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(Non-)integrability of Hamiltonian systems via differential Galois theory and the Painlevé property

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Abstract

In this thesis the Ziglin-Morales-Ramis theory for non-integrability of Hamiltonian systems is discussed as well as how the Painlevé property can be used to find completely integrable Hamiltonian systems via the Painlevé test. Furthermore differential Galois theory and preliminaries, Hamiltonian systems and the Painlevé equations are discussed.

Notation: Below we give an overview of the used notation:

$\partial :=$ a derivation.

$r^{(i)} := \partial^i :=$ the i^{th} composition of a derivation.

$R^\times :=$ group of units.

$\langle x \rangle :=$ the ideal generated by x.

$\natural :=$ the natural map.

$R\{\{x\}\} :=$ a differential polynomial ring.

$C_K :=$ the field of constants of a differential field K.

$\overline{K} :=$ the algebraic closure of K.

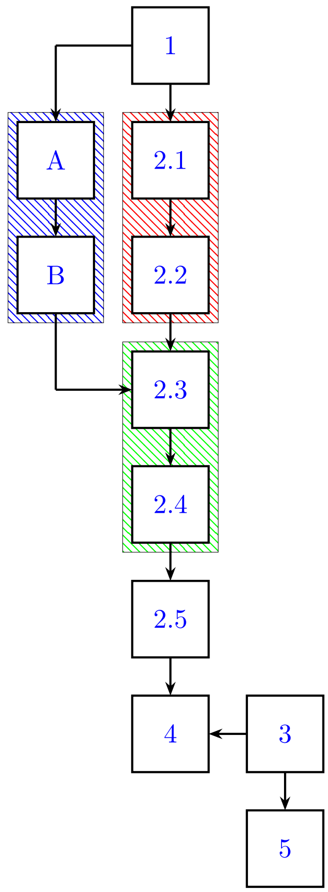
$L/K :=$ the field extension of L over K.

$\mathrm{DGal}(L/K) :=$ the differential Galois group of L/K.

$\mathbb{A}^n := n$ dimensional affine space.

Contents

Below we give a section dependence diagram:



: Rings are commutative.
: K is an algebraically closed field.
: C_K is an algebraically closed field of constants with characteristic 0.

Figure 1: Here the arrows denote dependencies, and the coloured area's denote assumptions.

Introduction

In this report we shall introduce differential Galois theory. To do so rigorously, we shall also discuss preliminaries from algebraic geometry and algebraic groups. We shall see how differential Galois theory can be used to show whether a differential equation is integrable or not, the ultimate purpose to apply differential Galois theory to the non-integrability of Hamiltonian systems. We do so via so-called Ziglin-Morales-Ramis theory, and apply the theory to completely determine the (non-)integrability of the spring pendulum. We shall show what it means for a Hamiltonian system to be completely integrable using Hamlitonian formalism, and what it means when a systems is not completely integrable, via the KAM and Nekhorosev theorems. We shall also discuss the Painleé equations and property, and show how this property relates to the complete integrability of Hamiltonian systems. As an example we find integrable cases of the Hénon-Heiles system using the (ARS) Painlevé test.

1 Preliminaries

In this section we discuss some preliminaries for the theory which will be discussed in this report.

1.1 General preliminaries

We start with general preliminaries, needed for both Section 2 and Section A.

Definition 1.1 (Multiplicative subset).

Let R be a ring. Let $S \subseteq R$ such that $s_1, s_2 \in S \implies s_1 \cdot s_2 \in S$ and such that $1 \in S$.

Example 1.2.

Let R be a commutative ring and let $I \trianglelefteq R$ be a prime ideal, then $R \setminus I$ is a multiplicative set.

Definition 1.3 (Localisation).

Let R be a commutative ring and $S \not\ni 0$. Define the following equivalence relation on $R \times S$. Let $r_i, s_i, t \in R$ $(r_1, s_1) \sim (r_2, s_2) \iff \exists t \in S$ s.t. $(r_1 s_2 - r_2 s_1)t = 0$. Then let $S^{-1}R := R \times S/{\sim}$, and denote its elements as r/s.

Remark 1.4.

We note the following things concerning localisations:

(i) Note that a localisation $S^{-1}R$ is a ring, with zero $0 = 0/r$, unit $1/1$ and multiplication $r_1/s_1 \cdot r_2/s_2 = \dfrac{r_1 r_2}{s_1 s_2}$ and addition $r_1/s_1 + r_2/s_2 = \dfrac{r_1 s_2 + r_2 s_1}{s_1 s_2}$.

(ii) The map $\psi : R \to S^{-1}R : r \mapsto r/1$, the localisation map, is a ring homomorphism.

(iii) If R is an integral domain, and we choose $S = R^\times$, then $S^{-1}R$ is a field of fractions, as the condition $(r_1 s_2 - r_2 s_1)t = 0 \implies r_1 s_2 = r_2 s_1$ in an integral domain, which corresponds to the condition used in the definition of a field of fractions.

Proposition 1.5 (Universal property of localisation).

Let $\phi : R \to Q$ be a ring homomorphism such that $\phi|_S : S \to Q^\times$. Let ψ be the localisation map, then there exists a unique homomorphism h such that the following diagram is commutative:

$$\begin{tikzcd}
R \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & Q \\
& S^{-1}R \arrow[ur, dashed, "\exists! h"'] &
\end{tikzcd}$$

This h will be given by $h : r/s \mapsto \phi(r)\phi(s)^{-1}$.

Proof. Let $h : r/s \mapsto \phi(r)\phi(s)^{-1}$, then clearly h makes the diagram commute, as $h \circ \psi(r) = h(r/1) = \phi(r)$.
Now for uniqueness: By way of contradiction, assume $g : S^{-1}R \to Q$ is a ring homomorphism distinct from h making the diagram commute, then

$$\phi(r) = g(r/1) \implies \phi(r)g(r/1)^{-1} = 1 \implies \phi(r)g(1/r) = 1 \implies g(1/r) = \phi(r)^{-1}$$

Thus $g(r/s) = g(r/1 \cdot 1/s) = g(r/1) \cdot g(1/s) = \phi(r)\phi(s)^{-1} = h(r/s)$. Thus h is unique. ■

Proposition 1.6.

Let $\phi : R \to S$ be a ring homomorphism for R a simple ring, then ϕ is injective.

Proof. Let $\phi : R \to S$ be a ring homomorphism. Then $\ker(\phi) \trianglelefteq R$ an ideal, thus $\ker(\phi) \in \{0, R\}$, since R is simple, but $\phi(1) = 1 \implies \ker(\phi) = 0$, thus ϕ is injective. ■

Definition 1.7 (Transcendence basis).

Let L/K be a field extension, a transcendence basis is a set $\{\alpha_i\}_{i \in I}$ such that $L/K(\alpha_i;\, i \in I)$ is an algebraic extension.

Proposition 1.8.

A transcendence basis always exists.

Proposition 1.9.

Consider the following short exact sequence of groups:

$$e \longrightarrow K \xhookrightarrow{\psi} G \xrightarrow{\phi} H \longrightarrow e.$$

If ψ has a retract, then $G \cong K \times H$. If ϕ has a sequent, then $G \cong H \ltimes K$.

1.2 Algebraic Geometry preliminaries

Here we discuss some preliminary results necessary for discussing algebraic geometry in A.

Definition 1.10 (Noetherian ring).

A ring R is Noetherian if every ideal is finitely generated.

Example 1.11.

Any principal ideal domain is Noetherian.

Proposition 1.12 (Equivalent definition for Noetherian rings).

Let R be a ring. Every ascending chain of ideals stabilises $\iff$ R is Noetherian. Said differently: Let $I_i \trianglelefteq R$. If

$$\forall\ I_0 \subseteq I_1 \cdots$$

$$\exists n \text{ such that } I_n = I_{n+1} = \cdots$$

$\iff$ R is Noetherian.

Proof. " $\Longleftarrow$ ": Let R be Noetherian, then every ideal is finitely generated, then any ascending chain of proper ideals must have an additional generating element. Let $I_0 \subseteq I_1 \subseteq \cdots$ be a chain of ideals. Then let $J = \cup_{i \in \mathbb{N}} I_i$ be an ideal, then $J = \langle r_j \rangle_{j \leq m}$, since J must be finitely generated. Then there is some I_{i_0} such that $J \subseteq I_{i_0}$, thus $I_{i_0} = I_{i_0+1} = \dots$, where again we use that each ideal must be finitely generated.
" $\Longrightarrow$ ": Let each ascending chain of ideals in R stabilise. By way of contradiction now assume that that there is some $I \trianglelefteq R$ such that I is not finitely generated. Then we may write $I = \langle r_\alpha \rangle_{\alpha \in A}$.

Then we pick some countable infinite subset $J := \langle r_i \rangle_{i\in\mathbb{N}}$, then

$$\langle r_1 \rangle \subseteq \langle r_1, r_2 \rangle \subseteq \cdots$$

is an infinite ascending chain which does not stabilise $\lightning$. Hence each ideal must be finitely generated. ■

Theorem 1.13 (Hilbert basis Theorem).

Let R be a Noetherian, then $R[X]$ is Noetherian.

Proof. Let R be an Noetherian ring. Now let $I \subseteq R[X]$ be an ideal. By way of contradiction assume I is not finitely generated. We now inductively define a process:
First for $f \in I$ let $\deg(f)$ be the degree of f (as a polynomial). Let f_0 be the polynomial of least degree in I (this choice need not be unique).
Then let f_1 be the polynomial of least degree in $I \setminus \langle f_0 \rangle$.
Let f_i be the polynomial of least degree in $I \setminus \langle f_0, f, \ldots, f_{i-1} \rangle$.
By our assumption this process will be non terminating. Now let a_i be le leading coefficient of the polynomial f_i. Then $\langle a_1, a_2, \ldots \rangle = \langle a_1, a_2, \ldots, a_n \rangle$, where we use that R is Noetherian.
Thus we can write $\sum_i{}^n r_i a_i = a_{n+1}$. Define

$$g := \sum_{j=0}^{n} r_i f_i x^{\deg(f_{n+1})-\deg(f_j)} \in \langle f_0, \ldots, f_n \rangle.$$

Clearly $f_{m+1} - g \notin \langle f_0, \ldots, f_n \rangle$, but $\deg(f_{n+1} - g) < \deg(f_{n+1})$, where we use that the leading coefficients of f_{n+1} and g cancel, using that $a_{n+1} = \sum_i{}^n r_i a_i$. This yields a contradiction. Thus I must be finitely generated. ■

Corollary 1.13.

Let R be a Noetherian ring, then $R[x_1, \ldots, x_n]$ is Noetherian.

Proof. By Theorem 1.13 $R[x_1]$ is Noetherian, thus $R[x_1, x_2]$ is Noetherian, thus applying the theorem iteratively yields that $R[x_1, \ldots, x_n]$ is Noetherian. ■

2 Differential Galois Theory

After Galois theory had been established for algebraic equations, Sophus Lie (1842 - 1899) was inspired to do the same for differential equations, which led him to the discovery of what we now call Lie groups. He did not end up making any direct contributions to the theory of differential Galois theory, but was the first of many interested in this field. Émile Picard (1859 - 1942) publishes an article establishing a theory for linear differential equations in 1877. Ernest Vessiot (1856 - 1952) also makes contributions to this theory of linear differential equations. The differential Galois theory of linear differential equations is now know as Picard-Vessiot theory. In 1898 Jules Drach (1871 - 1941) publishes his thesis, establishing differential Galois theory for non-linear differential equations, however Vessiot find mistakes in this thesis, and has worked on trying to re-establish the results from Drach's thesis, however, this was never fully accomplished. Ellis Kolchin (1916 - 1991) makes Picard-Vessiot theory rigorous, by making the connection to algebraic geometry, and publishes his results in his book from 1973. In this section we shall discuss Picard-Vessiot theory.

2.1 Basics of Differential Algebra

We will now introduce some basic concepts and results from differential algebra. In this section we'll will follow and take inspiration from [1, 2, 3, 4]. In the following we will take rings to be unitary and commutative.

Definition 2.1 (Derivation on a ring / differential ring).

(i) Let R be a ring. A map $\partial : R \to R$ is called a *derivation* on R if

$$\begin{aligned}(\forall r, s \in R) \quad \partial(r+s) &= \partial(r) + \partial(s),\\ (\forall r, s \in R) \quad \partial(rs) &= \partial(r)s + r\partial(s).\end{aligned}$$

Often we will write $r' := \partial r$, similarly $r'' := \partial\partial r$, etc. as well as $r^{(i)} := \underbrace{\partial\partial\ldots\partial}_{i \text{ times}} r := \partial^i r$ (and set $\partial^0(r) := r$).

(ii) Let R be a ring, and let ∂ be a derivation. Then the double (R, ∂) is a *differential ring*. If R is a field, then we call it a *differential field*.

Example 2.2 (Differential rings).

The following are examples of differential rings:

(i) Any ring with $\partial = 0$ is trivially a differential ring.

(ii) $R[X]$ with R a ring and ∂ the formal derivative is a differential ring.

(iii) $\mathbb{C}(X)$ with $\partial = \dfrac{\mathrm{d}}{\mathrm{d}x}$ is a differential field.

The following basic facts hold, as one would expect for a derivation on a ring.

Proposition 2.3.

Let (R, ∂) be a differential ring.

(i) $\partial(1) = 0$,

(ii) $(\forall s \in R^\times) \quad \partial(s^{-1}) = -s^{-1}s's^{-1}$,

(iii) $(\forall r \in R)(\forall n \in \mathbb{N}) \quad \partial r^n = nr^{n-1}r'$,

(iv) $(\forall s \in R^\times)(\forall n \in \mathbb{Z}) \quad \partial s^n = ns^{n-1}s'$,

(v) $(\forall r \in R)(\forall s \in R^\times) \quad \partial(rs^{-1}) = (r's - rs')s^{-2}$.

Proof.

(i) Note that

$$\partial(1) = \partial(1 \cdot 1) = \partial(1)1 + 1\partial(1) = \partial(1) + \partial(1).$$

Thus $\partial(1) = 2\partial(1) \implies \partial(1) = 0$.

(ii) Let $s \in R^\times$. Note that

$$\partial(1) = \partial(ss^{-1}) = s's^{-1} + s\partial(s^{-1}) = 0,$$

where in the final equality we used (i). This implies $\partial(s^{-1}) = s^{-1}s's^{-1}$.

(iii) Let $r \in R$. We will proceed via induction, note that the case for $n = 1$ is trivially true. Now assume that $\partial(r^n) = nr^{n-1}r'$, then

$$\partial(r^{n+1}) = \partial(r \cdot r^n) = r'r^n + r\partial(r^n) = r'r^n + r(nr^{n-1}r') = r^nr' + nr^nr' = (n+1)r^nr'.$$

(iv) Let $s \in R^\times$. Then by (iii) the formula holds for $n \in \mathbb{N}$. By (ii) $\partial(s^{-1}) = -s^{-2}s'$. Now

$$\partial(s^{-n}) = \partial[(s^{-1})^n] = n(s^{-1})^{n-1}\partial(s^{-1}) = ns^{-(n-1)}\partial(s^{-1}) \cdot -s^{-2}s' = -ns^{-(n+1)}s',$$

where in the third equality we applied (iii).

(v) Let $r \in R$ and $s \in R^\times$, then

$$\partial(rs^{-1}) = r's^{-1} + r\partial(s^{-1}) = r's^{-1} + r \cdot -s^{-2}s' = (r's - rs')s^{-2}.$$

■

Proposition 2.4.

Let (R, ∂) be a differential ring. Let $\natural : R[\epsilon]/\langle \epsilon^2 \rangle \twoheadrightarrow R$ be the natural map. There is a canonical bijection between derivations on R and sections of $\natural$ (homomorphisms s such that $\natural \circ s = \mathrm{id}_{R[\epsilon]/\langle \epsilon^2 \rangle}$). This bijection sends a section $s : r \mapsto \overline{r + \epsilon f(r)}$ to the derivation $\partial_s : r \mapsto f(r)$, who's inverse is the map that sends a derivation ∂ to a section $s_\partial : r \mapsto \overline{r + \epsilon \partial(r)}$.

Proof. First note that trivially $s : r \mapsto \overline{r + \epsilon f(r)}$ is indeed an inverse of the natural map. The given bijection (we prove it is a bijection below) maps s to $\partial_s : r \mapsto f(r)$, we now show that ∂_s is indeed a derivation. Let $\overline{x}, \overline{y} \in R[\epsilon]/\langle \epsilon^2 \rangle$, then

$$s(\overline{x + y}) = s(\overline{x}) + s(\overline{y}) = \overline{(x + y) + \epsilon \left(f(x) + f(y)\right)}.$$

Similarly

$$s(\overline{xy}) = s(\overline{x})s(\overline{y}) = \overline{xy + \epsilon \left(f(x)y + xf(y)\right)}.$$

Thus for $x, y \in R$ we have $\partial_s(x + y) = \partial_s(x) + \partial_s(y)$ and $\partial_s(x, y) = \partial_s(x)y + x\partial_s(y)$. Hence ∂_s is indeed a derivation.

Now we show $\partial \mapsto s_\partial$ is indeed a section.

$$s_\partial(0) = \overline{0} \text{ and } s_\partial(1) = \overline{1 + \epsilon\partial(1)} = \overline{1},$$

where in the final equality we applied Proposition 2.3, Item (i) in the final equality. Now we check that addition is preserved:

$$s_\partial(x + y) = \overline{(x + y) + \epsilon\partial(x + y)} = \overline{\left(x + \epsilon\partial(x)\right) + \left(y + \epsilon\partial(y)\right)} = s_\partial(x) + s_\partial(y).$$

Now we check that multiplication is preserved:

$$s_\partial(xy) = \overline{xy + \epsilon\partial(xy)} = \overline{xy + \epsilon\partial(x)y + x\partial(y)} = \overline{\left(x + \epsilon\partial(x)\right)\left(y + \epsilon\partial(y)\right)} = s_\partial(x)s_\partial(y).$$

It is easy to see that s_∂ is indeed a section of $\natural$. Now we show that the map is indeed a bijection, as claimed. Note that $s : x \mapsto \overline{x + f(x)}$ gets sent to ∂_s, which gets sent to $s_{\partial_s} : x \mapsto \overline{x + \epsilon\partial_s(x)} =$

$\overline{x+\epsilon f(x)}$. ∂ gets sent to s_∂, which gets sent to $\partial_{s_\partial} : x \mapsto \partial(x)$.
Thus the given maps are each others inverse and thus indeed yield a bijection. ■

Definition 2.5 (Differential ring homomorphism).

Let (R, ∂_R) and (S, ∂_S) be a differential rings. Let $\phi : R \to S$ be a ring homomorphism, such that

$$(\forall r \in R) \quad \phi(\partial_R r) = \partial_S \phi(r).$$

Then ϕ is a differential ring homomorphism. The concept of an isomorphism specializes analogously.

Definition 2.6 (Differential ideal).

Let (R, ∂) be a differential ring. Let $I \trianglelefteq R$ be an ideal. If $\partial I \subseteq I$, then I is a differential Ideal.
We will then call a *simple differential ring*, a ring in which the only differential ideals are R and $\langle 0 \rangle$.

Example 2.7.

Let $\phi : R \to S$ be an differential morphism. Let $r \in \ker(\phi)$, then $\phi(r) = 0 \implies 0 = \phi(r)' = \phi(r')$.
Thus $\ker(\phi)$ is a differential ideal.

Remark 2.8 (Differential quotient rings).

Let (R, ∂) be a differential ring and $I \trianglelefteq R$ be an differential ideal. Then ∂ induces a derivation on R/I as follows. Let $\partial : R/I \to R/I : \overline{r} \mapsto \overline{\partial(r)}$. This is well-defined, as $\partial(I) \subseteq I$ ensures that $a + I \neq b + I \implies \partial(a) + I \neq \partial(b) + I$.
Note now that by Example 2.7, we see that the first isomorphism theorem still holds for differential morphisms and differential ideals, i.e. we obtain the following diagram:

$$\begin{tikzcd} R \arrow[r, two heads, "\phi"] \arrow[d, "\natural"'] & \phi(R) \\ R/\ker(\phi) \arrow[ur, dashed, "\sim"] & \end{tikzcd},$$

for $\natural$ the natural map, and all maps differential morphisms (note that $\natural$ is indeed a differential morphism by the above).

Proposition 2.9.

Let (K, ∂) be a differential field of characteristic zero and let R/K be a ring extension. Then $\mathfrak{m}$ is a maximal differential ideal $\implies$ $\mathfrak{m}$ is a prime ideal.

Proof. Recall that I is prime $\iff$ R/I is an integral domain. Note that R has a maximal differential ideal, which can be shown in much the same way as for regular maximal ideals, via application of Zorn's lemma. Thus let $\mathfrak{m} \trianglelefteq R$ be a maximal differential ideal and by way of contradiction we assume that $R/\mathfrak{m}$ is not an integral domain. Note that $R/\mathfrak{m}$ is a differentially simple ring, thus we will now try to find proper non-trivial differential ideals as a contradiction. Thus let $s, r \in R/\mathfrak{m}$ such that $st = 0$. We shall show that then $\partial^n(r)s^{n+1} = 0$, via induction. The base case is trivial for $n = 0$. Now assume $\partial^n(r)s^{n+1} = 0$. Then

$$\partial[\partial^n(r)s^{n+1}] = 0 \implies \partial^{n+1}(r)s^{n+1} + (n+1)\partial^n(r)s^n = 0$$

$$\implies \partial^{n+1}(r)s^{n+1} \cdot s + (n+1)\partial^n(r)s^n s = 0 \implies \partial^{n+1}(r)s^{n+2} = 0,$$

where in the final implication we applied the induction hypothesis. Now we define $I = \langle r, r', \ldots \rangle \trianglelefteq R/\mathfrak{m}$, this is then clearly a differential ideal.

Now first we assume $(\forall n \in \mathbb{N})\ s^n \neq 0$. Then each element of I is a zero divisor. Thus $1 \notin I$, but $r \in I \neq \langle 0 \rangle$ ↯ (as $R/\mathfrak{m}$ should be simple).

Now assume $\exists n \in \mathbb{N}$ s.t. $s^n = 0$. Since the chosen zero divisors were arbitrary, this implies every zero divisor must have this property. In particular then a has this property. Let $m \in \mathbb{N}$ be the least number such that $a^m = 0$, then $\partial(a^m) = ma^{m-1}a' = 0 \implies a'$ is a zero divisor, again since a is an arbitrary zero-divisor the same result holds for $\partial^i(a)$ (they are also zero divisors). Now we can consider the same differential ideal I as before, $a \in I$ and $1 \notin I$, thus again I is a proper, non-trivial differential ideal ↯. ■

Proposition 2.10 (Derivation on a localisation).

Let (R, ∂) be a differential ring and let $S^{-1}R$ a localisation of R, then there is a unique extension of ∂ to $S^{-1}R$.

