac{\partial A_1}{\partial x^1} - \frac{A_1}{f}\frac{\partial f}{\partial x^1} + 2f^2 A_2\frac{\partial}{\partial x^2} + f^2\frac{\partial A_2}{\partial x^2} + A_2 f\frac{\partial f}{\partial x^2}\right)\\ &- b^2\left(A_1^2 + A_2^2 f^2\right)\end{aligned}$$

Proof. We use the decomposition in equation (4.2). We have already computed the first term, see Lemma 4.5. It suffices to compute the three remaining terms. We find that

$$\begin{aligned}d^*(A\psi) &= |f|\frac{\partial}{\partial x^1}\left(\frac{A_1\psi}{|f|}\right) + |f|\frac{\partial}{\partial x^2}\left(A_2|f|\psi\right)\\ &= \frac{\partial A_1}{\partial x^1}\psi + A_1\frac{\partial\psi}{\partial x^1} - \frac{A_1}{f}\frac{\partial f}{\partial x^1}\psi + f^2\frac{\partial A_2}{\partial x^2}\psi + f^2 A_2\frac{\partial\psi}{\partial x^2} + A_2 f\frac{\partial f}{\partial x^2}\psi,\end{aligned}$$

$$A^*(d\psi) = A_1\frac{\partial\psi}{\partial x^1} + f^2 A_2\frac{\partial\psi}{\partial x^2},$$

and

$$A^*A\psi = A_1^2\psi + A_2^2 f^2\psi.$$

This gives the required expression. ☐

4.2 Spectral theory

In this section, we introduce all of the preliminary spectral theory concepts. This is necessary because in order to define the magnetic Laplace-Beltrami operator on the singular set $\mathcal{Z}$, one must study the self-adjointness of the operator. This section is based on [24], [25].

Definition 4.8. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. A linear operator T : $\mathrm{dom}(T) \subset \mathcal{H} \to \mathcal{H}$ (we assume the domain of T to be a dense subspace of $\mathcal{H}$) is *self-adjoint* if

(i) T is *symmetric*, namely $\langle Tx, y\rangle_{\mathcal{H}} = \langle x, Ty\rangle_{\mathcal{H}}$ for any $x, y \in \mathrm{dom}(T)$,

(ii) The domain of T is equal to the domain of its adjoint T^* (i.e., $\mathrm{dom}(T) = \mathrm{dom}(T^*)$).

In physics, one usually refers to a symmetric operator as a Hermitian one. Not all operators are self-adjoint. However, there are certain ways to extend them to be.

Definition 4.9. An operator A is a *self-adjoint extension* of T if

(i) $\mathrm{dom}(T) \subseteq \mathrm{dom}(A) = \mathrm{dom}(A^*) \subseteq \mathrm{dom}(T^*)$,

(ii) $Ax = T^*x$ for any $x \in \mathrm{dom}(A)$.

For non-negative densely defined symmetric operators, one always has at least one self-adjoint extension, called the *Friedrichs extension*. In this case, one can show that there are two possibilities for the number of self-adjoint extensions that an operator admits, namely

- Only one self-adjoint extension exists. In this case, we call an operator *essentially self-adjoint*,
- There are infinitely many self-adjoint extensions.

A physical consequence of this in our analysis is that if the magnetic Laplace-Beltrami operator restricted to $C_c^\infty(M \setminus \mathcal{Z})$ is essentially self-adjoint, a quantum particle cannot cross the singular set $\mathcal{Z}$. Mathematically, this means that the domain of such an operator splits into a direct sum of parts of the almost-Riemannian manifold. Moreover, the operator splits into a direct sum of self-adjoint operators on these parts.

Remark 4.10. When we say that we will compute the spectrum of the magnetic Laplace-Beltrami operator, we will actually be computing the spectrum of $-\Delta_A^b$. This has to do with the fact that the standard Laplace-Beltrami operator is a negative operator. Namely, for all $f \in C^\infty(M)$ we have

$$\langle \Delta f, f \rangle_{L^2} \leq 0,$$

where $\langle \cdot, \cdot \rangle_{L^2}$ is the standard L^2 inner product on M. This can be shown by integration by parts. It therefore follows that $-\Delta$ is a non-negative operator. A convenient implication of this is that all the eigenvalues of $-\Delta$ are non-negative [26]. Moreover, the existence of the Friedrichs extension requires the operator to be non-negative.

Recall that the spectrum $\sigma(T)$ of an operator T that operates on a Hilbert space $\mathcal{H}$ is the set of all scalars λ such that the operator $T - \lambda I$ does not have a bounded inverse in $\mathcal{H}$. It turns out that $\sigma(T)$ can be separated into three parts. We discuss this below. We first introduce some preliminary measure theory (in particular, a refined version of the Lebesgue decomposition theorem), then define the three types of spectrum.

Definition 4.11. Let $(\Omega, \mathcal{A})$ be a measurable space, where Ω is a set and $\mathcal{A}$ is the σ-algebra on Ω, and let λ_1 and λ_2 be measures on that space. A measure λ_1 is said to be *absolutely continuous* with respect to a measure λ_2 if for all $A \in \mathcal{A}$ such that $\lambda_2(A) = 0$ implies $\lambda_1(A) = 0$. We then write $\lambda_1 \ll \lambda_2$.

Definition 4.12. Let $(\Omega, \mathcal{A})$ be a measurable space and let λ be a measure. The measure λ is said to be *concentrated* on a set $A \in \mathcal{A}$ if $\lambda(A^c) = 0$.

Definition 4.13. Let $(\Omega, \mathcal{A})$ be a measurable space and let λ_1 and λ_2 be measures. Then λ_1 and λ_2 are said to be *mutually singular* if λ_1 is concentrated on $A_1 \in \mathcal{A}$, λ_2 is concentrated on $A_2 \in \mathcal{A}$, and $A_1 \cap A_2 = \varnothing$. In that case, we write $\lambda_1 \perp \lambda_2$.

Definition 4.14. A Borel measure μ is said to be a *pure point measure* if for every Borel set Ω we have

$$\mu(\Omega) = \sum_{x \in \Omega} \mu(\{x\}).$$

We now decompose the measure into two parts. The result below is called the Lebesgue decomposition theorem.

Theorem 4.15. *Let $(\Omega, \mathcal{A})$ be a measurable space and let λ and μ be σ-finite measures. There is a unique pair of measures λ_{ac} and λ_{sing} such that $\lambda = \lambda_{\mathrm{ac}} + \lambda_{\mathrm{sing}}$, $\lambda_{\mathrm{ac}} \ll \mu$ and $\lambda_{\mathrm{sing}} \perp \mu$.*

Proof. A proof can be found in Theorem 9.14 of [27] or Theorem 19.42 of [28]. $\square$

As a consequence of the above, $\lambda_{\mathrm{ac}} \perp \lambda_{\mathrm{sing}}$ (i.e., the two measures are mutually singular). We call λ_{ac} the absolutely continuous part of the measure and λ_{sing} as the singular part of the measure. It is possible to decompose λ even further by decomposing λ_{sing} as a sum of two measures. This is done by taking out the pure point part. The result below is called the refined Lebesgue decomposition theorem.

Theorem 4.16. *Let $(\Omega, \mathcal{A})$ be a measurable space and let λ and μ be σ-finite measures. Then there are unique measures λ_{ac}, λ_{cs} and λ_{pp} such that $\lambda = \lambda_{\mathrm{ac}} + \lambda_{\mathrm{cs}} + \lambda_{\mathrm{pp}}$, $\lambda_{\mathrm{ac}} \ll \mu$, $\lambda_{\mathrm{cs}} \perp \mu$, $\lambda_{\mathrm{cs}}(\{x\}) = 0$ for all $x \in \Omega$, and λ_{pp} is a pure point measure.*

Proof. A proof can be found in Theorem 19.61 of [28]. $\square$

Like the standard Lebesgue decomposition theorem, it follows that the three measures are mutually singular. We call λ_{cs} the continuous singular part of the measure and λ_{pp} the pure point part of the measure. We now define the spectral measure of an operator T associated to a point in a Hilbert space.

Theorem 4.17. *Let $T : \mathrm{dom}(T) \subseteq \mathcal{H} \to \mathcal{H}$ be a self-adjoint linear operator (we again assume that $\mathrm{dom}(T)$ is a dense subspace of $\mathcal{H}$) on a Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Moreover, let f be a continuous bounded function in $\mathbb{R}$ and let $v \in \mathcal{H}$. Then there exists a unique finite Borel measure $\mu_{T,v} : \mathcal{B}(\mathbb{R}) \to [0, \infty)$ such that*

$$\langle f(T)v, v \rangle_{\mathcal{H}} = \int_{\mathbb{R}} f(x)\; d\mu_{T,v}(x).$$

We call this measure the spectral measure of T associated to $v \in \mathcal{H}$.

