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A Hopf algebra approach to q-Deformation of Physics

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Abstract

In this thesis we study the theory of Hopf algebras and their applications to q-deformation of physics. As an example of q-deformation we study the q-deformed quantum harmonic oscillator. We develop the basic theory of Hopf algebras, and generalize this to Hopf algebra objects in arbitrary braided categories. We also discuss the representation theory of Hopf algebras and their connections to algebraic topology; in particular braid- and knot theory. As applications we derive knot invariants from several particular Hopf algebras, and we demonstrate a Tannaka-Krein duality for Hopf algebras. We then discuss the systematic q-deformation of basic physics, including an explicit description of q-deformed Minkowski spacetime. Finally, we discuss the consequences of such a q-deformation for models of physics. This includes a dicussion of basic q-deformed field theories on q-Minkowski spacetime.

Contents

1 Introduction

We begin with some historical notes. The formalism of quantum field theory (QFT) was developed over the last century, and has given rise to the now well-known theories of quantum electrodynamics (QED) and the standard model of particle physics, to name a few. The applicability and accuracy of these theories are unprecedented, and they stand as our best descriptions of reality to date.

Nevertheless, even modern quantum field theories are plagued with inherent problems; essentially the same problems that the pioneers of QFT grappled with initially. In the 1930's, much of the foundations of QFT were already developed by Dirac, Heisenberg, etc. At the time they found that the theory contained insurmountable divergences, and they were ready to abandon it: in 1937 Paul Dirac had said [1]

> "Because of its extreme complexity, most physicists will be glad to see the end of quantum electrodynamics."

The founders of modern quantum theory (e.g. Bohr, Born, Dirac, Heisenberg) believed that the deeper reason for the divergences in QFT lies in an inadequate description of spacetime. After all, it had been demonstrated by Einstein with his theory of relativity that a correct description of spacetime geometry is essential to Physics. Hence it was believed that spacetime itself was also in need of a quantization scheme in order to resolve the problems of QFT: for instance this was proposed in 1938 by Heisenberg [2].

It took a new generation of physicists (Feynman, Schwinger, Tomonaga, etc.) to finally solve the divergences in the 1950's. Their approach is called *renormalization theory*, and it provides results of outstanding accuracy. With the success of renormalization, the ideas of a quantization scheme on spacetime faded.

New problems arise when we wish to introduce gravity to QFT: while QFT describes the world of particles and its forces very accurately, it does not contain any description of gravity. The reason for this is simple: gravity has approximately 10^{-30} times the strength of the strong nuclear force, and 10^{-25} that of the weak nuclear force. Thus gravity is completely negligible on the scale of quantum field theory! The reason that we notice gravity at all is that it is exclusively attractive and acts over arbitrary distances. This allows gravity to be the dominating force on very heavy (electrically neutral) objects, over very large distances; e.g. the dynamics of the solar system. For esoteric reasons, it is naturally believed that all of physics should be described by a single coherent theory, and hence we wish to include gravity into QFT. The problem with this is that *gravity is not renormalizable as a quantum field theory.* For a pedagogical exposition of this fact, see [3].

This brings us to reconsider renormalizability: renormalization allows us to neglect quantum processes that yield divergences, via a *perturbative* treatment. This is because, in a renormalizable theory, such processes are strongly suppressed at the energy scales normally under consideration. As such it is very well possible that renormalizable theories are a low-order approximation of something more fundamental, which could very well be non-renormalizable. More-over, the extreme relative weakness of gravity may be an indication that it is indeed a higher-order effect which should be described by a non-renormalizable theory [4].

This flies in the face of much of what was accomplished since the 1950's: if we are serious about calling renormalization into question for the development of a unifying theory, then we must return to the 30's and again face the divergences of QFT. Due to this consideration, the idea of a quantization scheme for spacetime (together with other related regularization schemes) has enjoyed renewed popularity in recent decades.

1.1 q-Deformation

One approach to regularization of physics and quantization of spacetime is that of *q-deformation*. This is a deformation of some physical system via a parameter q. The classical system is recovered in the limit $q \to 1$, and q is to be thought of as 'close to 1'. This is analogous to the role of $\hbar$ in deforming function algebras when canonical quantization is imposed, except that $\hbar$ is to be though of as close to 0. It is possible to impose a quantization scheme on spacetime (t, x, y, z) by q-deforming its function algebra, so that e.g. x and y no longer commute, but may instead satisfy $xy = qyx$. This noncommutativity of spacetime is analogous to the noncommutativity of position and momentum that is induced by caconical quantization. This is related to the more general development of *noncommutative geometry* concepts: another example of such concepts is found in the theory due to Alain Connes [5].

In principle, for any given system there are many ways to q-deform. There is no rule that tells us where to insert q's, so choosing any particular q-deformation may seen rather ad-hoc. A systematic prescription of q-deformation for flat (Minkowski) spacetime and much of its structure (e.g. derivatives and differential forms) is given by the formalism of *braided geometry*, developed by Shahn Majid in the mid-1990's [6]. An introduction to this is given in the next subsection. First, we sum up the concrete motivations for q-deformation [6]:

- A q-deformed theory of physics, in particular of QFT, may a step in the right direction towards developing a non-renormalizable, unified theory of quantum gravity. We should especially keep an open mind to this idea in light of the issues with renormalization, and the possibility that renormalizability is not a physical requirement.

- The $q \neq 1$ world may have less singularities than our usual geometry. For instance some of the infinities in QFT may actually be poles of the form $\frac{1}{q-1}$. There are two points of view here: either the q-deformation serves as a mathematical tool of q-regularization, or the $q \neq 1$ world really is a crude (perhaps first-order) model of quantum corrections to spacetime. Even if the q-deformation is only a non-phyisical regularization tool, then it is more systematic and less brutal than e.g. an arbitrary momentum cutoff.

- Not all systems are q-deformed equally elegantly. The requirement of q-deformalizability or continuity of physics at $q = 1$ may help as a tool to single out natural structures in physics, by degenerating accidental $q = 1$ isomorphisms.

- In the particular braided geometry theory of q-deformations, interesting unifications occur. Namely, we will see a unification of the ideas of continuous symmetry and of *grading*, i.e. supersymmetry in the case of a $\mathbb{Z}_2$-grading. We will also cast q-deformed Minkowski spacetime as a q-deformed enveloping algebra of the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$: an isomorphism that is not at all apparent for $q = 1$. This may have interesting implications for a possible development of q-deformed field theory, since $\mathrm{SU}(2) \oplus \mathrm{U}(1)$ is the symmetry group of electro-weak theory.

- If it is indeed the case that our $q = 1$ geometry is a special case of something more fundamental or general that works just as well mathematically, then there is no reason we should limit ourselves to $q = 1$ [6].

1.2 Braided Geometry

As stated in the last subsection, braided geometry is the formalism for a systematic approach to q-deformation. The study of braided geometry will take up most of this thesis. Its language is that of *braided categories* and *Hopf algebras*; both unfamiliar objects that are mathematically interesting in their own right, and require extensive introduction. Here we provide the basic ideas and intentions of braided geometry.

The key notion is that there are in fact *two* kinds of noncommutativity. First there is an *inner noncommutativity*, which is that between components of a single physical system. For instance, the noncommutativity of canonical quantization is between position and momentum of the same single system. Then there is also an *outer noncommutativity* possible; a noncommutativity between components of two separate systems. The simplest example is that of fermions: they are anti-symmetric under interchange, meaning they exhibit perfect anticommutativity. (Note that inner noncommutativity can occur as an example of outer noncommutativity occuring between two copies of the same system.)

The idea of braided geometry is to associate a q-deformation to this second, outer kind of noncommutativity. We shall see that this also implies some inner noncommutativity, but as a special case. This outer commutativity turns out to be the key for a systematic q-deformation of all of physics: the approach of braided geometry is to replace the ubiquitous category of vector spaces by a q-deformed version. This version has a q-deformed tensor product $\underline{\otimes}$, which introduces factors of q in a systematic way. Much of basic physics is constructed from vector spaces, so if we can succeed in deforming the entire *category* of vector spaces, then we obtain q-deformed versions of all our favorite constructions 'for free'.

The q-deformed category of vector spaces will turn out to be what is called a *braided category*, equipped with a *braided tensor product*. We will later see why the terminology 'braided' is used; for now it is just a name. A braided category is also equipped with a 'braiding' Ψ. This map is a generalization of the vector space map

$$\tau : V \otimes W \to W \otimes V : \sum v \otimes w \mapsto \sum w \otimes v.$$

If this map is no longer trivial, then it models the outer noncommutativity in our braided category (and the inner noncommutativity, if $V = W$). Thus the braiding Ψ is the origin of all the q-deformations coming out of braided geometry. For this reason, the terms 'q-deformed' and 'braided' are sometimes used interchangeably.

So far, we have seen no mention yet of *Hopf algebras*. These are algebraic structures built from vector spaces, that are related to the usual notion of an algebra. If we define Hopf algebras in a purely arrow-theoretic (i.e. categorical) way, then we can generalize them to *Hopf algebras in an arbitrary braided category* or more succinctly: *braided groups*. Considering braided groups as Hopf algebras (just ones that live in some braided category), we will find that Hopf algebras play two central roles in braided geometry: first recall that the category of vector spaces is deformed to a braided category. The individual vector spaces themselves are then also deformed, but we will be particularly interested in the function algebras on those vector spaces. These will be braided groups, and this is the first important appearance of Hopf algebras. The other appearance is more fundamental. It can be though of physically as a deformation of the symmetry groups of our vector spaces, e.g. the q-Lorentz group for q-Minkowski space. More precisely we will find that the braided category in which our braided groups reside is generated *by some Hopf algebra*. This Hopf algebra plays the background role of describing the braiding Ψ via its actions on the braided groups. Thus the theory of braided geometry hinges on Hopf algebras at a very fundamental level.

1.3 Why Hopf algebras?

So far we have thoroughly motivated the role of braided categories in the braided geometry approach to q-deformation: they allow us to q-deform the entire category of vector spaces. To finish this introduction, we finally motivate the use of Hopf algebras for braided geometry.

This motivation is partly due to Shahn Majid's personal view on the role of representation theory in physics. In [7] he argues that a theory of physics fundamentally describes two things: the physical systems and processes that occur externally from our perception, and the representations of those systems which we are able to measure. A complete theory of physics should, in his view, be able to describe both physical systems *and* their representations. This motivates the need to describe physics in terms of structures with a suitably nice representation theory and self-duality. Otherwise, suppose that we have a theory of physics X_0. The need to describe both systems and their representations then drives us to consider the representations $\hat{X}_0$. If X_0 and $\hat{X}_0$ are not described by the same theory, then this provides us with an 'engine' that drives the invention of a new theory X_1 that encompasses both. We can then consider $\hat{X}_1$, construct X_2 accordingly, etc. This process can only halt if $X_n = \hat{X}_n$ for some n, in which case we speak of a representation-theoretic self-duality [7].

It is also possible that the process we have described halts at a theory that is not general enough to describe reality. In this case we also have reason to generalize. A perfect example of this is classical mechanics in $\mathbb{R}^n$. The Hamiltonian formalism of classical mechanics elegantly puts position and momentum on an equal footing. Here the representations of position space are precisely momentum space, and the representation-theoretic self-duality manifests itself in the Born reciprocity [7]. This is invariance of Hamilton's equations under $x \to p$, $p \to -x$. Majid's larger landscape of structures in physics (that may or may not be self-dual) is depicted in figure 1.1. Here $\mathbb{R}^n$ fits under the heading of 'Abelian Groups', and in this thesis we will delve into the rectangle labelled 'Quantum Groups'.

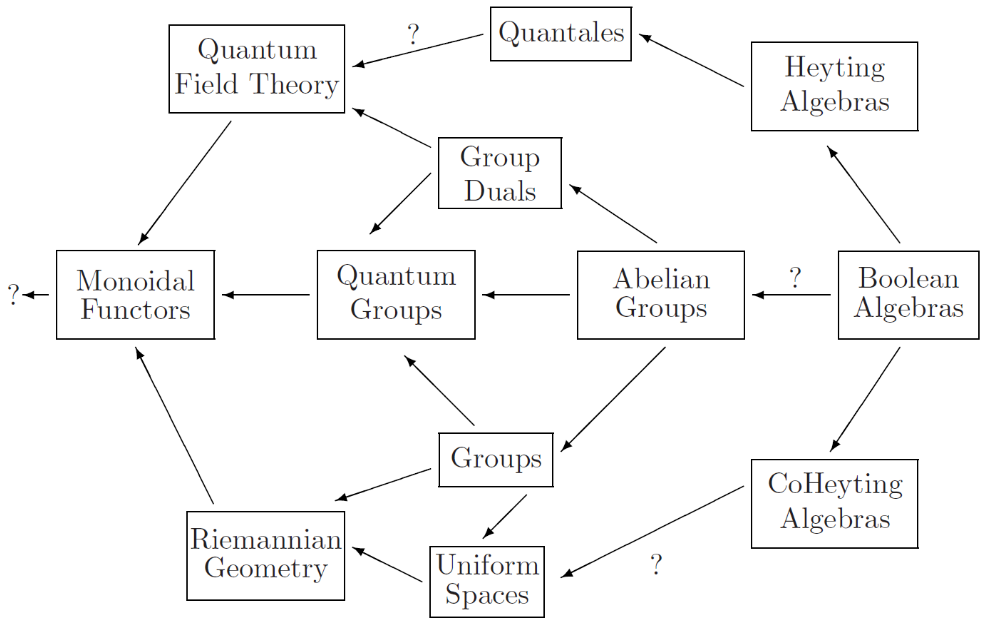


Figure 1.1: Structures along the central axis are representation-theoretically self-dual. Arrows indicate generalization; arrows with '?' marks are speculative. Taken from [7]. Here 'quantum groups' is used synonymously with 'Hopf algebras'.

In section 5 we will discuss the representation theory of Hopf algebras. We will find that the structure of a Hopf algebra is precisely right to allow for a very coherent representation theory: we can form tensor products and duals of representations, and in the special case that our Hopf algebra is *quasitriangular* the representations form a braided category. We will make use of this at a very fundamental level for the development of braided geometry. We also have that the dual of a (finite-dimensional) Hopf algebra is again a Hopf algebra, and that any Hopf algebra can be uniquely reconstructed from its representations. This all suggests that Hopf algebras indeed exhibit the representation-theoretic self-duality that Majid's philosophy demands for a complete theory of Physics. This does not mean, however, that Hopf algebras are sufficiently general to unify *all* of physics; we only suggest that they are a step in the right direction.

In conclusion, the reason for Hopf algebras in braided geometry is two-fold:

- The coherent representation theory of Hopf algebras satisfies the esoteric requirements of Majid's philosophy for unifying theories.
- As stated before, the theory of braided geometry hinges on Hopf algebras on a fundamental level: this is made possible precisely by the representation theory of Hopf algebras, mainly because Hopf algebra representations provide us with a broad class of braided categories. We will see that these braided categories are exactly braided versions of the category of vector spaces. Thus the representations of Hopf algebras are precisely what allows them to generate our deformed category of vector spaces.

1.4 Outlook

To finish the introduction, we give a brief outlook of what lies ahead:

The next section provides a first example of q-deformation: in it we discuss the q-deformed quantum harmonic oscillator, and many of its physical features. We derive the spectrum of the q-oscillator and show that it admits states of arbitrarily low position-momentum uncertainty. The q-oscillator is an 'inner noncommutativity' q-deformation, and hence it is not related to braided geometry. Instead it finds applications in modelling molecular spectra. For our purposes it is also a relatively simple and concrete example, to help introduce the ideas of q-deformation. This section is entirely physics-oriented (or slightly chemistry-oriented, at points).

Section three is an introduction to the basic mathematical structures that we will need in the rest of the discussion. Most central is the notion of *Hopf algebras* which we will encounter in all the following sections, in some incarnation. As such, this section is almost entirely mathematical.

In the fourth and fifth sections we delve deeper into the theory of Hopf algebras. We will need the theory developed here in subsequent sections, but both of the topics addressed in these sections have very interesting mathematical applications as well. In the fourth section we discuss the relationship between Hopf algebras and knots. We derive several knot invariants from specific Hopf algebras, and along the way we develop important concepts like quasitriangularity. We also discuss the 'braided diagrams' introduced in the second section in more formal detail. This section is completely mathematical. In the fifth section we discuss the representation theory of Hopf algebras and more generally of braided groups. We discuss the explicit construction of a class of braided categories, which are necessary for q-deformation. We also show how a Hopf algebra can be recovered from its 'category' of representations, proving a Tannaka-Krein duality for Hopf algebras. This section is also purely mathematical.

In the sixth section we apply all the theory developed so far to give a systematic q-deformation of the basic ingredients of physics. This section is the main body of the work: all preceding sections develop the context to carry out the q-deformations of this section, and the sections after it explores the implications of the theory developed here. In order, we give q-deformations of vectors and covectors, symmetry groups, differentiation, and exterior algebra. These q-deformed mathematical notions are the foundations for a development of q-Minkowski spacetime and q-deformed field theory. The intentions of this section are purely physics-oriented, but due to the relatively high level of mathematical sophistication this section is very mathematical in its approach.

In the seventh section we use the previously developed material to construct q-deformed Minkowski spacetime. To this end we move through the concept of *braided matrices*, casting regular Minkowski spacetime as a space of Hermitian matrices. This section is purely physics-oriented.

In the eighth we discuss the implications of the q-deformations developed in the preceding sections. We use the theory of q-deformed exterior algebra to give a basic q-deformed scalar field theory on q-Minkowski spacetime. Next we give a remarkable isomorphism between q-Minkowski space and a q-deformed enveloping algebra of the electro-weak symmetry group. Lastly, we suggest a unification of the fundamental ingredients of q-Lorentz symmetry and supersymmetry. This section is again purely physical.

Finally, the ninth section is a discussion of what is achieved in the thesis. We give a summary of the main results, and provide some context on the current state of research in the direction of q-deformation.

2 The q-Harmonic Oscillator

In this short section we discuss a first example of q-deformation: the q-deformed one-dimensional harmonic oscillator, or *q-oscillator*. Since the harmonic oscillator is one of the most basic physical systems, this is a relatively simple q-deformation. As such, there will be no need yet for the complicated formalism of Hopf algebras or braided groups. (Meaning that this deformation is relatively *ad-hoc*.) These are only necessary later on, for the q-deformation of e.g. Minkowski spacetime which we will do in section 6. However, we will encounter ideas and notation of q-deformation that will carry over to section 6.

After we have sufficiently described the q-oscillator and its spectrum, we will go on to consider some of the physical characteristics of the q-oscillator model. Most notable among these is the Heisenberg uncertainty principle: we will find that the q-deformed model allows for states whose q-deformed position and momentum operators have arbitrary low uncertainty.

2.1 The Quantum Harmonic Oscillator

Before we can deform to the case $q \neq 1$, we must thoroughly understand the classical $q = 1$ case that we wish to generalize. In our case, this is the quantum harmonic oscillator given by the following Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

Here ω is the oscillator's frequency, and $p = -i\hbar\frac{\partial}{\partial x}$ is the momentum operator. From now on we will work in units such that $\hbar = 1$; hence $p = -i\frac{\partial}{\partial x}$. To recast this Hamiltonian we write it in terms of two new operators $a_\pm$ defined as

$$a_\pm := \frac{1}{\sqrt{2m\omega}}(\mp ip + m\omega x). \tag{2.1}$$

Using the canonical commutation relation $[x, p] = i$ we find that

$$[a_-, a_+] = \frac{1}{2}(-i[x, p] + i[p, x]) = 1.$$

It is also straightforward to show that [8]:

$$\begin{aligned} a_+a_- &= \left(\frac{m\omega}{2}\right)\left(x^2 + \frac{p^2}{m^2\omega^2}\right) + \frac{i}{2}[x, p] \\ &= \frac{H}{\omega} - \frac{1}{2}. \end{aligned}$$

We may thus recast the Hamiltonian as

$$H = \omega\left(a_+a_- + \frac{1}{2}\right).$$

Using the commutator $[a_-, a_+] = 1$ we can also write this as

$$H = \frac{\omega}{2}\left(a_+a_- + a_-a_+\right).$$

The operators $a_\pm$ are referred to as 'creation' and 'annihilation' operators respectively. They are to be though of as creating or annihilating an energy ω in the system. This interpretation is sensible due to the following result:

Lemma 2.1. *Let H be the quantum harmonic oscillator Hamiltonian, and let ψ be an energy eigenfunction of H such that $H\psi = E\psi$ with $E \in \mathbb{R}$. Then $a_+\psi$ is an energy eigenfunction with eigenvalue $E + \omega$, and $a_-\psi$ is an eigenfunction with energy $E - \omega$.*

Proof. Using that $[a_-, a_+] = 1$, we compute that

$$\begin{aligned} H(a_+\psi) &= \omega \left(a_+a_- + \frac{1}{2} \right) a_+\psi \\ &= \omega a_+ \left(a_-a_+ + \frac{1}{2} \right) \psi \\ &= a_+\omega \left(a_+a_- + 1 + \frac{1}{2} \right) \psi \\ &= a_+(H+\omega)\psi = a_+(E+\omega)\psi = (E+\omega)a_+\psi. \end{aligned}$$

Similarly $H(a_-\psi) = (E-\omega)a_-\psi$. Alternatively we can note that

$$[H, a_+] = \omega a_+ \qquad \text{and} \qquad [H, a_-] = -\omega a_-.$$

□

Hence the operators a_+ and a_- give us many states of energy $E + n\omega$ for $n \in \mathbb{Z}$, assuming that we have some starting eigenstate. To find such a state, we postulate the existence of a 'vacuum state' $|0\rangle$ such that

$$a_-|0\rangle = 0.$$

This condition means exactly that the vacuum state is 'empty'; no energy can be removed from it by a_-. This condition is equivalent to the following differential equation:

$$\frac{1}{\sqrt{2m\omega}} \left(\frac{\partial}{\partial x} + m\omega x \right) |0\rangle = 0,$$

The normalized solution of this equation is given by [9]:

$$|0\rangle = \left(\frac{m\omega}{\pi} \right)^{\frac{1}{4}} e^{-\frac{m\omega}{2}x^2}.$$

We wish to note that the existence of the vacuum state is not an ad-hoc postulate: if we do not put in place a state that is annihilated by a_-, then we could apply a_- repeatedly to obtain eigenstates of H with *negative energies*. This is impossible: since the minimal value of the harmonic oscillator potential $\frac{1}{2}m\omega^2x^2$ is 0, such states are not normalizable. For a pedagogical explaination of this fact, see [9, Ch.2]. This argument necessitates the existence of $|0\rangle$. We shall see that a more sophisticated version of this argument also necessitates a vacuum state for certain values of $q \neq 1$.

Applying H, we find that $|0\rangle$ is an energy eigenstate with

$$H|0\rangle = \omega \left(a_+a_- + \frac{1}{2} \right) |0\rangle = \frac{1}{2}\omega|0\rangle.$$

Starting at the vacuum state, we can apply a_+ repeatedly to generate excited states $|n\rangle$, increasing the energy by ω each step. Thus the state $|n\rangle$ that is created after n applications of a_+ has energy $\left(n + \frac{1}{2}\right)\omega$. This process determines the states $|n\rangle$ up to normalization. The normalization can be calculated analytically [9], and in fact the creation and annihilation operators act on eigenstates as

$$\begin{aligned} a_+|n\rangle &= \sqrt{n+1}|n+1\rangle, \\ a_-|n\rangle &= \sqrt{n}|n-1\rangle. \end{aligned}$$

Hence the normalized n-th excited stated is given by

$$|n\rangle = \frac{1}{\sqrt{n!}} {a_+}^n |0\rangle.$$

Finally, the harmonic oscillator has the notion of a *coherent state*. This is defined to be an eigenstate $|\lambda\rangle$ of the annihilation operator, i.e. a state such that $a_-|\lambda\rangle = \lambda|\lambda\rangle$. An example of such a state is

$$|\lambda\rangle = c_0 e^{\lambda a_+}|0\rangle.$$

where c_0 is a normalization constant. Indeed, we compute that

$$\begin{aligned} a_-|\lambda\rangle &= c_0 a_- \sum_{n=0}^{\infty} \frac{(\lambda a_+)^n}{n!}|0\rangle = c_0 a_- \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}}|n\rangle \\ &= c_0 \sum_{n=1}^{\infty} \frac{\lambda^n}{\sqrt{n!}}\sqrt{n}|n-1\rangle = c_0\lambda \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{\sqrt{(n-1)!}}|n-1\rangle = \lambda c_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}}|n\rangle = \lambda|\lambda\rangle. \end{aligned}$$

Here the third equality uses that $a_-|0\rangle = 0$. In fact, the normalization constant is given by $c_0 = \exp(-|\lambda|^2/2)$, and $|\lambda\rangle$ satisfies the minimal Heisenberg uncertainty relation [8]

$$\Delta x \Delta p = \frac{1}{2}.$$

(Recall here that we set $\hbar = 1$.)

2.2 The Abstract q-Oscillator

We start by defining the q-oscillator abstractly in terms of creation and annihilation operators. For these operators, we stipulate q-deformed commutators, and derive the resulting spectrum. Only later will we give an explicit coordinate representation of the q-oscillator. We define the q-oscillator [10], [11] as the system that is described by the Hamiltonian

$$H = \frac{\omega}{2}(a_-a_+ + a_+a_-).$$

Here $a_\pm$ are postulated creation and annihilation operators, satisfying the following q-commutation relation:

$$[a_-, a_+]_q := a_-a_+ - qa_+a_- = 1.$$

(Recall here that we have set $\hbar = 1$.) The deformation parameter q is to be thought of as close to 1. Indeed, we recover the quantum harmonic oscillator for $q = 1$. We will mainly restrict to $q \in (0, 1)$, in which case we can find an explicit coordinate representation [10]. Using the q-commutator for $a_\pm$ we can rewrite the Hamiltonian as

$$H = \frac{\omega}{2}((1+q)a_+a_- + 1) = \omega\left(\frac{1+q}{2}a_+a_- + \frac{1}{2}\right).$$

As for the regular quantum oscillator, we define a vacuum state $|0\rangle$ such that $a_-|0\rangle = 0$. Since we do not yet have explicit coordinate representations for $a_\pm$ we cannot give $|0\rangle$ explicitly. This is postponed to the next subsection. However, we can already use $|0\rangle$ as before to find the spectrum of the q-oscillator. As before, $|0\rangle$ is an eigenfunction of H:

$$H|0\rangle = \omega\left(\frac{1+q}{2}a_+a_- + \frac{1}{2}\right)|0\rangle = \frac{1}{2}\omega|0\rangle.$$

Thus the ground state energy is $\omega/2$ and independent of q. The further spectrum of H can then be found by successive applications of a_+. We begin this process with the following definitions: we define the *q-integers* $[n]$ as

$$[n] := 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}.$$

where the latter equality is only for $q \neq 1$. In analogy with regular integers we let $[n]! = [n] \cdot [n-1] \cdot, \ldots, \cdot [1]$. In accordance with [12] we then describe the excited states as

$$|n\rangle = \frac{1}{\sqrt{[n]!}} {a_+}^n |0\rangle.$$

The operators $a_\pm$ then act on these states as

$$\begin{aligned} a_+|n\rangle &= \sqrt{[n+1]}|n+1\rangle, \\ a_-|n\rangle &= \sqrt{[n]}|n-1\rangle. \end{aligned}$$

As we might hope, the states $|n\rangle$ are eigenfunctions of the Hamiltonian. Hence we indeed obtain the spectrum of H by repeated applications of a_+ to $|0\rangle$. This is the content of the following lemma:

Lemma 2.2. The n-th excited state $|n\rangle$ is an energy eigenfunction of H with eigenvalue

$$E_n = \omega\left([n] + \frac{q^n}{2}\right) = \omega\left(\frac{[n] + [n+1]}{2}\right). \tag{2.2}$$

Proof. We proceed by induction: the case $n = 0$ is trivial, since $E_0 = \omega/2$ which agrees with the energy found for $|0\rangle$. For the inductive step, suppose that $|n\rangle$ is indeed an eigenfunction with energy E_n. Then we compute

$$\begin{aligned} H|n+1\rangle &= \omega\left(\frac{1+q}{2}a_+a_- + \frac{1}{2}\right)\frac{a_+}{\sqrt{[n+1]}}|n\rangle \\ &= \frac{\omega}{\sqrt{[n+1]}}a_+\left(\frac{1+q}{2}a_-a_+ + \frac{1}{2}\right)|n\rangle \\ &= \frac{\omega}{\sqrt{[n+1]}}a_+\left(\frac{q+q^2}{2}a_+a_- + \frac{1+q}{2} + \frac{1}{2}\right)|n\rangle \\ &= \frac{1}{\sqrt{[n+1]}}a_+\left(qH + \omega\right)|n\rangle = \frac{1}{\sqrt{[n+1]}}a_+(qE_n + \omega)|n\rangle = (qE_n + \omega)|n+1\rangle. \end{aligned}$$

The inductive step then follows from the identity

$$(qE_n + \omega) = \omega\left(q[n] + \frac{q^{n+1}}{2} + 1\right) = \omega\left([n+1] + \frac{q^{n+1}}{2}\right) = E_{n+1},$$

using that $q[n] + 1 = [n+1]$, by definition of $[n+1]$. □

This lemma gives us the spectrum of H, and hence completes our abstract analysis of the q-oscillator in terms of $a_\pm$. Note that all the results derived so far reduce to the regular quantum harmonic oscillator for $q \to 1$, as desired of a q-deformation.

As a final note, we wish to consider the vacuum state $|0\rangle$ in more detail for $q \neq 1$. For $q = 1$ we have seen that it is necessary to postulate $|0\rangle$; otherwise we obtain non-normalizable states with negative energy eigenvalues. We will now present a similar argument for general $q \in (0, 3)$. The emergence of the number 3 will be clear soon. In practise it is not a problem that our argument does not work for $q \geq 3$. After all, q is to be though of as close to 1, and in all applications of the q-oscillator cited below q is taken less than 1.

Let $q \in (0, 3)$. Suppose that we have some energy eigenstate $|\varepsilon\rangle$ of the q-oscillator such that $H|\varepsilon\rangle = \varepsilon|\varepsilon\rangle$ for some energy $\varepsilon > 0$. Furthermore suppose that there is *no* vacuum state that is annihilated by a_-. As a contradiction we shall now show that this implies one of two cases:

Case 1: We obtain eigenstates of H with negative energy.
Case 2: We do not obtain the quantum harmonic oscillator in the limit $q \to 1$.

To show this we compute that

$$
\begin{aligned}
Ha_-|\varepsilon\rangle &= \omega\left(\frac{1+q}{2}a_+a_- + \frac{1}{2}\right)a_-|\varepsilon\rangle \\
&= \omega\left(\frac{1+q}{2}\frac{a_-a_+ - 1}{q} + \frac{1}{2}\right)a_-|\varepsilon\rangle \\
&= a_-\omega\left(\frac{1}{q}\left[\frac{1+q}{2}a_+a_- + \frac{1}{2}\right] + \frac{1}{2} - \frac{3}{2q}\right)|\varepsilon\rangle \\
&= a_-\left(\frac{H}{q} + \omega\frac{q-3}{2q}\right)|\varepsilon\rangle \\
&= \frac{\varepsilon + \omega\frac{q-3}{2}}{q}a_-|\varepsilon\rangle.
\end{aligned}
$$

Analogously we find that

$$
Ha_-^2|\varepsilon\rangle = \frac{\frac{\varepsilon+\omega\frac{q-3}{2}}{q} + \omega\frac{q-3}{2}}{q}a_-^2|\varepsilon\rangle = \frac{\varepsilon + \omega\frac{q-3}{2} + q\omega\frac{q-3}{2}}{q^2}a_-^2|\varepsilon\rangle,
$$

and by a simple induction argument we thus have

$$
Ha_-^n|\varepsilon\rangle = \frac{\varepsilon + [n]\omega\frac{q-3}{2}}{q^n}a_-^n|\varepsilon\rangle.
$$

For $q > 3$ these energies are all positive, so we do not run into any contradiction. It is for this reason that we restrict this discussion to $q \in (0,3)$. For $q \in (1,3)$, the q-integers $[n]$ diverge since $q > 1$. There must thus be an N such that

$$
|\varepsilon| < [N]\omega\frac{q-3}{2}.
$$

Meanwhile, $\frac{q-3}{2} < 0$ so we find that the energy eigenvalue of $a_-^N|\varepsilon\rangle$ is negative. This gives our desired contradiction so that the necessity of $|0\rangle$ follows.

Next, for $q \in (0,1)$ the q-integers converge to $\frac{1}{1-q}$. We thus have that

$$
\varepsilon + [n]\omega\frac{q-3}{2} \to \varepsilon + \frac{1}{1-q}\omega\frac{q-3}{2}.
$$

Since we assumed that there is no vacuum state, we must have that

$$
\varepsilon > \frac{1}{1-q}\omega\frac{3-q}{2},
$$

because otherwise we arrive at the same contradiction as for $q \in (1,3)$. Thus under our assumptions the spectrum of the q-oscillator is bounded from below by

$$
E_{\min} = \frac{1}{1-q}\omega\frac{3-q}{2}.
$$

Now, as q approaches 1 from below we clearly have that $E_{\min} \to +\infty$. We thus violate the requirement that we obtain the non-deformed system as $q \to 1$, since the spectrum of the regular quantum harmonic oscillator contains $E_0 = \omega/2$. Again, we arrive at a contradiction! In conclusion, the existence of a vacuum state is required by physical considerations for $q \in (0,3)$, as claimed.

2.3 Coordinate Representation of the q-Oscillator

Now that we have derived the spectrum of the q-oscillator, we can discuss an explicit representation in terms of position-momentum phase space (x, p). Such a description for $q \in (0, 1)$ was derived in [10]. We will only consider this case, since it is the most useful and common case for applications [13]. In what follows we will assume $\omega = m = 1$, for simplicity.

The coordinate representation from [10] is given by

$$\begin{aligned} a_- &= \frac{\exp(-2i\alpha x) - \exp(i\alpha\frac{\partial}{\partial x})\exp(-i\alpha x)}{-i\sqrt{1-\exp(-2\alpha^2)}}, \\ a_+ &= \frac{\exp(2i\alpha x) - \exp(i\alpha x)\exp(i\alpha\frac{\partial}{\partial x})}{i\sqrt{1-\exp(-2\alpha^2)}}. \end{aligned}$$

Here $\alpha = \sqrt{-\ln(q)/2}$. This α is real and non-negative as long as $q \in (0, 1)$. The limit $q \to 1$ corresponds to $\alpha \to 0$. In the limit $\alpha \to 0$ these representations of $a_\pm$ indeed reduce to equation (2.1) [10], as can be shown using the calculus trick of L'Hôpital's rule. The operators $a_\pm$ given above indeed obey $[a_-, a_+]_q = 1$ [10], so that they indeed give a proper coordinate representation of the q-oscillator.

We define the associated q-deformed coordinate- and momentum operators analogously to the $q = 1$ case:

$$\hat{x} = \frac{a_- + a_+}{\sqrt{2}} \qquad \text{and} \qquad \hat{p} = \frac{a_- - a_+}{i\sqrt{2}}. \tag{2.3}$$

We can now find an explicit form of the q-deformed vacuum state $|0\rangle$. The equation $a_-|0\rangle = 0$ becomes

$$\left(\exp(-2i\alpha x) - \exp(i\alpha\frac{\partial}{\partial x})\exp(-i\alpha x)\right)|0\rangle = 0.$$

The normalized solution of this differential equation is given by [10]:

$$|0\rangle = \frac{1}{\pi^{\frac{1}{4}}}\exp\left(-\frac{x^2}{2} + \frac{3}{2}i\alpha x\right). \tag{2.4}$$

For $q = 1$ and $\alpha = 0$ we indeed recover the vacuum solution of the regular quantum harmonic oscillator. Using the coordinate form for a_+ we can then construct the states $|n\rangle$. The final result is [10]

$$|n\rangle = \frac{\exp\left(-\frac{x^2}{2} + \frac{3}{2}i\alpha x\right)}{\pi^{\frac{1}{4}} i^n \left(1-\exp(-2\alpha^2)\right)^{\frac{n}{2}}\sqrt{[n]!}} \sum_{k=0}^{n} \frac{(-1)^k [n]!}{[k]![n-k]!} \exp\left((n-k)2i\alpha x - k\alpha^2\right). \tag{2.5}$$

2.4 Consequences of the q-Oscillator Model

We now have a solid description of the q-oscillator. This allows us to discuss some of its features that are a consequence of the q-deformation, as well as applications of this q-oscillator model. With respect to applications we reiterate that this q-deformation is rather ad-hoc, and hence its applications are found mainly in modelling externally perturbed systems, using q as a perturbation modelling parameter.

First we consider the spectrum of the q-oscillator in more detail. Recall that it is given by equation (2.2). At $q = 1$ this becomes the standard linear spectrum of the harmonic oscillator

$$E_n = \omega\left(n + \frac{1}{2}\right).$$

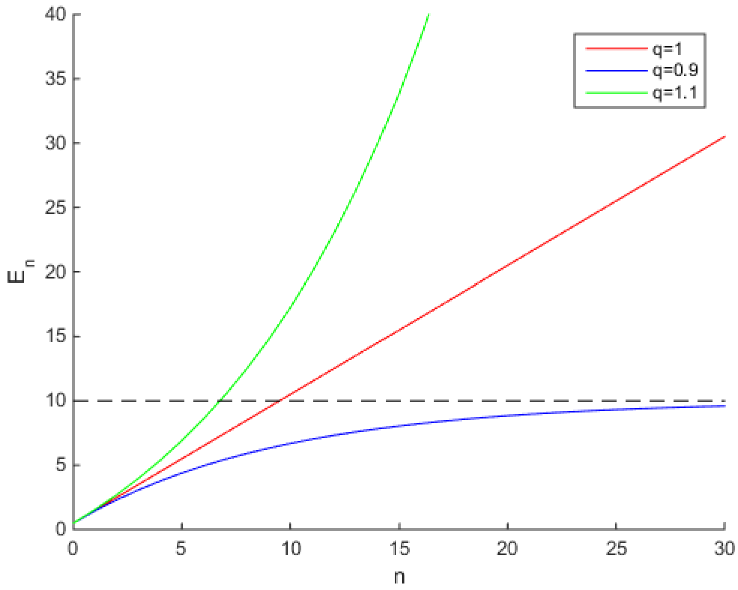


Figure 2.1: Spectra of the q-oscillator for $q \in \{0.9, 1, 1.1\}$, $\omega = 1$. The dashed line indicates E_∞.

Meanwhile, if $q \in (0,1)$ we have that $\lim_{n\to\infty} q^n = 0$, and hence

$$\lim_{n\to\infty} E_n = \lim_{n\to\infty} \omega\left(\frac{1-q^n}{1-q} + \frac{q^n}{2}\right) = \frac{\omega}{1-q} =: E_\infty.$$

In conclusion, for $q \in (0,1)$ the spectrum is bounded! Similarly, for $q > 1$ we have that q^n grows exponentially, and hence the spectrum is exponential. This is depicted for $q = 0.9$, $q = 1$, and $q = 1.1$ in figure 2.1. In this figure the black dashed line indicates E_∞ for $q = 0.9$.

This behaviour of E_n as a function of q indicates that the q-harmonic oscillator is strongly perturbed by q-deformation: the linear divergence of E_n becomes either convergent or exponentially divergent. Note here that the q-oscillator is intended to model anharmonicity, and thus we do not expect to retrieve a classical oscillator for large quantum numbers. An interesting case for applications is when $q = 1 - \epsilon$ for small $\epsilon > 0$. In this case we can expand the spectrum to first order in ϵ to find [10]:

$$\begin{aligned}
E_n &= \omega\left(\frac{1-(1-\epsilon)^n}{1-(1-\epsilon)} + \frac{(1-\epsilon)^n}{2}\right)\\
&= \omega\left(\frac{1-(1-n\epsilon+\frac{1}{2}n(n-1)\epsilon^2)}{\epsilon} + \frac{1-n\epsilon}{2} + \mathcal{O}(\epsilon^2)\right)\\
&= \omega\left(n - \frac{1}{2}(n^2-n)\epsilon - \frac{n\epsilon}{2} + \frac{1}{2} + \mathcal{O}(\epsilon^2)\right)\\
&= \omega\left(n + \frac{1}{2} - \frac{n^2}{2}\epsilon + \mathcal{O}(\epsilon^2)\right) \approx \omega\left(n + \frac{1}{2} - \frac{n^2}{2}\epsilon\right).
\end{aligned}$$

Hence for small deviations of q from unity, we find that the q-oscillator has an initially quadratic, but eventually bounded spectrum. This makes the q-oscillator with $q = 1-\epsilon$ a useful tool in mathematical chemistry. It can be used to describe the oscillatory- or rotational spectra of diatomic and multi-atomic molecules, modelling the anharmonicity of the potentials via ϵ [14], [15].

As another consequence of q-deformation we consider the excited states and ground state of the q-oscillator, and their limits as $q \to 1$ or $q \to 0$. Again, for convenience we set $m = \omega = 1$.

Of course, for $q \to 1$ we hope to recover the usual quantum harmonic oscillator. First, recall the coordinate representation for $|0\rangle$ from equation (2.4). The probability density of this state is given by

$$||0\rangle|^2 = \frac{1}{\sqrt{\pi}} \exp\left(-x^2\right).$$

This is a Gaussian shape, and is independent of q. Thus remarkably, the probability density of the ground state is always the same as that of the $q = 1$ case. In contrast, now recall the coordinate representations for $|n\rangle$ from equation (2.5). In the limit $q \to 1$, $\alpha \to 0$ these converge to the energy eigenfunctions of the standard harmonic oscillator [10]:

$$|n\rangle \xrightarrow{q\to 1} \frac{1}{\pi^{\frac{1}{4}} 2^{\frac{n}{2}} \sqrt{n!}} H_n \exp\left(-\frac{x^2}{2}\right).$$

Here H_n is the n-th Hermite polynomial. (These representations for $|n\rangle$, $q = 1$ can be found in e.g. [9].) This is as expected. However, in the limit $q \to 0$, $\alpha \to \infty$ all the excited states degenerate to the probability density of the ground state [10]:

$$|n\rangle \xrightarrow{q\to 0} \frac{1}{\pi^{\frac{1}{4}} i^n} \exp\left(-\frac{x^2}{2} + \left(2n + \frac{3}{2}\right) i\alpha x\right)$$

$$\implies ||n\rangle|^2 \xrightarrow{q\to 0} \frac{1}{\sqrt{\pi}} \exp\left(-x^2\right).$$

This again shows that the energy eigenfunctions of the harmonic oscillator are strongly perturbed by our q-deformation.

Next, we consider the q-deformed coherent states $|\lambda\rangle$. These have a very remarkable feature: for all $q \in (0, 1)$ they violate the Heisenberg uncertainty principle, in that they provide examples of states such that $\Delta x \Delta p < \frac{1}{2}$. This indicates that the q-deformed commutation relations lead to a change in the usual uncertainty relation, which was first proposed in [16].

As before, we define a coherent state of the q-oscillator to be a state $|\lambda\rangle$ such that

$$a_-|\lambda\rangle = \lambda|\lambda\rangle,$$

i.e. an eigenstate of the annihilation operator. Analogously to the $q = 1$ case, an example of coherent states is given by [10]

$$|\lambda\rangle = c_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{[n]!}} |n\rangle = c_0 \exp_q(\lambda a_+)|0\rangle.$$

Here c_0 is a normalization constant, and $\exp_q$ is the *q-exponential* defined by the usual power series for exp but with the familiar integers replaced by q-integers. The inclusion of a q-exponential means we must be careful: $|\lambda\rangle$ as described above will not be normalizable for all $\lambda \in \mathbb{R}$. To see this, note that the normalization constant as determined by the condition $\langle\lambda|\lambda\rangle = 1$ is given by

$$|c_0|^2 = \frac{1}{\sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{[n]!}} = \frac{1}{\exp_q(|\lambda|^2)}.$$

This expression for c_0 is only well-defined if the q-exponential indeed converges at $|\lambda|^2$. Whether or not this is the case can be deduced from the following lemma:

Lemma 2.3. Let $\exp_q(x)$ be given by the formal power series

$$\exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!}.$$

If $q \geq 1$, then $\exp_q$ converges everywhere. If $q \in (0,1)$ then $\exp_q$ has radius of convergence $\frac{1}{1-q}$, i.e. $\exp_q$ converges for all x such that

$$|x| < \frac{1}{1-q}.$$

Proof. In the case $q = 1$ we obtain the regular exponential, whose power series is well-known to converge everywhere. Thus assume that $q \in (0, \infty)$ and $q \neq 1$. We proceed using the ratio test. This states that the series $\sum_{n=0}^{\infty} a_n$ converges absolutely if $\lim_{n\to\infty}|a_{n+1}/a_n| < 1$, and diverges if this limit exceeds 1. For our purposes we take

$$a_n = \frac{x^n}{[n]!} = \frac{x^n(1-q)^n}{(1-q^n)(1-q^{n-1})\dots(1-q)}.$$

We then compute that

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{n+1}(1-q)^n}{(1-q^{n+1})\dots(1-q)} \cdot \frac{(1-q^n)\dots(1-q)}{x^n(1-q)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{x(1-q)}{(1-q^{n+1})}\right|.
\end{aligned}$$

If $0 < q < 1$, we have $q^{n+1} \xrightarrow{n\to\infty} 0$, and hence this limit equals $|x(1-q)|$. This is less than 1 if and only if

$$|x| < \frac{1}{1-q}.$$

Hence in this case we obtain a radius of convergence equal to $\frac{1}{1-q}$ as claimed. If $q > 1$ then $q^{n+1} \xrightarrow{n\to\infty} \infty$ and $\lim_{n\to\infty}|a_{n+1}/a_n| = 0$ for all x. Hence in this case we obtain an infinite radius of convergence. This completes the proof. $\square$

This result implies that the q-deformed coherent state $|\lambda\rangle$ only exists for

$$|\lambda| < \frac{1}{\sqrt{1-q}},$$

if $q \in (0,1)$. This is in contrast to the $q \geq 1$ case, where any value of λ is admissible. For $q \in (0,1)$, explicit coordinate representations of the admissible coherent states can be found in [10]. We now examine the earlier claim that these coherent states violate the Heisenberg uncertainty principle, in the sense that

$$\Delta x \Delta p = \sqrt{\langle x^2\rangle - \langle x\rangle^2}\sqrt{\langle p^2\rangle - \langle p\rangle^2} < \frac{1}{2}.$$

To show this we must compute $\langle x^2\rangle$ etc. for $|\lambda\rangle$. Recall here that x, p are q-deformed *operators*, as given by equation (2.3). First note that $a_+|\lambda\rangle = \lambda^*|\lambda\rangle$, where * denotes complex conjugation [10]. This is simply because $a_+ = (a_-)^*$. Using this we compute:

$$\begin{aligned}
\langle\lambda|H|\lambda\rangle &= \langle\lambda|\left(\frac{1}{2} + \frac{1+q}{2}a_+a_-\right)|\lambda\rangle = \frac{1}{2} + \frac{1+q}{2}|\lambda|^2 \\
\langle x\rangle &= \langle\lambda|\left(\frac{a_- + a_+}{\sqrt{2}}\right)|\lambda\rangle = \frac{\lambda+\lambda^*}{\sqrt{2}} \\
\langle p\rangle &= \langle\lambda|\left(\frac{a_- - a_+}{i\sqrt{2}}\right)|\lambda\rangle = \frac{\lambda-\lambda^*}{i\sqrt{2}}.
\end{aligned}$$

Next, using the q-commutator $a_-a_+ = qa_+a_- + 1$ we compute:

$$\langle x^2 \rangle = \langle \lambda | \left(\frac{a_- + a_+}{\sqrt{2}} \right)^2 | \lambda \rangle = \frac{1}{2} \left(\lambda^2 + \lambda^{*2} \right) + \langle \lambda | H | \lambda \rangle, \tag{2.6}$$

$$\langle p^2 \rangle = \langle \lambda | \left(\frac{a_- - a_+}{i\sqrt{2}} \right)^2 | \lambda \rangle = -\frac{1}{2} \left(\lambda^2 + \lambda^{*2} \right) + \langle \lambda | H | \lambda \rangle. \tag{2.7}$$

Proofs of equations (2.6) and (2.7) are given in appendix A.2.