Proof. Using Proposition 2.4 we first find a correspondence between ∂ and the section $s_\partial : R \to R[\epsilon]/\langle \epsilon^2 \rangle$. We now extend this map naturally to map to $S^{-1}R[\epsilon]/\langle \epsilon \rangle$, as follows:
$s'_\partial : x \mapsto \overline{x/1 + \epsilon\partial(x/1)}$. From now on we will omit the bar indicating the quotient. Let $x \in S$, then

$$s'_\partial(x)\frac{x - \epsilon\partial(x)}{x^2} = \frac{[x + \epsilon\partial(x)][x - \epsilon\partial(x)]}{x^2} = 1.$$

Thus we may invoke the universal property of the localisation mapping (see Proposition 1.5) to see there exists a unique map $s_{\widetilde{\partial}}$ making the following diagram commute:

$$\begin{tikzcd} R \arrow[rr, "s'_{\partial}"] \arrow[dr, "\psi"'] & & S^{-1}R[\epsilon]/\langle \epsilon^2 \rangle \\ & S^{-1}R \arrow[ur, dashed, "\exists! s_{\widetilde{\partial}}"'] & \end{tikzcd},$$

where ψ is the localisation mapping. Now

$$s_{\widetilde{\partial}}(x/y) = s_{\widetilde{\partial}}(x) \cdot s_{\widetilde{\partial}}(y)^{-1} = [x/1 + \epsilon\partial(x/1)] \cdot [\epsilon\partial(1/y)] = x/y + \epsilon\frac{x'y - xy'}{y^2}.$$

Clearly this is a section of $\natural : S^{-1}R[\epsilon]/\langle\epsilon^2\rangle \to S^{-1}R$, thus there is a bijection between $s_{\widetilde{\partial}}$ and a derivation on $S^{-1}R$ given by $\widetilde{\partial} : x/y \mapsto \dfrac{x'y - xy'}{y^2}$. ∎

Definition 2.11 (Differential extension).

Let (R, ∂_R) be a differential ring. A *differential extension* (S, ∂_S), a differential ring, of (R, ∂_R), is an extension of R (i.e. R is a subring of S) such that $\partial_S|_R = \partial_R$.

Proposition 2.12 (Separable field extensions).

Let K be a field, and $K(\alpha)/K$ be a finite separable extension. Then ∂ extends uniquely to $K(\alpha)$. If $f^\alpha = \sum_{i=0}^n a_i x^i \in K[X]$ is the minimal polynomial of α, then we get

$$\partial(\alpha) = -\frac{\sum_{i=0}^n \partial(a_i)\alpha^i}{\sum_{i=0}^n i a_i \alpha^{i-1}}.$$

Proof. We will denote the extension of ∂ to $K(\alpha)$ by $\widetilde{\partial}$. Let (K, ∂) be a differential field, then a differential extension from K to $K(\alpha)$ is completely determined by where it sends α, i.e. if we know how $\widetilde{\partial}$ acts on α, we know how it acts on the whole of $K(\alpha)$, to see this note that $\widetilde{\partial}\left(\sum_i b_i\alpha^i\right) = \sum_i \partial(b_i)\alpha^i + ib_i\alpha^{i-1}\widetilde{\partial}(\alpha)$. Let $f^\alpha = \sum_{i=0}^n a_i x^i$ be the minimal polynomial of α, we then have that

$$f^\alpha(\alpha) = 0 \implies \widetilde{\partial} f^\alpha(\alpha) = \sum_{i=0}^{n} \partial(a_i)\alpha^i + ia_i\alpha^{i-1}\widetilde{\partial}(\alpha) = 0$$

$$\implies \widetilde{\partial}(\alpha) = -\frac{\sum_{i=0} \partial(a_i)\alpha^i}{\sum_{i=0} ia_i\alpha^{i-1}\partial(\alpha)}.$$

Thus if $\widetilde{\partial}$ is a derivation, then it must be unique.
Now we check that $\widetilde{\partial}$ is indeed a derivation. We will do so by showing that that $s_{\widetilde{\partial}}$ is a section for various restrictions of $K(\alpha)$, playing the contents as well as using the notation from Proposition 2.4. We proceed by induction, for $K[\alpha]$, noting that $K[\alpha] = K(\alpha)$, since α is algebraic.

Trivially $\left.\widetilde{\partial}\right|_K$ is a section of $\natural|_K$ (for $\natural : K(\alpha)[\epsilon]/\langle\epsilon^2\rangle \to K(\alpha)$).
Now assume that for all polynomials in α, $\sum_i b_i\alpha^i$, of degree $\leq n$ we have that $\natural \circ s_{\widetilde{\partial}}\left(\sum_i b_i\alpha^i\right) = \sum_i b_i\alpha^i$ as our induction hypothesis. Then

$$\natural \circ s_{\widetilde{\partial}}\left(\sum_i^{n+1} b_i\alpha^i\right) = \natural \circ s_{\widetilde{\partial}}(b_{n+1}\alpha^{n+1}) + \sum_i^n b_i\alpha^i = \natural \circ s_{\widetilde{\partial}}(b_{n+1}\alpha^n) \cdot \natural \circ s_{\widetilde{\partial}}(\alpha) + \sum_i^n b_i\alpha^i = \sum_i^{n+1} b_i\alpha^i.$$

Thus $s_{\widetilde{\partial}}$ is indeed a section, so $\widetilde{\partial}$ is indeed a derivation on $K(\alpha)$. ∎

Example 2.13 (Non-separable field extension).

It is not hard to show that $\mathbb{F}_2(\sqrt{t})/\mathbb{F}_2(t)$ is a non-separable field extension. Note that $(\mathbb{F}_2(t), \partial)$ is a differential field, with ∂ acting as $\frac{d}{dt}$ does on rational polynomials. Assume ∂ extends to $\mathbb{F}_2(\sqrt{t})$, then $1 = \partial(t) = \partial(\sqrt{t} \cdot \sqrt{t}) = \overline{2}\partial(\sqrt{t})\sqrt{t} = 0$ ↯. Thus we see that Proposition 2.12 does not hold for general extensions, and that in general it is not even necessarily possible to make an arbitrary extension into a differential extension.

Definition 2.14 (Constant).

Let (R, ∂) be a differential ring. Let $c \in R$ such that $\partial c = 0$, then c is called a constant.

Remark 2.15 (Ring of constants).

Let (R, ∂) be a differential ring, then $C_R := \{r \in R; \partial r = 0\}$ is a subring of R. To see this note that trivially $0 \in \partial R$ and that by Proposition 2.3, (i), $1 \in R$. Additionally let $r, s \in C_R$, then $\partial(r + s) = \partial r + \partial s = 0$, and $\partial(rs) = r's + rs' = 0$.

Example 2.16.

We will now examine the ring of constants over certain differential rings.

(i) Let $(\mathbb{Z}, \partial)$ be a differential ring, then ∂ is the zero map.
To see this, first note that $(\forall n \in \mathbb{N}) \quad \partial(-n) = \partial(-1)n - \partial(n) = -1\partial(1) \cdot -1 - \partial(n) = -\partial(n)$, where we used (ii). Thus $(\forall z \in \mathbb{Z}) \quad \partial(z) = \pm\partial(\underbrace{1 + \cdots + 1}_{|z| \text{ times}}) = \pm\big(\partial(1) + \cdots + \partial(1)\big) = 0$, where we used Proposition 2.3, (i).

(ii) Let $(\mathbb{Q}, \partial)$ be a differential ring, then ∂ is the zero map. To see this, note that $(\forall q \in \mathbb{Q}) \quad \exists n, m \in \mathbb{Z}$ such that $\partial(q) = \partial(nm^{-1}) = \partial(n)m^{-1} + nm^{-1}\partial(m)m^{-1} = 0$.

(iii) Let $(\overline{\mathbb{Q}}, \partial)$ be a differential ring, then ∂ is the zero map.
Using that this extension can be written as a countable number of separable extensions over $\mathbb{Q}$, we can apply Proposition 2.12, from which it follows that ∂ must be the zero-map for each subsequent extension.

(iv) For transcendental extensions we need not obtain the zero-map as our only possible derivation.
Let t be a transcendental number, then $\mathbb{Q}(x) \cong \mathbb{Q}(t)$. Let ϕ be this isomorphism, then $\phi^{-1} \circ \frac{\mathrm{d}}{\mathrm{d}x} \circ \phi$ defines a derivation on $\mathbb{Q}(t)$, indeed any differential on $\mathbb{Q}(x)$ could thus be translated.

(v) Let $(\mathbb{R}, \partial)$ be a differential ring. From the previous result one might doubt whether ∂ needs to be the zero-map, but maybe the additional structure of $\mathbb{R}$ imposes enough conditions for the zero-map to indeed be the only possible derivation.
This turns out not to be the case, as we will now show. We do so using a similar approach to (iv). We need to make sure $\mathbb{R}$ has an algebraically independent basis, to exemplify this problem somewhat, note that: π and $\sqrt{\pi - 2}$ are both transcendental, but are not algebraically independent over $\mathbb{Q}$, as $x - y^2 + 2$ has a zero when evaluated at both those numbers, less trivially it is not known yet whether π and e are algebraically independent or not.
However we do not need to explicitly find an independent basis. From theory from transcendence basses, we know that for every field extension L/K there exists a set $\{l_\alpha\}_{\alpha \in A}$ (for A an indexing set) of elements of L such that $L/K(l_i;\, i \in I)$ is an algebraic extension, this however does require use of the axiom of choice.
Thus in particular $\exists \{t_\alpha\}_{\alpha \in A}$ for t_α transcendental, such that $\mathbb{R}/\overline{\mathbb{Q}}(t_\alpha;\, \alpha \in A)$ is algebraic. Thus letting t_{α_0} be some fixed element of $\{t_\alpha\}_{\alpha \in A}$ we have that $\exists \phi : \overline{\mathbb{Q}}(t_\alpha;\, \alpha \in A \setminus \alpha_0)(x) \xrightarrow{\sim} \overline{\mathbb{Q}}(t_\alpha;\, \alpha \in A) : x \mapsto t_{\alpha_0}$, thus again $\phi^{-1} \circ \frac{\mathrm{d}}{\mathrm{d}x} \circ \phi$ is a derivation on $\mathbb{R}$. A similar argument also shows that ∂ on $\mathbb{C}$ is again not necessarily the zero-map.

Note that if a ring R contains a subring S, then the ring of constants of the subring C_S, will be a subring of the ring of constants of R, i.e. $R \supseteq S \implies C_R \supseteq C_S$.

2.2 Picard-Vessiot Rings

In this section we will introduce solution spaces, and give a differential analogue to splitting fields for differential algebra, via Picard-Vessiot rings. In the following, unless stated explicitly, all modules will be assumed to be left modules.

Definition 2.17 (Derivation on a module).

Let M be a left R-module, and let ∂_R be a derivation on R. Let $\partial_M : M \to M$ such that

$$(\forall r \in R)(\forall m \in M) \quad \partial_M(rm) = \partial_R(r)m + r\partial_M(m).$$

This definition can be extended analogously for right R-modules. In general we will abuse notation slightly and denote ∂_M and ∂_R by the same symbol.

Definition 2.18 (Differential module).

Let M be an R-module, and ∂ be a derivation on M, then (M, ∂) is a *differential R-module*.

Example 2.19.

Let $(M, \partial_M), (N, \partial_N)$ be differential R-modules, for R commutative, then $(M \otimes_R N, \partial_{M\otimes N} : m \otimes n \mapsto m' \otimes n + m \otimes n')$ is a differential module.

Definition 2.20 (Differential module homomorphism).

Let (M, ∂_M) and (N, ∂_N) be differential R-modules. Let $\phi : M \to N$ be an R-module homomorphism. If

$$(\forall m \in M) \quad \phi(\partial_M m) = \partial_N \phi(m),$$

then ϕ is a D-module homomorphism. The concept of an isomorphisms specializes analogously.

Remark 2.21.

Proposition 1.6 also holds for differential modules, and differentially simple rings, with an analogous proof.

Remark 2.22 (Free D-module).

If (M, ∂) is a free differential module, then ∂ is completely determined by how it acts on the basis of M. Let $\{e_i\}_{i\in I}$ be a basis for M. Then

$$m \in M \implies m = \sum_{i\in I} r_i e_i,$$

with $r_i = 0$ for all but finitely many i. Now

$$\partial(m) = \partial\left(\sum_{i\in I} r_i e_i\right) = \sum_{i\in I} \partial(r_i)e_i + r_i\partial(e_i).$$

Thus $\partial e_i \mapsto \sum_{j\in I} s_{ij}e_j$, for $0 \neq s_{ij} \in R$ for only finitely many s_{ij} completely determines ∂. Furthermore note that

$$\partial(m) = \sum_{i\in I}\left(\partial(r_i)e_i + r_i \sum_{j\in I} s_{ij}e_j\right).$$

In general we will work of a differential field, and have $\dim(M) = n$. In this case the elements of M can be seen as vectors, and ∂ has a matrix representation, as M will now be a vector space, thus we obtain

$$\partial m = 0 \implies {r_i}' e_i + r_i \sum_{j\in I} s_{ij}e_j = 0 \implies -\sum_{i\in I}\partial(r_i)e_i = \sum_{i\in I} r_i \sum_{j\in I} s_{ij}e_j,$$

$$\implies -\begin{pmatrix} r_1' \\ r_2' \\ \vdots \\ r_n' \end{pmatrix} = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & \vdots & & \vdots \\ s_{n1} & s_{n2} & \dots & s_{nn} \end{bmatrix} \cdot \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$$

$$\implies \boldsymbol{r}' = (-S)\boldsymbol{r},$$

where we let ∂ act on the vector $\boldsymbol{r}$ component-wise.

Thus with every matrix differential equation $\boldsymbol{y}' = A\boldsymbol{y}$, we can associate a module corresponding to A, via the above given identification.

Remark 2.23 (Linear scalar homogenous differential equation).

Let $a_i \in K$, then $\mathcal{L}(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$ is a scalar homogenous differential equation. This can be represented as a differential matrix equation via

$$\begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-2)} \\ y^{(n-1)} \end{pmatrix}' = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-2)} \\ y^{(n-1)} \end{pmatrix}.$$

Proposition 2.24.

Let K be a field, let $A \in \mathrm{M}_n(K)$ and let $\boldsymbol{y}' = A\boldsymbol{y}$ be a matrix equation. Let $\{\boldsymbol{v}_i\}_{i\leq m}$ be the set of solutions to the matrix equation, then if the $\boldsymbol{v}_i$ are linearly dependent over K this implies that the $\boldsymbol{v}_i$ are linearly dependent over C_K.

Proof. Let $\{\boldsymbol{v}_i\}_{i\leq m}$ be a set of linearly dependent vectors. Let $\{\boldsymbol{v}_j\}_{j\leq q}$, for $q < m$ be a minimal linearly dependent subset, then $\{\boldsymbol{v}_j\}_{2\leq j\leq q}$ is linearly independent and we have that $\boldsymbol{v}_1 = \sum_{i=2}^{q} k_j \boldsymbol{v}_j$, for $k_j \in K$. Now we have that

$$0 = \boldsymbol{v}_1{}' - A\boldsymbol{v}_1 = \left(\sum_{j=2}^{q} \boldsymbol{v}_j\right)' - A\sum_{j=2}^{q} \boldsymbol{v}_j = \sum_{j=2}^{q}(k_j{}'\boldsymbol{v}_j + k_j\boldsymbol{v}_j{}') - \sum_{j=2}^{q} k_j\boldsymbol{v}_j{}' = \sum_{j=2}^{q} k_j{}'\boldsymbol{v}_j$$

$$\implies k_j{}' = 0 \implies k_j \in C_K.$$

This result can be extended to all of $\{\boldsymbol{v}_i\}_{i\leq m}$ by repeating the above for differing proper subsets of $\{\boldsymbol{v}_i\}_{i\leq m}$ (i.e. switching one vector from $\{\boldsymbol{v}_j\}_{2\leq j\leq q}$ with a linear dependent vector from $\{\boldsymbol{v}_i\}_{i\leq m} \setminus \{\boldsymbol{v}_j\}_{2\leq j\leq q}$). ■

Remark 2.25.

Let M be a differential module over K, related to the matrix differential equation $\boldsymbol{y} = A\boldsymbol{y}$, then $\ker(\partial)$ on M can be seen as the solution space of the matrix equation $\boldsymbol{y} = A\boldsymbol{y}$. To see this note that if $\{e_i\}_{i\leq n}$ is a basis on M, then

$$\sum_i \boldsymbol{v}_i e_i \in \ker(\partial) \iff \sum_i \boldsymbol{v}_i{}' e_i + \boldsymbol{v}_i e_i{}' = 0 \iff \sum_i \boldsymbol{v}_i{}' e_i = -e_i{}'\boldsymbol{v}_i \iff \boldsymbol{v}' = A\boldsymbol{v}.$$

Similarly if R/K is a ring extension we can use extension of a scalars and note that $R^n \cong R\otimes_K K^n \cong R \otimes_K M$, equipping it with the derivation from Example 2.19.

Remark 2.26 (Linear scalar inhomogeneous differential equation).

We shall now show that an inhomogeneous differential equation can be rewritten as a homogeneous differential equation of a higher order. Namely we will show that $\mathcal{L}(y) = k$ and $k[\frac{1}{k}\mathcal{L}(y)]' = 0$ span the same solution space, for $k \in K$ a differential field.

First suppose v satisfies $\mathcal{L}(v) = k$. Then

$$k[\tfrac{1}{k}\mathcal{L}(v)]' = k[\tfrac{1}{k}k]' = k(1)' = 0.$$

Conversely let v satisfy $k[\frac{1}{k}\mathcal{L}(v)]' = 0$, then

$$k[\tfrac{1}{k}\mathcal{L}(v)]' = 0 \implies [\tfrac{1}{k}\mathcal{L}(v)]' = 0 \implies \frac{1}{k}\mathcal{L}(v) = c \implies \mathcal{L}(v) = ck,$$

for $c \in C_K$, thus a solution to the homogenous differential equation is still in the span of the solution of the inhomogeneous differential equation.

Proposition 2.27.

Let K be a field, let $A \in \mathrm{M}_n(K)$ and let $\boldsymbol{y}' = A\boldsymbol{y}$ be a matrix equation. Then the set $V = \{\boldsymbol{v} \in K^n; \boldsymbol{v} = A\boldsymbol{v}\}$, called the solution space of $\boldsymbol{y}' = A\boldsymbol{y}$, is a vector space over C_K such that $\dim_K(V) = n$.

Proof. K is a field, thus C_K is a field, then obviously V is a vector field over C_K. For the statement on the dimension, we have that any set $\{\boldsymbol{v}_i\}_{i \leq n+1}$ of vectors must be linearly dependent over K and thus by Proposition 2.24 the same is true over C_K. ∎

Definition 2.28 (Wronskian matrix).

Let (K, ∂) be a differential field. Let $y_i \in K$, then

$$W(y_1, \dots, y_n) = \begin{bmatrix} y_1 & y_2 & \dots & y_n \\ {y_1}' & {y_2}' & \dots & {y_n}' \\ \vdots & \vdots & & \vdots \\ {y_1}^{(n-1)} & {y_2}^{(n-1)} & \dots & {y_n}^{(n-1)} \end{bmatrix}.$$

Is called the *Wronskian matrix* and $\mathrm{wr}(y_1, \dots, y_n) \coloneqq \det\big(W(y_1, \dots, y_n)\big)$ is called the *Wronskian*.

Proposition 2.29.

Let $\{y_i\}_{i \leq n} \subseteq K$, for (K, ∂) a differential field, then $\{y_i\}_{i \leq n}$ are linearly dependent over C_K $\iff$ $\mathrm{wr}(y_1, \dots, y_n) = 0$.

Proof. " $\Longrightarrow$ ": Follows immediately from the linear dependence.
" $\Longleftarrow$ ": $\mathrm{wr}(y_1, \dots, y_n) = 0$, thus the column vectors of $\mathrm{W}(y_1, \dots, y_n)$ are linearly dependent over K. Now we will find a matrix $A \in \mathrm{M}_n(K)$ such that the vectors $(y_i, \dots, {y_i}^{(n-1)})^{\mathrm{T}}$ satisfy the matrix equation $\boldsymbol{y}' = A\boldsymbol{y}$. If we find this matrix, we can employ Proposition 2.24, to obtain the desired result.
We define linear operator on K^n inductively.

$$\mathcal{L}_1(y) := \begin{cases} 0 & \text{if } y_1 = 0 \\ y' - \frac{y_1'}{y_1} y & \text{otherwise,} \end{cases}$$

$$\mathcal{L}_{i+1} := \begin{cases} 0 & \text{if } \mathcal{L}_i(y_{i+1}) = 0 \\ \mathcal{L}_i(y)' - \frac{\mathcal{L}_i(y_{i+1})'}{\mathcal{L}_i(y_{i+1})} \mathcal{L}_i(y) & \text{otherwise.} \end{cases}$$

Now we check via induction that $\mathcal{L}_n$ has the desired property. For $n = 1$, $\mathcal{L}_1(y_1) = 0$ as desired. Now as our induction hypothesis assume $(\forall i \leq m)\ \mathcal{L}_m(y_i) = 0$. Thus we evaluate $\mathcal{L}_{m+1}(y_{m+1})$ (the cases y_i for $i \leq m$ are trivially true by the induction hypothesis). If $\mathcal{L}_m(y_{m+1}) = 0$, we are done, else

$$\mathcal{L}_{m+1}(y_{m+1}) = \mathcal{L}_m(y_{m+1})' - \frac{\mathcal{L}_m(y_{m+1})'}{\mathcal{L}_m(y_{m+1})} \mathcal{L}_m(y_{m+1}) = 0.$$

Thus we are done. ■

Example 2.30.

(i) Let $\left(\mathbb{C}(x), \frac{\mathrm{d}}{\mathrm{d}x}\right)$ be a differential field. Let $\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}x} - 1$ be a differential operator. Then it is clear that e^x is a solution of $\mathcal{L}(Y) = 0$, and that the solution space is thus spanned by e^x. Now the corresponding matrix equation is just $y' = (1)y$ and the $\mathrm{W}(e^x) = \mathrm{wr}(e^x) = e^x$.

(ii) Let $\left(\mathbb{C}(x), \frac{\mathrm{d}}{\mathrm{d}x}\right)$ be a differential field. Let $\mathcal{L} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2$ be a differential operator. Then it is clear that $e^{i\sqrt{2}x}$ and $e^{-i\sqrt{2}x}$ is a solution of $\mathcal{L}(Y) = 0$, and that the solution space is thus spanned by $e^{i\sqrt{2}x}$ and $e^{-i\sqrt{2}x}$. Now the corresponding matrix equation is

$$\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}.$$

The Wronskian matrix is

$$\mathrm{W}(e^{i\sqrt{2}x}, e^{-i\sqrt{2}x}) = \begin{bmatrix} e^{i\sqrt{2}x} & e^{-i\sqrt{2}x} \\ i\sqrt{2}e^{i\sqrt{2}x} & -i\sqrt{2}e^{-i\sqrt{2}x} \end{bmatrix},$$

with corresponding determinant

$$\mathrm{wr}(e^{i\sqrt{2}x}, e^{-i\sqrt{2}x}) = -i\sqrt{2} - (i\sqrt{2}) = -i2\sqrt{2} \neq 0.$$

Thus $e^{i\sqrt{2}x}$ and $e^{-i\sqrt{2}x}$ are linearly independent over $\mathbb{C}$.

Definition 2.31 (Fundamental matrix).

Let R be a ring an K be a field. Let R/K be a differential extension such that $C_R = C_K$. Furthermore let $A \in \mathrm{M}_n(K)$. Then a matrix $F \in \mathrm{GL}_n(R)$ is called the fundamental matrix if $F' = AF$.

Remark 2.32.

In particular note that if a fundamental matrix exists, since it is invertible, the solution space of the corresponding differential matrix equation has full dimension.

Remark 2.33 (Uniqueness of fundamental matrices).

