The Maxwell algebra

Symmetries of a particle in an electromagnetic field

Abstract

The Maxwell algebra is an extension of the Poincaré algebra. When it is nonlinearly realised in a particle Lagrangian, the dynamics are that of a charged particle in a constant electromagnetic field. The algebra can be extended further, in an iterative way, up to an infinitely large $\mathbb{Z}$-graded algebra (with empty negative levels) containing a free Lie algebra. Truncations to a finite level of the infinite algebra can be considered, giving dynamics consistent with a particle travelling through a field to which a higher order term in a Taylor expansion is added every level, but this is not all it describes. After giving an overview of the Maxwell algebra and some of its quotients, we describe how to nonlinearly realise its symmetries in Lagrangians and interpret theories built in this way. We show that a Lagrangian nonlinearly realising Maxwell up to and including the third level describes an induced dipole in a linear electromagnetic field. This dipole has the property that its electric and magnetic polarisability are equal and opposite, which can be realised by a perfect superconductor. Additionally, an attempt is made to establish a classification of classical electromagnetic particle theories, by using an analogy between the soft limit classification of scalar effective field theories and the Maxwell level structure.

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1 Introduction: Guided by symmetry we commence

Symmetries play a major role in theoretical physics, both as a guide to new physics and to a more thorough understanding of 'known' physics. This role was first recognised in the beginning of the last century, when special relativity was developed by Einstein and the Lorentz transformations instrumental to it were shown by Poincaré to form a symmetry group [1]. Perhaps even more important was the role of symmetry in the context of quantum field theory and the standard model, which is in a sense defined by an $SU(3) \times SU(2) \times U(1)$ gauge symmetry. After the first big step toward this theory, the development of quantum electrodynamics up until the early fifties, applying similar quantum field theoretic techniques to the weak and strong interactions seemed a logical choice. However, for various reasons, this appeared much more difficult[1]. This struggle resulted in what some refer to as a crisis in quantum field theory in the fifties and sixties [2].

In the early sixties however, more natural ways of incorporating the large number of observed hadrons were found by Gell-Mann and Ne'eman, by introducing spontaneously broken symmetries ($SU(3) \times SU(3)$ broken to $SU(3)$[2]) [3, 4]. The theory of how symmetries spontaneously break and how that always propagates a massless particle was developed by Nambu and Goldstone [5, 6], first in the context of superconductivity, but quickly applied to particle physics as

[1]The strong interaction had too many particles associated with it to keep things simple, and the coupling constant was too large. The weak interaction appeared not to be renormalisable [2].

[2]We will shortly treat this in section 3.1.

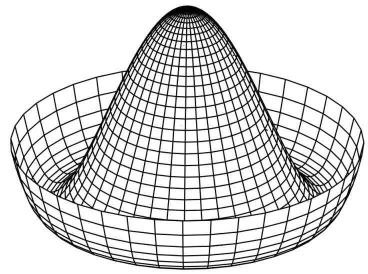

Figure 1: The Sombrero potential, with $U(1)$ symmetry broken down to the identity.

well. A famous example of a broken symmetry is given by the Sombrero potential (figure 1), in which the cylindrical symmetry of the potential is broken by the choice of a lowest energy state (somewhere in the valley). The possibility of transitioning between the nonequivalent 'vacuum states' introduces a massless particle, because the energy of the energies of the vacua are all the same. The Goldstone theorem says that for any internal spontaneous symmetry breaking, there must be a massless particle in the system [6]. In the work of Gell-Mann and Ne'eman, the breaking of symmetries introduces a set of (nearly) massless particles, exactly corresponding to part of the observed hadrons, among which the pions[3]. Soon after that the existence of quarks, making up the hadrons, was postulated. The successes of the in this way predicted omega particle, and the unification of the weak and electromagnetic forces ($SU(2) \times U(1)$), meant the end of the QFT crisis [2].

Around the same time, theorists had been working on an alternative to field theory all together. In S-matrix theory for example, one only looked at the scattering amplitudes, giving the likelihood of a state transitioning into another state. Symmetry also played a role in this approach. Coleman and Mandula in 1967 proved that, if the Poincaré group was a subgroup of the symmetry group of the S-matrix, it was the maximally allowed spacetime symmetry group, and that combinations with internal groups could only be direct products [7]. The theorem can be circumvented in a few ways however. Firstly, by the introduction of Fermionic generators, which allows to nontrivially combine internal and spacetime symmetry groups in supersymmetric theories [8], and secondly, by theories with additional broken spacetime symmetries, that would show up in the Lagrangian and equations of motion of a system, but would not be present in the S-matrix[4].

That last 'loop hole' is the one that is used in this work. In 1968, Callan, Coleman, Wess, and Zumino gave a hands-on description of how to use a particular set of broken symmetries, to create the most general Lagrangians possible with such symmetries [9][5], in a way generalising the work of Gell-Mann c.s. This essentially gives an opportunity to theorists to not only analyse systems in terms of their symmetries, or impose certain symmetries on systems, but actually start out by writing down the desired (broken) symmetries, and conclude what properties of the system must follow.

In this thesis, a particular symmetry group is studied, called the Maxwell group. This is an

[3]The other hadrons were all neatly organised in an octet and a decuplet [4]

[4]Also, if Poincaré is not a subgroup in the first place, but the spacetime group is some other group, like anti-de Sitter, the theorem does not apply.

[5]To be accurate, the method given by CCWZ is created for internal symmetries, though the results can largely be transferred to the case of spacetime symmetries.

extension of the Poincaré group, which is the symmetry group of special relativity. The extension has been studied for the first time decades ago [10] and produces an interesting result. When writing down a Lagrangian nonlinearly realising the symmetry, the equations of motion take a form identical to that of a massive, charged particle traversing a constant electromagnetic field, so experiencing a Lorentz force. Since Maxwell's name is inextricably linked to electromagnetism, the algebra received his name. It does not have any direct ties with his own equations.

More recently, it was shown that the Maxwell algebra can be extended itself in a natural, iterative way, giving rise to an infinite-parameter group that is dubbed Maxwell$_\infty$ [11, 12]. The consecutive extensions can be organised in a level structure, and are isomorphic to an algebra containing a so-called free Lie algebra. The question naturally arises whether the higher level extensions also describe physical phenomena. As claimed by Gomis and Kleinschmidt [12], the equations of motion exhibit forms that resemble the behaviour of multipoles in electromagnetic fields, but this has not been shown. It has been shown that the background field found in the higher level extensions forms a Taylor series, but this is not the only complication of the system introduced by realising the higher-level symmetries [12]. The purpose of this research is to determine if and how the systems with higher level Maxwell symmetries describe a physical system.

In recent years, the S-matrix programme from the sixties has made a comeback. It has been applied to dramatically simplify scattering amplitude calculations by bypassing the enormous amounts of Feynman diagrams sometimes needed to carry those out, and it manages to do so based only on Lorentz symmetry and consistency relations. Not only do calculations become less tedious, they also become more insightful, revealing structures that were hiding from view in earlier approaches [13]. The amplitude methods are especially powerful in the context of renormalisable interactions and particles with spin, but have also shown their worth for effective field theories. EFTs can be classified by and even constructed from their amplitudes' small momentum behaviour, characterised by so-called soft limits. As we will see, the different 'soft degrees' associated with the limits, signal the presence of symmetries. The generators of those symmetries can be related to each other by translation generators.

As in this work we also aim to interpret the theories we are working with as effective theories, it is an interesting question whether we can find an analogous classification for the theories described by Maxwell$_\infty$. Even more so, because Maxwell also has a level structure, where the different levels are related by translation generators. We will try to do this by starting from what can be thought of as the classical analog of the scattering amplitude: the action.

In section 2 of this text, we will shortly revisit Lie groups and algebras, before introducing the Maxwell algebra. Section 3 will be on how to construct nonlinearly symmetric actions according to Callan, Coleman, Wess and Zumino, with the Maxwell case treated specifically. After this, in section 4 we will look at the dynamics the Maxwell Lagrangians produce and interpret it. In section 5, we will look at the role of soft limits in effective field theory and try to establish an analogous limit for our effective particle theories before we conclude.

2 The Maxwell algebra

2.1 Lie groups, algebras and extensions

As made clear (hopefully) abundantly in the introduction, symmetries of physical systems and their description form an essential part of the toolbox for every physicist. Aside from discrete symmetries, such as time reversal symmetry, parity and charge conjugation, also continuous symmetries are highly relevant in physical theories. In Physics, we often study symmetries that are analytically dependent on their parameters, such that they can be described using Lie groups. Lie groups can roughly be thought of as a differentiable manifold (locally flat space) that is also a group, in which the group multiplication and the map of an element to its inverse are smooth (C^∞). In studying Lie groups, physicists generally restrict themselves to linear groups, which are those that can be represented by matrices [14]. This subsection will recap some concepts related to Lie groups and introduce the concept of extensions.

Lie groups can be studied for global properties, such as connectedness and compactness[6], as well as local ones. A Lie group in principle has an infinite number of elements, but it can be parametrised by a finite set of real parameters, at least in the neighbourhood of the identity, by virtue of it being a manifold. This finite set of parameters is called the dimension n of the group. Local properties of a Lie group can be analysed by their n-dimensional tangent space at the origin (or identity element). This Lie algebra L is a vector space with a Lie bracket $[.,.]$ which has the properties that it [14]

1. closes: $[a,b] \in L,\ \forall a,b \in L$,
2. is bilinear: $[\alpha a + \beta b, c] = \alpha[a,c] + \beta[b,c],\ \forall a,b,c \in L,\ \alpha,\beta \in \mathbb{R}$,
3. is anti-symmetric: $[a,b] = -[b,a],\ \forall a,b \in L$,
4. satisfies the Jacobi identity: $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0,\ \forall a,b,c \in L$.

The whole algebra can then be defined by giving the Lie brackets of n basis vectors of the vector space.

An example of a Lie algebra, is that of the Lorentz group in four dimensions $SO(1,3)$, which is $\mathfrak{so}(1,3)$. The Lorentz group consists of three rotations J_k and three boosts K_i, so must be six dimensional. The six generators can be given in anti-symmetric 4×4 matrices M_{ab} (such that $M_{0i} = K_i,\ M_ij = \epsilon_{ijk} J_k$), labelled by two indices in which they are also anti-symmetric. The defining relations are given by

$$[M_{ab}, M_{cd}] = \quad \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} + \eta_{ad} M_{bc}, \tag{1}$$

where the η is the Minkowski metric with $\eta_{00} = -1$.

Once we define an algebra of a Lie group, we can always obtain a group again, by exponentiating the algebra with a parameter for every generator. In our example, this would mean taking the exponent of the six generators, with the introduction of six parameters Ω_{ab}

$$\Lambda = e^{\frac{1}{2}\Omega_{ab} M^{ab}}. \tag{2}$$

For connected and compact groups, such as $SU(2)$ (the group of unitary matrices with determinant $\det(U) = 1$), this even gets us the original group. Furthermore, every element of the connected subgroup of a Lie group can be represented by a finite product of exponents of the algebra elements. In the case of the Lorentz group $O(1,3)$, which consists of four connected components, we get the connected subgroup also called the *proper orthochronous* Lorentz group $SO^+(1,3)$[7], which has $\det(\Lambda) = 1$ and ${\Lambda^0}_0 \leq 1$ [14]. This exponential parametrisation of groups

[6] Connectedness is the possibility of reaching every element of the group, by continuously varying its parameters. Compactness is the property that the parameters of a group (with a finite number of connected components) vary over a closed and bounded set.

[7] We will often refer simply to $SO(1,3)$ or 'the Lorentz group' when talking just about the connected subgroup.

is used a lot in Physics, because of the well-known properties of exponential functions making this a natural way to write down group elements.

In researching symmetric systems, one often looks for the irreducible representations (irreps), since objects transforming under those will be the objects the theory is built out of. Looking at the Lorentz group again, this reminds us of four-vectors and bi-spinors, two examples transforming under nonequivalent four dimensional representations.