Using the above computations, we compute that the Heisenberg uncertainty in the state $|\lambda\rangle$ is given by

$$\begin{aligned}
\Delta x \Delta p &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \sqrt{\frac{1}{2} (\lambda^2 + \lambda^{*2}) + \langle \lambda | H | \lambda \rangle - \frac{1}{2} (\lambda^2 + 2|\lambda|^2 + \lambda^{*2})} \\
&\qquad \cdot \sqrt{-\frac{1}{2} (\lambda^2 + \lambda^{*2}) + \langle \lambda | H | \lambda \rangle + \frac{1}{2} (\lambda^2 - 2|\lambda|^2 + \lambda^{*2})} \\
&= \sqrt{\langle \lambda | H | \lambda \rangle - |\lambda|^2} \sqrt{\langle \lambda | H | \lambda \rangle - |\lambda|^2} \\
&= \langle \lambda | H | \lambda \rangle - |\lambda|^2 \\
&= \frac{1}{2} - \frac{1-q}{2} |\lambda|^2.
\end{aligned}$$

For all $q \in (0, 1)$ and $\lambda \neq 0$ we thus have

$$\Delta x \Delta p < \frac{1}{2}.$$

Since the radius of convergence of $\exp_q$ is nonzero for all $q \in (0, 1)$, there are admissible nonzero values of λ. Hence we have shown the existence of q-oscillator states that violate the Heisenberg uncertainty principle, as required. Also note that the coherent states are no longer 'states of minimal uncertainty'. In fact, 'minimal uncertainty' has no meaning in the q-deformed case: as λ approaches the radius of convergence we find

$$|\lambda| \to \frac{1}{\sqrt{1-q}} \implies \Delta x \Delta p \to 0.$$

Thus we can construct states with arbitrarily low uncertainty! This indicates that the position and momentum operators are *very* strongly perturbed by our q-deformation, since the Heisenberg uncertainty principle can be arbitrarily strongly violated in the q-deformed case. It is not known whether there is a q-Heisenberg uncertainty principle that does not exhibit this phenomenon.

3 Preliminaries

The purpose of this section is to introduce and define the further mathematical concepts and structures that we will need. An undergraduate level of knowledge about abstract algebra is assumed.

We introduce the basic objects that are needed for our braided geometry approach to q-deformation; namely Hopf algebras, braided categories, and braided groups. Of these, the notion of a Hopf algebra is most important: a braided group is nothing but a generalization of this structure to arbitrary braided categories.

3.1 Hopf Algebras

In the following, all algebras referred to will be associative and unital. As such, we will present associativity and existence of a unit as defining axioms of algebras:

Definition 3.1. An *algebra* over a field k is a vector space $(A, +)$ with an associative, bilinear product. This product is denoted $\cdot : A \times A \to A$, and is assumed to have a unit 1_A. The requirement of compatibility is more compactly expressed by requiring that the map $\cdot : A \otimes A \to A$ induced by $\cdot$ is linear. This is equivalent to bilinearity due to the universal property of the tensor product.

We may express the existence of 1_A via a map $\eta : k \to A$ using the following trick: for any $a \in A$ we may define a map

$$\begin{aligned} \eta_a : & k \to A \\ & \lambda \mapsto \lambda a. \end{aligned}$$

Now let $\eta := \eta_{1_A}$. Then the statement that 1_A is a two-sided unit with respect to $\cdot$ may be expressed in equations as:

$$\cdot\left((\eta \otimes \mathrm{id})(\lambda \otimes a)\right) = \lambda \otimes a \equiv \lambda a \qquad \text{and} \qquad \cdot\left((\mathrm{id} \otimes \eta)(a \otimes \lambda)\right) = a \otimes \lambda \equiv \lambda a.$$

for $\lambda \in k$ and $a \in A$, noting that $k \otimes A \cong A$ under the isomorphism $\lambda \otimes a \mapsto \lambda a$ and similarly for $A \otimes k$. Here '$\equiv$' denotes correspondence under these isomorphisms.

Most concisely then, we may express an algebra as a vector space with two linear maps $(\cdot, \eta)$ such that the following diagrams commute:

$$\begin{tikzcd}
 & A \otimes A \otimes A \arrow[dl, "\cdot \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id} \otimes \cdot"] & \\
A \otimes A \arrow[dr, "\cdot"'] & & A \otimes A \arrow[dl, "\cdot"] \\
 & A &
\end{tikzcd}$$

$$\begin{tikzcd}
 & A \otimes A \arrow[dr, "\cdot"] & \\
k \otimes A \arrow[ur, "\eta \otimes \mathrm{id}"] \arrow[rr, phantom, "\cong"] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes A \arrow[dr, "\cdot"] & \\
A \otimes k \arrow[ur, "\mathrm{id} \otimes \eta"] \arrow[rr, phantom, "\cong"] & & A
\end{tikzcd}$$

Definition 3.2. An algebra is said to be *commutative* if $\cdot$ is commutative, which translates to commutativity of the following diagram:

$$\begin{tikzcd}
& A\otimes A \arrow[dr, "\cdot"] & \\
A\otimes A \arrow[ur, "\tau"] \arrow[rr, "\cdot"'] & & A
\end{tikzcd}$$

Here $\tau : A\otimes A \to A\otimes A$ is the 'twist' map, given by $a\otimes b \mapsto b\otimes a$.

This alternative definition of an algebra in terms of commutative diagrams may appear inconvenient. However, having defined algebras completely in terms of arrows and diagrams allows for an intuitive notion of 'coalgebra'. Namely, a coalgebra is defined precisely as we have defined an algebra above, but with all the arrows reversed:

Definition 3.3. A *coalgebra* over a field k is a vector space $(C, +)$ equipped with a linear comultiplication map $\Delta : C \to C\otimes C$ that is coassiciative, and a counit map $\epsilon : C \to k$. The relevant axioms are then that the following diagrams commute:

$$\begin{tikzcd}
& C\otimes C\otimes C & \\
C\otimes C \arrow[ur, "\Delta\otimes\mathrm{id}"] & & C\otimes C \arrow[ul, "\mathrm{id}\otimes\Delta"'] \\
& C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& C\otimes C \arrow[dl, "\epsilon\otimes\mathrm{id}"'] & \\
k\otimes C \arrow[rr, phantom, "\cong"] & & C \arrow[ul, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& C\otimes C \arrow[dl, "\mathrm{id}\otimes\epsilon"'] & \\
C\otimes k \arrow[rr, phantom, "\cong"] & & C \arrow[ul, "\Delta"']
\end{tikzcd}$$

The comultiplication can be thought of 'un-multiplying' or splitting an element; somewhat like a probability distribution. In the same vein as before, we say a coalgebra is *cocommutative* if the following diagram commutes:

$$\begin{tikzcd}
& C\otimes C \arrow[dl, "\tau"'] & \\
C\otimes C & & C \arrow[ul, "\Delta"'] \arrow[ll, "\Delta"]
\end{tikzcd}$$

To write an element Δc explicitly we need an expression that looks like:

$$\Delta c = \sum_i c_{(1)}^{(i)} \otimes c_{(2)}^{(i)}.$$

As a shorthand, we often write Δc in *Sweedler notation* as

$$\Delta c = \sum c_{(1)} \otimes c_{(2)}.$$

or sometimes just as $\Delta c = c_{(1)} \otimes c_{(2)}$, Note that both algebras and coalgebras allow for a tensor multiplication: the tensor product of two algebras A and B has $A \otimes B$ as the underlying vector space, with product $(a \otimes b)(c \otimes d) = (ac \otimes bd)$. Similarly, the tensor product of coalgebras C and D has $C \otimes D$ as underlying vector space, with coproduct $\Delta(c \otimes d) = \Delta(c) \otimes \Delta(d)$.

We are now in the position to define a Hopf algebra. As a last preliminary, we need the notion of a *bialgebra*:

Definition 3.4. A *bialgebra* over k is a vector space $(H, +)$ equipped with the structures of both an algebra and a coalgebra. These structures must more-over be compatible. Explicitely:

$$\begin{aligned} \Delta(h \cdot g) &= \Delta(h) \cdot \Delta(g), & \Delta(1) &= 1 \otimes 1, \\ \epsilon(h \cdot g) &= \epsilon(h)\epsilon(g), & \epsilon(1) &= 1. \end{aligned}$$

The multiplication on the right-hand side is done in the tensor product algebra $H \otimes H$. These requirements state exactly that Δ and ϵ are algebra homomorphisms. One may show that this is equivalent to the requirement that $\cdot$ and η are coalgebra homomorphisms. Note that since we are considering k to be a field, the requirement $\epsilon(1) = 1$ is autometically met. These additional bialgebra axioms are represented in the following commutative diagrams:

$$\begin{tikzcd} H \otimes H \arrow[rr, "\cdot"] \arrow[d, "\Delta\otimes\Delta"'] & & H \arrow[r, "\Delta"] & H \otimes H \\ H \otimes H \otimes H \otimes H \arrow[rrr, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"'] & & & H \otimes H \otimes H \otimes H \arrow[u, "\cdot\otimes\cdot"'] \end{tikzcd}$$

$$\begin{tikzcd} H \arrow[rr, "\epsilon"] & & k \\ & H \otimes H \arrow[ul, "\cdot"] \arrow[ur, "\epsilon\otimes\epsilon"'] & \end{tikzcd} \qquad \begin{tikzcd} k \arrow[rr, "\eta"] \arrow[dr, "\eta\otimes\eta"'] & & H \arrow[dl, "\Delta"] \\ & H \otimes H & \end{tikzcd}$$

Definition 3.5. A *Hopf algebra* $(H, +, \cdot, \eta, \Delta, \epsilon, S)$ over k is a bialgebra equipped with a linear *antipode* $S : H \to H$ such that:

$$\cdot(S \otimes \mathrm{id}) \circ \Delta = \cdot(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon.$$

These relations translate to the commutativity of the following diagram:

$$\begin{tikzcd} & H \otimes H \arrow[rr, "S\otimes\mathrm{id}"] & & H \otimes H \arrow[dr, "\cdot"] & \\ H \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & k \arrow[rr, "\eta"] & & H \\ & H \otimes H \arrow[rr, "\mathrm{id}\otimes S"'] & & H \otimes H \arrow[ur, "\cdot"'] & \end{tikzcd}$$

This antipode may be thought of as a generalized inverse, since the map $\eta \circ \epsilon : H \to H$ may be thought of as 'trivial' in some sense. However, S need not be invertible.

One characteristic that S shares with the group inverse is that it is an *anti-morphism*. This is the content of the following lemma:

Lemma 3.6. The antipode S of a Hopf algebra H is an algebra anti-morphism and a coalgebra anti-morphism. Explicitly, this means that

$$S(h \cdot g) = S(g) \cdot S(h) \qquad \text{and} \qquad (S \otimes S) \circ \Delta h = \tau \circ \Delta \circ Sh.$$

for all $h, g \in H$.

Proof. We will give a proof of the analogous statement for braided groups later. Hopf algebras are a special case of braided groups, so that this lemma follows. $\square$

Before continuing with more definitions, we give some basic examples of Hopf algebras:

Example 3.7. Let G be a finite group, and k a field. Let kG consists be the vector space with basis G, i.e. kG is the vector space consisting of formal sums of the form

$$\sum_{g\in G} \lambda_g g,$$

with $\lambda_g \in k$ for all g. We can give this vector space the structure of a Hopf algebra as follows: on basis elements $g, h \in G$ let $g \cdot h = gh$ (group multiplication in G), $\eta(1) = e$, $\Delta g = g \otimes g$, $\epsilon g = 1$, $Sg = g^{-1}$. Then extend these operations to linear maps.

Example 3.8. Again let G be a finite group, and k a field. Let $k(G)$ consists of functions $f : G \to k$. This set forms a vector space in a trivial way. We can again give this vector space the structure of a Hopf algebra: for $x, y \in G$ and $f, g \in k(G)$ let $(f \cdot g)(x) = f(x)g(x)$, $\Delta f(x, y) = f(xy)$, $\epsilon f = f(e)$, $(Sf)(x) = f(x^{-1})$.
In the definition of Δ we have identified $k(G) \otimes k(G) \cong k(G \times G)$. This is done via the isomorphism

$$(f(x) \otimes g(y)) \mapsto f(x)g(y).$$

Since G is finite this has the inverse

$$f(x, y) \mapsto \sum_{g\in G} f(x, g) \otimes \delta_g(y).$$

Thus we may alternatively write $\Delta f(x, y) = \sum_g f(xg) \otimes \delta_g(y)$. Here δ_g is a Kronecker delta function from G to k.

Example 3.9. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k. The well-known Universal enveloping algebra $U(\mathfrak{g})$ is the algebra generated by $1 \in k$ and a basis $\mathbf{b}$ for $\mathfrak{g}$, modulo the relations $[\xi, \eta] = \xi\eta - \eta\xi$ for all $\xi, \eta \in \mathbf{b}$. This algebra becomes a Hopf algebra with

$$\Delta\xi = \xi \otimes 1 + 1 \otimes \xi, \quad \epsilon\xi = 0, \quad S\xi = -\xi,$$

for $\xi \in \mathbf{b}$. These are extended to algebra morphisms for Δ, ϵ or to an algebra anti-morphism in the case of S. We also refer to this additive type of coproduct as a *coaddition*, as opposed to that of example 3.7 for instance.

Example 3.10. One of the first classes of Hopf algebras that were discovered are the *Drinfeld-Jimbo* type Hopf algebras [11]. These are also known as *quantum groups*. Let $\mathfrak{g}$ be a *semi-simple* Lie algebra over a field k. Then a quantum group is a certain Hopf algebra $U_q(\mathfrak{g})$, where q is a deformation parameter. As $q \to 1$ we recover the universal enveloping algebra $U(\mathfrak{g})$. As such, quantum groups are certain *q-deformations* of these algebras.

We will not need to discuss the specifics of quantum group theory, but we will see an application of them to knot theory later on. We also wish to mention the example of $U_q(\mathfrak{su}_2)$: this Hopf algebra finds applications in molecular physics and mathematical chemistry. For example, it is a hidden symmetry in the spectrum of certain diatomic molecules [17]. In these applications, it is more commonplace to refer to the Hopf algebra $U_q(\mathfrak{su}_2)$ as the quantum group $SU_q(2)$ [18].

The quantum group $SU_q(2)$ is abstractly given as the complex algebra generated by $1, J_x, J_y, J_z$ such that $J_{x,y,z}$ are self-adjoint and obey the relations [12]

$$[J_z, J_\pm] = \pm J_\pm \qquad \text{and} \qquad [J_+, J_-] = \frac{q^{2J_z} - q^{-2J_z}}{q - q^{-1}},$$

where $J_\pm = J_x \pm i J_y$. As the notation suggests, these can be though of as q-deformed angular momentum operators. To recover the usual angular momentum operators in the limit $q \to 1$, we let $q = e^s$ (we restrict to $q \in (0, \infty)$). We then find

$$\frac{q^{2J_z} - q^{-2J_z}}{q - q^{-1}} = \frac{\sinh(2sJ_z)}{\sinh(s)} \xrightarrow{s \to 0} 2J_z.$$

Thus indeed in the limit $q \to 1$, $s \to 0$ we recover the angular momentum operators and the universal enveloping algebra of $\mathfrak{su}_2$.

Interestingly (and admittedly, perhaps coincidentally) we can use the q-oscillator from the previous section to give a realization of $SU_q(2)$. This is due to [12]. To this end we postulate the q-deformed *number operator* N, which satisfies

$$N|n\rangle = n|n\rangle.$$

For $q = 1$ this operator is simply given by $N = a_+ a_-$. For $q \neq 1$ we have, however [12]:

$$a_+ a_- = \frac{q^N - q^{-N}}{q - q^{-1}}.$$

This indeed approaches N as $q \to 1$, via the same trick as before. An explicit description for N can be found in [12]. Now consider two independent copies of the q-oscillator, generated by (a_+^1, a_-^1) and (a_+^2, a_-^2) respectively, with associated number operators N^1, N^2. Then we define

$$J_+ = a_+^1 a_-^2,$$
$$J_- = a_+^2 a_-^1.$$

These are clearly self-adjoint, since $a_+^i = (a_-^i)^*$ for $i \in \{1, 2\}$. Since $a_+^1 a_-^1 = (q^{N^1} - q^{-N^1})/(q - q^{-1})$ and $a_+^2 a_-^2 = (q^{N^2} - q^{-N^2})/(q - q^{-1})$, we find that

$$[J_+, J_-] = \frac{q^{N^1 - N^2} - q^{N^2 - N^1}}{q - q^{-1}}.$$

Hence if we define $2J_z = N^1 - N^2$, we recover the correct commutator for J_+, J_-. This definition of J_z also obeys the other commutators, and is self-adjoint [12]. Hence we obtain realization of $SU_q(2)$ using two independent copies of the q-oscillator, as claimed.

3.1.1 Braided Diagrams and Self-Duality

The axioms defining bialgebras and Hopf algebras have a clear symmetry: if we invert all the arrows and replace $(\epsilon, \cdot)$ by (η, Δ) and vice versa, then we get exactly the same axioms back.

In fact, this is exactly why the dual $H^* := \mathrm{Lin}(H, k)$ of a Hopf algebra H over a field k is also a Hopf algebra: the algebra structure of H induces a coalgebra structure on H^*, and the coalgebra structure of H induces a compatible algebra structure (in the finite-dimensional case). The antipode of H induces an antipode on H^*; for details see [11], [19].

This self-duality of Hopf algebras is best exhibited by a different type of diagram that exhibits arrow-reversal duality as horizontal reflection.

To construct these diagrams, we represent an element of H by a vertical line, which is to be though of as a physical 'strand'. Horizontal juxtaposition of these strands means taking their tensor product. Later we will see that this is analogous to the knot-theoretic formalism of *tangles*. These diagrams are read from top to bottom (i.e. they are oriented downward). We call them *braided diagrams*. The reason for this name is two-fold: first, these diagrams are the most elegant formalism for developing the theory of *braided categories* and *braided groups* in the next subsection. Second, these diagrams can really be thought of as braided pieces of physical string: a proof of this assertion is given in section 4. In these diagrams, the white space around the strands can be thought of as copies of the field k, since horizontal juxtaposition corresponds to a tensor product, and $H \otimes k^I \cong H$ for any indexing set I.

We represent multiplication of Hopf algebras by a `Y` shape: we're putting two strands together, so to speak. Co-multiplication is represented by the horizontal mirror image; splitting a strand in two. The unit is represented by a `T`; starting a strand from the field k. The counit is again the mirror image, ending a strand into the field. Finally the antipode is represented by a circled S. The diagrams are read starting and the top and ending at the bottom.

The procedure is best explained by showing some examples. For instance, the axioms of associativity and coassociativity respectively translate to the following diagrammatic equalities:

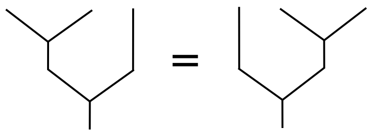


Figure 3.1: Associativity axiom.

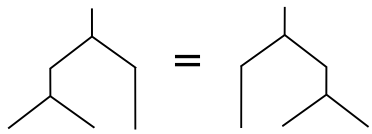


Figure 3.2: Coassociativity axiom.

Intuitively, the requirements of (co)associativity thus tell us that we can swap the order of branching paths in braided diagrams. The axioms for the unit and counit are represented by the following equalities:

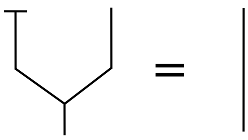


Figure 3.3: Unit axiom.

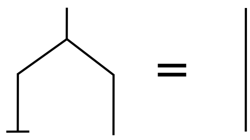


Figure 3.4: Counit axiom.

These axioms tell us that in a braided diagram we can always add new branches, provided they begin or end in the field k. Finally, the bialgebra and Hopf algebra axioms have the following associated diagrams:

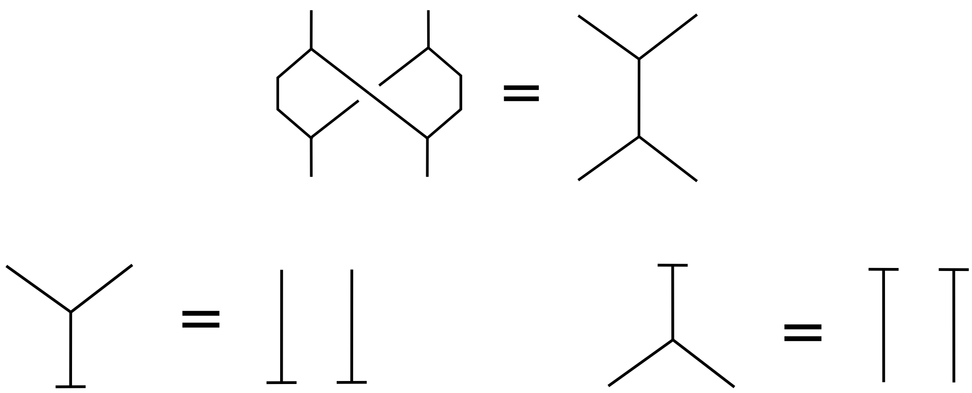


Figure 3.5: Comultiplication is an algebra homomorphism.

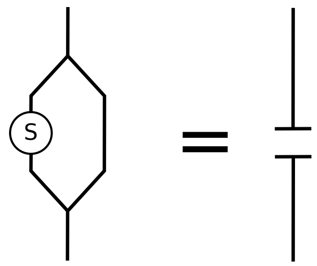


Figure 3.6: Antipode axiom.

Note how this diagrammic notation exhibits the self-duality of the Hopf algebra axioms: if all the diagrams are flipped horizontally (which corresponds to inverting all the arrows, and swapping $\cdot$ with Δ, η with ϵ) we obtain the same axioms.

We will make extensive use of braided diagrams in the next two subsections, which introduce *braided categories* and *braided groups* respectively.

3.2 Category Theory

For our purposes, it suffices to define a category somewhat informally as follows:

Definition 3.11. A category $\mathcal{C}$ consists of the following information:

- A 'collection' of *objects*, denoted $\operatorname{ob}\mathcal{C}$.
- For each $A, B \in \operatorname{ob}\mathcal{C}$ a collection $\operatorname{Hom}(A, B)$ of *morphisms* from A to B. These morphisms are often drawn in diagrams as arrows.
- For each $A, B, C \in \operatorname{ob}\mathcal{C}$ a *composition* function

$$\circ : \operatorname{Hom}(B, C) \times \operatorname{Hom}(A, B) \to \operatorname{Hom}(A, C) : (g, f) \mapsto g \circ f.$$

- For each $A \in \operatorname{ob}\mathcal{C}$ an *identity* element $\mathrm{id}_A \in \operatorname{Hom}(A, A)$.

The composition of morphisms must satify the following axioms:

- *Associativity*: for each $f \in \operatorname{Hom}(A, B)$, $g \in \operatorname{Hom}(B, C)$, $h \in \operatorname{Hom}(C, D)$ we have $(h \circ g) \circ f = h \circ (g \circ f)$.
- *Identity laws*: for each $f \in \operatorname{Hom}(A, B)$, we have

$$f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f.$$

The collection of objects need not be a set, and can in fact be 'larger'. Indeed: an example of a category is that of sets, with functions as morphisms. The collection of all sets does not form a set; otherwise one could construct a set of all sets that do not contain themselves, which is an obvious paradox (does this set contain itself?). We will not need the axiomatic mathematics involved in dealing with this properly, and for our purposes the intuitive idea of a 'collection' suffices. For a more formal treatment, see e.g. [20].

Definition 3.12. A morphism $f \in \operatorname{Hom}(A, B)$ in $\mathcal{C}$ is called an *isomorphism* if there exists some two-sided inverse $g \in \operatorname{Hom}(B, A)$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Example 3.13. Some examples of categories include:

- Groups form the objects of a category `Grp`, with group homomorphisms as morphisms.
- Fix a field k. Vector spaces over k form a category $\mathtt{Vec}_k$, with linear maps as morphisms. In what follows k will be implicitly fixed, and we will denote this category by `Vec`.
- The collection of all topological spaces forms a category `Top`, with continuous functions as morphisms.

The category concept captures the idea of a class of mathematical structures and their structure-preserving maps. The former serve the role of objects, and the latter that of morphisms. Categories themselves also obey this principle: there is a category of categories, with certain structure-preserving maps as morphisms. These maps are known as *functors:*

Definition 3.14. Let $\mathcal{C}, \mathcal{D}$ be categories. A *functor* $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ consist of a function $\operatorname{ob}\mathcal{C} \to \operatorname{ob}\mathcal{D}$ written as $C \mapsto \mathscr{F}(C)$ and for each $C, C' \in \operatorname{ob}\mathcal{C}$ a function $\operatorname{Hom}(C, C') \to \operatorname{Hom}(\mathscr{F}(C), \mathscr{F}(C'))$ written as $f \mapsto \mathscr{F}(f)$. This function must additionally satisfy:

- $\mathscr{F}(g \circ f) = \mathscr{F}(g) \circ \mathscr{F}(f)$ for all $C \xrightarrow{f} C' \xrightarrow{g} C''$.
- $\mathscr{F}(\mathrm{id}_C) = \mathrm{id}_{\mathscr{F}(C)}$ for all $C \in \mathcal{C}$.

A functor is also referred to as a *covariant functor*. In contrast, a *contravariant functor* is defined similarly, except that a contravariant functor is defined on morphisms as a function $\mathrm{Hom}(C, C') \to \mathrm{Hom}(\mathscr{F}(C'), \mathscr{F}(C))$ for each $C, C' \in \mathrm{ob}\,\mathcal{C}$. This function must satisfy

- $\mathscr{F}(g \circ f) = \mathscr{F}(f) \circ \mathscr{F}(g)$ for all $C \xrightarrow{f} C' \xrightarrow{g} C''$.

(Note the mnemonic in the termination 'contra'. A contravariant functor sends morphisms to ones going in the opposite directions: 'against' the flow of the original morphism.) Finally, a *bifunctor* is a map $\mathcal{C} \times \mathcal{C} \to \mathcal{D}$ that is a functor (covariant or contravariant) when either of its inputs are fixed.

Example 3.15. Many categories allow for the definition of a *fiber functor* or *forgetful functor*. An example is the forgetful functor $\mathtt{Grp} \to \mathtt{Set}$, that sends a group to its underlying set, and a group homomorphism to the same map, regarded as a function of sets. This functor simply 'forgets' the group structure.

Example 3.16. Let $\mathcal{C}$ be a category (such that all Hom collections are sets) and C an object in $\mathcal{C}$. The *covariant Hom functor* $\mathrm{Hom}(C, -) : \mathcal{C} \to \mathtt{Set}$ is defined by sending an object D of $\mathcal{C}$ to $\mathrm{Hom}(C, D)$, and by sending a morphism $f : X \to Y$ in $\mathcal{C}$ to

$$\mathrm{Hom}(C, f) : \mathrm{Hom}(C, X) \to \mathrm{Hom}(C, Y) : g \mapsto f \circ g.$$

It is immediate that this indeed defines a covariant functor. Similarly, the *contravariant Hom functor* $\mathrm{Hom}(-, C) : \mathcal{C} \to \mathtt{Set}$ is defined by sending an object D to $\mathrm{Hom}(D, C)$, and by sending a morphism $f : X \to Y$ to

$$\mathrm{Hom}(f, C) : \mathrm{Hom}(Y, C) \to \mathrm{Hom}(X, C) : g \mapsto g \circ f.$$

Again, it is immediate that this defines a contravariant functor. In total, this defines the *Hom bifunctor* $\mathrm{Hom}(-, -)$, which is contravariant in the first argument and covariant in the second.

There will be many more concepts or additional structures that we can attach to the concept of a category. Each new piece of additional structure that we add to a category will come with axioms in the form of diagrams, and the total picture may be daunting. However, there is an underlying principle that the definitions of structure on a category roughly adhere to. The idea is as follows:

1. **Machinery**: The addition of some structure to a category is often through requiring the existence of some object, morphism, or transformation. This addition should fulfill a certain role, in that it should obey some axioms that express its purpose.

2. **Coherence**: This new machinery will 'interact' with the machinery already in place. This may allow for several sequences of operations ('diagrammatic paths') that achieve the same basic goal. If we wish that our new machinery interacts coherently with the structure in place, we require that these paths are in fact the same. This will often boil down to requiring commutativity of certain diagrams.

As a first application of this principle, we introduce the concept of a *natural transformation*. We have seen functors, which are maps between categories. Natural transformations are maps between functors, and are conceptually similar to homotopies between continuous maps.

Definition 3.17. Let $\mathcal{C}, \mathcal{D}$ be categories and let

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathscr{F}"] \arrow[r, bend right, "\mathscr{G}"'] & \mathcal{D} \end{tikzcd}$$

be functors. A *natural transformation* $\alpha : \mathscr{F} \to \mathscr{G}$ consists of the information of a morphism $\alpha_C : \mathscr{F}(C) \to \mathscr{G}(C)$ in $\mathcal{D}$, for each $C \in \operatorname{ob}\mathcal{C}$. For coherence, we must consider the interaction of α with morphisms. For any morphism $f : C \to C'$ in $\mathcal{C}$ we naturally obtain a square diagram:

$$\begin{tikzcd} \mathscr{F}(C) \arrow[r, "\mathscr{F}(f)"] \arrow[d, "\alpha_C"'] & \mathscr{F}(C') \arrow[d, "\alpha_{C'}"] \\ \mathscr{G}(C) \arrow[r, "\mathscr{G}(f)"'] & \mathscr{G}(C') \end{tikzcd}$$

For α to be a natural transformation we require all of these squares to commute. We then alternatively say that the transformation is *functorial in C*.

Note that the coherence requirement is set up precisely such that for every morphism $f : C \to C'$ in $\mathcal{C}$ we can construct exactly one morphism $\mathscr{F}(C) \to \mathscr{G}(C')$ in $\mathcal{D}$. For any two categories $\mathcal{C}, \mathcal{D}$ one can make a category of the functors $\mathcal{C} \to \mathcal{D}$. The morphisms of this category are natural transformations. An isomorphism in this category is called a *natural isomorphism.* It is immediate to verify that a natural isomorphism is exactly a natural transformation α such that α_C is an isomorphism for all $C \in \mathcal{C}$.

The concept of a natural transformation allows us to introduce one more concept regarding functors: for this, recall the Hom functors of example 3.16.

Definition 3.18. A functor $\mathcal{C} \to \mathcal{D}$ is *representable* if it is naturally isomorphic to a Hom functor. In particular, a contravariant functor is representable if it is naturally isomorphic to $\operatorname{Hom}(-, C)$ for some object C in $\mathcal{C}$.

In the upcoming additions of structure to a category, we will see the introduction of several natural transformations. These will generally have several inputs of objects from the category at hand. The idea of *funtoriality* extends to such transformations analogously: varying a single input individually using morphisms generates square diagrams. For functoriality in this input we require that all these squares commute. Thus the coherence condition of these transformations when interacting with morphisms is expressed by requiring that they are functorial in all of their inputs.

The next structure we add to a category is a 'product' of objects, denoted by $\otimes$. This need not be related to the tensor product of vector spaces, but it is modelled precisely so that $\mathtt{Vec}$ equipped with the usual tensor product is an example of such a 'product' category.

Definition 3.19. A *monoidal category* consists of:

- A category $\mathcal{C}$.
- The information $(\otimes, \Phi)$ of a *product*: Here $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (also denoted $\otimes$) is a bifunctor which is *associative* in the sense that there exists a natural isomorphism $\Phi : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$, i.e. there are isomorphisms

  $$\Phi_{V,W,Z} : (V \otimes W) \otimes Z \to V \otimes (W \otimes Z),$$

  for each $V, W, Z \in \operatorname{ob}\mathcal{C}$, and these are functorial in V, W, Z.
- The information $(\underline{1}, l, r)$ of a *unit* with respect to this product: Here $\underline{1}$ is an object of $\mathcal{C}$ such that there are natural isomorphisms $l : \mathrm{id}_{\mathcal{C}} \to - \otimes \underline{1}$ and $r : \mathrm{id}_{\mathcal{C}} \to \underline{1} \otimes -$, where $\mathrm{id}_{\mathcal{C}}$ is the identity functor.

These new transformations (Φ, l, r) are moreover subject to the following *coherence conditions*:

- A condition on the interaction of Φ with itself, called the *pentagon condition*:

$$\begin{tikzcd}[column sep=small]
& (V\otimes W)\otimes(Z\otimes U) \arrow[dr, "\Phi"] & \\
((V\otimes W)\otimes Z)\otimes U \arrow[ur, "\Phi"] \arrow[d, "\Phi\otimes\mathrm{id}"'] & & V\otimes(W\otimes(Z\otimes U)) \\
(V\otimes(W\otimes Z))\otimes U \arrow[rr, "\Phi"'] & & V\otimes((W\otimes Z)\otimes U) \arrow[u, "\mathrm{id}\otimes\Phi"']
\end{tikzcd}$$

  must commute for all $V, W, Z, U \in \mathrm{ob}\,\mathcal{C}$.

- A condition on Φ interacting with l, r, called the *triangle condition*:

$$\begin{tikzcd}
(V\otimes \underline{1})\otimes W \arrow[rr, "\Phi"] & & V\otimes(\underline{1}\otimes W) \\
& V\otimes W \arrow[ul, "l\otimes\mathrm{id}"] \arrow[ur, "\mathrm{id}\otimes r"'] &
\end{tikzcd}$$

  must commute for all $V, W \in \mathrm{ob}\,\mathcal{C}$. (Note: all subscripts on Φ, l, r, id have been suppressed.)

As with categories, monoidal categories form a category of their own, with the corresponding notion of a structure-preserving map as morphisms. These maps between monoidal categories that preserve the monoidal structure are called *monoidal functors*:

Definition 3.20. Let $\mathcal{C}, \mathcal{D}$ be monoidal categories. Let $\otimes, \underline{1}, \Phi, l, r$ be denoted the same in both $\mathcal{C}$ and $\mathcal{D}$; there should be no confusion as to which product $\otimes$ refers to at any given instance. A *monoidal functor* $F : \mathcal{C} \to \mathcal{D}$ is a functor such that the bifunctors $F^2, F \circ \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{D}$ are naturally isomorphic. Here $F^2(V, W) = F(V) \otimes F(W)$ and $(F \circ \otimes)(V, W) = F(V \otimes W)$. Thus a monoidal functor is a functor together with isomorphisms $c_{V,W} : F(V) \otimes F(W) \cong F(V \otimes W)$ that are functorial in V and W.
The natural transformation c is subject to the following coherence conditions: the diagram

$$\begin{tikzcd}
(F(V)\otimes F(W))\otimes F(Z) \arrow[r, "c\otimes\mathrm{id}"] \arrow[d, "\Phi"'] & F(V\otimes W)\otimes F(Z) \arrow[r, "c"] & F((V\otimes W)\otimes Z) \arrow[d, "F(\Phi)"] \\
F(V)\otimes(F(W)\otimes F(W)) \arrow[r, "\mathrm{id}\otimes c"'] & F(V)\otimes F(W\otimes Z) \arrow[r, "c"'] & F(V\otimes(W\otimes Z))
\end{tikzcd}$$

must commute for all $V, W, Z \in \mathrm{ob}\,\mathcal{C}$. This is a coherence condition on the interaction of c with Φ. For coherence of the interaction of c with $(\underline{1}, l, r)$ we require that $F(\underline{1}) = \underline{1}$ and that the following diagrams commute for all $V \in \mathrm{ob}\,\mathcal{C}$:

$$\begin{tikzcd}
F(V) \arrow[r, "l_{F(V)}"] \arrow[dr, "F(l_V)"'] & F(V)\otimes\underline{1} = F(V)\otimes F(\underline{1}) \arrow[d, "c_{V,\underline{1}}"] \\
& F(V\otimes\underline{1})
\end{tikzcd}
\qquad
\begin{tikzcd}
F(V) \arrow[r, "r_{F(V)}"] \arrow[dr, "F(r_V)"'] & \underline{1}\otimes F(V) = F(\underline{1})\otimes F(V) \arrow[d, "c_{\underline{1},V}"] \\
& F(\underline{1}\otimes V)
\end{tikzcd}$$

In `Vec`, together with the tensor product functor $\otimes : (V, W) \mapsto V \otimes W$ there is also a functor $\otimes^{\mathrm{op}} : (V, W) \mapsto W \otimes V$. These functors are naturally isomorphic in a trivial way, namely via

$$V \otimes W \cong W \otimes V : \sum v \otimes w \mapsto \sum w \otimes v.$$

It is trivial that this latter assignment is linear and invertible (namely self-inverse). For general monoidal categories this is no longer the case. To emulate this isomorphism $\otimes \cong \otimes^{\mathrm{op}}$ in a monoidal category, we introduce a braiding Ψ, which is not necessarily self-inverse.

Definition 3.21. A *braided* monoidal category or *quasitensor* category consists of:

- A monoidal category $(\mathcal{C}, \otimes, \Phi)$.
- A *braiding* Ψ that communicates commutativity of $\otimes$: the braiding is a natural ismorphism $\Psi : - \otimes - \to - \otimes^{\mathrm{op}} -$, i.e. a collection of isomorphisms $\Psi_{V,W} : V \otimes W \to W \otimes V$ that are functorial in V, W.

The transformation Ψ is more-over subject to the coherence conditions called the *hexagon conditions*: the diagrams

$$\begin{tikzcd}
 & V \otimes (W \otimes Z) \arrow[dl, "\mathrm{id}\otimes\Psi"'] \arrow[dr, "\Phi^{-1}"] & \\
V \otimes (Z \otimes W) \arrow[d, "\Phi^{-1}"'] & & (V \otimes W) \otimes Z \arrow[d, "\Psi"] \\
(V \otimes Z) \otimes W \arrow[dr, "\Psi\otimes\mathrm{id}"'] & & Z \otimes (V \otimes W) \arrow[dl, "\Phi^{-1}"] \\
 & (Z \otimes V) \otimes W &
\end{tikzcd}$$

and

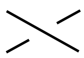
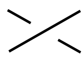
$$\begin{tikzcd}
 & (V \otimes W) \otimes Z \arrow[dl, "\Phi"'] \arrow[dr, "\Psi\otimes\mathrm{id}"] & \\
V \otimes (W \otimes Z) \arrow[d, "\Psi"'] & & (W \otimes V) \otimes Z \arrow[d, "\Phi"] \\
(W \otimes Z) \otimes V \arrow[dr, "\Phi"'] & & W \otimes (V \otimes Z) \arrow[dl, "\mathrm{id}\otimes\Psi"] \\
 & W \otimes (Z \otimes V) &
\end{tikzcd}$$

must commute for all $V, W, Z, U \in \mathrm{ob}\,\mathcal{C}$.

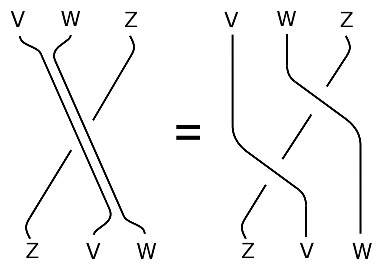
Again, Ψ need not be self-inverse. In the case that $\Psi^2 = \mathrm{id}$ we say that $(\mathcal{C}, \otimes, \Psi)$ is a *symmetric category*. The hexagon conditions simply say that transposing $V \otimes W$ past Z is the same as transposing W past Z and then V past Z, and similarly for transposing V past $W \otimes Z$. Using the hexagon conditions one may derive various other identities that one would expect Ψ to obey; for instance that Ψ is trivial on $\underline{1}$:

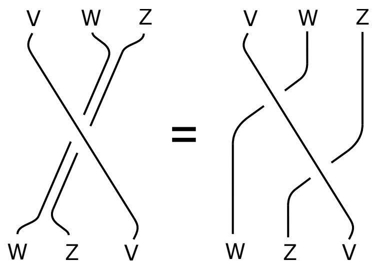
$$\Psi_{V,\underline{1}} = \mathrm{id} = \Psi_{\underline{1},V}.$$

Instead of using commutative diagrams (which are generally hard to read) we can also use the braided diagrams from 3.1.1. To do so we let Ψ be represented by a right-handed crossing and Ψ^{-1} by a left-handed crossing as follows:

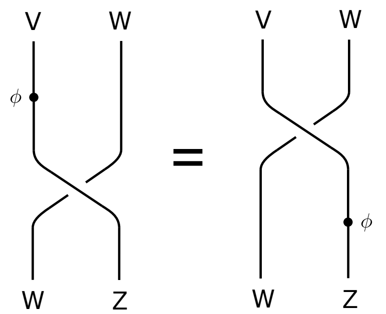
$\Psi =$  $\Psi^{-1} =$ 

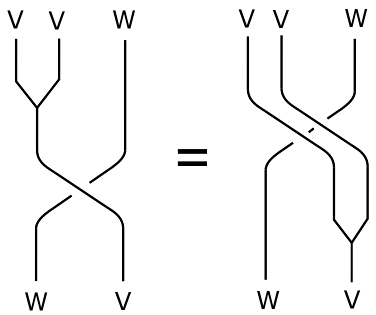
Recall that these diagrams are read from top to bottom (hence the left crossing is indeed right-handed), and that horizontal juxtaposition of strands indicates a tensor product. In this notation, the hexagon identities become



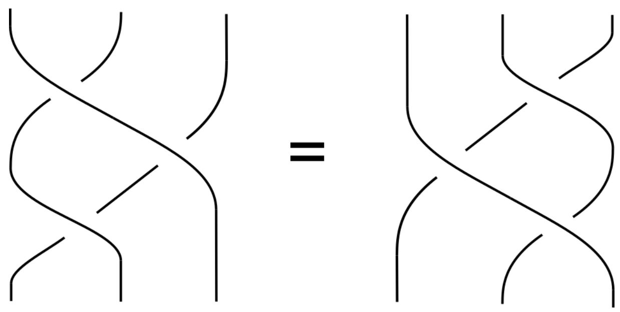


Here the close horizontal juxtapositions on the left-hand sides signify that we pass $(V \otimes W)$ or $(W \otimes Z)$ over or under a crossing as a whole. These diagrammatic identities essentially convey that the depiction of the tensor product as a horizontal juxtaposition is well-defined with respect to Ψ. Similarly we can express functoriality of Ψ in braided diagrams as follows:





Here the left diagram displays functoriality of the crossing Ψ, and the right diagram depicts the special case where Ψ is the multiplication of an algebra object in a braided category, which we will define properly in the next subsection. In this same vein, the hexagon identities and functoriality can be combined to derive that Ψ obeys the braid relation, depicted below [19]. This is the relation used to define the well-known *braid group*.



This relation is in fact why Ψ is called a braiding in the first place.

Functoriality of operations like multiplication, together with the braid relation, allow us to treat the above diagrams as physical strings sitting in three-dimensional space. (Again, a formal proof of this assertion is given in section 4; the argument indeed hinges on functoriality and the braid relation.) In the next subsection we will see that this remarkable fact allows us to do diagrammatic mathematics that lies on the border between category theory and knot theory.

An interesting example of a braided category is that of the *super vector spaces*. This category is of great interest to the study of *supersymmetry* [21].

Example 3.22. A vector space V is *graded* over a group G if it has a decomposition of the form

$$V = \bigoplus_{g \in G} V_g.$$

If $v \in V_g$ we say that v is of degree g, i.e. $|v| = g$. A *super vector space* is a $\mathbb{Z}_2$-graded vector space, where $\mathbb{Z}_2$ is the group of order 2, taken to be $\{0,1\} \subseteq k$ so that $\mathbb{Z}_2$ is also a ring over

k. Thus a super vector space decomposes as $V = V_0 \oplus V_1$. A map $f : V \to W$ of super vector spaces is *grade preserving* if $f(V_i) \subseteq W_i$ for all $i \in \mathbb{Z}_2$.
The category `SuperVec` is defined as the subcategory of `Vec` that has super vector spaces as objects and grade preserving linear maps as morphisms. This definition is justified because the composition of two grade preserving maps is clearly again grade preserving. Note that `SuperVec` is a subcategory of `Vec` simply because a super vector space is in particular a vector space.
If V, W are super vector spaces, then we have a natural grading on their tensor product as follows:

$$(V \otimes W)_i = \bigoplus_{j+k=i\in\mathbb{Z}_2} V_j \otimes W_k.$$

Thus for instance, $(V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0)$. With this tensor product, `SuperVec` forms a monoidal category. Finally, `SuperVec` becomes a braided category with

$$\Psi_{V,W}(x \otimes y) = (-1)^{|x|\cdot|y|} y \otimes x.$$

Clearly $\Psi^2 = \mathrm{id}$ which means that `SuperVec` is in fact a symmetric category.

Another construction that we can carry out in `Vec` is that of dualization: the construction of a vector space V^* from $V \in \mathrm{ob}\,\texttt{Vec}$. For a vector space V (over k), $V^* = \mathrm{Hom}(V, k)$ where k is regarded as a vector space over itself. This dual construction generates the canonical *evaluation* map

$$\mathrm{ev} : V \otimes V^* \to k : (v, f) \mapsto f(v).$$

If V is finite-dimensional, there is also a canonical *coevaluation* map in the other direction:

$$\mathrm{coev} : k \to V \otimes V^* : \lambda \mapsto \sum_i \lambda v_i \otimes v_i^*.$$

Here $\{v_i\}$ is a finite basis for V, and $\{v_i^*\}$ is the dual basis. It is not difficult to show that this map is independent of the chosen basis. The next construction generalized these aspects of dualization to monoidal categories.

Definition 3.23. Let $\mathcal{C}$ be a monoidal category. An object V of $\mathcal{C}$ has a *left dual* or is *rigid* if there exists an object V^* of $\mathcal{C}$ and maps $\mathrm{ev}_V : V^* \otimes V \to \underline{1}$, $\mathrm{coev}_V : \underline{1} \to V \otimes V^*$ such that the following diagrams commute:

$$\begin{tikzcd}
& V\otimes\underline{1} & V\otimes(V^*\otimes V) \arrow[l, "\mathrm{id}\otimes\mathrm{ev}"'] \\
V \arrow[loop left, "\mathrm{id}"] \arrow[ur, "l^{-1}"] \arrow[dr, "r"'] & & \\
& \underline{1}\otimes V \arrow[r, "\mathrm{coev}\otimes\mathrm{id}"'] & (V\otimes V^*)\otimes V \arrow[uu, "\Phi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
V^*\otimes(V\otimes V^*) \arrow[dd, "\Phi"'] & V^*\otimes\underline{1} \arrow[l, "\mathrm{id}\otimes\mathrm{coev}"'] & \\
& & V^* \arrow[loop right, "\mathrm{id}"] \arrow[ul, "l"'] \\
(V^*\otimes V)\otimes V^* \arrow[r, "\mathrm{id}\otimes\mathrm{ev}"'] & \underline{1}\otimes V^* \arrow[ur, "r^{-1}"'] &
\end{tikzcd}$$

In these diagrams, the subscripts are suppressed as before. Further suppressing $\underline{1}, \Phi, l, r$ these diagrams convey the identities

$$(\mathrm{id} \otimes \mathrm{ev}_V) \circ (\mathrm{coev}_V \otimes \mathrm{id}) = \mathrm{id}_V, \tag{3.1}$$
$$(\mathrm{ev}_V \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coev}_V) = \mathrm{id}_{V^*}. \tag{3.2}$$

If every object of $\mathcal{C}$ is rigid, then we say that $\mathcal{C}$ is *rigid*.

Lemma 3.24. Let $\mathcal{C}$ be a monoidal category. If an object V of $\mathcal{C}$ has a left dual $(V^*, \mathrm{ev}, \mathrm{coev})$ then it is unique up to isomorphism in $\mathcal{C}$.