We shall show that fundamental matrices are unique up to multiplication with a matrix of constants. Note that if F is a fundamental matrix for the equation $F' = AF$ over R/K, and $M \in \mathrm{GL}_n(C_R)$, then

$$(FM)' = F'M = AFM$$

is again a fundamental matrix.
Let G be a fundamental matrix satisfying $G' = AG$ distinct from F, then

$$F'F^{-1} = A \implies G' = F'F^{-1}G \implies F[(F')^{-1}G'] = G$$

then

$$\left(F[(F')^{-1}G']\right) = AF[(F')^{-1}G'] \implies F'[(F')^{-1}G'] + F[(F')^{-1}G']' = AF[(F')^{-1}G'] = F'[(F')^{-1}G']$$

$$\implies F[(F')^{-1}G']' = 0 \implies [(F')^{-1}G'] \in \mathrm{GL}_n(C_R).$$

Definition 2.34 (Picard-Vessiot ring and field / extension).

Let (R, ∂) be a differential ring and $\boldsymbol{y}' = A\boldsymbol{y}$ a matrix differential equation. Let the following conditions hold:

(i) R is a simple differential ring.

(ii) There is a fundamental matrix, corresponding to $\boldsymbol{y}' = A\boldsymbol{y}$, such that $F' = AF$.

(iii) As a ring R is finitely generated over K by $\det(F^{-1})$, and f_{ij}, the coefficients of F.

Then R is a Picard-Vessiot ring over $\boldsymbol{y}' = A\boldsymbol{y}$.
If in L/K is a field extension such that $L = \mathrm{frac}(R)$, then we call L/K a *Picard-Vessiot field* or *Picard-Vessiot extension*.
Often we will abbreviate Picard-Vessiot with "PV".

Remark 2.35.

Thus condition (ii) tells us the solution space of the matrix differential equation $\boldsymbol{y}' = A\boldsymbol{y}$, for $\boldsymbol{y}$ now having elements in R has full dimension.
Furthermore (iii) tells us that R is generated over K as the coefficients of the solutions of $\boldsymbol{y}' = A\boldsymbol{y}$. Thus in particular R is the smallest (up to isomorphism) ring extension over K to have a solution space full dimension. Thus we may think of a PV ring as the differential analogue of a splitting field.

Remark 2.36 (Field of fractions of a Picard-Vessiot ring).

Since a PV ring R is differentially simple, it's maximal differential ideal is trivial. Hence, since by Proposition 2.9 this maximal differential ideal is also prime $R/\langle 0\rangle \cong R$, thus R is an integral domain, and $\mathrm{frac}(R)$ is well defined.

Proposition 2.37.

Let $\boldsymbol{y}' = A\boldsymbol{y}$ be a matrix differential equation over a differential field K. Then the following hold:

(i) There exists a Picard-Vessiot ring R, where R/K is an extension.

(ii) Let R, S are two Picard-Vessiot rings for the matrix differential equation, then $R \cong S$.

(iii) Let R be a Picard-Vessiot ring, then $C_{\mathrm{Frac}(R)} = C_K$.

Proof.

(i) Let $F = (f_{ij})_{i,j\leq n}$ be a matrix of variables. Let $Q \coloneqq K[f_{ij}, \det(F)^{-1}; i,j \leq n]$. We extend the derivation to $K[f_{ij}; i,j \leq n]$ by letting $\partial(F) = AF$ and then extend to Q via the localisation with respect to $\det(F)$, where we apply Proposition 2.10. We then also see that by how we extended Q/K that $C_Q = C_K$. Note now that Q satisfies conditions (ii), with fundamental matrix F and (iii) of a Picard-Vessiot ring. However Q is not necessarily differentially simple, thus we quotient out a maximal differential ideal $\mathfrak{M} \trianglelefteq Q$, i.e. $R \coloneqq Q/\mathfrak{M}$, which is then necessarily differentially simple. Conditions (ii) and (iii) still remain true under this quotient, thus R is a Picard-Vessiot ring.

(ii) Let R_1, R_2 be two PV rings over the same differential matrix equation over a field K, with corresponding fundamental matrices F_i. Now equip $R_1 \otimes R_2$ with the differential given by $\partial : r_1 \otimes r_2 \mapsto (r_1)' \otimes r_2 + r_1 \otimes (r_2)'$. Define $S \coloneqq R_1 \otimes R_2/I$ for $I \trianglelefteq R_1 \otimes R_2$ a maximal differential ideal. Now let $\phi_1 : R_1 \to S : r_1 \mapsto r_1 \otimes 1 + I$ and define ϕ_2 similarly. Since R_i is a simple differential ring, by Remark 2.21 $\phi_i : R_i \xrightarrow{\sim} \mathrm{Im}(\phi_i)$ is an isomorphism. Then clearly $\mathrm{Im}(\phi_i)$ is generated over K by the coefficients of $\phi_i(F_i)$ and $\phi_i[\det(F_i)^{-1}]$. Now $\phi_i(F_i)$ are both fundamental matrices over S, for the same matrix differential equation as R_i. Since $C_{R_1} = C_{R_2}$, and by our choice of differential on S, we find that $C_S \cong C_{R_1} \otimes C_{R_2} \cong C_{R_i} \otimes_K C_{R_i} \cong C_{R_i}$, where we use that by Proposition 2.37, (iii), $C_{R_i} \subseteq C_K \subseteq K$. Hence by Remark 2.33 we have that there is some constant invertible matrix C such that $\phi_1(F_1) = \phi_2(F_2)C$. Thus $\mathrm{Im}(\phi_1) = \mathrm{Im}(\phi_2)$, as they are generated by the same variables over K. Thus we have that $R_1 \cong R_2$.

(iii) See [1, 3].

■

2.3 Differential Galois groups

In this section we will define the differential Galois group, show it is an linear algebraic group and go forth and establish the differential analogue to the fundamental theorem of Galois theory.

Definition 2.38 (Differential Galois group).

Let $\boldsymbol{y}' = A\boldsymbol{y}$ be a matrix differential equation over a differential field K, and let R be the PV ring over K, then we let $\mathrm{DGal}(R/K) \coloneqq \mathrm{DAut}_K(R)$, the group of differential K-algebra automorphisms.

Proposition 2.39.

The differential Galois group is a linear algebraic group over C_K.

Proof. We shall first show the differential Galois group is isomorphic to a subgroup of $\mathrm{GL}_n(C_K)$ and then show it is closed.

Let $\mathrm{DGal}(R/K)$ be the differential Galois group corresponding to the differential matrix equation $\boldsymbol{y}' = A\boldsymbol{y}$. Let $\sigma \in \mathrm{DGal}(R/K)$ and $F' = AF$, then, since σ is a differential morphism, $\sigma(F)$ is also the fundamental matrix for the same differential matrix equation. From Remark 2.33 we then see that $\sigma(F) = FC_\sigma$, for $C_\sigma \in \mathrm{GL}_n(C_K)$. Thus we have an injection $\mathrm{DGal}(R/K) \hookrightarrow \mathrm{GL}_n(C_K) : \sigma \mapsto C_\sigma$. Thus $\mathrm{DGal}(R/K)$ is isomorphic to a subgroup of $\mathrm{GL}_n(C_K)$. Recall that we obtained a PV ring by extending K to $K[1/\det(x_{ij}), x_{ij};\, i,j \leq n]$ and then factoring out a maximal differential ideal $\mathfrak{m} \trianglelefteq K[1/\det(x_{ij}), x_{ij};\, i,j \leq n]$, i.e. $R = K[1/\det(x_{ij}), x_{ij};\, i,j \leq n]/\mathfrak{m}$. An automorphism $\sigma \in \mathrm{DGal}(R/K)$ acts on $K[1/\det(x_{ij}), x_{ij};\, i,j \leq n]$, by the previous identification, i.e. $\sigma(x_{ij}) = (x_{ij})C_\sigma$. Then $\sigma(\mathfrak{m}) \subseteq \mathfrak{m}$, as $\mathfrak{m}$ is an ideal. By Hilbert's basis theorem $K[1/\det(x_{ij}), x_{ij};\, i,j \leq n]$ is Noetherian, thus $\mathfrak{m}$ is finitely generated. Let $\langle r_i \rangle_{i \leq m} = \mathfrak{m}$, and let $\{e_i\}_{i \leq n}$ be a basis for C_K. Now, since σ acts as a matrix, $\sigma(r_i) = \sum_j c_{ij} e_i$, i.e. a finite sum, where $c_{ij} \in C_K$ and depend on C_σ, thus we can see c_{ij} as a polynomial with entries in C_σ. Then $\sigma(\mathfrak{m}) \subseteq \mathfrak{m} \implies \sigma(r_i) = 0 \implies \sum_j c_{ij} e_i = 0 \implies c_{ij} = 0$, thus $\mathrm{Im}[\mathrm{DGal}(R/K) \hookrightarrow \mathrm{GL}_n(C_K)]$ is Zariski closed. ∎

Remark 2.40.

Note that $\mathrm{DGal}(R/K) \cong \mathrm{DGal}[\mathrm{frac}(R)/K]$. We have that $\mathrm{DGal}(R/K) \hookrightarrow \mathrm{DGal}[\mathrm{frac}(R)/K] : \sigma \mapsto [r/s \mapsto \sigma(r)/\sigma(s)]$. Now if $\big(\forall \tau \in \mathrm{DGal}[\mathrm{frac}(R)/K]\big)\ \tau(R) = R$, then each K-algebra automorphism of $\mathrm{frac}(R)$ is also an K-algebra automorphism of R, in which case the two groups will then be isomorphic. Now as R and $\tau(R)$ are both generated over K by the coefficients of the fundamental matrix and the inverse of the determinant, they are isomorphic, and the result follows.

Theorem 2.41 (Fundamental theorem of (linear) differential Galois theory, [1, 3]).

Let L/K be a PV field extension of K, then

(i) There is the following bijection between subgroups $H \subseteq \mathrm{DGal}(L/K)$, and intermediate differential fields $K \subseteq M \subseteq L$.

$$\begin{aligned} \{\text{Differential intermediate fields}\} &\longleftrightarrow \{\text{Closed subgroups of } \mathrm{DGal}(L/K)\} \\ M &\longrightarrow \mathrm{DGal}(L/M) \\ L^H &\longleftarrow H \end{aligned}$$

Note here that $L^H \coloneqq \{\alpha \in L; (\forall \sigma \in H)\ \sigma(\alpha) = \alpha\}$.

(ii) The above bijection is inclusion reversing.

(iii) $H \subseteq \mathrm{DGal}(L/K)$ is a normal subgroup $\iff M = L^H$.
Additionally we then have $\mathrm{DGal}(L/K)/\mathrm{DGal}(L/M) \cong \mathrm{DGal}(M/K)$, said differently $\mathrm{DGal}(L/K) \twoheadrightarrow \mathrm{DGal}(M/K) : \sigma \mapsto \sigma|_M$ is a surjective group homomorphism.

(iv) $L^{\mathrm{DGal}(L/K)^0}/K$ is a finite Galois extension, with Galois group $\mathrm{DGal}(L/K)/\mathrm{DGal}(L/K)^0$, and $L^{\mathrm{DGal}(L/K)^0} = \overline{K}$.

We will postpone some examples of differential Galois groups till the next section.

2.4 Integrability

We shall now show the differential analogue of radical extensions and their connection to solvable Galois groups. Here it will able us to characterise integrability, via the identity component of the differential Galois group.

Definition 2.42 (Integral / exponential integral elements).

Let R/K b a differential extension.

(i) Let $\alpha \in L$, if $\partial(\alpha) \in K$, then we call α *integral over* K, and write $\exists \beta \in K$ such that $\alpha = \int \beta$. We shall call differential extensions by integral elements, *integral extensions.*

(ii) Let $\alpha \in L$, if $\alpha \neq 0$ and $\partial(\alpha)/\alpha \in K$, then we call α *exponentially integral over* K, and write $\exists \beta \in K$ such that $\alpha = e^{\int \beta}$. We shall call differential extensions by exponential integral elements, *exponential integral extensions.*

(iii) If a differential equation is solvable by any combination of integrals, exponential integrals and visa verse, we call the differential equation *solvable by generalised quadratures.*

Definition 2.43 (Liouvillian extension).

Let L/K be a differential extension such that there exists a chain of intermediate differential extensions

$$K \coloneqq K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_{n-1} \subseteq K_n \coloneqq L$$

such that $\forall i\ K_{i+1} = K_i(\alpha_i)$ one of the three statements is true for α_i:

(i) α_i is algebraic over K_{i-1},

(ii) α_i is integral over K_{i-1},

(iii) α_i is exponentially integral over K_{i-1}.

Then we call L/K a *Liouvillian extension.*

Example 2.44.

(i) Consider the differential equation $y'' - \frac{a'}{a}y' = 0$. Clearly it has solutions 1 and $\int a$, which are linearly independent over C_K. Thus we can associate to it the matrix differential equation

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 0 & a'/a \end{bmatrix} \boldsymbol{y},$$

which has corresponding fundamental matrix

$$\begin{bmatrix} 1 & \int a \\ 0 & a \end{bmatrix}.$$

Then $K[\alpha_1]$ is differentially simple, as any non-trivial differential ideal contains a polynomial of $\int a$, which contains $(\int a)' = a$ a unit. Then it is easy to see that $K[\int a]$ is a PV ring, thus $K(\int a)$ is a PV extension. Let $\sigma \in \mathrm{DGal}[K(\int a)/K]$, then as σ is a differential K-automorphism, then $\sigma(\int a) = \int a + c$ for $c \in C_K$. Then $\mathrm{DGal}[K(\int a)/K] \cong \mathbb{G}_a$.

(ii) Consider the differential equation $y' = ay$, for $a \in K^\times$, with fundamental matrix $e^{\int a} \coloneqq \alpha$, then $K[\alpha, \alpha^{-1}]$ is again clearly a simple differential ring and thus a PV ring, then $K(\alpha)$ is a PV extension. Note that if $\sigma \in \mathrm{DGal}[K(\alpha)/K]$, then $\left(\frac{\sigma(\alpha)}{\alpha}\right)' = \frac{\sigma(\alpha)' - \sigma(\alpha)\alpha'}{\alpha^2} = \frac{\sigma(a\alpha)\alpha - \sigma(\alpha)a\alpha}{\alpha^2} = 0$. Hence $\sigma(\alpha)/\alpha = c \implies \sigma(\alpha) = c\alpha$, for $c \in C_K$. Thus $\mathrm{DGal}[K(\alpha)/K] = \mathbb{G}_m$.

Lemma 2.45.

Let L/K be a normal differential extension. If $\exists \alpha_i \in L$ for $i \leq n$ such that

$$[\forall \sigma \in \mathrm{DGal}(L/K)] \quad \exists b_{ij} \in C_L \text{ such that } \sigma(\alpha_j) = \sum_{i=1}^{j} b_{ij}\alpha_i \quad (j \leq n),$$

then $K(\alpha_i;\, i \leq n)$ is Liouvillian.

Proof. Without loss of generality we assume the $\alpha_i \neq 0$. We proceed by induction. For the base case consider

$$\sigma(\alpha_1) = b_{11}\alpha_1 \implies \sigma(\alpha_1') = b_{11}\alpha_1'.$$

Thus dividing both sides of the equality we obtain

$$\sigma(\alpha_1'/\alpha_1) = \alpha_1'/\alpha_1.$$

This holds for any σ, thus α_1'/α_1 is invariant under $\mathrm{DGal}(L/K)$, thus using the normality of the extension $\alpha_1'/\alpha_1 \in K$. Thus α_1 is an exponentially integral over K. As our induction hypothesis the property holds for any sum of length less than m. Now we divide $\sigma(\alpha_m)$ by $\sigma(\alpha_1)$, we then obtain

$$\sigma(\alpha_m/\alpha_1) = 1 + \sum_{i=2}^{m} \frac{b_{2j}}{b_{11}} \frac{\alpha_i}{\alpha_1}.$$

Now we differentiate both sides, yielding

$$\sigma[(\alpha_m/\alpha_1)'] = \sum_{i=2}^{m} \frac{b_{2j}}{b_{11}} \left(\frac{\alpha_i}{\alpha_1}\right)'.$$

This is of the same form as in the theorem, but for a sum of m terms. The induction hypothesis then gives that $M := K(\alpha_1,\, [\alpha_i/\alpha_1]';\, 2 \leq i \leq m)$ is a Liouvillian extension. Then $M(\alpha_i/\alpha_1;\, 2 \leq i \leq m)$ is a series of integral extensions and thus Liouvillian. However clearly $K(\alpha_i;\, i \leq m) = M(\alpha_i/\alpha_1;\, 2 \leq i \leq m)$. Thus the induction hypothesis holds, and in particular is true for $m = n$, thus we are done. ■

Theorem 2.46 (Liouville theorem).

Let L/K be a PV extension, then the following are equivalent:

(i) $\mathrm{DGal}(L/K)^0$ is solvable.

(ii) L is a Liouvillian extension of K.

(iii) L is contained in a Liouvillian extension of K.

Proof. "(i) $\implies$ (ii)": Set $M = L^{\mathrm{DGal}(L/K)^0}$, then M/K is a normal finite extension by the fundamental theorem, (iv). We have that $\mathrm{DGal}(L/M) \cong \mathrm{DGal}(L/K)^0$. Thus by the Lie-Kolchin theorem, $\mathrm{DGal}(L/K)^0$ has a basis such that its elements are upper triangular, this exactly means we may invoke Lemma 2.45 (realising that the lemma is written to be applied for lower triangular matrices, but that this doesn't matter), using that L is a PV extension. Thus we have that L/M is a Liouvillian extension. We also have that M/K is finite, thus M/K is algebraic and thus Liouvillian. Hence we have that L/K is Liouvillian.
"(ii) $\implies$ (iii)": Let M/K be Liouvillian for $K \subseteq L \subseteq M$. Then there exists a chain satisfying the conditions for a Liouvillian extension. Let the following be such a chain:

$$K = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = M.$$

Then there is some i such that $K_i \subseteq M \subseteq K_{i+1}$, where K_{i+1}/K_i is Liouvillian extension with a chain of length one. Then it is easy to see that if $m \in M \implies m \in K_{i+1} \implies m$ is algebraic, integral or exponential integral over K_i.
"(iii) $\implies$ (i)": Let M/K be Liouvillian and $K \subseteq L \subseteq M$. Then we can write $M = K(\alpha_1, \ldots, \alpha_n)$. We will proceed by induction on n, i.e. on the length of the chain of the Liouvillian extension. We take $M = K$ as our base case, then $K = L = M \implies \mathrm{DGal}(L/K) = \{e\}$ and thus has a solvable identity component. Now as our induction hypothesis we assume that if M/K is Liouvillian with a chain length n and $L \subseteq M$, then $\mathrm{DGal}(M/K)$ has a solvable identity component. Consider the extension $L(\alpha_1)/K(\alpha_1)$. It is not hard to see that $L(\alpha_1)/K(\alpha_1)$ is a PV extension. Let $H \coloneqq \mathrm{DGal}[L(\alpha_1)/K(\alpha_1)]$. Consider the group homomorphism $\mathrm{DGal}[L(\alpha_1)/K(\alpha_1)] \to \mathrm{DGal}(L/K) : \sigma \mapsto \sigma|_L$. Since σ fixes α_1, the homomorphism has a trivial kernel and is thus injective and isomorphic to some closed subgroup of $\mathrm{DGal}(L/K)$. Since $L^H = L(\alpha_1)^H \cap L = K(\alpha_1) \cap L$, we have that $H \cong \mathrm{DGal}\left(L/[L \cup K(\alpha_1)]\right)$. Clearly $L(\alpha_1) \subseteq M$, and $M/K(\alpha_1)$ is Liouvillian and with a shorter chain length than M/K, thus by our induction hypothesis we have that $\mathrm{DGal}[L(\alpha_1)/K(\alpha_1)]^0 = H^0$ is solvable. Now we make a case distinction on what kind of extension $K(\alpha_1)/K$ is, and in each case show that $\mathrm{DGal}(L/K)^0 = H^0$.
Let α_1 be algebraic over K. Then $[\mathrm{DGal}(L/K) : H]$ is finite, hence $\mathrm{DGal}(L/K)^0 \subseteq H^0$, but clearly $\mathrm{DGal}(L/K)^0 \supseteq H^0$, thus we have the desired equality and since H^0 is solvable, $\mathrm{DGal}(L/K)^0$ is solvable.
Let α_1 be transcendental over K, and thus integral or exponential integral. In both cases we will show that $\mathrm{DGal}[K(\alpha_1)/K]$ is abelian, in which case $L \cap K(\alpha)/K$ is normal and also has a abelian Galois group, by the fundamental theorem, (iii). Then H is normal in $\mathrm{DGal}(L/K)$ and $\mathrm{DGal}(L/K)/H \cong \mathrm{DGal}[(L \cap K(\alpha)/K]$ is abelian, then by Proposition 2.9, (ii), $\mathrm{DGal}(L/K)^0$ is solvable. First we show that if α_1 is integral over K, then $K(\alpha_1)/K$ is a PV extension. Let

$\alpha_1' = a \in K$. To do this we first show that $C_K = C_{K_{\alpha_1}}$. Then we need to show that any rational polynomial in α_1 must have a non-zero derivative. We proceed by induction on the degree of the denominator of the rational polynomial. For our base case we then consider a regular polynomial. By way of contradiction first assume there is polynomial f s.t. $f(\alpha) = \sum_i^n b_i \alpha_1^i$ is a constant. Then

$$f'(\alpha_1) = 0 \implies b_n na + b_{n+1}' = 0 \implies a = -\frac{b_{n-1}}{b_n \cdot n} = \left(\frac{-b_{n-1}}{nb_n}\right)',$$

where we use that since α_1 is transcendental, $\sum_i c_i \alpha^i = 0 \implies c_i = 0$. The above contradicts α_1 being transcendental, as $\alpha_1' = a = \left(\frac{-b_{n-1}}{nb_n}\right)' \implies \alpha_1 = \frac{-b_{n-1}}{nb_n} + c \in K$ ↯, for c a constant. Now as our induction hypothesis we assume $[f(\alpha)/g(\alpha)]'$ is non-zero for all rational polynomials with a denominator of degree m. Again we argue by contradiction. Thus we assume that there exists a rational polynomial f/g s.t. $[f(\alpha_1)/g(\alpha_1)]' = 0$ for g of degree $m+1$. Then

$$[f(\alpha_1)/g(\alpha_1)]' = 0 \implies \frac{f'(\alpha_1)ag(\alpha_1) - f(\alpha_1)g'(\alpha_1)}{g(\alpha^1)^2} = 0 \implies \frac{f'(\alpha_1)}{g'(\alpha_1)} = \frac{f(\alpha_1)}{g(\alpha_1)}.$$

But if $\frac{f'(\alpha_1)}{g'(\alpha_1)} = \frac{f(\alpha_1)}{g(\alpha_1)}$ is true, then since $g'(\alpha)$ is of degree m, this contradicts the induction hypothesis ↯. Thus we now have shown $K(\alpha_1)/K$ has no new constants. Note that $y'' - \frac{a'}{a}y' = 0$ has solutions 1 and α_1, which are linearly independent over C_K, then by Example 2.44, Item (i), we see that the corresponding differential Galois group is $\mathbb{G}_a$, which is abelian. Now for α_1 exponential integral. Then $\alpha_1'/\alpha_1 = a \in K^\times$, thus we can associate the matrix equation $y' = ay$, which by Example 2.44, (ii), has a corresponding differential Galois group $\mathbb{G}_m$, which is abelian. ■

Corollary 2.46.

DGal(L/K) a Galois group for the PV extension corresponding to the matrix equation $\boldsymbol{y}' = A\boldsymbol{y}$. Then

(i) DGal(L/K) is connected and solvable $\iff$ $\boldsymbol{y}' = A\boldsymbol{y}$ can be solved by quadratures.

(ii) DGal(L/K) is unipotent $\iff$ $\boldsymbol{y}' = A\boldsymbol{y}$ can be solved by integrals.