The Lorentz algebra can alternatively be given in terms of two combinations of boosts and rotations

$$C_i = \frac{1}{2}(J_i + iK_i), \qquad D_i = \frac{1}{2}(J_i - iK_i). \tag{3}$$

This complexification results in the commutation relations

$$\begin{aligned} [C_i, C_j] &= \quad i\epsilon_{ijk}C_k \\ [D_i, D_j] &= \quad i\epsilon_{ijk}D_k \\ [C_i, D_j] &= \quad 0, \end{aligned} \tag{4}$$

which are two separate $\mathfrak{su}(2)$ algebras, meaning $\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ (with all algebras complex). Using the eigenvalues (j_1, j_2) of the Casimir operators $C_iC^i = C^2$ and D^2, we can label each irrep. In this way, the defining four-vector representation gets $(\frac{1}{2}, \frac{1}{2})$, while the bi-spinor, as the name suggests, becomes two spinors $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. This is reducible for the algebra, but irreducible once we also introduce parity invariance, since this requires an interchange of j_1 and j_2 [14,15].

The direct sum of the two special unitary algebras is just one way of extending an algebra ($\mathfrak{su}(2)$) into a larger one ($\mathfrak{so}(1,3)$). Another example is taking the semi-direct sum of two algebras. In general, a Lie algebra extension is given by a *short exact sequence* [16]

$$\mathfrak{h} \overset{i}{\hookrightarrow} \mathfrak{e} \overset{s}{\twoheadrightarrow} \mathfrak{g}, \tag{5}$$

such that $\mathrm{Im}(\mathfrak{h} = \mathrm{Ker}(s)$, where the first arrow in the sequence represents an injective homomorphism, while the second is a surjective homomorphism. Here we say that $\mathfrak{h}$ is extending $\mathfrak{g}$, creating a larger algebra $\mathfrak{e}$ of which $\mathfrak{h}$ is an *invariant subalgebra* (or *ideal*). An ideal is a subalgebra that stays within itself under the action of any element from the total algebra. This means that the generators of the ideal transform under a representation of the extended algebra g. It is clear that $\mathfrak{g} \cong \mathfrak{e}/\mathfrak{h}$.

Extensions of algebras have their analog in groups of course. The Lorentz group can for example be extended into the Poincaré group $ISO(1,3)$, by taking the semi-direct product with the group of translations in 3+1 dimensions (meaning the translations are an invariant or *normal* subgroup of the full group, but the Lorentz group is not). The algebra of that group then amounts to the semi-direct sum of the translation and Lorentz algebras. And as we will see shortly, the translations transform under a representation of the Lorentz algebra.

2.2 Maxwell definition and background

Any relativistic theory will need to be symmetric under the transformations of the Poincaré group, since it is the spacetime symmetry group of special relativity. So if we want to build a relativistic theory with more symmetry, for reasons mentioned in the introduction, we need to extend the Poincaré group and therefore its algebra. The Maxwell algebra is such an extension. The group corresponding to this extended algebra can be spontaneously broken in a physical system, such that the symmetries of the S-matrix are still Poincaré or a smaller group (Lorentz for instance). This is one of the 'escapes' from the Coleman-Mandula theorem (another one being supersymmetry), in principle prohibiting a larger spacetime symmetry group than Poincaré.

The Poincaré algebra is given by the following commutation relations (following [12])

$$\begin{aligned}
[P_a, P_b] &= 0, \\
[M_{ab}, M_{cd}] &= \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} + \eta_{ad} M_{bc}, \\
[M_{ab}, P_c] &= -\eta_{ca} P_b + \eta_{bc} P_a.
\end{aligned} \tag{6}$$

Here, the translations P_a commute. Or in other words, it does not matter in which order one applies multiple translations. This means the space these operators are defined on is a flat spacetime (Minkowski space), and the translations can be given as differential operators that are simply the partial derivatives $P_a = \partial_a$. Note also that in the above relations, the bracket of Lorentz and translation generators gives translation generators, making the translations an ideal of the Poincaré algebra.

In the Maxwell algebra, a new symmetry generator is added, showing up in the commutation of translations [12]:

$$[P_a, P_b] = Z_{ab}, \tag{7}$$

while the other new relations are given by

$$\begin{aligned}
[P_a, Z_{ab}] &= 0, \\
[Z_{ab}, Z_{cd}] &= 0, \\
[M_{ab}, Z_{cd}] &= \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}.
\end{aligned} \tag{8}$$

The generator Z_{ab} is anti-symmetric in the exchange of its indices, as follows immediately from its definition.

Introducing a failure of the translations to commute, implies some curvature of the manifold making up the physical system; it is no longer flat. This means that we have to replace all partial derivatives by covariant derivatives. A covariant derivative is a way of taking a derivative with respect to vectors tangent to a manifold. Defining a set of coordinates $x^b(u)$ and taking the absolute derivative of a vector field λ^a with respect to the parameter u of a curve, we see

$$\frac{D\lambda^a}{du} = \partial_b \lambda^a \dot{x}^b + \Gamma^a{}_{bc} \lambda^c \dot{x}^b = (\partial_b \lambda^a + \Gamma^a{}_{bc} \lambda^c) \dot{x}^b, \tag{9}$$

where $\nabla_b \lambda^a \equiv \partial_b \lambda^a + \Gamma^a{}_{bc} \lambda^c$ is the covariant derivative. In the context of general relativity, this $\Gamma^a{}_{bc}$ is the Christoffel symbol, arising because of the freedom to choose whichever reference frame we like. In quantum electrodynamics, the connection is needed because of $U(1)$ gauge symmetry. Then the covariant derivative simply becomes $D_\mu = \partial_\mu - iqA_\mu$, with A_μ the four potential.

Since translations are partial derivatives in flat spacetime, which we need to change to covariant derivatives, the commutator $[P_a, P_b]$ will give the Riemann curvature tensor R^d_{abc} in GR and the field strength $F_{\mu\nu}$ in QED. In a sense, the tensor Z_{ab} is analogous to both these quantities. However, in the Maxwell algebra case, there is no local symmetry. As we will see briefly in section 2.4, the similarity to spacetime curvature can be established further. Moreover, section 4 shows how we can interpret a quantity dual to the generator Z_{ab} as a constant electromagnetic field.

The Maxwell extension is a non-central extension, because the generator does clearly not lie at the centre of the algebra: it does not commute with all other generators. In fact, every central extension of the Poincaré algebra is trivial [17], meaning it is a direct product with some other group and no mixing between generators occurs. This other group would then be the group of internal symmetries of the system. Since there is no other nontrivial extension we can do of the Poincaré group (at least in $D > 2$ dimensions), the Maxwell algebra is unique in this sense.

An example of an application of the Maxwell algebra is given in [18], which shows that by making the symmetry a gauge symmetry, one can obtain Einstein gravity, with a cosmological constant term. This approach is inspired by the anti-de Sitter algebra, which is also an adjusted Poincaré algebra, with the translations commuting into the Lorentz generators:

$$[P_a, P_b] = \frac{1}{R^2} M_{ab}. \tag{10}$$

Here the P_a are AdS translations and R is the de Sitter radius, determining the curvature of the spacetime. The cosmological constant is then identified to be $\lambda = \frac{1}{R^2}$.

In gauging the Maxwell algebra, one then introduces six vector fields A_μ^{ab} associated with the six Abelian generators Z_{ab} (in four dimensions). Again, this is just like how in QED the gauging of the $U(1)$ symmetry group results in the introduction of the vector potential A_μ. These vector fields are interesting themselves, since inflation can be driven by vector fields [18, 19].

The Maxwell algebra was first studied outside the context of quantum field theory however, in 1972, by Robert Schrader [10]. Schrader studied the symmetries of a relativistic particle in a constant electromagnetic field, building on the work of Bacry, Combe and Richards, who did the same for the case where the fields have a particular size and direction [20][8]. This will obviously be a smaller group, since the a preferred direction is chosen. The algebra associated with this is called BCR, after its discoverers, and it is a subalgebra of Poincaré. It is made up of the four translations, and two Lorentz generators that are defined as follows:

$$G = \frac{1}{2} F_{ab} M^{ab}, \qquad G^\star = \frac{1}{2} \epsilon_{abcd} F^{ab} M^{cd}, \tag{11}$$

with ϵ^{abcd} the fully anti-symmetric Levi-Civita tensor in four dimensions and F_{ab} the field strength tensor. Taking the electric and magnetic field in the same direction for example, these generate boosts along the direction of the fields and rotations around it.

Interestingly, though there is only one BCR algebra, there are two distinct BCR groups. Defining a quantity

$$S^2 = \frac{1}{2} F_{ab} F^{ab} + \frac{i}{2} \epsilon_{abcd} F^{ab} F^{cd}, \tag{12}$$

[8] Actually, Bacry, Combe and Richards mentioned the Maxwell algebra in passing as well, in the conclusion of their 1970 paper [20], though it did not have a name yet. The name has supposedly been given by Nobel laureate Sheldon Lee Glashow [10].

made up of the two invariants we can build with the field, it is possible to label the two groups. The case where $S^2 \neq 0$ can always be rotated and boosted such that the fields are parallel. A special case of this, is when S^2 is real, meaning the fields are perpendicular in some frames while in one frame one of the fields vanishes completely.

When the fields cannot be made parallel while the fields are nonzero, $S^2 = 0$. In this case, the fields are necessarily perpendicular and equal in magnitude. The remaining possibility with $F_{ab} = 0$ of course corresponds to the Poincaré group. The defining relations of the BCR algebra are

$$\begin{aligned} [P_a, P_b] &= \quad 0, \\ [G, G^\star] &= \quad 0, \\ [G, P_a] &= \quad F_{ab}P^b, \\ [G^\star, P_a] &= \quad F^\star_{ab}P^b, \end{aligned} \tag{13}$$

where $F^\star_{ab} = \frac{1}{2}\epsilon_{abcd}F^{cd}$. Choosing a specific case, for example the case in which the electric and magnetic field are parallel (say, in the z-direction), we can see this (now single) group splits up into two subgroups,

$$\begin{aligned} [J_z, P_x] &= \quad P_y, \\ [J_z, P_y] &= \quad -P_x, \\ [K_z, P_z] &= \quad P_t, \\ [K_z, P_t] &= \quad P_z, \end{aligned} \tag{14}$$

of which the first consists of rotations around the z-axis and the translations perpendicular to it (two dimensional Euclidean group $E(2)$), and the second of boosts in the z-direction and translations in the time and z-direction (two dimensional Poincaré group $E(1,1)$).

The last related algebra that we will treat here is the EBCR algebra, which is a central extension of BCR. A central extension is an extension in which the added generators, or charges, are at the centre of the new algebra, meaning they commute with the whole algebra. In the EBCR algebra,

$$[P_a, P_b] = \quad F_{ab}Z_e + F^\star_{ab}Z_m, \tag{15}$$

where the central charges have the interpretation of electric and magnetic charge. It is this algebra that will contain the symmetries of the system once a specific solution for the Maxwell equations of motion are chosen, not the BCR algebra. That is because, just like two Lorentz generators $(G, G^\star)$ survive the choosing of a solution, so will two out of the six Maxwell generators. A scheme of the mentioned algebras and their relation to each other can be seen in figure 2.2. One might notice that the Maxwell algebra does not contain magnetic charge itself, only something that can be interpreted as akin to a field strength multiplied by an electric charge (the generator Z_{ab}). Perhaps, assuming four spacetime dimensions, one can include a magnetic charge tensor in the Maxwell algebra by adding a term $\epsilon_{abcd}Z^{cd}_{\text{mag}}$ to the translation commutator.

2.3 Level structure

The Maxwell group holds a special place as the unique nontrivial extension of the Poincaré group and it has the interesting interpretation as the symmetry group of a relativistic particle in a constant electromagnetic field. This presents the question whether extending the Maxwell algebra itself yields a similarly interpretable result.