Proof. As with equations (3.1) and (3.2), this proof suppresses $\underline{1}, l, r$. This proof is based on [22]. Suppose that $(V_1^*, \mathrm{ev}_1, \mathrm{coev}_1)$ and $(V_2^*, \mathrm{ev}_2, \mathrm{coev}_2)$ are left duals of V. Then we have a morphism $\theta : V_1^* \to V_2^*$ given by the composition

$$V_1^* \xrightarrow{\mathrm{id}\otimes\mathrm{coev}_2} V_1^* \otimes (V \otimes V_2^*) \xrightarrow{\Phi^{-1}} (V_1^* \otimes V) \otimes V_2^* \xrightarrow{\mathrm{ev}_1\otimes\mathrm{id}} V_2^*.$$

Similarly we have a morphism $\theta^{-1} : V_2^* \to V_1^*$ given by the composition

$$V_2^* \xrightarrow{\mathrm{id}\otimes\mathrm{coev}_1} V_2^* \otimes (V \otimes V_1^*) \xrightarrow{\Phi^{-1}} (V_2^* \otimes V) \otimes V_1^* \xrightarrow{\mathrm{ev}_2\otimes\mathrm{id}} V_1^*.$$

As suggested by the notation, we claim that θ and θ^{-1} are mutually inverse. Indeed, consider the following diagram (in which we have further suppressed Φ):

$$\begin{tikzcd}
V_1^* \arrow[r, "\mathrm{id}\otimes\mathrm{coev}_1"] \arrow[d, "\mathrm{id}\otimes\mathrm{coev}_2"'] & V_1^* \otimes V \otimes V_1^* \arrow[d, "\mathrm{id}\otimes\mathrm{coev}_2\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\
V_1^* \otimes V \otimes V_2^* \arrow[r, "\mathrm{id}\otimes\mathrm{coev}_1"'] \arrow[d, "\mathrm{ev}_1\otimes\mathrm{id}"'] & V_1^* \otimes V \otimes V_2^* \otimes V \otimes V_1^* \arrow[r, "\mathrm{id}\otimes\mathrm{ev}_2\otimes\mathrm{id}"'] \arrow[d, "\mathrm{ev}_1\otimes\mathrm{id}"'] & V_1^* \otimes V \otimes V_1^* \arrow[d, "\mathrm{ev}_1\otimes\mathrm{id}"] \\
V_2^* \arrow[r, "\mathrm{id}\otimes\mathrm{coev}_1"'] & V_2^* \otimes V \otimes V_1^* \arrow[r, "\mathrm{ev}_2\otimes\mathrm{id}"'] & V_1^*
\end{tikzcd}$$

We claim that this diagram commutes. It is clear that the squares in the diagram commute: consider for instance the top left square. Written out explicitly, this square expresses the following identity of maps from $V_1^* \otimes \underline{1} \otimes \underline{1}$ to $V_1^* \otimes V \otimes V_2^* \otimes V \otimes V_1^*$:

$$(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{coev}_1) \circ (\mathrm{id} \otimes \mathrm{coev}_2 \otimes \mathrm{id}) = (\mathrm{id} \otimes \mathrm{coev}_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{coev}_1).$$

This equality trivially holds: since $\otimes$ is a bifunctor, both sides are equal to $\mathrm{id} \otimes \mathrm{coev}_2 \otimes \mathrm{coev}_1$. The other squares commute similarly. The commutativity of the triangle follows from equation (3.1). Thus, the whole diagram commutes. Now, by equation (3.2) the path from V_1^* to itself along the top and right of the diagram is equal to $\mathrm{id}_{V_1^*}$. Meanwhile the path along the left and bottom is equal to $\theta^{-1} \circ \theta$. Thus by commutativity, $\theta^{-1} \circ \theta = \mathrm{id}_{V_1^*}$. An analogous diagram shows that $\theta \circ \theta^{-1} = \mathrm{id}_{V_2^*}$. Thus by definition $\theta : V_1^* \to V_2^*$ is an isomorphism. $\square$

3.3 Braided Groups

Since we have already introduced braided monoidal categories and Hopf algebras, most of the work needed for introducing braided groups is already done! Essentially, a braided group is the analogue of a Hopf algebra in a braided category. Since we have introduced Hopf algebras in a completely element-free (i.e. diagrammatic) way, this generalization is immediate:

Definition 3.25. Let $(\mathcal{C}, \otimes)$ be a monoidal category. An algebra object B in the category $\mathcal{C}$ consists of

- An object B of $\mathcal{C}$.
- A product morphism $m : B \otimes B \to B$ (also denoted $\cdot$) obeying the associativity condition of figure 3.1.
- A unit morphism $\eta : \underline{1} \to B$ obeying the diagrammatic condition of figure 3.3.

Lemma 3.26. Let B, C be algebras in a *braided* monoidal category $(\mathcal{C}, \otimes, \Psi)$. Then $B \otimes C$ is also an algebra in $\mathcal{C}$, denoted $B \underline{\otimes} C$, with multiplication

$$m_{B \underline{\otimes} C} = (m_B \otimes m_C) \circ (\mathrm{id} \otimes \Psi_{C \otimes B}, \otimes \mathrm{id}),$$

and unit morphism $\eta_B \otimes \eta_C$.

Proof. The associativity is proven by the diagrammatic equalities in figure 3.7. The first and third equality are by functoriality of Ψ in m_C and m_B respectively. The second equality is by associativity of m_B and m_C.

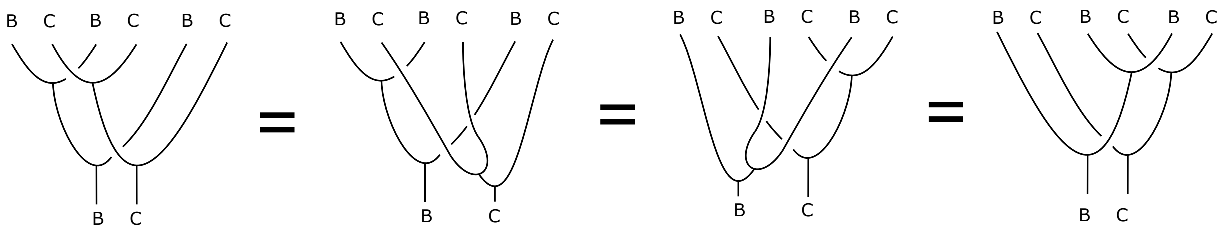


Figure 3.7: Proof of lemma 3.26.

The proof that $\eta_B \otimes \eta_C$ satisfies the unit axiom is omitted; it is similar, but more trivial. $\square$

The tensor product $\underline{\otimes}$ is also called the *braided tensor product*, and will be important in our applications of braided groups.

Recall that in defining a Hopf algebra, we needed the map $\tau : x \otimes y \to y \otimes x$ in `Vec` in several places. Thus to define a Hopf algebra in a monoidal category we indeed need a braiding Ψ on the category. The braiding Ψ plays the role of τ:

Definition 3.27. Let $(\mathcal{C}, \otimes, \Psi)$ be a braided monoidal category. A *braided group* or Hopf algebra object $(B, m, \eta, \Delta, \eta, S)$ in the category $\mathcal{C}$ consists of

- An algebra (B, m, η) in $\mathcal{C}$.
- Morphisms $\Delta : B \to B \underline{\otimes} B$, $\epsilon : B \to \underline{1}$ forming a coalgebra (B, Δ, ϵ) in $\mathcal{C}$. (I.e. Δ, ϵ obey the conditions of figures 3.2 and 3.4 respectively.) We also require that Δ, ϵ are algebra morphisms, i.e. that the conditions of figure 3.5 are satisfied. Note the appearance of Ψ in this figure, which was immaterial for the case of Hopf algebras in `Vec`, where $\Psi = \tau$.
- A morphism $S : B \to B$ obeying the condition of figure 3.6.

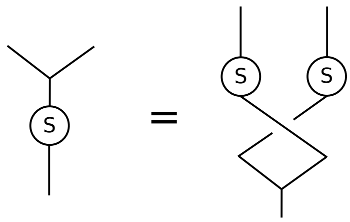
Note that the braiding Ψ is implicitly included in the data of a braided group: to specify B we must specify the category $\mathcal{C}$ of which it is an object, and for that we must specify Ψ.

For Hopf algebras, S is an algebra anti-morphism as well as a coalgebra anti-morphism. We deferred the proof of this fact to the more general case for braided groups. Note that Hopf algebras are braided groups in `Vec` with $\Psi = \tau$. The statement that the antipode of a braided group is also an anti-morphism is the content of the following lemma:

Lemma 3.28. The antipode S of a braided group B satisfies

$$S \circ \cdot = \cdot \circ \Psi_{B,B} \circ (S \otimes S) \qquad \text{and} \qquad \Delta \circ S = (S \otimes S) \circ \Psi_{B,B} \circ \Delta.$$

Proof. As an equality of braided diagrams, the first statement is:



In accordance with the self-dual nature of Hopf algebras that these diagrams exhibit, the second statement is represented by the same diagrams, but reflected in the horizontal axis.
The braided-diagrammatic proof of this identity is shown in figure 3.8. The procedure is quite complicated, but the idea is the same as for the inverse of a group: to show that $(gh)^{-1} = h^{-1}g^{-1}$ for g, h in a group G, we sneak in terms $h^{-1}h$ and $g^{-1}g$ to find

$$(gh)^{-1} = h^{-1}h(gh)^{-1} = h^{-1}g^{-1}gh(gh)^{-1} = (h^{-1}g^{-1})(gh)(gh)^{-1} = h^{-1}g^{-1}.$$

Note that this proof implicitly uses associativity.
The proof in figure 3.8 is exactly the same: we graft on two nested antipode loops, knowing that they disappear via the axioms of figures 3.3, 3.4, and 3.6. We then move string around using associativity and coassociativity, and subsequently cancel out added loops to obtain the desired result. In the proof, we frequently need to pull the Hopf algebra operations $\cdot, \Delta, \eta, \epsilon, S$ over a crossing Ψ. The fact that this is always allowed is precisely the statement of functoriality of Ψ! As stated before, this is exactly what allows us to handle these diagrams as if they were braids of physical string.

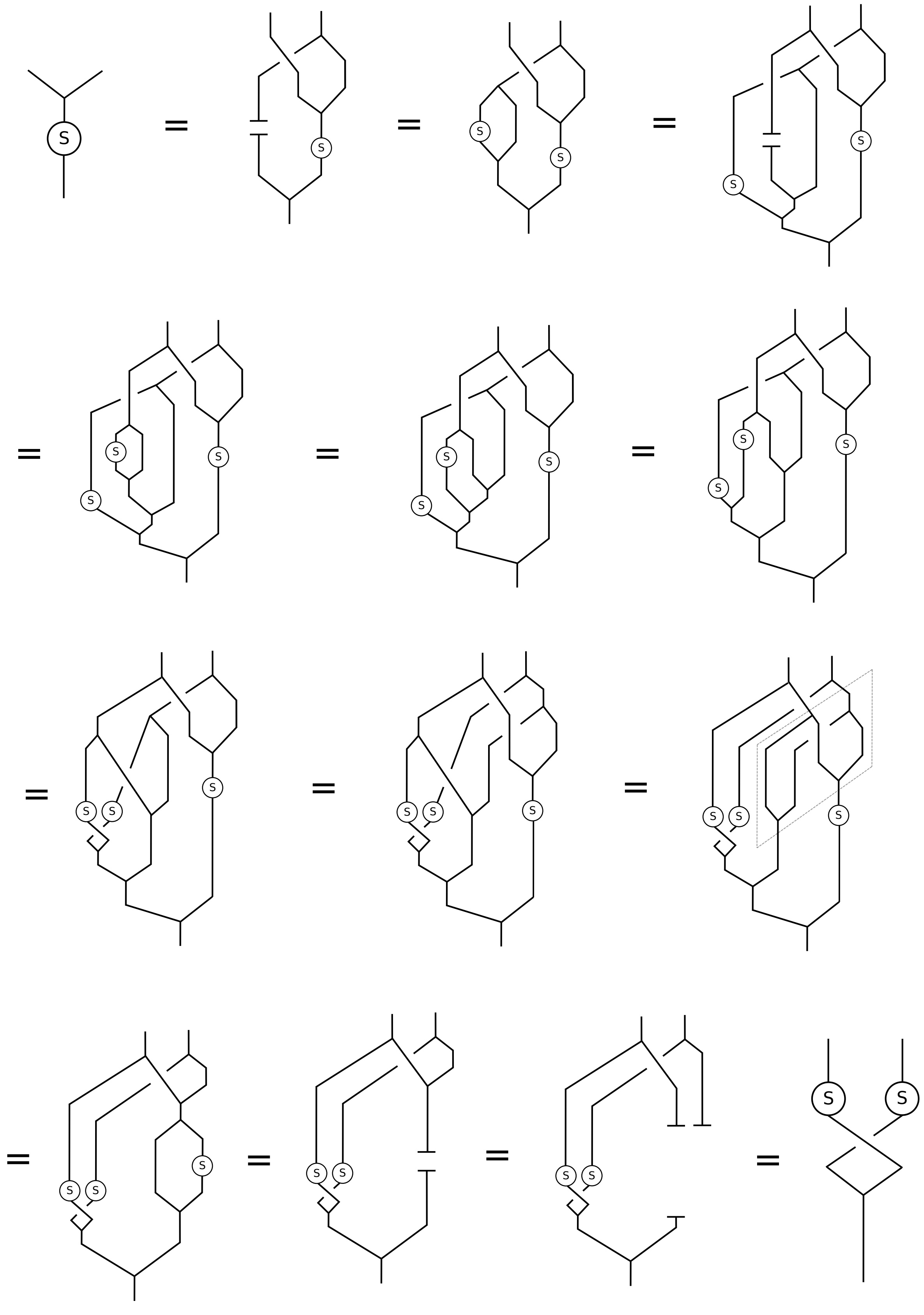


Figure 3.8: Proof of lemma 3.28.

In detail: the first and second equalities graft on a loop, first using the unit and counit axioms, and then using the antipode axioms. The third and fourth equalities do the same, grafting on a second loop inside the first. Equalities 5 and 6 use associativity to rearrange the diagram. Equality 7 pulls the inner antipode over the outer one, creating the application of Ψ that remains in the final result. Equalities 8 and 9 use coassiciativity to further rearrange the diagram. Equality 10 uses the first axiom of figure 3.5 to rearrange the indicated seqment, followed by an application of associativity to create the antipode loop on the right. Equalities 11, 12, and 13 cancel this loop using the axioms of figures 3.6, 3.5, and 3.4 respectively, to give the desired result.

The proof of the second statement is given by the same diagrammatic equalities, but reflected horizontally. $\square$

4 Hopf Algebras and Knots

In this section we consider applications of the basic theory of Hopf algebras (in the category $\texttt{Vec}$) to the theory of *knots*. Namely, we will derive a class of knot invariants related to the Hopf algebra kG from example 3.7, and we will formally show that braided diagrams can indeed be treated like physical tangles of string, as we have claimed earlier. We also return to the 'quantum groups' of Drinfeld and Jimbo, seen in the last section. We show how they can be used to construct 'quantum knot invariants'; in particular we consider the case of $U_q(\mathfrak{sl}_2)$.

In the first subsection we introduce important concepts like quasi-triangular structures and the quantum Yang-Baxter equation. These are paramount for applications in knot theory, and provide the first indication of a deep relationship between Hopf algebras and braids. We will also need these notions in later sections.

4.1 Quasitriangularity and the Drinfeld Double

Definition 4.1. Two Hopf algebras H, H' are dually paired if there is a linear map $\langle\, ,\,\rangle$: $H' \otimes H \to k$ called a *dual pairing* such that

$$\begin{gathered}\langle \phi\psi, h\rangle = \langle \phi \otimes \psi, \Delta h\rangle, \quad \langle , , 1, h\rangle = \epsilon(h),\\ \langle \phi, gh\rangle = \langle \Delta\phi, h \otimes g\rangle, \quad \langle \phi, 1\rangle = \epsilon(\phi),\\ \langle S\phi, h\rangle = \langle \phi, Sh\rangle,\end{gathered}$$

for all $\phi, \psi \in H'$, $h, g \in H$. Here we use the convention $\langle \phi \otimes \psi, h \otimes g\rangle = \langle \phi, h\rangle\langle \psi, g\rangle$.

Note that a Hopf algebra H and its dual H^* are clearly dually paired by the evaluation map (provided that H^* forms a Hopf algebra, which is canonically the case if H is finite-dimensional). As an example, we have the following result:

Lemma 4.2. There is a dual pairing between kG and $k(G)$. In fact $k(G)^* = kG$.

Proof. Consider the map $kG \to k(G)^* : \sum kg \to \sum kE_g$ where E_g is evaluation in g. This map is clearly linear and injective. Surjectivity then follows since $k(G)$ and kG are finite-dimensional; namely of equal dimension $|G|$.
In other words, the explicit dual pairing here is given by evaluation as

$$\left\langle \phi, \sum kg \right\rangle = \sum k\phi(g);$$

one easily check that this is indeed a dual pairing. □

The notion of a dual pairing allows us to construct a novel example of a Hopf algebra, called the Drinfeld double:

Example 4.3. Let H, H' be dually paired Hopf algebras with a dual pairing $\langle \cdot, \cdot\rangle$. Then we can give the vector space $H' \otimes H$ the structure of a Hopf algebra as follows: for $(\phi \otimes h), (\psi \otimes g) \in H' \otimes H$ we define:

$$\begin{gathered}(\phi \otimes h)(\psi \otimes g) = \sum \psi_{(2)}\phi \otimes h_{(2)}g\langle Sh_{(1)}, \psi_{(1)}\rangle\langle h_{(3)}, \psi_{(3)}\rangle,\\ \Delta(\phi \otimes h) = \sum(\phi_{(1)} \otimes h_{(1)}) \otimes (\phi_{(2)} \otimes h_{(2)}),\\ \epsilon(\phi \otimes h) = \epsilon(\phi)\epsilon(h),\\ S(\phi \otimes h) = (1 \otimes Sh)(S^{-1}\phi \otimes 1).\end{gathered}$$

Recall here the Sweedler notation for Δ. These Hopf algebra operations are extended linearly to arbitrary elements of $H' \otimes H$. This Hopf algebra is called the *Drinfeld double* of H and denoted $D(H)$. (The Hopf algebra H' is not reflected in the notation since we usually take H finite-dimensional and $H' = H^*$.) For a proof that this is indeed a valid Hopf algebra structure, see [19].

Definition 4.4. A bialgebra or Hopf algebra H is *quasitriangular* if there exists an element $\mathcal{R} \in H \otimes H$ such that

- $\mathcal{R}$ is invertible and obeys
$$\tau \circ \Delta h = \mathcal{R}(\Delta h)\mathcal{R}^{-1}, \tag{4.1}$$
for all $h \in H$.

- The following identities hold for $\mathcal{R}$:
$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}. \tag{4.2}$$
Here both sides of the equalities are in $H \otimes H \otimes H$. The suffixes on $\mathcal{R}$ denote the embedding of $\mathcal{R}$ into higher tensor powers of H. For instance, writing $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$, the notation $\mathcal{R}_{13} \in H \otimes H \otimes H$ means $\sum R^{(1)} \otimes 1 \otimes \mathcal{R}^{(2)}$

We then say that $(H, \mathcal{R})$ forms a *quasitriangular bialgebra / Hopf algebra*.

Lemma 4.5. If $(H, \mathcal{R})$ is a quasitriangular bialgebra. Then $\mathcal{R}$ obeys the so-called *Quantum Yang-Baxter equation*:
$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$
We will see more of this equation later on, in section 6. For now it suffices to say that this equation algebraically encodes the braid relation (i.e. the third Reidemeister relation).

Proof. We can prove this by computing $(\mathrm{id} \circ \tau \circ \Delta)\mathcal{R}$ in two ways. On the one hand, by equation (3.2) we have
$$(\mathrm{id} \circ \tau \circ \Delta)\mathcal{R} = (\mathrm{id} \otimes \tau)(\mathrm{id} \otimes \Delta)\mathcal{R} = (\mathrm{id} \otimes \tau)(\mathcal{R}_{13}\mathcal{R}_{12}) = \mathcal{R}_{12}\mathcal{R}_{13}.$$
On the other hand, first using equation (4.1) and then equation (4.2) gives
$$(\mathrm{id} \circ \tau \circ \Delta)\mathcal{R} = \mathcal{R}_{23}((\mathrm{id} \otimes \Delta)\mathcal{R})\mathcal{R}_{23}^{-1} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\mathcal{R}_{23}^{-1}.$$
Hence we conclude
$$\mathcal{R}_{12}\mathcal{R}_{13} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\mathcal{R}_{23}^{-1} \implies \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12},$$
as required. □

Example 4.6. If H is a finite-dimensional Hopf algebra, then the Drinfeld double $D(H)$ of H paired with H^* is quasitriangular, with
$$\mathcal{R} = \sum_a (f^a \otimes 1) \otimes (1 \otimes e_a),$$
where $\{e_a\}$ is a basis for H and $\{f^a\}$ is the corresponding dual basis of H^*.

A proof of this statement can be found in [19, p.45]. For our purposes it is only important to note that $\mathcal{R}$ is indeed invertible, with
$$\mathcal{R}^{-1} = \sum_a (S^{-1} f^a \otimes 1) \otimes (1 \otimes e_a), \tag{4.3}$$
where S^{-1} is the inverse antipode of H^*.

As an example and application of the above theory, we'll explicitly compute the structure of the Drinfeld double $D(kG)$, and explain how it can be used to generate a class of knot invariants related to the Wirtinger presentation of a knot.

Lemma 4.7. The Drinfeld double $D(kG)$ is given by the vector space $k(G)\otimes kG$ endowed with the structure of a Hopf algebra as follows:

$$(\phi\otimes h)(\psi\otimes g)(x)=\psi(h^{-1}xh)\phi(x)\otimes hg,$$

$$\Delta(\phi\otimes h)(x,y)=\sum_g(\phi(xg)\otimes h)\otimes(\delta_{gy}\otimes h),$$

with counit $\epsilon(\phi\otimes h)=\phi(e)$. Under the identification $k(G)\otimes kG\otimes k(G)\otimes kG\cong k(G\times G)\otimes k(G)\otimes k(G)$ this becomes

$$\Delta(\phi\otimes h)(x,y)=\phi(xy)\otimes h\otimes h.$$

The antipode is given by

$$S(\phi\otimes h)(x)=(1\otimes h^{-1})(\phi\otimes e)(x)=\phi(hxh^{-1})\otimes h^{-1}.$$

Proof. For the dual pairing, we use $H'=k(G)$, since $k(G)=kG^*$ as shown in lemma 4.2. The only non-trivial component of the given structure is the computation of the product: the coproduct structure follows immediately from the definition of the Drinfeld double and the alternative form of Δf given in example 3.8. The antipode structure is immediate using the product structure.

To arrive at the given structure for the product, first note that the product of the Drinfeld double is by definition given by

$$(\phi\otimes h)(\psi\otimes g)=\sum\psi_{(2)}\phi\otimes h_{(2)}g\langle Sh_{(1)},\psi_{(1)}\rangle\langle h_{(3)},\psi_{(3)}\rangle.$$

To reduce this to a manageable form first note that

$$\left(\sum\psi_{(1)}\otimes\psi_{(2)}\otimes\psi_{(3)}\right)(x,y,z)=\psi(xyz),$$

in $k(G\times G\times G)$. This corresponds in $k(G)\otimes k(G)\otimes k(G)$ to

$$\sum_{\alpha\in G}\sum_{\beta\in G}\delta_\alpha(x)\psi(\alpha y\beta)\delta_\beta(z)$$

(among other equivalent possible forms). Furthermore using that $\Delta h=h\otimes h$ and $Sh=h^{-1}$ we thus evaluate

$$\begin{aligned}(\phi\otimes h)(\psi\otimes g)(x)&=\sum_{\alpha\in G}\sum_{\beta\in G}\psi(\alpha x\beta)\phi(x)\otimes hg\langle h^{-1},\delta_\alpha(x)\rangle\langle h,\delta_\beta(x)\rangle\\&=\sum_{\alpha\in G}\sum_{\beta\in G}\psi(\alpha x\beta)\phi(x)\otimes hg\delta_\alpha(h^{-1})\delta_\beta(h)\\&=\psi(h^{-1}xh)\phi(x)\otimes hg.\end{aligned}$$

□

From the above we see immediately that the product of $D(kG)$ on basis elements δ_a,δ_b,h,g is

$$(\delta_a\otimes h)(\delta_b\otimes g)(x)=\delta_a\delta_{hbh^{-1}}\otimes hg.$$

Since kG is finite-dimensional, example 4.6 tells us that $D(kG)$ is a quasitriangular Hopf algebra with

$$\mathcal{R}=\sum_{g\in G}(\delta_g\otimes e)\otimes(1\otimes g).$$

4.2 The knot invariant I_G

In this subsection we will derive a class of *knot invariants* from $D(kG)$. To this end we first give a brief description of knots and knot diagrams. This will mostly be an informal introduction; for a formal dicussion see e.g. [23].

Definition 4.8. A *parametrized knot* is the image of a smooth embedding (injective smooth map) of S^1 into $\mathbb{R}^3$. An *oriented* parametrized knot is a smooth embedding of S^1 with a counterclockwise orientation into $\mathbb{R}^3$. The orientation of an oriented parametrized knot is indicated in pictures by an arrow that is the image of a counterclockwise-pointing arrow along S^1.

A knot is almost the same as a parametrized knot, except that we wish to consider parametrized knots up to an equivalence: we essentially consider two parametrized knots to be equivalent if we can continuously deform one into the other with crossing any strands. This is encoded mathematically by an *ambient isotopy*:

Definition 4.9. We impose an equivalence on the parametrized knots by saying that two parametrized knots $a, b : S^1 \to \mathbb{R}^3$ are *equivalent* if there is a continuous function $F : \mathbb{R}^3 \otimes [0, 1] \to \mathbb{R}^3$ such that:

1. $F_0 = \mathrm{id}_{\mathbb{R}^3}$, so that $F_0 \circ a = a$,
2. F_t is a homeomorphism for all $t \in [0, 1]$,
3. $F_1 \circ a = b$.

Here F_t means F with the input from $[0, 1]$ fixed at t.

A *knot* is an equivalence class of parametrized knots under this equivalence.

The map F is to be thought of as a family of continuous deformations of $\mathbb{R}^3$ indexed by t, and is called an *ambient isotopy*. It should eventually deform a to b, and the requirement that all F_t are isomorphisms encodes that no strands of the oriented strings are ever crossed in the deformation.

Knots are usually depicted using *knot diagrams*. These are projections of a parametrized knot onto the plane, such that each intersection of the projection only consists of two intersecting strands, and at every such intersection there is an indication which each strand went 'over' and which went 'under' in the original knot. An example of a knot is depicted in a knot diagram below, in figure 4.1.

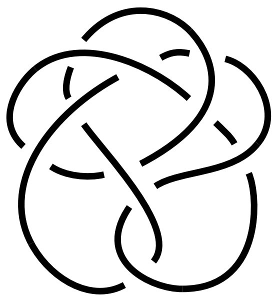


Figure 4.1: Knot diagram of the knot 10_{132}.

It is paramount to discuss how the ambient isotopy equivalence of oriented knot translates to their knot diagrams. This is detailed by the famous *Reidemeister theorem*:

Theorem 4.10. (Reidemeister): Two parametrized knots are equivalent if and only their knot diagrams can be related by a sequence of *Reidemeister moves*. The Reidemeister moves are depicted below:

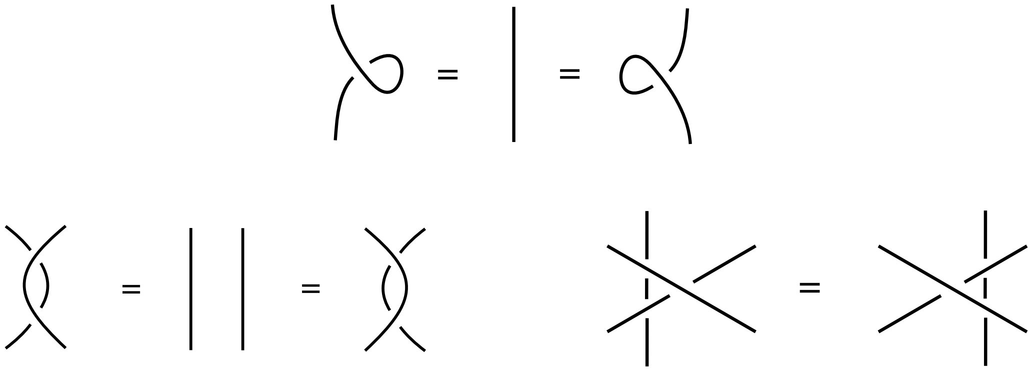


Figure 4.2: The three Reidemeister moves.

These are referred to as the Reidemeister-1,2,3 moves respectively. It is of note that the Reidemeister-3 move is equivalent to the *braid relation*.

In the case of oriented knot the same statement holds, but now with the *oriented* Reidemeister moves. The oriented Reidemeister moves are exactly the usual ones, but with all the possible orientations of the involved strands. Thus, for example the second Reidemeister-2 move gives rise to *four* oriented Reidemeister-2 moves.

It is a nontrivial matter to distinguish whether or not two knots are identical. The tool for doing so is the concept of a *knot invariant*:

Definition 4.11. A *knot invariant* is a map that assigns to each parametrized knot (or knot diagram) some object. This can be a group, a number, a polynomial, etc. The assigned object must be invariant under equivalence of parametrized knots.

There is a wide array of knot invariants, and we will use Hopf algebras to construct one class of such invariants below. To show that the assignment of an object to a knot indeed gives rise to a knot invariant, one only needs to show invariance of this assignment under the Reidemeister moves.

Let G be a finite group. We use the quasitriangular pair $(D(kG), \mathcal{R})$ to construct knot invariants as follows: say that we are given an oriented knot diagram K. We can open this diagram at one point by cutting one of its line segments. This creates a 'tangle' diagram with begin- and end-points. An example for the trefoil knot is shown below:

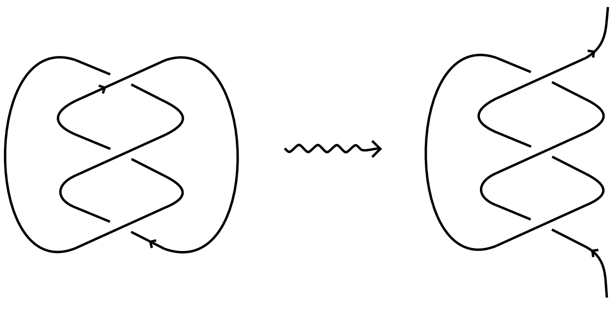


Figure 4.3: Opening a knot diagram.

Now, to construct a knot invariant from $D(kG)$ we place at each right-handed crossing a copy of $\mathcal{R}$, and at each left-handed crossing a copy of $\mathcal{R}^{-1}$. These copies are taken to be distinct: the sum over G should be in a different variable for each copy. Since $\mathcal{R}$ is an element of $D(kG) \otimes D(kG)$, it has a left and a right component in the tensor product: $\delta_g \otimes e$ and $1 \otimes g$ respectively. Similarly for $\mathcal{R}^{-1}$. At each crossing (where we have now placed a copy of $\mathcal{R}$ or

$\mathcal{R}^{-1}$) we place the left component on the strand going under, and the right component on the strand going over. For our trefoil knot example this is shown below:

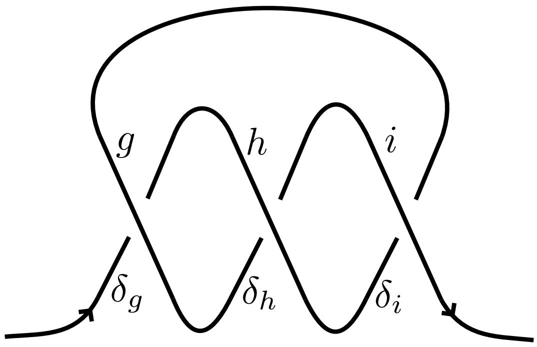


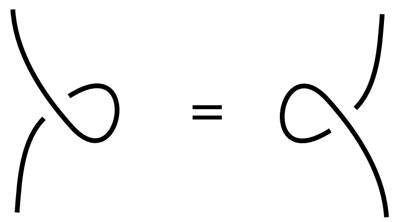
Figure 4.4: Construction of I_G.

To finally construct a knot invariant I_G, we move along the knot from beginning to end, multiplying the elements of $D(kG)$ we have placed in the order we encounter them. For the example of the trefoil knot this gives

$$\begin{aligned} I_G &= \sum_{g,h,i} (\delta_g \otimes e)(1 \otimes h)(\delta_i \otimes e)(1 \otimes g)(\delta_h \otimes e)(1 \otimes i) \\ &= \sum_{g,h,i} \delta_g \delta_{hih^{-1}} \delta_{hghg^{-1}h^{-1}} \otimes hgi. \end{aligned}$$

This construction is taken from [24].

To be exact, this does not actually yield an invariant of knots, but of *framed* knots. These are knots which have a certain 'thickness', like a thin strip of paper. For framed knots we do not require the first Reidemeister condition, but a weaker version:



To verify that I_G is indeed an invariant of framed knots, we must verify that it is invariant under the weakened first Reidemeister move, and the second and third Reidemeister moves. Invariance under the weakened first Reidemeister move is trivial: on the both sides of the corresponding condition we have a single right-handed crossing. The only difference is that on the left-hand side we first go under the crossing and then over, while on the right-hand side we first go over and then under. Invariance of I_G then follows from

$$(\delta_g \otimes e)(1 \otimes g) = \delta_g \otimes g = (1 \otimes g)(\delta_g \otimes e).$$

Invariance of I_G under type 2 Reidemeister moves is similarly trivial, since we place $\mathcal{R}$ on right-handed crossings and its inverse on left-handed crossing. Note here that we are considering *oriented knots*, for which there are *four* Reidemeister-2 moves depending on the orientations. However, all four indeed consist of one positive- and one negative crossing. Invariance under type 3 Reidemeister moves follows from the fact that $\mathcal{R}$ obeys the quantum Yang-Baxter equation: see remark 6.8 for the explicit connection between the Yang-Baxter equation and the braid relation.

Finally, we can give an interpretation of I_G using the Wirtinger presentation: for a knot (or knot diagram) K we define the knot group as $\pi_1(\mathbb{R}^3 - K)$, the fundamental group of the knot complement. The knot group is clearly a knot invariant. The Wirtinger presentation of K provides a *presentation* of this group. That is, it describes $\pi_1(\mathbb{R}^3 - K)$ as a free group modulo a number of relations on the generators of this free group. For a concise and self-contained exposition on the knot group and Wirtinger presentation, see [25]. Here it suffices to say that the relations are all of the form $g = hih^{-1}$.

Meanwhile, note that the $k(G)$ component of I_G consists of a number of δ functions multiplied with eachother: $\delta_g \delta_{hih^{-1}} \delta_{hghg^{-1}h^{-1}}$ for the trefoil knot, for instance. This element of $k(G)$ must vanish, unless all the subscripts are equal. Thus (for the trefoil) the only choices of $g, h, i \in G$ that yield a nonzero contribution to I_G are those such that

$$g = hih^{-1} = hghg^{-1}h^{-1}.$$

These are precisely the relations for the Wirtinger presentation of the trefoil knot. This is no coincidence: the multiplication in $D(kG)$

$$(\delta_a \otimes h)(\delta_b \otimes g)(x) = \delta_a \delta_{hbh^{-1}} \otimes hg.$$

acts on the δ's in a way that precisely emulates the construction of the Wirtinger presentation seen in [25]. Upon close comparison with the construction of I_G, this is an easy exercise to verify. In conclusion, a choice (g, h, i) yields a nonzero term in I_G if and only if this choice gives a representation of the knot group into G. Thus I_G contains the information of

$$\#\mathrm{Hom}(\pi_1(\mathbb{R}^3 - K), G),$$

which is evidently a knot invariant.

Interestingly, the kG-component of I_G can be interpreted to contain the information of the *framed longitude* ℓ of our knot [24]. This is one of the generators for the *peripheral subgroup* of the space-subspace system $K \subset \mathbb{R}^3$. For an introduction of this topic, see [26]. In [27] it was shown that the peripheral subgroup generates a *complete* knot invariant when used in conjunction with the knot group. That is, it provides an invariant that identifies knot *perfectly.* By varying G the collection of invariants I_G thus contains the information of all representations of $\pi_1(\mathbb{R}^3 - K)$ and ℓ into finite groups.

From the above discussion it is clear why $(D(kG), \mathcal{R})$ is a natural choice to construct knot invariants: the quasitriangular structure of $\mathcal{R}$ ensures that we can maintain invariance under type 3 Reidemeister moves, and the group conjugation character of multiplication in $D(kG)$ allows us to relate our construction to the well-known construction of the Wirtinger presentation.

4.3 Braided Diagrams and Tangles

As already stated before in subsection 3.2, for a braided group with braiding Ψ the braid relation and functoriality of Ψ allow us to treat the 'strings' of braided groups as physical pieces of string. We now give a knot-theoretical formulation and proof of this statement. This required the introduction of a generalization of knots, called *tangles.* To introduce tangles we follow [23].

Definition 4.12. A *parametrized tangle* is a smooth embedding of a one-dimensional compact smooth manifold X into a box

$$\{(x, y, z) \mid x \in [w_0, w_1], y \in [-1, 1], z \in [h_0, h_1]\}.$$

Here $[w_0, w_1]$ and $[h_0, h_1]$ describe the *width* and *height* of the box, respectively. Rougly speaking a tangle is a knot with beginning- and end-points, such that all the end-points lie in one rectangle, and all beginning-points lie in an opposing, congruent rectangle. Two parametrized tangles are *equivalent* if one can be mapped onto the other by an ambient isotropy of boxes, followed by a re-scaling of the form

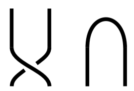
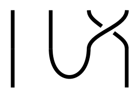
$$(x, y, z) \mapsto (f(x), y, g(z)),$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are strictly increasing. A *tangle* is an equivalence class of parametrized tangles. In analogy with oriented knots, an *oriented tangle* is a tangle such that X is oriented.

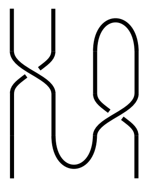
Intuitively, a tangle consists of a set of strings or loops that run from the bottom of the box to the top. In analogy with knots, we can consider *tangle diagrams* as projections of a tangle with crossings that the indicate 'over'- and 'under'-going strands. This immediately raises the question whether there is a counterpart of the Reidemeister theorem for tangles. There is, and we will call it the *Turaev theorem.* Before discussing it, we introduce the decomposition of tangles into *simple tangles* and *elementary tangles.*

Definition 4.13. Suppose that T_1 and T_2 are parametrized tangles whose boxes are of equal height and width. If the beginning-points of T_2 coincide with the end-points of T_1 on the faces of their respective boxes, then we define the product $T_1 \cdot T_2$ as putting T_1 on top of T_2. We define the tensor product $T_1 \otimes T_2$ as a horizontal juxtaposition of T_1 and T_2. These operations extend to well-defined operations on the equivalence classes of parametrized tangles, so that we obtain operations $(\cdot, \otimes)$ on tangles.

Example 4.14. Let T_1 and T_2 be given by

$T_1 =$  $T_2 =$ 

Then $T_1 \cdot T_2$ and $T_1 \otimes T_2$ are given by

$T_1 \cdot T_2 =$ 

$T_1 \otimes T_2 =$ 

Definition 4.15. An *elementary tangle* is a tangle represented by one of the following five minimal tangle diagram 'elements':

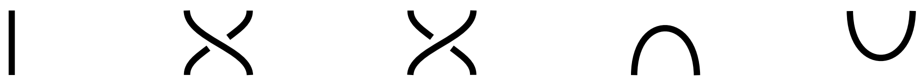


Figure 4.5: The elementary tangles.

We define a simple tangle as a finite tensor product of elementary tangles.

It is a fact that every tangle diagram can be decomposed into a product of simple tangle diagrams.

From example 4.14, the similarities between tangle diagrams and braided diagrams should be clear. We already know from the definition of ambient isotopies that we can treat the strings of tangles and tangle diagrams as pieces of physical string, just as with knots. This is precisely the property we now wish to show for braided diagrams. To this end we can just show that braided diagrams behave the same as tangle diagrams, i.e. that they obey the same Reidemeister-like relations as tangle diagrams. For this we need a tangle-analogue of Reidemeister moves, which is given by the *Turaev moves*. Since braided diagrams are oriented (downward), we consider the oriented case here. For the unoriented case see [23].

Theorem 4.16. (Turaev): Two oriented parametrized tangles are equivalent if and only if their tangle diagrams are related by a sequence of *Turaev moves*. The Turaev moves are depicted below, in terms of products of simple tangles:

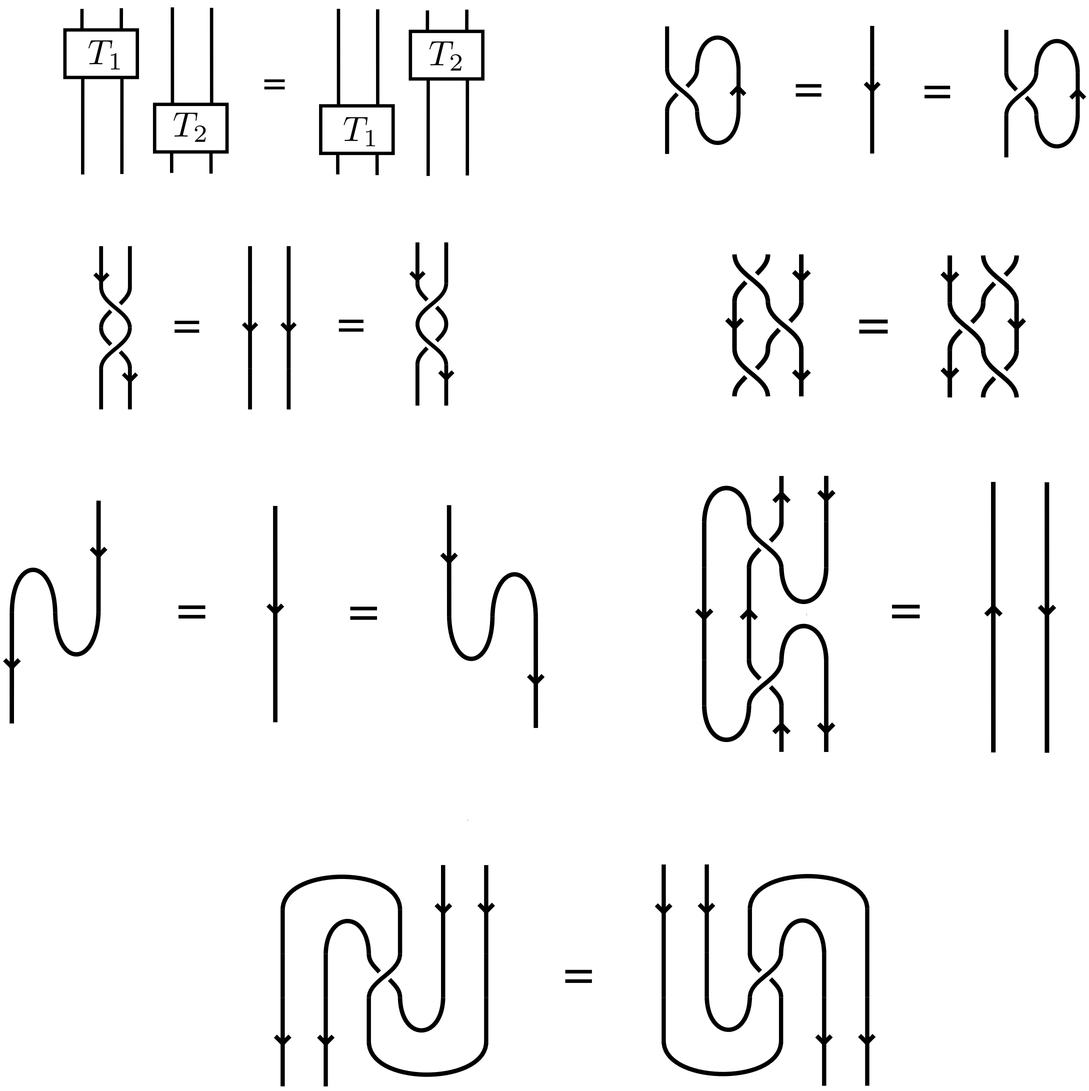


Figure 4.9: The seven Turaev moves.

Here the boxes labelled T_1, T_2 represent arbitrary tangles with two beginning- and two end-points. The first Turaev move must hold for arbirary orientations of the strings. Moves 5,6, and 7 must also hold with reversed orientations.

To relate the Turaev theorem to braided diagrams, we still need incarnations of the elementary tangles $\cup$ and $\cap$ for braided diagrams. To this end, recall that the empty space in a braided diagram corresponds to copies of the field k, while a down-ward flowing string is a copy of a braided group B. Now suppose that we are working in a *rigid* braided category $\mathcal{C}$. Then we declare that an upward-flowing string corresponds to the dual object B^* of a braided group.

With this, the need for $\cup$ and $\cap$ is translated to the need for maps $B^* \otimes B \to k$ and $k \to B \otimes B^*$ respectively. These are provided by the rigidity of $\mathcal{C}$: we let

$$\cup := \mathrm{ev}_B \qquad \text{and} \qquad \cap := \mathrm{coev}_B.$$

Functoriality of Ψ in the morphisms ev_B and coev_B tells us that we can pull them over and under crossings. (The addition of $\cap$ and $\cup$ to braided diagrams is not just to complete the analogy with tangles; we'll need them in section 6.) Since braided diagrams represent oriented tangles, we have to prescribe $\cap$ and $\cup$ with both possible orientations. The definitions above give one of these orientations. The other orientations must be given by maps $B \otimes B^* \to k$ and $k \to B^* \otimes B$, so they are naturally given by $\mathrm{ev}_{B^*}, \mathrm{coev}_{B^*}$. (Here we take for granted a natural identification $B^{**} \cong B$; see [19] for such an isomorphism.) By the uniqueness of duals from lemma 3.24, it turns out we can express $\mathrm{ev}_{B^*}, \mathrm{coev}_{B^*}$ in terms of $\mathrm{ev}_B, \mathrm{coev}_B$ very easily; this is shown in figure 4.10 below.

Figure 4.10: Dual evaluations and coevalutaions expressen in terms of $\mathrm{ev}_V, \mathrm{coev}_V$.

For a neat categorical proof of this, see [28]. Equipped with oriented braid diagram versions of $\cap$ and $\cup$, we can now prove that the braided group diagrams can indeed be treated as tangles, and hence as physical pieces of string:

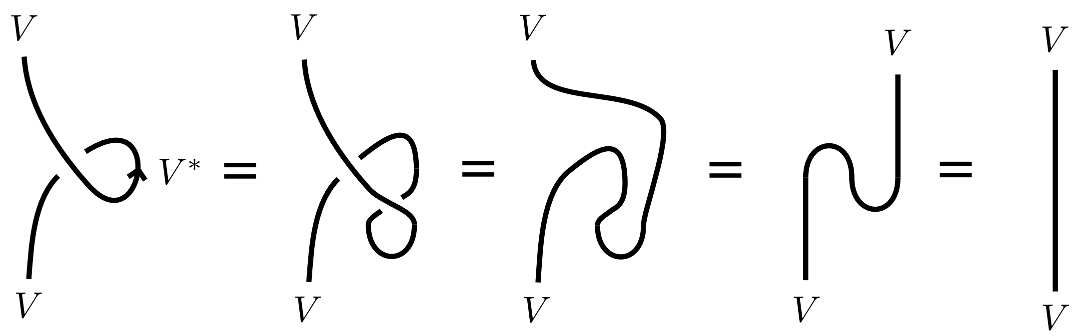
Corollary 4.17. Braided diagrams obey the Turaev relations. They can hence be thought of as tangle diagrams, albeit equipped with extra operations like S.

Proof. The third Turaev relation holds trivially since $\Psi_{V,W} \circ \Psi^{-1}_{V,W} = \mathrm{id} = \Psi^{-1}_{V,W} \circ \Psi_{V,W}$. The fourth Turaev move is exactly the braid relation, which we know to hold for braided diagrams. The fifth 'bend-straightening' Turaev moves are satisfied by the definition of a rigid category. Indeed, they represent the equations

$$(\mathrm{id} \otimes \mathrm{ev}_V) \circ (\mathrm{coev}_V \otimes \mathrm{id}) = \mathrm{id}_V,$$
$$(\mathrm{ev}_V \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coev}_V) = \mathrm{id}_{V^*}.$$

which are precisely the requirements of equations (3.1) and (3.2). The first and sixth turaev move hold trivially using functoriality, as well as the fifth Turaev move in the case of the sixth move. The seventh move also clearly holds by functoriality, after an application of the equations in figure 4.10.

Finally, for the second Turaev relation we also use the equalities from figure 4.10. Using the fifth Turaev move we compute for example:



This shows one of the Turaev-2 moves, the other is proven analogously. ☐

4.4 Quantum Knot Invariants

As a final application of Hopf algebras to knot theory, we consider the construction of *quantum knot invariants*. This will use everything we've discussed so far: knots, tangles, braided diagrams, and Hopf algebras. This discussion is based on [23]. Quantum knot invariant can be derived for any Drinfeld-Jimbo Hopf algebra (i.e. quantum group) $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a semisimple Lie algebra. In fact this was one of the first applications of quantum groups. Following [23], here we will only consider the case of $U_q(\mathfrak{sl}_2)$. We have already seen a Drinfeld-Jimbo Hopf algebra in the last section, namely $SU_q(2)$; now we see a concrete application of this type of Hopf algebra. The main idea for the construction of knot invariants from general $U_q(\mathfrak{g})$ should be clear from the example of $U_q(\mathfrak{sl}_2)$; only considering this single example will allow us to detail the construction without having to discuss almost any of the theory of Drinfeld-Jimbo Hopf algebras [23].

The only non-trivial statement from the theory of quantum groups that we will need is this: Let $\mathfrak{g}$ be a semisimple Lie algebra, and V a finite-dimensional Lie algebra representation of $\mathfrak{g}$. (Here we take for granted the notion of a Lie algebra- and group representations. See the next section for a formal introdcution to representations of Hopf algebras; they are quite similar to those of groups.) Then V is also a representation of $U_q(\mathfrak{g})$, and it always gives rise to an inverstible n-dimensional solution $R : V \otimes V \to V \otimes V$ of the *Yang-Baxter equation* (YBE):

$$(R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) = (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R).$$

The relation between the YBE and the *quantum* YBE from subsection 4.1 is explored in section 6. Instead of proving these statements in general, we content ourselves with the line taken in [23] and simply give (V, R) for $\mathfrak{g} = \mathfrak{sl}_2$.