(iii) DGal(L/K) is diagonalisable $\iff$ $\boldsymbol{y}' = A\boldsymbol{y}$ can be solved by exponentials.

Proof.

(i) Same as the proof of the theorem, but now for " $\impliedby$ " the proof simplifies, as, since the Galois group is connected the fundamental theorem gives us that K is algebraically closed, so there is no algebraic extension.

(ii) " $\implies$ ": If DGal(L/K) is unipotent, then the proof of Lemma 2.45 can easily be modified to see that now any extension is an integral one, using the notation from the proof of

Lemma 2.45, just let $b_{jj} = 1$ (and note that the automoprhism in the proof is of the form of a lower-triangular matrix, but this doesn't matter).

" $\Longleftarrow$ ": We can write $M/L/K$ with $M = K(\beta_1, \ldots, \beta_m)$, where each β_j is integral over K. Let $K_0 = K$ and $K(\beta_j;\, j \leq i) \coloneqq K_i$. Now since we can write $\mathrm{DGal}(M/K)$ as a chain of groups which are extensions of unipotent groups, Proposition 2.17 yields that $\mathrm{DGal}(M/K)$ is unipotent.

(iii) " $\Longrightarrow$ ": If $\mathrm{DGal}(L/K)$ is diagonalisable, then the proof of Lemma 2.45 can easily be modified to see that any extension is an exponential integral one, using the notation from the proof of Lemma 2.45, just let $b_{ij} = 0$ for $i \neq j$.

" $\Longleftarrow$ ": If $\boldsymbol{y}' = A\boldsymbol{y}$ can be solved by only exponential extensions, then we can write $K \subseteq K(\alpha_1) \subseteq K(\alpha_1, \alpha_2) \subseteq K(\alpha_1, \ldots, \alpha_n) = L$, for $K_{i+1} \coloneqq K(\alpha_1, \ldots, \alpha_{i+1})/K(\alpha_1, \ldots, \alpha_i) \eqqcolon K_i$ an integral extension. Then we know that $G_i/G_{i+1} \cong \mathbb{G}_m$, where the G_i correspond to the fields extensions K_i. We shall now prove that then $G_n \cong \mathbb{G}_m^n$, via induction. For the base case note that $G_0 = e$, thus $G_1/G_0 \cong \mathbb{G}_m \implies G_1 \cong \mathbb{G}_m$. For the induction hypothesis assume $G_i \cong \mathbb{G}_m^i$. Consider the following exact sequence of groups:

$$e \longrightarrow G_{i+1} \xhookrightarrow{\iota} G_i \xrightarrow{\natural} G_i/G_{i+1} \cong \mathbb{G}_m \longrightarrow e.$$

Since G_n is connected G_i is connected, then by the Lie-Kolchin theorem, we may assume G_i is upper triangular, then $r : G_i \to G_{i+1}$, which sends a matrix to its diagonal defines a retract for ι, thus the sequence is split exact. Then we have that $G_i \cong G_{i+1} \times \mathbb{G}_m \cong \mathbb{G}_m^{i+1}$. Thus the induction hypothesis holds in general, and we are finished.

■

Remark 2.47.

Being algebraic / integral / exponential integral over a field over $\mathbb{C}(x)$ with the usual derivation yields all the standard elementary functions. For example $\int 1/x = \ln(x)$, $\int \frac{1}{x^2+1} = \arctan(x)$ are integral extensions, e^x is an exponential integral extension, hence trigonometric functions are also exponential integral ($\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$), and $\sqrt[n]{x}$ is an algebraic extension. Being Liouvillian allows for all of these kinds of extensions, and their compositions.

For example $\sin^{1/2}(x)$ is also Liouvillian over $\mathbb{C}(x)$ and $\sin^{1/2}(x)$ is a solution to the differential equation $\frac{4}{-2\sin^2(x) - \cos^2(x)} y'' - \frac{2}{\cos(x)\sin(x)} y' = 0$ over $\mathbb{C}(x, e^{ix})$.

Remark 2.48.

We note some extra things from the above proofs. Let G denote the differential Galois group.

(i) If the identity component of the Galois group is of the form $\mathbb{D}_n \times \mathbb{U}_m$, then $\mathbb{D}_n$ and $\mathbb{U}_m$ are both extensions of the base field, i.e. we get no integrals of integral extensions (no "$\int e^{\int}$").

(ii) If $\mathbb{U}_m \cong \mathbb{G}_a^m$, then each integral extension, is an extension over the base field, i.e. we get no compositions of integral extensions.

(iii) We can also specify there to be algebraic and integral or algebraic and exponential integral equations, by letting G not be solvable and respectively G^0 be either diagonal or unipotent.

To summarise this a Liouvillian extension can be written as

$$\underbrace{\overbrace{K \longrightarrow L^{G^0}}^{G/G^0}}_{\text{algebraic,}} \overbrace{\underbrace{\overbrace{\longrightarrow M}^{\mathbb{G}_m^d}}_{\text{exponential integral}} \underbrace{\overbrace{\longrightarrow L}^{\mathbb{U}}}_{\text{integral}}}^{G^0}$$

Example 2.49.

Consider the differential equation

$$y'' + 2xy' = 0$$

over $\mathbb{C}(x)$, with the usual derivation. It has solutions given by $\int e^{x^2}$ and 1. Thus the corresponding PV extension is $\mathbb{C}(x, \int e^{x^2}, e^{x^2})$. The Galois group is then determined completely by where the automorphism sends $\int e^{x^2}$ and e^{x^2}. Let σ be such an automorphism, then

$$\sigma = \begin{cases} e^{x^2} \mapsto \lambda e^{x^2} \\ \int e^{x^2} \mapsto \lambda \int e^{x^2} + c, \end{cases}$$

for $\lambda, c \in \mathbb{C}$. Clearly this corresponds with the matrix group

$$\begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}.$$

It is not hard to verify that the commutator subgroup is given by

$$\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix},$$

which is solvable. This means that $y'' + 2xy' = 0$ is solvable by quadratures, of course the solutions we gave showed that this would already be the case. Mover note that this group is not unipotent, diagonal or abelian, thus not isomorphic to $\mathbb{D}_n \times \mathbb{U}_m$. It is in fact easy to see that this is the usual semidirect product $\mathbb{G}_a \ltimes \mathbb{G}_m$ This can be stated differently: The differential equation is not solvable by elementary functions, i.e. we needed to add an integral of an exponential. Thus in particular $\int e^{x^2}$ is not an elementary function.

2.5 Differential Galois theory over $\mathbb{C}(x)$.

We shall now introduce kovacic's algorithm, for determining if a linear differential equation of order 2 is Liouvillian or not. There exist variants for higher order equations as well.

Proposition 2.50.

Let $\mathcal{L}(y) = \partial^n y + a_{n-1}\partial^{n-1}y + \cdots + a_1\partial y + a_0 y$ be a linear differential operator. Then $\mathcal{L}$ can be transformed to the form $\widetilde{\mathcal{L}} = \partial^n + b_{n-1}\partial^{n-2} + \cdots + b_0$, via $y = vz$, for v a solution of $nv' + a_{n-1} = 0$. The solutions of $\mathcal{L}$ are Liouvillian $\iff$ the solutions of $\widetilde{\mathcal{L}}$ are Liouvillian.

Proof. Let $\mathcal{L}(y) = \partial^n y + a_{n-1}\partial^{n-1}y + \cdots + a_1\partial y + a_0 y$. If we substitute vz for y, the only ∂^{n-1} term we can get, comes from the terms $\partial^n y + a_{n-1}\partial^{n-1}y$.

$$\partial^n(vz)+a_{n-1}\partial^{n-1}(vz) = v\partial^n z+nv'\partial^{n-1}z+\cdots+a_{n-1}(\partial^{n-1}z+\dots) = \partial^n z+(nv'+a_{n-1})\partial^{n-1}(z)+\dots,$$

thus by definition of v the ∂^{n-1} term cancels.

Note that $v' = -a_{n-1}/n \in K$, thus v is Liouvillian. Let α be a liouvillian solution of $\mathcal{L}$, then $K(\alpha, v)$ is Liouvillian. Now α/v is a solution of $\widetilde{\mathcal{L}}$ and $K(\alpha/v) \subseteq K(\alpha, v)$, thus α/v is Liouvillian. Let β be a Liouvillian solution of $\widetilde{\mathcal{L}}$, then $K(v, \beta)$ is a Liouvillian extension, furthermore $v\beta$ is a solution of $\mathcal{L}$, and $K(v\beta) \subseteq K(v, \beta)$, thus $v\beta$ is Liouvillian. ■

The transformation outlined above does not preserve the differential Galois group in general.

Proposition 2.51.

Let F be the fundamental matrix corresponding to the matrix differential equation $\boldsymbol{y}' = A\boldsymbol{y}$, then $\det(F)' = \operatorname{tr}(A)\det(F)$.

Proof. Let $F' = AF$, with columns labelled $\{f_i\}_{i\leq n}$ and coefficients labelled $F = (f_{ij})_{i,j\leq n}$. Let $A = (a_{ij})_{i,j\leq n}$, then $f'_{ij} = \sum_k^n a_{ik}f_{kj}$. Then

$$\det(F)' = \begin{vmatrix} f'_{11} & f'_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix} + \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f'_{21} & f'_{22} & \cdots & f'_{2n} \\ \vdots & \vdots & & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & & \vdots \\ f'_{n1} & f'_{n2} & \cdots & f'_{nn} \end{vmatrix}.$$

Now we substitute the expression for f'_{ij} into the above expression and note that due to linear dependence we then obtain for example that

$$\begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ \vdots & \vdots & & \vdots \\ f'_{i1} & f'_{i2} & \cdots & f'_{in} \\ \vdots & \vdots & & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix} = \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ \vdots & \vdots & & \vdots \\ \sum_k^n a_{ik}f_{k1} & \sum_k^n a_{ik}f_{k2} & \cdots & \sum_k^n a_{ik}f_{kn} \\ \vdots & \vdots & & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix} = \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ \vdots & \vdots & & \vdots \\ a_{ii}f_{i1} & a_{ii}f_{i2} & \cdots & a_{ii}f_{in} \\ \vdots & \vdots & & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix} = a_{ii}\det(F).$$

Thus $\det(F)' = \operatorname{tr}(A)\det(F)$. ■

Remark 2.52.

Using the matrix representation of a linear differential operator $\mathcal{L}$ of order n (see Remark 2.23), we then see that by the above proposition, $\det(F)' = -a_{n-1}\det(F)$.

Proposition 2.53.

Let L/K be a PV extension with corresponding fundamental matrix F, then $\mathrm{DGal}(L/K) \subseteq \mathrm{SL}_n(C_k) \iff \det(F) \in K$.

Proof. " $\implies$ ": Let $\sigma \in \mathrm{DGal}(L/K)$, then we have seen that σ acts on F by matrix multiplication with a matrix C_σ with constant coefficients, for instance $\sigma(F) = FC_\sigma$. Then σ acts on $\det(F)$ by multiplication by $\det(C_\sigma) = 1$, so $\det(F)$ is invariant under σ, thus $\det(F) \in K$.
" $\impliedby$ ": Let $\det(F) \in K$, then $\det(F)$ is invariant under $\mathrm{DGal}(L/K)$, but σ acts on $\det(F)$ by multiplication by $\det(C_\sigma)$, thus $\det(C_\sigma) = 1 \implies \sigma \in \mathrm{SL}_n(C_K)$. ■

Now we will discuss Kovacic's Algorithm, for how to perform the algorithm see [5]. Let $y'' + p(x)y = 0$ be a differential equation over $\mathbb{C}(x)$, with corresponding fundamental matrix F. Note that any differential equation of degree 2 can be written in this form by Proposition 2.50. Now, since there is no y'-term, we have that by Remark 2.52 $\det(F)' = 0$ by, then $\det(F) \in \mathbb{C}$, then by Proposition 2.53 the corresponding Galois group, is a subgroup of $\mathrm{SL}_2(\mathbb{C})$.

We shall now, up to conjugation give the connected subgroups of $\mathrm{SL}_2(\mathbb{C})$, as given in [6].

Proposition 2.54.

Let G be an algebraic subgroup of $\mathrm{SL}_2(\mathbb{C})$, then, up to conjugation, G is one of the following groups listed below.

1. The Galois group is finite, and the identity component is trivial.
2. $$G = G^0 = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}.$$
3. $$\forall k \qquad G_k = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}; a \text{ a } k\text{-root of unity, } b \in \mathbb{C} \right\},$$
   with
   $$G^0 = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}.$$
4. The Galois group a subgroup of
   $$\left\{ \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix}; c \in \mathbb{C}^\times \right\} \cup \left\{ \begin{bmatrix} 0 & c \\ -c^{-1} & 0 \end{bmatrix}; c \in \mathbb{C}^\times \right\},$$

where

$$G^0 = \left\{ \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} ; c \in \mathbb{C}^\times \right\}.$$

5.

$$G = G^0 = \left\{ \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} ; c \in \mathbb{C}^\times \right\}.$$

6.

$$G = G^0 = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} ; a \in \mathbb{C}^\times,\, b \in \mathbb{C} \right\}.$$

7. $G = G^0 = \mathrm{SL}_2(\mathbb{C})$.

Here G has solvable identity components in cases 1 through 6.

Using this Kovacic's algorithm can determine the differential Galois group, determine integrability, and give solutions to the given differential equation, if the algorithm fails, the differential equation is not integrable. For example Mathematica uses this algorithm to solve certain differential equations, and the algorithm can directly be invoked using the "`kovacicsols`" command in Maple.

With the following conditions, it can be easy to sometimes show a differential equation is not Liouvillian.

Proposition 2.55 ([3]).

One of the following cases is true for the differential equation $y'' + p(x)y = 0$.

1. Let $f \in \mathbb{C}(x)$. $e^{\int f}$ is a solution of the above equation.
2. Let $[\mathbb{C}(x, f) : \mathbb{C}(x, f)] = 2$. $e^{\int f}$ is a solution of the above equation.
3. All solutions of the above equation are algebraic over $\mathbb{C}(x)$.
4. The differential equations has no Liouvillian solution.

Furthermore we have that the above are respectively necessary conditions for:

1. Every pole of $p(x)$ has an order equal to 1, or divisible by 2 and the order of $p(x)$ at ∞ must be 2 or of odd order greater than 2.
2. $p(x)$ has at least one pole of order 2, or of odd order greater than 2.
3. The orders of poles of $p(x)$ do not exceed 2, and the order of $p(x)$ at ∞ must be at least 2.
4. None of the above is true.

Example 2.56.

Consider the following differential equation: $y'' - xy = 0$, this is called the Airy equation. $p(x) = x$ has order -1 at ∞, and no further poles, thus by Proposition 2.55 the Airy equation is not Liouvillian and has a differential Galois group corresponding to $\mathrm{SL}_2(\mathbb{C})$.

3 Symplectic geometry, Hamiltonian systems and (non-)integrability

We shall discuss some Hamiltonian formalism as to define what it means for a Hamiltonian system to be completely integrable. This section is based on the contents of [7, 8, 9, 10, 11].

Definition 3.1 (Symplectic form / manifold).

Let M be a $\mathcal{C}^\infty$ manifold of finite dimension. Let ω be a closed non-degenerate differential 2-form, then ω is a *symplectic form*. We call a manifold equipped with a symplectic form a *symplectic manifold*.

Remark 3.2.

Recall that ω is closed if $\mathrm{d}\omega = 0$ and non-degenerate if

$$(\forall m \in M)(\forall v \neq 0 \in \mathrm{T}_p M) \quad \exists w \text{ s.t. } \omega_m(v, w) \neq 0.$$

Example 3.3.

Consider $(\mathbb{R}^2, \mathrm{d}x \wedge \mathrm{d}y)$, then clearly ω is closed 2-form, and is non-degenerate.

Proposition 3.4.

Symplectic manifolds have even dimension.

Proposition 3.5 (Musical isomorphism).

Let (M, ω) be a symplectic manifold, then $\omega^\sharp : \mathrm{T}M \to \mathrm{T}^*M : v \mapsto \omega(v, \text{-})$ is an isomorphism.

Proof. ω is bilinear, thus $\omega(v, \text{-})$ is linear. Now, since for finite dimensional vector spaces we have that $\dim V = \dim V^*$, we see that $\omega^\sharp$ is an isomorphism. ∎

Theorem 3.6 (Darboux's theorem).

Let (M, ω) be a symplectic manifold of dimension $2n$, then $\forall m \in M$ there are coordinates $p_1, \ldots, p_n, q_1, \ldots, q_n$, which we will refer to as the canonical coordinates, in an neighbourhood open U_m of m, s.t. $\omega = \sum_i^n \mathrm{d}p_i \wedge \mathrm{d}q_i$).

Remark 3.7.

Said differently, every symplectic manifold is locally diffeomorphic to $(\mathbb{R}^{2n}, \sum_i^n \mathrm{d}p_i \wedge \mathrm{d}q_i)$.

Definition 3.8 (Hamiltonian vector fields).

Let (M, ω) be a symplectic manifold, $H \in \mathcal{C}^\infty(M)$ and let X_H be a vector field such that for all vector fields Y

$$\omega(X_H, Y) = \mathrm{d}H(Y).$$

Then we call X_H, the *Hamiltonian vector field of H*.

Proposition 3.9.

The Hamiltonian vector field X_H, for H exists and is unique.

Proof. It is easy to check that $X_H = (\omega^\sharp)^{-1}(\mathrm{d}f)$, which, since by Proposition 3.5 $\omega^\sharp$ is an isomorphism, exists and is unique. ■

Example 3.10.

Consider the symplectic manifold $(\mathbb{R}^2, \omega = \mathrm{d}x \wedge \mathrm{d}y)$. Let $f \in \mathcal{C}^\infty$, then $\mathrm{d}f = \partial_x f \mathrm{d}x + \partial_y f \mathrm{d}y$. Now $\omega^\sharp(\partial_x) = \mathrm{d}y$ and $\omega^\sharp(\partial_y) = -\mathrm{d}x$, then

$$X_f = (\omega^\sharp)^{-1}(\mathrm{d}f) = (\omega^\flat)(\partial_x f \mathrm{d}x + \partial_y f \mathrm{d}y) = \partial_x \omega^\flat(\mathrm{d}x) + \partial_y \omega^\flat(\mathrm{d}y) = \partial_y f \partial_x - \partial_x f \partial_y.$$

Definition 3.11 (Poisson brackets).

Let (M, ω) be a symplectic manifold, then $\{F, G\} \coloneqq \omega(X_F, X_G)$ is the *Poisson bracket of F and G*.

Example 3.12.

Let $(\mathbb{R}, \omega = \mathrm{d}x \wedge \mathrm{d}y)$ be a symplectic manifold. Let $f, g \in \mathcal{C}^\infty(\mathbb{R})$, then $\{F, G\} = \omega(X_F, X_G) = \partial_y f \partial_x H - \partial_x f \partial_y H$.

This shows that for a general symplectic manifold (M, ω) of dimension $2n$, using Darboux's theorem, we can take ω to be in canonical coordinates and find that for $f, g \in \mathcal{C}^\infty(M)$

$$\{f, g\} = \sum_i^n (\partial_{q_i} f \partial_{p_i} g - \partial_{p_i} f \partial_{q_i} g).$$

Definition 3.13 (Hamiltonian flow).

Let X_H be a Hamiltonian vector field. Let (M, ω) be a symplectic manifold. Let $\phi_t : M \to M$ for $t \in (-\epsilon, \epsilon)$, be a parametrised smooth function satisfying $\phi_t^*(\omega) = \omega$ and

$$(\forall x \in M) \quad \frac{\mathrm{d}}{\mathrm{d}t}\phi_t(x) = X_H[\phi_t(x)] \text{ and } \phi_0(x) = x.$$

Then ϕ_t *is the flow generated by* X_H.

Remark 3.14.

The Hamiltonian flow conserves energy, which follows from that $H \circ \phi_t$ is constant in t. To see this, note that

$$\frac{\mathrm{d}}{\mathrm{d}t} H \circ \phi_t = \mathrm{d}H \circ \phi_t \cdot X_H \circ \phi_t = \iota(X_H \circ \phi_t)\omega(X_H \circ \phi_t) = \omega(X_H \circ \phi_t,\, X_H \circ \phi_t) = 0.$$

Proposition 3.15.

Let ϕ_t be the flow along the Hamiltonian vector field X_H and $F \in \mathcal{C}^\infty$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(F \circ \phi_t) = \{F \circ \phi_t, H \circ \phi_t\}.$$

Proof. Let

$$\frac{\mathrm{d}}{\mathrm{d}t}(F \circ \phi_t) = \mathrm{d}F \circ \phi_t \cdot \frac{\mathrm{d}}{\mathrm{d}t}\phi_t = \mathrm{d}F \circ \phi_t \cdot X_H \circ \phi_t = \{F \circ \phi_t, H \circ \phi_t\},$$

where in the first equality we apply the chain rule, and in the second equality we used that ϕ_t is the flow of X_H, and in the last equality we use that X_F is a Hamiltonian vector field. ∎

Remark 3.16.

Let (M, ω) be a symplectic manifold of dimension $2n$. By Darboux's theorem we may assume we are working in the canonical coordinates $\omega = \sum_i^n \mathrm{d}p_i \wedge \mathrm{d}q_i$. Let ϕ_t be the flow of the Hamiltonian vector field X_H. Note that we then use the form of the Poisson bracket as described in Example 3.12. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} p_i(\phi_t) = \{p_i \circ \phi_t, H \circ \phi_t\} = \partial_{q_i} p_i \circ \phi_t \cdot \partial_{q_i} H \circ \phi_t - \partial_{p_i} p_i \circ \phi_t \cdot \partial_{q_i} H \circ \phi_t = -\partial_q H \circ \phi_t$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} q_i(\phi_t) = \{q_i \circ \phi_t, H \circ \phi_t\} = \partial_{q_i} q_i \circ \phi_t \cdot \partial_{q_i} H \circ \phi_t - \partial_{p_i} q_i \circ \phi_t \cdot \partial_{q_i} H \circ \phi_t = \partial_{p_i} H \circ \phi_t.$$

Thus the Hamilton equations of motion hold on a symplectic manifold.

Definition 3.17 (Hamiltonian system / first integral / completely integrable).

(i) Let (M, ω) be a symplectic manifold, and $H \in \mathcal{C}^\infty(M)$, then the triple (M, ω, H) is a *Hamiltonian system.*

(ii) (M, ω, H) be a Hamiltonian system. Let $f \in \mathcal{C}^\infty(M)$, then if $\{f, H\} = 0$, then f *is a constant of motion* or a *first integral.*

(iii) Let (M, ω, H) be a Hamiltonian system with f, g first integrals of this system, then they *commute* if $\{f, g\} = 0$.

(iv) Let (M, ω, H) be a Hamiltonian system of dimension $2n$, let $f_i \in \mathcal{C}^\infty(M)$ s.t. $i \leq n-1$ such that $H, f_1, \ldots, f_{n-1}$ all commute pairwise. In addition let $\{\mathrm{d}f_i\}_{i \leq n}$ be linearly independent on some open dense subset of M, then we call (M, ω, H) *completely integrable.*

Example 3.18.

(i) Since $\{H, H\} = 0$, H is trivially a first integral, this corresponds to energy conservation, thus any one degree of freedom Hamiltonian is completely integrable.

(ii) Consider the spherical pendulum, with mass m, length l, where θ is the angle of the pendulum with respect to its position at rest (the angle a 2 dimensional pendulum would make), and ϕ is a rotation around this rest position.