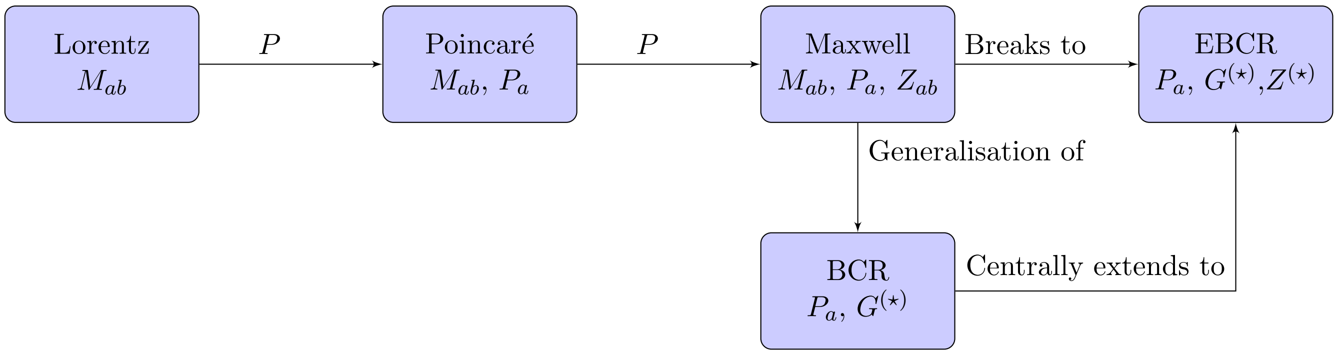

Figure 2: A summary of the relations between the mentioned algebras, noting the generators present in each.

The simplest way to extend the Maxwell algebra, is by introducing a new set of generators in the following way

$$[P_a, Z_{bc}] = \quad Y_{abc}, \tag{16}$$

with the other relations as before. This new tensor must be anti-symmetric in the last two indices due to the anti-symmetry of Z_{ab}, and because of the Jacobi identity (one of the axioms of a Lie Algebra)[9], its completely anti-symmetric part vanishes [10]:

$$\epsilon^{abcd} Y_{abc} = 0. \tag{17}$$

Note that this kind of relation holds for all numbers of dimensions higher than or equal to three. The additional non-vanishing commutation relation of the new algebra is given by

$$[M_{ab}, Y_{cde}] = \quad \eta_{be} Y_{cad} - \eta_{ae} Y_{cbd} + \eta_{bd} Y_{cea} - \eta_{ad} Y_{ceb} + \eta_{bc} Y_{aed} - \eta_{ac} Y_{bed}. \tag{18}$$

Notice that $[P_a, Z_{bc}] = [P_a, [P_b, P_c]] = Y_{abc}$, and the other relations are also algorithmically similar to the original Maxwell group. One easily sees that this extension procedure can be done indefinitely, creating an ever larger group. The structure underlying this iterative extension of the Poincaré group, was uncovered in [12], whose explanation we will largely follow in the remainder of this section.

The structure is as follows. When assigning a level to each of the generators in Poincaré, namely, $l = 0$ to the Lorentz generators M_{ab} and $l = 1$ to the translations P_a, commutators give combinations into generators of their added level:

$$[G'_{l=i}, G''_{l=j}] = G_{l=i+j}. \tag{19}$$

So the $Z_{ab} = [P_a, P_b]$ in the first extension, gets $l = 2$, giving the algebra containing this as the only extension the name Maxwell$_2$. The generator Y_{abc} then has level $l = 3$, naming the algebra with this as the highest level extension Maxwell$_3$. The sequence is schematically shown in figure 2.3.

Every application of a Lie bracket with P to a generator results in a higher level generator. One can continue this up to any level l, making it the Maxwell$_l$ algebra.

[9] The Jacobi identity says that $[P_a, [P_b, P_c]] + [P_b, [P_c, P_a]] + [P_c, [P_a, P_b]] = 0$.

[10] So in four dimensions the number of generators is $4 \cdot 6 - 4 = 20$.

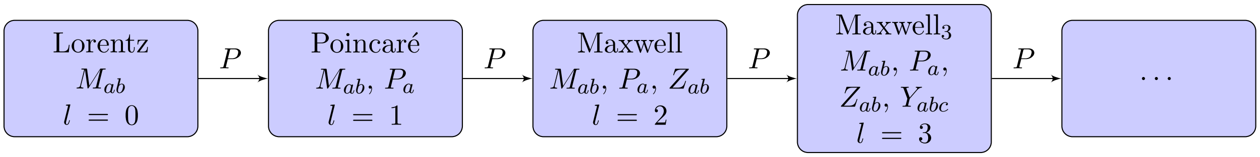

Figure 3: The sequence of extensions, starting from the Lorentz algebra, to Maxwell$_\infty$, noting the generators present and the level of the extension.

The extension structure is related to what is called a free Lie algebra. That is an algebra, generated by a set, in our case the translations P_a, and has the minimal requirements of a Lie algebra: the product is anti-symmetric and satisfies the Jacobi identity. This means the algebra encompasses all multi-commutators $\left[\left[[P_{a_1}, P_{a_2}], \ldots, P_{a_{l-1}}\right], P_{a_l}\right]$. Importantly, because the anti-symmetry and Jacobi identity guarantee all multi-commutators can be written in the given form, and the number of P_a in a multi-commutator cannot change, we can assign a level based on the number of P_a in the expression and write products like (19).

Since the level l parts of the free Lie algebra do not mix, we can write the full algebra as the direct sum of all these parts

$$\mathfrak{f} = \bigoplus_{l>0} \mathfrak{f}_l. \tag{20}$$

The full Maxwell$_\infty$ algebra is then isomorphic to the semi-direct sum of the Lorentz algebra and a free Lie algebra $\mathfrak{f}$ generated by the translations:

$$\text{Maxwell}_\infty \cong \mathfrak{so}(1, D-1) \oplus \mathfrak{f}. \tag{21}$$

Of the free Lie algebra, we can also construct ideals, which are subalgebras $\mathfrak{i} \subseteq \mathfrak{f}$ of the total algebra that have the property

$$[\mathfrak{f}, \mathfrak{i}] \subseteq \mathfrak{i}. \tag{22}$$

Obvious ideals are then the subalgebras in which only the generators above a certain level are included:

$$\mathfrak{i}_l = \bigoplus_{k>l} \mathfrak{f}_k, \tag{23}$$

which make quotient algebras

$$\mathfrak{q}_l = \mathfrak{f}/\mathfrak{i}_l, \tag{24}$$

that only contain the generators up to and including level l. This means we can write the finite Maxwell extensions as the semi-direct sum of the Lorentz algebra and such a quotient:

$$\text{Maxwell}_l \cong \mathfrak{so}(1, D-1) \oplus \mathfrak{q}_l. \tag{25}$$

These truncations of the infinite algebra generate the symmetries studied in this thesis.

A helpful way to represent the generators of the free Lie algebra, is by Young Tableaux [12]. We can see the set of all P_a as the fundamental representation of the general linear algebra $\mathfrak{gl}(D)$ and write

$$\{P_a\} \longleftrightarrow \square\,. \tag{26}$$

The next level, being the anti-symmetric product of two fundamental representations is then

$$\{Z_{ab}\} \longleftrightarrow \ydiagram{1,1}\,. \tag{27}$$

Applying the rules of young tableau multiplication, we see that the next level has two options

$$\ydiagram{1,1} \otimes \ydiagram{1} = \ydiagram{1,1,1} \oplus \ydiagram{2,1}\,, \tag{28}$$

of which the first one is discarded because it is completely anti-symmetric, which our representation cannot be. So we see

$$\{Y_{abc}\} \longleftrightarrow \ydiagram{2,1}\,, \tag{29}$$

giving the correct symmetry properties of the tensor. This process can be continued for higher levels, while constantly checking the found representations for consistency with anti-symmetry and the Jacobi identity. In this representation, the number of boxes in this representation exactly corresponds to the level of the generator. An important remark to make presently, is that the produced irreducible representations are of the general linear algebra $\mathfrak{gl}(D)$, while the representations we are after are those of the Lorentz algebra, since all generators which extend an algebra will be in the carrier space of representations of the original algebra. This concretely means we can contract indices of the generators with the Minkowski metric η_{ab}, which is invariant under Lorentz transformations, and in that way decompose representations into a traced and a traceless one [12]:

$$Y_{abc} = \widetilde{Y}_{abc} + \frac{1}{D-1}\left(\eta_{ab}Y_c - \eta_{ac}Y_b\right), \tag{30}$$

in which the tilde shows the tensor is traceless and $\eta^{ab}Y_{abc} = Y_c$. In young tableau form, this reads

$$\ydiagram{2,1} \longrightarrow \widetilde{\ydiagram{2,1}} \oplus \ydiagram{1}\,. \tag{31}$$

Consequently, at every Maxwell level $l \geq 3$, there are irreps and therefore generators that have one or multiple traces taken, leaving ones in which those traces are not taken. This causes the irreps of $\mathfrak{so}(1,3)$ in general to have a different number of boxes than their level.

2.4 Quotients of Maxwell$_\infty$

We already looked at one possible quotient of the Maxwell$_\infty$ algebra, describing formally what it means to limit ourselves to a finite number of levels. However, there are more ways to construct interesting algebras, with their own consequences for the physical systems realising their symmetries.

An example is one equal to the AdS algebra, as was shown in [21], which we will use as a basis for the explanation in this section. Taking a quotient essentially amounts to identifying generators with each other in a certain way. To find the AdS algebra though, we need to make sure the quotient is consistent, in the sense that the identifications we make form themselves an ideal of the total algebra (on the group level, the coset needs to be a quotient). This means simply equating $Z_{ab} = M_{ab}$ will not be sufficient, because the difference $M_{ab} - Z_{ab}$ is not an ideal.

The first step in obtaining such a quotient is considering only the generators with one or two indices. This is itself a consistent quotient, as was shown (for a slightly different but related case) in [22]. We find these generators at alternating levels, as can be seen by considering the Young Tableaux of the different levels. As we have seen, considering the tableaux as representations of the Lorentz algebra introduces the possibility of having less boxes in a representation than the level number, by allowing traces. This first becomes a possibility at the third level, where there is a representation Y_a. At the level after that, the we have the products

$$
\begin{aligned}
\ydiagram{2,1} \otimes \ydiagram{1} &= \ydiagram{3,1} \oplus \ydiagram{2,1,1} \oplus \ydiagram{1,1} \\
\ydiagram{1} \otimes \ydiagram{1} &= \ydiagram{1,1},
\end{aligned}
\tag{32}
$$

where diagrams with the same or impossible symmetries are not included. Clearly, if we discard all representations with a higher number of boxes than two, even levels will have anti-symmetric tensors with two indices, while odd levels have vectors. Defining these generators $M_{ab}^{(m)}$ and $P_a^{(m)}$ for $m \geq 0$, we have commutation relations [21]

$$
\begin{aligned}
\left[M_{ab}^{(m)}, M_{cd}^{(n)}\right] &= \eta_{bc} M_{ad}^{(m+n)} - \eta_{ac} M_{bd}^{(m+n)} - \eta_{bd} M_{ac}^{(m+n)} + \eta_{ad} M_{bc}^{(m+n)}, \\
[P_a^{(m)}, P_b^{(n)}] &= M_{ab}^{(m+n+1)}, \\
[M_{ab}^{(m)}, P_c^{(n)}] &= -\eta_{ca} P_b^{(m+n)} + \eta_{bc} P_a^{(m+n)}.
\end{aligned}
\tag{33}
$$

This algebra has been called Poincaré$_\infty$ [21], as seems appropriate because of the infinite number of generalised Lorentz transformations and translations. Notice that in this algebra, because it only alters the generator content of Maxwell$_\infty$ at levels $l \geq 3$, the first Maxwell extension is still present as a consistent truncation, since the generators of level higher than three obviously form an ideal. The AdS algebra

$$
\begin{aligned}
[M_{ab}, M_{cd}] &= \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} + \eta_{ad} M_{bc}, \\
[M_{ab}, P_c] &= -\eta_{ca} P_b + \eta_{bc} P_a, \\
[P_a, P_b] &= \frac{1}{R^2} Mab,
\end{aligned}
\tag{34}
$$

is then given by taking the quotient with the ideal generated by [21]

$$
P_a^{(0)} - P^{(m)}, \qquad M_{ab}^{(0)} - M_{ab}^{(m)}, \qquad \forall m > 0,
\tag{35}
$$

which amounts to setting equal all even level generators and all odd level generators. So the AdS algebra is a quotient of the Poincaré$_\infty$ algebra and therefore of the Maxwell$_\infty$. In fact, truncating Poincaré$_\infty$ at finite levels gives approximations of the AdS algebra, forming an expansion up to the order of the level in terms of the inverse of the length scale associated with the curvature of spacetime in AdS. The construction given above is entirely analogous to the way the Galilean algebra (with nonrelativistic boosts) can be extended to an infinite algebra, containing in its infinite limit all special relativistic corrections, such that the algebra contains a quotient isomorphic to Poincaré [23].