Proof. A proof using the Riesz-Markov representation theorem can be found on pages 4 and 5 of [29]. $\square$

As a consequence, one may also decompose the Hilbert space $\mathcal{H}$ as a direct sum of spaces. Let λ be the Lebesgue measure. We define

$$\mathcal{H}_{\mathrm{ac}} := \{v \in \mathcal{H} : \mu_{T,v} \ll \lambda\},$$

$$\mathcal{H}_{\mathrm{cs}} := \{v \in \mathcal{H} : \mu_{T,v} \text{ continuous singular}\},$$

$$\mathcal{H}_{\mathrm{pp}} := \{v \in \mathcal{H} : \mu_{T,v} \text{ pure point}\}.$$

We then have $\mathcal{H} = \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{cs}} \oplus \mathcal{H}_{\mathrm{pp}}$. We may thus define three different types of spectrum as follows:

Definition 4.18. We define the *absolutely continuous*, *continuous singular* and *pure point spectrum* as follows:

$$\sigma_{\mathrm{ac}}(T) := \sigma(T|_{\mathcal{H}_{\mathrm{ac}}}),$$

$$\sigma_{\mathrm{cs}}(T) := \sigma(T|_{\mathcal{H}_{\mathrm{cs}}}),$$

$$\sigma_{\mathrm{pp}}(T) := \sigma(T|_{\mathcal{H}_{\mathrm{pp}}}) = \overline{\sigma_{\mathrm{p}}(T)}.$$

where $\sigma_{\mathrm{p}}(T)$ is the point spectrum of T (i.e., the set of eigenvalues of T) and the bar above the set represents closure.

One then finds the decomposition

$$\sigma(T) = \sigma_{\mathrm{ac}}(T) \cup \sigma_{\mathrm{cs}}(T) \cup \sigma_{\mathrm{pp}}(T) = \sigma_{\mathrm{ac}}(T) \cup \sigma_{\mathrm{cs}}(T) \cup \overline{\sigma_{\mathrm{p}}(T)}.$$

Note that in most cases (including the cases we will consider later), the continuous singular spectrum of an operator is empty. We therefore define the continuous spectrum by

$$\sigma_{\mathrm{c}}(T) \coloneqq \sigma_{\mathrm{ac}}(T) \cup \sigma_{\mathrm{cs}}(T).$$

Hence, in our analysis it suffices to compute the continuous spectrum and the eigenvalues of the operator to obtain a full description of the spectrum. We will use the terms 'discrete spectrum' (which we will denote by $\sigma_{\mathrm{d}}(T)$) and 'point spectrum' interchangeably. Physically, the discrete spectrum of the Hamiltonian represents the bound states of a quantum system, which are normalizable, and the continuous spectrum of the Hamiltonian represents its scattering states, which are non-normalizable.

Lastly, we provide a result for the spectrum of a direct sum of linear operators, which will be useful when finding the spectrum of the magnetic Laplace-Beltrami operator on the Grushin cylinder.

Theorem 4.19. *Let $T = \bigoplus_{k\in\mathbb{Z}} T_k$, where each T_k is a linear operator on a Hilbert space $\mathcal{H}$ over a field $\mathbb{K}$. Then, we have the following relations [30]:*

$$\sigma_{\mathrm{p}}(T) = \bigcup_{k\in\mathbb{Z}} \sigma_p(T_k),$$

$$\sigma_{\mathrm{c}}(T) = \left\{ \left(\bigcup_{k\in\mathbb{Z}} \sigma_p(T_k) \right)^c \cap \left(\bigcup_{k\in\mathbb{Z}} \sigma_r(T_k) \right)^c \cap \left(\bigcup_{k\in\mathbb{Z}} \sigma_c(T_k) \right) \right\} \cup \left\{ \lambda \in \bigcap_{k\in\mathbb{Z}} \rho(T_k) : \sup_{k\in\mathbb{Z}} \|R_\lambda(T_k)\| = +\infty \right\},$$

where $\rho(T)$ is the resolvent set

$$\rho(T) \coloneqq \{\lambda \in \mathbb{K} : T - \lambda I \text{ is invertible}\},$$

$R_\lambda(T)$ is the resolvent operator

$$R_\lambda(T) = (T - \lambda I)^{-1},$$

and $\sigma_r(T)$ is the residual spectrum of T, which is the set of all $\lambda \in \mathbb{K}$ such that $T - \lambda I$ does not have dense range but is injective. For normal operators (in particular, self-adjoint operators), this spectrum is empty.

Proof. A proof can be found in Theorem 2.3 of [31]. □

4.3 Spectrum in the two-dimensional Euclidean plane

We consider the concrete situation as in Figure 2.1. In this case, the one-form corresponding to the magnetic vector potential outside the solenoid is given by

$$A = \frac{1}{2} a^2 B_0 \, d\phi,$$

where we fix the Coulomb gauge $d^*A = 0$. Note that due to the symmetry of the infinite solenoid, it suffices to consider the case where the particle is constrained to a plane orthogonal to the solenoid (and outside it).

Remark 4.20. Strictly speaking, we are not computing the spectrum on the plane, but on a plane without the cross section of the solenoid. This manifold is topologically equivalent to a punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$.

We now find a partial differential equation satisfied by a wave function $\psi \in C^\infty(\mathbb{R}^2 \setminus \{\text{cross section of solenoid}\})$ which vanishes at the origin. For the sake of simplicity, in the following we let $\gamma = (1/2)a^2 B_0$ so that $A = \gamma\ d\phi$. To evaluate the Hodge dual, we need knowledge of the metric for polar coordinates. It is well-known that

$$[g_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad [g^{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}.$$

Therefore, it follows that $g(dr, dr) = 1$, $g(d\phi, d\phi) = 1/r^2$ and all other components vanish. It thus follows that $g(dr \wedge d\phi, dr \wedge d\phi) = 1/r^2$. The Riemannian volume form is given by $\omega_g = r\ dr \wedge d\phi$. A simple computation then shows that

$$\star(dr) = r\ d\phi, \quad \star(d\phi) = -(1/r)\ dr, \quad \star(dr \wedge d\phi) = 1/r.$$

We use the decomposition in equation (4.2). It follows that

$$\begin{aligned}
\Delta\psi &= d^* d\psi \\
&= d^* \left(\frac{\partial \psi}{\partial r} dr + \frac{\partial \psi}{\partial \phi} d\phi \right) \\
&= \star d \left(\frac{\partial \psi}{\partial r} r\ d\phi - \frac{\partial \psi}{\partial \phi} \frac{1}{r}\ dr \right) \\
&= \star \left(\frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right)\ dr \wedge d\phi - \frac{1}{r} \frac{\partial^2 \psi}{\partial \phi^2}\ d\phi \wedge dr \right) \\
&= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} \\
&= \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2},
\end{aligned}$$

and

$$d^*(A\psi) = \star d \left(-\frac{\gamma \psi}{r}\ dr \right) = \star \left(\frac{\gamma}{r} \frac{\partial \psi}{\partial \phi}\ d\phi \wedge dr \right) = \frac{\gamma}{r^2} \frac{\partial \psi}{\partial \phi}.$$

It can similarly be seen in this case that $d^*(A\psi) = A^*(d\psi)$. Finally, we have

$$A^* A\psi = \star A \left(-\frac{\gamma \psi}{r}\ dr \right) = \star \left(-\frac{\gamma^2 \psi}{r}\ d\phi \wedge dr \right) = \frac{\gamma^2 \psi}{r^2}.$$

Therefore, the magnetic Laplace-Beltrami operator is given by

$$\Delta_A^b = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} - \frac{2ib\gamma}{r^2} \frac{\partial}{\partial \phi} - \frac{b^2 \gamma^2}{r^2},$$

or alternatively, we can write the Hamiltonian operator

$$\mathcal{H} = -\frac{\hbar^2}{2m} \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left(\frac{\partial}{\partial \phi} - i \frac{q a^2 B_0}{2\hbar} \right)^2 \right].$$

The problem now rests on solving the eigenvalue equation $-\Delta_A^b \psi = \lambda \psi$. To do so, we use separation of variables with the ansatz $\psi(r, \phi) = R(r)\Phi(\phi)$. Omitting function arguments, we then obtain

$$\Delta_A^b \psi = R''\Phi + \frac{1}{r} R'\Phi + \frac{1}{r^2} R\Phi'' - \frac{2ib\gamma}{r^2} R\Phi' - \frac{b^2\gamma^2}{r^2} R\Phi = -\lambda R\Phi.$$

Multiplication with $r^2/(R\Phi)$ and a rearrangement of the terms gives

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \lambda r^2 = -\frac{\Phi''}{\Phi} + 2ib\gamma\frac{\Phi'}{\Phi} + b^2\gamma^2.$$

Since the two sides depend on different independent variables, there must be a constant μ to which they are equal. We then obtain that Φ (the angular term) satisfies the ordinary differential equation

$$\Phi'' - 2ib\gamma\Phi' + (\mu - b^2\gamma^2)\Phi = 0,$$

which has solutions

$$\Phi_n(\phi) = c_1 e^{in\phi}, \quad \mu = (n - b\gamma)^2,$$

where c_1 is an arbitrary constant. Due to 2π-periodicity of Φ, n has to be an integer, which constrains μ to take discrete values $\mu_n = (n - b\gamma)^2$. The equation for R (the radial term) is given by

$$r^2R'' + rR' + (\lambda r^2 - \mu_n)R = 0.$$

For $\lambda \neq 0$, the change of variable $s = \sqrt{\lambda}r$ leads to the Bessel differential equation of order $\sqrt{\mu_n}$

$$s^2R'' + sR' + (s^2 - \mu_n)R = 0,$$

which has general solution

$$R(s) = c_2 J_{\sqrt{\mu_n}}(s) + c_3 Y_{\sqrt{\mu_n}}(s),$$

where $J_\nu(\cdot)$ and $Y_\nu(\cdot)$ are Bessel functions of the first and second kind of order ν respectively (for more information about the Bessel functions, see Appendix C.1) and both c_2 and c_3 are arbitrary constants. Hence, solutions to the radial equation are given by

$$R_n(r) = c_2 J_{|n-b\gamma|}(\sqrt{\lambda}r) + c_3 Y_{|n-b\gamma|}(\sqrt{\lambda}r).$$