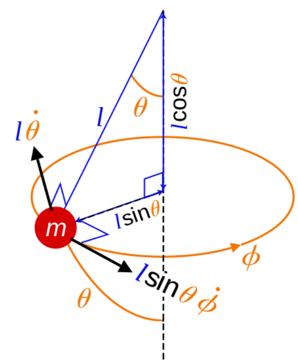


Figure 2: The coordinate system used for the spherical pendulum. Image from [12].

Such a pendulum has a Hamiltonian

$$H = 1/2ml^2(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2) - mlg\cos(\theta),$$

which can be rewritten in canonical momenta $p_\phi = ml^2\sin^2(\theta)\dot{\phi}$ and $p_\theta = ml^2\dot{\theta}$, as

$$H = \frac{1}{2ml^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2(\theta)}\right) - mgl\cos(\theta).$$

This is then a Hamiltonian system $(\mathbb{R}^4, \omega = \mathrm{d}\theta \wedge \mathrm{d}p_\theta + \mathrm{d}\phi \wedge \mathrm{d}p_\phi, H)$. The equations of motions are then

$$\dot{p}_\theta = -\partial_\phi H = 0 \text{ and } \dot{p}_\theta = -\partial_\theta H = \frac{p_\phi^2\cos(\theta)}{ml^2\sin^3(\theta)} - mgl\sin(\theta).$$

From $\dot{p}_\theta = 0$, we can see that $\{p_\phi, H\} = 0$ and is a first integral, clearly the corresponding conserved quantity is angular momentum. Since then $ml^2\sin^2(\theta)\dot{\phi} = \text{const}$, we can then integrate to find an expression for ϕ.

Theorem 3.19 (Arnold-Liouville theorem).

Let (M, ω, H) be a completely integrable Hamiltonian system, with n degrees of freedom, then

(i) setting each first integral equal to some constant value defines a smooth surface / $\mathcal{C}^\infty$ manifold in $2n$-dimensional phase space. Each solution starting on the surface must remain on the surface indefinitely. If in addition the surface is bounded and connected, it defines an n-dimensional torus $D \times \mathbb{T}^n$, for $D \subseteq R^n$.

(ii) if the surface discuss above is indeed a torus, there exist canonical "action-angle" coordinates $(I_1, \ldots, I_n; \theta_1, \ldots, \theta_n) \in D \times \mathbb{T}^n$ in which the Hamiltonian becomes a function of simply $(I_1, \ldots, I_n)$, i.e. $H(p_i, q_i; i \leq n) = \widetilde{H}(I_i; i \leq n)$, where

$$\frac{\mathrm{d}I_i}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i \quad (i \leq n)$$

$$I_i(t) = I_i(t_0), \quad \theta_i(t) = \theta_i(t_0) + \omega_i \cdot t \bmod 2\pi,$$

for $\omega_i := \frac{\partial \widetilde{H}}{\partial I_i}$, which are referred to as frequencies. Here the symplectic form becomes $\sum_i \mathrm{d}I_i \wedge \mathrm{d}\theta_i$.

(iii) the equations of motion can be solved by quadratures.

Definition 3.20 (Liouville Torus / (non-)resonance).

(i) The torus described in Liouville-Arnold theorem, (ii), is called the *Liouville torus.*

(ii) Consider a Liouville torus T from a Hamiltonian system. We use the notation from Liouville-Arnold theorem, (ii). If there exist $\{k_i \in \mathbb{Z}\}_{i \leq n}$ such that $\sum_i k_i \omega_i \bmod 2\pi = 0$, for k_i not all 0, then we call $(\omega_i; i \leq n)$ and T *resonant*, otherwise we call $(\omega_i; i \leq n)$ and T *non-resonant.*

Example 3.21 ([13]).

Consider the Hamiltonian corresponding to a one dimensional harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2},$$

with frequency ω and mass m. It is completely integrable, as it has only one degree of freedom. We set $E = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} \implies 1 = \frac{p^2}{2mE} + \frac{m\omega^2 q^2}{2E}$, from which we see that the trajectories though phase space from ellipses. Consider the change of variables

$$q = \sqrt{\frac{2I}{m\omega}}\sin(\theta) \quad \text{and} \quad p = \sqrt{2Im\omega}\cos(\theta).$$

Substituting this change of variables for H yields $\widetilde{H} = I\omega$. Then for the Hamilton equations for $\widetilde{H}$ it clearly follows that $\frac{\mathrm{d}}{\mathrm{d}t}I = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}\theta = I$. We also see that the frequency ω is resonant, if $\omega \in \mathbb{Q}$ and non-resonant otherwise.

Proposition 3.22.

Let T be a Liouville torus. If T is resonant, the flow on it will be periodic. If T is non-resonant, the flow is non-periodic.

Remark 3.23.

The Liouville-Arnold theorem tells us that completely integrable systems behave well. Their motion through phase space is confined to some surface and cannot traverse all of phase space. In the case the surface is a Liouville torus, there orbits can only go though a finite amount of phase space and the motion will be similar at all times due to this, a small change of initial conditions cannot lead to a big change in the final conditions of the system. Thus completely integrable Hamiltonian systems are non-chaotic.

3.1 Non-integrability

We shall now discuss some results on not completely integrable systems. This section is based on [14, 15]. In honour of Oscar II, King of Norway's 60th birthday in 1889 a scientific competition was held, with four prize problems, one of which was on the study of the behaviour of a system of arbitrarily many bodies attracting each other according to Newton's law, and in particular the study of the stability of the solar system. Henri Poincaré (1854 - 1912) was the prize winner, for his results on the n body problem. In his prize winning paper he worked on Hamiltonian perturbation theory and the non-integrability of Hamiltonian systems. After this he kept working on elaborating on the ideas he laid out in his prize winning paper.

Poincaré did this by considering perturbed n-degrees of freedom Hamiltonians of the form

$$H(I_i.\,\theta_i,\,\epsilon;\,i \leq n) = H_0(I_i;\,i \leq n) + \epsilon f(I_i,\,\theta_i,) \quad (i \leq n),$$

where H_0 is completely integrable and in action-angle coordinates and where f is some analytic function, and ϵ controls the size of the perturbation.

Using this the following result was proved:

Proposition 3.24.

The completely integrable Hamiltonian systems with ≥ 2 degrees of freedom form a set of measure 0 is the space of Hamiltonian systems.

This result might seem to indicate that most Hamiltonian systems might display some chaotic motion, as the lack of constants of motion, indicates that there would not be invariant surfaces, thus the motion would not be constrained to only one part of phase space. Historically this idea of non-integrability implying chaos in some sense was taken to far, and untrue statements were proved. This idea of Hamiltonian systems being chaotic was desired, for its use in statistical mechanics [14]. We shall now discuss the results, which showed, along with the back up of numerical simulations, that this is not the case, which in some sense caused a paradigm shift.

Definition 3.25 ((non-)Degenerate Hamiltonian).

Let H be a completely integrable system with action-angle coordinates $(I_1,\,\ldots,\,I_n;\,\theta_1,\,\ldots,\,\theta_n) \in D \times \mathbb{T}^n$, then $H(I_i;\,i \leq n)$ is *non-resonant* if the hessian matrix of $H(I_i;\,i \leq n)$ has non-zero determinant for each point in D. Otherwise we call H degenerate.

Proposition 3.26.

For a non-degenerate Hamiltonian system, the set of non-resonant Liouville tori form a set of full measure and are everywhere dense. The set of resonant tori from a dense set of measure 0.

Theorem 3.27 (KAM-theorem).

Let (H_0, M) be a completely integrable non-degenerate Hamiltonian system, with $n \geq 2$ degrees of freedom, with action-angle coordinates $(I_1, \dots, I_n; \theta_1, \dots, \theta_n)$. Consider the Hamiltonian system

$$H(I_i, \theta_i, \epsilon; i \leq n) = H_0(I_i; i \leq n) + \epsilon f(I_i, \theta_i, \epsilon; i \leq n),$$

where f is smooth and $\epsilon \in \mathbb{R}$, then for some sufficiently small value for ϵ, the perturbed Hamiltonian H has a non-empty set S of n-dimensional tori in it's phase space on which the flow of H is quasi periodic. Furthermore as $\epsilon \to 0$, the measure of S becomes full.

Remark 3.28.

The following result can be interpreted as follows. For small enough perturbations, many of a non-degenerate Hamiltonian system's non-resonant tori are only slightly deformed, whereas the resonant tori, and some close to resonant tori break up chaotically. The slightly deformed non-resonant tori not breaking up entails that if an orbit in phase space starts on the surface of the torus, then it remains there, so these orbits remain stable.

Example 3.29.

Below is depicted the phase portraits of a kicked rotor for increasing kicking strengths. The increased kicking strengths can be interpreted as increased perturbations. As can then be seen, in most of the Poincaré sections there are still regions of stability.

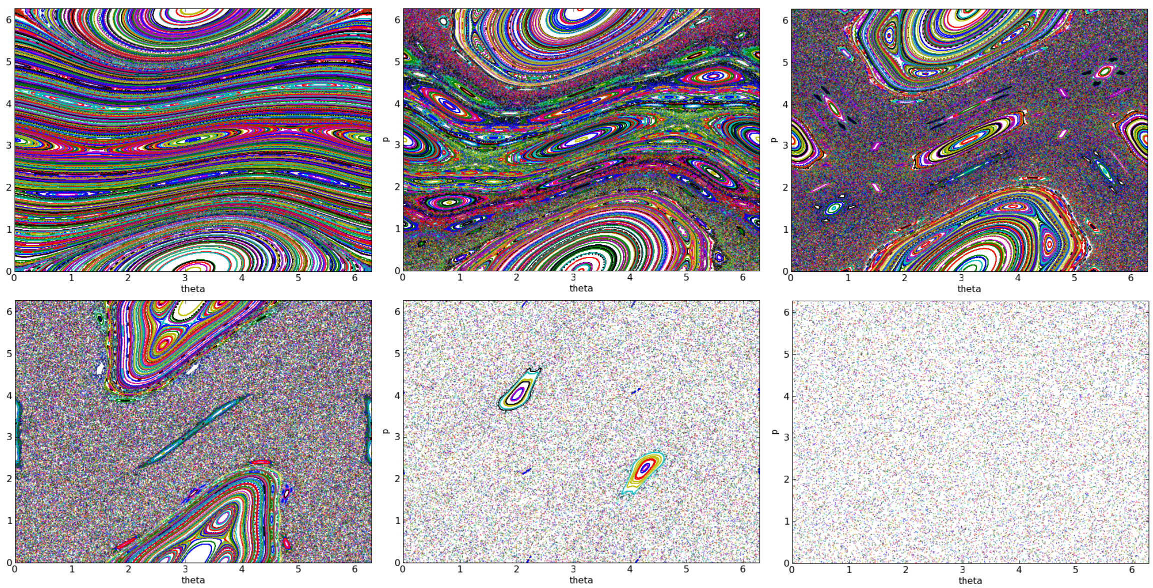


Figure 3: Image from [16].

The KAM-theorem gives that certain orbits remain eternally stable under small enough perturbations, however, the following result gives a similar result for non-eternally stable orbits:

Definition 3.30 (Steep function).

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be an analytic function, then f is *steep* if it has no stationary points and its restriction to any plane has only isolated stationary points.

Theorem 3.31 (Nekhoroshev theorem).

Let (H_0, M) be a completely integrable non-degenerate steep Hamiltonian system, with $n \geq 2$ degrees of freedom, with action-angle coordinates $(I_1, \ldots, I_n; \theta_1, \ldots, \theta_n)$. Consider the Hamiltonian system

$$H(I_i, \theta_i, \epsilon; i \leq n) = H_0(I_i; i \leq n) + \epsilon f(I_i, \theta_i, \epsilon; i \leq n),$$

where f is smooth and $\epsilon \in \mathbb{R}^+$, then there exist $a, b, c, d \in \mathbb{R}$ such that for sufficiently small ϵ, we have that

$$\|(I_i; i \leq n)(t) - (I_i; i \leq n)(0)\| \leq a\epsilon^b \text{ for } |t| \leq c \exp(\epsilon^{-a}).$$

Remark 3.32.

In a sense the above theorem can be interpreted as follows: For the KAM-theorem we saw that for small perturbations there are still eternally stable orbits. The Nekhoroshev theorem proves something similar, but for only finite stability. However, the theorem is more general is the sense that it holds for all initial conditions, whereas the KAM theorem gives eternal stability only for initial conditions coinciding with one of the deformed non-resonant tori.

Remark 3.33.

The non-degeneracy and steepness conditions from the KAM and Nekhoroshev theorems are required to assure certain values remain sufficiently bounded in their respective proofs.

What conclusions can be drawn from these theorems? How readily can they be applied to Physics? For the KAM-theorem the regime for sufficiently small ϵ in the proof is disappointingly small. Famous mathematician and astronomer Michel Hénon calculated how small the perturbation should be for the restricted three body problem, taking the ratio between the two non-zero masses to be the perturbation. He obtained two estimated perturbation strengths [14], from two different proofs: $\epsilon = 10^{-333}$ and $\epsilon = 10^{-48}$, both to small to be applicable, however since then work has been done to make the estimate more generous (when the theorem was proved, the concentration was on showing ϵ to be positive). However from numerical experiments, it has been shown that the stable orbits remain for very strong perturbations. The estimate on the perturbation from the Nekhoroshev theorem are more generous, and the theorem has seen some direct application in celestial mechanics and in rigid body dynamics for example. The general thing we should take away is that completely integrable systems are not chaotic, and that non-integral systems may display some degree of chaos, but can also still permit regular motion.

3.2 Two degree of freedom Hamiltonians

The simplest possible non-integrable systems are systems with two degrees of freedom. We shall now discuss two such systems which admit chaotic motion.

3.2.1 The Hénon-Heiles Hamiltonian

Michel Hénon and astrophysicist Carl Heiles, who were interested in the question of whether a star moving in a weakly disturbed cylindrical potential could have a third constant of motion. As previous numerical evidence seemed to indicate this. They chose the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3$$

The Hamiltonian is supposed to model the motion of stars around a galactic centre, when restricted to a plane. It was not necessarily chosen for it's realism, but for it's simplicity, which allowed for easy numerical investigation. The chosen potential is triangular:

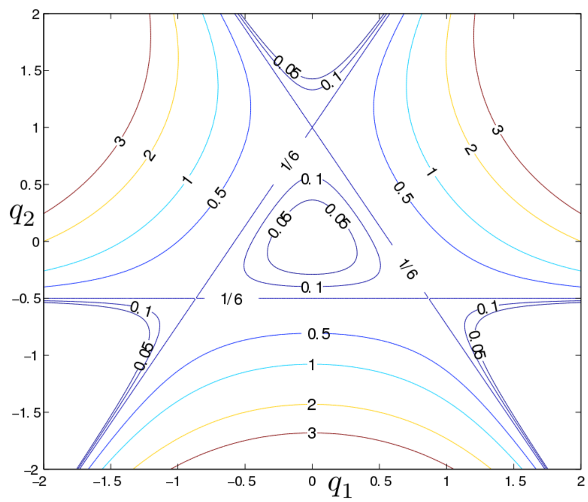


Figure 4: Image from [17].

The potential was chosen to be constant along the lines $x - \frac{y-1}{\sqrt{3}} = 0$, $x + \frac{y-1}{\sqrt{3}} = 0$ and $y + 1/2 = 0$ [18], yielding a potential

$$U(x, y) = (y + 1/2)\left(x - \frac{y-1}{\sqrt{3}}\right)\left(x + \frac{y-1}{\sqrt{3}}\right) = \frac{1}{2}(x^2 + y^2)x^2 y - y^3/3 - 1/6.$$

Here the constant $-1/6$ is left out, setting the local minimum of the potential to $1/6$. In spite of derivation being somewhat ad-hoc, for a Hamiltonian supposedly modelling reality. It was later found, however, that the potential of the system gives the approximation of a gravitational potential of a black hole with a halo [19].

It was shown numerically that at low energies the motion is non-chaotic, but that at higher energies there are areas in phase space displaying chaotic motion.

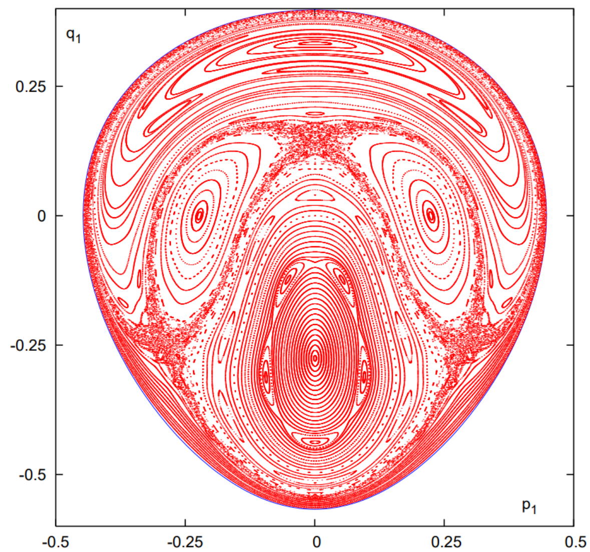


Figure 5: Image from [20]. We can clearly see that here the Hénon-Heiles system has chaotic and non-chaotic area's in it's phase space.

From the above image we suspect that the Hénon-Heiles system is not completely integrable. This was later proved to indeed be the case. We shall later study the generalised Hénon-Heiles Hamiltonian, which introduced 4 parameters into the original Hamiltonian, given below, via Painlevé analysis.

$$H = \frac{1}{2}(p_x + p_y) + \frac{1}{2}(ax^2 + by^2) + dx^2y - \frac{e}{3}y^3.$$

3.2.2 Planar spring pendulum

Another two-degrees of freedom system, showing chaotic behaviour is that of the planar spring pendulum. In Cartesian coordinates it has the Hamiltonian [21]:

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}\left[(x+1)^2 + y^2\right] - (1-a)\sqrt{(x+1) + y^2} - a(x+1) \quad a := \frac{mg}{mg + kl_0},$$

where x is the vertical coordinate, starting at the rest point of the mass, the spring has a spring constant k, a rest length l_0 and the pendulum has a mass m. Changing to circular coordinates $x + 1 = r\cos(\theta)$ and $r\sin(\theta)$ yields

$$H = \frac{1}{2}(p_r^2 + p_\theta^2/r^2) + r^2/2 + r[(a-1) - a\cos(\theta)].$$

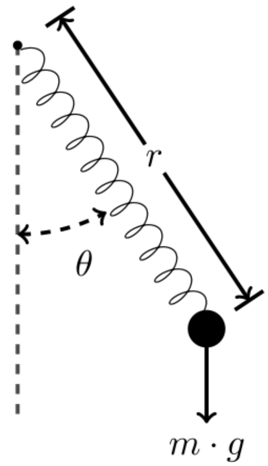


Figure 6: The spring pendulum in circular coordinates. Image from [22].

Below we see a Poincaré section for the spring-pendulum, which signals that it is probably not completely integrable. We shall prove this in the next section using Ziglin-Morales-Ramis theory.

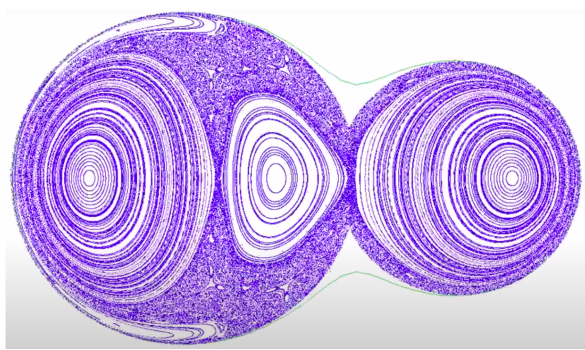


Figure 7: Image from [23].

The spring pendulum has been analysed as a model for atmospheric balance [24], and has, according to [25] been used as a classical analogue for the quantum phenomenon of Fermi resonance in the infra-red spectrum of carbon dioxide.

4 Ziglin-Ramis-Morales theory

In 1982 S. L. Ziglin establishes an effective criterion for the non-integrability of Hamiltonian systems, via the monodromy group of the linearised system. At the turn of the century the connection to differential Galois theory was made, resulting in what is now called Ziglin-Morales-Ramis theory. This will take the form of a criterion for the complete integrability of a Hamiltonian system. Using this new theory, many results of non-integrability, which used Ziglin's theory, were able to be simplified. In this section we will introduce this theory and apply it to some examples. This criterion will be based on what the people who developed this theory call the "guiding principle".

Guiding principle: *Assume a Hamiltonian system is "integrable", then we expect the linearised system to also be "integrable".*

What "integrable" exactly means here exactly is left open.

The following theorem establishes that the "guiding principle" is true, and for what sense of integrability.

Definition 4.1 (Meromorphic function).

Let f be a complex function, which is holomorphic everywhere, except at a set of isolated points. Then we call f *meromorphic*.

For a n degrees of freedom the linearised system of a Hamiltonian $H(q_i, p_i;\, i \leq n)$ along a solution $\phi(t)$ is given by

$$\begin{bmatrix} \dot{\xi}_1 \\ \vdots \\ \dot{\xi}_n \\ \dot{\xi}_{n+1} \\ \vdots \\ \dot{\xi}_{2n} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 H}{\partial p_1 \partial q_1}\big(\phi(t)\big) & \frac{\partial^2 H}{\partial p_1 \partial q_2}\big(\phi(t)\big) & \dots & \frac{\partial^2 H}{\partial p_1 \partial q_n}\big(\phi(t)\big) & \frac{\partial^2 H}{\partial p_1^2}\big(\phi(t)\big) & \frac{\partial^2 H}{\partial p_1 \partial p_2}\big(\phi(t)\big) & \dots & \frac{\partial^2 H}{\partial p_1 \partial p_n}\big(\phi(t)\big) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial^2 H}{\partial p_n \partial q_1}\big(\phi(t)\big) & \frac{\partial^2 H}{\partial p_n \partial q_2}\big(\phi(t)\big) & \dots & \frac{\partial^2 H}{\partial p_n \partial q_n}\big(\phi(t)\big) & \frac{\partial^2 H}{\partial p_n \partial p_1}\big(\phi(t)\big) & \frac{\partial^2 H}{\partial p_n \partial p_2}\big(\phi(t)\big) & \dots & \frac{\partial^2 H}{\partial p_n^2}\big(\phi(t)\big) \\ \frac{-\partial^2 H}{\partial q_1^2}\big(\phi(t)\big) & \frac{-\partial^2 H}{\partial q_1 \partial q_2}\big(\phi(t)\big) & \dots & \frac{-\partial^2 H}{\partial q_1 \partial q_n}\big(\phi(t)\big) & \frac{-\partial^2 H}{\partial q_1 \partial p_1}\big(\phi(t)\big) & \frac{-\partial^2 H}{\partial q_1 \partial p_2}\big(\phi(t)\big) & \dots & \frac{-\partial^2 H}{\partial q_1 \partial p_n}\big(\phi(t)\big) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{-\partial^2 H}{\partial q_n \partial q_1}\big(\phi(t)\big) & \frac{-\partial^2 H}{\partial q_n \partial q_2}\big(\phi(t)\big) & \dots & \frac{-\partial^2 H}{\partial q_n^2}\big(\phi(t)\big) & -\frac{\partial^2 H}{\partial q_n \partial p_1}\big(\phi(t)\big) & -\frac{\partial^2 H}{\partial q_n \partial p_2}\big(\phi(t)\big) & \dots & \frac{-\partial \partial H}{\partial q_n \partial p_n}\big(\phi(t)\big) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \\ \xi_{n+1} \\ \vdots \\ \xi_{2n} \end{bmatrix}$$

Definition 4.2 (Reduced linearised system).