An example closer to the questions posed in this project is the quotient found in [12], taking only those generators transforming under representations having no more than two rows in their Young Tableaux, and no more than one box in the second row. This yields representations like

$$\begin{array}{|c|}\hline a \\ \hline\end{array}\,, \quad \begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array}\,, \quad \begin{array}{|c|c|c|c|c|}\hline a & c_1 & c_2 & \cdots & c_n \\ \hline \multicolumn{1}{|c|}{b} & \multicolumn{4}{c}{} \\ \cline{1-1}\end{array}\,, \tag{36}$$

with $n > 0$ in $\mathfrak{gl}(D)$. The reason to consider this quotient was to exclude terms in the Euler-Lagrange equations following from Lagrangians realising the symmetry that block integrability in terms of only spacetime coordinates. Gomis and Kleinschmidt noticed that including only the representations described above, they could write down a Lagrangian leading to equations only explicitly dependent on spacetime coordinates (and not on coordinates associated with other generators than the translations P_a). In this way, they found a particle governed only by a Lorentz force, due to a general electromagnetic field, given as a Taylor series in powers of spacetime coordinates x^a. However, as we will see in section 4, the Lorentz force and a general field is not all that the above quotient can describe.

3 Constructing nonlinear Maxwell realisations

In the introduction it was already mentioned that for Maxwell extensions, which have Poincaré as a subgroup, to be allowed by the Coleman-Mandula theorem, they need to be spontaneously broken. This means that the vacuum of the system is not invariant under the full group transformation leaving the Lagrangian invariant. Referring to the example given in the introduction, the theory (the Sombrero potential) was invariant under a rotation, but choosing a lowest energy state somewhere in the valley spontaneously breaks the symmetry. The angle around the hat then parametrises the Goldstone boson, transforming with a constant shift under the circular rotation. Lagrangians spontaneously break symmetries when the symmetries are realised nonlinearly in them. Therefore, we will look at the way Maxwell symmetry can be nonlinearly realised in this section, by first considering nonlinear realisations in general and treating the example of chiral symmetry breaking before continuing with the specific case of Maxwell.

3.1 Nonlinearly realising symmetries in general

How to construct an action nonlinearly realising a symmetry, was worked out by Callan, Coleman, Wess and Zumino [9]. Realising a symmetry nonlinearly means the vacuum states are not invariant under the action of certain generators. These generators are the generators of the broken symmetries and have parameters associated with them that correspond to a massless mode: the Goldstone boson. If we want to write down something that is invariant under the total group action, but breaks some subset of the symmetries when choosing a vacuum, these Goldstone modes are a good place to start. This is exactly what is done in the CCWZ construction, where we take a group G, having a subgroup H which is not broken. Since we can parametrise a group element

$$g = e^{\alpha_i A^i} e^{\alpha_j V^j}, \tag{37}$$

with V_j all generators forming the subgroup H, we see that taking the coset G/H allows us to only keep the Goldstone modes α_i. Then we need to make sure to form a combination of elements from the coset that actually is invariant under the full group action. To do this, it is important to first consider how such a coset element transforms. Let the coset element be given by

$$U = e^{\pi_i A^i}, \tag{38}$$

then the transformation under a generic group element is

$$gU = e^{\alpha_i A^i} e^{\alpha_j V^j} e^{\pi_k A^k} = e^{\pi_i' A^i} e^{\alpha_j' V^j} = U' h, \tag{39}$$

with $h \in H$. Or rewriting slightly, $U' = gUh^{-1}$.

CCWZ showed that a specific differential form, the Maurer-Cartan 1 form, is useful in constructing an invariant, using an expression with the above transformation. The MC form is given by $\Omega = U^{-1}dU$, and can be parametrised in terms of the generators of the group as

$$\Omega = c_i A^i + c_j V^j. \tag{40}$$

Using the transformation of U under a specific generic element g, we see

$$\begin{aligned} \Omega' = \left(U^{-1}dU\right)' &= (hUg^{-1})d(gUh^{-1}) \\ &= hU^{-1}g^{-1}g(dU)h^{-1} + hU^{-1}g^{-1}gU(dh^{-1}) \\ &= h\left(U^{-1}dU\right)h^{-1} + hdh^{-1}, \end{aligned} \tag{41}$$

since h is dependent on the Goldstones π_i. Expanding in broken and unbroken generators again gives

$$c_i' A^i + c_j' V^j = h(c_i A^i + c_j V^j)h^{-1} + hdh^{-1} = hc_i A^i h^{-1} + h(c_j - d\alpha_j')V^j h^{-1}. \tag{42}$$

This shows that the coefficients of the broken generators of the MC form transform like a linear representation of the subgroup H, under the full group transformation. Taking the expression $c_i c^i$ we can create a scalar, which will then be invariant, while being composed of things not leaving the vacuum invariant.

3.2 Chiral symmetry breaking

We will illustrate the process of constructing nonlinear realisations by considering the example of the nonlinear chiral symmetry, breaking $SU_L(3) \times SU_R(3)$ to $SU_V(3)$. The three lightest quarks have this approximate symmetry (approximate because their masses are not exactly equal), corresponding to transformations of left-handed and right-handed quarks of which only the vector combinations (as opposed to the axial-vectors) are unbroken.

A generic element $g \in G = SU_L(3) \times SU_R(3)$ can be rewritten as a left-handed and a right-handed part, both individually elements of $SU(3)$. The fact that this is a unitary group means we can introduce an identity element $I = L^\dagger L$ and write [24]

$$g = (L, R) = (L, RL^\dagger L) = (1, RL^\dagger)(L, L), \tag{43}$$

in which we can identify $(L, L) \in H = SU(3)_V$, so that $(1, RL^\dagger) = U \in G/H$ is a coset element containing Goldstones and broken generators. The transformation of U is given by

$$(\tilde{L}, \tilde{R})(1, RL^\dagger) = (\tilde{L}, \tilde{R}RL^\dagger \tilde{L}^\dagger \tilde{L}) = (1, \tilde{R}RL^\dagger \tilde{L}^\dagger)(\tilde{L}, \tilde{L}), \tag{44}$$

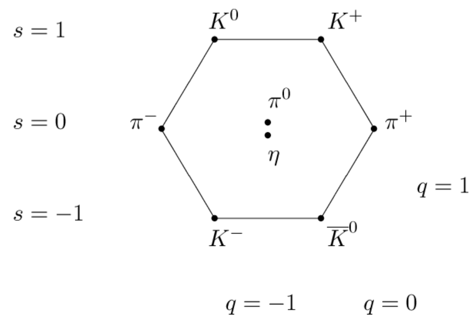

Figure 4: The meson octet resulting from the spontaneous breaking of $SU(3) \times SU(3)$ breaking to $SU(3)$ [26].

which indeed has the form $U' = gUh^{-1}$. Considering only the second part $\Sigma = RL^\dagger$ of U for a moment (and noting the fields are functions of spacetime), we see the MC form itself transforms linearly:

$$\begin{aligned}\Omega'_\Sigma = (\Sigma^\dagger \partial_\mu \Sigma)' &= \tilde{L}LR^\dagger \tilde{R}^\dagger \partial_\mu \left(\tilde{R}RL^\dagger \tilde{L}^\dagger\right) \\ &= \tilde{L}(LR^\dagger)\partial_\mu (RL^\dagger)\tilde{L}^\dagger \\ &= \tilde{L}\Sigma^\dagger \partial_\mu \Sigma^\dagger \tilde{L}^\dagger.\end{aligned} \tag{45}$$

This implies the trace of the square will be invariant

$$\begin{aligned}\mathrm{Tr}\Big\{\left(\Sigma^\dagger \partial_\mu \Sigma\right)\left(\Sigma^\dagger \partial^\mu \Sigma\right)\Big\}' &= \mathrm{Tr}\Big\{\tilde{L}\left(\Sigma^\dagger \partial_\mu \Sigma\right)\tilde{L}^\dagger \tilde{L}\left(\Sigma^\dagger \partial^\mu \Sigma\right)\tilde{L}^\dagger\Big\} \\ &= \mathrm{Tr}\Big\{\left(\Sigma^\dagger \partial_\mu \Sigma\right)\left(\Sigma^\dagger \partial^\mu \Sigma\right)\Big\}.\end{aligned} \tag{46}$$

Since integrating one of the factors by parts results in a term with $\partial_\mu(\Sigma^\dagger\Sigma) = 0$, the prevailing term that gives the lowest order term in the Lagrangian will be

$$L_{Ch} = f\,\mathrm{Tr}\Big\{(\partial_\mu \Sigma^\dagger)(\partial^\mu \Sigma)\Big\}, \tag{47}$$

where f is some constant.

An analysis of this symmetry breaking pattern was done in 1961, resulting in a low energy description of these three quarks in terms of a set of interacting mesons and baryons [25]. The breaking of eight of the generators is associated with Goldstones, the mesons depicted in figure 4. The three pions in the middle row can also be seen as resulting from chiral symmetry breaking from $SU(2) \times SU(2)$ to $SU(2)$, completely analogously to the above example, only with the lightest two quarks instead of three. Because the $SU(3)$ is also broken explicitly, the masses of these mesons are small, but not zero.

This explicit symmetry breaking occurs through a term like

$$L_{mass} = \bar{\psi}M\psi = m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s, \tag{48}$$

in which the ψs are the up, down and strange quark, and M is a diagonal matrix with their masses. If we apply an infinitesimal $SU(3)$ operation (keeping only terms up to first order), say

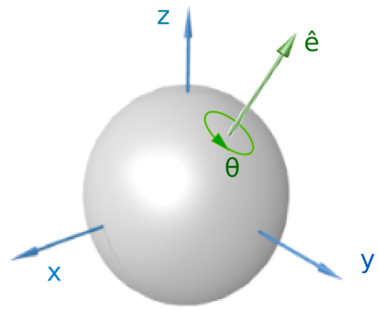

Figure 5: The rotations of a sphere, parametrised by an axis and a rotation angle [27].

$\delta \exp\{\frac{i}{2}\alpha_1\lambda_1\} = \frac{i}{2}\delta\alpha_1\lambda_1$, with the first of the Gell-Mann matrices, we see

$$\begin{aligned}
L'_{mass} &= \bar{\psi}(1 - \frac{i}{2}\delta\alpha_1\lambda_1)M(1 + \frac{i}{2}\delta\alpha_1\lambda_1)\psi \\
&= L_{mass} + \frac{i\delta\alpha_1}{2}\begin{pmatrix}\bar{u} & \bar{d} & \bar{s}\end{pmatrix}\begin{pmatrix}m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s\end{pmatrix}\begin{pmatrix}0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}u \\ d \\ s\end{pmatrix} \\
&- \frac{i\delta\alpha_1}{2}\begin{pmatrix}\bar{u} & \bar{d} & \bar{s}\end{pmatrix}\begin{pmatrix}0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s\end{pmatrix}\begin{pmatrix}u \\ d \\ s\end{pmatrix} \\
&= L_{mass} + \frac{i\delta\alpha_1}{2}(\bar{d}m_d u + \bar{u}m_u d - \bar{u}m_d d + \bar{d}m_u u),
\end{aligned} \tag{49}$$

which shows the symmetry is explicitly broken, if the masses $m_u \neq m_d$. Similar considerations with other Gell-Mann matrices show that all three masses must be equal to not break $SU(3)$. Note that we did not need any distinction between left- and right-handed quarks for this derivation, this is simply a result of the interchange of quark flavours.

3.3 Nonlinearly realising Maxwell symmetry

In constructing nonlinearly realised symmetries, one chooses a coset. In the case of Maxwell symmetry, this means choosing both the level of the extension (that is, take a certain quotient of the total group), and the subgroup H. This determines what generators are present in the coset element U of which invariant terms are constructed.