Example 4.18. Let $\mathfrak{g} = \mathfrak{sl}_2$. It is well-known [29] that we can represent $\mathfrak{sl}_2$ as 2×2 matrices using the generators

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

This gives us a two-dimensional representation of $\mathfrak{sl}_2$, i.e. with $V = \mathbb{C}^2$. Namely, for an element $X \in \mathfrak{sl}_2$ we represent it as a matrix, and then send $v \in V$ to $Xv \in V$. This is known as the *standard representation* of $\mathfrak{sl}_2$. (Again, for a full introduction of representations of Hopf algebras, see the next section.) The map $R : V \otimes V \to V \otimes V$ associated to V by the theory of $U_q(\mathfrak{g})$ is then given by

$$R : \begin{cases} e_1 \otimes e_1 \mapsto q^{1/4} e_1 \otimes e_1, \\ e_1 \otimes e_2 \mapsto q^{-1/4} e_2 \otimes e_1, \\ e_2 \otimes e_1 \mapsto q^{-1/4} e_1 \otimes e_2 + (q^{1/4} - q^{3/4}) e_2 \otimes e_1, \\ e_2 \otimes e_2 \mapsto q^{1/4} e_2 \otimes e_2, \end{cases}$$

where $\{e_1, e_2\}$ is the standard basis for V.

Armed with this information we are ready to construct our quantum knot invariants. This is facilitated by tangles: consider an oriented knot diagram K. This is in particular a tangle diagram, where the box of this tangle is some sufficiently large box that contains the knot. We now draw any generic horizontal line through K (that does not cross any crossings of the diagram at their precise intersection point). To every intersection of this line with K we either assign a copy of V or V^*. Here V^* is the *dual representation* of V. We shall prove in the next section that for a Hopf algebra, the dual vector space V^* of any representation V is indeed again a representation. If the knot diagram is directed *up* at the point of intersection with our line, we place a copy of V there. If it is directed *down* then we place a copy of V^*. If our line does not intersect the knot diagram at all, then we associate a copy of the ground field $\mathbb{C}$ to it.

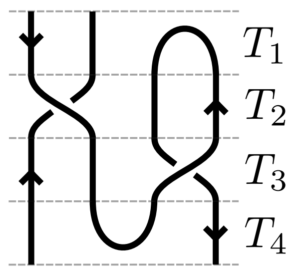
Now, regarding our knot diagram K as a tangle diagram, use a finite amount of horizontal lines to decompose it into a product of *simple* tangles. Attach copies of $V, V^*, \mathbb{C}$ to these lines according to the above prescription. A part of K between our two adjacent lines then represents a simple tangle. We now assign a linear transformation $\theta(K) : \mathbb{C} \to \mathbb{C}$ to K as follows:

- To each horizontal line, associate the vector space given by the ordered tensor product of copies of V, V^* we have attached. Say that this associates the space W_i to the i-th horizontal line.
- To each simple tangle T_i between the $i - th$ and $(i + 1)$-th line, associate a linear map $\theta(T_i) : W_i \to W_{i+1}$.
- Let θ distribute over products and tensor products of tangles via the rules

$$\theta(T_1 \cdot T_2) = \theta(T_1) \circ \theta(T_2) \qquad \text{and} \qquad \theta(T_1 \otimes T_2) = \theta(T_1) \otimes \theta(T_2).$$

- We then obtain $\theta(K)$ as $\theta(T_1 \cdot T_2 \cdot \cdots \cdot T_n) = \theta(T_1) \circ \cdots \circ \theta(T_n)$.

This procedure is depicted in figure 4.11 below, for the tangle $T = T_1 \cdot T_2$ from example 4.14.



$$\Longrightarrow \qquad \begin{tikzcd} V^* \otimes V \\ V^* \otimes V \otimes V^* \otimes V \arrow[u, "\theta(T_1)"] \\ V \otimes V^* \otimes V^* \otimes V \arrow[u, "\theta(T_2)"] \\ V^* \otimes V \otimes V \otimes V^* \arrow[u, "\theta(T_3)"] \\ V \otimes V^* \arrow[u, "\theta(T_4)"] \arrow[uuuu, bend right=60, "\theta(T)"'] \end{tikzcd}$$

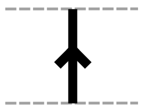
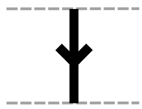
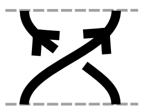
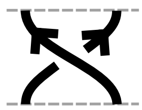
Figure 4.11: The construction of the linear map $\theta(T)$.

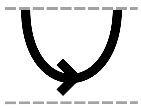
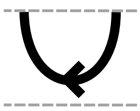
Since we declare that θ distributes over products, we only need to define θ on simple tangles. Since it also distributes over tensor products of tangles, we *really* only need to define it on the *elementary tangles*. In the oriented case there are eight of these. We essentially design θ on these elementary tangles to eventually yield a knot invariant, and this is where the structures of $(U_q(\mathfrak{sl}_2), V, R)$ come in: it is particularly important that R satisfies the YBE. Especially the values on $\cap$ and $\cup$ depend non-trivially on these structures. The values of θ on the eight elementary tangles are given below in figure 4.12: the first four are defined similarly for all $U_q(\mathfrak{g})$, the last four mimic ev and coev as maps $\mathbb{C} \to V^* \otimes V$ etc. They depend non-trivially on the specifics of $U_q(\mathfrak{g})$, but can be constructed generally [23].

Note that, in principle, we can construct a map $\theta(T)$ for any tangle T in this way. For a knot, i.e. a closed tangle, we end up with a map $\theta(K) : \mathbb{C} \to \mathbb{C}$. This map is associated to the number $(\theta(K))(1)$. By linearity this is a unique association. Hence the procedure above leaves us with a complex number associated to each knot diagram K. We claim that this function / number $\theta(K)$ is an *invariant of framed knots*.

To show that $\theta(K)$ is a framed knot-invariant, we only need to show its invariance under the weakened Reidemeister-1 move and the Reidemeister-2,3 moves. For the weakened Reidemeister-1 relation, applying θ to both sides of the equation results in

$$(\mathrm{id} \otimes \theta(\cap)) \circ (R^{-1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \theta(\cup)),$$

	$V \xleftarrow{\mathrm{id}_V} V$ (i.e. $V \overset{\mathrm{id}_V}{\uparrow} V$)		$V^* \overset{\mathrm{id}_{V^*}}{\uparrow} V^*$
	$V \otimes V \overset{R}{\uparrow} V \otimes V$		$V \otimes V \overset{R^{-1}}{\uparrow} V \otimes V$

	$V^* \otimes V \uparrow \mathbb{C}$	$q^{-1/2} f^1 \otimes e_1 + q^{1/2} f^2 \otimes f_2 \upmapsto 1$
	$V \otimes V^* \uparrow \mathbb{C}$	$e_1 \otimes f^1 + e_2 \otimes f^2 \upmapsto 1$

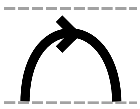
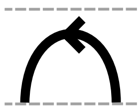
	$\mathbb{C} \uparrow V \otimes V^*$	$q^{1/2} \upmapsto e_1 \otimes f^1$	$0 \upmapsto e_1 \otimes f^2$	$0 \upmapsto e_2 \otimes f^1$	$q^{-1/2} \upmapsto e_2 \otimes f^2$
	$\mathbb{C} \uparrow V^* \otimes V$	$1 \upmapsto e_1 \otimes f^1$	$0 \upmapsto e_1 \otimes f^2$	$0 \upmapsto e_2 \otimes f^1$	$1 \upmapsto e_2 \otimes f^2$

Figure 4.12: The values of θ on the oriented elementary tangles. Here $\{f^i\}$ is the basis of V^* dual to $\{e_i\}$.

and

$$(\theta(\cap) \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R^{-1}) \circ (\theta(\cup) \otimes \mathrm{id}),$$

respectively; both of these are maps $V \to V$. Noting that $\theta(\cap)$ is a map into $\mathbb{C}$, these expressions are certainly equal (up to the isomorphism $\mathbb{C} \otimes V \cong V \cong V \otimes \mathbb{C}$). As for the Reidemeister-2 relation; applying θ to both sides results in the requirement that

$$R \circ R^{-1} = \mathrm{id} \otimes \mathrm{id} = R^{-1} \circ R.$$

Clearly this holds, by the definition of an inverse. Finally, applying θ to both sides of the Reidemeister-3 relation yields the requirement

$$(R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) = (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R).$$

This is exactly the YBE, which we know to hold for R. Hence $\theta(K)$ is indeed an invariant of framed knots, as claimed.

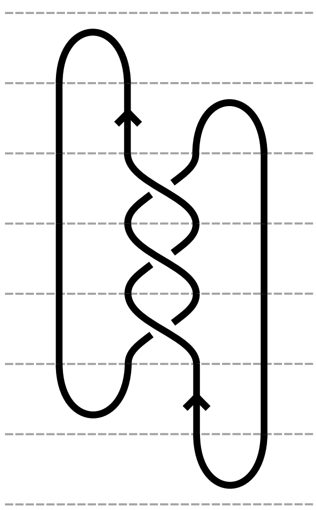
Now that we have a complete description of θ and a proof that it provides a knot invariant, we are ready to consider some examples. The simplest example to consider is the unknot O, oriented counter-clockwise as a knot diagram. As a tangle diagram, this is simply a product of $\cup$ and $\cap$. Thus using figure 4.12 we immediately find

$$\begin{aligned} \theta(O) : \mathbb{C} &\to V^* \otimes V \to \mathbb{C} \\ 1 &\mapsto q^{-1/2} f^1 \otimes e_1 + q^{1/2} f^2 \otimes e_2 \mapsto q^{-1/2} + q^{1/2}. \end{aligned}$$

Thus as a number, $\theta(O) = q^{-1/2} + q^{1/2}$. If we wish to obtain a *normalized* invariant of framed knots (i.e. an invariant that equals 1 on the unknot), then we can consider

$$\widetilde{\theta} := \frac{\theta}{q^{1/2} + q^{-1/2}}.$$

As a final example we give θ on the trefoil knot K. The relevant tangle diagram is depicted in figure 4.13 below.



$$\Longrightarrow \quad \begin{array}{c} \mathbb{C} \\ \uparrow \\ V^* \otimes V \\ \uparrow \\ V^* \otimes V \otimes V \otimes V^* \\ \uparrow \scriptstyle \mathrm{id}_{V^*} \otimes R^{-1} \otimes \mathrm{id}_{V^*} \\ V^* \otimes V \otimes V \otimes V^* \\ \uparrow \scriptstyle \mathrm{id}_{V^*} \otimes R^{-1} \otimes \mathrm{id}_{V^*} \\ V^* \otimes V \otimes V \otimes V^* \\ \uparrow \scriptstyle \mathrm{id}_{V^*} \otimes R^{-1} \otimes \mathrm{id}_{V^*} \\ V^* \otimes V \otimes V \otimes V^* \\ \uparrow \\ V \otimes V^* \\ \uparrow \\ \mathbb{C} \end{array}$$

Figure 4.13: Computation of $\theta(K)$ for the trefoil knot.

Using figures 4.12 and 4.13 we can compute that

$$\theta(K) = q^{-7/4} + q^{-3/4} + q^{1/4} - q^{9/4},$$

and

$$\widetilde{\theta}(K) = q^{-5/4} + q^{3/4} - q^{7/4}.$$

For this computation, see [23].

5 Actions and Representations

In this section we will construct an important class of braided categories: representation categories. These are subcategories of `Vec` equipped with a nontrivial braiding. Before we can define these categories and discuss their properties, we must first discuss what is meant by a *representation* of a (Hopf) algebra, or equivalently an *action* of that algebra.

After discussing the construction of braided categories from representations of Hopf algebras, which is one of the main applications of Hopf algebras to braided geometry, we show that a Hopf algebra can be uniquely reconstructed from its representations. We have already argued in subsection 3.1.1 that the dual of a Hopf algebra is again a Hopf algebra. Thus this reconstruction proves our claims of representation-theoretic self-duality from subsection 1.3.

5.1 Actions

Definition 5.1. Let H be an algebra over a field k. An *action* of H on a vector space consists of a pair (α, V), where V is a vector space over k and $\alpha : H \otimes V \to V$ is a linear map. We can write $\alpha(h \otimes v) := \alpha_h(v)$, which makes $\alpha_h : V \to V$ a linear map. For α we require that $\alpha_{hg} = \alpha_h \circ \alpha_g$ and $\alpha_1 = \mathrm{id}$. These axioms are shown as commutative diagrams below:

$$\begin{tikzcd} & H \otimes H \otimes V \arrow[dl, "\cdot\otimes\alpha"'] \arrow[dr, "\mathrm{id}\otimes\alpha"] & \\ H \otimes V \arrow[dr, "\alpha"'] & & H \otimes V \arrow[dl, "\alpha"] \\ & V & \end{tikzcd} \qquad \begin{tikzcd} H \otimes V \arrow[dr, "\alpha"] & \\ k \otimes V \arrow[u, "\eta\otimes\mathrm{id}"] \arrow[r, phantom, "\cong"] & V \end{tikzcd}$$

Instead of writing α explicitly, we can denote an action by $\triangleright$. The definitions then become:

$$\alpha(h \otimes v) := h \triangleright v, \quad (hg) \triangleright v = h \triangleright (g \triangleright v)), \quad 1 \triangleright v = v,$$

for $h, g \in H$, $v \in V$.

A representation of a group G is nothing but an action of G on a vector space. Indeed, an action of G on V gives a set of automorphisms $\alpha_g : V \to V$ for all $g \in G$, such that $\alpha_{gh} = \alpha_g \alpha_h$. In other words, the action gives a homomorphism

$$\alpha : G \to \mathrm{Aut}(V) : g \mapsto \alpha_g.$$

This is precisely what we know as a group representation. Similarly, a *representation of an algebra H* is the same as an action of H on a vector space. From now on, an *action* of an algebra will always mean an action on some vector space, and the terms ‘action’ and ‘representation’ will be used interchangeably.

Remark 5.2. Analagously to an action, there is also the concept of a *coaction* or *corepresentation* of an algebra. This is just the arrow-reversed dual of an action, i.e. a pair (β, V) where V is a vector space and $\beta : V \to H \otimes V$ a linear map. This map β must then satisfy the same commutative diagrams as α, but with all the arrows reversed and $(\cdot, \eta)$ interchanged with (Δ, ϵ). We will make extensive use of coactions in subsection 6.2, but in this section we will only discuss actions since they are a more familiar notion. It will suffice to keep in mind that any result about actions has an equally valid arrow-reversed dual statement. The proof of such a dual statement can be obtained via arrow-reversal of the proof for the statement about actions.

If H is a bialgebra or Hopf algebra, we can also have actions of H on an *algebra*; not just a vector space. To define an action on an algebra, we naturally wish that the action respects the algebra structure in some way. For this we need H to be a bialgebra or Hopf algebra, and not just a regular algebra:

Definition 5.3. Let H be a bialgebra or Hopf algebra. Then H acts on an algebra A if:

- H acts on A as a vector space via an action $\alpha : H \otimes A \to A$.
- The product $\cdot$ of A commutes with α in the sense that

$$h \triangleright (ab) = \sum (h_{(1)} \triangleright a)(h_{(2)} \triangleright b).$$

  (Recall here the Sweedler notation for Δ.)
- The unit η of A commutes with α in the sense that

$$h \triangleright 1 = \epsilon(h) \cdot 1 \in A.$$

These additional axioms are shown as commutative diagrams below:

$$\begin{tikzcd}
H \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes\cdot"] \arrow[d, "\Delta\otimes\mathrm{id}\otimes\mathrm{id}"'] & H \otimes A \arrow[r, "\alpha"] & A & & A \otimes A \arrow[ll, "\cdot"'] \\
H \otimes H \otimes A \otimes A \arrow[rrrr, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"] & & & & H \otimes A \otimes H \otimes A \arrow[u, "\alpha\otimes\alpha"']
\end{tikzcd}$$

$$\begin{tikzcd}
H \arrow[r, "\epsilon"] \arrow[dr, "\mathrm{id}\otimes\eta"'] & k \arrow[r, "\eta"] & A \\
& H \otimes A \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

Again note that we need the bialgebra structure (Δ, ϵ) of H in these axioms. If H acts on an algebra A, we also refer to A as an *H-module algebra*.

We can also consider linear maps between different representations of the same algebra H, that commute with the action. These are also known as *intertwiners*. For our purposes it will suffice to only consider intertwiners of vector space representations:

Definition 5.4. Let $(V, \triangleright_V)$ and $(W, \triangleright_W)$ be representations of an algebra H. Suppose $f : V \to W$ is a linear map. Then f is said to be an *intertwiner* if it commutes with the actions on V, W in the sense that

$$h \triangleright_W f(v) = f(h \triangleright_V v),$$

for all $h \in H$, $v \in V$.

It is trivial to see that the composition of two intertwiners is again an intertwiner. This (along with other similarly trivial considerations) allows us to define a *category* of representations over an algebra H:

Definition 5.5. Fix an algebra H. We define the *representation category* ${}_H\mathcal{M}$ as the category whose objects are representations $(V, \triangleright_V)$ of H, and whose morphisms are intertwiners of these representations. Clearly ${}_H\mathcal{M}$ is a subcategory of `Vec`, since a representation is in particular a vector space and an intertwiner is a linear map.

We now present some concrete examples of algebra- and Hopf algebra representations.

Example 5.6. Let H be a bialgebra or Hopf algebra. Then H acts on itself as a vector space via the *left regular action* L given by

$$L_h(g) = hg,$$

for $h, g \in H$; i.e. $h \triangleright g = hg$.

Lemma 5.7. Let H be any Hopf algebra. Then H acts on itself as an algebra, i.e. H is an H-module algebra, via the *adjoint action*

$$\mathrm{Ad} : \mathrm{Ad}_h(g) = \sum h_{(1)} g S h_{(2)},$$

for $h, g \in H$.

Proof. First we verify that Ad indeed defines an action. Using lemma 3.6 we compute that

$$\begin{aligned} h \triangleright (g \triangleright a) &= h \triangleright (g_{(1)} a S g_{(2)}) \\ &= h_{(1)} g_{(1)} a S g_{(2)} S h_{(2)} \\ &= (hg)_{(1)} a S (hg)_{(2)} \\ &= (hg) \triangleright a, \end{aligned}$$

for any $h, g, a \in H$. Here the third equality uses that Δ distributes over multiplication by definition. Next note that for the antipode S of any Hopf algebra we have

$$S(1) = 1 \cdot S(1) = 1_{(1)} \cdot S(1_{(2)}) = \epsilon(1) = 1,$$

using that $\Delta(1) = 1 \otimes 1$ by definition. Hence we also have $1 \triangleright g = 1 \cdot g \cdot S(1) = g$. Hence Ad is indeed an action. Next we verify that Ad indeed turns H into an H-module algebra. To this end we compute for $h, a, b \in H$:

$$\begin{aligned} h \triangleright (ab) &= h_{(1)} ab S h_{(2)} \\ &= h_{(1)} a \epsilon(h_{(2)}) b S h_{(3)} \\ &= h_{(1)} a (S h_{(2)}) h_{(3)} b S h_{(4)} \\ &= (h_{(1)} \triangleright a)(h_{(2)} \triangleright b). \end{aligned}$$

In this computation we use the Sweedler notation $h_{(1)}, \ldots, h_{(4)}$, which is unambiguous by virtue of cocommutativity. The second equality uses the counit axiom to sneak in a factor $\epsilon(h_{(2)})$ that commutes with a (since it is simply a multiple of $1 \in H$). This factor is expanded into $(Sh_{(2)})h_{(3)}$ in the third equality, using the antipode axiom. To complete the proof, we finally compute using the antipode axiom that

$$h \triangleright 1 = h_{(1)} 1 S h_{(2)} = h_{(1)} S h_{(2)} = \epsilon(h) \cdot 1,$$

as required. ☐

A particularly interesting example for Physics is that of $H = k\mathbb{Z}_2$; the Hopf algebra kG from example 3.7 with $G = \mathbb{Z}_2$:

Example 5.8. Suppose $H = k\mathbb{Z}_2$ with $\mathrm{Char}(k) \neq 2$, where $\mathbb{Z}_2$ is the cyclic group of order 2 with generator g such that $g^2 = 1$. Then we claim that an action of H on a vector space V is nothing but a $\mathbb{Z}_2$-grading of V. (Recall the definition of a grading from example 3.22.)
Indeed, suppose that H acts on a vector space V and consider the element $p = \frac{1-g}{2}$ of H. We find

$$p^2 = \left(\frac{1-g}{2}\right)^2 = \frac{1 - 2g + g^2}{4} = \frac{2 - 2g}{4} = p,$$

so that p acts as a projection on V. By [30, Thm. 6.18], this decomposes V into

$$V = \mathrm{Ker}(p) \oplus \mathrm{Im}(p) := V_0 \oplus V_1,$$

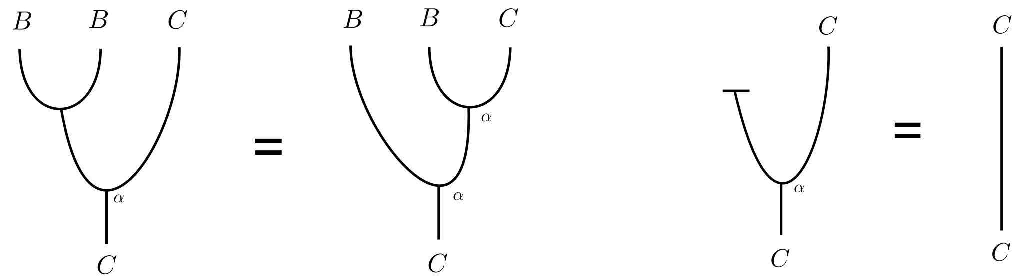
i.e. V decomposes as a $\mathbb{Z}_2$-graded vector space with the grading given by the eigenvalue of p acting on v (which is 0 if $v \in \mathrm{Ker}(p)$ and 1 if $v \in \mathrm{Im}(p)$). Conversely, a $\mathbb{Z}_2$-grading on a vector space V gives rise to a projection operator p onto V_1. Interpreting $p = \frac{1-g}{2}$ this grading then determines the action of p, which uniquely determines the action of g. Since g is the only nontrivial generator of $k\mathbb{Z}_2$ this uniquely determines an action of $k\mathbb{Z}_2$ on V.

In conclusion, a representation of $k\mathbb{Z}_2$ is equivalent to a $\mathbb{Z}_2$-graded vector space, i.e. a super vector space. Thus there is a relation between ${}_{k\mathbb{Z}_2}\mathcal{M}$ and `SuperVec`. We will discuss this in more detail in subsection 8.3.

5.2 Actions of Braided Groups

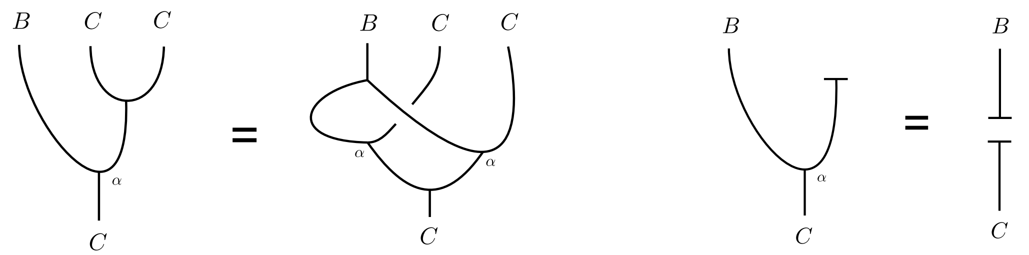
The axioms for algebra actions extend naturally to braided categories using braided diagrams, in the same way that the axioms of a Hopf algebra extend to braided groups:

Definition 5.9. Let B be an algebra object in a braided category $\mathcal{C}$. Then an action of B on an object V of $\mathcal{C}$ consist of a morphism $\alpha : B \otimes V \to V$. In a braided diagram we depict this α as a multiplication between B and C, labelled with an α for clarity. We require α to obey the following braided diagram equalities, that are analogous to the commutative diagrams from definition 5.1:



The generalization of actions on algebras to braided categories is similar, but slightly less straight-forward. As in the case of `Vec`, to define actions on an algebra it does not suffice for B to be an algebra object. Instead B needs to have the additional structure of a braided group (i.e. Hopf algebra object in $\mathcal{C}$).

Definition 5.10. Let B be a braided group in a braided category $\mathcal{C}$. Then B acts on an algebra object C of $\mathcal{C}$ if B acts on C as a regular object of $\mathcal{C}$. This action $\alpha : B \otimes C \to C$ must moreover commute with the multiplication and unit of C. These requirements are depicted in the following braided diagram equalities:



Again, these are analogous to the commutative diagrams of the $\mathcal{C} = $ `Vec` case. If these requirements are fulfilled, we also refer to C as a *braided B-module algebra*.

Remark 5.11. As stated earlier, the above definition is not an entirely straight-forward generalization. Namely, the τ that switches B and C in the case of `Vec` is replaced by $\Psi_{B,C}$. However, in principle we could also choose to define braided actions on an algebra using the reversed braiding $\Psi^{-1}_{B,C}$. This yields an equally valid definition of a braided module algebra. If this definition with reversed braiding is used, we will indicate this for clarity.

As a concrete example, we can naturally extend the adjoint action Ad of a Hopf algebra on itself to braided groups. This generalization is due to [31].

Example 5.12. Let B be a braided group in a braided category $\mathcal{C}$ with antipode S. Then we define the adjoint action $\mathrm{Ad} : B \otimes B \to B$ as follows:

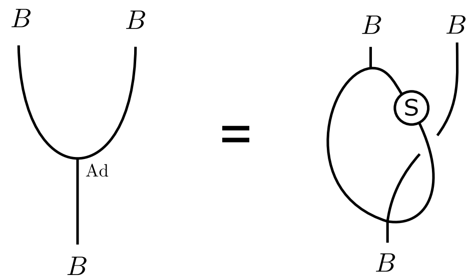


Figure 5.1: The adjoint action of a braided group B on itself.

(This picture includes a multiplication of three elements, which is justified by commutativity.) This action turns B into a braided B-module algebra. Clearly, in the case of $\mathcal{C} = $ `Vec` this reduces to the action from lemma 5.7.

Proof. We only prove the property $(hg) \triangleright a = h \triangleright (g \triangleright a)$. This proof is depicted in figure 5.2 below. The other properties can be proven similarly with the formalism of braided diagrams.

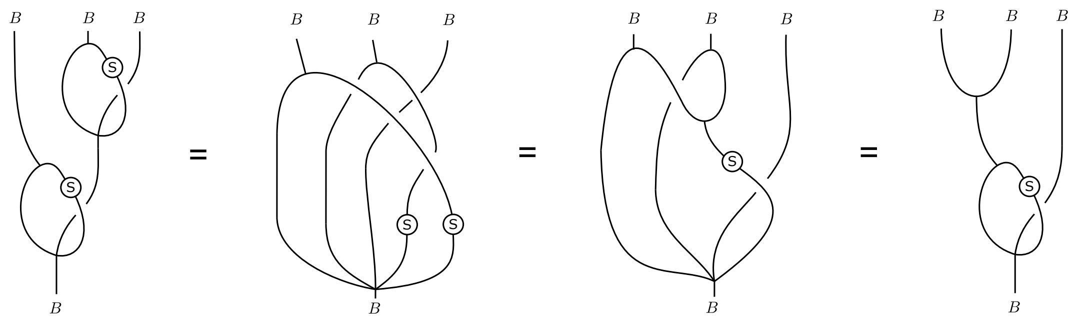


Figure 5.2: Proof that Ad defines an action.

□

The first equality extends the Ad loops, using functoriality to pull the multiplications, comultiplications, and antipodes under the braidings Ψ as necessary. The second equality follows immediately from lemma 3.28 (and another application of functoriality to rearrange). The final equality follows by using the braided group axiom that Δ is an algebra morphism, depicted in figure 3.5.

5.3 Construction of Braided Categories

Recall the representation category ${}_H\mathcal{M}$ of an algebra H from definition 5.5. It turns out that these categories give us large classes of monoidal, braided, or rigid categories, provided that the algebra H has certain additional structure.

Proposition 5.13. If H is a bialgebra, then ${}_H\mathcal{M}$ is a monoidal category with $-\otimes-$ defined by

$$h \triangleright (v \otimes w) = \sum h_{(1)} \triangleright v \otimes h_{(2)} \triangleright w,$$

for $V, W \in \mathrm{ob}\, {}_H\mathcal{M}$, $h \in H$, $v \in V$, $w \in W$; extended linearly to arbitrary elements of $V \otimes W$. The unit object $\underline{1}$ is given by the trivial module k, with

$$h \triangleright \lambda = \epsilon(h)\lambda \qquad \text{for } h \in H,\ \lambda \in k.$$

Since ${}_H\mathcal{M}$ is a subcategory of `Vec`, we can simply let Φ, l, r be the corresponding canonical transformations in `Vec`.
Moreover, the forgetful functor $F : {}_H\mathcal{M} \to$ `Vec` is a monoidal functor.

Proof. We only need to show that $h \triangleright (v \otimes w)$ as prescribed indeed defines a valid action of H on $V \otimes W$. By the counit axiom it is then trivial that k provides a two-sided unit object. To see that we indeed have a valid action, we compute

$$\begin{aligned}(hg) \triangleright (v \otimes w) &= (hg)_{(1)} \triangleright v \otimes (hg)_{(2)} \triangleright w \\ &= (h_{(1)}g_{(1)}) \triangleright v \otimes (h_{(2)}g_{(2)}) \triangleright w \\ &= h_{(1)} \triangleright (g_{(1)} \triangleright v) \otimes h_{(2)} \triangleright (g_{(2)} \triangleright w) \\ &= h \triangleright (g_{(1)} \triangleright v \otimes g_{(2)} \triangleright w) \\ &= h \triangleright (g \triangleright (v \otimes w)).\end{aligned}$$

Here the second equality uses that Δ distributes over multiplication, and the third equality uses that $\triangleright$ is an action on V and W separately. We also have

$$1 \triangleright (v \otimes w) = 1_{(1)}v \otimes 1_{(2)}w = v \otimes w.$$

Recall here that $\Delta(1) = 1 \otimes 1$ since Δ is an algebra morphism. Thus the definition of an action is indeed satisfied, as required.

Finally, we must show that F is a monoidal functor. This requires functorial isomorphisms

$$c_{V,W} : F(V) \otimes F(W) \to F(V \otimes W),$$

for all objects V, W of ${}_H\mathcal{M}$. Since $F(V) \otimes F(W) = V \otimes W = F(V \otimes W)$ as vector spaces, $c_{V,W} := \mathrm{id}_{V \otimes W}$ suffices. Since k is the identity object in both ${}_H\mathcal{M}$ and `Vec` we have that $F(\underline{1}) = \underline{1}$. With this the other defining properties of a monoidal functor are trivial, since $c_{V,W}$ is an identity map. $\square$

Recall the definition of a *quasitriangular structure* from definition 4.4. We have already seen in lemma 4.5 that if a bialgebra or Hopf algebra is equipped with a quasitriangular structure $\mathcal{R}$, then $\mathcal{R}$ obeys the quantum Yang-Baxter equation. This allowed us to conclude invariance of I_G under the Reidermeister-3 relation, i.e. the braid relation (see remark 6.8 for details). We have also seen in section 3.2 that the braiding Ψ of a braided category satisfies this same relation. This link between the braid relation, quasitriangular Hopf algebras, and braided categories is completed by the following proposition:

Proposition 5.14. If H is a quasitriangular bialgebra, then ${}_H\mathcal{M}$ is braided, with

$$\Psi_{V,W}(v \otimes w) = \sum \mathcal{R}^{(2)} \triangleright w \otimes \mathcal{R}^{(1)} \triangleright v,$$

for $v \in V$, $w \in W$ extended linearly to arbitrary elements of $V \otimes W$. Here the superscripts $(1), (2)$ indicate that we take one of the tensor components of $\mathcal{R} \in H \otimes H$, i.e. we write $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$.

Proof. We must show three things: that $\Psi_{V,W}$ is indeed a morphism in ${}_H\mathcal{M}$, that Ψ obeys the hexagon conditions, and that $\Psi_{V,W}$ is functorial in V, W.
To show that $\Psi_{V,W}$ is a morphism we must verify it is an intertwiner of the action of H on $V \otimes W$. If we use $\triangleright$ to denote the action of $H \otimes H$ on $V \otimes W$ built naturally from the actions of H on V, W then Ψ becomes

$$\Psi_{V,W}(v \otimes w) = \tau(\mathcal{R} \triangleright (v \otimes w)),$$

where τ is the transposition map $v \otimes w \mapsto w \otimes v$. Now let $h \in H$. In the above notation we then have

$$\Psi(h \triangleright (v \otimes w)) = \Psi(\Delta h \triangleright (v \otimes w)) = \tau(\mathcal{R}(\Delta h) \triangleright (v \otimes w)).$$

Now recall the quasitriangularity axiom of equation 4.1: from it we find that $\mathcal{R}(\Delta h) = (\tau \circ \Delta h)\mathcal{R}$. Applying this to the above we find

$$\begin{aligned}
\Psi(h \triangleright (v \otimes w)) &= \tau((\tau \circ \Delta h)\mathcal{R} \triangleright (v \otimes w)) \\
&= \tau\left(\sum (h_{(2)}\mathcal{R}^{(1)}) \triangleright v \otimes (h_{(1)}\mathcal{R}^{(2)}) \triangleright w\right) \\
&= \sum (h_{(1)}\mathcal{R}^{(2)}) \triangleright w \otimes (h_{(2)}\mathcal{R}^{(1)}) \triangleright v \\
&= \sum h_{(1)} \triangleright (\mathcal{R}^{(2)} \triangleright w) \otimes h_{(2)} \triangleright (\mathcal{R}^{(1)} \triangleright v) \\
&= h \triangleright \left(\sum \mathcal{R}^{(2)} \triangleright w \otimes \mathcal{R}^{(1)} \triangleright v\right) \\
&= h \triangleright \Psi_{V,W}(v \otimes w).
\end{aligned}$$

Here the fourth equality uses that $\triangleright$ is an action. This shows that Ψ is an intertwiner, as required.

Next we show that Ψ obeys the hexagon conditions. The first hexagon condition amounts to $\Psi_{V \otimes W, Z} = (\Psi_{V,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Psi_{W,Z})$. To verify this, let $v \otimes w \otimes z \in V \otimes W \otimes Z$. Then we find

$$\Psi_{V \otimes W, Z}(v \otimes w \otimes z) = \mathcal{R}^{(2)} \triangleright z \otimes \mathcal{R}^{(1)} \triangleright (v \otimes w) = \mathcal{R}^{(2)} \triangleright z \otimes \mathcal{R}^{(1)}{}_{(1)} \triangleright v \otimes \mathcal{R}^{(1)}{}_{(2)} \triangleright w.$$

Here the summations are left implicit. Using $\triangleright$ in the same way as before for the action of $H \otimes H \otimes H$ on $V \otimes W \otimes Z$, we thus have

$$\Psi_{V \otimes W, Z}(v \otimes w \otimes z) = \tau_{12}\tau_{23}((\Delta \otimes \mathrm{id})\mathcal{R} \triangleright (v \otimes w \otimes z)).$$

Here the indices on τ denote on which tensor powers it is acting; see definition 4.4. We can then use the first quasitriangularity axiom from 4.2 to find

$$\begin{aligned}
\Psi_{V \otimes W, Z}(v \otimes w \otimes z) &= \tau_{12}\tau_{23}(\mathcal{R}_{13}\mathcal{R}_{23} \triangleright (v \otimes w \otimes z)) \\
&= \tau_{12}\tau_{23}((\mathcal{R}^{(1)} \otimes \mathcal{R}'^{(1)} \otimes \mathcal{R}^{(2)}\mathcal{R}'^{(2)}) \triangleright (v \otimes w \otimes z)) \\
&= \mathcal{R}^{(2)}\mathcal{R}'^{(2)} \triangleright z \otimes \mathcal{R}^{(1)} \triangleright v \otimes \mathcal{R}'^{(1)} \triangleright w \\
&= (\Psi_{V,Z} \otimes \mathrm{id})(v \otimes \mathcal{R}'^{(2)} \triangleright z \otimes \mathcal{R}'^{(1)} \triangleright w) \\
&= (\Psi_{V,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Psi_{W,Z})(v \otimes w \otimes z),
\end{aligned}$$

where $\mathcal{R}'$ denotes a second copy of $\mathcal{R}$. This extends to arbitrary elements of $V \otimes W \otimes Z$ by linearity. Thus the first hexagon condition holds. The second hexagon condition is verified analogously, now using the second axiom from 4.2.

Finally, we show functoriality of $\Psi_{V,W}$. We only show functoriality in V; the proof of functoriality in W is entirely analogous. To this end, let $f : V_1 \to V_2$ be a morphism in ${}_H\mathcal{M}$, i.e. an intertwiner. Then for $v \otimes w \in V_1 \otimes W$ we have

$$\begin{aligned}\Psi_{V_2,W}(f(v) \otimes w) &= \sum \mathcal{R}^{(2)} \triangleright w \otimes \mathcal{R}^{(1)} \triangleright f(v) \\ &= \sum \mathcal{R}^{(2)} \triangleright w \otimes f(\mathcal{R}^{(1)} \triangleright v) \\ &= (\mathrm{id} \otimes f) \left(\sum \mathcal{R}^{(2)} \triangleright w \otimes \mathcal{R}^{(1)} \triangleright v\right) \\ &= (\mathrm{id} \otimes f) \circ \Psi_{V_1,W}(v \otimes w),\end{aligned}$$

using that f is an intertwiner. This extends to arbitrary elements of $V_1 \otimes W$ by linearity, hence we conclude that the following diagram commutes:

$$\begin{tikzcd} V_1 \otimes W \arrow[r, "\Psi_{V_1,W}"] \arrow[d, "f\otimes\mathrm{id}"'] & W \otimes V_1 \arrow[d, "\mathrm{id}\otimes f"] \\ V_2 \otimes W \arrow[r, "\Psi_{V_2,W}"'] & W \otimes V_2 \end{tikzcd}$$

Thus $\Psi_{V,W}$ is functorial in V by definition, completing the proof. $\square$

In this proof we see that the definitions of a quasitriangular structure are precisely enough to ensure that Ψ is a braiding: equation 4.1 ensures that Ψ is an intertwiner, and the equations in 4.2 lead to the hexagon conditions.

So far we have only considered the case where H is a bialgebra. If H is a Hopf algebra we can deduce even more structure on ${}_H\mathcal{M}$, as long as we restrict our attention to *finite-dimensional* representations:

Proposition 5.15. If H is a Hopf algebra then ${}_H\mathcal{M}_F$, the monoidal subcategory of ${}_H\mathcal{M}$ consisting of finite-dimensional representations of H, is rigid. The dual object $(V^*, \mathrm{ev}_V, \mathrm{coev}_V)$ is the same as in the category of finite-dimensional vector spaces. The dual V^* (consisting of all linear maps $f : V \to k$) is a representation via the action

$$(h \triangleright \phi)(v) = \phi((Sh) \triangleright v),$$

for $h \in H$, $\phi \in V^*$, $v \in V$.

Proof. We only need to show that $\triangleright$ as prescribed is a valid action, and that $\mathrm{ev}_V, \mathrm{coev}_V$ are morphisms in ${}_H\mathcal{M}_F$ (as well as in $\mathtt{Vec}$).

To verify that $\triangleright$ is an action, we compute for $h, g \in H$, $\phi \in V^*$:

$$\begin{aligned}(h \triangleright (g \triangleright \phi))(v) &= (g \triangleright \phi)((Sh) \triangleright v) \\ &= \phi(Sg \triangleright (Sh \triangleright v)) \\ &= \phi((Sg)(Sh) \triangleright v) \\ &= \phi(Shg \triangleright v) = (hg \triangleright \phi)(v).\end{aligned}$$

Here the third equality uses the definition of an action and the fourth equality uses lemma 3.6. Since $S(1) = 1$ as seen in the proof of lemma 5.7, we also have

$$(1 \triangleright \phi) = \phi(S(1) \triangleright v) = \phi(1 \triangleright v) = \phi(v).$$

Thus $\triangleright$ indeed defines an action on V^*, as required. Next we show that $\mathrm{ev}_V, \mathrm{coev}_V$ are intertwiners. First, recall that for $f \in V^*$, $v \in V$ we have $\mathrm{ev}_V(f \otimes v) = f(v) \in k$. Also recall that the action of H on k is $h \triangleright \lambda = \epsilon(h)\lambda$ for $\lambda \in k$. Thus we must show that $\mathrm{ev}_V(h \triangleright (f \otimes v)) = \epsilon(h)\mathrm{ev}_V(f \otimes v)$. To this end we compute:

$$\begin{aligned}
\mathrm{ev}_V(h \triangleright (f \otimes v)) &= \mathrm{ev}_V(h_{(1)} \triangleright f \otimes h_{(2)} \triangleright v) \\
&= (h_{(1)} \triangleright f)(h_{(2)} \triangleright v) \\
&= f((Sh_{(1)})h_{(2)} \triangleright v) \\
&= \epsilon(h)f(v) = \epsilon(h)\mathrm{ev}_V(f \otimes v).
\end{aligned}$$

Here the final equality uses the antipode axioms, as well as linearity of f. Next, recall that coev_V is defined by $\mathrm{coev}_V(1) = \sum e_a \otimes f^a$, extended linearly, where $\{e_a\}$ is any basis of V and $\{f^a\}$ is the corresponding dual basis of V^*. Using this we compute:

$$\begin{aligned}
h \triangleright \mathrm{coev}_V(1) &= h \triangleright \left(\sum e_a \otimes f^a(\dots)\right) \\
&= \sum h_{(1)} \triangleright e_a \otimes (h_{(2)} \triangleright f^a)(\dots) \\
&= \sum h_{(1)} \triangleright e_a \otimes f^a(Sh_{(2)} \triangleright (\dots)) \\
&= h_{(1)}(Sh_{(2)}) \triangleright \sum e_a \otimes f_a(\dots) \\
&= \epsilon(h) \sum e_a \otimes f^a(\dots) \\
&= \epsilon(h)\mathrm{coev}_V(1) = \mathrm{coev}_V(h \triangleright 1).
\end{aligned}$$

Here the $(\dots)$ acts as a placeholder for a function input. (Essentially, we verify that the involved functions are equal by checking that they agree on all inputs.) Also see [11]. The above computation extends to arbitrary elements of k by linearity. Thus ev_V and coev_V are intertwiners by definition, completing the proof. □

Proposition 5.15 highlights an important role of the antipode S, and a way in which S is like a generalized inverse: for a representation V of a Hopf algebra H the antipode structure endows the dual V^* with the structure of a representation. This same role is fulfilled by the inverse of a group, in the case of group representations. To enforce this remark, note that the definition of an antipode is precisely what is needed for the proof of proposition 5.15 to work: we need both the left- and right-sided antipode axioms to ensure that $\mathrm{ev}_V, \mathrm{coev}_V$ are intertwiners.

5.4 Reconstructions

We have now proven that the representation category of a Hopf algebra H is monoidal and rigid, has a monoidal forgetful functor, and is more-over braided if H is quasitriangular. Recall that the axioms of monoidal, braided, and rigid categories essentially serve to emulate certain characteristics of `Vec`. This immediately raises the question: why do we model *these* characteristics? Why not others, and why do we consider braided categories instead of e.g. symmetric categories? Is there even more `Vec`-like structure that we can derive for ${}_H\mathcal{M}$?

The answer to these questions is provided by the theory of representation-theoretic *reconstructions*. As it turns out, we cannot just create rigid monoidal categories from Hopf algebras; we can also build a Hopf algebra from *any* rigid monoidal category with a monoidal forgetful functor! The *reconstruction theorem* states that the latter process is inverse to the first: if we construct a Hopf algebra H' from the category ${}_H\mathcal{M}$, then $H' \cong H$ (also in the quasitriangular / braided case). This means that a (braided) rigid monoidal strucutre is *all* we can hope to have on the representation category of a (quasitriangular) Hopf algebra. In fact, this duality between rigid monoidal categories and Hopf algebras is an example of *Tannaka-Krein* duality, named after the first instance of such a duality which was for noncommutative compact topological groups [32].

We begin by stating the reconstruction theorem for algebras. Recall that in this text, we assume all algebras to be unital and associative. As a first taste of reconstruction, we show that the underlying set of an algebra can be reconstructed from its representations. We then discuss how to make an algebra from any monoidal category equipped with a suitable notion of a forgetful functor.

Lemma 5.16. Let A be an algebra, and ${}_A\mathcal{M}$ its representation category. Consider the forgetful functor $F : {}_A\mathcal{M} \to \mathtt{Vec}$. Denote the set of natural transformations from F to itself by $\mathrm{Nat}(F, F)$. Then $\mathrm{Nat}(F, F)$ is in one-to-one correspondence with with elements of (the underlying set of) A. That is, $\mathrm{Nat}(F, F) \cong A$ as sets.

Proof. We prove this by explicitly giving a bijection $\theta : A \to \mathrm{Nat}(F, F)$. For $a \in A$ define the natural tranformation Φ by

$$\Phi_V(v) := a \triangleright v. \tag{5.1}$$

It is easy to check that this indeed defines a natural transformation, i.e. that Φ_V is functorial in V: the requirement of functoriality becomes commutativity of the diagram

$$\begin{tikzcd} V_1 \arrow[r, "\triangleright"] \arrow[d, "f"'] & V_1 \arrow[d, "f"] \\ V_2 \arrow[r, "\triangleright"'] & V_2 \end{tikzcd}$$

for some morphism $f \in \mathrm{Hom}(V_1, V_2)$ in the category ${}_A\mathcal{M}$. This diagram amounts to the requirement that f is an intertwiner, which is indeed the case since f is a morphism in ${}_A\mathcal{M}$. Thus, we let $\theta(a) = \Phi$.

Coversely, suppose $\Phi \in \mathrm{Nat}(F, F)$. Consider A as an object of ${}_A\mathcal{M}$ via the left regular representation (i.e. multiplication). Then we obtain a linear map $\Phi_A : A \to A$. We then define $\theta^{-1}(\Phi) = \Phi_A(1)$.

The notation suggests that θ and θ^{-1} are mutually inverse. We now show that this is indeed the case: first we compute that

$$\theta^{-1}(\theta(a)) = \Phi_A(1) = a \triangleright 1 = a \cdot 1 = a,$$

where Φ is as given by equation 5.1. We thus conclude $\theta^{-1} \circ \theta = \mathrm{id}_A$. Converesly, suppose $\Phi \in \mathrm{Nat}(F, F)$ and let $a := \theta^{-1}(\Phi) = \Phi_A(1)$. Then we find that $\theta(a)$ is the natural transformation Φ' given by

$$\Phi'_V(v) = a \triangleright v = \Phi_A(1) \triangleright v.$$

If we let $\phi_v : A \to V$ be the morphism $a \mapsto a \triangleright v$ in ${}_A\mathcal{M}$ (it is trivial that ϕ_v is indeed an intertwiner), then the above becomes

$$\Phi'_V(v) = (F(\phi_v) \circ \Phi_A)(1).$$

Note here that we really need to apply F to ϕ_v, since we need it to be a morphism of vector spaces. Now, since Φ is a natural transformation, we obtain by functoriality in A that

$$F(\phi_v) \circ \Phi_A = \Phi_V \circ F(\phi_v), \tag{5.2}$$

since ϕ_v is a morphism in ${}_A\mathcal{M}$. Using this we find

$$\Phi'_V(v) = (\Phi_V \circ F(\phi_v))(1) = \Phi_V(1 \triangleright v) = \Phi_V(v).$$

Hence $\Phi' = \Phi$. This allows us to conclude that $\theta \circ \theta^{-1} = \mathrm{id}_{\mathrm{Nat}(F,F)}$. Thus θ is a bijection, finishing the proof. $\square$

We now want to recover the algebra structure on A. For this we have to able to recover an algebra from any monoidal category $\mathcal{C}$ with an analogue of the forgetful functor F. We first describe the kind of forgetful functor analogue that is required on $\mathcal{C}$.

Lemma 5.17. Let $\mathcal{C}$ be a category, and let $F : \mathcal{C} \to \mathtt{Vec}$ be a functor. Then we also have a functor $V \otimes F : \mathcal{C} \to \mathtt{Vec}$ that acts on objects as $(V \otimes F)(X) = V \otimes F(X)$ and on morphisms as $(V \otimes F)(f) = \mathrm{id} \otimes F(f)$. This allows us to define a functor

$$G : \mathtt{Vec} \to \mathtt{Set} : V \mapsto \mathrm{Nat}(V \otimes F, F).$$

This functor is contravariant.