If $\lambda = 0$, the radial equation reduces to a Cauchy-Euler equation, which has solutions

$$R_n(r) = c_4 r^{|n-b\gamma|} + c_5 r^{-|n-b\gamma|},$$

where c_4 and c_5 are arbitrary constants. Hence, the general class of separable solutions to the eigenvalue problem $-\Delta_A^b\psi = \lambda\psi$ is given by

$$\psi(r, \phi) = \sum_n C_n R_n(r)\Phi_n(\phi),$$

where C_n are arbitrary constants. Since ψ must vanish at the origin, the asymptotic behaviour of the radial functions (in the limit $r \to 0^+$) implies that they must be of the form (for ψ to be an eigenstate of $-\Delta_A^b$)

$$R_n(r) = \begin{cases} c_2 J_{|n-b\gamma|}(\sqrt{\lambda}r), & \text{if } \lambda \neq 0, \\ c_4 r^{|n-b\gamma|}, & \text{if } \lambda = 0, \end{cases}$$

which are all non-normalizable (no matter the choice of c_2 and c_4). Therefore, the discrete spectrum is empty and the continuous spectrum is $[0, \infty)$.

Discretization of the spectrum can be done by localizing the particle to a set of finite Lebesgue measure, which will remove the continuous part of the spectrum. To this end, we consider a particle outside the solenoid moving in an annulus with $r \in [\alpha, \beta]$, where the wave function is assumed to vanish on the inner and outer walls of the annulus (we impose Dirichlet boundary conditions). We now compute the eigenvalues. The only part of the wave function dependent on λ is the radial part. It therefore suffices to compute λ such that $R_n(r)$ is non-zero. Since the wave function vanishes at $r = \alpha$ and $r = \beta$, it follows that for $\lambda \neq 0$

$$\begin{cases} c_2 J_{|n-b\gamma|}(\sqrt{\lambda}\alpha) + c_3 Y_{|n-b\gamma|}(\sqrt{\lambda}\alpha) = 0, \\ c_2 J_{|n-b\gamma|}(\sqrt{\lambda}\beta) + c_3 Y_{|n-b\gamma|}(\sqrt{\lambda}\beta) = 0, \end{cases}$$

and for $\lambda = 0$
$$\begin{cases} c_4\alpha^{|n-b\gamma|} + c_5\alpha^{-|n-b\gamma|} = 0, \\ c_4\beta^{|n-b\gamma|} + c_5\beta^{-|n-b\gamma|} = 0. \end{cases}$$

This has non-trivial solutions if and only if the determinants of the matrix corresponding to the linear system is zero. Hence, for $\alpha < \beta$, one sees that $\lambda = 0$ is not an eigenvalue (unless $n = b\gamma$) and that $\lambda \neq 0$ is an eigenvalue if and only if

$$J_{|n-b\gamma|}(\sqrt{\lambda}\alpha)Y_{|n-b\gamma|}(\sqrt{\lambda}\beta) - Y_{|n-b\gamma|}(\sqrt{\lambda}\alpha)J_{|n-b\gamma|}(\sqrt{\lambda}\beta) = 0. \tag{4.3}$$

Let the l-th root of this transcendental equation be given by λ_{ln}. From the above, one sees that the spectrum is composed of an empty continuous spectrum and discrete spectrum $\{\lambda_{ln} : l \in \mathbb{N}, n \in \mathbb{Z}\}$. Hence, the eigenvalues of the Hamiltonian are the energies

$$E_{ln} = \frac{\hbar^2}{2m}\lambda_{ln}, \quad l \in \mathbb{N},\ n \in \mathbb{Z}. \tag{4.4}$$

In the limit that $\alpha \to \beta^-$ (i.e., When one restricts that particle to a circular ring outside the solenoid), the physical phenomena which appear become even simpler to understand. One obtains the ordinary differential equation

$$\Delta_A^b\psi = \frac{1}{\beta^2}\left(\frac{d^2\psi}{d\phi^2} - 2ib\gamma\frac{d\psi}{d\phi} - b^2\gamma^2\psi\right) = -\lambda\psi,$$

which (similarly to how one solves the angular equation) has general solution

$$\psi = ce^{in\phi}, \quad \lambda = \frac{(n-b\gamma)}{\beta^2},$$

where $n \in \mathbb{Z}$ due to 2π-periodicity of ψ and c is a normalization constant. This results in the eigenvalues
$$\lambda_n = \frac{(n-b\gamma)^2}{\beta^2}, \quad n \in \mathbb{Z}.$$

Hence, the spectrum is composed of discrete spectrum $\{\lambda_n : n \in \mathbb{Z}\}$ and empty continuous spectrum. The eigenvalues of the Hamiltonian are the energies

$$E_n = \frac{\hbar^2}{2m\beta^2}\left(n - \frac{qa^2B_0}{2\hbar}\right)^2, \quad n \in \mathbb{Z}. \tag{4.5}$$

This leads to the following physical consequence: Assuming that q is positive, a particle travelling in the same direction as the current (this represents positive n) in the solenoid has lower energy than a particle travelling in the other direction (negative n). In other words, we have $E_n < E_{-n}$. Moreover, the eigenvalues are non-degenerate. If one compares this to the case where a charged particle is restricted to a circular ring without the solenoid ($B_0 = 0$), one has two-fold degeneracy instead. Hence, the non-simply connectedness of the region defined to be where the magnetic field is zero leads to the splitting of energy levels [13].

4.4 Spectrum in the two-dimensional Euclidean unit cylinder

Let $(x, \phi) \in \mathbb{R} \times [0, 2\pi)$ be the standard cylindrical coordinates on a unit cylinder of infinite length. We again choose the vector potential such that

$$A = \frac{1}{2}a^2B_0\ d\phi = \gamma\ d\phi,$$

where we use the Coulomb gauge. It is well-known that the metric is flat

$$[g_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad [g^{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, we have $\star(dx) = d\phi$ and $\star(d\phi) = -dx$. Let $\psi \in C^\infty(\mathbb{R} \times \mathbb{S}^1)$. A simple computation shows that

$$\Delta\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial\phi^2}, \quad d^*(A\psi) = A^*(d\psi) = \gamma\frac{\partial\psi}{\partial\phi}, \quad A^*A = \gamma^2\psi.$$

Hence, the magnetic Laplace-Beltrami operator is given by

$$\Delta_A^b = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\phi^2} - 2ib\gamma\frac{\partial}{\partial\phi} - b^2\gamma^2.$$

We now solve the eigenvalue equation $-\Delta_A^b\psi = \lambda\psi$. To do so, we proceed in a similar way to the planar case: we use separation of variables with ansatz $\psi(x,\phi) = X(x)\Phi(\phi)$. The separation procedure then gives us (omitting function arguments)

$$\frac{X''}{X} + \lambda = -\frac{\Phi''}{\Phi} + 2ib\gamma\frac{\Phi'}{\Phi} + b^2\gamma^2 = \mu,$$

where μ is the separation constant. This results in the following ordinary differential equations

$$X'' + (\lambda - \mu)X = 0,$$

$$\Phi'' - 2ib\gamma\Phi' + (\mu - b^2\gamma^2)\Phi = 0.$$

The equation for Φ has already been solved previously when computing the spectrum on the plane. The solutions are

$$\Phi_n(\phi) = c_1 e^{in\phi}, \quad \mu = (n - b\gamma)^2,$$

where c_1 is an arbitrary constant and n is an integer due to 2π-periodicity of Φ. Therefore μ is constrained to take discrete values $\mu_n = (n - b\gamma)^2$. We have $\lambda - \mu \in \mathbb{R}_+$. Thus, if we let $k_n^2 := \lambda - \mu_n > 0$, it follows that the equation for X is that of a free particle in one dimension which has solution

$$X_n(x) = c_2\cos(k_n x) + c_3\sin(k_n x),$$

where c_2 and c_3 are arbitrary constants. The general class of separable solutions to the eigenvalue problem $-\Delta_A^b\psi = \lambda\psi$ are hence given by

$$\psi(x,\phi) = \sum_n e^{in\phi}\left(A_n\cos(k_n x) + B_n\sin(k_n x)\right),$$

where A_n and B_n are arbitrary constants. Imposing the boundary condition that the wave function vanishes at $x \to \pm\infty$ for all ϕ, it is clear that the discrete spectrum is empty. However, the continuous spectrum is non-empty. Since $k_n^2 > 0$ for all n, it follows that the continuous spectrum is

$$\sigma_c(-\Delta_A^b) = \bigcup_{n\in\mathbb{Z}}\{\lambda \in \mathbb{R} : \lambda > \mu_n\} = [z_{b,\gamma}, \infty),$$

where $z_{b,\gamma} := \min\{(n - b\gamma)^2 : n \in \mathbb{Z}\} \in [0, 1/4]$. An expression for $z_{b,\gamma}$ without involving n can be found using the floor and ceiling functions $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ respectively. One has

$$z_{b,\gamma} = \min\{b\gamma - \lfloor b\gamma\rfloor, \lceil b\gamma\rceil - b\gamma\}^2 = (1/2 - |1/2 + \lfloor b\gamma\rfloor - b\gamma|)^2.$$

Clearly, if $b\gamma \in \mathbb{Z}$, we have $z_{b,\gamma} = 0$, so that $\sigma_c(-\Delta_A^b) = [0,\infty)$. Hence, the magnetic flux makes a difference to the spectrum, as expected due to the topology of the Aharonov-Bohm potential.