What it means to take a coset, can be illustrated using some well-known groups with obvious geometric interpretations. For example, take the group $SO(3)$, with the subgroup $SO(2)$. The former of course corresponding to the three parameter group[11] giving all rotations in $\mathbb{R}^3$, while the latter corresponds to rotations around a fixed axis, so having one parameter.

[11] Two for specifying the axis, since we have three axes and a relation $x^2 + y^2 + z^2 = 1$ guaranteeing a unit vector. Another one for the size of the rotation.

Taking the coset $SO(3)/SO(2)$ will in our example amount to retaining only the generators specifying the direction of rotation axis in three dimensions, since all group elements related to each other by a transformation from $SO(2)$ are identified. This means what remains from the picture we had before, is the spherical shell S^2.

Most often in nonlinearly realising Maxwell, the coset with the Lorentz group as subgroup is taken

$$\mathrm{Maxwell}_l/\mathrm{Lorentz}, \tag{50}$$

where l is the level of truncation of the infinite group. This makes sure we are working with a relativistic theory. Taking a different coset could for example be interesting in the study of the theory in a medium, when the boosts are broken because different velocities are no longer equivalent.

The coset element is given by:

$$U = e^{x^a P_a} e^{\frac{1}{2}\theta^{ab} Z_{ab}} e^{\frac{1}{2}\xi^{abc} Y_{abc}} \cdots , \tag{51}$$

where the dots represent exponents of higher level generators, up to and including level l. The fact that we are not including $\exp\left(\frac{1}{2} r^{ab} M_{ab}\right)$, means we've chosen the Lorenz gauge to work with. The factors of $\frac{1}{2}$ in front of the Z and Y are not strictly necessary, but simplify numerical coefficients later on. Notice that in this equation we also introduce the parameters for every generator, adding a degree of freedom to system related to each of them. These coordinates are the Goldstone modes related to the nonlinear generators. Spacetime coordinates x^a too have this interpretation, as a consequence of the translations being broken.

Choosing extensions up to and including the third level, so taking $\mathrm{Maxwell}_3$, the 1-form is given by

$$\Omega_3 = dx^a P_a + \frac{1}{2}\left(d\theta^{ab} + dx^a x^b\right) Z_{ab} + \frac{1}{2}\left(d\xi^{abc} + \theta^{bc} dx^a + \frac{1}{3} x^a x^b dx^c\right) Y_{abc}, \tag{52}$$

which is the expansion in terms of the symmetry generators. The coefficients in front of the generators are the building blocks for Lorentz scalars having $\mathrm{Maxwell}_3$ symmetry. The symmetries of the generators themselves project out the same symmetries in the coefficients, giving

$$\begin{aligned} c_1^a &= dx^a \\ c_2^{ab} &= d\theta^{ab} + \frac{1}{2}(dx^a x^b - dx^b x^a) \\ c_3^{abc} &= d\xi^{abc} + \frac{1}{3}(2\theta^{bc} dx^a - \theta^{ca} dx^b - \theta^{ab} dx^c) + \frac{1}{6} x^a (x^b dx^c - x^c dx^b). \end{aligned} \tag{53}$$

Since these coefficients nonlinearly realise $\mathrm{Maxwell}_3$, but linearly realise Lorentz (once their indices are contracted), the symmetries other than boosts and rotations will be spontaneously broken by the dynamics following from a Lagrangian built out of them.

4 Dynamics of Maxwell Lagrangians

4.1 Constructing lowest order Lagrangians

Now we have seen how to find building blocks for a Lagrangian nonlinearly realising Maxwell symmetry, we would like a physical interpretation of a Lagrangian constructed with them. One option is to construct a particle system, in which the coordinates x^a describes the position of a particle, or in the case of a particle of nonzero extend, the centre of mass.

Taking all parameters related to the broken generators as functions of a world-line parameter τ, by the chain rule the differentials become derivatives with respect to it and multiplied by $d\tau$. For the first coefficient in the MC form from the Maxwell$_3$/Lorentz coset given in the previous section this would mean so $c_1^a = \dot{x}^a d\tau \equiv \omega_1^a d\tau$.

To lowest order in derivatives, possible terms in a Maxwell$_3$ action are

$$\begin{aligned} \omega_{1a}\omega_1^a &= \omega_1^2, \\ \omega_{2ab}\omega_2^{ab} &= \omega_2^2, \\ \omega_{3abc}\omega_3^{abc} &= \omega_3^2, \end{aligned} \tag{54}$$

for any number of dimensions. In a specific number of dimensions, say four, ϵ^{abcd} can be used as well to contract indices, giving in addition

$$\epsilon^{abcd}\omega_{2ab}\omega_{2cd}. \tag{55}$$

The other possible contractions vanish due to (anti-)symmetry and the Jacobi identity. Higher order terms are of course possible, but disregarded as we aim to interpret the theory as an effective theory. More on this is found in 4.2.

The simplest symmetry group in the sequence of Maxwell extensions is the ordinary Poincaré group. To lowest order, we have a free particle Lagrangian

$$L_{M_1} = m\omega_1^2, \tag{56}$$

giving $m\ddot{x}^a = 0$ as equation of motion. This is motion in a straight line, without any acceleration.

At the next level, where we have the classic Maxwell group as studied by Schrader [10],

$$L_{M_2} = m\omega_1^2 + \frac{a}{2}\omega_2^2, \tag{57}$$

gives that ω_2^{ab} is constant, and

$$m\dot{\omega}_1{}^a = a\omega_2^{ba}\omega_{1b} \tag{58}$$

reduces to the Lorentz force law once we introduce the interpretation $qF_{ab} = -a\omega_{2ab}$:

$$m\ddot{x}^a = qF^{ab}\dot{x}_b. \tag{59}$$

This means that compared to the previous level either an electromagnetic field has been turned on and the particle has received an electric charge, as illustrated in figure 6.

The Maxwell$_3$ Lagrangian is given by

$$L_{M_3} = m\omega_1^2 + \frac{a}{2}\omega_2^2 + \frac{b}{2}\omega_3^2. \tag{60}$$

Using the Euler-Lagrange equations, we find that it is again possible to substitute two equations into the last one to obtain an equation only dependent on the parameters x^a, the Goldstones

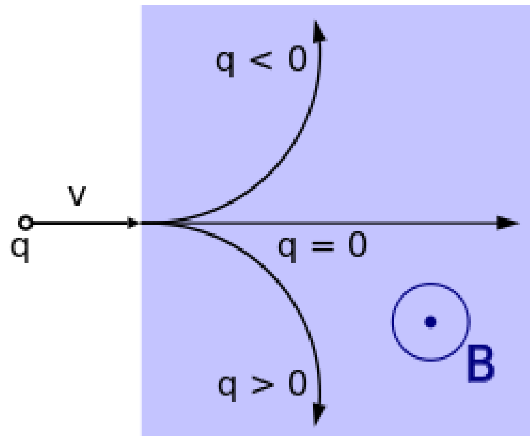

Figure 6: A charged particle entering a constant magnetic field, where it would be described by a Lagrangian having Maxwell symmetry [28].

of the translations P_a. Let us now explicitly derive the equation of motion for this Lagrangian where

$$\begin{aligned} \omega_1^a &= \dot{x}^a \\ \omega_2^{ab} &= \dot{\theta}^{ab} + \dot{x}^{[a}x^{b]} \\ \omega_3^{abc} &= \dot{\xi}^{abc} + \theta^{<bc}\dot{x}^{a>} + \frac{1}{3}x^a x^{[b}\dot{x}^{c]}, \end{aligned} \tag{61}$$

in which the brackets are representing different symmetries, namely:

$$\begin{aligned} \dot{x}^{[a}x^{b]} &= \frac{1}{2}(\dot{x}^a x^b - \dot{x}^b x^a) \\ \theta^{<bc}\dot{x}^{a>} &= \frac{1}{3}(2\theta^{bc}\dot{x}^a - \theta^{ca}\dot{x}^b - \theta^{ab}\dot{x}^c). \end{aligned} \tag{62}$$

The variations of the Lagrangian with respect to ξ^{abc}, θ^{ab} and x^a give

$$\begin{aligned} \delta\xi^{abc} &: \frac{d}{d\tau}(b\omega_3^{abc}) = 0 \\ \delta\theta^{ab} &: \frac{d}{d\tau}(a\omega_2^{ab}) = b\omega_3^{def}\frac{\partial\omega_{3def}}{\partial\theta_{ab}} \\ \delta x^a &: \frac{d}{d\tau}\left(2m\omega_1^a + a\omega_2^{bc}\frac{\partial\omega_{2bc}}{\partial\dot{x}_a} + b\omega_3^{bcd}\frac{\partial\omega_{3bcd}}{\partial\dot{x}_a}\right) = a\omega_2^{bc}\frac{\partial\omega_{2bc}}{\partial x_a} + b\omega_3^{bcd}\frac{\partial\omega_{3bcd}}{\partial x_a}, \end{aligned} \tag{63}$$

which, noting the partials are

$$\begin{aligned} \frac{\partial\omega_{3bcd}}{\partial x_a} &= \frac{1}{3}(\delta_b^a x_{[c}\dot{x}_{d]} + x_b\delta^a_{[c}\dot{x}_{d]}) \quad \frac{\partial\omega_{2bc}}{\partial x_a} = \dot{x}_{[b}\delta^a_{c]} \\ \frac{\partial\omega_{3bcd}}{\partial\dot{x}_a} &= \theta_{<cd}\delta^a_{b>} + \frac{1}{3}(x_b x_{[c}\delta^a_{d]}) \quad \frac{\partial\omega_{2bc}}{\partial\dot{x}_a} = \delta^a_{[b}x_{c]} \\ \frac{\partial\omega_{3def}}{\partial\theta_{ab}} &= \delta^a_{<e}\delta^b_f\dot{x}_{d>}, \end{aligned} \tag{64}$$

becomes

$$
\begin{aligned}
&2m\dot{\omega}_1^a + b\omega_3^{def}\delta^b_{<e}\delta^c_f\dot{x}_{d>}(\delta^a_{[b}x_{c]}) + a\omega_2^{bc}(\delta^a_{[b}\dot{x}_{c]}) + b\omega_3^{bcd}(\dot{\theta}_{<cd}\delta^a_{b>} + \frac{1}{3}\dot{x}_b x_{[c}\delta^a_{d]} + \frac{1}{3}x_b\dot{x}_{[c}\delta^a_{d]}) \\
&= a\omega_2^{bc}(\dot{x}_{[b}\delta^a_{c]}) + \frac{b}{3}\omega_3^{bcd}(\delta^a_b x_{[c}\dot{x}_{d]} + x_b\delta^a_{[c}\dot{x}_{d]}).
\end{aligned}
\tag{65}
$$

Rearranging and renaming some indices, while using the symmetry properties of the MC form coefficients, this becomes

$$
2m\dot{\omega}_1^a + 2a\omega_2^{ab}\omega_{1b} + b\omega_3^{cab}x_b\dot{x}_c + b\omega_3^{abc}\dot{\theta}_{bc} + \frac{b}{3}\omega_3^{bca}\dot{x}_b x_c + \frac{b}{3}\omega_3^{bca}x_b\dot{x}_c - \frac{b}{3}\omega_3^{abc}x_b\dot{x}_c - \frac{b}{3}\omega_3^{cab}\dot{x}_b x_c = 0.
\tag{66}
$$

Now, using $\dot{\theta}_{bc} = \omega_{2bc} + x_{[b}\dot{x}_{c]}$ and $\omega_3^{bca}\dot{x}_b x_c = -\omega_3^{cab}x_b\dot{x}_c$ we find

$$
2m\dot{\omega}_1^a + 2a\omega_2^{ab}\omega_{1b} + b\omega_3^{abc}\omega_{2bc} + \frac{2b}{3}(\omega_3^{abc} + \omega_3^{bca} + \omega_3^{cab})x_b\dot{x}_c = 0
\tag{67}
$$

But $\omega_3^{abc} + \omega_3^{bca} + \omega_3^{cab} = 0$, so

$$
\dot{\omega}_1^a + a\omega_2^{ab}\omega_{1b} + \frac{b}{2}\omega_3^{abc}\omega_{2bc} = 0.
\tag{68}
$$

Calling the integration constant for $b\omega_3^{abc} = I_1^{abc}$, and $a\omega_2^{ab} = I_1^{cab}x_c + I_2^{ab}$, this becomes

$$
m\ddot{x}^a + (I_1^{cab}x_c + I_2^{ab})\dot{x}_b + \frac{1}{2a}I_1^{abc}(I_{1dbc}x^d + I_{2bc}) = 0.
\tag{69}
$$

Interpreting $-qF^{ab} = I_1^{cab}x_c + I_2^{ab}$ as a field strength, this equation of motion shows the familiar Lorentz force (the $\dot{x}$ term) and a term with the product of the field strength and its gradient. This type of term is what one would expect of a dipole moment, generated by an external field, since it will be proportional to the field, and interact with the gradient of it. This term is new compared to earlier studies of Maxwell$_3$ [11, 12], and has been found by Tonnis ter Velthuis for the vector representation and verified by the author for the tensor representation above[12].