Let $\dot{\boldsymbol{\xi}} = A\boldsymbol{\xi}$ be a linearised Hamiltonian system with n degrees of freedom, then this system may be reduced to the form

$$\dot{\boldsymbol{\eta}} = \begin{bmatrix} 0 & I_{m\times m} \\ -I_{m\times m} & 0 \end{bmatrix} S(t)\boldsymbol{\eta},$$

where $2m = n - 2$. We shall call this the *reduced linearised system*.

Theorem 4.3 (Ziglin-Ramis-Morales).

Let $(\mathbb{C}^{2n}, \omega, H)$ be a Hamiltonian system, then if $(\mathbb{C}^{2n}, \omega, H)$ is completely integrable by respectively meromorphic / rational first integrals in the neighbourhood of a particular, non-stationary solution $\phi(t)$, such that the reduced linear system has respectively regular / irregular singularities at infinity, we have that the differential Galois groups with base field $\mathbb{C}(t)$ of the reduced linearised system has an abelain identity component.

Remark 4.4.

In general the previous theorem can be extended from the case of the complete integrability by rational first integrals to complete integrability by meromorphic first integrals, by working over the base field $\mathcal{M}(\Gamma)$, defined as the field meromorphic functions over the Riemann surface Γ defined as the maximal analytic continuation of our particular solution $\phi(t)$.

Proposition 4.5.

Let $y'' + p(t)y' + q(t)y = 0$ be a differential equation, then t_0 is a *regular singular point* if in the limit $t \to t_0$, neither $(t - t_0)q(t)$ or $(t - t_0)^2 q(t)$ diverges. Note that ∞ may also be a regular singular point, which we test by first applying the substitution $w = 1/t$ and then checking $\lim_{w \to 0}$.

Remark 4.6.

The Kovacic's algorithm is very useful for the application of Theorem 4.3, as Hamiltonian systems are systems of second order differential equations, and non-solvability is stronger than being non-abelian.

Example 4.7.

We shall now use Ziglin-Ramis-Morales theory to prove the Hamiltonian $H = \frac{p_1^2+p_2^2}{2} + q_1q_2^2$, is non-integrable. The Hamilton equations of motion are given by

$$\dot{q}_1 = p_1, \quad \dot{q}_2 = p_2$$

$$\dot{p}_1 = q_2^2, \quad \dot{p}_2 = 2q_1q_2.$$

Now as our particular solution we pick $q_2 = p_2 = 0 \implies \dot{p}_1 = 0$ and $\ddot{q}_1 = 0 \implies p_1 = a$ and $q_1 = at + b$, for $a, b \in \mathbb{C}$. The corresponding linearised system is then

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 2(at+b) & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix}.$$

We now see that the reduced linearised system is a second order differential equation of the form $2(at + b)\xi_2 = \dot{\xi}_4 = \ddot{\xi}_2$, which for $a \neq 0$ is clearly similar to Example 2.56, this equation is not integrable, and has a Galois group $\mathrm{SL}_2(\mathbb{C})$, which is not abelian, then Theorem 4.3 gives that the above Hamiltonian is not completely integrable by rational first integrals. The above is an example of a Hamiltonian corresponding to the Hénon-Heiles generalised system with all parameters set to 0.

Remark 4.8.

One might wonder if the abelian identity components of the differential Galois groups we might encounter in general could be classified. Often we will choose a particular solution in such a way that we work over $\mathbb{C}(t)$ as a base field, and such that the reduced linear system becomes a homogenous linear differential equation. We assume this is the case. Assume the identity component of a differential Galois group is abelian, then the identity component is solvable, thus by the Lie-Kolchin theorem we may assume the identity component is upper triangular. Since we assumed to be working with a differential equation of order 2, the identity component will be a matrix group of dimension 2. The identity component can be written as a series of group extensions, corresponding to algebraic, integral and exponential integral extension. These group extensions will all be semi-direct products. A semi-direct product is only abelian if the product is trivial and both groups in the product are abelian. Then the differential Galois groups (not necessarily only the identity components) we encounter will consist of products of finite groups, unipotent or diagonal groups. In this case, where we are dealing with matrix groups of dimension two, the groups in the chain will be finite groups, $\mathbb{G}_a$ or $\mathbb{G}_m$. The direct product of two of these matrix groups has dimension 4, and this would not correspond to a differential equation of degree 2. The direct product of a finite group with one of these matrix groups has identity component equal to the identity component of the matrix group. We thus find that the only abelian identity groups one will encounter this way are

$$\{e\},\ \mathbb{G}_a \text{ and } \mathbb{G}_m.$$

This can be verified somewhat, by noting that the only abelian identity components occurring in the list of Proposition 2.54 (cases 1 through 5) are of these forms.

For certain equations the abelianity of the identity of the differential Galois group has been characterised. This has for example been done for the Riemann-Papperitz equation, the Lamé equation, the confluent hypergeometric equation [5].

We shall discuss this for the Riemann-Papperitz equation.

Definition 4.9 (Riemann-Papperitz equation).

Consider the equation of the form:

$$y'' + \left[\frac{1-\alpha-\widetilde{\alpha}}{z} + \frac{1-\beta-\widetilde{\beta}}{z-1}\right] + \left[\frac{\alpha\beta}{z^2} + \frac{\widetilde{\alpha}\widetilde{\beta}}{(z-1)^2} + \frac{\gamma\widetilde{\gamma}-\alpha\widetilde{\alpha}-\beta\widetilde{\beta}}{z(z-1)}\right] y = 0, \tag{1}$$

for $\alpha+\widetilde{\alpha}+\beta+\widetilde{\beta}+\gamma+\widetilde{\gamma}=1$ and $\alpha,\widetilde{\alpha},\beta,\widetilde{\beta},\gamma,\widetilde{\gamma}\in\mathbb{C}$. This is known as the *Riemann-Papperitz equation*, with singularities at $0,1$ and ∞.

Proposition 4.10 ([26]).

Let $y'' + a(t)y' + b(t)y = 0$ be a complex differential equation, with $a(t),\, b(t) \in \mathbb{C}(t)$, then if this equation has three regular singular points at $0, 1, \infty$, then it can be transformed into (1) for certain parameters which are determined as follows:
Let $c_s \coloneqq \lim_{t\to s}(t-s)a(t)$ and $d_s \coloneqq \lim_{t\to s}(t-s)^2 b(t)$, for $s \in \{0,\, 1,\, \infty\}$. Note that for $s = \infty$, we first make a change of variables $t \to 1/w$ and then go on a prescribed checking for a singularity at 0. If any of the limits diverge, we are dealing with a singular point which is not regular. Then we find the roots of $r(r-1) + c_s r + d_s = 0$, called the indicial equation, and denote these as r_0^s and r_1^s. Then using the notation from (1), we obtain that the differential equation is equivalent to (1) with parameters

$$\alpha = r_0^0,\ \widetilde{\alpha} = r_1^0,\ \beta = r_1^1,\ \widetilde{\beta} = r_1^1,\ \gamma = r_0^\infty,\ \widetilde{\gamma} r_1^\infty.$$

Proposition 4.11 ([5]).

The transformation of a differential equation to the Riemann-Papperitz equation as outlined in the previous proposition, preserves the identity component of the differential Galois group.

The following theorem shows how the discussed transformation gives us useful results for non-integrability.

Theorem 4.12 (Kimura's theorem, [5]).

The identity component of the Riemann-Papperitz equation with singularities at $0, 1, \infty$ is solvable $\iff$ One of the following is true, for $\widehat{\lambda} \coloneqq \alpha - \widetilde{\alpha}$, $\widehat{\mu} \coloneqq \gamma - \widetilde{\gamma}$, $\widehat{\nu} \coloneqq \beta - \widehat{\beta}$.

(i) $\widehat{\lambda} + \widehat{\mu} + \widehat{\nu}$ is an odd integer.

(ii) $-\widehat{\lambda} + \widehat{\mu} + \widehat{\nu}$ is an odd integer.

(iii) $\widehat{\lambda} - \widehat{\mu} + \widehat{\nu}$ is an odd integer.

(iv) $\widehat{\lambda} + \widehat{\mu} - \widehat{\nu}$ is an odd integer.

(v) ($\widehat{\lambda}$ or $-\widehat{\lambda}$) and ($\widehat{\mu}$ or $-\widehat{\mu}$) and ($\widehat{\nu}$ or $-\widehat{\nu}$), belong to one of the fifteen families denoted in Table 1.

1	$1/2+l$	$1/2+m$	z	-
2	$1/2+l$	$1/3+m$	$1/3+q$	-
3	$2/3+l$	$1/3+m$	$1/3+q$	$2 \mid (l+m+q)$
4	$1/2+l$	$1/3+m$	$1/4+q$	-
5	$2/3+l$	$1/4+m$	$1/4+q$	$2 \mid (l+m+q)$
6	$1/2+l$	$1/3+m$	$1/5+q$	-
7	$2/5+l$	$1/3+m$	$1/3+q$	$2 \mid (l+m+q)$
8	$2/3+l$	$1/5+m$	$1/5+q$	$2 \mid (l+m+q)$
9	$1/2+l$	$2/5+m$	$1/5+q$	$2 \mid (l+m+q)$
10	$3/5+l$	$1/3+m$	$1/5+q$	$2 \mid (l+m+q)$
11	$2/5+l$	$2/5+m$	$2/5+q$	$2 \mid (l+m+q)$
12	$2/3+l$	$1/3+m$	$1/5+q$	$2 \mid (l+m+q)$
13	$4/5+l$	$1/5+m$	$1/5+q$	$2 \mid (l+m+q)$
14	$1/2+l$	$2/5+m$	$1/3+q$	$2 \mid (l+m+q)$
15	$3/5+l$	$2/5+m$	$1/3+q$	$2 \mid (l+m+q)$

Table 1: Here $l, m, q \in \mathbb{Z}$ and $z \in \mathbb{C}$.

4.1 Non-integrability of the spring-pendulum

Now we tackle the spring-pendulum, following [27, 28]. Recall that its Hamiltonian is given by

$$H = \frac{1}{2}(p_r^2 + p_\theta^2/r^2) + r^2/2 + r[(a-1) - a\cos(\theta)],$$

with equations of motion

$$\dot{r} = p_r,\ \dot{p}_r = \frac{p_\theta^2}{r^3} - r - (a-1) + a\cos(\theta),$$
$$\dot{\theta} = p_\theta/r^2,\ \dot{p}_\theta = -ra\sin(\theta).$$

We pick as our particular solution $\theta = p_\theta = 0$. We then obtain the relation $\ddot{r} = \dot{p}_r = 1 - r$. By choosing $H = E = 0$, we also obtain $\dot{r}^2 = p_r^2 = r(2-r)$. We linearise the system to obtain:

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/r^2 \\ -1 & 0 & 0 & 0 \\ 0 & -ar & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix}.$$

Thus we obtain

$$\xi_2'' = (\xi_4/r^2)' = -2\frac{r'}{r^3}\xi_4 + \frac{1}{r^2}\xi_4' = -2\frac{r'}{r}\xi_2' - \frac{a}{r}\xi_2,$$

which we rewrite as

$$y'' + 2\frac{r'}{r}y' + \frac{a}{r}y = 0, \tag{2}$$

as our reduced linearised system. We now apply the transform $t \mapsto z = r(t)/2$. Note that then

$$\frac{\mathrm{d}y(z)}{\mathrm{d}t} = \frac{\mathrm{d}y(z)}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}t} = y' \cdot \frac{\dot{r}}{2}$$

and

$$\frac{\mathrm{d}^2y(z)}{\mathrm{d}t^2} = \frac{\mathrm{d}^2}{\mathrm{d}t}\left(y' \cdot \frac{\dot{r}}{2}\right) = y''\frac{\dot{r}^2}{4} + \frac{\ddot{r}}{2}y',$$

where we let $\cdot$ denote a derivation w.r.t. t and $'$ w.r.t. z. Substituting these identities in (2) we then obtain

$$\frac{\dot{r}^2}{4}y'' + \left(\frac{\ddot{r}}{2} + 2\frac{\dot{r}}{r} \cdot \frac{\dot{r}}{2}\right)y' + \frac{a}{r}y = 0.$$

Substituting $\ddot{r} = 1 - r$ and $\dot{r}^2 = r(2-r)$ gives

$$\begin{aligned}
&\frac{r(2-r)}{4}y'' + \left(\frac{1-3r}{2} + 2\right)y' + \frac{a}{r}y = 0 \\
\implies\ & y'' + \frac{10-6r}{r(2-r)}y' + \frac{4a}{r^2(2-r)}y = 0 \\
\implies\ & y'' + \frac{5-6z}{2z(1-z)}y' + \frac{a}{2z^2(1-z)}y = 0.
\end{aligned}$$

We now try to transform this equation to the form of (1).

$$\lim_{z\to 0} z\frac{5-6z}{2z(1-z)} = \lim_{z\to 0}\frac{5-6z}{2(1-z)} = 5/2,$$

$$\lim_{z\to 0} z^2\frac{a}{2z^2(1-z)} = \lim_{z\to 0}\frac{a}{2(1-z)} = a/2,$$

for which the indicial equation has roots $r^0_{0,1} = -3/4 \pm \frac{1}{4}\sqrt{9-8a}$.

$$\lim_{z\to 1}(z-1)\frac{5-6z}{2z(1-z)} = \lim_{z\to 1}\frac{6z-5}{2z} = 1/2,$$

$$\lim_{z\to 1}(z-1)^2\frac{a}{2z^2(1-z)} = \lim_{z\to 1}\frac{a(z-1)}{2z^2} = 0,$$

for which the indicial equation has roots $r^1_0 = 1/2$ and $r^1_1 = 0$. To handle ∞ we make the substitution $w = 1/z$, this gives us

$$\frac{\mathrm{d}^2y}{\mathrm{d}w^2} + \frac{2-w}{2w(w-1)}y'\frac{\mathrm{d}y}{\mathrm{d}w} + \frac{1}{2w(w-1)}y = 0.$$

Then

$$\lim_{w\to 0} w\frac{2-w}{2w(w-1)} = \lim_{w\to 0}\frac{2-w}{2(w-1)} = -1,$$

$$\lim_{w\to 0} w^2\frac{1}{2w(w-1)} = \lim_{w\to 0}\frac{w}{2(w-1)} = 0,$$

for which the indicial equation has roots $r_0^\infty = 2$ and $r_1^\infty = 0$. Thus we find the Riemann-Papperitz equation

$$y'' + \left[\frac{5/2}{z} + \frac{1/2}{z-1}\right] y' + \left[\frac{a/2}{z^2} - \frac{a/2}{z(z-1)}\right] y = 0.$$

We now apply Kimura's theorem for $\widehat{\lambda} = \frac{1}{2}\sqrt{9-8a}$, $\widehat{\mu} = \frac{1}{2}$, $\widehat{\nu} = 2$. From here it can then be found that the planar spring pendulum system is not completely integrable by meromorphic first integrals unless $a = \frac{1}{2}(2 - b(b+1))$ for b an integer [28]. We now use that $a \in [0,1]$, for the system to be physical, we then find that the system can only be integrable if $a \in \{0,1\}$. For the case $a = 1$, recall that the planar spring pendulum in Cartesian coordinates is

$$H = \frac{1}{2}(p_x + p_y) + \frac{1}{2}\left[(x+1)^2 + y^2\right] - (1-a)\sqrt{(x+1)^2 + y^2} - a(x+1),$$

which splits for $a = 0$, and is thus completely integrable. For the case $a = 0$, recall that $a \coloneqq \frac{mg}{mg+kl_0}$, we can then deal with this case as follows. Note that $\lim_{k\to\infty} a = 0$, if $k = \infty$, then the spring becomes just a rigid rod, and we obtain just a planar-pendulum, which has one degree of freedom, and is thus completely integrable by meromorphic first integrals.

Thus we have fully characterised the integrability of the system.

Note that choosing the right initial condition for a three dimensional spring-pendulum, i.e. giving it no angular momentum, results in only planar movement, thus the above argument can be extended to characterize the integrability of the three dimensional spring pendulum.

5 The Painlevé equations and property

Paul Painlevé (1863 - 1933) was interested in defining new functions. Many new functions are defined as a solution to a given differential equation. Notable examples are the exponential, hypergeometric and elliptic functions. This led Painlevé to formulate and investigate functions as solutions of differential equations satisfying the Pianlevé property. In this section we shall discuss the how the Painlevé equations were found, their general properties as well as how the Painlevé property relates to integrability. This section is based on [29, 30, 31, 32, 33].

Definition 5.1 (Movable singularity / Painlevé property).

Let

$$y^{(n)} + F(y^{(i)};\, i \leq n-1) = 0 \tag{3}$$

be an ODE, for F a polynomial with coefficients in $\mathbb{C}(t)$.

Let f be the general solution of the (3), which has singularities at coordinates dependent on undetermined constants of integration, then f has *movable singularities.*

If f has no movable critical singularities, i.e. singularities at which multivaluedness occurs, then f has the *Painlevé property.*

Example 5.2.

Consider the differential equation

$$y'' + (y')^2 = 0.$$

It has as a solution $\ln(z+c)$ for c a constant of integration. The complex logarithm has a critical singularity at the origin, thus the equation has a movable critical singularity

Painlevé studied second order equations having the Painlevé property, for details see [30]. This eventually led to the discovery of the six Painlevé equations, which we shall further discuss in this section.

Definition 5.3 (The Painlevé equations).

The following are the six *Painlevé equations*:

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 6y^2 + t \tag{$\mathrm{P_I}$}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 2y^3 + ty + \alpha \tag{$\mathrm{P_{II}}$}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}y}{dx} + \frac{\alpha y^2+\beta}{t} + \gamma y^3 + \frac{\delta}{y} \tag{$\mathrm{P_{III}}$}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{1}{2y}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \frac{\beta}{y} \tag{$\mathrm{P_{IV}}$}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1} \tag{$\mathrm{P_V}$}$$

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} + \\ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{y} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\right)\end{aligned} \tag{$\mathrm{P_{VI}}$}$$

Remark 5.4.

All of the Painlevé equations have a singularity at ∞. Equations $\mathrm{P_{III}}$, $\mathrm{P_V}$ and $\mathrm{P_{VI}}$ have 0 as a singularity, and $\mathrm{P_{VI}}$ has 1 as a singularity. All the equations may have movable poles, and have no other singularities.

5.1 Symmetry in the Painlevé equations

We shall consider the symmetry properties of $\mathrm{P_{IV}}$ in detail, following the discussion from [29], and then state the analogue of these properties for the other Painlevé equations. To start we consider the system of equations below:

$$\begin{cases} f_0' &= f_0(f_1 - f_2) + \alpha_0, \\ f_1' &= f_1(f_2 - f_0) + \alpha_1, \\ f_2' &= f_2(f_0 - f_1) + \alpha_2. \end{cases} \tag{4}$$

We note that under the transform $t \mapsto t/c$, $f_j \mapsto cf_j$ and $a_j \mapsto c^2a_j$, for $c \neq 0$ a constant, the system (4) is invariant, as under this transform $\frac{\mathrm{d}f_j}{\mathrm{d}t}$ gets sent to $\frac{\mathrm{d}c\cdot f_j(t/c)}{\mathrm{d}(t/c)} = c^2\frac{\mathrm{d}f_j(t/c)}{\mathrm{d}t} = c^2\frac{\mathrm{d}f_j(t)}{\mathrm{d}(t/c)}\frac{\mathrm{d}t/c}{\mathrm{d}t} = c^2f_j'$, similarly, we get a $\cdot c^2$ for the RHS of the system, thus we can divide both sides by c^2 to obtain the invariance.

Now note that

$$(f_0 + f_1 + f_2)' = \alpha_0 + \alpha_1 + \alpha_2,$$

i.e. constant. hence we obtain

$$f_0 + f_1 + f_2 = (\alpha_0 + \alpha_1 + \alpha_2)t + c.$$

Now assume $\alpha_0 + \alpha_1 + \alpha_2 \neq 0$ and $c = 0$. Since we may rescale the system by a constant, we may now assume that $\alpha_0 + \alpha_1 + \alpha_2 = 1$ and that $f_0 + f_1 + f_2 = t$.

Proposition 5.5.

The following system of equation is equivalent to $\mathrm{P_{IV}}$.

$$\begin{cases} f_0' &= f_0(f_1 - f_2) + \alpha_0, \\ f_1' &= f_1(f_2 - f_0) + \alpha_1, \\ f_2' &= f_2(f_0 - f_1) + \alpha_2, \end{cases}$$

where $\alpha_0 + \alpha_1 + \alpha_2 = 1$ and that $f_0 + f_1 + f_2 = t$.

Proof. From the relation $f_0 + f_1 + f_2 = t$ we can reduce the system of three equations to a system of two equations. Eliminating f_0 then yields

$$\begin{cases} f_1' &= f_1(f_1 + 2f_2 - t) + \alpha_1, \\ f_2' &= f_2(t - 2f_1 - f_2) + \alpha_2. \end{cases} \tag{5}$$

We use the first equation from (5) to get an expression for f_2:

$$f_2 = \left(\frac{f_1' - \alpha_1}{f_1} - f_1 + t \right). \tag{6}$$

Now we differentiate the first equation from (5), yielding

$$f_1'' = f_1'(2f_1 + 2f_2 - t) + f_1(2f_2 - 1).$$

In this equation we first eliminate f_2' using the second equation of (5), and then eliminate f_2 from the resulting equation using (6). Setting $f_1 := y$, we then obtain

$$y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 2y^2 t + y(t^2/2 + \alpha_1 + 2\alpha_2 - 1) - \frac{a^2}{2y}.$$

(We also apply the equality $\alpha_0 + \alpha_1 + \alpha_2 = 1$) Now we define $\alpha = \alpha_2 - \alpha_0$ and $\beta = -2\alpha_1^2$, and apply the change of variables $t \mapsto \sqrt{2}t$ and $y \mapsto y/\sqrt{2}$, from which we obtain P_{IV}. ■

Definition 5.6 (Auto-Bäcklund transformations).

Let $y^{(n)} = \sum_i^{n-1} a_i (y^{(i)})^{b_i}$ be a differential equation, then a transformation of the dependent variables, which leaves the form of the equation unchanged, but may change the parameters, is called an *auto-Bäcklund transformation*.

Example 5.7.

The system (4) has an inherent symmetry, given by $\rho(f_j) = f_{j+1}$ and $\rho(\alpha_j) = \alpha_{j+1}$, for $j \in \mathbb{Z}/3\mathbb{Z}$. Then $\rho(f_j) = f_{j+1}$ and $\rho(\alpha_j) = \alpha_{j+1}$, for $j \in \mathbb{Z}/3\mathbb{Z}$ is an auto-Bäcklund transformation.

From now on let the subscripts of α_j and f_j lie in $\mathbb{Z}/3\mathbb{Z}$.

Remark 5.8.

When viewed in its symmetric form, we can easily determine some particular solutions of P_{IV}. For example assuming $f_0 = f_1 = f_2$ and $\alpha_0 = \alpha_1 = \alpha_2$, it is not hard to verify that $(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{t}{3}, \frac{t}{3}, \frac{t}{3}\right)$ is a solution. Another particular solution can be found by choosing $f_0 = \alpha_0 = 0$, yielding $f_1 + f_2 = t$, then from (4) we obtain

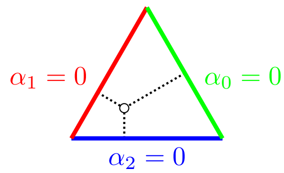
$$f_1' = f_1(f_2 - f_0) + \alpha_1 = f_1 f_2 + \alpha_1 = f_1(t - f_1) + \alpha_1.$$

Applying the transform $f_1 = \frac{u'}{u}$ we obtain $u'' - tu' - \alpha_1 u = 0$, now rescaling $t \mapsto \sqrt{2}t$ yields $u'' - 2tu' - \sqrt{2}\alpha_1 u = 0$. The above differential equation is Hermite's differential equation, who's general solution can be expressed in terms of the hypergeometric series and Hermite polynomials. Let g be such a solution, then it can easily be checked that

$$f_0 = 0,\ f_1 = \frac{g'}{g},\ f_2 = t - \frac{g'}{g},$$

is a solution to (4).