Note that if we restrict the particle to a finite surface area of the cylinder (similarly to the way we did on the Euclidean plane) and impose the condition that the wave function vanishes at the boundaries, then the continuous spectrum disappears and the spectrum is purely discrete. For instance, if we restrict to $x \in [0, L]$ (which can be done without loss of generality due to translational symmetry) with $L > 0$, one can verify that the discrete spectrum is given by

$$\sigma_{\mathrm{d}}(-\Delta_A^b) = \{(l\pi/L)^2 + (n - b\gamma)^2 : l \in \mathbb{Z}, n \in \mathbb{Z}\},$$

which results in the energy spectrum

$$E_{l,n} = \frac{\hbar^2}{2m}\left(\left(\frac{l\pi}{L}\right)^2 + \left(n - \frac{qa^2B_0}{2\hbar}\right)^2\right), \quad l, n \in \mathbb{Z}.$$

Similarly to the case where one restricts a particle to a circular ring (as studied at the end of Section 4.3), the term $(n - (qa^2B_0)/(2\hbar))^2$ lifts the two-fold degeneracy of the energy spectrum.

4.5 Spectrum on the Grushin cylinder

This section is based on [8], [32]. We use the same coordinates (x, ϕ) as the Euclidean unit cylinder. Choosing the vector potential such that $A = \gamma\ d\phi$, a computation using Theorem 4.7 yields the magnetic Laplace-Beltrami operator

$$\Delta_A^b = \frac{\partial^2}{\partial x^2} - \frac{1}{x}\frac{\partial}{\partial x} + x^2\left(\frac{\partial^2}{\partial \phi^2} - 2ib\gamma\frac{\partial}{\partial \phi} - b^2\gamma^2\right).$$

Let M be the Grushin cylinder. Through a slight extension of the proof in [33], one finds that the magnetic Laplace-Beltrami operator with domain $C_c^\infty(M \setminus \mathcal{Z})$ is essentially self-adjoint on $L^2(M, \omega_g)$ and separates in the direct sum of its restrictions to $M_\pm = \mathbb{R}_\pm \times \mathbb{S}^1$ (for more information on the separation, see Appendix D). Hence, the evolutions on the two sides of the singularity are independent of each other. Hence, without loss of generality, one may restrict Δ_A^b to act on M_+. We may separate this further. Cylindrical symmetry motivates the use of a Fourier transform with respect to the variable ϕ. One then obtains the separation of spaces

$$L^2(M_+, \omega_g) = \bigoplus_{k \in \mathbb{Z}} H_k, \quad H_k \cong L^2(\mathbb{R}_+, (1/x)dx),$$

and the transformation of operators

$$\Delta_A^b = \bigoplus_{k \in \mathbb{Z}} \widehat{\Delta}_{A,k}^b, \quad \widehat{\Delta}_{A,k}^b = \frac{\partial^2}{\partial x^2} - \frac{1}{x}\frac{\partial}{\partial x} - x^2(k - b\gamma)^2.$$

Since Δ_A^b is essentially self-adjoint, each $\widehat{\Delta}_{A,k}^b$ is self-adjoint on the closure (with respect to the graph norm of $C_c^\infty(\mathbb{R}_+)$). By the transformation $U : L^2(\mathbb{R}_+, (1/x)dx) \to L^2(\mathbb{R}_+, dx)$ defined by $U\psi(x) := \frac{\psi(x)}{\sqrt{x}}$, one obtains that $L_{A,k}^b := U\widehat{\Delta}_{A,k}^b U^{-1}$ when acted on $\psi \in H_k$ takes the form

$$\begin{aligned}
L_{A,k}^b\psi &= U\widehat{\Delta}_{A,k}^b(\sqrt{x}\psi) \\
&= U\left(\left(\sqrt{x}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{\sqrt{x}}\frac{\partial\psi}{\partial x} - \frac{\psi}{4x^{3/2}}\right) - \left(\frac{1}{\sqrt{x}}\frac{\partial}{\partial x} + \frac{\psi}{2x^{3/2}}\right) - x^{5/2}(k - b\gamma)\psi\right) \\
&= \frac{\partial^2\psi}{\partial x^2} - \frac{3\psi}{4x^2} - x^2(k - b\gamma)^2\psi.
\end{aligned}$$

This leads to the fact that

$$L_{A,k}^b = \frac{\partial^2}{\partial x^2} - \frac{3}{4}\frac{1}{x^2} - x^2(k - b\gamma)^2, \quad \mathrm{dom}(L_{A,k}^b) = U(\mathrm{dom}(\widehat{\Delta}_{A,k}^b)).$$

Note that the spectrum does not change under unitary transformations. Namely, we have

$$(L^b_{A,k} - \lambda)\psi = 0 \iff (\widehat{\Delta}^b_{A,k} - \lambda)U^{-1}\psi = 0. \tag{4.6}$$

This is easy to show: Assume $(L^b_{A,k} - \lambda)\psi = 0$, then

$$\begin{aligned}
(\widehat{\Delta}^b_{A,k} - \lambda)U^{-1}\psi &= (U^{-1}L^b_{A,k}U - \lambda)U^{-1}\psi \\
&= (U^{-1}L^b_{A,k} - \lambda U^{-1})\psi \\
&= U^{-1}U(U^{-1}L^b_{A,k} - \lambda U^{-1})\psi \\
&= U^{-1}(L^b_{A,k} - \lambda)\psi = 0.
\end{aligned}$$

The converse is shown similarly. It therefore suffices to study the spectral properties of this operator, since one has a formula for the direct sum of operators (recall Theorem 4.19). Now let $k \neq b\gamma$. Using the equivalence in equation (4.6) and letting $\widetilde{\psi} = U^{-1}\psi$, the problem rests on solving the ordinary differential equation

$$\frac{d^2\widetilde{\psi}}{dx^2} - \frac{1}{x}\frac{d\widetilde{\psi}}{dx} - x^2(k - b\gamma)^2\widetilde{\psi} - \lambda\widetilde{\psi} = 0.$$

The change of variable $z = |k - b\gamma|x^2$ results in the equation

$$4z|k - b\gamma|\frac{d^2\widetilde{\psi}}{dz^2} - z|k - b\gamma|\widetilde{\psi} - \lambda\widetilde{\psi} = 0.$$

Division by $4z|k - b\gamma|^2$ results in a specific case of Whittaker's equation

$$\frac{d^2\widetilde{\psi}}{dz^2} + \left(-\frac{1}{4} - \frac{\lambda}{4z|k - b\gamma|}\right)\widetilde{\psi} = 0,$$

which has solutions $M_{\lambda/(4|k-b\gamma|),1/2}(z)$ and $W_{\lambda/(4|k-b\gamma|),1/2}(z)$. For more information on the Whittaker equation and functions, see Appendix C.2. It thus follows that solutions to the eigenvalue problem are linear combinations of the functions

$$\psi_1(x) = \frac{1}{\sqrt{x}}M_{\lambda/(4|k-b\gamma|),1/2}(|k - b\gamma|x^2), \quad \psi_2(x) = \frac{1}{\sqrt{x}}W_{\lambda/(4|k-b\gamma|),1/2}(|k - b\gamma|x^2).$$

By definition of the Whittaker functions, one can see that $\psi_1 \notin L^2(\mathbb{R}_+, dx)$ for all choices of λ and $\psi_2 \in L^2(\mathbb{R}_+, dx)$ if and only if $\frac{\lambda}{4|k-b\gamma|}$ is a non-negative integer. It thus follows that the eigenvalues of $L^b_{A,k}$ are

$$\lambda^{b,\gamma}_{n,k} = 4n|k - b\gamma|, \quad n \in \mathbb{Z},$$

with corresponding eigenfunctions

$$\psi^{b,\gamma}_{n,k}(x) = \frac{1}{\sqrt{x}}W_{n,1/2}(|k - b\gamma|x^2).$$

Now consider the case where $k = b\gamma$, [30] shows that $-L^b_{A,b\gamma}$ has purely continuous spectrum $[0, \infty)$. It is a Schrödinger operator with a Calogero potential of strength $3/4$.