4.2 Note on mass dimensions

When building the algebra, one has a freedom to choose pre-factors for the generators newly added to it. For example, going from level 1 to level 2, one adds

$$
[P_a, P_b] = \frac{1}{L^\alpha}Z_{ab},
\tag{70}
$$

with the pre-factor $1/L^\alpha$, which can be dimensionful. In order to have the interpretation of the translations intact, meaning P_a give rise to momentum, we need the total mass dimension of the right hand side to be 2. This means that to have Z_{ab} dimensionless, and letting L be a measure of length, it is required that $\alpha = 2$. Similarly, for every next level, the generator G constructed by

$$
[P_{i_1}, [P_{i_2}, \dots P_{i_n}]] = \frac{1}{L^{\alpha_n}}G_{i_1 i_2 \cdots i_n},
\tag{71}
$$

[12] Actually, since tracelessness of the tensor $\widetilde{Y}^{abc}$ and therefore the quantity ω_{3abc} is never assumed, the derivation above holds for both irreps.

will have dimension zero if $\alpha_n = n$. This requirement is a natural one, as the parameter associated with each generator will have the negative mass dimension of the generator, and whereas the parameter x^a of P_a has the interpretation of length and consistently mass dimension $[x] = -1$, no such physical interpretation is available for the parameters of the added symmetries. The conclusion is that the mass dimension of these parameter should vanish.

For the other commutators, this choice has the consequence that for every commutator with P_a present, a factor $1/L$ appears, e.g.

$$[P_a, Z_{bc}] = \frac{1}{L} Y_{abc}. \tag{72}$$

This gives the conventions that all translations, as usual have $[P] = 1$, while the other free Lie algebra generators have dimension $[Z, Y, \dots] = 0$, with their associated parameters similarly dimensionless.

In the Maurer-Cartan coefficients this results in

$$\begin{aligned} \omega_2^{ab} &= \dot{\theta}^{ab} + \frac{1}{L^2} \dot{x}^{[a} x^{b]} \\ \omega_3^{abc} &= \dot{\xi}^{abc} + \frac{1}{3L}(2\theta^{bc}\dot{x}^a - \theta^{ca}\dot{x}^b - \theta^{ab}\dot{x}^c) + \frac{1}{6L^3} x^a (x^b \dot{x}^c - x^c \dot{x}^b), \end{aligned} \tag{73}$$

such that the coefficients have mass dimension $[\omega_{2,3}] = 1$. It is not hard to see that for every higher order coefficient, since the first term is always the derivative of a dimensionless parameter, this will be the case.

Using this knowledge to analyse the dimensions of the different terms of the Lagrangian given earlier, we see $[\dot{x}^2] = 0$ such that the constant has $[m] = 1$. For the other terms, $[\omega_{2,3}^2] = 2$, meaning $[a] = [b] = -1$. Combinations of more MC form coefficients will have even higher dimension, suppressing their importance to the low energy dynamics.

The equations of motion will then be slightly altered. For example, Maxwell$_3$ for the new choice of commutation relations shows explicitly the length scale L introduced into it:

$$m\ddot{x}^a + \frac{1}{L^2}(I_1^{cab} x_c + I_2^{ab})\dot{x}_b + \frac{1}{2a} I_1^{abc}(I_{1dbc} x^d + I_{2bc}) = 0. \tag{74}$$

Considerations like these allow us to interpret the theories we build up to a certain order of derivatives (or order in MC form coefficients) as effective theories.

4.3 Dipole interaction

In order to develop the connection suggested previously between the particle Lagrangian non-linearly realising Maxwell$_3$ symmetry and electrodynamics, we will now look at the interaction of a dipole with an external electromagnetic field. The connection given in this section has been made by Tonnis ter Velthuis.

In index notation, the interaction of a dipole with an electromagnetic field has the following Lagrangian [29]

$$L = \frac{m}{2}\dot{x}^2 + qA_\mu \dot{x}^\mu - \frac{1}{2} F_{\mu\nu} D^{\mu\nu}. \tag{75}$$

The first term is simply the kinetic term of the dipole, with m being the mass, whereas the second term shows the four potential A_μ combined with charge q, to form the interaction of the charge with the field. The third term is where the dipole couples to the electromagnetic field $F_{\mu\nu}$ via the dipole tensor $D^{\mu\nu}$.

The dipole moment tensor can be taken to be anti-symmetric, without loss of generality [29]. It can be written as a separate electric and magnetic part, as in

$$D^{\mu\nu} = P^{[\mu}\dot{x}^{\nu]} + \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}M_\kappa \dot{x}_\lambda. \tag{76}$$

Here the electric and magnetic dipole moments are chosen to be proportional to the external field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$:

$$\begin{aligned} P^\mu &= \alpha F^{\mu\nu}\dot{x}_\nu \\ M_\kappa &= \frac{1}{2}\beta\epsilon_{\mu\nu\kappa\lambda}F^{\mu\nu}\dot{x}^\lambda. \end{aligned} \tag{77}$$

Taking the frame in which the 3-velocity is zero, such that $\dot{x}_0 = -1$, we see that since $-E^i = F^{0i}$, $P^i = \alpha E^i$ indeed picks out the electric field, giving a dipole moment proportional to it with polarisability α. Similarly, $M_i = \beta B_i$, with β the magnetic polarisability[13].

After substituting these back into the dipole tensor, we find

$$D^{\mu\nu} = -(\alpha+\beta)F^{\lambda[\mu}\dot{x}^{\nu]}\dot{x}_\lambda + \frac{\beta}{2}F^{\mu\nu}, \tag{78}$$

where we used that $\dot{x}_\mu \dot{x}^\mu = -1$. When choosing the particular case of $\alpha = -\beta$, the dipole Lagrangian becomes

$$L_{dip} = -\frac{\beta}{4}F_{\mu\nu}F^{\mu\nu}. \tag{79}$$

Assuming a field linear in x, so $F^{\mu\nu} = a_0^{\mu\nu} + b_0^{\lambda\mu\nu}x_\lambda$, varying the complete Lagrangian with respect to it then gives the equation of motion

$$m\ddot{x}^\mu + qF^{\nu\mu}\dot{x}_\nu + \frac{\beta}{2}F_{\nu\lambda}b_0^{\mu\nu\lambda} = 0. \tag{80}$$

In other words, a massive particle theory having nonlinearly realised Maxwell$_3$ symmetry is equivalent to a theory describing a particle having nonzero magnetic and electric dipole moments, travelling through a linear external field, as shown in figure 7. The magnetic and electric polarisability of this particle are related in a particular way, in which their magnitude is the same, but their direction opposite with respect to their respective fields. In nature, induced electric dipoles always have polarisation in the direction of the field, but magnetic dipoles can be induced in a direction opposing the magnetic field. This is the phenomenon of diamagnetism. Diamagnetic materials such as Bismuth exist in nature, but their diamagnetic susceptibility pales in comparison to that of superconductors. In fact, when a superconductor is in its superconducting state, the Meissner effect occurs (see figure 8), completely expelling all magnetic field lines from its interior [30], just like in a perfect conductor the electric field inside the conductor is completely cancelled by shifting charges to the boundary. Therefore, we conclude that the

[13] In electromagnetism, polarisability is used for the polarisation response of an object, like a molecule, to a field. This is the electric or magnetic susceptibility, multiplied by the dimensions of the object.

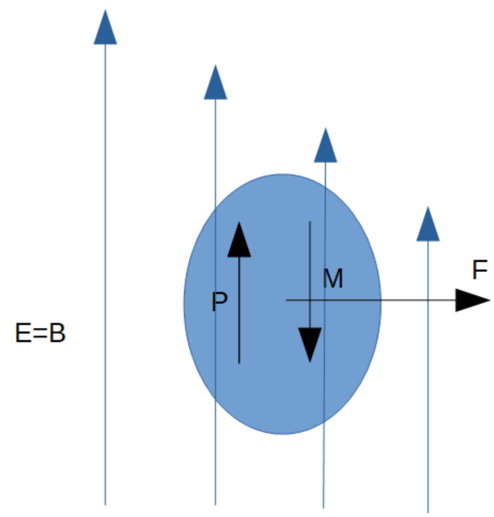

Figure 7: A dipole of the type described in the text, with opposing magnetic and electric dipole moment, in a linear field. The electric and magnetic fields are parallel.

dipole we are describing here could be a perfect superconductor, with $\alpha = -\beta = 1$ in the correct units. Of course, since in principle the size of the polarisabilities does not matter, it could also describe other materials. However, as the typical size of electric susceptibility is much larger than that of magnetic susceptibility, it is valid to assume that the only real-world materials coming close to our description are superconductors.

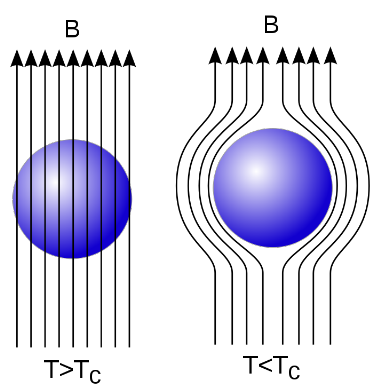

Figure 8: The Meissner effect, in which a superconductor excludes the external magnetic field when the temperature gets below the transition temperature [31]

4.4 Beyond Maxwell$_3$

Having seen the sequence of increasing complexity from Poincaré to Maxwell and Maxwell$_3$, it seems natural to continue the analysis for higher level truncations. As shown in [12], the infinite Maxwell algebra contains an ideal, of which the quotient describes a theory simply giving the

Lorentz force equation with a general electric field written as a Taylor series[14]. And since the particle itself simply travels straight at Poincaré, gets a charge at Maxwell$_2$ and becomes a dipole at level three, the idea of the multipole expansion continuing with a quadrupole at level four is not far-fetched. Similar ideas were suggested already in [11] and again in [12], though they were not worked out in detail and proposed the dipole interaction to be at level four instead of three.

The trouble in this interpretation stems from the lack of integrability encountered from the fourth level onward. This introduces a dependence on the θ^{ab} parameter, related to the Z_{ab} generator, which hampers the integration of the equations in terms of the spacetime coordinates x^a. Moreover, as the degrees of freedom θ^{ab} do not have a clear interpretation themselves, proceeding seems fruitless.

As mentioned in section 2.4, the quotient found in [12] excludes terms blocking integrability and simply allows one to find the Lorentz equation with a general, Taylor series-expanded electromagnetic field. To remind the reader, this quotient only contains representations like

$$\begin{ytableau} a \end{ytableau}\,, \quad \begin{ytableau} a \\ b \end{ytableau}\,, \quad \begin{ytableau} a & c_1 & c_2 & \cdots & c_n \\ b \end{ytableau}\,, \tag{81}$$

with $n > 0$ in $\mathfrak{gl}(D)$. The attentive reader will notice that the levels studied so far indeed only have generators in the carrier space of representations like these, while still giving a nontrivial multipole interaction. An interesting problem would therefore be to look at level four Maxwell while only including the first of the newly introduced generators

$$\ydiagram{3,1}\,, \quad \ydiagram{2,1,1}\,. \tag{82}$$

Could it be that the difference in constructing the Lagrangian between [12] and this text, that caused us to find a dipole already at level three, also allows us to find more interesting behaviour at level four, when taking the quotient as described above?