Since the parameter space for P_{IV} is two dimensional, and we have that $\alpha_0 + \alpha_1 + \alpha_2 = 1$ for P_{IV} in the form of (4), we can view the real parameter space of (P_{IV}) as an equilateral triangle with sides of length $2/\sqrt{3}$. Consider the following diagram.



Along each of the coloured lines $\alpha_j = 0$. From some basic geometry we then have that for any point the length of the dotted lines in the above image add up to 1. Note then that in this coordinate system, the centre of the triangle corresponds with the solution $(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{t}{3}, \frac{t}{3}, \frac{t}{3}\right)$, and the edges correspond to the solution in terms of solutions to Hermite's equation.

We will now define another auto-Bäcklund transformation on the system (4).

Proposition 5.9.

Take the system (4), then

$$\sigma_0 : (\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2) \mapsto \left(-\alpha_0, \alpha_1 + \alpha_0, \alpha_2 + \alpha_0; f_0, f_1 + \frac{\alpha_0}{f_0}, f_2 - \frac{\alpha_0}{f_0}\right)$$

defines an auto-Bäcklund transformation, and $\sigma_0^2 = \mathrm{id}$.

Proof. We need to show that $\sigma_0(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2)$ satisfies the system of equations (4). For the first equation we obtain

$$\sigma_0(f_0)' = f_0' = f_0(f_1 - f_2) + \alpha_0$$

$$= \sigma_0(f_0)\left(\sigma_0(f_1) - \sigma_0(f_2) - \frac{2\alpha_0}{\sigma_0(f)}\right) + \alpha_0 = \sigma_0(f_0)\left(\sigma_0(f_1) - \sigma_0(f_2)\right).$$

For the second equation we obtain

$$\sigma_0(f_1)' = \left(f_1 + \frac{\alpha_0}{f_0}\right)' = f_1' - \frac{\alpha_0}{f_0^2} = f_1(f_2 - f_0) + \alpha_1 - \frac{\alpha_0}{f_0}(f_1 - f_2) - \frac{\alpha_0^2}{f_0^2}$$

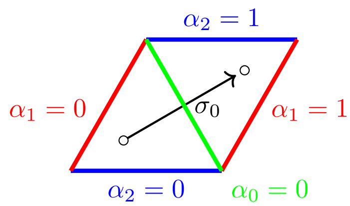
$$= \left(\sigma_0(f_1) - \frac{\alpha_0}{f_0}\right)\left(\sigma_0(f_2) - \sigma_0(f_0) + \frac{\alpha_0}{\sigma_0(f_0)}\right) + \alpha_1 - \frac{\alpha_0}{\sigma_0(f_0)}\left(\sigma_0(f_1) - \sigma_0(f_2) - 2\frac{\alpha_0}{\sigma_0(f_0)}\right) - \frac{\alpha_0^2}{\sigma_0(g_0)^2}$$

$$= \sigma_0(f_1)\left(\sigma_0(f_2) - \sigma_0(f_0)\right) + \alpha_1 + \alpha_0.$$

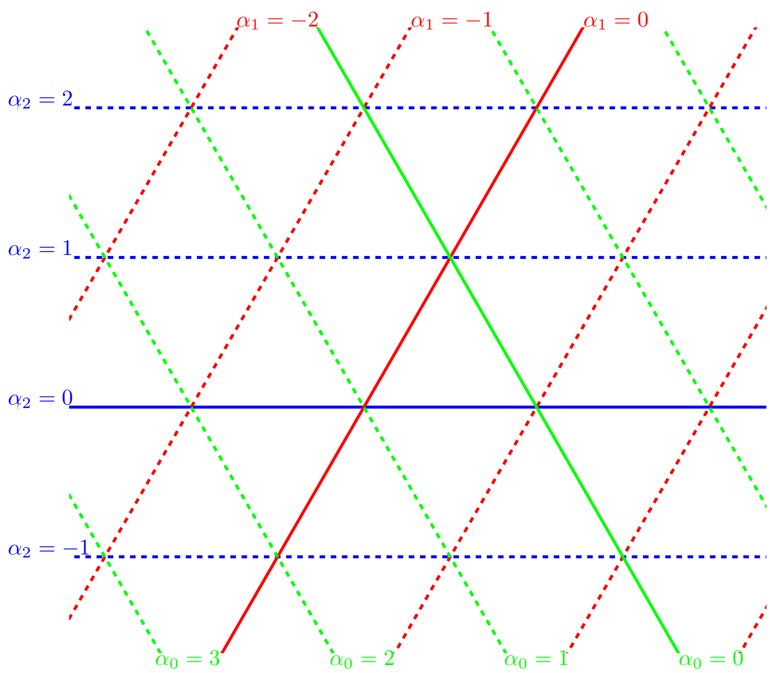
The result for the third equation in the system is analogous. Thus the transformed equations indeed satisfy the same system.
It is not hard to see that $\sigma_0^{=}\mathrm{id}$. ■

We can now define new auto-Bäcklund transforms from compositions of σ_0 and ρ. Namely let $\sigma_1 =$

$\rho\sigma_0\rho^{-1}$ and $\sigma_2 = \rho\sigma_1\rho^{-1}$. Note that these σ_j transforms, move us outside of our original triangle parameter space (for real α_j). We can visualise this with the following image:



We see that we move into a different triangle of the parameter space by the transformation, where again the centre and edges of the triangle correspond to spacial solutions, thus by applying successive auto-Bäcklund transforms we obtain the following parameter space.



Proposition 5.10.

The auto-Bäcklund transforms from Example 5.7 and Proposition 5.9 of $\mathrm{P_{IV}}$, form the group $\langle h_1, h_2, h_3;\, h_j^2 = e\rangle \ltimes \langle g;\, g^3 = e\rangle$.

Proof. It is clear via the given geometric interpretation that these transforms will be a group. Let G be the group generated by these transforms, then $G = \langle \rho, \sigma_j;\, j \le 2\rangle \cong \langle g, h_1, h_2, h_3;\, g^3 = h_j^2 = e, gh_jg^{-1} = h_{j+1}\rangle$, for $j \in \mathbb{Z}/3\mathbb{Z}$. Since $\{e, g, g^2\}$ is normal in G, we easily see that $G \cong \langle h_1, h_2, h_3;\, h_j^2 = e\rangle \ltimes \langle g;\, g^3 = e\rangle$. Note that $\langle h_1, h_2, h_3;\, h_j^2 = e\rangle \cong A_2$, the Coxeter group. The interior bounded edges where P_{IV} has a particular solution is referred to as a Weyl chamber. ■

Remark 5.11.

Note that $\sigma_i(f_j) = f_j + \frac{\alpha_i}{f_i} b_{ij}$, where $(b_{ij})_{i,j\leq 3}$ is given by the matrix

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

For P_{IV} we derived the structure of the auto-Bäcklund transformations and how these related to the parameter space of the equations. A similar result is true for the other equations, there will be auto-Bäcklund transformations which will form an extended affine Weyl groups. Again the edges and centres of the Weyl chambers will form special solutions of the equations. These special solutions along the edges will be in terms of the corresponding degeneration of the Gauss hypergeometric equation as we shall see later in Figure 9. These facts are given in Table 2.

Painlevé equations	Number of parameters	Auto-Bäcklund Group	Hypergeometric solution
P_I	0	-	-
P_{II}	1	A_1	Airy
P_{III}	2	$A_1 \oplus A_1$	Bessel
P_{IV}	2	A_2	Hermite
P_V	3	A_3	Kummer
P_{VI}	4	D_4	Gauss

Table 2: Auto-Bäcklund structure of the Painlevé equations.

5.2 Hamiltonian structure of the Painlevé equations

We shall use the previous results for P_{IV} to define a Hamiltonian structure for the system (4).

Proposition 5.12.

The following Hamiltonian system is equivalent to P_{IV}:

$$H = (t - q - p)pq + \alpha_2 p - \alpha_1 p - \alpha_1 q + \frac{1}{3}(\alpha_1 - \alpha_2)t.$$

Proof. First we note that (4) is equivalent to P_{IV}. Using the notation from Remark 5.11, let $\{f_i, f_j\} := b_{ij}$ and $\{g, h\} = \sum_{i,j=0}^{2} \frac{\partial g}{\partial f_i} b_{ij} \frac{\partial h}{\partial f_j}$. Furthermore let $H := f_0 f_1 f_2 + b_0 f_0 + b_1 f_1 + b_2 f_2$. Then we then find that

$$\{H, f_j\} = f_j(f_{j+1} - f_{j+2}) + (b_{j+2} - b_{j+1}).$$

Then, setting $b_2 - b_1 = \alpha_0 - 1$, $b_0 - b_2 = \alpha_1$ and $b_1 - b_0 = \alpha_2$, we have that the b_j satisfy $\alpha_0 + \alpha_1 + \alpha_2 = 1$ and that $b_0 + b_1 + b_2 = 0$. Then

$$b_0 = \frac{1}{3}(\alpha_1 - \alpha_2), \quad b_1 = \frac{1}{3}(\alpha_1 + 2\alpha_2), \quad b_2 = \frac{-1}{3}(2\alpha_1 + \alpha_2).$$

We then see that $f_0' = \{H, f_0\} + 1$, $f_1' = \{H, f_1\}$ and $f_2' = \{H, f_2\}$. Now we set $f_1 = p$ and $f_2 = q$, and recall that $f_0 + f_1 + f_2 = t$. from which it follows that $\{p, q\} = 1$ and $\{p, t\} = \{q, t\} = 0$. Then the Poisson bracket becomes $\{g, h\} = \frac{\partial g}{\partial p}\frac{\partial h}{\partial q} - \frac{\partial g}{\partial q}\frac{\partial h}{\partial p}$. This is the Poisson bracket in standard coordinates for a complex symplectic manifold. Now by substituting $f_0 = t - q - p$, $f_1 = p$ and $f_2 = q$ we obtain that

$$H = (t - q - p)pq + \alpha_2 p - \alpha_1 p - \alpha_1 q + \frac{1}{3}(\alpha_1 - \alpha_2)t,$$

which is then the equivalent Hamiltonian system of P_{IV}. ■

There are equivalent Painlevé Hamiltonian systems for all Painlevé equation. We denote these below.

Below we give the Painlevé equivalent Hamiltonian systems as stated in [33] for all but P_{IV}, for which we give the equivalent system as derived above.

Definition 5.13 (Hamiltonian Painlevé systems).

The *Painlevé Hamiltonian systems* are

$$H = q^2/2 - 2p^3 - tp \tag{$\mathrm{H_I}$}$$

$$H = q^2/2 - (p^2 + t/2)q - (\alpha + 1/2)p \tag{$\mathrm{H_{II}}$}$$

$$H = \frac{1}{t}\left[2q^2p^2 - \Big(2\eta_\infty tp^2 + (2\kappa_0 + 1)p - 2\eta_0 t\Big)\, q + \eta_\infty(\kappa_0 + \kappa_\infty)tp\right], \tag{$\mathrm{H_{III}}$}$$

where $(\alpha,\, \beta,\, \gamma,\, \delta) = (-4\eta_\infty\kappa_\infty,\, 4\eta_0(\kappa_0 + 1), 4\eta_\infty^2,\, -4\eta_0^2)$.

$$H = (t - q - p)pq + (\alpha_2 - \alpha_1)p - \alpha_1 q + \frac{1}{3}(\alpha_1 - \alpha_2)t, \tag{$\mathrm{H_{IV}}$}$$

where we use the notation from (4).

$$H = \frac{1}{t}\left[p(p-1)^2q^2 - \Big(\kappa_0(p-1)^2 + \kappa_t p(p-1) - \eta tp\Big)\, q + \kappa(p-1)\right], \tag{$\mathrm{H_V}$}$$

where $(\alpha,\, \beta,\, \gamma,\, \delta,\, \kappa) = \Big(\kappa_\infty^2/2,\, -\kappa_0^2,\, -\eta(1+\kappa_t),\, -\eta^2/2,\, (\kappa_0 + \kappa_t)^2/4 - k_\infty^2\Big)$.

$$\begin{aligned} H = \frac{1}{t(t-1)} \Big[& p(p-1)(p-t)q^2 - \big(\kappa_0(p-1)(p-t) + \\ & \kappa_1 p(p-t) + (\kappa_t - 1)p(p-1)\big)\, q + \kappa(p-t)\Big], \end{aligned} \tag{$\mathrm{H_{VI}}$}$$

where $(\alpha,\, \beta,\, \gamma,\, \delta,\, \kappa) = (\kappa_\infty^2/2, \kappa_0^2/2\,,\, \kappa_1^2/2, \kappa_t^2/2, [(\kappa_0 + \kappa_1 + \kappa_t - 1)^2 - \kappa_\infty]/4)$.

5.3 Other properties of the Painlevé equations

In this section we describe some of the other general theory relating to the Painlevé equations.

Proposition 5.14.

The following diagram shows the so called convalescence cascade, where the arrows denote that it is possible to go from one equation to another via limits and substitutions.

$$\begin{tikzcd}
(\mathrm{P_{VI}}) \arrow[r] & (\mathrm{P_{V}}) \arrow[r] \arrow[dr] & (\mathrm{P_{IV}}) \arrow[dr] & & \\
& & (\mathrm{P_{III}}) \arrow[r] & (\mathrm{P_{II}}) \arrow[r] & (\mathrm{P_{I}})
\end{tikzcd}$$

Figure 8: Coalescence cascade of the Painlevé equations.

Example 5.15.

We shall now see how we can go from $\mathrm{P_{II}}$ to $\mathrm{P_{I}}$, as stated in the above proposition. We first let $t \mapsto \epsilon^2 t - 6\epsilon^{-10}$, $y \mapsto \epsilon y + \epsilon^{-5}$, $\alpha \mapsto 4\epsilon^{-15}$. Applying these substitution to $\mathrm{P_{II}}$ yields

$$\epsilon^{-3} y'' = \epsilon^3(2y^3 + ty) + \epsilon^{-3}(6y^2 + t) \implies y'' = \epsilon^6(2y^3 + ty) + (6y^2 + t).$$

Now letting $\epsilon \to 0$ gives $\mathrm{P_{I}}$, of course there needs to be a justification for this to be true at possible singularities of $2y^3 + ty$. For an example of how this can be made rigorous is given, see [32].

We shall now find a particular solution to $\mathrm{H_{VI}}$. First consider the following:

Proposition 5.16.

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = a(t)y^2 + b(t)y + c(t)$, then the substitution $y = \frac{-1}{a(t)}\frac{\mathrm{d}}{\mathrm{d}t}\ln(u)$ applied to the differential equation yields a differential equation of the form

$$u'' + \left[\frac{a(t)'}{a(t)} - b(t)\right] u' + a(t)c(t)u = 0.$$

Proof. The substitution yields

$$\frac{a'}{a}\frac{u'}{u} - \frac{1}{a}\frac{u''}{u}^2 + \frac{u'^2}{u^2} = \frac{1}{a}\left(\frac{-}{u}{}'u\right)^2 + b/a\frac{u'}{u} + c$$

Then collecting all the terms to one side, and multiplying by a and u yields

$$u'' + \left[\frac{a(t)'}{a(t)} - b(t)\right] u' + a(t)c(t)u = 0$$

■

Proposition 5.17.

The Hamiltonian system H_{VI} has a particular solution

$$\begin{cases} p(t) = A^{-1}t(t-1)\frac{\mathrm{d}}{\mathrm{d}t}\ln[(t-1)^{\kappa_0}u(t)] \\ q(t) = 0 \end{cases},$$

where $A \coloneqq \kappa_0 + \kappa_1 + \kappa_t - 1 \neq 0$ and where we pick $\kappa = 0$ and where $u(t)$ is the hypergeometric function with parameters $1 - \kappa_1$, $\kappa_t + 1$ and $\kappa_0 + \kappa_t - 1$.

Proof. From the Hamilton equations we obtain that $\dot{q} = 0$ and that

$$\dot{p} = -[\kappa_0(p-1)(p-t) - \kappa_1 p(p-t) - (\kappa_t - 1)p(p-1)]/[t(t-1)],$$

which can be brought to a form, where we may apply Proposition 5.16 for the substitution $p = A^{-1}t(t-1)\frac{\mathrm{d}}{\mathrm{d}t}\ln[(t-1)^{\kappa_0}u(t)]$. This substitution would yield

$$t(1-t)\ddot{u} + [\kappa_0 + \kappa_t - 1 - (1 - \kappa_1 + \kappa_t + 2)t]\dot{u} - (1-\kappa_1)(\kappa_t + 1)u = 0,$$

which is the Gauss hypergeometric equation with the desired parameters. Thus $p(t)$ has the desired solution. ∎

Similarly we will find that the particular solutions of the Painlevé equations 2 up to 6 follow the degeneration of the Gauss' hypergeometric function. To illustrate this consider the following diagram, which is similar to the convalescence cascade of the Painlevé equations shown previously.

Proposition 5.18.

The degeneration of the Gauss hypergeometric function follows the confluence cascade of the Painlevé equations, as is depicted below, and the degenerations show up in the respective (with respect to Figure 8) particular solutions of the Painlevé equations (as showed for P_{VI} above).

$$\begin{tikzcd}
\text{Gauss} \arrow[r] & \text{Kummer} \arrow[r] \arrow[dr] & \text{Hermite-Weber} \arrow[dr] & & \\
& & \text{Bessel} \arrow[r] & \text{Airy} \arrow[r] & \text{None}
\end{tikzcd}$$

Figure 9: Degeneration of Gauss' hypergeometric function.

Remark 5.19.

The above are the equations referred to in Table 2.

We shall illustrate such a degeneration in the following example.

Example 5.20.

We shall show that an Airy function is just a special case of a Bessel function. As such note that that the Bessel function is a solution of

$$t^2\ddot{y} + t\dot{y} + (t^2 - \alpha^2)y = 0$$

and that an Airy function is a solution to

$$\ddot{y} + ty = 0.$$

We call these the Bessel and Airy functions respectively. We now apply the transform $z = \frac{2}{3}t^{3/2}$, $u(z) = \sqrt{t}y$ to the Airy equation, we then obtain

$$t^2\ddot{y} + t\dot{y} + (t^2 - 1/3^2)y,$$

which shows the restricting the parameter of the Bessel function yields the Airy function.

5.4 The Painlevé property and Integrability

We shall show in this section how the Painlevé property is related to integrability, and use it to determine integrable parameter choices for the Hénon-Heiles Hamiltonian system. We will use a Painlevé test to do so, the ARS algorithm, which we describe below. We start with some background. The Painlevé property had already been linked to integrability of Hamiltonian systems by Sofya Kovalevskaya (1850 - 1891), who was the first to find a new completely integrable solution to the heavy top problem after Euler and Lagrange. She used, what is now referred to as the Kovalevskaya-test, which needs to be passed for a differential equations to posses the Painlevé property. For this she won the Prix Bordin in 1888, for which the jury was so impressed that the prize money was more than doubled. Since then the Painlevé property has been connected to integrability in a general way. However having the Painlevé property is neither sufficient nor necessary for many types of integrability [34].

We shall now introduce an algorithm, referred to as the ARS-algorithm, as outlined and justified in [35, 36, 37], who's success is a necessary condition for an ODE having the Painlevé property.

5.5 Painlevé test

Step 1: Let $\tau := (t - t_0)$. Try $u = A\tau^{-p}$ as a solution, for $\mathrm{Re}(p) > 0$ and $z_0 \in \mathbb{C}$. Determine all the possible values of (p, A) for which at least two terms balance.

If p can take non-integer values, then the equation does not have the Painlevé property, if p is an integer, continue to step 2.

Step 2: Substitute $A\tau^{-p} + B\tau^{r-p}$ into the terms that balanced in step 1. Then equate the leading terms in B. This should reduce to $Q(r)B\tau^{q}$, for Q a polynomial of degree n and $q \geq r - (p+n)$.

Determine the roots of $Q(r)$.

It is the case that -1 is always a root.

If A was arbitrary, then 0 is a root.

Ignore any roots with $\mathrm{Re}(r) < 0$.

If $Q(r)$ has any other roots r with $\mathrm{Re}(r) > 0$, but $r \notin \mathbb{N}$, then the equation does not have the Painlevé property. If for all pairs (p, A) $Q(r)$ does have only roots in $\mathbb{N}$ (excluding -1 or 0), continue to step 3. If for every pair (p, A), $Q(r)$ has less than $n-1$ non-negative roots, this indicates that u misses an essential part of the solution and the algorithm is inconclusive.

Step 3: Order the positive integer roots as $r_1 \leq \cdots \leq r_m$. Substitute $v = A\tau^{-p} + \sum_{j=1}^{\max(r)} c_j \tau^{j-p}$ into our original ODE, where $\max(r)$ denotes the maximal integer value of r found as a root of $Q(r)$. Equate powers of τ. Use this to determine the coefficients c_j in order.

If at any point for the determined coefficients the equalities can't hold, then the test has failed.

End.

Example 5.21.

We shall apply the algorithm as outlined above to the first Painlevé equation: $y'' = 6y^2 + t$.

Step 1: We substitute $A(t-t_0)^{-p} := A\tau^{-p}$ into $\mathrm{P_I}$, this gives

$$Ap(p+1)\tau^{-(p+2)} = 6A^2\tau^{-2p} + t.$$

We set $p+2 = 2p$ yielding $p = 2$. Then setting $Ap(p+1) = A^2 6 \implies A = 1$.

Step 2: Now we substitute $\tau^{-2} + B\tau^{-(r+2)}$ into the terms of $\mathrm{P_I}$, which balanced in step 1, yielding

$$6\tau^{-4} + B(r-2)(r-3)\tau^{r-4} = 6\tau^{-4} + 12B\tau^{r-4} + 6B^2\tau^{2(r-2)}.$$

Equation the terms with are first order in B gives $(r-2)(r-3) = 12 \implies r^2 - 5r - 6 = 0 \implies (r+1)(r-6) = 0$, which has roots $r = -1$ as expected, as well as $r = 6$.

Step 3: We substitute

$$\tau^{-2} + C_1\tau^{-1} + C_2 + C_3\tau + C_4\tau^2 + C_5\tau^3 + C_6\tau^4$$

into $\mathrm{P_I}$, and equate powers of τ, yielding:

$$\begin{aligned}
\tau^{-4} &: 6 &&= 6,\\
\tau^{-3} &: 2C_1 &&= 12C_1,\\
\tau^{-2} &: 0 &&= 6C_1^2 + 12C_2,\\
\tau^{-1} &: 0 &&= 12C_1C_2 + 12C_3,\\
\tau^{0} &: 2C_4 &&= 6C_2^2 + 12C_1C_3 + 12C_4,\\
\tau^{1} &: 6C_5 &&= 1 + 12C_2C_3 + 12C_1C_4 + 12C_5,\\
\tau^{2} &: 12C_6 &&= 6C_3^2 + 12C_2C_4 + 12C_1C_5 + 12C_6.
\end{aligned}$$

Solving this system of equations yields

$$C_1 = C_2 = C_3 = C_4 = 0 \text{ and } C_5 = -1/6.$$

End

Hence we see that as expected, $\mathrm{P_I}$ has passed the Painlevé test.

The ARS algorithm can also be applied to systems of equations, in much a similar way as for just equations.

For the first step we just substitute $A_i(t-t_0)^{-p_i}$ into the system of for each variable x_i, and go on as described before, requiring that all the p_i are integers and then determine the A_i.