We now reconsider the original magnetic Laplace-Beltrami operator on M_+. It follows from Theorem 4.19 and the above that the discrete spectrum is given by

$$\sigma_{\mathrm{d}}(-\Delta^b_A) = \left\{\lambda^{b,\gamma}_{n,k} = 4n|k - b\gamma| : n \in \mathbb{N}, k \in \mathbb{Z} \setminus \{b\gamma\}\right\}.$$

If $b\gamma \in \mathbb{Z}$, the spectrum has continuous part $[0, \infty)$. Otherwise, the spectrum is purely discrete. Via the Fourier transform used previously and the definition of U, it follows that the corresponding eigenfunctions for any selection of $b\gamma$ are $\psi^{b,\gamma}_{n,k}(x,\phi) = \frac{e^{ik\phi}}{x} W_{n,1/2}(|k-b\gamma|x^2)$. Clearly, the spectrum changes significantly depending on $b\gamma$, which from a physical point of view depends on the electric charge and the magnetic flux through the solenoid. This is even more apparent when computing the degeneracy of the spectrum. Let $d(n)$ be the number of divisors of n. We claim that the following holds for all $b\gamma \in \mathbb{R}$:

- If $b\gamma$ is irrational, the spectrum is simple (i.e., non-degenerate).
- If $b\gamma$ is rational, each eigenvalue λ has multiplicity less than or equal to $2d(\lambda/4)$.
- If $b\gamma$ is an integer, the eigenvalues have multiplicity $2d(\lambda/4)$ if $\lambda/4$ is odd and $2d(\lambda/4)-2$ if $\lambda/4$ is even.

Proof. Suppose that $b\gamma$ is irrational. Then $x := k - b\gamma$ is irrational. Fix an arbitrary (n,k), suppose that $\lambda^{b,\gamma}_{n',k'} = \lambda^{b,\gamma}_{n,k}$ for some pair $(n',k') \neq (n,k)$ and let $x' := k' - b\gamma$. Note that k' will differ from k by an nonzero integer, say m so that $k' = k - m$. It then follows that $n'|x-m| = n|x|$. There are now three cases, where both $x-m$ and x are positive, where $x-m$ is negative and the other positive, and where both are negative. For simplicity, we only consider the case where both are positive, as the other cases are proved similarly. In this case, we obtain the equality $n'(x-m) = nx$, which is equivalent to $x = \frac{n'm}{n'-n}$. However, this implies that x is rational, which is a contradiction. Hence $\lambda^{b,\gamma}_{n',k'} = \lambda^{b,\gamma}_{n,k}$ if and only if $(n',k') = (n,k)$, which shows that the spectrum is simple.

Now suppose that $b\gamma$ is rational. Then there exists p, q such that $b\gamma = p/q$ with $p \in \mathbb{Z}$, $q \in \mathbb{N}$ such that $\gcd(p,q) = 1$. Fix an arbitrary (n,k) and suppose that $\lambda^{b,\gamma}_{n',k'} = \lambda^{b,\gamma}_{n,k}$ for some pair (n',k'). It then follows that

$$4n'|qk' - p| = q\lambda^{b,\gamma}_{n,k}.$$

Now assume without loss of generality that $qk' - p$ is non-negative. Since $\gcd(p,q) = 1$, we have $4n'|qk'-p| \nmid q$, hence $4n'|qk'-p| \mid \lambda^{b,\gamma}_{n,k}$. If $b\gamma$ is not an integer, then $q \neq 1$, so that $\{|qk'-p| : k \in \mathbb{Z}\} \subseteq (q\mathbb{Z} - p) \subsetneq \mathbb{Z}$. Hence, it follows that the number of (n',k') such that $\lambda^{b,\gamma}_{n',k'} = \lambda^{b,\gamma}_{n,k}$ can be at most $2d(\lambda^{b,\gamma}_{n,k}/4)$, as required.

Lastly, suppose that $b\gamma$ is an integer. Assume without loss of generality that k is non-negative. Then one has

$$\lambda^{b,\gamma}_{k,n+b\gamma} = \lambda^{b,\gamma}_{n,k+b\gamma} = \lambda^{b,\gamma}_{n,-k+b\gamma} = \lambda^{b,\gamma}_{k,-n+b\gamma}.$$

In the case that $n|k| = \lambda^{K,\gamma}_{n,k+K\gamma}/4$ is even (where equality holds for arbitrary $K \in \mathbb{R}$), then two of the combinations repeat themselves (when $n = k$). If $\lambda^{K,\gamma}_{n,k+K\gamma}/4$ is odd, then this does not happen, as required. $\square$

5 Extensions and related effects

In the following, we discuss extensions and modifications to the canonical example (namely, a charged particle moving in opposite directions around a infinitely long cylindrical solenoid in Euclidean space without any relativistic effects) discussed in the context of the Aharonov-Bohm effect. Long derivations will be omitted in this section and we will refer to some sources for the interested reader.

5.1 The relativistic Aharonov-Bohm effect

In this section, we derive an analogous effect for a relativistic Dirac particle (a massive particle of spin-1/2, such as an electron) via the Dirac equation with electromagnetic effects (which we derive first in this section). We use the Einstein summation convention throughout and employ the $(+,-,-,-)$ metric signature for the Minkowski metric η. We let I_n be the $n \times n$ identity matrix. This section is based on [34]–[36].

We let ψ be a (four-component) Dirac spinor and denote its Dirac adjoint by $\overline{\psi} := \psi^\dagger \gamma^0$. The Lagrangian density for the standard Dirac equation is known to be

$$\mathcal{L}_1 = \overline{\psi}(i\hbar c \gamma^\mu \partial_\mu - mc^2)\psi,$$

where $\partial_\mu = ((1/c)(\partial/\partial t), \nabla)$ and γ^μ are the Dirac matrices which satisfy the anti-commutation relation $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_4$ and

$$(\gamma^0)^2 = I_4, \quad (\gamma^k)^2 = -I_4,$$

where $k \in \{1,2,3\}$. In the following, we take the Dirac basis

$$\gamma^0 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}, \quad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix},$$

where σ^k are the Pauli matrices given by

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

For an electromagnetic field with four-vector potential $A^\mu = (V/c, \mathbf{A})$, the Lagrangian density is

$$\mathcal{L}_2 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - qA_\mu j^\mu,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor and $j^\mu = c\overline{\psi}\gamma^\mu \psi$ is the current density (this can be derived using $U(1)$ symmetry, otherwise the Lagrangian density is not gauge invariant). The QED Lagrangian density is therefore given by the sum

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = \overline{\psi}\left(i\hbar c\gamma^\mu \left(\partial_\mu + \frac{iq}{\hbar} A_\mu\right) - mc^2\right)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

Using the Euler-Lagrange equation for fields, one obtains the following equation of motion by varying with respect to $\overline{\psi}$:

$$i\hbar c\gamma^\mu \left(\partial_\mu + \frac{iq}{\hbar} A_\mu\right)\psi - mc^2\psi = 0. \tag{5.1}$$

This is one way to write the Dirac equation with an electromagnetic field. However, it is more convenient to separate the time and spatial components for our analysis. Multiplying by γ^0 from the left and using $(\gamma^0)^2 = I_4$ gives

$$i\hbar \frac{\partial \psi}{\partial t} + i\hbar c \gamma^0 \gamma^k \left(\partial_k + \frac{iq}{\hbar} A_k\right)\psi - mc^2 \gamma^0 \psi - qV\psi = 0.$$

Recall that $\mathbf{p} \to -i\hbar\nabla$, it therefore follows that $p_k = i\hbar\partial_k$. Hence,

$$i\hbar\frac{\partial\psi}{\partial t} = c\gamma^0\gamma^k(-p_k + qA_k)\psi + mc^2\gamma^0\psi + qV\psi,$$

which can be written as

$$i\hbar\frac{\partial\psi}{\partial t} = \mathcal{H}_D\psi, \quad \mathcal{H}_D := c\boldsymbol{\alpha}\cdot(\mathbf{p} - q\mathbf{A}) + \beta mc^2 + qVI_4. \tag{5.2}$$

where $\alpha^k := \gamma^0\gamma^k$, $\beta := \gamma^0$ and $\boldsymbol{\alpha} := (\alpha^1, \alpha^2, \alpha^3)$ is a vector with matrix entries. One may also express α^k and β in terms of the Pauli matrices. We obtain

$$\alpha^k = \begin{bmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}.$$

Both matrices are Hermitian and they satisfy the involutive property, namely $(\alpha^k)^2 = I_4$ and $\beta^2 = I_4$. Additionally, they satisfy the anti-commutation relations

$$\{\alpha^k, \alpha^l\} = 2\eta^{kl}I_4, \quad \{\alpha^k, \beta\} = 0,$$

where $l \in \{1, 2, 3\}$. In particular, the matrices are mutually anti-commutative if $k \neq l$.

Equation (5.2) is the electromagnetic Dirac equation. Like in the non-relativistic case, assume there is no electric field and that $V = 0$ due to gauge freedom. If $\mathbf{A}$ is time-independent, it can be shown analogously to the proof of equation (2.5) that the solution to this equation is given by

$$\psi(\mathbf{r}, t) = e^{ig(\mathbf{r})}\psi_0(\mathbf{r}, t), \quad g(\mathbf{r}) = \frac{q}{\hbar}\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}',$$

where $\psi_0(\mathbf{r}, t)$ is the solution to the free Dirac equation (when $\mathbf{A} = 0$). Notice that the phase factor in the relativistic case is exactly the same as the non-relativistic case, the only difference being that the standard wave function gets replaced by a Dirac spinor.