Proof. To begin we must specify G on morphisms. This is done similarly to the contravariant Hom functor: recall that $h \in \mathrm{Nat}(V \otimes F, F)$ is equivalent to a family $h_X \in \mathrm{Hom}(V \otimes F(X), F(X))$ functorial in X. Thus for a morphism $f : V \to W$ in $\mathtt{Vec}$ we define

$$G(f) : \mathrm{Nat}(W \otimes F, F) \to \mathrm{Nat}(V \otimes F, F) : h \mapsto h \circ (f \otimes \mathrm{id}).$$

Here $(h \circ (f \otimes \mathrm{id}))_X := h_X \circ (f \otimes \mathrm{id})$. It is trivial that this indeed gives a contravariant functor G, with one caveat: we must check that $h \circ (f \otimes \mathrm{id})$ is indeed an element of $\mathrm{Nat}(V \otimes F, F)$. In other words we must verify functoriality of $h_X \circ (f \otimes \mathrm{id})$ in X. To this end let $g : X \to Y$ be a morphism in $\mathtt{Vec}$. Then we must check that

$$\begin{tikzcd} V \otimes F(X) \arrow[r, "h_X \circ (f \otimes \mathrm{id})"] \arrow[d, "\mathrm{id} \otimes F(g)"'] & F(X) \arrow[d, "F(g)"] \\ V \otimes F(Y) \arrow[r, "h_Y \circ (f \otimes \mathrm{id})"'] & F(Y) \end{tikzcd}$$

commutes. Thus let $\sum v \otimes x$ be an arbitrary element of $V \otimes F(X)$. Then the path of this element along the bottom of the diagram reads:

$$\sum v \otimes x \mapsto \sum v \otimes g(x) \mapsto h_Y \left(\sum f(v) \otimes g(x) \right),$$

while the path along the top reads:

$$\sum v \otimes x \mapsto h_X \left(\sum f(v) \otimes x \right) \mapsto F(g) \left(h_X \left(\sum f(v) \otimes x \right) \right).$$

Functoriality of h_X in X gives that $F(g) \circ h_X = h_Y \circ (\mathrm{id} \otimes F(g))$. This means

$$F(g) \left(h_X \left(\sum f(v) \otimes x \right) \right) = h_Y \left((\mathrm{id} \otimes F(g)) \left(\sum f(v) \otimes x \right) \right) = h_Y \left(\sum f(v) \otimes g(x) \right),$$

so that the diagram indeed commutes. Thus by definition $h \circ (f \otimes \mathrm{id})$ is a natural transformation, so that $G(f)$ is well-defined. □

Recall that natural transformations are 'maps between functors', while functors are maps between categories. We are now at the level of sets of natural transformations, and functions between *them*; a third level of maps. This is truly at the edge of what we can accomplish in category theory without getting serious about so-called 2-categories or ∞-categories. It is therefore no surprise that the proofs get extraordinarily involved in what follows, and hence we shall only give references for most of them.

Lemma 5.18. Suppose A is an algebra. Let $F : {}_A\mathcal{M} \to \mathtt{Vec}$ be the forgetful functor. Then the contravariant functor G from lemma 5.17 is representable, with representing object $F(A)$.

Proof. Recall that for G to be representable, there must exist some vector space H such that G and $\mathrm{Hom}(-, H)$ are naturally isomorphic functors. This is equivalent to the existence of $\mathtt{Set}$-isomorphisms (i.e. bijections)

$$\theta_V : \mathrm{Hom}(V, H) \cong G(V) = \mathrm{Nat}(V \otimes F, F),$$

that are functorial in V. In fact, we claim $H = F(A)$, or just A as a vector space.

To show this, we construct the isomorphisms θ_V explicitly. For $\Phi \in \mathrm{Nat}(V \otimes F, F)$ we can naturally construct an element of $\mathrm{Hom}(V, A)$ by considering A as an A-representation via the left regular action. We can then construct our map $V \to A$ using

$$\Phi_A \in \mathrm{Hom}((V \otimes F)(A), F(A)) = \mathrm{Hom}(V \otimes A, A).$$

Namely, if we fix the A-input of Φ_A to be 1, we obtain a map

$$\begin{aligned} \theta_V^{-1}(\Phi) := \Phi_A((\dots) \otimes 1) : V &\to A \\ v &\mapsto \Phi_A(v \otimes 1). \end{aligned}$$

Since Φ_A is linear, this map is also linear so that it is indeed an element of $\mathrm{Hom}(V, A)$.
Conversely, suppose $\phi \in \mathrm{Hom}(V, A)$. Then we define $\theta_V(\phi) \in \mathrm{Nat}(F, F)$ on a representation X of A as:

$$\begin{aligned} (\theta_V(\phi))_X : V \otimes F(X) &\to F(X) \\ v \otimes x &\mapsto \phi(v) \triangleright x. \end{aligned}$$

This indeed defines a natural transformation: naturality of $(\theta_V(\phi))_X$ in X is clear since any morphism $X_1 \to X_2$ is an intertwiner and hence commutes with the action $\phi(v)\triangleright$.
Next we check that θ and θ^{-1} are mutually inverse, as is suggested by the notation. First, we compute for $\phi \in \mathrm{Hom}(V, A)$:

$$\theta_V^{-1}(\theta_V(\phi))(v) = (\theta_V(\phi))_A((v \otimes 1) = \phi(v) \triangleright 1 = \phi(v).$$

Note here that $\phi(v) \in A$, so that the last $\triangleright$ denotes the left regular action, i.e. multiplication.
Conversely, we compute for $\Phi \in \mathrm{Nat}(V \otimes F, F)$:

$$(\theta_V(\theta_V^{-1})(\Phi))_X(v \otimes x) = (\theta_V^{-1})(\Phi))(v) \triangleright x = \Phi_A(v \otimes 1) \triangleright x = (F(\phi_x) \circ \Phi_A)(v \otimes 1).$$

Here $\phi_x : A \to X$ is in the same notation as ϕ_v from lemma 5.16. We now apply the same functoriality trick as equation (5.2) to find

$$(\theta_V(\theta_V^{-1})(\Phi))_X(v \otimes x) = (\Phi_X \circ (\mathrm{id} \otimes F(\phi_x)))(v \otimes 1) = \Phi_X(v \otimes 1 \triangleright x) = \Phi_X(v \otimes x).$$

This shows that

$$\theta_V(\theta_V^{-1})(\Phi) = \Phi.$$

We have thus shown that $\theta_V \circ \theta_V^{-1} = \mathrm{id}_{\mathrm{Hom}(V,A)}$ and $\theta_V^{-1} \circ \theta_V = \mathrm{id}_{\mathrm{Nat}(V \otimes F, F)}$, hence θ and θ^{-1} are mutually inverse, as required.

To complete the proof, we check functoriality of θ_V in V. This amounts to commutativity of the following diagram:

$$\begin{tikzcd} \mathrm{Hom}(V_1, F(A)) \arrow[r, "\theta_{V_1}"] & \mathrm{Nat}(V_1 \otimes F, F)) \\ \mathrm{Hom}(V_2, F(A)) \arrow[u, "- \circ f"] \arrow[r, "\theta_{V_2}"'] & \mathrm{Nat}(V_2 \otimes F, F) \arrow[u, "- \circ (f \otimes \mathrm{id})"'] \end{tikzcd}$$

To verify this commutativity, start with $\phi \in \mathrm{Hom}(V_2, F(A))$. Going along the top path of the diagram gives $\theta_{V_1}(\phi \circ f)$ which is given on an object X of ${}_A\mathcal{M}$ as

$$(\theta_{V_1}(\phi \circ f))_X(v \otimes x) = (\phi \circ f)(v) \triangleright x,$$

for $v \otimes x \in V_1 \otimes F(X)$. Meanwhile, going along the bottom path yields $\theta_{V_2}(\phi) \circ (f \otimes \mathrm{id})$ which on X is given by

$$(\theta_{V_2}(\phi)_X) \circ (f \otimes \mathrm{id})(v \otimes x) = (\theta_{V_2}(\phi)_X)(f(v) \otimes x) = \phi(f(v)) \triangleright x.$$

This concides with $(\theta_{V_1}(\phi \circ f))_X$, so we conclude the diagram indeed commutes. □

Definition 5.19. Let $(\mathcal{C}, \otimes)$ be a monoidal category. We define a *generalized forgetful functor* to be a monoidal functor $F : \mathcal{C} \to \mathtt{Vec}$ such that G from lemma 5.17 is representable.

Again, representability of G entails that there is some $H \in \mathrm{ob}\,\mathtt{Vec}$ such that G is naturally isomorphic to $\mathrm{Hom}(-, H)$, i.e. there are isomorphisms

$$\theta_V : \mathrm{Hom}(V, H) \cong \mathrm{Nat}(V \otimes F, F),$$

functorial in V.

We are completely now in the position to construct algebras from categories. We will only give the constructions themselves in detail. Proofs of these constructions can be found in [11, Ch.9].

Proposition 5.20. Let $(\mathcal{C}, \otimes)$ be a monoidal category equipped with a generalized forgetful functor F. Let A be the vector space such that G is naturally isomorphic to $\mathrm{Hom}(-, A)$. Then we can define an algebra structure on A as follows:

Recall that an algebra structure on a vector space consists of two linear maps: $\cdot : A \otimes A \to A$ and $\eta : k \to A$. Now consider the structure we have on $\mathcal{C}$, induced by G; namely a set of isormorphisms θ_V that allow us to turn elements of $\mathrm{Nat}(V \otimes F, F)$ into elements of $\mathrm{Hom}(V, A)$. Thus if we wish to construct a map $\cdot \in \mathrm{Hom}(A \otimes A, A)$, it is natural to do so by starting with an appropriate element of $\mathrm{Nat}(A \otimes A \otimes F, F)$.

We first consider $\theta_A(\mathrm{id}) = \mathrm{Nat}(A \otimes F, F)$. In the case of lemma 5.18, $\mathcal{C} = {}_A\mathcal{M}$ and objects X are representations of A. In this case we find $\theta_A(\mathrm{id})_X(a \otimes x) = a \triangleright x$. Thus $\theta_A(\mathrm{id})$ is nothing but the representation map α by which A acts on representations X. In this same vein, for general $\mathcal{C}$ we denote the linear maps $(\theta_A(\mathrm{id}))_X$ by $\alpha_X : A \otimes F(X) \to F(X)$. This is the natural transformations on which we base the rest of our reconstructions.
To obtain a multiplication $\cdot$, consider

$$\alpha_X \circ (\mathrm{id} \otimes \alpha_X) : A \otimes A \otimes F(X) \to F(X).$$

Under $\theta^{-1}_{A \otimes A}$ this corresponds to a map $A \otimes A \to A$, which we define to be the multiplication on A. Similarly, to obtain the unit $\eta : k \to A$ we take the image of $\mathrm{id} \in \mathrm{Nat}(F, F)$ under θ^{-1}_k. This multiplicative structure is more-over such that the maps $\alpha_X : A \otimes F(X) \to F(X)$ are indeed actions of A on the vector spaces $F(X)$.

For the next theorems, we need a more general version of the representability that a generalized forgetful functor exhibits. Namely, in the same spirit as G we can define a functor

$$G^n : \mathtt{Vec} \to \mathtt{Set} : V \mapsto \mathrm{Nat}(V \otimes F^n, F^n),$$

where $F^n(X_1, \ldots X_n) = F(X_1) \otimes \cdots \otimes F(X_n)$. In what follows, we may need to assume that these functors are representable by $H^{\otimes n}$, where H is the representing object of G. This amounts to functorial isomorphisms

$$\theta^n_V : \mathrm{Hom}(V, H^{\otimes n}) \to \mathrm{Nat}(V \otimes F^n, F^n).$$

From now on, we define a generalized forgetful functor to be such that *each* G^n is indeed representable by $H^{\otimes n}$, i.e. such that each θ^n exists. For our applications it is important to note in the spirit of lemma 5.18 that the forgetful functor $F : {}_A\mathcal{M} \to \mathtt{Vec}$ satisfies this condition. Namely, the appropriate functorial isomorphisms are given by:

$$(\theta^n_V(\phi))_{X_1,\ldots,X_n}(v, x_1, \ldots x_n) = \sum \phi(v)^{(1)} \triangleright x_1 \otimes \cdots \otimes \phi(v)^{(n)} \triangleright x_n.$$

Here $\phi \in \mathrm{Hom}(V, H^{\otimes n})$, and $X_1, \ldots X_n$ are objects of ${}_A\mathcal{M}$. As before, $\phi(v)^{(i)}$ denotes the i-th tensor component of $\phi(v)$. The proof of this statement is analogous to that of lemma 5.18, but more cumbersome. Hence it is omitted.

This extra assumption allows us to state the analogue of proposition 5.20 for bialgebras, which means the reconstruction result corresponding to proposition 5.13:

Proposition 5.21. Let $(\mathcal{C}, \otimes)$ be a monoidal category equipped with a generalized forgetful functor F, and let H be the representing object of G. Since F is a monoidal functor, we have a collection of functorial isomorphisms

$$c_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y).$$

Now define the natural transformation $\Delta' \in \mathrm{Nat}(H \otimes F^2, F^2)$ as

$$\Delta'_{X,Y} = c^{-1}_{X,Y} \circ \alpha_{X \otimes Y} \circ c_{X,Y} : H \otimes F(X) \otimes F(Y) \to F(X) \otimes F(Y).$$

Then H becomes a bialgebra with the coproduct $\Delta = (\theta^2_H)^{-1}(\Delta')$.

Next we state the reconstruction result concerning braided categories and quasitriangularity; this is the result corresponding to proposition 5.14:

Proposition 5.22. Let $(\mathcal{C}, \otimes, \Psi)$ be a braided monoidal category equipped with a generalized forgetful functor F, and let H be the representing object of G equipped with the bialgebra structure from proposition 5.21. Define the natural transformation $\mathcal{R}'_{X,Y} \in \mathrm{Nat}(k \otimes F^2, F^2)$ as

$$\mathcal{R}'_{X,Y} = \tau^{-1}_{F(X),F(Y)} \circ c^{-1}_{Y,X} \circ F(\Psi_{X,Y}) \circ c_{X,Y}.$$

Then H is quasitriangular with $\mathcal{R} = (\theta^2_k)^{-1}(\mathcal{R}') : k \to H \otimes H$. Recall here that a morphism $k \to H \otimes H$ just corresponds to an element of $H \otimes H$, so that we can indeed think of $\mathcal{R}$ as an element of $H \otimes H$.

Finally, for a rigid category we revover an antipode. This corresponds to proposition 5.15.

Proposition 5.23. Let $(\mathcal{C}, \otimes, \Psi, \cdot^*)$ be a rigid braided monoidal category equipped with a generalized forgetful functor F, and let H be the representing object of G equipped with the bialgebra structure from proposition 5.21. Using that F is a monoidal functor, it is immediate to check that the triplet

$$\Big(\quad F(X^*) \quad , \quad F(\mathrm{ev}_X) \circ c_{X^*,X} \quad , \quad c^{-1}_{X,X^*} \circ F(\mathrm{coev}_X) \quad \Big)$$

forms a dual for $F(X)$. (Here $\mathrm{ev}_X, \mathrm{coev}_X$ are morphisms in $\mathcal{C}$.) Hence by lemma 3.24 we have isomorphisms

$$d_X : F(X^*) \to F(X)^*,$$

namely d_X is explicitely given by

$$d_X = (F(\mathrm{ev}_X) \circ c_{X^*,X} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coev}_{F(X)}).$$

Now define the natural transformation $S' \in \mathrm{Nat}(F, F)$ as

$$
\begin{aligned}
S'_X = &(\mathrm{id}_{F(X)} \otimes \mathrm{ev}_{F(X)}) \circ (\mathrm{id}_{F(X)} \otimes d_X \otimes \mathrm{id}_{F(X)}) \circ (\mathrm{id}_{F(X)} \otimes \alpha_{X^*} \otimes \mathrm{id}_{F(X)}) \\
&\circ (\mathrm{id}_H \otimes \mathrm{id}_{F(X)} \otimes d_X \otimes \mathrm{id}_{F(X)}) \circ (\tau_{H \otimes F(X)} \otimes \mathrm{id}_{F(X^*)} \otimes \mathrm{id}_{F(X)}) \circ (\mathrm{id}_H \otimes \mathrm{coev}_{F(X)} \otimes \mathrm{id}_{F(X)}),
\end{aligned}
$$

where $\alpha_{X^*} = (\theta_H(\mathrm{id}))_X$ as in proposition 5.20. For clarity, this map is shown as a commutative diagram below (with the id's omitted):

$$
\begin{tikzcd}
 & H \otimes F(X) \otimes F(X)^* \otimes F(X) \arrow[r, "\tau_{H,F(X)}"] & F(X) \otimes H \otimes F(X)^* \otimes F(X) \arrow[d, "d^{-1}"] \\
H \otimes F(X) \arrow[ur, "\mathrm{coev}_{F(X)}"] \arrow[dd, "S'_X"'] & & F(X) \otimes H \otimes F(X^*) \otimes F(X) \arrow[d, "\alpha_{X^*}"] \\
 & & F(X) \otimes F(X^*) \otimes F(X) \arrow[dl, "d"] \\
F(X) & F(X) \otimes F(X)^* \otimes F(X) \arrow[l, "\mathrm{ev}_{F(X)}"] &
\end{tikzcd}
$$

With S' as such, H is a Hopf algebra with the antipode S given by $(\theta_H)^{-1}(S') : H \to H$.

Now that we can construct categories from Hopf algebras and vice versa, we are finally in the position to state the *reconstruction theorem*:

Theorem 5.24. ***(The Reconstruction Theorem)***: Let H be a Hopf algebra, and take $\mathcal{C} = {}_H\mathcal{M}$ in the above to construct a Hopf algebra H'. Then $H \cong H'$ as Hopf algebras, and as quasitriangular Hopf algebras in the case that ${}_H\mathcal{M}$ is braided (i.e. if H is quasitriangular).

This theorem is the culmination of all our work in this subsection. Again, a proof is given in [11]. The reconstruction theorem essentially tells us that a braided rigid monoidal structure is all we can hope to have on the representation category of a quasitriangular Hopf algebra, in general. In this sense the reconstruction theorem gives a rather powerful characterization.

6 Braided Geometry and q-Deformation

In this section we will discuss in detail how braided groups can be used for a systematic q-deformation of physics, resulting in the formalism of *braided geometry*. In theory, any number of q-deformations of a given algebra are possible, and there is no clear choice for any single deformation. The formalism of braided geometry gives us a structural approach to naturally pick a q-deformation. This is because braided geometry essentially provides a simultaneous q-deformation of the entire *category* of vector spaces. After this entire category has been deformed we can carry out our favourite vector space constructions: we will discuss the construction of familiar objects in this q-deformed vector spaces, such as differentiation and exterior derivatives.

6.1 Braided Vectors and Co-vectors

We start by giving a q-deformation of the standard coordinates of k^n, where k is an arbitrary field. More precisely, we will give a deformation of the algebra of coordinates that is generated by the coordinates of space. This is just the free algebra $\langle 1, x_1, \dots x_n \rangle$ modulo the relations $x_i x_j = x_j x_i$ for all i, j. In other words, this is the function algebra of polynomials in $x_1, \dots, x_n$. We will make no restriction on the field k, but to model spacetime we will eventually take $k = \mathbb{R}$ or $k = \mathbb{C}$. Our q-deformed algebras of interested will be Hopf algebra objects in a braided category. The construction of these objects will be facilitated by solutions of the *quantum Yang-Baxter equation*:

Definition 6.1. Let R be a linear automorphism of $V \otimes V$. We say that R is a solution of the *Yang-Baxter equation* (YBE) if

$$(R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) = (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R), \tag{6.1}$$

as automorphisms of $V \otimes V \otimes V$.
We say that R is a solution of the *Quantum* Yang-Baxter equation (QYBE) if

$$R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12}, \tag{6.2}$$

as automorphisms of $V \otimes V \otimes V$. Here $R_{ij} \in \mathrm{Aut}(V \otimes V \otimes V)$ means R acting on the i-th and j-th component; this is all as seen before in definition 4.4. A solution of the QYBE is also called an *R-matrix*.

The following lemma from [33] is useful to check the QYBE:

Lemma 6.2. Let $P \in \mathrm{Aut}(V \otimes V)$ be given by $P(v \otimes w) = w \otimes v$. Then R solves the QYBE if and only if $P \circ R$ solves the YBE.

Example 6.3. Let $V = \langle x \rangle$. Then $x \otimes x$ is a basis for $V \otimes V$. Let R be given by $R(x \otimes x) = qx \otimes x$ for $q \in \mathbb{R}$. Then R solves both the YBE and the QYBE.

Example 6.4. Let $V = \langle x, y \rangle$, and let $\{x, y\}$ be the standard basis for V. Then $\mathbf{b} = \{x \otimes x, x \otimes y, y \otimes x, y \otimes y\}$ is a basis for $V \otimes V$. Let R be the automorphism whose matrix with respect to $\mathbf{b}$ given by

$$R = \begin{bmatrix} q^2 & 0 & 0 & 0 \\ 0 & q & q^2-1 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q^2 \end{bmatrix},$$

where $q \in \mathbb{R}$. Then R satisfies the QYBE. Indeed: the matrix of PR is given by

$$PR = \begin{bmatrix} q^2 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q & q^2-1 & 0 \\ 0 & 0 & 0 & q^2 \end{bmatrix}.$$

It is trivial to check (by computer or by hand) that $(PR \otimes I_2)(I_2 \otimes PR)(PR \otimes I_2) = (I_2 \otimes PR)(PR \otimes I_2)(I_2 \otimes R)$ as matrices, where I_2 is the matrix of id_V and $\otimes$ is the Kronecker product. This directly implies that $P \circ R$ is a solution of equation (6.1).

Remark 6.5. The matrix R from example 6.4 is a 4×4 matrix that represents an automorphism of $V \otimes V$, where V is two-dimensional. This means that R actually represents a Kronecker product of two 2×2 matrices. Thus R should not have two indices $R^i{}_j$ that run from 1 to 4 like a regular matrix, but should instead have four indices running from 1 to 2. As such, we adopt the following notation:

$$R = \begin{bmatrix} R^1{}_1{}^1{}_1 & R^1{}_1{}^1{}_2 & R^1{}_2{}^1{}_1 & R^1{}_2{}^1{}_2 \\ R^1{}_1{}^2{}_1 & R^1{}_1{}^2{}_2 & R^1{}_2{}^2{}_1 & R^1{}_2{}^2{}_2 \\ R^2{}_1{}^1{}_1 & R^2{}_1{}^1{}_2 & R^2{}_2{}^1{}_1 & R^2{}_2{}^1{}_2 \\ R^2{}_1{}^2{}_1 & R^2{}_1{}^2{}_2 & R^2{}_2{}^2{}_1 & R^2{}_2{}^2{}_2 \end{bmatrix}.$$

By design, this is reminiscent of the Kronecker product. The notation is extended to $n^2 \times n^2$ matrices analogously. Note in the case $n = 2$ that the matrix of P is given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies PR = \begin{bmatrix} R^1{}_1{}^1{}_1 & R^1{}_1{}^1{}_2 & R^1{}_2{}^1{}_1 & R^1{}_2{}^1{}_2 \\ R^2{}_1{}^1{}_1 & R^2{}_1{}^1{}_2 & R^2{}_2{}^1{}_1 & R^2{}_2{}^1{}_2 \\ R^1{}_1{}^2{}_1 & R^1{}_1{}^2{}_2 & R^1{}_2{}^2{}_1 & R^1{}_2{}^2{}_2 \\ R^2{}_1{}^2{}_1 & R^2{}_1{}^2{}_2 & R^2{}_2{}^2{}_1 & R^2{}_2{}^2{}_2 \end{bmatrix}.$$

Hence $(PR)^a{}_i{}^b{}_j = R^b{}_i{}^a{}_j$. This also holds for $n > 2$.

Now that we have developed the machinery of R-matrices, we are in the position to define *braided covector-algebras*. These are q-deformations of n-dimensional coordinate algebras.

Definition 6.6. Let $V = \langle x_1, \ldots, x_m \rangle$. Suppose that R is invertible and solves the Quantum YBE, and let $R' \in \mathrm{Aut}(V \otimes V)$ be such that the following equations are satisfied:

$$R_{12} \circ R_{13} \circ R'_{23} = R'_{23} \circ R_{13} \circ R_{12}, \quad R_{23} \circ R_{13} \circ R'_{12} = R'_{12} \circ R_{13} \circ R_{23}, \tag{6.3}$$

$$(P \circ R + I)(P \circ R' - I) = 0, \tag{6.4}$$

$$R_{21} \circ R' = R'_{21} \circ R. \tag{6.5}$$

Here I is the identity, and R_{21} means R with swapped inputs: it can also be read as $R \circ P$. The equations of (6.3) are also called the *mixed Yang-Baxter equations*.

If (R', R) satisfy these conditions, then we can define the *braided covector-algebra* $V\check{}(R', R)$ as $\langle 1, x_1, \ldots, x_m \rangle$ modulo the relations

$$x_i x_j = x_b x_a R'^a{}_i{}^b{}_j.$$

This algebra of covectors then becomes a braided group with the following coproduct, antipode, and braiding:

$$\Delta x_i = x_i \otimes 1 + 1 \otimes x_i, \quad \epsilon x_i = 0, \quad S x_i = -x_i,$$
$$\Psi(x_i \otimes x_j) = x_b \otimes x_a R^a{}_i{}^b{}_j.$$

A coproduct of this additive form is also referred to as a *coaddition*. We will see later that it is the crucial ingredient needed to define q-deformed differentiation.

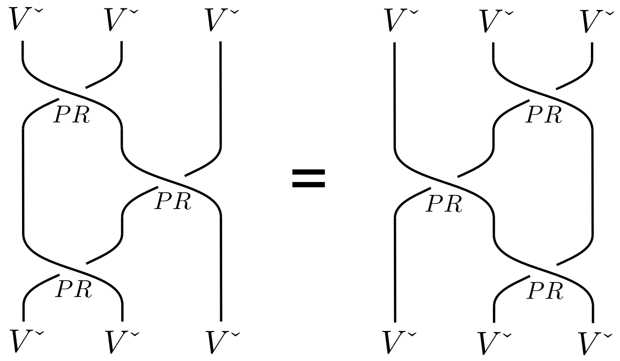
The proof that this indeed defines a braided group comes down to simple calculations. The most crucial of these calculations is postponed until we have developed some more notation, at the end of this subsection.

Remark 6.7. The braided covector-algebra $V\breve{}(R', R)$ is built from two matrices: R' specifies the multiplication rules for the q-deformed coordinates, and R specifies the braiding on $V\breve{}(R', R) \otimes V\breve{}(R', R)$. Note that aside from this possibly nontrivial braiding, $V\breve{}(R', R)$ is essentially a vector space. Thus $V\breve{}(R', R)$ lives in a braided category that is at least very similar to `Vec`. Of course, to properly define $V\breve{}(R', R)$ we must specify exactly what this braided category is. However, this is a complicated matter, and we will not currently need this detail. We will discuss the matter of the underlying braided category in the next subsection, after we have seen some examples.

Remark 6.8. It is not immediately clear why R and R' must satisfy the relations that they do. It turns out that equations (6.3), (6.4), and (6.5) are needed for the multiplication laws and the prescribed braiding to satisfy the definition of a braided group. However, we do not seem to need the fact that R solves the QYBE anywhere. This is a subtle matter: it turns out that we need R to solve the QYBE in order to define a proper braided category. Again, this is deferred to the next subsection. However, we can already discuss one role that the YBE plays in definition 6.6: since $(PR)^a{}_i{}^b{}_j = R^b{}_i{}^a{}_j$ we have that

$$\Psi(x_i \otimes x_j) = x_a \otimes x_b (PR)^a{}_i{}^b{}_j.$$

Hence PR is the matrix of the braiding Ψ. The requirement that R solves the Quantum YBE implies that PR solves the YBE. Since PR is the matrix of Ψ we can treat every application of PR as an application of Ψ. The YBE thus translates to a braided diagrams as:



This is precisely the braid relation. As such, the YBE already indicates that the Ψ we've defined on $V\breve{}(R', R)$ is indeed a valid braiding.

Now for some concrete examples of braided covector-algebras:

Example 6.9. *The braided line*: The simplest example of a braided group is the braided line B. This is the algebra $\langle 1, x\rangle$, endowed with the following braided group structure:

$$\Delta x = x \otimes 1 + 1 \otimes x, \quad \epsilon(x) = 0, \quad Sx = -x, \quad \Psi(x^m \otimes x^n) = q^{mn} x^n \otimes x^m,$$

extended linearly in the case of Δ, ϵ, and anti-linearly in the case of S. This braided group structure is that of a braided covector-algebra.

Proof. We take the matrix $R = [q]$ from example 6.3 and let $R' = [1]$, then substitute this into definition 6.6. $\square$

Example 6.10. *The quantum plane*: The standard quantum plane $\mathbb{C}_q^{2|0}$ is the algebra $\langle 1, x, y\rangle$ modulo the relation $yx = qxy$, endowed with the following braided group structure:

$$\Delta x = x \otimes 1 + 1 \otimes x, \quad \Delta y = y \otimes 1 + 1 \otimes y,$$

$$\epsilon x = \epsilon y = 0, \quad Sx = -x, \quad Sy = -y,$$

$$\Psi(x\otimes x)=q^2x\otimes x,\quad \Psi(x\otimes y)=qy\otimes x,\quad \Psi(y\otimes y)=q^2y\otimes y,$$
$$\Psi(y\otimes x)=qx\otimes y+(q^2-1)y\otimes x.$$

This braided group structure is again that of a braided covector-algebra.

Proof. We take the matrix R from example 6.4 and let $R'=q^{-2}R$, then substitute this into definition 6.6. In example 6.4 we have seen that R satisfies the quantum YBE. This immediately implies equations (6.3) since R,R' are proportional. Equations (6.4) and (6.5) are easily checked in matrix form by computer like in example 6.4, since we know the matrices of R, R', and P. Note that

$$x_ix_j=x_bx_aR'^a{}_i{}^b{}_j=x_ax_b(PR')^b{}_i{}^a{}_j.$$

This relation can be written more concisely as a covector-matrix multiplication:

$$\begin{bmatrix}xx & xy & yx & yy\end{bmatrix}=\begin{bmatrix}xx & xy & yx & yy\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&q^{-1}&0\\0&q^{-1}&1-q^{-2}&0\\0&0&0&1\end{bmatrix}.$$

(Incidentally, this is why we refer to $V\check{}(R',R)$ as a covector algebra.) In this way, PR' serves as the matrix for multiplication: in precisely the same way that PR is the matrix for the braiding. Let's see that we indeed recover the relation $yx=qxy$: from the second column of the covector-matrix multiplication we read off $xy=q^{-1}yx$ which indeed implies $yx=qxy$. From the third column we obtain

$$yx=q^{-1}xy+(1-q^{-2})yx\implies q^{-2}yx=q^{-1}xy\implies yx=qxy,$$

which is indeed consistent.

The braiding as given is obtained analogously from R. □

Example 6.11. *The Fermionic Quantum Plane*: Related to the previous example is the Fermionic quantum plane $\mathbb{C}_q^{0|2}$, which is the braided covector-algebra with

$$R=-q^{-2}\begin{bmatrix}q^2&0&0&0\\0&q&q^2-1&0\\0&0&q&0\\0&0&0&q^2\end{bmatrix},\quad R'=q^2R.$$

This generates an algebra $\langle 1,\theta,\phi\rangle$ modulo the relations

$$\theta^2=\phi^2=0,\qquad \phi\theta=-q^{-1}\theta\phi.$$

Hence the name *fermionic* quantum plane: as $q\to 1$ this algebra is anti-commutative.

Note that the matrices R,R' here are the same as for the standard quantum plane, but with $R\leftrightarrow -R'$. In this case, if (R',R) solves them then so does $(-R,-R')$, simply because R' is proportional to R. Thus the equations for R',R are symmetric under $R\leftrightarrow -R'$. We will see later that this same symmetry is also used to construct braided algebras of differential forms.

Dually to a braided covector-algebra we can just as well define a braided algebra of vectors v^i. Note that these have upper indices, as opposed to the lower indices of covectors.

Definition 6.12. Let $V=\langle v^1,\dots,v^m\rangle$, and let R,R' be the same as for $V\check{}(R',R)$, satisfying the same relations. Then we define the *braided vector-algebra* $V(R',R)$ as $\langle 1,v^1,\dots,v^m\rangle$ modulo the relations

$$v^iv^j=R'^i{}_a{}^j{}_bv^bv^a$$

This is a braided group as follows:

$$\Delta v^i=v^i\otimes 1+1\otimes v^i,\quad \epsilon v^i=0,\quad Sv^i=-v^i.$$
$$\Psi(v^i\otimes v^j)=R^i{}_a{}^j{}_bv^b\otimes v^a.$$

Remark 6.13. Note that for the vector algebra we have $v^i v^j = R'^i{}_a{}^j{}_b v^b v^a$: we write the matrix R' on the left, as opposed to on the right for covectors. This signifies that we write vectors as column vectors, and that the braided multiplication is to be read as a matrix-vector multiplication. This is as opposed to the case of covectors, which are written as column vectors and whose multiplication statistics are read as covector-matrix multiplication (as in the proof of example 6.10). Also note analogously to remark 6.5 that

$$v^i v^j = R'^i{}_a{}^j{}_b v^b v^a = (R'P)^i{}_a{}^j{}_b v^a v^b.$$

Hence $R'P$ is the matrix of multiplication with respect to $\{v^1, \dots, v^m\}$.

Example 6.14. Consider the braided vector-algebra $V(R', R)$ with the same R and R' as the regular quantum plane 6.10. It is simple to check that this generates the algebra $\mathbb{C}^{2|0}_{q^{-1}}$: if we denote the generators of $V(R', R)$ by v, w then we have $wv = q^{-1}vw$ and

$$\Psi(v \otimes v) = q^2 v \otimes v, \quad \Psi(w \otimes v) = qv \otimes w, \quad \Psi(w \otimes w) = q^2 w \otimes w,$$

$$\Psi(v \otimes w) = qw \otimes v + (q^2 - 1)v \otimes w.$$

Remark 6.15. To ease the process of working with many indices, we introduce a compact notation for our braided vectors/covectors. This ties into the interpretation of PR' and $R'P$ as the matrices of multiplication for $V\check{}(R', R)$ and $V(R', R)$ respectively.
First, instead of considering separate coordinates x_i, v^j we can arrange them into covectors $\mathbf{x}$ and vectors $\mathbf{v}$:

$$\mathbf{x} = [x_1, \dots, x_n] \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}.$$

The Kronecker product of $\mathbf{x}$ with itself is then

$$\mathbf{x} \otimes \mathbf{x} = [x_1x_1, x_1x_2, \dots, x_1x_n, x_2x_1, \dots, x_nx_{n-1}, x_nx_n].$$

We then define $\mathbf{x}_1$ as the $1 \times n^2$ covector consisting of the first terms of the products in $\mathbf{x} \otimes \mathbf{x}$, and we let $\mathbf{x}_2$ be the $1 \times n^2$ covector consisting of all the second terms. Thus:

$$\mathbf{x}_1 := [x_1, x_1, \dots, x_1, x_2, \dots, x_n, x_n], \qquad \mathbf{x}_2 := [x_1, x_2, \dots, x_n, x_1, \dots, x_{n-1}, x_n].$$

This notation is continued analogously for higher tensor powers of $\mathbf{x}$: in the case of $\mathbf{x}^{\otimes m}$ we let $\mathbf{x}_i$ be the $1 \times n^m$ covector containing the i-th product term of $\mathbf{x}^{\otimes m}$. Finally, we let products like $\mathbf{x}_1\mathbf{x}_2$ denote the Hadamard (i.e. element-wise) product, so that $\mathbf{x}_1\mathbf{x}_2 = \mathbf{x} \otimes \mathbf{x}$. Similarly we let $\mathbf{x}_1 \otimes \mathbf{x}_2$ be the covector of pointwise tensor products:

$$\mathbf{x}_1 \otimes \mathbf{x}_2 := [x_1 \otimes x_1, x_1 \otimes x_2, \dots, x_n \otimes x_{n-1}, x_n \otimes x_n].$$

We can consider the map R as a matrix, and write expression like the braiding on $V\check{}(R', R) \otimes V\check{}(R', R)$ more compactly as:

$$\Psi(\mathbf{x}_1 \otimes \mathbf{x}_2) = \mathbf{x}_2 \otimes \mathbf{x}_1 R,$$

and similarly for e.g. the multiplication relations on $V(R', R)$, etc.

The compact notation of remark 6.15 above allows us to compactly prove that the braided vector- and covector-algebras as we have defined them are indeed valid braided groups. There is much to check here: we only show that Δ as defined is a well-defined algebra homorphism. This is the most non-trivial computation, and also the most relevant since the coaddition form of Δ is the most novel ingredient of the braided covector-algebra construction [6].

For the braided covectors $V\check{}(R', R)$, the statement that Δ is a well-defined algebra homomorphism is exactly that the elements Δx_i obey the same multiplicative relations as x_i [6], [34]. In our compact notation, this means

$$(\Delta\mathbf{x}_1)(\Delta\mathbf{x}_2) = (\Delta\mathbf{x}_2)(\Delta\mathbf{x}_1)R'.$$

To show this, we compute that

$$\begin{aligned}(\Delta\mathbf{x}_1)(\Delta\mathbf{x}_2) &= (\mathbf{x}_1 \otimes 1 + 1 \otimes \mathbf{x}_1)(\mathbf{x}_2 \otimes 1 + 1 \otimes \mathbf{x}_2)\\ &= \mathbf{x}_1\mathbf{x}_2 \otimes 1 + 1 \otimes \mathbf{x}_1\mathbf{x}_2 + \mathbf{x}_1 \otimes \mathbf{x}_2 + \Psi(\mathbf{x}_1 \otimes \mathbf{x}_2)\\ &= \mathbf{x}_1\mathbf{x}_2 \otimes 1 + 1 \otimes \mathbf{x}_1\mathbf{x}_2 + \mathbf{x}_1 \otimes \mathbf{x}_2 + \mathbf{x}_2 \otimes \mathbf{x}_1 R.\end{aligned}$$

Recall here that the multiplication in $V\check{}(R', R) \otimes V\check{}(R', R)$ is as defined by lemma 3.26, hence the Ψ in the fourth term. Next we compute

$$\begin{aligned}(\Delta\mathbf{x}_2)(\Delta\mathbf{x}_1)R' &= (\mathbf{x}_2 \otimes 1 + 1 \otimes \mathbf{x}_2)(\mathbf{x}_1 \otimes 1 + 1 \otimes \mathbf{x}_1)R'\\ &= \mathbf{x}_2\mathbf{x}_1 R' \otimes 1 + 1 \otimes \mathbf{x}_2\mathbf{x}_1 R' + \mathbf{x}_2 \otimes \mathbf{x}_1 R' + \mathbf{x}_1 \otimes \mathbf{x}_2 R_{21}R'.\end{aligned}$$

These two computations give equal results if and only if

$$\mathbf{x}_1 \otimes \mathbf{x}_2(R_{21}R' - 1) = \mathbf{x}_2 \otimes \mathbf{x}_1(R - R').$$

In turn, this is satisfied if we have

$$R_{21}R' - 1 = P(R - R'),$$

which is exactly the condition of equation 6.4. This completes our proof. The proof for braided vectors is analogous, except that the matrices R, R' now appear on the left of terms like $\mathbf{v}_1 \otimes \mathbf{v}_2$ in accordance with their vector nature. For more details see [35].

6.2 Background Quantum Groups

In this subsection we will discuss the braided categories in which all of our q-deformed constructions will reside. So far we have only discussed braidings Ψ on two copies of the same braided group. With respect to the natural transformation Ψ on the whole category they are of the form $\Psi_{B,B}$. Knowledge of the category we work in is particularly necessary to deduce the braid statistics that exist between different braided groups; for instance between braided vectors and covectors. These are 'external braidings' of the form $\Psi_{B,C}$.

In short, the braided category of $V\check{}(R', R)$ and $V(R', R)$ is the category of corepresentations of a Hopf algebra. This Hopf algebra is the algebra of *quantum matrices* $A(R)$, where R is the same R-matrix that prescribes the braidings on $V\check{}(R', R)$ and $V(R', R)$ that we have now put in manually. It turns out that this corepresentation category is braided precisely because $A(R)$ is *dual quasi-triangular*. In the context of q-deformed physics, $A(R)$ is also known as a *background quantum group*. Everything the background quantum group coacts on is endowed with so-called *braid statistics* (which is just another term for the specifics of Ψ).

The reason for the nomenclature of 'background quantum group' is two-fold: on the one hand, this background group is a Hopf algebra. Hopf algebras are also referred to as quantum groups; in particular the Drinfeld-Jimbo type Hopf algebras. More importantly, the background quantum group is an underlying structure that is intrinsically connected to the q-deformed vector spaces we consider. In fact, it turns out that the background quantum group generates the underlying q-deformed symmetry group of these spaces [6], [36]! Thus the background quantum group parallels the role of the underlying symmetry group of e.g. $\mathbb{R}^n$ or Minkowski space. In this analogy the quantum matrices $A(R)$ play the same role as the matrices $\mathrm{SL}(2, \mathbb{C})$ do for Minkowski space. We will see more of this in section 7.

There is a lot to unpack in the above: we have to

- Define corepresentations / coactions.
- Define dual quasitriangular structures.
- Show that the corepresentation category of a dual quasitriangular Hopf algebra is braided.
- Define $A(R)$, show that it is dual quasitriangular, and compute the braiding.

Definition 6.16. A *coaction* or *corepresentation* of a coalgebra H consists of a vector space V together with a morphism $\beta : V \to V \otimes H$ such that

$$(\beta \otimes \mathrm{id}) \circ \beta = (\mathrm{id} \otimes \Delta) \circ \beta \qquad \text{and} \qquad (\mathrm{id} \otimes \epsilon) \circ \beta = \mathrm{id}.$$

These axioms are depicted as commutative diagrams below:

$$\begin{tikzcd}
 & V \otimes H \otimes H & \\
V \otimes H \arrow[ur, "\beta\otimes\mathrm{id}"] & & V \otimes H \arrow[ul, "\mathrm{id}\otimes\Delta"'] \\
 & V \arrow[ul, "\beta"] \arrow[ur, "\beta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
V \otimes H \arrow[d, "\mathrm{id}\otimes\epsilon"'] & \\
V \otimes k \arrow[r, Rightarrow] & V \arrow[ul, "\beta"']
\end{tikzcd}$$

Note that these are indeed just the axioms of an action by an algebra, but with the arrows reversed and $(\cdot, \eta)$ interchanged with (Δ, ϵ).

If we write β explicitly as $\beta(v) = \sum v^{(\bar{1})} \otimes v^{(\bar{2})}$ then these axioms become

$$\sum v^{(\bar{1})(\bar{1})} \otimes v^{(\bar{1})(\bar{2})} \otimes v^{(\bar{2})} = \sum v^{(\bar{1})} \otimes v^{(\bar{2})}{}_{(1)} \otimes v^{(\bar{2})}{}_{(2)},$$

$$\sum v^{(\bar{1})} \epsilon\left(v^{(\bar{2})}\right) = v.$$

The notion of a dual quasitriangular structure is a similarly arrow-reversed notion, dual to that of a quasitriangular structure: 'we think of a quasitriangular structure as a map $k \to H \otimes H$ and revert all the arrows' [19, p. 47].

Definition 6.17. An element $\mathcal{R} : H \otimes H \to k$ is *convolution-invertible* if there exists a morphism $\mathcal{R}^{-1} : H \otimes H \to k$ such that

$$\sum \mathcal{R}^{-1}(h_{(1)} \otimes g_{(1)}) \mathcal{R}(h_{(2)} \otimes g_{(2)}) = \epsilon(h)\epsilon(g) = \sum \mathcal{R}(h_{(1)} \otimes g_{(1)}) \mathcal{R}^{-1}(h_{(2)} \otimes g_{(2)}), \qquad (6.6)$$

for all $h, g \in H$.

A bialgebra or Hopf algebra H is *dual quasitriangular* if there exists a morphism $\mathcal{R} : H \otimes H \to k$ such that for all $h, g, f \in H$

- $\mathcal{R}$ is convolution-invertible and obeys

$$\sum b_{(1)} a_{(1)} \mathcal{R}(a_{(2)} \otimes b_{(2)}) = \sum \mathcal{R}(a_{(1)} \otimes b_{(1)}) a_{(2)} b_{(2)}. \qquad (6.7)$$

- The following identities hold for $\mathcal{R}$:

$$\mathcal{R}(hg \otimes f) = \sum \mathcal{R}(h \otimes f_{(1)}) \mathcal{R}(g \otimes f_{(2)}), \quad \mathcal{R}(h \otimes gf) = \sum \mathcal{R}(h_{(1)} \otimes f) \mathcal{R}(h_{(2)} \otimes g). \quad (6.8)$$

(Recall here the Sweedler notation for Δ.)

Analogously to the representation category ${}_H\mathcal{M}$ of a Hopf algebra of bialgebra H we can define the corepresentation category $\mathcal{M}^H$. This category has the vector spaces V of coactions as objects, hence $\mathcal{M}^H$ is a subcategory of `Vec`. Similarly to ${}_H\mathcal{M}$, the morphisms are linear maps that suitably commute with the coaction β.

Lemma 6.18. Let H be a dual quasitriangular Hopf algebra or bialgebra. Then the category $\mathcal{M}^H$ is braided with braiding

$$\Psi_{V,W}(v \otimes w) = \sum w^{(\bar{1})} \otimes v^{(\bar{1})} \mathcal{R}(v^{(\bar{2})} \otimes w^{(\bar{2})}),$$

extended linearly.

Proof. This statement is precisely the dual statement of lemma 5.14. Thus the proof is dual to that of lemma 5.14: it is obtained by reverting all the arrows and interchanging $(\cdot, \eta)$ with (Δ, ϵ). $\square$

Next, we define the background quantum group $A(R)$:

Definition 6.19. Let R be an element of $\mathrm{Aut}(V \otimes V)$, where $V = \langle x_1, \ldots x_n \rangle$, so that $V \otimes V$ is the algebra generated by n^2 indeterminates $\{t^i{}_j\}$. Then the bialgebra $A(R)$ is the algebra generated by 1 and $\{t^i{}_j\}$ modulo the relations

$$R^i{}_a{}^k{}_b t^a{}_j t^b{}_l = t^k{}_b t^i{}_a R^a{}_j{}^b{}_l.$$

A bialgebra structure on $A(R)$ is provide by Δ and ϵ as follows:

$$\Delta(t^i{}_j) = t^i{}_a \otimes t^a{}_j, \quad \epsilon t^i{}_j = \delta^i{}_j.$$

$A(R)$ is usually defined simply as a bialgebra, but can be made into a proper Hopf algbra with an antipode S [37]. We will not need an explicit formulation of S, but when handling vectors we will make use of its existence.

The quantum matrices $A(R)$ indeed coact on braided vectors and covectors, namely via the coactions

$$x_i \mapsto x_a \otimes t^a{}_i \qquad \text{and} \qquad v^i \mapsto v^a \otimes St^i{}_a, \tag{6.9}$$

with the Einstein summation convention on the repeated indices. To define an exterior algebra, we will only need the former. To deduce a braided tensor product $\underline{\otimes}$ between vectors and covectors, we need a dual quasitriangular structure on $A(R)$. This is the content of the following theorem:

Theorem 6.20. Let R be an invertible solution to the quantum YBE. Then $A(R)$ is dual quasitriangular with

$$\mathcal{R}(t^i{}_j, t^k{}_l) = R^i{}_j{}^k{}_l. \tag{6.10}$$

Proof. This is theorem 4.1.5 in [11]. $\square$

This theorem is why we required the matrix R in $V\check{}(R', R)$ to be a solution to the quantum YBE in the first place! Without this assumption, we cannot give a dual quasitriangular on $A(R)$ and hence $\mathcal{M}^{A(R)}$ is not braided. As stated before, we *define* the braided vector- and covector algebras to be Hopf algebra elements of the category $\mathcal{M}^{A(R)}$. We now show that the braiding on $V\check{}(R', R) \otimes V\check{}(R', R)$ induced by $A(R)$ agrees with the one put in place by definition 6.6:

Lemma 6.21. The braiding Ψ on $V\check{}(R', R) \otimes V\check{}(R', R)$ in the category $\mathcal{M}^{A(R)}$ is given by

$$\Psi(x_i \otimes x_j) = x_b \otimes x_a R^a{}_i{}^b{}_j.$$

Proof. Applying lemma 6.18 to the coaction of $A(R)$ on $V\check{}(R', R)$ given in (6.9), we find the induced braiding to be

$$\begin{aligned}\Psi(x_i \otimes x_j) &= x_b \otimes x_a \mathcal{R}(t^a{}_i, t^b{}_j) \\ &= x_b \otimes x_a R^a{}_i{}^b{}_j,\end{aligned}$$

where the second equality follows from theorem 6.20. $\square$

Next we give the two external braidings between $V^{\check{}}(R', R)$ and $V(R', R)$. These need some additional work; first we must have that the R-matrix of $A(R)$ is *bi-invertible*:

Definition 6.22. An $n^2 \times n^2$ matrix $[R^i{}_j{}^k{}_l]$ is *bi-invertible* if it:

- Has an inverse, i.e. a matrix R^{-1} such that
$$(R^{-1})^i{}_a{}^k{}_b R^a{}_j{}^b{}_l = \delta^i{}_j \delta^k{}_l = R^i{}_a{}^k{}_b (R^{-1})^a{}_j{}^b{}_l.$$
- Has a *second inverse*, i.e. a matrix $\widetilde{R}$ such that
$$\widetilde{R}^i{}_a{}^b{}_l R^a{}_j{}^k{}_b = \delta^i{}_j \delta^k{}_l = R^i{}_a{}^b{}_l \widetilde{R}^a{}_j{}^k{}_b.$$

  Here $\delta^i{}_j \delta^k{}_l$ is really just a product of two delta functions, but the notation of (i, j, k, l) is precisely such that this can also be read as the kronecker product of two delta function matrices, which is simply the four-dimensional identity matrix.