5 Toward a Maxwell 'soft limit'

As should be clear by now, Maxwell realising particle Lagrangians describe a particle in an electromagnetic field. Also, higher level Maxwell theories seem to describe more complicated behaviour than lower ones. That is clear from the fact that the field gets Taylor expanded order-by-order going up in Maxwell level, while the particle also receives some new property, at least in the first few levels, where it first gets a mass, then a charge and thirdly a dipole moment. Could it be that the infinite algebra describes a general theory of an electromagnetic particle? If so, then the Maxwell levels give a natural ordering to such theories, since they are formed in a strict hierarchy: we cannot have third level Maxwell in our algebra without having the second.

This motivates an exploration of the question whether we can relate electromagnetically interacting particle theories to their Maxwell level, giving a classification of all such theories. In this

[14] Although it should be noted that the method of constructing the Lagrangian from which these equations of motion are derived, are significantly different.

section we make an attempt to find such a classification, drawing inspiration from the classification of scalar effective field theories (EFTs) based on their soft limit. Ideally, we would be able to distil a kind of index from electromagnetic particle theories, directly related to the minimum level of Maxwell symmetry we would need to describe an equivalent particle.

Firstly, we will have a look at the soft limit approach in the context of the scattering amplitude programme. After that, we will analyse the particle theories of which we now the Maxwell levels, so that we can try to find a way of relating general electromagnetic particle theories to their level.

5.1 Soft limits in scalar EFTs and the scattering amplitude programme

As in the next section we plan to draw a parallel between the soft limit classification and construction of Effective Field Theories and the effective particle theory we are describing with the Maxwell extensions, this section will establish some context and motivation for, and results from the soft limit approach in the field theory setting. The soft limit fits into a larger effort of approaching field theories and the calculations of their predictions from a different angle, bypassing their action and going directly to the observables: the scattering amplitudes. This section relies largely on the lecture notes on the modern scattering amplitude programme by Cheung [13].

Scattering amplitudes are a way of writing down the link between two quantum mechanical states. One state at a time long ago, when all particles were beyond each others reach, and one far into the future, when the particles are once again outside each others sphere of influence. These states are called asymptotically free states, for $t \to -\infty$ and $t \to \infty$. In interacting theories, particles of one kind can sometimes transition into particles of another kind with a certain probability. So if we want to write down all possible transitions of some incoming state to an outgoing states at once, we need to have a scattering matrix, specifying the amplitudes for the different processes. The square of any of these individual amplitudes then gives the probability.

The amplitudes are conventionally calculated from Feynman diagrams, which through the integrals they represent are inextricably linked to the action of the theory. Starting from the action, in an algorithmic manner one can obtain the probability amplitude for a process, by writing down the possible diagrams and applying the Feynman rules. But in the action, there is a considerable redundancy, in the form of field redefinitions and gauge or diffeomorphism invariance. This redundancy is clear when we consider the path integral formulation of QFT, since there we are using the fields as an integration variable.

This introduces a few drawbacks of the Feynman diagram method. Firstly, it requires a quickly growing number of diagrams going from simple processes to more complicated ones, while a rewriting of the final answers in some cases allows dramatic simplifications, showing that this method is at times *needlessly* difficult. This was illustrated convincingly by a set of articles by Parke and Taylor in 1986 [32, 33]. In these articles, Parke and Taylor calculated a six-gluon process, involving 220 diagrams, and were able to reduce it to a single term, which was a combination of the momenta of the particles (summed over permutations of the external legs).

Aside from the time lost on calculations of this kind, physicists using the Feynman diagram approach also miss out on recognising the physical structure enabling the cancellations leading to

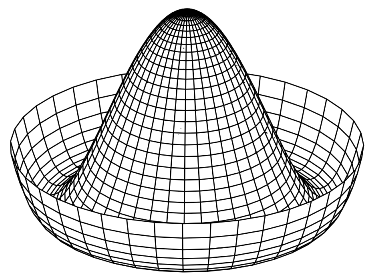

Figure 9: The Sombrero potential, with $U(1)$ symmetry.

the simple answers. This is perhaps the main motivation behind the modern scattering amplitude programme. An example of this, is that one can show that the only possible interacting massless vector theory is Yang-Mills theory, and similarly, that the only possible massless tensor theory is general relativity [13]. Moreover, they are related through so-called colour-kinematics duality: the fact that, when written in a certain way, kinematic factors in the amplitudes satisfy the same relations as colour factors[15].

Clearly, the amplitude programme constrains amplitudes enormously. It does so by relying on dimensional analysis, Lorentz invariance and locality[16]. This set of arguments however is not sufficient for scalar EFTs, since it requires more physical input than a complete theory. This additional input comes in the form of the symmetry breaking pattern, as we have seen determined by the coset, which relies on the construction of an action. It turns out, that in the case of scalar EFTs, one can classify and reconstruct possible theories based on another piece of physical information, known as the soft limit. The soft limit looks at the behaviour of a tree level amplitude in the limit of small momentum of an external leg [13]

$$\lim_{p\to 0} A(p) \propto p^{\sigma}, \tag{83}$$

where σ is called the soft degree. For tree level amplitudes, this will be an integer. If amplitudes of a theory all have a certain soft limit, or have limits relating to lower-point amplitudes, the amplitudes are said to satisfy a soft theorem. There is a history of soft theorems in QFT going back to the sixties, with Steven Weinberg's soft theorems for photons and gravitons, and Adler's zero for the Nonlinear Sigma Model (of which the model with chiral symmetry breaking treated earlier is an example). However, in these cases, the theorems were derived from the action, whereas in the inverted logic of modern scattering amplitudes, theorems like this can be used to construct the S-matrix [13].

A simple example of a soft limit theorem is the limit for the Nambu-Goldstone Boson. This theory has the simple symmetry breaking pattern $U(1)/I$, with I the identity group. This can be seen by considering the internal potential, shaped like a Mexican Sombrero (as in figure 9), which has $U(1)$ symmetry, until one of the vacua is chosen and the symmetry is broken. There is a shift symmetry associated with the possibility of transitioning between the different angels $\phi \to \phi' = \phi + c$.

[15]This is the Jacobi identity of course, because of the Lie algebra underlying the colour structure. From the point of view of the amplitudes, the Jacobi identity arises as a consistency relation from factorisation of the four-particle amplitude into two three-particle amplitudes [13]

[16]The statement that objects can only be influenced by their direct surroundings.

The NGB is described by a general action

$$L_{NGB} = \frac{1}{2}(\partial\phi)^2 + \frac{\lambda_4}{4!}(\partial\phi)^4 + \frac{\lambda_6}{6!}(\partial\phi)^6 + \dots, \tag{84}$$

where the scalar field ϕ is always accompanied by a derivative. This means that the vertices always carry a factor of the momentum being sent to zero, since the derivatives bring down a momentum from the exponent when going to momentum space. Therefore, the amplitudes vanish as p^1 as the momentum goes to zero and the soft degree is $\sigma = 1$. Notice that assuming this soft degree does not get us very far in constraining the theory, as the coupling constants λ_n are still infinitely many and free to choose.

Assuming a higher soft degree, $\sigma = 2$, implies that all terms in the amplitude of order p must vanish. It can then be shown that, since every order p contribution from an n-point vertex must be related to a lower point vertex in a special way, only one parameter in the theory remains free [34]. The terms then become a Taylor expansion of a square root [35]

$$\begin{aligned} L_{DBI} &= \lambda(-\frac{1}{2}(\partial\phi)^2 + \frac{1}{4}(\partial\phi)^4 - \frac{3}{8}(\partial\phi)^6 + \dots) \\ &= -\lambda\sqrt{1 + (\partial\phi)^2} + \lambda, \end{aligned} \tag{85}$$

which is the scalar mode of Dirac-Born-Infeld theory. The coset corresponding to this Lagrangian is $ISO(1,4)/ISO(1,3)$ and the nonlinearly realised symmetry acts as $\delta\phi = c + b_a x^a + b_a \phi \partial^a \phi$ [36]. The field dependent terms here are needed to cancel terms given by the linear shift. This can be seen by considering the first two terms in the Taylor expansion, which under the given symmetry both give terms like $b_a \partial^a \phi (\partial\phi)^2$. The kinetic term also gives a term $b_a \partial^a \phi$, but this is a total derivative and therefore does not contribute to the dynamics. Similarly, the $(\partial\phi)^4$ term under the symmetry action produces terms like $b_a \partial^a (\partial\phi)^4$, which in turn get cancelled by the same terms coming from the $(\partial\phi)^6$ term in the original Lagrangian. That cancellation scheme continues to infinity.

The next step of course would be to assume a soft degree of $\sigma = 3$, but this will simply additionally fix the parameter λ_4 to zero, making a free theory. The conclusion is that DBI is the simplest interacting theory, with one derivative per field. If we allow more than one derivative per field, we can find a theory with soft degree $\sigma = 3$. This theory is called the Special Galileon

$$L_{SG} = -\frac{1}{2}(\partial\phi)^2 + \frac{1}{12\Lambda^6}(\partial\phi)^2 \left[(\Box\phi)^2 - (\partial_a\partial_b\phi)^2\right], \tag{86}$$

and it has an additional symmetry making the total transformation $\delta\phi = c + b_a x^a + s_{ab} x^a x^b + \frac{1}{\Lambda^6} s_{ab} \partial^a \phi \partial^b \phi$ [37].

Interestingly, it is possible to connect the generators from the soft limit levels to each other [38]. Since the generator of translations is just a partial derivative $P_a = \partial_a$, applying it to the generator of the shift will bring down the power of spacetime coordinates by one. If we call the generator creating the squared shift term G_2, its commutator with a translation gives

$$[P_a, G_2] = G_1 + \dots, \tag{87}$$

with G_1 the generator of the linear shift and the dots representing a combination of linearly realised generators. In the same way, the translations bring G_1 down to G_0, which generates the constant shift.

Clearly, the larger the soft degree, the more symmetric the corresponding theory. Also, applying the translation operator brings us down from one shift symmetry to a lower one. This gives us the sequence[17] in figure 5.1. In terms of Young Tableaux, you can see the generator G_2 as a symmetric tensor representation, the G_1 as a vector and G_0 as a scalar, giving

$$\square\!\square \xrightarrow{P} \square \xrightarrow{P} \bullet \tag{88}$$

The sequence given here is similar to the sequence of Maxwell extensions, in the sense that a translation symmetry generates a next step, but reverse in the sense that the applied translation yields a less symmetric theory in this case, going down in soft degree.

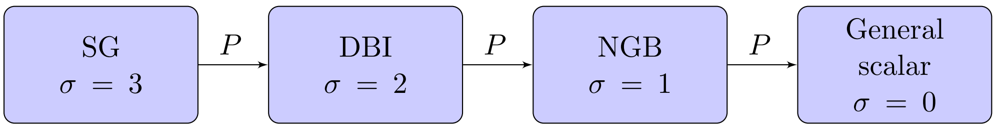

Figure 10: The sequence from most constrained to least constrained. If you want, you can imagine this sequence starting with a free scalar theory, which does not have scattering amplitudes and therefore they vanish infinitely quickly, with soft degree $\sigma = \infty$.

5.2 Integrating truncated Maxwell actions

In the following section we aim to integrate Lagrangians corresponding to different truncations of the Maxwell algebra, to try to establish a pattern in the obtained actions. This approach is inspired by the soft limit classification of effective field theories, in the sense that it tries to determine an index of complexity, similar to the soft degree in the previous section.

In our case, with every step upwards in the level ladder, the truncations of the Maxwell$_\infty$ algebra and the corresponding particle dynamics become more complicated, allowing more interactive possibilities for the massive particle. The steps are linked by the translations: every new generator is a multi-commutator with an extra translation compared to the previous level. This is just as in the soft limit sequence, except here the translation brings us up in the ladder of symmetry instead of down. In Young Tableaux the successive translations look like

$$\square \xrightarrow{P} \begin{array}{c}\square\\ \square\end{array} \xrightarrow{P} \begin{array}{l}\square\!\square\\ \square\end{array} \oplus \square \xrightarrow{P} \ldots \tag{89}$$

In our classical particle theory, the integrated particle action would play the role of scattering amplitude in the soft-limit analogy. This is a logical choice, since the action of the particle system, similarly to the scattering matrix of the quantum field system propagates the system from one state to the other. The goal of the section will then be to identify an analog to the soft-limit parameter, in its limit discriminating between actions of different levels.