5.2 The Aharonov-Bohm effect for the hydrogen atom with magnetic monopole field

In the following we discuss a case where the vector potential $\mathbf{A}$ differs from the standard infinite solenoid example. To this end, we consider the case where an electron of charge $q = -e$ is bound to a hydrogen nucleus with Coulomb potential energy $V(r) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$, when both the Aharonov-Bohm field and magnetic monopole field are present. This section is based on [37], [38].

It is well-known that there is currently no experimental evidence that magnetic monopoles exist. Nevertheless, it is interesting to analyse what would happen assuming they did, as this could provide evidence for their existence/non-existence. An attempt of this was made by Dirac, who showed that if magnetic monopoles exist, then electric charge must be quantized (this is called the Dirac quantization condition). As a matter of fact, this is indeed the case. Had this been false, we would be able to conclude that they do not exist.

The magnetic field of a point magnetic monopole of strength g analogous to a point electric charge is given by (in spherical coordinates $(r, \theta, \varphi) \in [0, \infty) \times [0, \pi] \times [0, 2\pi)$)

$$\mathbf{B} = \frac{g}{r^2}\hat{\mathbf{r}}.$$

It can be shown that the vector potential $\mathbf{A}_N$ defined by

$$\mathbf{A}_N := \frac{g(1 - \cos(\theta))}{r\sin(\theta)}\hat{\boldsymbol{\varphi}},$$

satisfies $\mathbf{B} = \nabla \times \mathbf{A}_N$ everywhere except when $\theta = \pi$. This line of singularity is called a Dirac string. To avoid this singularity, Wu and Yang showed that for $\delta \in (0, \pi/2]$, the two vector potentials $\mathbf{A}_N$ defined for $\theta \in [0, \pi/2 + \delta)$ (given above) and $\mathbf{A}_S$ defined for $\theta \in (\pi/2 - \delta, \pi]$ by

$$\mathbf{A}_S := -\frac{g(1+\cos(\theta))}{r\sin(\theta)}\hat{\boldsymbol{\varphi}},$$

differ by a gauge transformation in the overlap region $(\pi/2 - \delta, \pi/2 + \delta)$, with $\mathbf{B} = \nabla \times \mathbf{A}_S$. Namely, we have [39]

$$\mathbf{A}_N - \mathbf{A}_S = \frac{2g}{r\sin(\theta)}\hat{\boldsymbol{\varphi}} = 2g\nabla\varphi.$$

Hence, we may assume without loss of generality that the vector potential due to the magnetic monopole is $\mathbf{A}_g := \mathbf{A}_N$. In spherical coordinates, the Aharonov-Bohm potential is given by

$$\mathbf{A}_{\mathrm{AB}} = \frac{a^2 B_0}{2r\sin(\theta)}\hat{\boldsymbol{\varphi}},$$

so that the total vector potential is given by $\mathbf{A} = \mathbf{A}_g + \mathbf{A}_{\mathrm{AB}}$. This vector potential clearly has different functional dependence from the standard Aharonov-Bohm potential, assuming that $g \neq 0$. We thus obtain a different partial differential equation for the (time-independent) wave function which can be solved by separation of variables, namely

$$-\frac{\hbar^2}{2m}\left(\nabla + \frac{ie}{\hbar}\mathbf{A}\right)^2\psi - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}\psi = E\psi.$$

In this case, the result depends heavily on the values of two dimensionless parameters ξ and η defined by

$$\xi = ge/\hbar, \quad \eta = \frac{a^2 B_0 e}{2\hbar} + ge/\hbar + k_\varphi,$$

where $k_\varphi \in \mathbb{Z}$ is a constant arising from the fact that eigenstates are of the form $\psi(r,\theta,\varphi) = f(r,\theta)e^{ik_\varphi\varphi}$, where f is a smooth function (which can be shown using separation of variables with ansatz $\psi(r,\theta,\varphi) = f(r,\theta)h(\varphi)$). The energy spectrum is then shown to be

$$E_{l,n} = -\frac{1}{2\hbar^2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{me^4}{\left(l + \sqrt{(n+1/2 - \max\{|\xi|,|\eta|\})^2 - \eta^2 + 1/2}\right)^2},$$

where $l \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{Z}$.

This leads to the following surprising physical consequence: In the case that $|\eta| < |\xi|$, the energy spectrum is independent of the magnetic flux of the solenoid. Hence, the Aharonov-Bohm effect is absent for these quantum states. The same phenomenon has been found to occur in the case of a relativistic spin-0 particle [37] and a relativistic spin-1/2 particle [40].

5.3 The Aharonov-Casher effect

An effect related to the Aharonov-Bohm effect is the Aharonov-Casher effect. In this case, a (neutral) magnetic dipole $\boldsymbol{\mu}$ is affected by an electric field $\mathbf{E}$. Recall that a magnetic moment is generated by a current distribution

$$\mathbf{J}_m = \nabla \times \mathbf{M}, \quad \boldsymbol{\mu} = \iiint \mathbf{M}\, dV,$$

where $\mathbf{M}$ is the magnetization and $\mathbf{J}_m$ is the contribution to current density due to magnetization. In the Euclidean case, one has an acquired phase shift due to the non-simply connected nature of the setup (similar to the Aharonov-Bohm effect). The effect was first predicted in

1984 by Yakir Aharonov and Aharon Casher [41] and was observed experimentally in 1989 using neutron interferometry [42]. In fact, the two effects are electromagnetically dual. Roughly speaking, the roles of the solenoid and moving electrons switch. Instead, we have neutral particles possessing a magnetic dipole moment (such as neutrons) moving around a line charge.

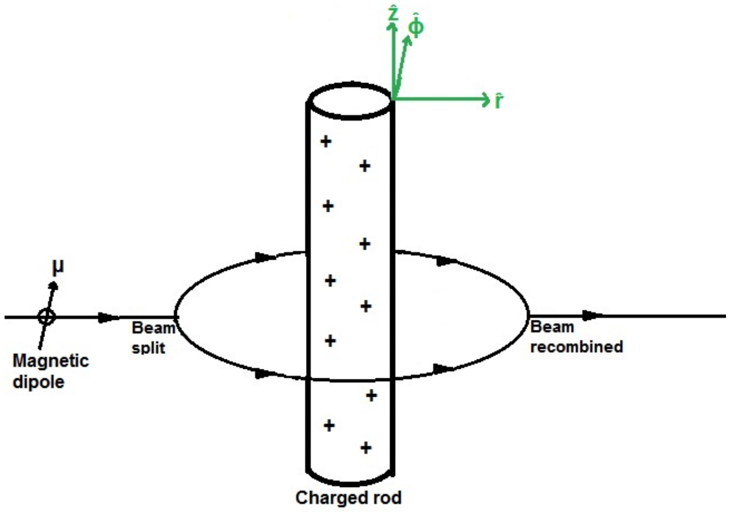

Figure 5.1: The Aharonov-Casher effect (compare with Figure 2.1)

Aharonov and Casher showed that the Hamiltonian for this system (with vanishing scalar potential) is given by

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{\mathbf{E}\times\boldsymbol{\mu}}{c^2}\right)^2 - \frac{\mu^2 E^2}{mc^4},$$

where c is the speed of light. In the case that $\mu E/c^2 \ll mv$ (here E is the electric field strength, not the energy), which happens in most physical cases, one can neglect the term $\mu^2E^2/(mc^4)$. Hence, in the following we drop this term [43]. Analogous to the derivation of the Aharonov-Bohm effect, the substitution $\mathbf{p} \to -i\hbar\nabla$ promotes this Hamiltonian into an operator $\mathcal{H}$ on $L^2(\mathbb{R}^3)$, which allows for the transition to quantum mechanics.

$$i\hbar\frac{\partial\psi}{\partial t} = \mathcal{H}\psi = -\frac{\hbar^2}{2m}\left(\nabla - \frac{i}{\hbar c^2}\mathbf{E}\times\boldsymbol{\mu}\right)^2\psi. \tag{5.3}$$

The computation of the phase factor is thus analogous for static electromagnetic fields and works out to be (cf. equation (2.5))

$$g_{\mathrm{AC}}(\mathbf{r}) = \frac{1}{\hbar c^2}\int_{\mathcal{O}}^{\mathbf{r}} (\mathbf{E}\times\boldsymbol{\mu})\cdot d\mathbf{r}'.$$

Due to the similar form of the Schrödinger equations (cf. equation (2.4)), we suspect that one could perform a similar spectral analysis on various almost-Riemannian manifolds for the Aharonov-Casher effect as we have done in Section 4 for the Aharonov-Bohm effect. The transition to differential forms can be made by noting that the Hodge dual of the wedge product of two vectors in $\mathbb{R}^3$ is just the cross product and that the magnetic moment is a axial vector (like the magnetic field), which results in the correspondence

$$\boldsymbol{\mu} = \mu_1\hat{\mathbf{x}} + \mu_2\hat{\mathbf{y}} + \mu_3\hat{\mathbf{z}} \longleftrightarrow \mu = \mu_1\ dy\wedge dz + \mu_2\ dz\wedge dx + \mu_3\ dx\wedge dy.$$

This will not be discussed here, but it is a possible topic for future research.