In the case that R is indeed bi-invertible and that we have an antipode S on $A(R)$, we have the following results for $\mathcal{R}$:

Lemma 6.23. The dual quasi-triangular structure on $A(R)$ given in equation (6.10) obeys
$$\mathcal{R}(St^i{}_j, t^k{}_l) = (R^{-1})^i{}_j{}^k{}_l \qquad \text{and} \qquad \mathcal{R}(t^i{}_j, St^k{}_l) = (\widetilde{R})^i{}_j{}^k{}_l. \tag{6.11}$$

Proof. We must first prove the following auxiliary statement about $\mathcal{R}$:
$$\mathcal{R}(a \otimes 1) = \epsilon(a) = \mathcal{R}(1 \otimes a),$$
for all $a \in A(R)$. This holds generally for dual quasitriangular structures. The proof of the first equality is as follows:
$$\begin{aligned}
\mathcal{R}(a \otimes 1) &= \mathcal{R}(\epsilon(a_{(1)})a_{(2)} \otimes 1) = \epsilon(a_{(1)})\mathcal{R}(\epsilon(a_{(2)} \otimes 1) \\
&= \sum \mathcal{R}^{-1}(a_{(1)} \otimes 1)\mathcal{R}(a_{(2)} \otimes 1)\mathcal{R}(a_{(3)} \otimes 1) \\
&= \sum \mathcal{R}^{-1}(a_{(1)} \otimes 1)\mathcal{R}(a_{(2)} \otimes 1 \cdot 1) \\
&= \epsilon(a)\epsilon(1) = \epsilon(a).
\end{aligned}$$

The first equality uses the counit axioms for Hopf algebras. The second equality is by linearity of $\mathcal{R}$. The third and fifth equalities use equation (6.6), and the fourth uses equation (6.7). The final equality follows from $\epsilon(1) = 1$, which holds since ϵ is an algebra morphism by the definition of a Hopf algebra. The proof of the second equality is analogous. Now, for the first stament of the lemma, we compute
$$\mathcal{R}(St^i{}_a, t^k{}_b)\mathcal{R}(t^a{}_j, t^b{}_l) = \mathcal{R}\left((St^i{}_a) \cdot t^a{}_j, t^k{}_l\right).$$

This follows by equation (6.7) since the coproduct in $A(R)$ is given by $\Delta t^k{}_l = t^k{}_b \otimes t^b{}_l$. By the antipode axiom from the definition of a Hopf algebra, we have that $(St^i{}_a) \cdot t^a{}_j = \epsilon(t^i{}_j) = \delta^i{}_j$. We thus conclude that
$$\mathcal{R}(St^i{}_a, t^k{}_b)\mathcal{R}(t^a{}_j, t^b{}_l) = \mathcal{R}(\delta^i{}_j, t^k{}_l).$$

On the right-hand side, if $i \neq j$ the result must be zero since $\mathcal{R} : H \otimes H \to k$ is linear. If $i = j$ the result is $\mathcal{R}(1, t^k{}_l) = \epsilon(t^k{}_l) = \delta^k{}_l$ by the auxiliary statement. Thus in total the right-hand side equals $\delta^i{}_j \delta^k{}_l$. Meanwhile on the left-hand side we have that $\mathcal{R}(t^a{}_j, t^b{}_l) = R^a{}_j{}^b{}_l$ by definition of $\mathcal{R}$. Thus in total we find
$$\mathcal{R}(St^i{}_a, t^k{}_b)R^a{}_j{}^b{}_l = \delta^i{}_j \delta^k{}_l.$$

From definition 6.22 we thus identify

$$\mathcal{R}(St^i{}_a, t^k{}_b) = (R^{-1})^i{}_a{}^k{}_b,$$

as required. The proof of the other statement is completely analogous, but with the indices arranged differently so that we need the second inverse $\widetilde{R}$ instead of R^{-1}. □

Note that as stated before, we do not need any expression for S in this proof. These are exactly the computations that we need to deduce the braidings between $V\check{}(R', R)$ and $V(R', R)$:

Lemma 6.24. The braiding induced by $\mathcal{M}^{A(R)}$ on $x_i \in V\check{}(R', R)$ and $v^j \in V(R', R)$ is given by

$$\Psi(x_i \otimes v^j) = \widetilde{R}^a{}_i{}^j{}_b v^b \otimes x_a. \tag{6.12}$$

Similarly,

$$\Psi(v^i \otimes x_j) = x_b \otimes v^a (R^{-1})^i{}_a{}^b{}_j. \tag{6.13}$$

We can compress these braidings into the compact notation of remark 6.15 as

$$\Psi(\mathbf{x}_1 \otimes (R\mathbf{v}_2)) = \mathbf{v}_2 \otimes \mathbf{x}_1 \qquad \text{and} \qquad \Psi(\mathbf{v}_1 \otimes \mathbf{x}_2) = \mathbf{x}_2 \otimes (R^{-1}\mathbf{v}_1),$$

respectively.

Proof. The braidings are given by lemma 6.18. In the case of $V\check{}(R', R)$ and $V(R', R)$ we use the actions from equation (6.9). The given expressions for Ψ then follow immediately from lemma 6.23.

The compact notations are immediate. Note that the R-matrices now appear *between* $\mathbf{x}$ and $\mathbf{v}$. This is already necessary to make the expressions well-defined, as $\mathbf{x}$ is a row vector and $\mathbf{v}$ a column vector. It also says something about the indices: $\mathbf{x}_1 \otimes (R\mathbf{v}_2)$ translates to $x_a \otimes R^a{}_i{}^j{}_b v^b$. Note here that R has two free indices: one upper and one lower, hence we must use the second inverse $\widetilde{R}$ to contract them. □

Analogously we can find the braiding on $V(R', R) \otimes V(R', R)$ to be the one stated in definition 6.12. There is one more braiding that we will be of particular interest in the next subsection:

Lemma 6.25. The inverse braiding on $V(R', R) \otimes V\check{}(R', R)$ is given by

$$\Psi^{-1}(v^i \otimes x_j) = x_b \otimes v^a R^b{}_j{}^i{}_a.$$

In compact notation, this is expressed as

$$\Psi^{-1}(\mathbf{v}_1 \otimes \mathbf{x}_2) = \mathbf{x}_2 \otimes (R\mathbf{v}_1).$$

Proof. This follows immediately from equation (6.12): we apply Ψ^{-1} to both sides, and multiply both sides with $R^i{}_a{}^b{}_j$ to contract $\widetilde{R}$. We then relabel to find the required expression. □

6.3 Braided Calculus

In this subsection we define braided differentiation. To this end it is helpful to first consider braided differentiation on the simplest braided group: the braided line B. In the case of covector-algebras with more than one coordinate, this extends to an algebra of braided partial derivatives.

Example 6.26. We define braided differentiaion $\partial_q : B \to B$ on the braided line B as

$$\partial_q f = \mathrm{coeff}_{x\otimes}(\Delta f), \qquad \text{i.e. } \Delta f = 1 \otimes f + x \otimes \partial_q f + \dots,$$

for $f \in B$, i.e. f a polynomial in x. Here $\dots$ denotes the higher powers of x in Δf, and $\mathrm{coeff}_{x\otimes}$ denotes 'picking out the component linear in x'. This alone bears some resemblance to the Taylor expansion. More-over, since Δ is a coaddition we can interpret it as an infinitesimal addition as follows: if we write $\underline{x} = x \otimes 1$ and $\underline{y} = 1 \otimes x$ then $\Delta x = \underline{x} + \underline{y}$. For any polynomial $f \in B$ we clearly have that $\Delta f(x) = f(\Delta x) = f(\underline{x} + \underline{y})$ since Δ distributes over products by definition. Similarly it is clear that $1 \otimes f(x) = f(\underline{y})$ by definition of multiplication in $B \otimes B$. In this notation, we thus find

$$f(\underline{x} + \underline{y}) - f(\underline{y}) = x \otimes \partial_q f + \dots.$$

If we let $x^{-1}(\cdot)|_{x=0}$ denote picking out the coefficient in x, then we recover

$$\partial_q f = x^{-1}(f(\underline{x} + \underline{y}) - f(\underline{y}))|_{x=0}.$$

This is very reminiscent of the familiar notion of a derivative: we can interpret Δ as an infinitesimal addition of $\underline{x}$ and then 'divide out' x in the Taylor expansion, neglecting higher order terms, to obtain $\partial_q f$.

We now claim that we recover ordinary differentiation on B in the limit $q \to 1$. To this end, let us compute ∂_q on B explicitly. First recall from section 2 that we define the q-integers as follows:

$$[m]_q = 1 + q + q^2 + \dots + q^{m-1} = \frac{1 - q^m}{1 - q},$$
$$\binom{m}{r}_q = \frac{[m]_q!}{[r]_q![m-r]_q!}.$$

Then for arbitrary $m \geq 1$ we have that

$$\Delta(x^m) = \sum_{r=0}^{m} \binom{m}{r}_q x^r \otimes x^{m-r}. \tag{6.14}$$

(In the notation $\Delta x = \mathbf{x} + \mathbf{y}$ this follows immediately from the *q-binomial formula* [19, Ch. 7].) The proof is by induction. To get an idea of the proof we show the case $m = 2$. The induction step of the general proof proceeds analogously, but is more notationally cumbersome. For $m = 2$ the statement is

$$\Delta(x^2) = 1 \otimes x^2 + (1 + q)x \otimes x + x^2 \otimes 1.$$

To show this we use that Δ distributes over multiplication. In a commutative diagram, this statement amounts to

$$\begin{tikzcd}
B \otimes B \arrow[r, "\cdot"] \arrow[d, "\Delta\otimes\Delta"'] & B \arrow[r, "\Delta"] & B \otimes B \\
B \otimes B \otimes B \otimes B \arrow[rr, "\mathrm{id}\otimes\Psi\otimes\mathrm{id}"'] & & B \otimes B \otimes B \otimes B \arrow[u, "\cdot\otimes\cdot"']
\end{tikzcd}$$

Starting at $x \otimes x \in B \otimes B$, going along the top path gives $\Delta(x^2)$. Recall that $\Psi(x^m \otimes x^n) = q^{mn} x^n \otimes x^m$ in B. Using this one computes that going along the bottom path gives $1 \otimes x^2 + (1 + q)x \otimes x + x^2 \otimes 1$. Thus the claim holds by commutativity of this diagram.

Now note that

$$\binom{m}{1}_q = \frac{[m]_q!}{[1]_q![m-1]_q!} = \frac{[m]_q!}{[m-1]_q!} = [m]_q.$$

Thus equation (6.14) immediately tells us that

$$\partial_q x^m = [m]_q x^{m-1} \quad \xrightarrow{q\to 1} \quad m x^{m-1},$$

by definition of ∂_q. It is clear from k-linearity of Δ that ∂_q is also k-linear. Thus the above shows that we recover ordinary differentiation in the limit $q \to 1$ for all $f \in B$. This also allows us to express ∂_q on B explicitly as

$$\partial_q f(y) = \frac{f(qy) - f(y)}{(q-1)y}.$$

(Coincidentally, this is a familiar q-analogue: the *Jackson derivative* introduced in [38].)

The exact same idea of a braided derivative that works on the braided line also works for general braided covector-algebras:

Definition 6.27. Let $V\check{}(R', R)$ be a braided covector-algebra. The i-th *braided derivative* $\partial_q^i : V\check{}(R', R) \to V\check{}(R', R)$ is defined as

$$\partial_q^i f = \mathrm{coeff}_{x_i \otimes}(\Delta f) = x_i^{-1}(f(\underline{\mathbf{x}} + \underline{\mathbf{y}}) - f(\underline{\mathbf{y}}))|_{\underline{\mathbf{x}}=0}.$$

for $f \in V\check{}(R', R)$ i.e. f a polynomial in $x_1, \ldots, x_m$. Here the notation $\underline{\mathbf{x}}$ is an amalgamation of the boldface notation from remark 6.15 and the underline that indicates implicit tensor products with 1. Thus e.g. $\underline{\mathbf{y}}$ represents $[1 \otimes x_1, 1 \otimes x_2]$ if $n = 2$.

By definition of ∂_q^i it follows that

$$\Delta f = 1 \otimes f + x_1 \otimes \partial_q^1 f + x_2 \otimes \partial_q^2 f + \cdots + x_m \otimes \partial_q^m f + \ldots .$$

We may also simply denote ∂_q^i by ∂^i, if it is clear from context that the differentials are q-deformed.

Example 6.28. Let us compute the values of ∂_q^i on monomials of n-dimensional $V\check{}(R', R)$. This is most conveniently done in the compact notation of remark 6.15. In this notation we can construct the object $\mathbf{x}_1\mathbf{x}_2 \ldots \mathbf{x}_m$, which is a $1 \otimes n^m$ covector consisting of all possible monomials in $V\check{}(R', R)$ of total degree m. Following [34] we can now compute the action of δ_q^i on this object, allowing us to compute ∂_q^i on all monomials of total degree m at once. By definition,

$$\partial_q^i(\mathbf{x}_1\mathbf{x}_2 \ldots \mathbf{x}_m) = \mathrm{coeff}_{a_i \otimes}((\underline{\mathbf{a}}_1 + \underline{\mathbf{x}}_1)(\underline{\mathbf{a}}_2 + \underline{\mathbf{x}}_2) \ldots (\underline{\mathbf{a}}_m + \underline{\mathbf{x}}_m)).$$

Here we choose to denote the infinitesimal addition by $\mathbf{a}$ instead of $\mathbf{x}$. We then expand brackets. Note that we are only interested in terms that are 'linear' in $\mathbf{a}$, so we can ignore all the other terms. Hence we obtain

$$\partial_q^i(\mathbf{x}_1\mathbf{x}_2 \ldots \mathbf{x}_m) = \mathrm{coeff}_{a_i \otimes}(\underline{\mathbf{a}}_1\underline{\mathbf{x}}_2 \ldots \underline{\mathbf{x}}_m + \underline{\mathbf{x}}_1\underline{\mathbf{a}}_2 \ldots \underline{\mathbf{x}}_m + \cdots + \underline{\mathbf{x}}_1\underline{\mathbf{x}}_2 \ldots \underline{\mathbf{a}}_m).$$

We can then move all of the $\underline{\mathbf{a}}$ factors to the left. This uses the braiding on $V\check{}(R', R)$: recall that $\underline{\mathbf{a}} = \mathbf{a} \otimes 1$ and $\underline{\mathbf{x}} = 1 \otimes \mathbf{x}$. Thus switching an $\underline{\mathbf{a}}$ and a $\underline{\mathbf{x}}$ amounts to acting with Ψ. Recalling that the matrix of Ψ is PR, our expression then becomes

$$\mathrm{coeff}_{a_i \otimes}(\underline{\mathbf{a}}_1\underline{\mathbf{x}}_2 \ldots \underline{\mathbf{x}}_m(I + (PR)_{12} + (PR)_{12}(PR)_{23} + \cdots + (PR)_{12}(PR)_{23} \ldots (PR)_{m-1,m})).$$

Recall here that the subscripts on PR indicate on which entries of the m-th tensor power Ψ acts. We now define the m-fold Kronecker product matrix as *generalized braided integers*:

$$[m; R]_{1,\ldots,m} := I + (PR)_{12} + (PR)_{12}(PR)_{23} + \cdots + (PR)_{12}(PR)_{23} \ldots (PR)_{m-1,m}.$$

Thus we now have

$$\partial_q^i(\mathbf{x}_1\mathbf{x}_2\dots\mathbf{x}_m) = \mathrm{coeff}_{a_i\otimes}(\underline{\mathbf{a}}_1\underline{\mathbf{x}}_2\dots\underline{\mathbf{x}}_m[m;R]_{1,\dots,m}).$$

Finally, we can evaluate this to give

$$\partial_q^i(\mathbf{x}_1\mathbf{x}_2\dots\mathbf{x}_m) = \mathbf{e}_1^i\mathbf{x}_2\dots\mathbf{x}_m[m;R]_{1,\dots,m}, \tag{6.15}$$

where $\mathbf{e}^i$ denotes the i-th basis covector of size n.

Unpacking the compact boldface notation, we have now obtained ∂_q^i on monomials as

$$\partial_q^i x_{i_1}x_{i_2}\dots x_{i_m} = \delta_{j_1}^i x_{j_2}\dots x_{j_m}[m;R]^{j_1}{}_{i_1}{}^{j_2}{}_{i_2}{}^{\cdots}{}_{\cdots}{}^{j_m}{}_{i_m}.$$

It is illustrative to consider the case $m=2$. We have that

$$\begin{aligned}[2;R] = I + (PR)_{12} \implies [2;R]^{j_1}{}_{i_1}{}^{j_2}{}_{i_2} &= \delta^{j_1}{}_{i_1}\delta^{j_2}{}_{i_2} + (PR)^{j_1}{}_{i_1}{}^{j_2}{}_{i_2} \\ &= \delta^{j_1}{}_{i_1}\delta^{j_2}{}_{i_2} + R^{j_2}{}_{i_1}{}^{j_1}{}_{i_2}\end{aligned}$$

Hence:

$$\partial_q^i(x_{i_1}x_{i_2}) = \delta_{i_1}^i x_{i_2} + x_{j_2}R^{j_2}{}_{i_1}{}^{i}{}_{i_2}. \tag{6.16}$$

This equation will be useful for explicit computations later on.

Since ∂_q^i has upper incides, the notation suggests that these differential operators have the braid statistics of vectors. This is exactly the content of the following lemma:

Lemma 6.29. Let $V\check{}(R',R)$ be a braided covector-algebra. Then the braided derivatives ∂_q^i obey the braid statistics of $V(R',R)$:

$$\partial_q^i\partial_q^j = R'^i{}_a{}^j{}_b\partial_q^b\partial_q^a.$$

Proof. We show this to be true on arbitrary monomials of total degree m, in the notation of example 6.28. The general statement then follows by linearity of ∂_q^i. To proceed we need the following identity:

$$[m-1;R]_{2,\dots,m}[m;R]_{1,\dots,m} = (PR')_{12}[m-1;R]_{2,\dots,m}[m;R]_{1,\dots,m}. \tag{6.17}$$

This is proved in appendix A.3. By equation 6.15 we have

$$\begin{aligned}\partial_q^i\partial_q^j\mathbf{x}_1\mathbf{x}_2\dots\mathbf{x}_m &= \partial_q^i\mathbf{e}_1^j\mathbf{x}_2\dots\mathbf{x}_m[m;R]_{1,\dots,m} \\ &= \mathbf{e}_1^j\left(\partial_q^i\mathbf{x}_2\dots\mathbf{x}_m\right)[m;R]_{1,\dots,m} \\ &= \mathbf{e}_1^j\mathbf{e}_2^i\mathbf{x}_3\dots\mathbf{x}_m[m-1;R]_{2,\dots,m}[m;R]_{1,\dots,m}.\end{aligned}$$

Similarly we have that

$$\begin{aligned}R'^i{}_a{}^j{}_b\partial_q^b\partial_q^a\mathbf{x}_1\mathbf{x}_2\dots\mathbf{x}_m &= R'^i{}_a{}^j{}_b\mathbf{e}_1^a\mathbf{e}_2^b\mathbf{x}_3\dots\mathbf{x}_m[m-1;R]_{2,\dots,m}[m;R]_{1,\dots,m} \\ &= \mathbf{e}_1^j\mathbf{e}_2^i\mathbf{x}_3\dots\mathbf{x}_m(PR')_{12}[m-1;R]_{2,\dots,m}[m;R]_{1,\dots,m}.\end{aligned}$$

By virtue of equation (6.17) we thus conclude

$$\partial_q^i\partial_q^j\mathbf{x}_1\mathbf{x}_2\dots\mathbf{x}_m = R'^i{}_a{}^j{}_b\partial_q^b\partial_q^a\mathbf{x}_1\mathbf{x}_2\dots\mathbf{x}_m,$$

as required. □

This implies that $v^i \mapsto \partial_q^i$ forms an action of $V(R', R)$ on $V\breve{}(R', R)$, namely $v^i \mapsto \partial_q^i$. In particular, it means that we have an algebra of braided partial derivatives generated by $1, \partial_q^1, \ldots, \partial_q^n$ that is isomorphic to $V(R', R)$. This realization of the algebra of q-derivatives as a copy of $V(R', R)$ allows us to give a q-deformed version of the usual Leibniz rule for differentiation. For this we need the braiding from lemma 6.25:

Lemma 6.30. For $a, b \in V\breve{}(R', R)$ arbitrary, the partial derivatives ∂_q^i obey the *braided Leibniz rule*

$$\partial_q^i(ab) = (\partial_q^i a)b + \cdot(\Psi^{-1}(\partial_q^i \otimes a)b).$$

Proof. This is shown by direct computation analogous to lemma 6.29 in [34]. A more sophisticated diagrammatic proof can be found in [31]. $\square$

The braided Leibniz rule has an interesting immediate consequence:

Lemma 6.31. The action $\alpha : v^i \mapsto \partial_q^i$ of $V(R', R)$ turns $V\breve{}(R', R)$ into a braided $V(R', R)$-module algebra (with reversed braiding), i.e. the action commutes with the algebra structure of $V\breve{}(R', R)$.

This proof becomes almost trivial with the formalism of braided group diagrams:

Proof. In a braided group diagram, a derivative ∂_q^i is denoted using the action $\alpha : v^i \mapsto \partial_q^i$. As before, this action is depicted in a braided diagram as a multiplication between $V(R', R)$ and $V\breve{}(R', R)$, marked with an α for clarity. With this notation, by definition 5.10 the statement of the lemma is depicted as a diagram in figure 6.1:

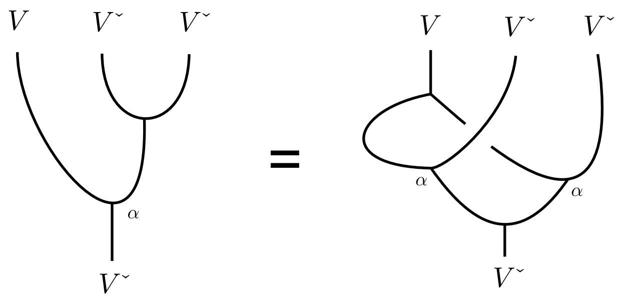


Figure 6.1: Statement of lemma 6.31 as a braided diagram.

To prove this, we can simply insert the braided Leibniz rule as a diagrammatic equality. This is depicted in figure 6.2:

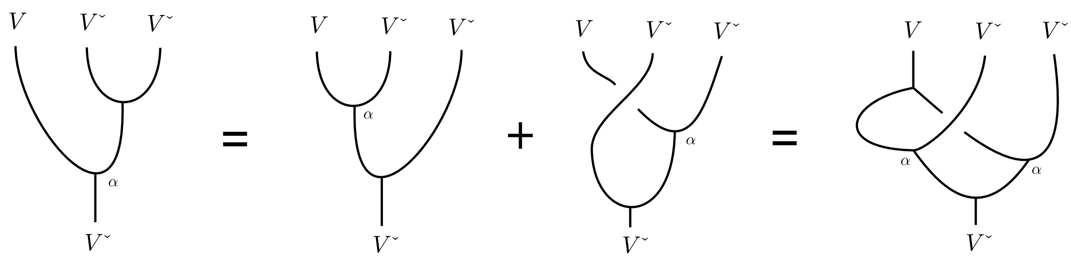


Figure 6.2: Proof of lemma 6.31.

Here the first equality is the braided Leibniz rule, and the second equality translates the $+$ into a Δ. This is possible precisely because Δ is a coaddition type coproduct. $\square$

6.4 Braided Exterior Algebra

In the previous subsection we saw the introduction of braided differentiation. To continue this development of q-deformed calculus, we now wish to introduce an algebra of braided differential forms and to develop exterior algebra on our braided groups. This proceeds analogously to the familiar notion of exterior algebra as follows:

- We define an algebra Λ of braided differential forms, in which multiplication is to be interpreted as the wedge product $\wedge$.
- We define the (graded) exterior algebra $\Omega = \Lambda\underline{\otimes}V\check{}$.
- We define the braided exterior derivative on Ω, and show that it behaves like a classical exterior derivative, i.e. that it is an anti-derivation of degree 1 and provides us with a differential complex.

Here the notation $\underline{\otimes}$ indicates the braided tensor product of lemma 3.26. Note that the object Λ is a new braided group in $\mathcal{M}^{A(R)}$. Hence to properly define Ω we will need to consider the induced braiding from $\mathcal{M}^{A(R)}$.

Recall the assumptions we've made on R, R' to construct $V\check{}(R', R)$, given by equations (6.2), (6.3), (6.4), and (6.5). We've already seen the solutions that yield the regular and fermionic quantum planes. The latter consisted of q-deformed coordinates θ, ϕ that were approximately anti-commutative. It was obtained by the substitution $R \leftrightarrow -R'$ which was possible in the case $R' \propto R$. To define an anti-commutative algebra built from $V\check{}(R', R)$ more generally, we impose the further conditions

$$R'_{12} \circ R'_{13} \circ R'_{23} = R'_{23} \circ R'_{13} \circ R'_{12}, \tag{6.18}$$

$$R'_{12} \circ R'_{13} \circ R_{23} = R_{23} \circ R'_{13} \circ R'_{12}, \quad R'_{23} \circ R'_{13} \circ R_{12} = R_{12} \circ R'_{13} \circ R'_{23}. \tag{6.19}$$

This ensure that the combined system of equations for R, R' exhibits an $R \leftrightarrow -R'$ symmetry.

Definition 6.32. Under the above assumptions on R, R', we define the *braided group of forms* as

$$\Lambda(R', R) := V\check{}(-R, -R').$$

We denote the generators of $\Lambda(R', R)$ by $1, \theta_1, \ldots, \theta_n$.

To interpret the generators θ_i as forms $\mathrm{d}x_i$ on $V\check{}(R', R)$ we must first cast Λ as an object of $\mathcal{M}^{A(R)}$. Note that this is not by default the case: Λ is an object of $\mathcal{M}^{A(-R')}$. For our convenience, we assume that $A(-R') = A(R)$. This holds quite generally: for example when $R' \propto R$ or when $PR' = f(PR)$ for some polynomial f [39]. In particular it is true if $R' \propto R$.

Under this assumption, we can stick the algebra of forms Λ and the algebra of covariant coordinates $V\check{}$ together to form a right-handed exterior algebra. For this purpose we consider Λ as an object of $\mathcal{M}^{A(R)}$, via the covector-algebra coaction $\theta_i \mapsto \theta_a \otimes t^a{}_i$.

Definition 6.33. We define the *right-handed exterior algebra* as

$$\Omega_R := \Lambda\underline{\otimes}V\check{}.$$

(This algebra is also just denoted by Ω.) The braiding is as induced by the category $\mathcal{M}^{A(R)}$ via the braided tensor product $\underline{\otimes}$. In other words, since Λ and $V\check{}$ are both covector-algebras with $A(R)$ as background quantum group, the induced braiding is given by

$$\Psi_{\Lambda,V\check{}}(\theta_i, x_j) = x_b \otimes \theta_a R^a{}_i{}^b{}_j.$$

Remark 6.34. When it comes to further applications of Λ in developing the theory of Ω, we'll explicitly use the braiding induced from $A(R)$ via $\underline{\otimes}$. This means in particular that when we discuss $\Lambda\underline{\otimes}\Lambda$ in the context of $\Omega\underline{\otimes}\Omega$, we will use the braiding

$$\Psi_{\Lambda,\Lambda}(\theta_i,\theta_j)=\theta_b\otimes\theta_a R^a{}_i{}^b{}_j.$$

This is a different braiding from that of the braided covector structure on Λ, which would require using $-R'$ instead. For the purposes of working with Ω, we forget about this latter braiding. Essentially this is done to keep all the constructions on Ω in the category $\mathcal{M}^{A(R)}$; see [39, p. 11].

We have a natural bi-graded structure on Ω, as well as two natural gradings, namely [39]:

$$\Omega^{p|q}=\mathrm{span}\{\theta_{i_1}\theta_{i_2}\dots\theta_{i_p}x_{j_1}x_{j_2}\dots x_{j_q}\},\quad \Omega^p=\bigoplus_{q=0}^{\infty}\Omega^{p|q},\quad \Omega^{|q}=\bigoplus_{p=0}^{\infty}\Omega^{p|q}.$$

This indeed gives

$$\Omega=\bigoplus_{p=0}^{\infty}\bigoplus_{q=0}^{\infty}\Omega^{p|q}.$$

Similarly, we have a grading on Λ given by $\Lambda^p=\mathrm{span}\{\theta_{i_1}\dots\theta_{i_p}\}$.

We are now ready to complete our interpretation of Ω as a genuine exterior algebra construction by defining the exterior derivative:

Definition 6.35. We define the exterior derivative $\overleftarrow{d}$ as

$$\overleftarrow{d}:\Omega^p\to\Omega^{p+1},\quad (\theta_{i_1}\theta_{i_2}\dots\theta_{i_p}f(x_1,\dots,x_n))\overleftarrow{d}=\sum_a\theta_{i_1}\theta_{i_2}\dots\theta_{i_p}\theta_a\frac{\partial}{\partial x_a}f(x_1,\dots,x_n).$$

With the interpretation of multiplication in Λ as the wedge product, this is directly analogous to the classical definition of the exterior derivative.

We now claim that $\overleftarrow{d}$ indeed behaves like a familiar exterior derivative. Recall that this amounts to the following two properties:

- $\overleftarrow{d}$ is an anti-derivation of degree of degree 1: for any $f\in\Omega^p, g\in\Omega$ we have

$$(fg)\overleftarrow{d}=f(g\overleftarrow{d})+(-1)^p f\overleftarrow{d}g.$$

  In other words, $\overleftarrow{d}$ obeys a $\mathbb{Z}_2$-graded Leibniz rule [6].

- $\overleftarrow{d}$ forms a differential complex:

$$\overleftarrow{d}^2=0.$$

The former statement is the content of the following theorem:

Theorem 6.36. The exterior derivative is an anti-derivation of degree 1.

Proof. This proof is best handled using braided diagrams. To do so we first need to introduce a new ingredient into our diagrams. To this end recall the coevaluation map of subsection 3.2. In particular recall that in `Vec` it is given by

$$\mathrm{coev}:k\to V\otimes V^*:\lambda\mapsto\sum_i\lambda v_i\otimes v_i^*.$$

For depicting $\overleftarrow{d}$ in a diagram we use a similar coevaluation in $\mathcal{M}^{R(A)}$, that pulls an element of $\Lambda\underline{\otimes}V$ out of the field:

$$\mathrm{coev}:k\to\Lambda\underline{\otimes}V:\lambda\mapsto\lambda\sum_a\theta_a\otimes v^a.$$

(Note: calling this a coevaluation requires a suitable duality between braided vectors and covectors. It exists [6], but discussing it here would take us too far off track.) In our braided group diagrams we then write coev(1) as a turn: $\cap$. Next recall that q-differentiation is an action of V on $V\check{}$. Denoting this action by α, the action of $\overleftarrow{d}$ is then depicted in figure 6.3:

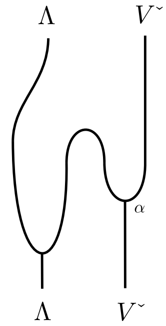


Figure 6.3: The exterior derivative.

Next recall the braided Leibniz rule of the action α: it states that for $a, b \in V\check{}$,

$$\partial^i(ab) = (\partial^i a)b + \Psi^{-1}(\partial^i \otimes a)b.$$

As a diagrammatic equality in terms of the action α, this is depicted in figure 6.4.

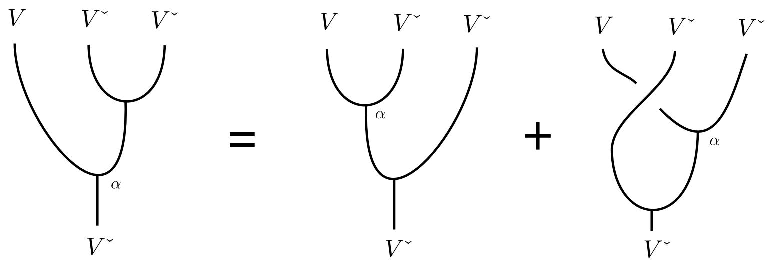


Figure 6.4: The braided Leibniz rule for ∂^i.

This diagrammatic equality is precisely what we need to prove the theorem: the proof is depicted in figure 6.5.

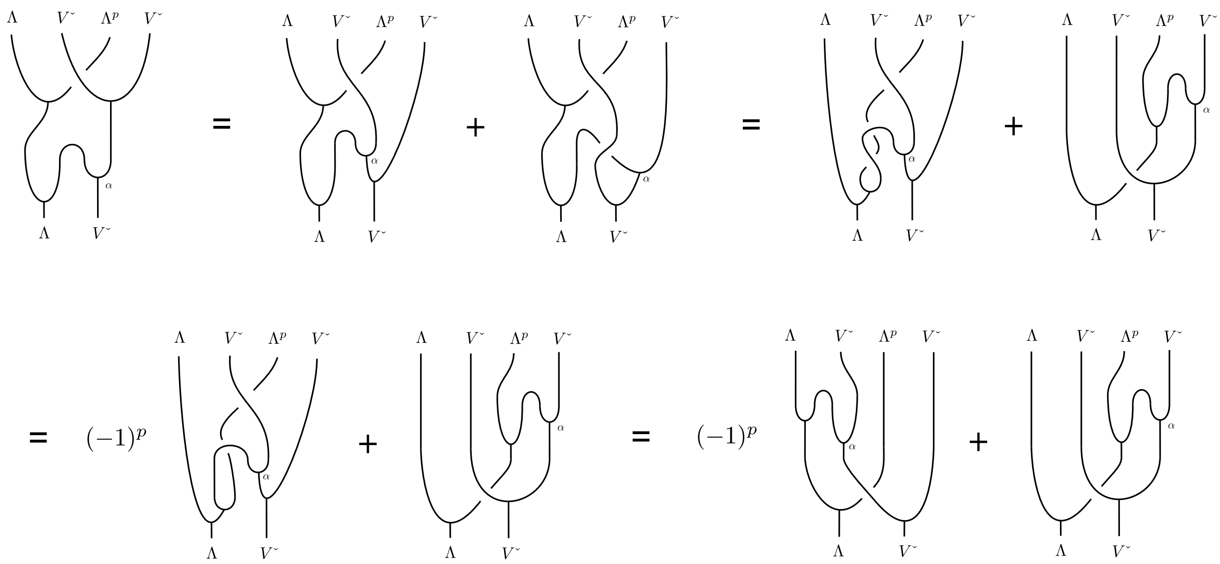


Figure 6.5: Proof of theorem 6.36.

The first equality in figure 6.5 is by the braided Leibniz rule for the action α. The second and fourth equalities are standard rearrangements of diagrams using braided associativity and functoriality of Ψ, Ψ^{-1}. The third equality is more interesting, and it is also the origin of the factor $(-1)^p$ needed to make $\overleftarrow{d}$ into a proper anti-derivation. All that happens in the third equality is that we undo a braiding, which also reverses the multiplication below it. Evidently this only gives rise to a factor $(-1)^p$ and not to more complicated braidings.

To explain why this is the case, note that the strands we are untangling represent elements of Λ. Recall now from remark 6.34 that we use the braiding induced by $\mathcal{M}^{A(R)}$ on $\Lambda\underline{\otimes}\Lambda$. As noted in remark 6.34, this means that the braiding we are undoing is given by R. In undoing this braiding we also invert the multiplication below, but the non-commutativity of this multiplication is governed by $-R$. Thus except for the minus signs (of where there are p, since the right strand represents an element of Λ^p), all the nontrivial factors of R created in undoing the braiding are swallowed up by reverting the multiplication below. In an equation:

$$\cdot \circ \Psi(\theta_i \otimes \theta_{i_1} \ldots \theta_{i_p}) = (-1)^p \theta_i \theta_{i_1} \ldots \theta_{i_p}.$$

This explains the third equality in figure 6.5, and finishes the proof. □

The requirement that $\overleftarrow{d}$ forms a differential complex is not always met, but it holds in nice cases. From the definition of $\overleftarrow{d}$ it is clear that to obtain $\overleftarrow{d}^2$ it suffices to show that

$$(\theta_1\theta_2)(\partial_2\partial_1) = 0.$$

This is in the compact notation from remark 6.15, i.e. $\theta_1\theta_2$ refers to the Kronecker product of the entire row vector of $\{\theta_i\}$ with itself. This equation is satisfied for instance if $PR' = f(PR)$ for a polynomial f such that $f(-1) \neq 1$. Indeed, since PR' is the matrix of the non-commutative multiplication in V we have

$$\partial_2\partial_1 = (PR')\partial_2\partial_1.$$

Similarly, since $-PR$ is the matrix of multiplication on Λ we have

$$\theta_1\theta_2 = (-PR)\theta_1\theta_2.$$

Thus in total we compute

$$(\theta_1\theta_2)(\partial_2\partial_1) = (\theta_1\theta_2)PR'(\partial_2\partial_1) = (\theta_1\theta_2)f(PR)(\partial_2\partial_1) = f(-1)(\theta_1\theta_2)(\partial_2\partial_1).$$

Here the last equality holds since f is a polynomial. Since $f(-1) \neq 1$ by assumption, from the above we conclude that $(\theta_1\theta_2)(\partial_2\partial_1) = 0$ so that $\overleftarrow{d}^2 = 0$. In particular, if $R' \propto R$ (for e.g. the quantum plane) we thus have that $\overleftarrow{d}$ forms a differential complex.

7 Braided Matrices and q-Minkowski Space

In this section we will discuss the construction of a q-deformed version of Minkowski space-time, via the formalism of braided geometry developed in the previous section. To this end we must first re-cast Minkowski space as a space of Hermitian matrices, which we can deform using braided geometry. We then recast these braided matrices back into a four-dimensional braided covector-algbra, for which we use an Octave program `contractor.m`. The process is depicted in figure 7.1. Note that this approach is quite particular to 4-dimensional space: e.g. 5D space cannot be modelled easily as a space of (Hermitian) matrices, and would require a different approach.

$$
\begin{tikzcd}
\text{Minkowski space } \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} \arrow[rr, shift left, "\check{V}(\mathbf{R}',\mathbf{R})"] \arrow[d, leftrightarrow, "\cong"'] & & \begin{matrix} q\text{-Minkowski} \\ \text{space} \\ \langle 1, t, x, y, z \rangle \end{matrix} \arrow[ll, shift left, "q=1"] & \in \mathcal{M}^{A(\mathbf{R})} \\
\text{Hermitian matrices } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \arrow[rr, shift right, "B(R)"'] & & \text{Braided matrices} \arrow[ll, shift right, "q=1"'] \arrow[u, rightsquigarrow, "\texttt{contractor.m}"'] & \in \mathcal{M}^{A(R)}
\end{tikzcd}
$$

Figure 7.1: Overview of the construction of q-Minkowski space.

To make a q-deformed version of Minkowski space, note that we could simply find a suitable pair (R', R) and use it to construct a four-dimensional braided covector-algbra. However, R and R' are then represented by 16×16 matrices. Presumably, there are many matrices of this size that satisfy the requirements for forming a braided algebra, which makes it difficult to pick a natural braiding. More-over, simply deforming a four-dimensional space does not give us a natural way to introduce a metric on our space, which is what we need to identify our construction as a bona fide q-Minkowski space.

We shall see that passing the construction through braided matrices solves both of these problems.

7.1 Classical Minkowski space

Before we can properly discuss the q-deformed version of Minkowski space, we must understand the classical $q = 1$ case that we wish to deform:

Minkowski spacetime $\mathbb{R}^{1,3}$ is given by four-dimensional space, equipped with the constant pseudo-Riemannian metric tensor

$$\eta^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

(With units such that $c = 1$.) This gives the distance of a point $p = (t, x, y, z)$ from the origin as

$$\|p\|^2 = t^2 - x^2 - y^2 - z^2.$$

Alternatively we can define a coordinate transformation $(t, x, y, z) \to (a, b, c, d)$ and give the point p as a Hermitian 2×2 matrix

$$X_p = \begin{bmatrix} a & b \\ c & d \end{bmatrix} := \begin{bmatrix} t - z & x + iy \\ x - iy & t + z \end{bmatrix}. \tag{7.1}$$

It is then easy to check that the map $p \mapsto X_p$ is a bijection to the space of Hermitian matrices, and that we have

$$\det(X_p) = t^2 - x^2 - y^2 - z^2 = \|p\|^2. \tag{7.2}$$

Thus we obtain a bijection

$$\text{Minkowski space} \quad \cong \quad \text{Hermitian matrices}$$

that preserves the notion of the metric. This bijection is familiar from defining the spinor map to the restricted Lorentz group: $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}^+(1, 3)$. Namely, we have an action α of $\mathrm{SL}(2, \mathbb{C})$ on the Hermitian matrices given by

$$\alpha_P : X \mapsto PXP^*,$$

for $P \in \mathrm{SL}(2, \mathbb{C})$. Note that $(PXP^*)^* = PX^*P^* = PXP^*$ so that $\alpha_P(X)$ is indeed again Hermitian. Since $P \in \mathrm{SL}(2, \mathbb{C})$ we have $\det(P) = \det(P^*) = 1$. Thus

$$\det\left(\alpha_P(X)\right) = \det(PXP^*) = \det(X),$$

so that α_P preserves the metric for all $P \in \mathrm{SL}(2, \mathbb{C})$. This allows us to identify any P with a Lorentz transformation, which yields the well-known spinor map.

One way to obtain $\eta^{\mu\nu}$ from the Minkowski distance $\|p\|^2$ is the (perhaps familiar) equation

$$\eta^{\mu\nu} = \frac{1}{2} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} \|p\|^2. \tag{7.3}$$

The factor $1/2$ appears because for the diagonal elements the derivative of a square gives a factor 2, and the metric tensor is symmetric so that all off-diagonal elements appear twice, also giving a factor 2.

Using this equation on $\det(X_p)$ with respect to the coordinates $\{a, b, c, d\}$ gives the *matrix metric* $\eta_M^{\mu\nu}$:

$$\det(X_p) = ad - bc \implies \eta_M^{\mu\nu} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Note that this metric no longer exhibits a $(+, -, -, -)$ signature: this is because we have applied a complex coordinate transformation. Thus the change in signature amounts to something like a Wick rotation. After we have braided the Hermitian matrices we will Wick-rotate back to braided spacetime coordinates, once again obtaining a (now q-deformed) $(+, -, -, -)$ signature.

7.2 Braided Matrices

In subsection 6.1 we have already seen braided versions of vector and covector algebras equipped with a coaddition. The objective is now to do the same for matrices.

Definition 7.1. Let R be as in definitions 6.6 and 6.12. We define the *braided matrices* $B(R)$ as the algebra generated by $1, u^i{}_j$ modulo the relations

$$R^k{}_b{}^i{}_a u^a{}_c R^c{}_j{}^b{}_d u^d{}_l = u^k{}_b R^b{}_c{}^i{}_a u^a{}_d R^d{}_j{}^c{}_l. \tag{7.4}$$

Here $i, j \in \{1, \ldots, n\}$ and thus R must be an $n^2 \times n^2$ matrix.

For applications to q-Minkowski space, we are interested in the case $n = 2$.

To make the multiplication relations manageable, we introduce the following compact notation: let $\mathbf{u}$ be the matrix of generators

$$\mathbf{u} = \begin{bmatrix} u^1{}_1 & u^1{}_2 \\ u^2{}_1 & u^2{}_2 \end{bmatrix}.$$

Then we define $\mathbf{u}_1$ and $\mathbf{u}_2$ as follows:

$$\mathbf{u}_1 = \begin{bmatrix} u^1{}_1 & u^1{}_1 & u^1{}_2 & u^1{}_2 \\ u^1{}_1 & u^1{}_1 & u^1{}_2 & u^1{}_2 \\ u^2{}_1 & u^2{}_1 & u^2{}_2 & u^2{}_2 \\ u^2{}_1 & u^2{}_1 & u^2{}_2 & u^2{}_2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{u}_2 = \begin{bmatrix} u^1{}_1 & u^1{}_2 & u^1{}_1 & u^1{}_2 \\ u^2{}_1 & u^2{}_2 & u^2{}_1 & u^1{}_2 \\ u^1{}_1 & u^1{}_2 & u^1{}_1 & u^2{}_2 \\ u^2{}_1 & u^2{}_2 & u^2{}_1 & u^2{}_2 \end{bmatrix}.$$

These matrices are chosen such that their entry-wise product (i.e. Hadamard product) equals the Kronecker product $\mathbf{u} \otimes \mathbf{u}$. This is in analogy with the compact notation for braided covectors from example 6.15. With this notation we can compactly state all the relations described by equation (7.4) in a single managable equation:

$$R_{21}\mathbf{u}_1 R \mathbf{u}_2 = \mathbf{u}_2 R_{21} \mathbf{u}_1 R.$$

Here we recall from remark 6.13 that R_{21} simply equals RP.

We now further demand that R obeys the q-Hecke condition:

$$(PR - q)(PR + q^{-1}) = 0. \tag{7.5}$$

Then we can give $B(R)$ the structure of a coadditive briaded group as follows:

Lemma 7.2. *$B(R)$ forms a braided group with*

$$\Delta u^i{}_j = u^i{}_j \otimes 1 + 1 \otimes u^i{}_j, \quad \epsilon u^i{}_j = 0, \quad S u^i{}_j = -u^i{}_j,$$

$$\Psi(R^{-1}\mathbf{u}_1 \otimes R\mathbf{u}_2) = \mathbf{u}_2 R_{21} \otimes \mathbf{u}_1 R.$$

For the braiding, we use the same shorthand notation as introduced for definition 7.1. Writing Ψ out in components is done in analogy with definition 7.1.

A detailed proof of this braided group structure on $B(R)$ can be found in [36] where it was first introduced in general R-matrix form. The q-Hecke condition is needed to show that Δ is an algebra morphism [6].

Remark 7.3. Technically, to properly define $B(R)$ we must be concerned with the braided category in which it resides, i.e. the background quantum group under which it is *covariant*. For our purposes this is not extremely important however, since we will cast $B(R)$ in a covector-algebra form with a different background quantum group.

It turns out that $B(R)$ is not only covariant under $A(R)$, but more generally under the *double cross product bialgebra* $A(R) \bowtie A(R)$. The double cross product bialgebra can be constructed generally for bialgebras A, H if A acts on H and H coacts on A. See [19, p. 134] for details. For our purposes it suffices to say that $A(R) \bowtie A(R)$ is the algebra $A(R) \otimes A(R)$ modulo the relations

$$R\mathbf{s}_1\mathbf{s}_2 = \mathbf{s}_2\mathbf{s}_1 R, \quad R\mathbf{t}_1\mathbf{t}_2 = \mathbf{t}_2\mathbf{t}_1 R, \quad R\mathbf{t}_1\mathbf{s}_2 = \mathbf{s}_2\mathbf{t}_1 R.$$

Here $\mathbf{s}, \mathbf{t}$ are the generators for the left and right copies of $A(R)$, respectively. The braided matrices $B(R)$ are then an object of $\mathcal{M}^{A(R) \bowtie A(R)}$ via the coaction

$$u^i{}_j \mapsto u^a{}_b \otimes (S s^i{}_a \otimes t^b{}_j).$$

(Assuming that we have an antipode S on $A(R)$.) If we wish to view $B(R)$ as living in the category $\mathcal{M}^{A(R)}$ then we may note that there is a homomorphism

$$A(R) \bowtie A(R) \to A(R),$$

given by multiplication. This casts $B(R)$ as an object of $\mathcal{M}^{A(R)}$ via the coaction

$$u^i{}_j \mapsto u^a{}_b \otimes (St^i{}_a t^b{}_j).$$

Remarkably, we can give $B(R)$ the structure of a braided group in a different way, with a coproduct of 'comultiplication type'. Since we are mainly concerned with braided covector-algebras and coaddition type coproducts, this braided group structure will be considered as secondary. However, it will still be useful; in particular for finding a natural q-deformed determinant in $B(R)$. We will use a dot to distinguish this secondary braided structure; $\Psi.$ for example.