The first level truncation of Maxwell, which is just Poincaré, corresponds to the Lagrangian

$$L = \frac{m}{2}\dot{x}^2, \tag{90}$$

[17]This is not to say that these theories can be obtained from one another. In fact, the Special Galileon can be obtained as a special case of the Galileon fourth and fifth order terms, which itself has a soft degree $\sigma = 2$ [34]

which because of the constancy of the velocity in the free particle case is simply integrated over proper time giving the action. Assuming the particle is initially at the origin, with velocity $\dot{x}^\mu(0) = u^\mu$, we see this becomes

$$S_{M_1} = \frac{m}{2} u^2 \tau, \tag{91}$$

with $u^2 = u_\mu u^\mu$.

Going one level higher, the Lagrangian corresponds to that of a particle in a constant EM field, or

$$L = \frac{m}{2} \dot{x}^2 + q A_\mu \dot{x}^\mu. \tag{92}$$

To integrate this, we need the explicit τ-dependence of the velocity and four potential. This can be found from the equation of motion

$$m \ddot{x}^\mu + q F^{\nu\mu} \dot{x}_\nu = 0, \tag{93}$$

together with the assumption of a form of the potential, in which

$$\partial_\mu A_\nu = -\partial_\nu A_\mu, \tag{94}$$

such that

$$A_\mu = \frac{1}{2} f_{\nu\mu} x^\nu \tag{95}$$

gives

$$F_{\mu\nu} = f_{\mu\nu}. \tag{96}$$

It is allowed without loss of generality to assume that the electromagnetic field tensor is empty, except for maximally two components. This is possible, because the system has one of the two BCR groups mentioned in section 2.2 as symmetry group. In the case $S^2 = \frac{1}{2} F_{ab} F^{ab} + \frac{i}{2} \epsilon_{abcd} F^{ab} F^{cd} \neq 0$, the system can be reduced to a setting with parallel electric and magnetic fields, while in the case $S^2 = 0$ the fields are perpendicular and equal in size.

To probe the dynamics of the second level, we will calculate the action for the case that the fields are parallel[18]. Let us choose the x components $F_{10} = f_1$ and $F_{23} = f_2$ of the fields. The equations of motion in this case are given by

$$\begin{aligned} m\ddot{x}_0 + q f_1 \dot{x}_1 &= 0, \\ m\ddot{x}_1 + q f_1 \dot{x}_0 &= 0, \\ m\ddot{x}_2 + q f_2 \dot{x}_3 &= 0, \\ m\ddot{x}_3 - q f_2 \dot{x}_2 &= 0, \end{aligned} \tag{97}$$

where the one minus sign comes from the fact that lowering a time index gives a minus, while lowering a space index does not. These are two separate systems of equations which are easily

[18]In the other case, where the fields are perpendicular and equal in size, the action should be equivalent, since the only difference is the kind of BCR group the action realises, but the algebra is the same.

solved to give

$$
\begin{aligned}
x_0(\tau) &= \frac{m}{qf_1}\left[\frac{u_0+u_1}{2}\left(e^{\frac{qf_1\tau}{m}}-1\right)+\frac{u_0-u_1}{2}\left(e^{\frac{-qf_1\tau}{m}}-1\right)\right],\\
x_1(\tau) &= \frac{m}{qf_1}\left[\frac{u_0+u_1}{2}\left(e^{\frac{qf_1\tau}{m}}-1\right)-\frac{u_0-u_1}{2}\left(e^{\frac{-qf_1\tau}{m}}-1\right)\right],\\
x_2(\tau) &= \frac{m}{qf_2}\left[\frac{u_2+iu_3}{2}\left(e^{\frac{iqf_2\tau}{m}}-1\right)+\frac{u_2-iu_3}{2}\left(e^{\frac{-iqf_2\tau}{m}}-1\right)\right],\\
x_3(\tau) &= \frac{m}{qf_2}\left[\frac{u_2+iu_3}{2i}\left(e^{\frac{iqf_2\tau}{m}}-1\right)-\frac{u_2-iu_3}{2i}\left(e^{\frac{-iqf_2\tau}{m}}-1\right)\right],
\end{aligned}
\tag{98}
$$

where we used $\dot{x}_\mu(0) = u_\mu$ and $x_\mu(0) = 0$ as initial conditions.

The action then becomes

$$
S_{M_2} = \frac{m^2}{2qf_1}(-u_0^2+u_1^2)\sinh\frac{qf_1\tau}{m} + \frac{m^2}{2qf_2}(u_2^2+u_3^2)\sin\frac{qf_2\tau}{m}. \tag{99}
$$

In the limit that $qf \to 0$, this expression reduces to the previous action S_{M_1}, as it should. The Taylor expansion around $\tau = 0$ of the above expression is given by

$$
S_{M_2} = \frac{m}{2}u^2\tau + \frac{m}{2}\left(\frac{qf_1}{m}\right)^2(-u_0^2+u_1^2)\frac{\tau^3}{3!} - \frac{m}{2}\left(\frac{qf_2}{m}\right)^2(u_2^2+u_3^2)\frac{\tau^3}{3!} + \mathcal{O}(\tau^5). \tag{100}
$$

While the action does not terminate when expanded, or start at a higher order of τ, which would be a most obvious similarity to the soft limit in the effective field theory case, we do see that the charge of the particle only starts to contribute at third order, along with the constant fields. This suggests the properties of the particle opening up at higher levels might lead to higher order corrections in the expansion of the action.

To further develop this hunch, we continue with the next level. Here the Y_{abc} generators are introduced, and the action only containing kinetic terms is equivalent to the electrodynamics action including dipole interaction, with a linear electromagnetic field (as seen in section 4):

$$
L_{M_3} = \frac{1}{2}m\dot{x}^2 + qA_\mu\dot{x}^\mu - \beta F_{\mu\nu}F^{\mu\nu}. \tag{101}
$$

Let us start with a simple field tensor, empty except for one term linear in the first space coordinate:

$$
F_{01} = g_{101}x^1 \equiv gx_1. \tag{102}
$$

This implies the following equations of motion

$$
\begin{aligned}
m\ddot{x}_0 + qgx_1\dot{x}_1 &= 0\\
m\ddot{x}_1 - qgx_1\dot{x}_0 - 4\beta g^2 x_1 &= 0\\
m\ddot{x}_2 = m\ddot{x}_3 &= 0.
\end{aligned}
\tag{103}
$$

These equations, although looking simple, have solutions that are hard to work with for the first two components, as a check with Mathematica showed. However, to establish the analogy with the soft limit, we only need infinitesimal information. Therefore, we make a Taylor expansion, using the same initial conditions as before.

$$\begin{aligned}
x_0(\tau) &= u_0\tau - \frac{qg}{m}\left(u_1^2\frac{\tau^3}{3!} + 4\left(\frac{qg}{m}u_0u_1 + 4\frac{\beta g^2}{m}u_1\right)\frac{\tau^5}{5!} + \mathcal{O}(\tau^7)\right), \\
x_1(\tau) &= u_1\tau + \left(\frac{qg}{m}u_0u_1 + 4\frac{\beta g^2}{m}u_1\right)\frac{\tau^3}{3!} + \left(\frac{qg}{m}\left(\frac{qg}{m}u_0u_1 + 4\frac{\beta g^2}{m}u_1\right)u_0 + 3u_1^3\right)\frac{\tau^5}{5!} \\
&\quad + \mathcal{O}(\tau^7).
\end{aligned} \tag{104}$$

Note that evaluating the nth derivative of the positions at $\tau = 0$ always results in zero when n is even, since for these derivatives, the other terms in the equations of motion always have an even derivative of lower order (which drops out because the zeroth order derivative is zero by assumption).

The expansion of the action is then found by substituting in the found coordinates and their derivatives

$$S_{M_3} = \frac{m}{2}u^2\tau + \left(\frac{qg}{2}(-u_1^2u_0 + u_0u_1^2 + u_1^2u_0) + 8\beta g^2u_1^2\right)\frac{\tau^3}{3!} + \mathcal{O}(\tau^5). \tag{105}$$

Naively, one would expect a certain delay in the influence of the dipole moment (and therefore the polarisability β) on the dynamics of the centre of mass with respect to the effect of the charge. After all, the charge is present in the system from the onset, whereas the dipole moment is induced by the field. This would make the influence of the dipole moment an effect of higher order than the Lorentz force on the charge. However, as we can see from the above action, both parameters contribute at the same order in τ. In the expansion of $x_0(\tau)$, the parameter β does enter at a higher order than the charge, but this is simply an artefact from the choice of coordinate dependence for $F_{\mu\nu}$. The expressions above therefore seem to give no indication of a classification in Maxwell theories analogous to the soft limit classification in effective field theories. Perhaps the analogy was no more than that and there is no limit we can take to distinguish classical electromagnetic particle theories on the basis of their corresponding Maxwell level (other than by relating them explicitly). Or perhaps we have simply identified the wrong quantities as analogous to the scattering amplitude and momentum.

6 Conclusion

In this project, the construction and physical interpretation of theories obeying Maxwell symmetry have been studied. In addition, an attempt has been made at drawing a parallel between soft limit classifications in effective field theory and the effective particle theories described by different level Maxwell symmetries.

Maxwell symmetry is a spacetime symmetry extension of the Poincaré group and as such has to be nonlinearly realised. It can itself be iteratively extended by introduction of new generators equal to a successively higher number of translations in multi-commutators. This can be continued indefinitely up to an algebra called Maxwell$_\infty$. Finite-level truncations of the algebra also form consistent quotients, ensuring our ability to talk about different levels as their own group. In enlarging the spacetime symmetry group, Maxwell extensions also introduce new degrees of freedom, causing theories with higher level symmetries (and therefore larger groups) not to behave more constrained, but less. An appropriate quotient of the full Maxwell$_\infty$ is compatible with a particle experiencing a Lorentz force from a general field written as a Taylor expansion, each new term of the Taylor expansion generated by a higher level generator. However, this is not all it can describe. The proposition that a similar expansion in terms of multipoles for the particle takes place, has not been proven. Nonetheless, the third level has been shown to be equivalent to a theory of a particular induced dipole in a linear electromagnetic field. This dipole has opposing electric and magnetic polarisability, which is a property realised by a perfect superconductor. Previous work suggested terms similar to a dipole arose at the fourth level [12].

Effective field theories can be classified according to their soft limits. For a given number of derivatives in a theory, the soft degree gives an index of complexity. The higher the index, the more symmetric the theory. Moreover, the generators of the higher level symmetries can be related to the lower level generators by commutation with a translation generator. The added shift symmetries distinguishing theories of different complexity index from each other in this way mirror the translations in Maxwell, only reverse: the more added symmetry, the simpler the theory, and translations bring the symmetries down in level instead of up. This sequence ends at the free theory, where the Maxwell sequence starts. The (anti)-parallel found in this comparison inspired the search for a soft limit parameter in the Maxwell theories constructed in this text.

It would be interesting to find a concrete index in electromagnetic particle theories, linking them to the level of Maxwell needed to describe an equivalent particle, since this would give a natural classification. The 'soft limit' approach used in this text has not resulted in such a classification. The reason this did not succeed may be that there is no similar limit to the soft limit in effective field theory scattering amplitudes in classical particle theories. Another explanation would be that we have simply not identified the right quantities in our Maxwell theories to play the role of scattering amplitude and momentum in effective field theory. Taking a step toward the original soft limit classification method, by considering Maxwell realising quantum field theories, would possibly help in constructing a classification.

The results in this work can be extended by taking different quotients from the complete Maxwell$_\infty$ algebra. As we have seen, this might lead to a variety of different physically interesting theories, for which the Maxwell algebra approach might provide a novel perspective. In particular, and close to the subject of this research, it would be interesting to truncate the quotient suggested in 5.2 at finite levels larger than three, while constructing the Lagrangian by simply taking squares of the Maurer-Cartan 1-form coefficients. This might give equations of motion integrable in only spacetime coordinates, making physical interpretation more feasible. Aside from this being interesting in and of itself, it might also pave the way to more clarity on how to classify general electromagnetic particle theories based on their Maxwell level.

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