6 Discussion

In Section 2, we found in the Euclidean case that if one splits an electron beam in two around a long cylindrical solenoid and bring them back to a point with non-zero magnetic flux through the area between the paths, an interference pattern will appear. This comes from the fact that the wave function of a charged particle travelling in different directions around the solenoid acquires a change in phase independent of time (it depends only on the path taken) called geometric phase. This shows that the Aharonov-Bohm effect is a non-local phenomenon, otherwise the fact that the magnetic field is zero where the electron beam passes would not affect them in any way.

In Section 4, we found that if a charged particle is bound to a plane (without the long cylindrical solenoid's cross section), one finds the spectra

$$\sigma_{\mathrm{d}}(\mathcal{H}) = \varnothing, \quad \sigma_{\mathrm{c}}(\mathcal{H}) = [0, \infty).$$

Surprisingly, this is independent of the magnetic flux through the solenoid. However, if one restricts the particle to a subset of finite area, then the continuous part of the spectrum disappears and the spectrum depends on the magnetic flux through the solenoid.

When one restricts the charged particle to a cylinder around the solenoid, one finds the spectra

$$\sigma_{\mathrm{d}}(\mathcal{H}) = \varnothing, \quad \sigma_{\mathrm{c}}(\mathcal{H}) = \left[\frac{\hbar^2}{2m}\left(\frac{1}{2} - \left|\frac{1}{2} + \left\lfloor\frac{qa^2B_0}{2\hbar}\right\rfloor - \frac{qa^2B_0}{2\hbar}\right|\right)^2, \infty\right).$$

In contrast to the Euclidean plane, the magnetic flux through the solenoid affects the spectrum in both cases, whether one does or does not restrict the particle to a subset of finite area.

Finally, on the Grushin cylinder, it was found that

$$\sigma_{\mathrm{d}}(\mathcal{H}) = \left\{\frac{2\hbar^2 n}{m}\left|k - \frac{qa^2B_0}{2\hbar}\right| : n \in \mathbb{N}, k \in \mathbb{Z} \setminus \left\{\frac{qa^2B_0}{2\hbar}\right\}\right\}, \quad \sigma_{\mathrm{c}}(\mathcal{H}) = \begin{cases} [0, \infty), & \frac{qa^2B_0}{2\hbar} \in \mathbb{Z}, \\ \varnothing, & \text{otherwise.} \end{cases}$$

The main difference between the effect on Riemannian manifolds and purely almost-Riemannian manifolds is that the nature itself of the spectrum (continuity and/or discreteness) is affected purely by a change in the magnetic flux on almost-Riemannian manifolds, whereas in Riemannian manifolds it is independent of the magnetic flux. It is also quite easy to see that the Grushin cylinder has infinite (almost-Riemannian) area $\int_{M\setminus\mathcal{Z}} \omega_g$. Hence, it is quite unusual that the continuous part of the spectrum disappears for some magnetic fluxes, since in most physical situations where a particle is not localized to a region of finite measure (scattering state), the energy spectrum is expected to be purely continuous. For instance, a one-dimensional free quantum particle due to the lack of boundary conditions is well-known to have continuous energy spectrum

$$E_k = \frac{\hbar^2 k^2}{2m}, \quad k \in \mathbb{R},$$

with non-normalizable (time-independent) eigenstates $\psi_k(x) = e^{ikx}$. Ultimately, doing this analysis on almost-Riemannian structures improves our mathematical understanding on how singular spaces (in the almost-Riemannian sense) affect the spectrum of the Laplace-Beltrami operator. It also improves our understanding on the control of quantum mechanical systems.

In Section 5, it was found that the phase factor for a relativistic massive spin-1/2 particle obtained via the magnetic flux is the same as the non-relativistic case. However, this does not necessarily imply that the spectrum of the Dirac operator $\mathcal{H}_D$ (see equation (5.2)) is the same as that of the Hamiltonian in the non-relativistic case. This may be worth investigating, as in [44], the authors have found an alternative explanation of the Aharonov-Bohm effect in terms of special relativity. We have also investigated some of the phenomena that could occur if magnetic monopoles were to exist. In particular, we focused on the case where an electron bound to a proton (a hydrogen atom) is subject to a magnetic monopole field and Aharonov-Bohm potential. It was found that despite the non-simply connected topology, the Aharonov-Bohm effect is absent (i.e., no difference in geometric phase is acquired) for some quantum states. Lastly, it was found that the Aharonov-Casher effect in a non-relativistic limit and under some approximation has similar properties to the Aharonov-Bohm effect, but with a difference in magnitude of the phase factor. A similar spectral study may thus be performed (this is likely to be easier than the relativistic extension).

Some techniques in this thesis can be used to compute the spectrum on n-dimensional versions of the almost-Riemannian manifolds covered here (for an arbitrary dimensional definition of an almost-Riemannian manifold, we refer to Section 3.1.3 of [5]). Unfortunately, the techniques used to derive the spectra in this thesis rely heavily on the freeness and explicit nature of the almost-Riemannian manifolds and the integrability of the geodesic equation of the almost-Riemannian structures and they are thus not suitable to derive spectral properties associated with the Aharonov-Bohm effect on generic almost-Riemannian manifolds.

7 Conclusion

In this thesis, we have investigated the theory and results regarding the spectrum of the magnetic Laplace-Beltrami operator on various two-dimensional almost-Riemannian manifolds with the vector potential produced by a infinitely long solenoid.

For the cases where the manifold is also Riemannian (with Euclidean metric), we find for both the plane (see Remark 4.20) and the unit cylinder that the discrete part of the spectrum vanishes if one does not constrain a particle to a subset of finite measure. In that case, the continuous part of the spectrum may vary with magnetic flux, which is expected due to the non-simply connectedness of the manifolds. The type of spectrum was found to be independent of the magnetic flux. In the purely almost-Riemannian case we considered (the Grushin cylinder), we found that the above does not hold. Both the types of spectrum and degeneracies are extremely sensitive to the magnetic flux.

We have also discussed what would happen if one considers more complex physical systems or if one adds certain effects. For instance, we have found that the existence of magnetic monopoles implies that the Aharonov-Bohm effect may not appear in some cases despite non-simply connected topology. Additionally, we have found in a QFT framework that the addition of relativity has no effect on the geometric phase, but we suspect that the energy spectrum may differ from the standard quantum mechanical case.

A possibility for future research could be spectral analysis of the electromagnetic Dirac operator (instead of the magnetic Laplace-Beltrami operator) on various almost-Riemannian manifolds. This is particularly useful for constructing path integrals in quantum field theory (this formulation is equivalent to canonical quantization) [45]. A similar analysis could also be performed on the related Aharonov-Casher effect.

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Appendices

A Simply connectedness and homotopy

In the following, we introduce the notion of simply connectedness and what it means for two paths to be homotopic.

Definition A.1. Let X be a topological space. A *path* from a point x to a point y in X is a continuous function $f : [0, 1] \to X$ with $f(0) = x$ and $f(1) = y$.

Definition A.2. A topological space X is called *path-connected* if there is a path joining any two points in X.

Definition A.3. Let $p : [0, 1] \to X$ and $q : [0, 1] \to X$ be two paths with the same start and end points ($p(0) = q(0)$ and $p(1) = q(1)$). A topological space X is *simply connected* if it is path-connected and there exists a continuous map $F : [0, 1] \times [0, 1] \to X$ such that $F(x, 0) = p(x)$ and $F(x, 1) = q(x)$ (F is then called a *homotopy* and the two paths are said to be *homotopic*). Intuitively, this means that p can be continuously deformed into q while staying in the topological space and keeping the endpoints fixed.

Example A.4. Below are examples of topological spaces in $\mathbb{R}^2$ and $\mathbb{R}^3$ (with the Euclidean topology) which are either simply connected or not simply connected.

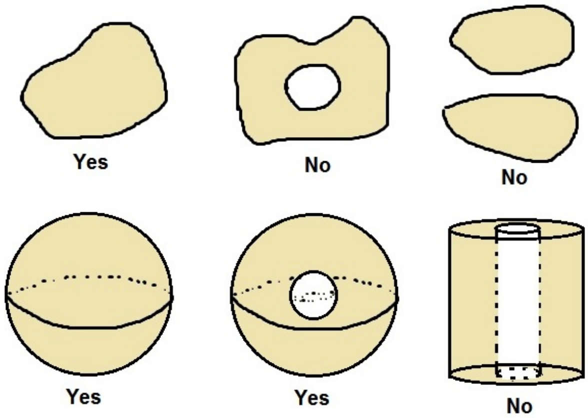

Figure A.1: Examples of (non-)simply connected domains

Considering the domains of $\mathbb{R}^2$, the second (from left to right) is not simply connected, because if one lets p and q be two paths that go around the inner circle in opposite directions, one cannot continuously deform p into q without leaving the domain. The third is not simply connected, as it is not path-connected. For domains in $\mathbb{R}^2$, one can (roughly speaking) identify simply connected domains as those which do not have holes in them.

Regarding the domains of $\mathbb{R}^3$, the concept is more subtle. For instance, the second domain has a hole inside the sphere, but is simply connected. The third domain is not simply connected (for a similar reason as the second domain of $\mathbb{R}^2$) and is of particular importance for the study of the Aharonov-Bohm effect (with respect to the solenoid example of Figure 2.1).