Lemma 7.4. The algebra $B(R)$ (with the same multiplication relations as before) forms a braided group with

$$\Delta.(u^i{}_j) = u^i{}_a \otimes u^a{}_j, \quad \epsilon.(u^i{}_j) = \delta^i{}_j,$$
$$\Psi.(R^{-1}\mathbf{u}_1 \otimes R\mathbf{u}_2) = \mathbf{u}_2 R^{-1} \otimes \mathbf{u}_1 R.$$

An antipode is not provided, so that technically this is a braided bialgebra, instead of a proper braided group.

For a detailed proof of this braided group structure, see [40]. We wish to note that the q-Hecke condition is not required for this structure.

The coproduct $\Delta.$ is referred to as a *comultiplication* for the same reason that we call Δ a coaddition: when u is viewed as a matrix we have that

$$\cdot \circ (\Delta.u^i{}_j) = (u \cdot u)^i{}_j.$$

This is analogous to how $\cdot(\Delta u^i{}_j) = u^i{}_j + u^i{}_j$ for the coaddition Δ. We also refer to $\Psi.$ as the *multiplicative braiding* on $B(R)$ as opposed to Ψ, the *additive braiding.*

Note that in both lemmas 7.2 and 7.4 the braidings are give implicitly. If R is bi-invertible in the sense of definition 6.22, we can give explicit formulae for the multiplicative relations and braidings. This is the content of the following lemma:

Lemma 7.5. Suppose R is bi-invertible. Then

$$R_{21}\mathbf{u}_1 R\mathbf{u}_2 = \mathbf{u}_2 R_{21}\mathbf{u}_1 R \iff u^{j_0}{}_{j_1} u^{l_0}{}_{l_1} = u^{k_0}{}_{k_1} u^{i_0}{}_{i_1} (R^{-1})^d{}_{k_0}{}^{j_0}{}_a R^{k_1}{}_b{}^a{}_{i_0} R^{i_1}{}_c{}^b{}_{l_1} \widetilde{R}^c{}_{j_1}{}^{l_0}{}_d.$$

Proof. Recall that we can write the multiplication relations in components as

$$R^k{}_b{}^i{}_a u^a{}_c R^c{}_j{}^b{}_d u^d{}_l = u^k{}_b R^b{}_c{}^i{}_a u^a{}_d R^d{}_j{}^c{}_l.$$

Note here that we have indices a on both the left- and right-hand sides. However they are not free indices, so they are distinct. Keeping this in mind, we carefully relabel the expression to obtain

$$R^{k_0}{}_d{}^a{}_{j_0} u^{j_0}{}_{j_1} R^{j_1}{}_c{}^d{}_{l_0} u^{l_0}{}_{l_1} = u^{k_0}{}_{k_1} R^{k_1}{}_b{}^a{}_{i_0} u^{i_0}{}_{i_1} R^{i_1}{}_c{}^b{}_{l_1}.$$

Now, to remove the term $R^{k_0}{}_d{}^a{}_{j_0}$ from the left-hand side, we can multiply by $(R^{-1})^d{}_{k_0}{}^{j_0}{}_a$. This contracts the indices k_0, a that are free on the left-hand side, and leaves a term $\delta^d{}_d \delta^{j_0}{}_{j_0}$ which is simply 1. Thus we obtain

$$u^{j_0}{}_{j_1} R^{j_1}{}_c{}^d{}_{l_0} u^{l_0}{}_{l_1} = (R^{-1})^d{}_{k_0}{}^{j_0}{}_a u^{k_0}{}_{k_1} R^{k_1}{}_b{}^a{}_{i_0} u^{i_0}{}_{i_1} R^{i_1}{}_c{}^b{}_{l_1}.$$

Next, to remove the $R^{j_1}{}_c{}^d{}_{l_0}$ term we must use the second inverse: indeed, we need to contract the free indices c, d. One of these is an upper index while the other is lower, so we require $\widetilde{R}$ to contract them simultaneously. Accordingly, multiplying with $\widetilde{R}^c{}_{j_1}{}^{l_0}{}_d$ yields

$$u^{j_0}{}_{j_1} u^{l_0}{}_{l_1} = \widetilde{R}^c{}_{j_1}{}^{l_0}{}_d (R^{-1})^d{}_{k_0}{}^{j_0}{}_a u^{k_0}{}_{k_1} R^{k_1}{}_b{}^a{}_{i_0} u^{i_0}{}_{i_1} R^{i_1}{}_c{}^b{}_{l_1}.$$

Rearranging the right-hand side then gives

$$u^{j_0}{}_{j_1} u^{l_0}{}_{l_1} = u^{k_0}{}_{k_1} u^{i_0}{}_{i_1} (R^{-1})^d{}_{k_0}{}^{j_0}{}_a R^{k_1}{}_b{}^a{}_{i_0} R^{i_1}{}_c{}^b{}_{l_1} \widetilde{R}^c{}_{j_1}{}^{l_0}{}_d,$$

as required. Clearly all the manipulations carried out are invertible, so the claimed equivalence indeed holds. □

Similarly one can give an explicit formula for the braidings as

$$\Psi(R^{-1}\mathbf{u}_1 \otimes R\mathbf{u}_2) = \mathbf{u}_2 R_{21} \otimes \mathbf{u}_1 R \iff \Psi(u^{j_0}{}_{j_1} \otimes u^{l_0}{}_{l_1}) = u^{k_0}{}_{k_1} \otimes u^{i_0}{}_{i_1} R^{j_0}{}_a{}^d{}_{k_0} R^{k_1}{}_b{}^a{}_{i_0} R^{i_1}{}_c{}^b{}_{l_1} \widetilde{R}^c{}_{j_1}{}^{l_0}{}_d,$$

$$\Psi.(R^{-1}\mathbf{u}_1 \otimes R\mathbf{u}_2) = \mathbf{u}_2 R^{-1} \otimes \mathbf{u}_1 R \iff \Psi(u^{j_0}{}_{j_1} \otimes u^{l_0}{}_{l_1}) = u^{k_0}{}_{k_1} \otimes u^{i_0}{}_{i_1} R^{j_0}{}_a{}^d{}_{k_0} (R^{-1})^a{}_{i_0}{}^{k_1}{}_b R^{i_1}{}_c{}^b{}_{l_1} \widetilde{R}^c{}_{j_1}{}^{l_0}{}_d.$$

The proofs of these statements are completely analogous to that for the multiplicative relations, so to avoid some tedium they are omitted.

We can compress these results with some notational devices: let $u^{i_0}{}_{i_1} := u_I$ and let

$$\mathbf{R}'^I{}_J{}^K{}_L = (R^{-1})^d{}_{k_0}{}^{j_0}{}_a R^{k_1}{}_b{}^a{}_{i_0} R^{i_1}{}_c{}^b{}_{l_1} \widetilde{R}^c{}_{j_1}{}^{l_0}{}_d,$$

$$\mathbf{R}^I{}_J{}^K{}_L = R^{j_0}{}_a{}^d{}_{k_0} R^{k_1}{}_b{}^a{}_{i_0} R^{i_1}{}_c{}^b{}_{l_1} \widetilde{R}^c{}_{j_1}{}^{l_0}{}_d,$$

$$\mathbf{R}.^I{}_J{}^K{}_L = R^{j_0}{}_a{}^d{}_{k_0} (R^{-1})^a{}_{i_0}{}^{k_1}{}_b R^{i_1}{}_c{}^b{}_{l_1} \widetilde{R}^c{}_{j_1}{}^{l_0}{}_d.$$

This notation lets us view $B(R)$ as a space with four generators u_I. The explicit multiplicative relations and braiding now become:

$$R_{21}\mathbf{u}_1 R\mathbf{u}_2 = \mathbf{u}_2 R_{21}\mathbf{u}_1 R \iff u_I u_J = u_B u_A \mathbf{R}'^A{}_I{}^B{}_J,$$

$$\Psi(R^{-1}\mathbf{u}_1 \otimes R\mathbf{u}_2) = \mathbf{u}_2 R_{21} \otimes \mathbf{u}_1 R \iff u_I \otimes u_J = u_B \otimes u_A \mathbf{R}^A{}_I{}^B{}_J,$$

$$u_I \otimes . \, u_J = u_B \otimes . \, u_A \mathbf{R}.^A{}_I{}^B{}_J.$$

Note the similarities between these expressions (in particular the first two) and the multiplication and braiding of definition 6.6. Indeed: as the notation suggests, $\mathbf{R}$ and $\mathbf{R}'$ allow us to view $\langle 1, u_I \rangle$ as a braided covector-algebra. For this we require the following lemma:

Lemma 7.6. The pair $(\mathbf{R}', \mathbf{R})$ satisfies the mixed Yang-Baxter equations required to define a braided covector-algebra, and so does the pair $(-\mathbf{R}, -\mathbf{R}')$.

Proof. The proof is given in [36]. Since we will only need a specific example for q-Minkowski space, there is no need to go into the details of the proof: we can simply let a computer verify all the required relations once we have determined the pair $(\mathbf{R}', \mathbf{R})$ explicitly. □

The last statement of this lemma is only to facilitate the construction of q-deformed differential forms on our braided geometry. Note how the construction of $\mathbf{R}$ reduces the complexity in q-deforming a four-dimensional spacetime: instead of needing to find a 16×16 solution to the quantum YBE, we can form $\mathbf{R}$ from any 4×4 solution!

Example 7.7. Consider the following matrix:

$$R = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix}.$$

This is simply the two-dimensional covector-algbra matrix associated to the quantum plane, normalized to satisfy the q-Hecke condition of equation (7.5). Here we note that it is also the QYBE solution associated to the quantum group $U_q(\mathfrak{su}_2)$ in the context of the quantum knot invariants from subsection 4.4 (after a reparametrization $q^{1/2} \to q$ and a normalization). We will see that, as it turns out, the braided matrices $B(R)$ associated to this matrix are precisely what we need to form a q-deformation of Minkowski space. Thus we will carry out the formalism described above to construct $\mathbf{R}$ and $\mathbf{R}'$:

First off, R is bi-invertible. Indeed, it is easy to verify that

$$R^{-1} = \begin{bmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & q^{-1}-q & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{bmatrix} \quad \text{and} \quad \widetilde{R} = \begin{bmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & q^{-3}-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{bmatrix}.$$

The latter is most easily found by noting that $\widetilde{R} = ((R^{T_2})^{-1})^{T_2}$, where R^{T_2} denotes transposition of the second tensor product factor of R, i.e.

$$(R^{T_2})^i{}_j{}^k{}_l = R^i{}_j{}^l{}_k.$$

We then put these into the equations for $\mathbf{R}$, $\mathbf{R}'$, and $\mathbf{R}.$, which yields 16×16 matrices. This is a tedious process, so it is best done by computer.

In appendix B we have provided some example Octave code (called `contractor.m`) that computes a single entry of $\mathbf{R}$. The program carries out the computations for $q = 10$. This is so that the entries, which are polynomials in q, can be read off easily from the resulting computations (assuming that the polynomials have low coefficients, which must clearly be the case). For example, if a matrix entry equals 99 we conclude it represents $q^2 - 1$. This code is more-over useful for verifying that $\mathbf{R}, \mathbf{R}'$ satisfy the the quantum YBE and other requirements: since we have a way to generate the matrices in Octave, we can also use Octave to check these relations for us. This justifies deferring the proof of lemma 7.6.

The resulting matrices $\mathbf{R}$, $\mathbf{R}'$, and $\mathbf{R}.$ are given in appendix A.1. Note that $\mathbf{R}$, $\mathbf{R}'$, $\mathbf{R}.$ are 16×16 matrices, so that they are indeed suited to encode a four-dimensional spacetime coordinate algebra.

7.3 q-Minkowski space

We are now finally in the position to define q-deformed Minkowski space. As stated before, q-Minkowski space will be a space of braided matrices regarded as a braided covector-algebra. This is in q-deformed analogy to the interpretation of classical Minkowski space as Hermitian matrices. Thus to properly interpret a space of braided matrices as q-Minkowski space, we must still specify what is meant by a *Hermitian* braided matrix. This is done via the introduction of a so-called $*$-structure:

Definition 7.8. Let B be a braided group. For simplicity, suppose it is an object of a subcategory of $\mathtt{Vec}_{\mathbb{C}}$; the vector spaces of $\mathbb{C}$. (This subcategory may, of course, have a non-trivial braiding.) A *$*$-structure* on B is an operation $* : B \to B$ such that:

- $*$ is anti-linear, i.e. $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$ for all $a, b \in B$, $\lambda, \mu \in \mathbb{C}$.

- $*^2 = \mathrm{id}$.
- $*$ is an algebra anti-morphism, i.e. $(ab)^* = b^* a^*$ for all $a, b \in B$.
- $(* \otimes *) \circ \Delta = \tau \circ \Delta \circ *, \quad \epsilon \circ * = \circ \epsilon, \quad * \circ S = S \circ *$.

These last three requirements are quite specific to braided groups: this definition of a $*$-structure for braided groups was introduced in [41].

While $*$-structures occur more generally in mathematics [42], for our purposes we can think of this definition to emulate the essential properties of complex conjugation. As such we are particularly interested in the Hermitian type $*$-structure on $B(R)$:

Example 7.9. Consider the braided group $B(R)$ over the field $k = \mathbb{C}$. On this braided group, we can choose to model the Hermitian transpose by requiring

$$(u^i{}_j)^* = u^j{}_i.$$

On $B(R)$, this Hermitian $*$ operator satisfies the requirements of a $*$-structure. In fact, this gives a valid $*$-structure both in the case of a multiplicative and an additive braiding on $B(R)$.

This $*$-structure is taken from [41]; a proof that $(u^i{}_j)^* = u^j{}_i$ indeed defines a $*$-structure on $B(R)$ can be found there as well. This is under some conditions on R, which are satisfied in the case of example 7.7 which we are interested in [6].

What this means in the case of 2×2 braided matrices over $k = \mathbb{C}$, for instance, is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* := \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Thus, if we interpret the $*$-structure as complex conjugation we retrieve exactly the requirement that the matrices in question are Hermitian. For this reason, we say that a quantity x in $B(R)$ is *real* if $x^* = x$.

Definition 7.10. We define q-Minkowski space $\mathbb{R}_q^{1,3}$ to be the braided covector-algebra $V\check{}(\mathbf{R}', \mathbf{R})$ over $k = \mathbb{C}$. Here $\mathbf{R}', \mathbf{R}$ are as in example 7.7. This covector-algebra is also a space of braided matrices with matrix generators a, b, c, d:

$$\mathbf{u} = \begin{bmatrix} u^1{}_1 & u^1{}_2 \\ u^2{}_1 & u^2{}_2 \end{bmatrix} := \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{7.6}$$

We define $\mathbb{R}_q^{1,3}$ to have the $*$-structure given on $B(R)$ by example 7.9:

$$\begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Evidently, $\mathbb{R}_q^{1,3}$ is an object in the category $\mathcal{M}^{A(\mathbf{R})}$, so that the background quantum group of $\mathbb{R}_q^{1,3}$ is taken to be $A(\mathbf{R})$ instead of $A(R) \bowtie A(R)$.

Remark 7.11. We have claimed before that the approach of braided matrices helps us pick out a single q-deformation for $\mathbb{R}^{1,3}$. Indeed, the choice of a q-deformation for Minkowski space is now reduced to the choice of a single two-dimensional solution of the QYBE. This must more-over be a one-parameter solution. Such solutions have been classified, and their are eight families of them [19]. The further requirements of the q-Hecke condition and a braided metric that reduces to a $(+, -, -, -)$ signature for $q \to 1$ then pick out the matrix of example 7.7 amongst these eigth families.

We now have a long list of conditions on $R, R', \mathbf{R}, \mathbf{R}'$ to ensure that $\mathbb{R}_q^{1,3}$ is well-defined and properly equipped with the structure of q-derivatives, a q-exterior algebra, etc. For instance R must obey the q-Hecke condition and $\mathbf{R}'$ must solve the QYBE: all of the requirements encountered so far are met for $\mathbb{R}_q^{1,3}$. This is most easily checked by computer, since we have explicit expressions for all matrices involved. For explicit computation, the matrices $\mathbf{R}, \mathbf{R}'$ can be generated by running the Octave code `matrices.m` given in appendix B.

Explicitly, $\mathbb{R}_q^{1,3} = \langle 1, a, b, c, d\rangle$ modulo the following relations [6]:

$$\begin{gathered} ba = q^2 ab, \quad ac = q^2 ca, \quad da = ad, \\ bc = cb + (1-q^{-2})a(d-a), \quad db = bd + (1-q^{-2})ab, \\ cd = dc + (1-q^{-2})ca. \end{gathered} \tag{7.7}$$

This can be read directly from the columns of $\mathbf{R}'$, using the formula

$$\begin{bmatrix} aa & ab & ac & \dots & dc & dd \end{bmatrix} = \begin{bmatrix} aa & ba & ca & \dots & cd & dd \end{bmatrix} \cdot \mathbf{R}'$$

which is part of the definition of $V\check{}(\mathbf{R}', \mathbf{R})$. (Note that this reduces to a commutative algebra of coordinates if $q \to 1$.) Similarly, the equations for the braiding can be read directly from the equation

$$\begin{bmatrix} a\otimes a & a\otimes b & \dots & d\otimes c & d\otimes d \end{bmatrix} = \begin{bmatrix} a\otimes a & b\otimes a & \dots & c\otimes d & d\otimes d \end{bmatrix} \cdot \mathbf{R}$$

With some minor rearrangement to reduce the expressions, this gives the following braid statistics [36]:

$$\begin{gathered} \Psi(a\otimes a) = q^2 a\otimes a, \quad \Psi(a\otimes b) = b\otimes a, \quad \Psi(b\otimes b) = q^2 b\otimes b, \quad \Psi(c\otimes a) = a\otimes c, \quad \Psi(c\otimes c) = q^2 c\otimes c, \\ \Psi(a\otimes c) = q^2 c\otimes a + (q^2-1)a\otimes c, \quad \Psi(a\otimes d) = d\otimes a + (q^2-1)b\otimes c + (q-q^{-1})^2 a\otimes a, \\ \Psi(b\otimes a) = q^2 a\otimes b + (q^2-1)b\otimes a, \quad \Psi(c\otimes b) = b\otimes c + (1-q^{-2})a\otimes a, \\ \Psi(b\otimes c) = c\otimes b + (1-q^{-2})(d\otimes a + a\otimes d) + (q-q^{-1})^2 b\otimes c - (2-3q^{-2}+q^{-4})a\otimes a, \\ \Psi(b\otimes d) = d\otimes b + (q^2-1)b\otimes d(q^{-2}-1)b\otimes a + (q-q^{-1})^2 a\otimes b, \\ \Psi(c\otimes d) = q^2 d\otimes c + (q^2-1)c\otimes a, \quad \Psi(d\otimes a) = a\otimes d + (q^2-1)b\otimes c + (q-q^{-1})^2 a\otimes a, \\ \Psi(d\otimes b) = q^2 b\otimes d + (q^2-1)a\otimes b, \quad \Psi(d\otimes c) = c\otimes d + (q^2-1)d\otimes c + (q-q^{-1})^2 c\otimes a + (q^{-2}-1)a\otimes c, \\ \Psi(d\otimes d) = q^2 d\otimes d + (q^2-1)c\otimes b + (q^{-2}-1)b\otimes c - -(1-q^{-2})^2 a\otimes a. \end{gathered}$$

Note that we need to specify many more relations than for the multiplication. This is because unlike with multiplication, we cannot deduce $\Psi(b\otimes a)$ from $\Psi(a\otimes b)$ etc. Also note that for $q=1$ we obtain the trivial braiding $\Psi = \tau$.

There is also a multiplicative braiding on $\mathbb{R}_q^{1,3}$, which can be deduced analogously from the matrix $\mathbf{R}$. given in appendix A.1. Since we will be primarily concerned with braided matrices viewed as a covector algebra, we will not give the relations explicitly.

By the formalism developed so far, $\mathbb{R}_q^{1,3}$ is automatically equipped with:

- An associated space $V(\mathbf{R}', \mathbf{R})$ of q-Minkowski vectors.
- A space of partial q-derivatives $\langle \frac{\partial}{\partial a}, \dots, \frac{\partial}{\partial d} \rangle$ with the braid statistics of $V(\mathbf{R}', \mathbf{R})$.
- A space of differential forms $\Lambda(-\mathbf{R}, -\mathbf{R}')$ and an exterior algebra $\Omega(\mathbb{R}_q^{1,3})$.

We now give some details of this braided calculus explicitly. As for the q-derivative relations, the multplication relations for vectors can be read off immediately from the rows of $\mathbf{R}'$ using the compact form of the definition of $V(\mathbf{R}', \mathbf{R})$:

$$\begin{bmatrix} \partial^a \partial^a \\ \partial^a \partial^b \\ \vdots \\ \partial^d \partial^d \end{bmatrix} = \mathbf{R}' \begin{bmatrix} \partial^a \partial^a \\ \partial^b \partial^a \\ \vdots \\ \partial^d \partial^d \end{bmatrix}.$$

(Note: after a correspondence with the author, these relations appear to be given incorrectly in [6], [11]. The correct relations are those given there, but transposed; e.g. $\partial^c \partial^d \leftrightarrow \partial^d \partial^c$.)

Similarly, we can recover the relations for the braided covector algebra of differential forms using

$$\begin{bmatrix} \mathrm{d}a\mathrm{d}a & \mathrm{d}a\mathrm{d}b & \dots & \mathrm{d}d\mathrm{d}d \end{bmatrix} = \begin{bmatrix} \mathrm{d}a\mathrm{d}a & \mathrm{d}b\mathrm{d}a & \dots & \mathrm{d}d\mathrm{d}d \end{bmatrix} \cdot (-\mathbf{R}).$$

Explicitly, after rearrangement to reduce the equations this gives:

$$\mathrm{d}a\mathrm{d}a = 0, \quad \mathrm{d}b\mathrm{d}b = 0, \quad \mathrm{d}c\mathrm{d}c = 0, \quad \mathrm{d}d\mathrm{d}d = (1 - q^{-2})\mathrm{d}b\mathrm{d}c,$$

$$\mathrm{d}b\mathrm{d}a = -\mathrm{d}a\mathrm{d}b, \quad \mathrm{d}c\mathrm{d}a = -\mathrm{d}a\mathrm{d}c, \quad \mathrm{d}c\mathrm{d}b = -\mathrm{d}b\mathrm{d}c,$$

$$\mathrm{d}d\mathrm{d}c = -q^{-2}\mathrm{d}c\mathrm{d}d + (1 - q^{-2})\mathrm{d}a\mathrm{d}c, \quad \mathrm{d}d\mathrm{d}b = -q^2\mathrm{d}b\mathrm{d}d - (q^2 - 1)\mathrm{d}a\mathrm{d}b,$$

$$\mathrm{d}d\mathrm{d}a = -\mathrm{d}a\mathrm{d}d - (q^2 - 1)\mathrm{d}b\mathrm{d}c.$$

We now verify that $\mathbb{R}_q^{1,3}$ is indeed a q-deformed Minkowski space, i.e. that we recover $\mathbb{R}^{1,3}$ if $q = 1$. For this we only need to recover the familiar $(+, -, -, -)$ metric signature of Minkowski spacetime. This requires a q-deformed version of a determinant on our braided matrices. Some reasonable requirements for such a *braided determinant* are the following:

- The braided determinant reduces to the usual determinant $ad - bc$ for $q = 1$.
- The braided determinant behaves like a scalar: it must be central in $\mathbb{R}_q^{1,3}$ (i.e. it must commute with all other elements of $\mathbb{R}_q^{1,3}$).
- The braided determinant distributes over multiplication.

It turns out that we can find a simple braided determinant element $\underline{\det}$ that satisfies all of these properties:

Lemma 7.12. Let the braided determinant be given by $\underline{\det} = ad - q^2 cb \in \mathbb{R}_q^{1,3}$. This element is central in $\mathbb{R}_q^{1,3}$, and more-over *group-like* with respect to the multiplicative braiding $\Psi.$ in the sense that

$$\Delta.\underline{\det} = \underline{\det} \otimes . \underline{\det}.$$

(The term group-like is in reference to the Hopf algebra kG.)
Moreover, $\underline{\det}$ is bosonic with respect to $\Psi.$, meaning that $\Psi.(\underline{\det} \otimes x) = x \otimes \underline{\det}$ for all $x \in \mathbb{R}_q^{1,3}$.

For a proof of this statement, see [40].

Remark 7.13. The fact that $\underline{\det}$ is grouplike with repsect to $\Psi.$ essentially entails that $\underline{\det}$ distributes over multiplication. To see this we arrange the generators of $\mathbb{R}_q^{1,3}$ into the matrix $\mathbf{u}$ as in equation (7.6), and let $\mathbf{u}'$ be another such copy of $\mathbb{R}_q^{1,3}$. We can then express $\Delta.$ on the generators as

$$\Delta. \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix},$$

where $\cdot$ is a matrix multiplication. In this notation we have for instance that aa' represents $a \otimes . a$. Hence we compute:

$$\begin{aligned}\Delta.\underline{\det}(\mathbf{u}) = \underline{\det}(\mathbf{u}) \otimes \underline{\det}(\mathbf{u}) &\implies \underline{\det}(\Delta.\mathbf{u}) = \underline{\det}(\mathbf{u}) \otimes \underline{\det}(\mathbf{u}) \\ &\implies \underline{\det}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) = \underline{\det}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \underline{\det}\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}.\end{aligned}$$

In this sense we see that $\underline{\det}$ indeed distributes multiplicatively over two copies of $\mathbb{R}_q^{1,3}$, as required.

In addition, there is another interesting central element in $\mathbb{R}_q^{1,3}$; the *quantum trace*:

Lemma 7.14. We define the *quantum* or *braided trace* as $\underline{\mathrm{Tr}} = qd + q^{-1}a \in \mathbb{R}_q^{1,3}$. This element is again central, as well as bosonic with respect to the multiplicative braiding.

Proof. To see that $\underline{\mathrm{Tr}}$ is central, we can use the multiplication relations (7.7) to compute the commutators $[\underline{\mathrm{Tr}}, a], \ldots, [\underline{\mathrm{Tr}}, d]$. The commutators with a and d are clearly zero, since $ad = da$. For the other two commutators, we compute:

$$\begin{aligned}\underline{\mathrm{Tr}} \cdot b &= qdb + q^{-1}ab \\ &= q(bd + (1 - q^{-2})ab) + q^{-1}ab \\ &= qbd + qab - q^{-1}ab + q^{-1}ab \\ &= qbd + q^{-1}ba = b \cdot \underline{\mathrm{Tr}},\end{aligned}$$

and

$$\begin{aligned}\underline{\mathrm{Tr}} \cdot c &= qdc + q^{-1}ac \\ &= q(cd + (q^{-2} - 1)ca) + qca \\ &= qcd + q^{-1}ca - qca + qca \\ &= qcs + q^{-1}ca = c \cdot \underline{\mathrm{Tr}}.\end{aligned}$$

Thus $\underline{\mathrm{Tr}}$ is indeed central. Again, for a proof that $\underline{\mathrm{Tr}}$ is bosonic we refer to [40]. □

Recall that we originally cast Minkowski spacetime as Hermitian matrices via equation (7.1). We are now ready to make the inverse transformation back to genuine space-time coordinates (t, x, y, z).

Definition 7.15. We define the braided spacetime coordinates in the braided covector-algebra $\mathbb{R}_q^{1,3}$ as follows:

$$t = \frac{qd + q^{-1}a}{2}, \quad x = \frac{b + c}{2}, \quad y = \frac{b - c}{2i}, \quad z = \frac{d - a}{2}. \tag{7.8}$$

Here $i \in \mathbb{C}$ is the usual imaginary constant.

Note that because of the braided Hermitian $*$-structure, we have that $t^* = t$, et cetera. This ensures that t, x, y, z are real. The Hermitian structure on the braided matrices of $\mathbb{R}_q^{1,3}$ is essentially the same as for regular matrices. Thus the transformation $(a, b, c, d) \mapsto (t, x, y, z)$ is essentially the same as for the $q = 1$ case, except for the definition of t. This particular form of t is chosen so that t is central, by lemma 7.14. Note that for $q = 1$ we still recover the correct transformation that is inverse to 7.1. Choosing t central is useful for physical theories on q-Minkowski spacetime, since it allows us to treat the t coordinate as 'special': e.g. we can still split t off of (x, y, z) in a Hamiltonian formalism without the noncommutativity of $\mathbb{R}_q^{1,3}$ causing new problems.

From the braided spacetime coordinates we can recover the metric signature of $\mathbb{R}_q^{1,3}$ using $\det_q$. First we compute the (braided) matrix metric $\eta_M^{\mu\nu}$:

Lemma 7.16. The matrix metric on $\mathbb{R}_q^{1,3}$ is given by

$$\eta_M^{\mu\nu} = \frac{1}{2}\begin{bmatrix} q^{-2}-1 & 0 & 0 & 1 \\ 0 & 0 & -q^2 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Here we use the index convention $(x_0, x_1, x_2, x_3) = (a, b, c, d)$.

Proof. By definition, we have

$$\eta_M^{\mu\nu} = \frac{1}{2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\underline{\det} = \frac{1}{2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}(ad - q^2 cb).$$

Here the partial derivatives are q-deformed, so they have the statistics of braided vectors. Thus we must apply the braided Leibniz rule to compute e.g. $\partial^b\partial^c(bc)$. As an example we will compute the entry η^{00}: equation (6.16) gives us:

$$\begin{aligned}\partial^i(x_{i_1}x_{i_2}) &= \delta^i{}_{i_1}x_{i_2} + x_{j_2} + \delta^i{}_{j_1}x_{j_2}(PR)^{j_1}{}_{i_1}{}^{j_2}{}_{i_2} \\ &= \delta^i{}_{i_1}x_{i_2} + x_{j_2}R^{j_2}{}_{i_1}{}^{i}{}_{i_2},\end{aligned}$$

using that $(PR)^i{}_j{}^k{}_l = R^k{}_j{}^i{}_l$. Here P is the 16×16 matrix of τ. We can compute these quantities using `contractor.m` in appendix B, if we write $a = u_{(1,1)}$, $b = u_{(1,2)}$ etc. (in accordance with the notation $u^{i_0}{}_{i_1} = u_I$). Doing so yields the following computations:

$$\begin{aligned}\partial^a(ad) &= \partial^{(1,1)}(u_{(1,1)}u_{(2,2)}) \\ &= u_{(2,2)} + u_I R^I{}_{(1,1)}{}^{(1,1)}{}_{(2,2)} \\ &= u_{(2,2)} + u_{(1,1)}(q-q^{-1})^2 \implies \partial^a\partial^a(ad) = (q-q^{-1})^2,\end{aligned}$$

where the final implication uses $\partial^I u_J = \delta^I{}_J$. Similarly:

$$\begin{aligned}\partial^a(cb) &= \partial^{(1,1)}(u_{(2,1)}u_{(1,2)}) \\ &= u_I R^I{}_{(2,1)}{}^{(1,1)}{}_{(1,2)} \\ &= u_{(1,1)}(1-q^{-2}) \implies \partial^a\partial^a(cb) = 1-q^{-2}\end{aligned}$$

Thus in total:

$$\begin{aligned}\partial^a\partial^a(\underline{\det}) &= (q-q^{-1})^2 - q^2(1-q^{-2}) \\ &= q^2 - 2 + q^{-2} - q^2 + 1 \\ &= q^{-2} - 1,\end{aligned}$$

as required. The other entries of $\eta_M^{\mu\nu}$ are computed analogously. □

As we would hope, $\eta_M^{\mu\nu}$ agrees with the matrix metric of classical Minkowski spacetime $\mathbb{R}^{1,3}$ for $q = 1$. Similarly, we can find a q-deformed version of equation (7.2). This is the content of the following lemma:

Lemma 7.17. The braided determinant $\underline{\det}$ is expressed in braided spacetime coordinates as

$$ad - q^2 cb = \frac{4q^2}{(q^2+1)^2}t^2 - q^2x^2 - q^2y^2 - \frac{2(q^4+1)q^2}{(q^2+1)^2}z^2 + 2q\left(\frac{q^2-1}{q^2+1}\right)^2 tz. \tag{7.9}$$

This reduces to equation (7.2) for $q = 1$.

Proof. This proof amounts to a computation, which is provided in appendix A.4. □

Lemma 7.17 is the definitive justification that $V\check{}(\mathbf{R}', \mathbf{R})$ with Hermitian $*$-structure is a q-deformed version of $\mathbb{R}^{1,3}$, since it reduces to a $(+,-,-,-)$ signature as $q \to 1$. This equation also allows us to read off the braided metric η immediately, namely by using equation 7.3

8 Consequences of q-Deformation

In this section we discuss some implications of q-deformation for theories of high-energy Physics. First we turn to somewhat concrete considerations, and discuss the q-deformed field theory of a free scalar field in q-Minkowski space. Next we will discuss a realization of q-Minkowski space as a universal enveloping algebra of a q-deformed version electroweak symmetry algebra, and finally we show how the braided geometry approach unifies the most basic ingredients for Lorentz symmetry and supersymmetry.

8.1 q-Scalar Fields

As a first application of q-deformation we derive a q-deformed field theory in q-Minkowski space. We will only consider the most basic field theory in detail: that of a scalar field φ. This discussion is largely based on [43]. As we shall see, most of the basic needed machinery is already provided by the braided derivatives and braided exterior algebra of subsections 6.3 and 6.4.

As before, it is paramount that we first have a thorough understanding of the $q = 1$ case. In this case, we consider a scalar field $\varphi(t, x, y, z)$ to be a 0-form on Minkowski spacetime that satisfies the *d'Alembert equation*

$$\Delta\varphi = 0.$$

Here Δ is the generalized *Laplace-Beltrami operator*. More on the definition of this operator later. As suggested by the notation, on $\mathbb{R}^3$ the operator Δ is equal to the gradient of the divergence, i.e. $\Delta = \nabla^2$. On Minkowski space it is given by

$$\Delta = \partial_\mu\partial_\nu\eta^{\mu\nu} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2},$$

which is the familiar *d'Alembertian operator* $\Box$. The Laplace-Beltrami operator can be defined algebraically on any oriented pseudo-Riemannian manifold, using the *Hodge $*$-operator*. We introduce this now for pseudo-Riemannian vector spaces.

Suppose M is an n-dimensional pseudo-Riemannian vector spaces, i.e. a vector space equipped with a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$. Then this bilinear form induces an inner product on k-forms via the *Gram determinant*: for k-forms $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k$ and $\beta = \beta_1 \wedge \cdots \wedge \beta_k$ we define

$$\langle\alpha, \beta\rangle := \det(\langle\alpha_i, \beta_j\rangle).$$

Using this, we define the *Hodge $*$-operator* to be a map $* : \Omega_k \to \Omega n - k$, with the following property for k-forms α, β:

$$\alpha \wedge *\beta = \langle\alpha, \beta\rangle\omega.$$

Here ω is the standard volume form $e_1 \wedge \cdots \wedge e_n$ for an orthonormal oriented basis $\{e_1, \ldots, e_n\}$. This defines $*\beta$ uniquely: existence and uniqueness of $*\beta$ is shown in [44]. An explicit computation of $*$ is given in [45], in terms of the metric and the *Levi-Civita ϵ-tensor*, which is the completely anti-symmetric tensor on $\mathbb{R}^n$. Using the Hodge $*$-operator, we define the *coderivative* of k-forms. This is the natural counterpart of the exterior derivative $\mathrm{d} : \Omega_k \to \Omega_{k+1}$, in that it is a map from Ω_k to Ω_{k-1}. It is defined by

$$\delta := * \,\mathrm{d}\, * .$$

This indeed defines a map $\Omega_k \to \Omega_{k-1}$, since it is a composition of maps that follows the route

$$\Omega_k \to \Omega_{n-k} \to \Omega_{n-k+1} \to \Omega_{n-(n-k+1)} = \Omega_{k-1}.$$

Finally, we are ready to give an algebraic definition of the Laplace-Beltrami operator, generalized to pseudo-Riemannian vector spaces. (In this context it is also called the *Laplace-de Rham operator.*) It is defined as

$$\Delta := \mathrm{d}\delta + \delta\mathrm{d} : \Omega_k \to \Omega_k.$$

Now that we have a sufficiently thorough understanding of the $q = 1$ case, we can move on to considering a q-deformed 0-form φ on q-Minkowski space that satisfies a q-deformed d'Alembert equation. For this we must develop a q-deformed Laplace-Beltrami operator, and all the associated machinery. Luckily, we have already done most of the work in subsection 6.4 where we developed braided exterior algebra, and we already have a braided metric η on q-Minkowski space from section 7. Thus we proceed for the specific case of of q-Minkowski spacetime.

First, to give a properly q-deformed Hodge $*$-operator we need a braided version of the q-epsilon tensor. For this we can use the braided differential forms of the braided group $\Lambda(R', R)$. For an n-dimensional braided covector-algebra this gives a top form $\theta_1\theta_2\ldots\theta_n$ in $\Lambda(R', R)$ [6]. We then define the rank n q-epsilon tensor as

$$\epsilon^{i_1 i_2 \ldots i_n} = \frac{\partial}{\partial\theta_{i_1}} \cdots \frac{\partial}{\partial\theta_{i_n}} \theta_1 \ldots \theta_n.$$

Since the multiplication in $\Lambda(R', R)$ is given by $-R$, the $-R$-symmetry of this tensor is equivalent to complete R-antisymmetry [6], [39]. Using the braided Leibniz rule, one can find ϵ to be given explicitly by [6], [39]:

$$\epsilon^{i_1 i_2 \ldots i_n} = ([n; -R']!)^{i_n \ldots i_1}_{12 \ldots n},$$

where $[\,\cdot\,]$ indicates the generalized braided integers, as before. Now that we are equipped with a braided ϵ tensor and a braided metric, we can define the *braided Hodge $*$-operator*. We define it as a map on the braided exterior algebra Ω of a braided covector-algebra $V\check{}(R', R)$ of dimension n:

$$*(\theta_{i_1} \ldots \theta_{i_k}) := \epsilon^{a_1 \ldots a_m b_n \ldots b_{m+1}} \eta_{a_1 i_1} \ldots \eta_{a_m i_m} \theta_{b_{m+1}} \ldots \theta_{b_n}.$$

Here the lower indices on η indicate that it is the inverse of the braided metric for q-Minkowski spacetime. From the above definition it is clear that $*$ is indeed a map from Ω_k to Ω_{n-k}. This expression for $*$ reduces to that of [45] for $q = 1$ (up to a normalization factor), and is taken from [37], [43]. For the further development of q-scalar field theory we proceed along the lines of [43].

Now that we have a braided Hodge $*$-operator we proceed to define a *braided coderivative* $\delta : \Omega_k \to \Omega_{k-1}$ and a *braided Laplace-Beltrami operator* $\Delta : \Omega_k \to \Omega_k$ exactly as before, namely:

$$\delta := *\mathrm{d}* \qquad \text{and} \qquad \Delta := \delta\mathrm{d} + \mathrm{d}\delta.$$

We are now again in the position to define the d'Alembert equation; this time for q-deformed spacetime:

Definition 8.1. We define a solution of the q-d'Alembert equation to be a braided 0-form φ of $\mathbb{R}^{1,3}_q$ such that

$$\Delta\varphi = 0,$$

where Δ is the braided Laplace-Beltrami operator.

One can show that for $\mathbb{R}^{1,3}_q$ we have $\Delta = \Box$, where $\Box$ is the *q-d'Alembertian* [43]:

$$\Box = \partial_\mu \partial_\nu \eta^{\mu\nu},$$

in analogy with $q = 1$ Minkowski space. In what follows we will stick to Δ, for clarity.

Of course, on a 0-form we have $\delta\varphi = 0$, hence the q-d'Alembert equation reduces to

$$\delta \mathrm{d}\varphi = 0.$$

To conclude our brief analysis of q-scalar field theory, we construct a family of plane-wave solutions to the q-d'Alembert equation. Clearly, since all the maps in the construction of Δ are linear, we have that any linear combination of such plane-wave solutions is again a solution to the q-d'Alembert equation. This opens up a wide class of solutions to the equation, in a very concrete analogy to the Fourier transform.

First, we take a copy of the braided vector-algebra $V(\mathbf{R}', \mathbf{R})$, and regard it as q-Minkowski momentum space, denoting its generators by p^a. Here $\mathbf{R}', \mathbf{R}$ are of course the R-matrices that generate $\mathbb{R}_q^{1,3}$. We then define the *q-deformed (momentum) light cone* P_0 to be $V(\mathbf{R}', \mathbf{R})$ modulo the relation

$$\eta_{\mu\nu} p^\mu p^\nu = 0.$$

Here η is the braided metric corresponding to equation (7.9). This clearly reduces to the usual light cone requirement as η reduces to $\mathrm{diag}(1, -1, -1, -1)$ for $q = 1$. The coaction $\beta : P_0 \to V(\mathbf{R}', \mathbf{R}) \otimes A(\mathbf{R})$ descends to a coaction $\beta : P_0 \to P_0 \otimes A(\mathbf{R})$, which means that the q-deformed light cone is covariant under the q-Lorentz symmetry induced by $A(\mathbf{R})$ [43]. (More details on $A(\mathbf{R})$ as q-Lorentz symmetry is given in subsection 8.3.)

Next, we define a *q-exponential* on $\mathbb{R}_q^{1,3} \underline{\otimes} P_0$. This q-exponential will be slightly different from that of section 2, but the underlying idea will be the same: namely to express $\exp_q$ as a formal power series, but with integers replaced by braided integers.

Definition 8.2. We define the (complex) *q-exponential* on $\mathbb{R}_q^{1,3} \underline{\otimes} P_0$ as a function that takes in a row vector $\mathbf{x}$ representing a q-Minkowski covector, and column vector $\mathbf{p}$ representing a vector of q-Minkowski differential operators. This function is given by the power sum

$$\exp_q(i\mathbf{x} \otimes \mathbf{p}) := \sum_{n=0}^{\infty} \frac{i^n}{[n]!} \mathbf{x}_1 \mathbf{x}_2 \ldots \mathbf{x}_n \otimes \mathbf{p}_n \mathbf{p}_{n-1} \ldots \mathbf{1}.$$

where i is the imaginary unit. After inputs from $\mathbb{C}$ are chosen for $\mathbf{x}, \mathbf{p}$ we obtain an element of $\mathbb{C} \otimes \mathbb{C}$, which is identified with a complex scalar under the isomorphism $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$.

In the above definition we use the compact notation from remark 6.15. Thus the vectors and covectors in the power sum increase in size n-fold every term. Note that the terms $\mathbf{x}_i$ are *column vectors*, while the terms $\mathbf{p}_i$ are *column vectors*. Thus each $\mathbf{x}_i$ contracts against $\mathbf{p}_i$, and $\exp_q$ returns a number for every input $\mathbf{x} \otimes \mathbf{p}$. As one would hope, the q-exponential is a scalar under the coaction of $A(\mathbf{R})$ [43]. For $q = 1$ we simply recover the usual power sum for an exponential.

In [43] it is shown that a solution to the q-d'Alembert equation indexed by P_0 is given by

$$\varphi_{\mathbf{p}} := \exp_q(i\mathbf{x} \otimes \mathbf{p}).$$

Solutions of this type are the *plane-wave solutions* to the q-d'Alembert equation, in analogy with the $q = 1$ case. In fact, these solution *only* exists on the q-deformed momentum light-cone, indicating that this wave equation must describe a *massless* field [43]. This may be interpreted to mean that our deformation parameter q induces some natural mass scale, in comparison to which the masses of usual particles are negligible. We come back to this point in section 9.

Analogously to the q-d'Alembert equation, we can go on to define different field equations of q-Minkowski spacetime using the q-deformed exterior derivative and Laplace-Beltrami operator. For instance, a 0-form φ is said to solve the q-deformed Klein-Gordon equation if it satisfies

$$(\Delta + m^2)\varphi = 0,$$

where m is some number. Similarly a braided 1-form on q-Minkowski space is said to be a solution to the *q-Maxwell equation* if

$$\delta \mathrm{d}A = 0.$$

For more details on these field equations see [43]. Here it suffices to note two final results from [43]: first, the q-Maxwell equation can be rewritten as

$$\Box A_\nu - \partial_\nu \partial^\mu A_\mu = 0.$$

Here $\partial_\mu = \eta_{\mu\nu}\partial^\mu$ using the braided metric η. This shows that we recover the classical equation $\partial^\mu\partial_\mu A_\nu - \partial^\mu\partial_\nu\partial_\mu$ in the limit $q \to 1$. Finally, it is of note that the q-deformed solutions to the q-Maxwell equations still have a gauge freedom

$$A \mapsto A + \mathrm{d}\varphi,$$

where φ is a solution of the q-d'Alembert equation. This indicates that the q-deformed theory is still well-behaved mathematically, and algebraically not at all unlike the classical equations.

8.1.1 Singularities in Quantum Field Theory

In relation to the discussion of q-deformed fields, in this subsection we quickly describe one of the most persistent problems in modern quantum field theory: Ultra-Violet (UV) divergences. This is one of the problems that is hopefully solved by a more advanced, perhaps q-deformed, version of field theory. In this subsection we make no attempt to develop q-deformed quantum field theory or to solve the arising divergences; we only intend to shed light onto the issue at hand.

We consider the simplest example of UV divergence in high-energy particle physics: the vacuum energy of a free, real scalar field $\phi(t,\vec{x})$. This discussion is based on [1]. The Lagrangian density $\mathcal{L}$ of ϕ is given by

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 \\ &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2.\end{aligned}$$

In the Hamiltonian formalism this gives the canonical momentum $\pi = \partial\mathcal{L}/\partial\dot{\phi} = \dot{\phi}$. This yields the Hamiltonian density

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2,$$

and the Hamiltonian $H = \int \mathcal{H}\, d^3x$. Note that the Hamiltonian formalism is not manifestly Lorentz invariant and requires the determination of a time t. This is not in contradiction with our q-deformations if $q \neq 1$ since t is central in $\mathbb{R}_q^{1,3}$, hence there is no novel time-uncertainty is introduced by the noncommutativity of $\mathbb{R}_q^{1,3}$.

To proceed, we impose the canonical quantization on ϕ, π:

$$[\phi,\phi] = [\pi,\pi] = 0, \qquad [\phi(\vec{x}), \pi(\vec{\pi})] = i\delta(\vec{x}-\vec{y}).$$

Next we expand ϕ and π via the *modal expansions* in terms of new operators $a_{\vec{p}}, a_{\vec{p}}^\dagger$ labelled by a 3-vector $\vec{p}$:

$$\begin{aligned}\phi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}}\right), \\ \pi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}(-i)\frac{\omega_{\vec{p}}}{\sqrt{2}}\left(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}}\right).\end{aligned}$$

Here $\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$, and the integrals are implied to be over all of $\mathbb{R}^3$. To recover the correct commutators for ϕ, π we must have

$$[a_{\vec{p}}^\dagger, a_{\vec{q}}^\dagger] = [a_{\vec{p}}, a_{\vec{q}}] = 0, \qquad [a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta(\vec{p} - \vec{q}).$$

The operators $a_{\vec{p}}^\dagger$ and $a_{\vec{p}}$ are known as *creation operators* and annihilation operators respectively, in analogy with $a_\pm$ from section 2. It is not immediately clear that these operators are useful, but they have many wonderful properties. For instance:

$$[H, a_{\vec{p}}^\dagger] = \omega_{\vec{p}} a_{\vec{p}}^\dagger \qquad \text{and} \qquad [H, a_{\vec{p}}] = -\omega_{\vec{p}} a_{\vec{p}}.$$

This allows us to interpret $a_{\vec{p}}^\dagger$ as 'creating' a particle of energy $\omega_{\vec{p}}$ in the field. Similarly $a_{\vec{p}}$ destroys such a particle, if one is present. In terms of these operators the Hamiltonian takes on the simple form

$$H = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left(a_{\vec{p}} a_{\vec{p}}^\dagger + a_{\vec{p}}^\dagger a_{\vec{p}} \right).$$

For the field ϕ, we define the *vacuum* or *group state* $|0\rangle$ to be a state of ϕ such that $a_{\vec{p}}|0\rangle = 0$. We define the *single-particle state* $|\vec{p}\rangle$ by

$$|\vec{p}\rangle := a_{\vec{p}}^\dagger |0\rangle.$$

This is to be interpreted as a state of ϕ containing a single particle of momentum $\vec{P}$ and energy $\omega_{\vec{p}}$. So far we have only treated $\vec{p}$ as a label for the modal expansions, but as the notation suggests $\vec{p}$ can be interpreted as a genuine momentum. This is done using the *momentum operator* $\vec{P}$ whose 3-vector components are defined by

$$P^i = \int \frac{d^3p}{(2\pi)^3} p^i a_{\vec{p}}^\dagger a_{\vec{p}}.$$

For $\vec{P}$ we then have that $\vec{P}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle$, so that $|\vec{p}\rangle$ is indeed a momentum eigenstate with eigenvalue $\vec{p}$. To see that $\vec{P}$ indeed represents the momentum, one can show that $\vec{P}^i$ acts on states of ϕ as

$$P^i : \varphi \mapsto -i \frac{\partial \varphi}{\partial x^i},$$

which is in accordance with the familiar canonically quantized momentum operator. More-over, one can show that the conserved Noether charges associated to translation invariance of $\mathcal{L}$ are given by

$$j^i = -\int d^3x \pi \nabla \phi = \int \frac{d^3p}{(2\pi)^3} p^i a_{\vec{p}}^\dagger a_{\vec{p}} = P^i.$$

Since the Noether charge associated to translation symmetry is classically exactly the same as momentum, this shows that P^i indeed gives the momentum of a state $|\vec{p}\rangle$.