B An alternative definition for the Laplace-Beltrami operator

It is possible to define the Laplace-Beltrami operator without use of the Hodge star operator. This section is based on [18].

Definition B.1. Let $f \in C^\infty(M)$ and (M, g) be a n-dimensional orientable Riemannian manifold with volume form as in equation (3.5). We define the *Laplace-Beltrami operator* to be

$$\Delta := \operatorname{div} \circ \nabla,$$

where the gradient operator $\nabla : C^\infty(M) \to \mathcal{X}(M)$ is defined such that for all $X \in \mathcal{X}(M)$

$$g(\nabla f, X) = df(X),$$

and the divergence operator $\operatorname{div} : \mathcal{X}(M) \to C^\infty(M)$ is defined such that for all $X \in \mathcal{X}(M)$

$$d(\iota_X \omega_g) = (\operatorname{div}(X))\omega_g,$$

where ι_X is interior multiplication by X.

The Laplace-Beltrami operator is thus dependent on the choice of Riemannian metric.

Theorem B.2. *An expression for the Laplace-Beltrami operator on a coordinate chart* $(U, x^1, \dots, x^n)$ *is*

$$\Delta = \frac{1}{\sqrt{|\det(g_{ij})|}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{|\det(g_{ij})|} g^{ij} \frac{\partial}{\partial x^j} \right). \tag{B.1}$$

Proof. We first find an expression in local coordinates for the gradient operator. We assume first that $\nabla f = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}$ for some $a^i \in C^\infty(U)$. Choosing in particular the vector field $X = \frac{\partial}{\partial x^j}$, we have

$$\frac{\partial f}{\partial x^j} = df\left(\frac{\partial}{\partial x^j}\right) = g\left(\nabla f, \frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a^i \cdot g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n a^i g_{ij}.$$

Fixing some $k \in \{1, \dots, n\}$, we find that

$$\sum_{j=1}^n g^{jk} \frac{\partial f}{\partial x^j} = \sum_{j=1}^n \sum_{i=1}^n a^i g_{ij} g^{jk} = \sum_{i=1}^n a^i \delta_i^k = a^k.$$

Therefore, an expression in local coordinates for the gradient is

$$\nabla f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^i}.$$

We now find an expression in local coordinates for the divergence. Writing $X = \sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}$ with $\xi^j \in C^\infty(U)$ and letting $X_2, \dots, X_n$ be arbitrary vector fields, we find using a cofactor expansion along the first column of the determinant that

$$\begin{aligned}
(\iota_X \omega_g)(X_2, \dots, X_n) &= \omega_g(X, X_2, \dots, X_n) \\
&= \sqrt{|\det(g_{ij})|}(dx^1 \wedge \dots \wedge dx^n)(X, X_2, \dots, X_n) \\
&= \sqrt{|\det(g_{ij})|} \cdot \det \begin{bmatrix} dx^1(X) & dx^1(X_2) & \cdots & dx^1(X_n) \\ dx^2(X) & dx^2(X_2) & \cdots & dx^2(X_n) \\ \vdots & \vdots & \ddots & \vdots \\ dx^n(X) & dx^n(X_2) & \cdots & dx^n(X_n) \end{bmatrix} \\
&= \sqrt{|\det(g_{ij})|} \sum_{i=1}^n (-1)^{i-1} dx^i(X)(dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n)(X_2, \dots, X_k),
\end{aligned}$$

where the hat over a symbol means that the symbol is absent in the sum. Since $dx^i(X) = \xi^i$ for fixed i, it follows that

$$\iota_X \omega_g = \sum_{i=1}^{n} (-1)^{i-1} \sqrt{|\det(g_{ij})|}\xi^i \; dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

We now compute its exterior derivative. We find that

$$\begin{aligned} d(\iota_X \omega_g) &= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial}{\partial x^i} \left(\sqrt{|\det(g_{ij})|}\xi^i \right) \; dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{|\det(g_{ij})|}\xi^i \right) dx^1 \wedge \cdots \wedge dx^n \\ &= \frac{1}{\sqrt{|\det(g_{ij})|}} \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{|\det(g_{ij})|}\xi^i \right) \omega_g. \end{aligned}$$

Therefore, by definition we have the following coordinate expression for the divergence:

$$\operatorname{div}(X) = \frac{1}{\sqrt{|\det(g_{ij})|}} \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{|\det(g_{ij})|}\xi^i \right).$$

Hence, it follows that

$$\Delta f = \frac{1}{\sqrt{|\det(g_{ij})|}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{|\det(g_{ij})|} g^{ij} \frac{\partial f}{\partial x^j} \right),$$

which agrees with equation (B.1). □

Example B.3. In n-dimensional Euclidean space, we have $g_{ij} = \delta_{ij}$, thus the Laplace-Beltrami operator reduces to the standard Laplacian

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial (x^i)^2}.$$

C Special functions

C.1 Bessel functions

This section is based on [46], [47]. Consider the linear second order differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0.$$

This equation has two linearly independent solutions, which we denote by $J_\nu(x)$ and $Y_\nu(x)$. They are referred to as Bessel functions of the first and second kind of order ν respectively. Especially in physics, one sometimes refers to the latter function as a Neumann function $N_\nu(x)$ of order ν. The two functions are defined by the following

$$J_\nu(x) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \nu + 1)} \left(\frac{x}{2} \right)^{2m+\nu},$$

where $\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} \, dx$ is the well-known Gamma function, a (shifted) extension of the factorial for non-integer values, and

$$Y_\nu(x) := \begin{cases} \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}, & \nu \notin \mathbb{Z}, \\ \lim_{\alpha \to \nu} Y_\alpha(x), & \nu \in \mathbb{Z}. \end{cases}$$

The Bessel function of the first kind $J_\nu(x)$ is zero at the origin, finite when $x > 0$, and diverges as $x \to 0$ for negative non-integer ν. In contrast, the Bessel function of the second kind $Y_\nu(x)$ is singular at $x = 0$ for all $\nu \in \mathbb{R}$.

C.2 Whittaker functions

This section is based on [46]. Consider the differential equation

$$\frac{d^2y}{d^2x} + \left(-\frac{1}{4} + \frac{\kappa}{x} + \frac{1/4 - \mu^2}{x^2}\right) = 0.$$

This equation has two linearly independent solutions, which we denote by $M_{\kappa,\mu}(x)$ and $W_{\kappa,\mu}(x)$. They are defined by

$$M_{\kappa,\mu}(x) := x^{\mu+1/2}e^{-x/2}\sum_{n=0}^{\infty}\frac{(\mu-\kappa+1/2)_n}{n!(2\mu+1)_n}x^n,$$

$$W_{\kappa,\mu}(x) := \frac{e^{-x/2}x^{\kappa}}{\Gamma(1/2-\kappa+\mu)}\int_0^{\infty} t^{-\kappa-1/2+\mu}\left(1+\frac{t}{2}\right)^{\kappa-1/2+\mu} e^{-t}\, dt.$$

where $(z)_n$ is the Pochhammer symbol, defined by

$$(z)_n := \frac{\Gamma(z+n)}{\Gamma(z)}.$$

D The splitting of the magnetic Laplace-Beltrami operator on the Grushin cylinder

This section is based on [32]. Let M be the Grushin cylinder with volume form ω_g on $M \setminus \mathcal{Z}$. In the following, we use the definitions of the gradient and Laplace-Beltrami operator as in Appendix B. We define the space

$$H^1(M,\omega_g) = \{u \in L^2(M,\omega_g) : |\nabla u| \in L^2(M,\omega_g)\},$$

with norm

$$\|u\|_{H^1} = \left(\int_M |u|^2 + |\nabla u|^2\right)^{1/2}.$$

From this, we can define the space

$$H_0^1(M,\omega_g) = \overline{C_c^\infty(M)},$$

where the overline represents the closure with respect to the H^1 norm. Finally, we define

$$H_0^2(M,\omega_g) = \{u \in H_0^1(M,\omega_g) : \Delta u \in L^2(M,\omega_g)\}.$$

Since the magnetic Laplace-Beltrami operator Δ_A^b with domain $C_c^\infty(M \setminus \mathcal{Z})$ is essentially self-adjoint on $L^2(M,\omega_g)$, the only self-adjoint extension is the Friedrichs extension $[\Delta_A^b]_F$ which is well-defined and self-adjoint and has domain

$$\mathrm{dom}([\Delta_A^b]_F) = H_0^2(M,\omega_g).$$

Since $L^2(M,\omega_g) = L^2(M_+,\omega_g) \oplus L^2(M_-,\omega_g)$ and $H_0^1(M,\omega_g) = H_0^1(M_+,\omega_g) \oplus H_0^1(M_-,\omega_g)$, it follows that

$$\mathrm{dom}([\Delta_A^b]_F) = H_0^2(M_+,\omega_g) \oplus H_0^2(M_-,\omega_g).$$

Therefore, the operator splits into a direct sum of its restrictions to $M_\pm$, as required. Note that H^1, H_0^1 and H_0^2 are all examples of Sobolev spaces, which are very important in the study of partial differential equations.