We are now ready to give the simplest UV divergence in quantum field theory: the vacuum energy. To this end we return to H and compute $H|0\rangle$:

$$\begin{aligned}
H|0\rangle &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left(a_{\vec{p}} a_{\vec{p}}^\dagger + a_{\vec{p}}^\dagger a_{\vec{p}} \right) |0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left(a_{\vec{p}} a_{\vec{p}}^\dagger + \frac{1}{2}(2\pi)^3 \delta(0) \right) |0\rangle \\
&= \left(\int d^3p \frac{1}{2} \omega_{\vec{p}} \delta(0) \right) |0\rangle \\
&= \infty |0\rangle.
\end{aligned}$$

Thus we find that the vacuum energy E_0 diverges. This is somewhat of a two-fold divergence: we integrate $\delta(0) = \infty$ over all of $\mathbb{R}^3$. The first divergence due to $\delta(0)$ is easy enough to get rid of; it occurs because we compute the *total* energy of an infinite spacetime. If we instead consider the energy *density* $\mathcal{E}_0$ then we find

$$\mathcal{E}_0 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\omega_{\vec{p}} = \frac{1}{2(2\pi)^3}\int \sqrt{\vec{p}^2 + m^2}\, d^3p.$$

This integral still diverges, since $\omega_{\vec{p}} \xrightarrow{\vec{p}\to\infty} \infty$. This is exactly the UV divergence of the vacuum energy. Normally in physics this is handled by arguing that we are only interested in energy *differences*. Hence we can simply recast the Hamiltonian as

$$H = \int \frac{d^3p}{(2\pi)^3}\omega_{\vec{p}} a^\dagger_{\vec{p}} a_{\vec{p}}.$$

This essentially subtracts off the UV divergence infinity to give $H|0\rangle = 0$ so that $E_0 = 0$.

One (rather unphysical) way to solve this problem is to remark our hubris, and assume that our field theory must break down at some energy scale. We then choose to neglect energies above this scale, and correspondingly impose a *momentum cutoff* $|\vec{p}|_{\max}$ on the integral. This allows us to find a finite vacuum energy of

$$\begin{aligned}
\mathcal{E}_0 &= \frac{1}{2(2\pi)^3}\int_B \sqrt{\vec{p}^2 + m^2}\, d^3p \\
&= \frac{1}{2(2\pi)^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{|\vec{p}|_{\max}} r^2\sin(\theta)\sqrt{r^2+m^2}\, dr d\theta d\phi \\
&= \frac{1}{2\pi^2}\int_0^{|\vec{p}|_{\max}} r^2\sqrt{r^2+m^2}\, dr \\
&= \frac{1}{(4\pi)^2}\left(\sqrt{|\vec{p}|^2_{\max} + m^2}(2|\vec{p}|^3_{\max} + m^2|\vec{p}|_{\max}) - m^4 \operatorname{arcsinh}\left(\frac{|\vec{p}|_{\max}}{m}\right)\right) < \infty.
\end{aligned}$$

Here $B \subseteq \mathbb{R}^3$ is a ball of radius $|\vec{p}|_{\max}$ centered at the origin. This approach to regularization is quite crude: it completely destroys any symmetry of the system under the Poincaré group. It is hoped that a well-developed theory of q-deformed field theory on q-Minkowski spacetime can impose a more natural regularization of UV divergences. For instance a q-deformation of the Poincaré group may allow for a variation of bounded momentum regularization; without the destruction of q-Poincaré symmetry. For more modern steps in this direction, see [46].

8.2 q-Minkowski Space as a Electroweak Braided Lie Algebra

In this subsection we will discuss a unification between Minkowski space and the symmetry group of the electroweak interaction, when both are q-deformed. Namely, we will find that there is a so-called *braided Lie group* $\mathfrak{gl}_{q,2}$ and a suitable braided notion $U(\cdot)$ of universal enveloping algebra such that

$$\mathbb{R}^{1,3}_q \cong BM_q(2) \cong U(\mathfrak{gl}_{q,2}).$$

Here $BM_q(2)$ are the q-deformed braided matrices; recall that these become q-Minkowski space under the addition of a braided $*$-structure. We will find that as $q \to 1$, $\mathfrak{gl}_{q,2}$ reduces to $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ which is the Lie algebra of the electroweak symmetry group $SU(2) \oplus U(1)$. Meanwhile as $q = 1$ we recover Minkowski space on the left-hand side. In conclusion:

$$\mathbb{R}^{1,3} \xleftarrow{q=1} \mathbb{R}^{1,3}_q \cong BM_q(2) \cong U(\mathfrak{gl}_{q,2}) \xrightarrow{q=1} U(\mathfrak{su}(2) \oplus \mathfrak{u}(1)).$$

This unification is a 'purely quantum' phenomenon [6]; it is not apparent for $q = 1$.

In this subsection we will be mathematically rigorous, although the necessary detailed proofs are deferred to references [47], [48]. To this end, we must define what is meant by a *braided* Lie algebra:

Definition 8.3. Let $(\mathcal{C}, \otimes, \Psi)$ be a braided category. A braided Lie algebra $(\mathcal{L}, \Delta, \epsilon, [\cdot,\cdot])$ is a coalgebra object $(\mathcal{L}, \Delta, \epsilon)$ in $\mathcal{C}$ equipped with a *bracket* morphism $[\cdot,\cdot] : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$. The bracket must obey the conditions of figure 8.1, and it must be a coalgebra morphism in the sense of figure 8.2.

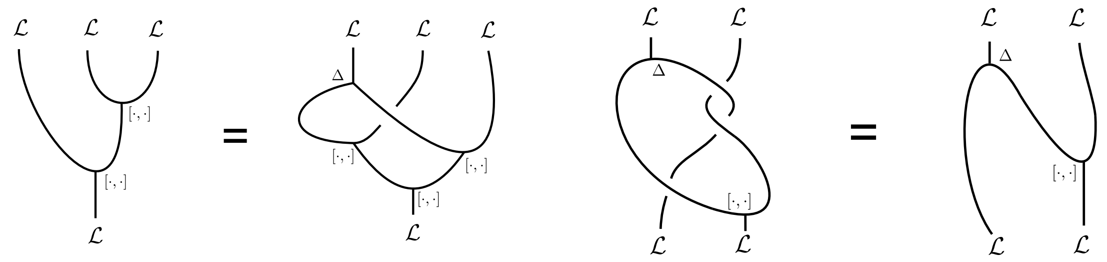


Figure 8.1: Axioms for a braided Lie algebra.

Figure 8.2: The bracket $[\cdot,\cdot]$ forms a coalgebra morphism.

It is not immediately clear that this definition is a sensible generalization of a Lie algebra to braided categories. The defining properties of a braided Lie algebra are motivated in [47] using properties of the adjoint representation in braided categories; also see subsection 5.2. As a motivation we note that the structure of a braided Lie algebra (with trivial braiding) is implicit for any Lie algebra [48]:

Example 8.4. Let $\mathfrak{g}$ be an ordinary Lie algebra. Then we can construct a braided Lie algebra $\mathcal{L}$ in `Vec` as

$$\mathcal{L} := \mathbb{C} \oplus \mathfrak{g}.$$

This indeed becomes a braided Lie algebra with the following coalgebra structure and bracket defined for a generator ξ of $\mathfrak{g}$ as:

$$\Delta\xi = \xi \otimes 1 + 1 \otimes \xi, \qquad \Delta 1 = 1 \otimes 1,$$

$$\epsilon\xi = 0, \qquad \epsilon 1 = 1,$$

$$[1, \xi] = \xi, \qquad [\xi, 1] = 0.$$

On two generators ξ, η we take the bracket $[\xi, \eta]$ to be that of $\mathfrak{g}$.

Before we can move on to explicit examples of braided Lie algebras we must introduce the notion of the *braided universal enveloping algebra*:

Proposition 8.5. Let $(\mathcal{L}, \Delta, \epsilon)$ be a braided Lie algebra in an *Abelian* braided category (here Abelian means that we have direct sums in the category; see [20] for details). Then there is a braided bialgebra $U(\mathcal{L})$ generated by 1 and $\mathcal{L}$ modulo the relations shown in figure 8.3. We refer to $U(\mathcal{L})$ as the *braided universal enveloping algebra* of $\mathcal{L}$.

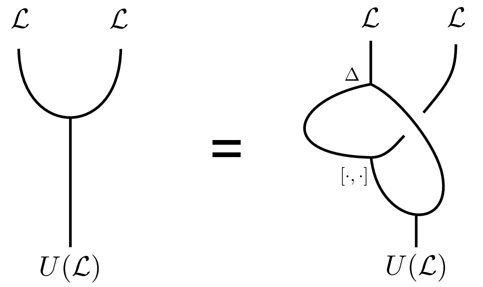


Figure 8.3: The relations for the braided universal eveloping algebra.

Proof. This construction is proven in [47], where it is proposition 4.2. □

An important class of braided Lie algebras is that of the *matrix braided Lie algebras* from [47]:

Example 8.6. Suppose R is a bi-invertible solution to the n-dimensional QYBE. Let $\mathcal{L}$ be the vector space generated by the matrix generators $\{u_I\}$, where $I = (i_0, i_1)$ denotes a double index as in subsection 7.2. Then define the tensors

$$\mathbf{R}.^I{}_J{}^K{}_L = R^{j_0}{}_a{}^d{}_{k_0}(R^{-1})^a{}_{i_0}{}^{k_1}{}_b R^{i_1}{}_c{}^b{}_{l_1}\widetilde{R}^c{}_{j_1}{}^{l_0}{}_d,$$

$$c^K{}_{IJ} = \widetilde{R}^a{}_{i_1}{}^{j_0}{}_b(R^{-1})^b{}_{k_0}{}^{i_0}{}_c R^{k_1}{}_e{}^c{}_d R^d{}_a{}^e{}_{j_1}.$$

These define a braided Lie algebra with the following coalgebra structure

$$\Delta u^i{}_j = u^i{}_k \otimes u^j{}_j, \qquad \epsilon u^i{}_j = \delta^i{}_j,$$

and with the following braiding and bracket:

$$\Psi(u_J \otimes u_L) = u_K \otimes u_I \mathbf{R}.^I{}_J{}^K{}_L,$$

$$[u_I, u_J] = c^K{}_{IJ} u_K.$$

We immediately recognize this matrix $\mathbf{R}.$ as the matrix from subsection 7.2 that gives the multiplicative braiding $\Psi.$ on the braided matrices $B(R)$. Filling in the relations in figure 8.3 of proposition 8.5 gives [48]:

$$R^k{}_b{}^i{}_a u^a{}_c R^c{}_j{}^b{}_d u^d{}_l = u^k{}_b R^b{}_c{}^i{}_a u^a{}_d R^d{}_j{}^c{}_l.$$

This is exactly equation (7.4). Thus we recover exactly the structure of the multiplicative braided matrices $B(R)$ from lemma 7.4. In conclusion, the braided universal enveloping algebra of the matrix braided Lie algebra $\mathcal{L}$ is given by [47]:

$$U(\mathcal{L}) = B(R).$$

In particular, we can use the R-matrix that generates q-Minkowski space from example 7.7 to construct a braided Lie algebra whose braided universal enveloping algebra is isomorphic to $B(R) = BM_q(2) \cong \mathbb{R}^{1,3}_q$, equipped with the multiplicative braiding. This braided Lie algebra is called $\mathfrak{gl}_{q,2}$:

Example 8.7. In example 8.6 take the R-matrix from example 7.7. This generates the 4-dimensional braided Lie algebra $\mathfrak{gl}_{q,2}$ from [48]: it has basis $\{h, x_+, x_-, \gamma\}$. The braided Lie bracket is given by

$$[h, x_+] = (q^{-2}+1)q^{-2}x_+ = -q^{-2}[x_+, h], \qquad [h, x_-] = -(q^{-2}+1)x_- = -q^{-2}[x_-, h],$$

$$[x_+, x_-] = q^{-2}h = -[x_-, x_+], \qquad [h,h] = (q^{-4}-1)h,$$
$$[\gamma, h] = (1-q^{-4})h, \qquad [\gamma, x_+] = (1-q^{-4})x_+, \qquad [\gamma, x_-] = (1-q^{-4})x_-.$$

This can be shown by immediate computation of the tensor $c^K{}_{IJ}$, analogously to the computations done in appendix A.1. We see that in the limit $q \to 1$, the generator γ decouples completely into the generator for a copy of $\mathfrak{u}(1)$. In total, as $q \to 1$ we find that

$$\mathfrak{gl}_{q,2} \to \mathfrak{su}(2) \oplus \mathfrak{u}(1).$$

Hence we obtain q-Minkowski space with the multiplicative braiding as a q-deformed enveloping algebra of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$. This suggests a novel unification between two fundamental ingredients of electroweak theory, which is unique to the braided approach of q-deformation [6].

This unification possibly gives a motivation for working with $\mathfrak{gl}_{q,2}$ as a quantum gauge symmetry for the unification of electroweak interactions in a q-deformed Yang-Mills theory in q-Minkowski space [49].

8.3 Supersymmetry and Lorentz Covariance

As a final application of q-deformation, we give a cursory overview of the most basic mathematical supersymmetry concepts in relation to braided geometry.

In the braided geometry approach to q-deformation, q-Minkowski space ends up being a braided group in the braided category $\mathcal{M}^{A(\mathbf{R})}$. Here $A(\mathbf{R})$ is a Hopf algebra of quantum matrices. As seen in remark 7.3, these matrices coact on the braided Hermitian matrices of $\mathbb{R}^{1,3}_q$ analogously to how $\mathrm{SL}(2,\mathbb{C})$ acts on the Hermitian matrices of $\mathbb{R}^{1,3}$. In this sense, $A(\mathbf{R})$ generates the q-deformed version of the Lorentz group [36]. This means that the q-Lorentz symmetries of q-Minkowski space manifest themselves through the fact that $\mathbb{R}^{1,3}_q$ is an object of $\mathcal{M}^{A(\mathbf{R})}$.

As it turns out, we can obtain the notion of supersymmetry in exactly the same way: supersymmetry is manifested through a $\mathbb{Z}_2$-grading. Elements of degree 0 are 'bosonic', and elements of degree 1 are 'fermionic'. Note that such a boson-fermion distinction is already present in usual particle physics. There the boson-fermion distinction is between irreducible reprentations of the Poincaré group (which we associate with particles). In contrast, the $\mathbb{Z}_2$-grading we mean here is *of the vector space itself*: this means a vector space with a prescribed number of bosonic (symmetric) and fermionic (antisymmetric) degrees of freedom. In fact, the underlying supermanifold of super Minkowski space is of this form, being isomorphic to a direct sum of usual Minkowski space and a number N of spinor representations of the Lorentz algebra [50]. For instance the $N = 1$ super Minkowski space is known in mathematics as the super vector space $M^{4|4}$, and $N = 2$ supersymmetry corresponds to $M^{4|8}$.

In what follows we will derive that such a $\mathbb{Z}_2$-grading is arrived at in exactly the same way as q-Lorentz symmetry. Namely, supersymmetric objects such as super vector spaces turn out to be objects of the representation category ${}_{k\mathbb{Z}_2}\mathcal{M}$.

Supersymmetry in objects thus manifests itself through the fact that that object lies in the category ${}_{k\mathbb{Z}_2}\mathcal{M}$. This casts supersymmetry and q-Lorentz symmetry as being of exactly the same type: the only difference is the Hopf algebra under consideration. For q-Lorentz symmetry this is the q-Lorentz group generated by $A(\mathbf{R})$, and for supersymmetry this is the Hopf algebra $k\mathbb{Z}_2$ built from $\mathbb{Z}_2 = \{0, 1\}$.

Recall the example of `SuperVec`; the braided category of super vector spaces defined in example 3.22. We have already seen in example 5.8 that a super vector space is equivalent to a representation of the Hopf algebra $k\mathbb{Z}_2$. This suggest an equivalence of ${}_{k\mathbb{Z}_2}\mathcal{M}$ and `SuperVec`. Since the formalism of subsection 5.3 allows us to cast representation categories as braided categories, we can use this formalism to explore such an equivalence.

The following is due to [40]:

Lemma 8.8. Let k be a field such that $\mathrm{Char}(k) \neq 2$. Then $k\mathbb{Z}_2$ is quasitriangular with

$$\mathcal{R} = 2^{-1}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g).$$

Proof. First, it is clear that $\mathcal{R}$ is invertible; namely $\mathcal{R}$ is self-inverse. Indeed, we compute

$$\begin{aligned}\mathcal{R}^2 = 2^{-2}(&(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g) \\ &+ (1 \otimes g + 1 \otimes 1 + g \otimes g - g \otimes 1) \\ &+ (g \otimes 1 + g \otimes g + 1 \otimes 1 - 1 \otimes g) \\ &- (g \otimes g + g \otimes 1 + 1 \otimes g - 1 \otimes 1)) \\ = 2^{-2}&(1 \otimes 1 + 1 \otimes 1 + 1 \otimes 1 + 1 \otimes 1) \\ = 1 \otimes& 1.\end{aligned}$$

Next we verify equation (4.1). It suffices to verify this on the nontrivial generators of $k\mathbb{Z}_2$, of which there is only one: g. For this we compute

$$\tau \circ \Delta g = g \otimes g,$$

and

$$\mathcal{R}(\Delta g)\mathcal{R}^{-1} = \mathcal{R}(g \otimes g)\mathcal{R} = g \otimes g,$$

since $k\mathbb{Z}_2$ is commutative. Thus indeed equation (4.1) is satisfied. Finally we must verify equations (4.2). We only show the first equality, the second being analogous. To this end we compute that

$$(\Delta \otimes \mathrm{id})\mathcal{R} = 2^{-1}(1 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes g + g \otimes g \otimes 1 - g \otimes g \otimes g),$$

and

$$\begin{aligned}\mathcal{R}_{13}\mathcal{R}_{23} &= 2^{-2}(1 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes g + g \otimes 1 \otimes 1 - g \otimes 1 \otimes g) \\ &\qquad \cdot (1 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes g + 1 \otimes g \otimes 1 - 1 \otimes g \otimes g) \\ &= 2^{-1}(1 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes g + g \otimes g \otimes 1 - g \otimes g \otimes g).\end{aligned}$$

Here the computation of the last equality is omitted; this computation is trivial and analogous to that of $\mathcal{R}^2$. This shows that equations (4.2) hold, completing the proof. $\square$

This means that by proposition 5.14, the representation category ${}_{k\mathbb{Z}_2}\mathcal{M}$ is braided. Using the formalism of example 5.8 we can derive the braiding that this quasitriangular structure induces on ${}_{k\mathbb{Z}_2}\mathcal{M}$:

Lemma 8.9. Suppose that V, W are super vector spaces. Then the braiding induced by the quasitriangular structure from lemma 8.8 is given by

$$\Psi_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v.$$

Here $|v|, |w|$ denote the $\mathbb{Z}_2$-gradings induced by the actions of $k\mathbb{Z}_2$ on V, W as in example 5.8.

Proof. Recall the element $p = \frac{1-g}{2}$ from example 5.8. Using this we can write $\mathcal{R}$ as

$$\mathcal{R} = 2^{-1}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g) = 1 \otimes 1 - 2p \otimes p.$$

Using 5.14 we then compute $\Psi_{V,W}$ to be

$$\begin{aligned}\Psi_{V,W}(v \otimes w) &= \tau(\mathcal{R} \triangleright (v \otimes w)) \\ &= (1 \otimes 1 - 2p \otimes p) \triangleright (v \otimes w) \\ &= (1 - 2|v||w|)(w \otimes v) \\ &= (-1)^{|v||w|} w \otimes v,\end{aligned}$$

as claimed. $\square$

This is exactly the braiding on `SuperVec` given in example 3.22. It is also clear that under the equivalence of $\mathbb{Z}_2$-gradings and actions by $k\mathbb{Z}_2$, there is an equivalence between grade-preserving maps and intertwiners. Hence the objects, morphisms, and braidings on `SuperVec` and ${}_{k\mathbb{Z}_2}\mathcal{M}$ all coincide, meaning that we indeed have the expected isomorphism of braided categories

$$\texttt{SuperVec} \cong {}_{k\mathbb{Z}_2}\mathcal{M}.$$

From here we can go on to consider algebra objects in the braided category ${}_{k\mathbb{Z}_2}\mathcal{M}$, correctly recovering the notion of a super algebra:

Definition 8.10. We define a *super algebra* to be an algebra object in the braided category ${}_{k\mathbb{Z}_2}\mathcal{M}$. Suppose that A is such an algebra object. Then $\cdot : A \otimes A \to A$ is a morphism ${}_{k\mathbb{Z}_2}\mathcal{M}$, i.e. a grade-preserving map. This amounts to the requirement that

$$A_i \cdot A_j \subseteq A_{i+j},$$

for $i, j \in \mathbb{Z}_2$. We thus recover the familiar notion of a super algebra (as seen in [21], for instance). We also recover the familiar notion of *supercommutativity* as a form of braided group commutativity: we define a braided group B to be *commutative* if the following diagram commutes:

$$\begin{tikzcd} & B \otimes B \arrow[dr, "\cdot"] & \\ B \otimes B \arrow[ur, "\Psi"] \arrow[rr, "\cdot"'] & & B \end{tikzcd}$$

This is the natural extension of definition 3.2 to braided groups. In the case of ${}_{k\mathbb{Z}_2}\mathcal{M}$ the requirement of commutativity becomes

$$xy = (-1)^{|x||y|} yx,$$

for x, y elements of a super algebra A. This is precisely the familiar supercommutativity, as claimed.

In conclusion, we can indeed recover the basic structures of supersymmetry as a structure induced by the representation category ${}_{k\mathbb{Z}_2}\mathcal{M}$. This suggests that supersymmetry can be cast as a symmetry of the same general type as q-Lorentz symmetry, via the theory of braided groups.

9 Discussion & Conclusion

In this thesis we have broadly looked at an approach for perturbing standard physical systems, known as q-deformation. The process of q-deformation can be applied in several ways: either to model perturbed quantum mechanical systems such as multi-atomic molecules, or to model conjectured quantum corrections on the geometry of spacetime. As an example of the former approach, we started our discussion with an analysis of the q-deformed quantum harmonic oscillator. While we have seen that this system has many interesting physical consequences, the q-deformation applied here is what may be described as 'ad hoc': we have inserted a factor q somewhere in the system with no physical motivation other than that the results are interesting and applicable in mathematical chemistry.

In response to this conceptual problem of q-deformation, in this thesis we have worked towards a more systematic formalism for q-deformation of physics called *braided geometry*. This systematic approach is facilitated by braided categories, which for our purposes amount to q-deformations of the category of vector spaces. To further develop q-deformed physics, we have seen that it is not enough to just consider vector space objects in this braided category: to define our braided version of differentiation we needed the structure of a co-addition on the braided vector spaces. This is one of the motivations for the introduction of Hopf algebras into the braided geometry approach. We have also seen that Hopf algebras are fundamentally the objects that generate the braided categories that we wish to consider in the context of q-deformation. Thus we see that braided geometry is necessarily written in the language of Hopf algebras.

This fact necessitated a formal introduction to the theory of Hopf algebras and their intricacies. This turned us onto a mathematical discussion of quasitriangularity and the representation theory of Hopf algebras. Each of these aspects of Hopf algebra theory, while necessary for q-deformation, also have interesting applications in mathematics: as side-paths in the discussion we have included expositions of Hopf algebras in relation to knot invariants, and a Tannaka-Krein reconstruction result for Hopf algebras.

As a result, this thesis is rather broad in scope. On the physical side we discuss q-deformations, in particular the q-oscillator and braided geometry, and we have considered some of the implications of these q-deformed models. This ended with an explicit description of q-Minkowski spacetime, and a description of the most basic q-deformed field theories in this spacetime. The latter showed that the entire mechanism of braided geometry is reasonably well-behaved mathematically: many of the constructions of exterior algebra go through well in the braided case. This has the pleasant physical implication that we retain some familiar constructions in q-deformed spacetime, such as the gauge freedom of the q-Maxwell equation, as well as a well-defined q-light-cone.

Meanwhile on the mathematical side we gave given a thorough introduction to the general theory of Hopf algebras. This opened up a discussion of knot invariants and furthermore on representation theory. In particular we discussed the construction of knot invariant I_G and quantum knot invariants. We also showed how to reconstruct a Hopf algebra from its category of representations. A final addition was a formal discussion of braided diagrams, which are ubiquitous in the development of braided categories and braided geometry. We used the Turaev theorem to prove that these diagrams are equivalent to tangle diagrams that are equipped with certain extra operations.

Of course, this one thesis cannot possibly cover all that there is to be said about Hopf algebras and braided geometry. In the grand scheme of things this only amounts to a basic introduction. There are still many topics in basic braided geometry that we have not touched upon such as braided integration and braided Fourier theory [51].

Since the introduction of braided spacetime in the 1990's, there have also been many modern developments in this direction of research. For more recent examples, there are the books

Calculus Revisited by Robert W Carroll [52] and *Quantum Riemannian Geometry* by Shahn Majid and Edward J Beggs [53]. The former book give a very comprehensive discussion of ideas from calculus recast in the context of Hopf algebras and braided categories. The latter book details Majid's efforts towards developing braided geometry that is more general than flat spacetime. For instance, it contains a discussion of a braided Schwarzschild metric and quantum corrections to black holes. This indicates that the braided approach to physics has come a long way since the introduction of basic structures like q-Minkowski spacetime.

It is difficult to say whether q-deformed physics and braided geometry are really the 'right' approach towards a theory of quantum gravity. However, the insurmountable rigidity of the mathematical problems posed by quantum gravity and renormalization theory suggest that we cannot go wrong in keeping an open mind. As a final note, we wish to share a personal view on this topic: it is our belief that the linear theory of braided geometry here cannot be more than only a first step towards a unified theory of physics. In other words, the q-deformations derived in this thesis can only be first-order quantum corrections to the theory of spacetime at best. As a motivation for this opinion, it is argued in [43] that scalar field theories in the q-Minkowski spacetime we have developed should be *massless*. This may be a problem for more general q-deformed field theories, since some of the particles we are familiar with do tend to have mass. However, these masses tend to be negligibly small in comparison to the Planck mass. Thus if our braided geometry is only a *first-order* deformation model for Planck-scale physics, then it is permissible that the theory describes all particles to have zero mass. Hence if braided geometry is indeed a physically realistic model at all, then it must only be a crude approximation for something more fundamental. This view appears to be shared in [6].

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References

[1] D. Tong, *Quantum field theory*, 2006.

[2] W. Heisenberg, 'Über die in der theorie der elementarteilchen auftretende universelle länge', *Annalen der Physik*, vol. 32, 20 1938.

[3] A. Shomer, 'A pedagogical explanation for the non-renormalizability of gravity', *arXiv preprint arXiv:0709.3555*, 2007.

[4] H. Wachter, 'Towards a q-deformed quantum field theory', in *Quantum Field Theory*, Springer, 2009, pp. 261–283.

[5] A. Connes, 'Noncommutative geometry', *San Diego*, 1994.

[6] S. Majid, 'Introduction to braided geometry and q-minkowski space', *arXiv preprint hep-th/9410241*, 1994.

[7] S. Majid, M. Heller, R. Penrose *et al.*, *On space and time*. Cambridge University Press, 2008.

[8] J. J. Sakurai and E. D. Commins, *Modern quantum mechanics, revised edition*, 1995.

[9] D. J. Griffiths and D. F. Schroeter, *Introduction to quantum mechanics*. Cambridge University Press, 2018.

[10] V. V. Eremin and A. A. Meldianov, 'The q-deformed harmonic oscillator, coherent states, and the uncertainty relation', *Theoretical and mathematical physics*, vol. 147, no. 2, pp. 709–715, 2006.

[11] S. Majid, *Foundations of quantum group theory*. Cambridge university press, 2000.

[12] A. Macfarlane, 'On q-analogues of the quantum harmonic oscillator and the quantum group $SU(2)_q$', *Journal of Physics A: Mathematical and general*, vol. 22, no. 21, p. 4581, 1989.

[13] M. Angelova, 'Applications of classical and quantum algebras to molecular thermodynamics', *arXiv preprint cond-mat/0201240*, 2002.

[14] R. S. Johal and R. K. Gupta, 'Two parameter quantum deformation of $U(2) \supset U(1)$ dynamical symmetry and the vibrational spectra of diatomic molecules', *International Journal of Modern Physics E*, vol. 7, no. 05, pp. 553–557, 1998.

[15] D. Bonatsos, C. Daskaloyannis and P. Kolokotronis, 'Coupled q-oscillators as a model for vibrations of polyatomic molecules', *The Journal of chemical physics*, vol. 106, no. 2, pp. 605–609, 1997.

[16] V. Man'ko, G. Marmo, S. Solimeno *et al.*, 'Physical nonlinear aspects of classical and quantum q-oscillators', *International Journal of Modern Physics A*, vol. 8, no. 20, pp. 3577–3597, 1993.

[17] J. Esteve, C. Tejel and B. Villarroya, '$SU_q(2)$ quantum group analysis of rotational spectra of diatomic molecules', *The Journal of chemical physics*, vol. 96, no. 8, pp. 5614–5617, 1992.

[18] L. Biedenharn, 'The quantum group $SU_q(2)$ and a q-analogue of the boson operators', *Journal of Physics A: Mathematical and General*, vol. 22, no. 18, p. L873, 1989.

[19] S. Majid, *A quantum groups primer*. Cambridge University Press, 2002, vol. 292.

[20] S. Mac Lane, *Categories for the working mathematician*. Springer Science & Business Media, 2013, vol. 5.

[21] V. S. Varadarajan, *Supersymmetry for Mathematicians: An Introduction*. American Mathematical Soc., 2004, vol. 11.

[22] P. Etingof, S. Gelaki, D. Nikshych *et al.*, *Tensor categories.* American Mathematical Soc., 2016, vol. 205.

[23] S. Chmutov, S. Duzhin and J. Mostovoy, *Introduction to Vassiliev knot invariants.* Cambridge University Press, 2012.

[24] D. Bar-Natan and R. van der Veen, 'A polynomial time knot polynomial', *arXiv preprint arXiv:1708.04853*, 2017.

[25] L. Linov, *An introduction to knot theory and the knot group*, 2014.

[26] R. H. Fox, 'A quick trip through knot theory', *Topology of 3-manifolds and related topics*, 1962.

[27] F. Waldhausen, 'On irreducible 3-manifolds which are sufficiently large', *Annals of Mathematics*, pp. 56–88, 1968.

[28] D. Bulacu, S. Caenepeel, F. Panaite *et al.*, *Quasi-Hopf Algebras: A Categorical Approach.* Cambridge University Press, 2019.

[29] B. Hall, *Lie groups, Lie algebras, and representations: an elementary introduction.* Springer, 2015, vol. 222.

[30] T. S. Blyth, *Module Theory: An approach to linear algebra.* University of St Andrews, 2018.

[31] S. Majid, 'Beyond supersymmetry and quantum symmetry (an introduction to braided-groups and braided-matrices)', pp. 231–282, 1993.

[32] T. Tannaka, 'Über den dualitätssatz der nichtkommutativen topologischen gruppen', *Tohoku Mathematical Journal, First Series*, vol. 45, pp. 1–12, 1939.

[33] F. Nichita, 'Introduction to the yang-baxter equation with open problems', *Axioms*, vol. 1, no. 1, pp. 33–37, 2012.

[34] S. Majid, 'Free braided differential calculus, braided binomial theorem, and the braided exponential map', *Journal of mathematical physics*, vol. 34, no. 10, pp. 4843–4856, 1993.

[35] ——, 'Braided momentum in the q-poincaré group', *Journal of mathematical physics*, vol. 34, no. 5, pp. 2045–2058, 1993.

[36] U. Meyer, 'q-lorentz group and braided coaddition on q-minkowski space', *Communications in mathematical physics*, vol. 168, no. 2, pp. 249–264, 1995.

[37] S. Majid, 'Quantum and braided linear algebra', *Journal of mathematical physics*, vol. 34, no. 3, pp. 1176–1196, 1993.

[38] F. H. Jackson, 'On q-functions and a certain difference operator', *Earth and Environmental Science Transactions of the Royal Society of Edinburgh*, vol. 46, no. 2, pp. 253–281, 1909.

[39] S. Majid, 'q-epsilon tensor for quantum and braided spaces', *Journal of Mathematical Physics*, vol. 36, no. 4, pp. 1991–2007, 1995.

[40] ——, 'Examples of braided groups and braided matrices', *Journal of Mathematical Physics*, vol. 32, no. 12, pp. 3246–3253, 1991.

[41] ——, 'The quantum double as quantum mechanics', *Journal of Geometry and Physics*, vol. 13, no. 2, pp. 169–202, 1994.

[42] J.-P. Tignol, 'Algebras with involution and classical groups', in *European Congress of Mathematics*, Springer, 1998, pp. 244–258.

[43] U. Meyer, 'Wave equations on q-minkowski space', *Communications in mathematical physics*, vol. 174, no. 3, pp. 457–475, 1996.

[44] H. Flanders, *Differential Forms with Applications to the Physical Sciences by Harley Flanders.* Elsevier, 1963.

[45] T. Frankel, *The geometry of physics: an introduction.* Cambridge university press, 2011.

[46] R. Oeckl, ‘Braided quantum field theory’, *Communications in Mathematical Physics*, vol. 217, no. 2, pp. 451–473, 2001.

[47] S. Majid, ‘Quantum and braided-lie algebras’, *Journal of Geometry and Physics*, vol. 13, no. 4, pp. 307–356, 1994.

[48] ——, ‘Lie algebras and braided geometry’, *arXiv preprint hep-th/9311109*, 1993.

[49] T. Brzeziński and S. Majid, ‘Quantum group gauge theory on quantum spaces’, *Communications in Mathematical Physics*, vol. 157, no. 3, pp. 591–638, 1993.

[50] P. Deligne and J. Morgan, ‘Notes on supersymmetry (following joseph bernstein)’, *Quantum fields and strings: a course for mathematicians*, pp. 41–97, 1999.

[51] A. Kempf and S. Majid, ‘Algebraic q-integration and fourier theory on quantum and braided spaces’, *Journal of Mathematical Physics*, vol. 35, no. 12, pp. 6802–6837, 1994.

[52] R. W. Carroll, *Calculus revisited.* Springer Science & Business Media, 2013, vol. 554.

[53] E. J. Beggs and S. Majid, *Quantum Riemannian Geometry.* Springer, 2020.

Appendices

A Computations

A.1 Matrices

$$\mathbf{R}=\begin{bmatrix}
q^2 & 0 & 0 & (q-q^{-1})^2 & 0 & 0 & 3q^{-2}-q^{-4}-2 & 0 & 0 & 1-q^{-2} & 0 & 0 & (q-q^{-1})^2 & 0 & 0 & -(1-q^{-2})^2\\
0 & 1 & 0 & 0 & q^2-1 & 0 & 0 & q^{-2}-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2-1 & 0 & 0 & (q-q^{-1})^2 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 1-q^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & q^2 & 0 & 0 & (q-q^{-1})^2 & 0 & 0 & 0 & 0 & 0 & q^2-1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2-1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & q^2-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & q^{-2}-1 & 0\\
0 & 0 & 0 & q^2-1 & 0 & 0 & (q-q^{-1})^2 & 0 & 0 & 1 & 0 & 0 & q^2-1 & 0 & 0 & q^{-2}-1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2 & 0 & 0 & q^2-1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1-q^{-2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2-1 & 0 & 0 & 0 & 0 & 0 & q^2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2
\end{bmatrix}$$

$$\mathbf{R}'=\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & q^{-2}-1 & 0 & 0 & 1-q^{-2} & 0 & 0 & (q-q^{-1})^2 & 0 & 0 & -(q-q^{-1})^2\\
0 & q^{-2} & 0 & 0 & 1-q^{-2} & 0 & 0 & q^{-4}-q^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 & 1-q^2 & 0 & 0 & 1-q^{-2} & 0 & 0 & q^2-q-q^{-2} & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 1-q^{-2} & 0 & 0 & q^{-2}-1 & 0 & 0 & -(q-q^{-1})^2 & 0 & 0 & (q-q^{-1})^2\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2-1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1-q^2 & 0 & 0 & 1-q^{-2}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1-q^2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & q^{-2}-1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & q^2-1 & 0 & 0 & q^{-2}-1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1-q^{-2} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-2} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$\mathbf{R}.=\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & q^{-2}-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & q^2-1 & 0 & 0 & q^{-2}-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 1-q^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & q^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & q^2-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & q^{-2}-1 & 0\\
1-q^2 & 0 & 0 & 1-q^{-2} & 0 & 0 & (1+q^2)(1-q^{-2})^2 & 0 & 0 & q^{-2} & 0 & 0 & 1-q^{-2} & 0 & 0 & q^{-2}(q^{-2}-1)\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1-q^{-2} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1-q^{-2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1-q^2 & 0 & 0 & 1-q^{-2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & q^{-2}-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

A.2 Proof of equations (2.6) and (2.7)

Recall the q-commutator $a_-a_+ = qa_+a_- + 1$. Using this we compute

$$\begin{aligned}
\langle x^2 \rangle &= \langle \lambda | \left(\frac{a_- + a_+}{\sqrt{2}} \right)^2 | \lambda \rangle \\
&= \langle \lambda | \left(\frac{1}{2}(a_-^2 + a_-a_+ + a_+a_- + a_+^2) \right) | \lambda \rangle \\
&= \langle \lambda | \left(\frac{1}{2}(a_-^2 + a_+^2) + \left(\frac{1}{2} + \frac{1+q}{2} a_+a_- \right) \right) | \lambda \rangle \\
&= \langle \lambda | \left(\frac{1}{2}(a_-^2 + a_+^2) + H \right) | \lambda \rangle \\
&= \frac{1}{2} \left(\lambda^2 + \lambda^{*2} \right) + \langle \lambda | H | \lambda \rangle,
\end{aligned}$$

as required. Similarly we compute

$$\begin{aligned}
\langle p^2 \rangle &= \langle \lambda | \left(\frac{a_- - a_+}{i\sqrt{2}} \right)^2 | \lambda \rangle \\
&= \langle \lambda | \left(-\frac{1}{2}(a_-^2 - a_-a_+ - a_+a_- + a_+^2) \right) | \lambda \rangle \\
&= \langle \lambda | \left(-\frac{1}{2}(a_-^2 + a_+^2) + \frac{1}{2}(a_-a_+ + a_+a_-) \right) | \lambda \rangle \\
&= \langle \lambda | \left(-\frac{1}{2}(a_-^2 + a_+^2) + H \right) | \lambda \rangle \\
&= -\frac{1}{2} \left(\lambda^2 + \lambda^{*2} \right) + \langle \lambda | H | \lambda \rangle,
\end{aligned}$$

completing the proof.

A.3 Proof of equation (6.17)

This computation is due to [34]. We proceed by induction on m. In the case $m = 1$ equation (6.17) reduces to

$$1 + PR = PR'(I + PR) \implies (PR + I)(PR' - I) = 0.$$

This is exactly equation 6.4. Next, assume that (6.17) holds for $m - 1$. We then wish to show that

$$((PR')_{12} - I)[m-1; R]_{2,\dots,m}[m; R]_{1,\dots,m} = 0.$$

From the definition of $[m; R]_{1,\dots,m}$ we have that

$$[m; R]_{1,\dots,m} = I + (PR)_{12}[m-1; R]_{2,\dots,m}. \tag{A.1}$$

Using this we compute:

$$\begin{aligned}
&((PR')_{12} - I)[m-1; R]_{2,\dots,m}[m; R]_{1,\dots,m} \\
&\quad = ((PR')_{12} - I)[m-1; R]_{2,\dots,m} \left(I + (PR)_{12}[m-1; R]_{2,\dots,m} \right) \\
&\quad = ((PR')_{12} - I) \left((1 + (PR)_{12})[m-1; R]_{2,\dots,m}([m-1; R]_{2,\dots,m} - I)(PR)_{12}[m-1; R]_{2,\dots,m} \right).
\end{aligned}$$

Here the second equality follows from adding and subtracting $(PR)_{12}[m-1; R]_{2,\dots,m}$. We can then apply equation (6.4), yielding

$$\begin{aligned}
&((PR')_{12} - I)[m-1; R]_{2,\dots,m}[m; R]_{1,\dots,m} \\
&\qquad = ((PR')_{12} - I)([m-1; R]_{2,\dots,m} - I)(PR)_{12}[m-1; R]_{2,\dots,m}.
\end{aligned}$$

We can then apply the same trick as in equation (A.1) to the term $([m-1;R]_{2,\dots,m}-I)$ to find

$$\begin{aligned}((PR')_{12}-I)&[m-1;R]_{2,\dots,m}[m;R]_{1,\dots,m}\\ &=((PR')_{12}-I)(PR)_{23}[m-2;R]_{3,\dots,m}(PR)_{12}[m-1;R]_{2,\dots,m}\\ &=((PR')_{12}-I)(PR)_{23}(PR)_{12}[m-2;R]_{3,\dots,m}[m-1;R]_{2,\dots,m}.\end{aligned}$$

Note here that everything in the term $[m-2;R]_{3,\dots,m}$ acts only on tensor powers 3 or higher, while $(PR)_{12}$ acts on the first two. Hence these two terms commute, justifying the second equality. To proceed, I claim that

$$(PR')_{12}(PR)_{23}(PR)_{12}=(PR)_{23}(PR)_{12}(PR')_{23}. \tag{A.2}$$

Indeed, by lemma 6.2 this is equivalent to $R'_{12}R_{13}R_{23}=R_{23}R_{13}R'_{12}$, which is precisely equation (6.3). Using equation (A.2) we find

$$\begin{aligned}((PR')_{12}-I)&[m-1;R]_{2,\dots,m}[m;R]_{1,\dots,m}\\ &=(PR)_{23}(PR)_{12}((PR')_{23}-I)[m-2;R]_{3,\dots,m}[m-1;R]_{2,\dots,m}.\end{aligned}$$

This last expression vanishes by the induction assumption. This completes the inductive step, and therefore equation (6.17) holds by induction.

A.4 Proof of lemma 7.17

The inverse transformations to (7.8) are given by

$$a=\frac{2q}{q^2+1}t-\frac{2}{1+q^{-2}}z,\quad b=x+iy,\quad c=x-iy,\quad d=\frac{2q}{q^2+1}t+\frac{2}{q^2+1}z.$$

We will also need the commutator $[x,y]$. For this we compute:

$$\begin{aligned}[x,y]&=\left[\frac{b+c}{2},\frac{b-c}{2i}\right]=\frac{1}{4i}\left([b,b]+[c,b]-[b,c]-[c,c]\right)\\ &=\frac{i}{2}[b,c]=\frac{i}{2}(1-q^{-2})a(d-a)=i(1-q^{-2})az.\end{aligned}$$

Now we can compute $\underline{\det}$. First:

$$\begin{aligned}ad&=\left(\frac{2q}{q^2+1}t-\frac{2}{1+q^{-2}}z\right)\left(\frac{2q}{q^2+1}t+\frac{2}{q^2+1}z\right)\\ &=\frac{4q^2}{(q^2+1)^2}t^2-\frac{4}{(q^2+1)(1+q^{-2})}z^2+\left(\frac{4q}{(q^2+1)^2}-\frac{4q}{(q^2+1)(1+q^{-2})}\right)tz\\ &=\frac{4q^2}{(q^2+1)^2}t^2-\frac{4q^2}{(q^2+1)^2}z^2+\frac{4(q-q^3)}{(q^2+1)^2}tz.\end{aligned}$$

Here the second equality uses that $[t,z]=0$ since t is taken to be central in $\mathbb{R}_q^{1,3}$. Similarly:

$$\begin{aligned}q^2cb&=q^2(x-iy)(x+iy)\\ &=q^2\left(x^2+y^2-iyx+ixy\right)=q^2\left(x^2+y^2+i[x,y]\right)\\ &=q^2x^2+q^2y^2-(q^2-1)az\\ &=q^2x^2+q^2y^2-(q^2-1)\left(\frac{2q}{q^2+1}tz-\frac{2}{1+q^{-2}}z^2\right)\\ &=q^2x^2+q^2y^2+\frac{2(q^2-1)}{1+q^{-2}}z^2-\frac{2(q^3-q)}{q^2+1}tz.\end{aligned}$$

In total, we thus have

$$
\begin{aligned}
ad - q^2cb &= \frac{4q^2}{(q^2+1)^2}t^2 - q^2x^2 - q^2y^2 + \left(\frac{2(1-q^2)}{1+q^{-2}} - \frac{4q^2}{(q^2+1)^2}\right)z^2 + \left(\frac{4(q-q^3)}{(q^2+1)^2} + \frac{2(q^3-q)}{q^2+1}\right)tz \\
&= \frac{4q^2}{(q^2+1)^2}t^2 - q^2x^2 - q^2y^2 - 2\frac{(q^2-1)(q^2+1)+2}{(q^2+1)^2}q^2z^2 + 2q\frac{2(1-q^2)+(q^2-1)(q^2+1)}{(q^2+1)^2}tz \\
&= \frac{4q^2}{(q^2+1)^2}t^2 - q^2x^2 - q^2y^2 - \frac{2(q^4+1)q^2}{(q^2+1)^2} + 2q\left(\frac{q^2-1}{q^2+1}\right)^2 tz.
\end{aligned}
$$

as required.

B Matlab Code

B.1 contractor.m

```
% Computing entries of R for q-Minkowski space

function[ res ] = bigcontractor(I,J,K,L)
q=10;

r=[q,0,0,0 ; 0,1,q-q^(-1),0 ; 0,0,1,0 ; 0,0,0,q];
ri = [q^(-1),0,0,0 ; 0,1,q^(-1)-q,0 ; 0,0,1,0 ; 0,0,0,q^(-1)];
rtil = [q^(-1),0,0,0 ; 0,1,q^(-3)-q^(-1),0 ; 0,0,1,0 ; 0,0,0,q^(-1)];

n=2;
R = @(i,j,k,l) r( n*(i-1) + k , n*(j-1) + l );
Rtil = @(i,j,k,l) rtil( n*(i-1) + k , n*(j-1) + l );
Ri = @(i,j,k,l) ri( n*(i-1) + k , n*(j-1) + l );

i0=I(1);
i1=I(2);
j0=J(1);
j1=J(2);
k0=K(1);
k1=K(2);
l0=L(1);
l1=L(2);

res=0;
for a=1:2
  for b=1:2
    for c=1:2
      for d=1:2
        res = res + R(j0,a,d,k0) * R(k1,b,a,i0) * R(i1,c,b,l1) * ...
            Rtil(c,j1,l0,d) ;
      end
    end
  end
end
```

B.2 matrices.m

```
% Computing R, R' for q-Minkowski space

clear all
clc
R=[];
Rp=[];

for a=1:2
for b=1:2
for c=1:2
for d=1:2

q=10;
z=0;
for i0=1:2
    for i1=1:2
        for k0=1:2
            for k1=1:2
            I=[i0,i1];
```

```
            J=[a,b];
            K=[k0,k1];
            L=[c,d];
            z=z+1;
            row1(z)=contractor(I,J,K,L);
            row2(z)=contractor2(I,J,K,L);
            end
        end
    end
end

col1=row1';
col2=row2';

R = [R,col1];
Rp = [Rp,col2];

end
end
end
end
```