For the second step we substitute $A_i(t-t_0)^{-p_i} + B_i(t-t_0)^{r-p}$, and only consider the leading terms in B_i (so no cross terms). These should give a a system of equations, which can be written as $\boldsymbol{Q}(r)\boldsymbol{B}$, where Q is a matrix who's entries are polynomials in r and $\boldsymbol{B} = [B_1, B_2, \ldots, B_n]^T$.

Now we find the roots of $\det[\boldsymbol{Q}(r)]$ in terms of r, from where we continue as in the case for only one equation.

For the third step we continue as done in the case for just one equation, and substitute $A\tau^{-p_i} + A_{i,1}\tau^{1-p_i} + \cdots + A_{i,\max(r)}\tau^{\max(r)-p_i}$ into our original system of equations into the variables x_i respectfully, and check that the resulting system of equations give no contradictions.

5.6 The generalised Hénon-Heiles system

Now we apply the above algorithm to an the generalised Hénon-Heiles system, following [38, 39, 40]. This will also serve as an example of how to apply the ARS-algorithm to a system of equations. Let

$$H = \frac{1}{2}(\dot{x} + \dot{y}) + \frac{1}{2}(ax^2 + by^2) + dx^2y - \frac{e}{3}y^3,$$

which has equations of motion

$$\ddot{x} = -ax - 2dxy \quad \text{and} \quad \ddot{y} = -by - dx^2 + ey^2.$$

Following step 1 of the algorithm, we set $\tau = (t - t_0)$ and substitute $x = A\tau^{-p}$ and $y = B\tau^{-q}$ into the equations of motion. We then obtain

$$Ap(p+1)\tau^{-(p+2)} = -Aa\tau^{-p} - 2dAB\tau^{-(p+q)}$$

and

$$Bq(q+1)\tau^{-(q+2)} = -Bb\tau^{-q} - dA^2\tau^{-2p} + eB^2\tau^{-2q}.$$

Then clearly the only way to obtain two or more terms with equal powers is to set $p + 2 = p + q$ for the first equation of motion and

$$\begin{cases} q + 2 = 2p + 2q & \text{if } p = q, \\ q + 2 = 2q & \text{otherwise.} \end{cases},$$

for the second equation of motion. We then determine that $q = 2$ is a possibility, then if $p = q$, balancing the coefficients leads to

$$p = q = 2, \quad A = \pm(3/d)\sqrt{2 + e/d},\ B = -d/3.$$

In the case $p \neq q$ we obtain

$$p = \frac{-1 \pm \sqrt{1 - 4 \cdot 12d/e}}{2}, \; q = 2, \quad A \in \mathbb{C}, \; B = 6/e.$$

For the algorithm not to immediately terminate p needs to be an integer, which already restrict the possible values for d and e quite heavily. We continue to step 2 and substitute $x = A\tau^{-p} + C\tau^{r-p}$ and $y = B\tau^{-q} + D\tau^{r-q}$. For the first case in leading orders of C and D we then obtain the system of equations

$$\begin{cases} C[(r-2)(r-3) + 2dB] + 2dAD = 0 \\ D[(r-2)(r-3) - 2eB] + 2dAC = 0 \end{cases}.$$

Now we Need to find the roots w.r.t. r of the determinant of

$$\begin{bmatrix} (r-2)(r-3) + 2dB & 2dA \\ 2dA & (r-2)(r-3) - 2eB \end{bmatrix}.$$

Via some algebra we then obtain that $(r^2 - 5r - 6)((r^2 - 5r + 12)d + 6e) = 0$. Then if $r^2 - 5r + 6 + 2eB = 0 \implies r = \frac{5 \pm \sqrt{25 - 4(6 - 2eB)}}{2} = \frac{5 \pm \sqrt{1 - 24(1 + e/d)}}{2}$. As other roots we obtain -1 and 6. Similarly for the other case, $p \neq q$, we find roots $r \in \{-1, 0, 6, \pm\sqrt{1 - 48d/e}\}$.

Now for the test not to fail it is required that $\frac{5 \pm \sqrt{1 - 24(1 + e/d)}}{2}$ and $\pm\sqrt{1 - 48d/e}$ are positive integers. This restricts possible values to $e = -6d$, $e = -d$, and $e = -2d$.

Now we substitute $A\tau^{-2} + A_1\tau^{-1} + \cdots + A_6\tau^4$ and $B\tau^{-2} + B_1\tau^{-1} + \cdots + B_6\tau^4$ for x and y respectfully into the equations of motion, and solve the systems we then obtain in such a way that no contradictions occur. If this is done we find that in the case $e = -d$ we have that $a = b = 2d(A_3 - B_3)$. Similarly for the case $e = -2d$, we find that $a = b$, but a and b remain arbitrary for the case $e = -6d$. Thus for these three values it remain to show whether they are integrable or not, which we discuss below.

For the case $e = -d$, $a = b$ we obtain the Hamiltonian

$$H = \frac{1}{2}(\dot{x} + \dot{y}) + \frac{a}{2}(x^2 + y^2) + ex^2y - \frac{e}{3}y^3,$$

Now substitute $x = (v - w)$ and $y = (v + w)$, then

$$H(v, w) = av^2 + \frac{4ev^3}{3} + aw^2 - \frac{4ew^3}{3},$$

i.e. the Hamiltonian completely splits it two one degree of freedom Hamiltonians and thus is completely integrable. Hence the system is completely integrable for these parameters.

Similarly for a specific change of variables the Hamiltonian for the case $e = -6d$ splits [41]. The case $e = -2d$ was shown to be non-integrable in [42].

Though further analysis one other integrable case was found for $e = -16d$ and $a = 16b$ [41]. Furthermore in [5] it was shown in that all other choices of parameters lead to non-integrable systems.

Summary and discussion

We have discussed differential Galois theory for matrix differential equations, discussing the formalism required from algebraic geometry and algebraic groups. We've seen that differential Galois theory gives a powerful way of studying the integrability of matrix differential equations. Picard-Vessiot rings can always be shown to exist for a given matrix differential equation and are unique up to isomorphism. The group of differential automorphisms acts as a matrix group on the fundamental matrix of a differential equation and it can be shown that this automorphism group is in fact a linear algebraic group, with coefficients in the field of constants. To develop this theory, we needed to assume we were working over a field with characteristic 0 and a algebraically closed field of constants. There are linear differential analogues of the classical Galois correspondence and theory of radical extensions, where here we deal with Liouville extensions, which can be thought of to encapsulate integrability, and Liouville's theorem establishes that this integrability in completely determined the be identity component of the differential Galois group. Finding the differential Galois groups and it's identity component may be hard in general, but for second order equations over $\mathbb{C}(x)$ Kovacic's algorithm may be used, which gives a necessary and sufficient condition for integrability of the equation. We then developed the Hamiltonian formalism and discussed (non-)integrability via the Liouville-Arnold, KAM and Nekhorosev theorems. Afterwards we discussed Ziglin-Moralis-Ramis theory, which gives a criterion for complete integrability and gave a detailed example of its application to the planar spring pendulum system. After this we introduced the Painlevé property and equations, giving an overview of basic results related to the Painlevé equations, discussing how the Painlevé property is related to complete integrability, and giving a test for the Painlevé property in the form of the ARS test, which we applied to the generalised Hénon-Heiles system. Both the Ziglin-Morales-Ramis theory and the Painlevé property give two different views on integrability.
Possibly interesting questions for further research could be the following:

In the same way the exponential function was given a special place in the Liouville theorem (being contained in a Liouvillian extension), it might be interesting to see if the same is possible for other known functions resulting from linear differential equations, for example could we also give the Airy function such a special role?
Monodromy finds application in the scattering of black holes, can the relation between differential Galois theory and monodromy be exploited in a meaningful way for this application?
There are examples of first integrals being constructed using Painlevé analysis. Can some algorithm be made to accomplish this?
Ziglin-Morales-Ramis theory gives an effective criterion for non-integrability of Hamiltonian systems. Since for many low dimensional parametrised Hamiltonian systems, all the non-integrable parameter values can be determined, the criterion is quite sharp. For parametrised Hamiltonians the Painlevé

test can be used to selected possible values for which complete integrability can be accomplished. Both the Painlevé property as well as the Ziglin-Moralis-Ramis theory go some way to answering the same question: Is it possible to show a Hamiltonian system is completely integrable, without constructing the actual first integrals, i.e. is it possible to deduce integrability from just the properties of the Hamiltonian system? How sharp can the criteria be made?

To put all the discussed topics in their historical context we conclude with the following summarising timeline.

1877:	Picard publishes an article establishing differential Galois theory for linear differential equations.
1888:	Kovalevskaya uses the Painleve property to find a new integrable case of the Heavy Top problem.
1889:	Poincaré wins Oscar II, King of Norway prize for his results on non-integrability of Hamiltonian systems, relating to the n body problem.
1900 - 1906:	The six Painlevé transcendents are found.
1954:	Kolmogorov gives first proof of the KAM theorem.
1973:	Kolchin makes Picard-Vessiot rigorous.
1982:	Ziglin gives a non-integrability criterion via the monodromy group of a linearised Hamiltonian system.
1986:	Kovacic develops Kovacic's algorithm for second order linear differential equations.
1990+:	Ziglin-Morales-Ramis theory is developed.

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Appendix A Algebraic Geometry

We shall in this section go through some of the preliminaries of algebraic Geometry. In this section we will follow / take inspiration from [3, 43, 44, 45, 46].

We assume we are dealing with K a algebraically closed field.

Definition 1.1 (Affine space / variety).

(i) Let K be a field and let $n \in \mathbb{N}$. Then we let $\mathbb{A}^n_K \coloneqq K^n$ be the *affine space*.

(ii) Let $S \subseteq K[x_1, \ldots, x_n]$ be a polynomial ring. Then

$$\mathcal{V}(S) \coloneqq \{(a_1, \ldots, a_n) \in \mathbb{A}^n;\ (\forall f \in S)\ f[(a_1, \ldots, a_n)] = 0\},$$

is called the *variety of* S.

Example 1.2.

The following are examples of varieties:

(i) Let $\mathcal{V}(x^2 + y^2 + 1) = \{(a, b) \in \mathbb{A}^2;\ a^2 + b^2 = 1\}$, the unit circle in $\mathbb{A}^2$.

(ii) Let $I \trianglelefteq K[x]$, then as $K[x]$ is a principal ideal domain, so $I = \langle f \rangle$, for some $f \in K[x]$. Let f have a set of roots $\{a_1, \ldots, a_n\}$ now

$$\mathcal{V}(I) = \mathcal{V}(f) = \{b \in \mathbb{A};\ f(b) = 0\} = \{b \in \mathbb{A};\ [(a_1 - x) \cdots (a_n - x)](b) = 0\} = \{a_1, \ldots, a_n\},$$

where we use that f can be factorised, since K is algebraically closed.

Proposition 1.3.

(i) Let $S \subseteq K[x_1, \ldots, x_n]$ and $I = \langle S \rangle$, the ideal generated by S, then $\mathcal{V}(S) = \mathcal{V}(I)$.

(ii) $\mathcal{V}(\langle 0 \rangle) = \mathbb{A}^n$ and $\mathcal{V}(\langle 1 \rangle) = \emptyset$.

(iii) Let $\{S_\alpha\}_{\alpha \in A}$ for some indexing set A, be a collection of subsets of $K[x_1, \ldots, x_n]$, then

$$\mathcal{V}\left(\bigcup_{\alpha \in A} S_\alpha\right) = \bigcap_{\alpha \in A} \mathcal{V}(S_\alpha).$$

(iv) Let $I, J \trianglelefteq K[x_1, \ldots, x_n]$, then $\mathcal{V}(IJ) = \mathcal{V}(I) \cup \mathcal{V}(J)$.

(v) $\mathcal{V}$ is inclusion reversing.

Proof.

(i) Note first that $\mathcal{V}(S) \subseteq \mathcal{V}(\langle S \rangle)$ is trivial, as $S \subseteq \langle S \rangle$. Now we show $\mathcal{V}(S) \supseteq \mathcal{V}(\langle S \rangle)$. $\langle S \rangle = \sum_i a_i f_i$, for $a_i \in K[x_1, \ldots, x_n]$ and $f_i \in S$. Now let $P \in \mathcal{V}(S)$, then

$$\implies \forall f \in S f(P) = 0 \implies \left(\sum_i a_i f_i \right)(P) = 0 \implies P \in \mathcal{V}(\langle S \rangle).$$

Thus $\mathcal{V}(S) \supseteq \mathcal{V}(\langle S \rangle)$.

(ii) $\mathcal{V}(\langle 0 \rangle) = \{P \in \mathbb{A}^n ;\, 0(P) = 0\} = \mathbb{A}^n$ and $\mathcal{V}(\langle 1 \rangle) = \{P \in \mathbb{A}^n ;\, (\forall f \in \langle 1 \rangle)\ f(P) = 0\} \subseteq \{P \in \mathbb{A}^n ;\, 1(0) = 0\} = \emptyset$.

(iii) $\mathcal{V}(\cup_{\alpha \in A} S_\alpha) = \{P \in \mathbb{A}^n ;\, (\forall S_\alpha)(\forall f \in S_\alpha)\ f(P) = 0\} = \cap_\alpha \big(\mathcal{V}(S_\alpha)\big)$.

(iv) Let I and J be ideals and let f denote some arbitrary element in I and g some arbitrary element in J, then

$$\mathcal{V}(IJ) = \{P \in \mathbb{A}^n ;\, (\forall fg \in IJ)\ (fg)(P) = 0\} = \{P \in \mathbb{A}^n ;\, (\forall fg \in IJ)\ f(P) = 0 \text{ or } g(P) = 0\}$$
$$= \mathcal{V}(I) \cup \mathcal{V}(J).$$

(v) If $S \subseteq T$, then $\mathcal{V}(S)$ has less restrictions on what elements can be part of it than $\mathcal{V}(T)$, thus $\mathcal{V}(S) \supseteq \mathcal{V}(T)$.

■

Note that by Corollary 1.13 any ideal of $K[x_1, \ldots, x_n]$ is Noetherian, thus we can take any ideal to be finitely generated.

Definition 1.4 (Zariski topology).

Let $\{\mathcal{V}(S);\, S \subseteq K[x_1, \ldots, x_n]\}$ form the basis for the closed sets of a topology $\mathcal{T}_Z$. The generated topology is the so called *Zariski topology*.

Remark 1.5.

Note that by Proposition 1.3, the above definition is justified, i.e. this does indeed yield a topology. We show this using the definition of a topological space, via closed sets.

(i) By (ii), $\mathbb{A}^n$ and $\emptyset$ are closed.

(ii) By (iii), $\cap_{\alpha\in A}\mathcal{V}(S_\alpha) = \mathcal{V}(\cup_{\alpha\in A}S_\alpha)$, thus arbitrary intersections of closed sets are closed.

(iii) By first applying (i) and subsequently (iv), $\mathcal{V}(S)\cup\mathcal{V}(T) = \mathcal{V}(\langle S\rangle)\cup\mathcal{V}(\langle T\rangle) = \mathcal{V}(\langle S\rangle\langle T\rangle)$. Thus the finite union of closed sets is closed.

Thus this is indeed a topology (note that the Roman numbers refer to Proposition 1.3).

Example 1.6.

The varieties from Example 1.2 are closed sets. In particular from (ii) we can identify the Zariski topology on $\mathbb{A}^1$. Any closed set, is the set of roots of a polynomials and thus finite, thus it is easy to see that the topology is the cofinite topology.
This will not be the case for $\mathbb{A}^n$, $n > 1$, as for as $\mathcal{V}(x_1x_2) = \{P \in \mathbb{A};\ x_1x_2(P) = 0\} = \{(0,\ x_2,\ \dots, x_n)\}\cup$ $\{(x_1, 0,\ x_3,\ \dots, x_j,\ \dots,\ x_n)\}$, which in general is not finite.

Remark 1.7 (Separation properties).

Clearly $(\mathbb{A}^n,\ \mathcal{T}_Z)$ is T_1, as $(\forall P \in \mathbb{A}^n)\quad \mathcal{V}\left((\sum_i^n x_i) - P\right) = P$ is closed.
Now from Example 1.6 we know that $\mathbb{A}^1$ has the cofinite topology on it, which is not Hausdorff. Now note that $\mathbb{A}\times(0,\ \dots,\ 0)$ equipped with the Zariski topology inherits the cofinite topology as its subspace topology. Thus $\mathbb{A}^n$ is not Hausdorff, for $n \geq 1$.

Definition 1.8 (Vanishing ideal).

Let $X \subseteq \mathbb{A}^n$, then we let $\mathcal{I}(X) \coloneqq \{f \in k[x_1,\ \dots,\ x_n];\ (\forall P \in X)\ f(P) = 0\}$, and call it *the vanishing ideal of X*.

Remark 1.9.

Note the following basic results about $\mathcal{I}$:

(i) $\mathcal{I}(X)$ is an ideal.

(ii) $\mathcal{I}$ is inclusion reversing, as if $Y \subseteq Z \subseteq \mathbb{A}^n$, $\mathcal{I}(Y)$ has less restrictions on its elements, thus $\mathcal{I}(Y) \supseteq \mathcal{I}(Z)$.

Proposition 1.10.

Let $X \subseteq \mathbb{A}^n \implies \mathcal{V}[\mathcal{I}(X)] = \overline{X}$.

Proof. "$\supseteq$": It is not hard to see that $X \subseteq \mathcal{V}[\mathcal{I}(X)]$. $\mathcal{V}[\mathcal{I}(X)]$ is closed, thus $\overline{X} \subseteq \mathcal{V}[\mathcal{I}(X)]$.
"$\subseteq$": $\overline{Y}$ is closed, thus $\exists J \trianglelefteq K[x_1, \ldots, x_n]$ s.t. $\mathcal{V}(J) = \overline{X}$. Then using that $\mathcal{I}$ is inclusion reversing we have that $\mathcal{I}(X) \subseteq \mathcal{I}[\mathcal{V}(J)]$. It is not hard to see that $J \subseteq \mathcal{I}[\mathcal{V}(J)] = \mathcal{I}(\overline{X}) \subseteq (X)$. Hence $\overline{X} = \mathcal{V}(J) \supseteq \mathcal{V}[\mathcal{I}(X)]$, using the inclusion reversing property of $\mathcal{V}$. Thus we have the desired equality. ■

Definition 1.11 (Coordinate ring).

Let $X \subseteq \mathbb{A}^n$, and define $A(X) \coloneqq K[x_1, \ldots, x_n]/\mathcal{I}(X)$, we call this the *coordinate ring*.

Definition 1.12 (Radical ideal).

Let R be a commutative ring, and $I \trianglelefteq R$ an ideal, then

$$\sqrt{I} \coloneqq \{r \in R; \exists m > 0 \text{ s.t. } r^m \in I\}$$

is called the *radical of* I.
If $I = \sqrt{I}$, I is called a *radical ideal.*

Remark 1.13.

We note the following basic facts about the radical of some ideal I.

(i) $\sqrt{I} \supseteq I$.

(ii) $\sqrt{\sqrt{I}} = \sqrt{I}$.

(iii) Note that $\sqrt{I}$ is an ideal. Let $r \in R$ and $s \in \sqrt{I}$ such that $s^m \in I$, then $(rs)^m = r^m s^m \in I$ using that $s^m \in I$. Let $s, t \in \sqrt{I}$ such that $s^n, t^m \in I$, then $(s+t)^{n+m}$ gives a sum of products of s and t using the binomial expansion. In this expansion, in each term there will be a term s^{n+i} or t^{m+j}, thus each term separately will contained in I, thus $(s+t)^{n+m} \in I$, hence $(s+t) \in \sqrt{I}$. Thus $\sqrt{I}$ is an ideal.

(iv) For any $X \subseteq \mathbb{A}^n$, $\mathcal{I}(X)$ is a radical ideal.

(v) Prime ideals are radical ideals.

Theorem 1.14 (Weak Nullstellensatz, [47]).

The maximal ideals of $K[x_1, \dots, x_n]$ have the form $\langle (x_1 - a_1, \dots, x_n - a_n) \rangle$, i.e. there is a bijection $\mathbb{A}^n \leftrightarrow \{\text{maximal ideals of } K[x_1, \dots, x_n]\}$, where $\mathbb{A}^n \ni P = (a_1, \dots, a_n) \mapsto \langle (x_1 - a_1, \dots, x_n - a_n) \rangle$.

Theorem 1.15 (Hilbert's Nullstellensatz, [47]).

Let $I \subseteq K[X_1, \dots, x_n]$ be an ideal, then $\mathcal{I}[\mathcal{V}(I)] = \sqrt{I}$.

Corollary 1.15.

The map $\mathbb{A}^n \supseteq X \mapsto \mathcal{I}(X)$, has a two sided inverse, where $K[x_1, \dots, x_n] \trianglerighteq I \mapsto \mathcal{V}(I)$ and gives a bijection

$$\{\text{closed subsets of } \mathbb{A}^n\} \leftrightarrow \{\sqrt{I};\, I \trianglelefteq K[x_1, \dots, x_n]\}.$$

Proof. Follows directly from Proposition 1.10 and Hilbert's Nullstellensatz. ■

Definition 1.16 ((Ir)reducible subset / component).

(i) Let $V \neq \emptyset$ be a subset of a topological space X, if $\nexists V_1, V_2 \subsetneq V$ closed sets such that $V = V_1 \cup V_2$, then V is called *irreducible*. If V is not irreducible, it is called *reducible*.

(ii) We call a maximal irreducible subset a *irreducible component*.

Proposition 1.17.

The following are equivalent:

(i) V is irreducible.

(ii) For all non-empty opens $U_1, U_2 \subseteq V$ we have that $U_1 \cap U_2 \neq \emptyset$.

(iii) Any open subset of V is dense in V.

Proof. "(i) $\implies$ (ii)": Let V be irreducible. By way of contradiction assume there exists non-empty opens U_1, U_2 s.t. $U_1 \cap U_2 \neq \emptyset$. Then U_1^c and U_2^c are closed proper subsets of V, where $U_1^c \cup U_2^c = V$ ↯.
"(ii) $\implies$ (i)": By way of contradiction let V_1, V_2 be proper closed non-empty subsets of V, who's union is V, then V_1^c, V_2^c are open non-empty subsets with a non-empty intersection ↯.
"(iii) $\iff$ (ii)": A non-empty open set U is dense in V $\iff$ U intersects every non-empty open set. ■

Example 1.18.

The following are examples of (ir)reducible set:

(i) $\mathbb{A}^1$ is irreducible. All its closed sets are finite, but $\mathbb{A}$ is infinite.

(ii) Any non-empty closed set of $\mathbb{A}^1$, which is not a singleton is reducible.

(iii) Any irreducible space is connected (as there doesn't exist a partition).

Proposition 1.19.

(i) Let $Y \subseteq X$, then Y is irreducible $\iff$ $\overline{Y}$ is irreducible.

(ii) Irreducible components are closed.

Proof.

(i) " $\implies$ ": Any open set intersecting Y, also intersects $\overline{Y}$. Thus any two open set in $\overline{Y}$ have a non-empty intersection, thus $\overline{Y}$ is irreducible.

" $\impliedby$ ": By way of contradiction assume $\overline{Y}$ is irreducible and Y is not. Then $Y = V_1 \cup V_2 \implies \overline{Y} = \overline{V}_1 \cup \overline{V}_2$ ↯.

(ii) Follows directly from (i).

■

Proposition 1.20.

Let $X \subseteq \mathbb{A}^n$. A variety X is irreducible $\iff$ $A(X)$ is an integral domain.

Proof. " $\implies$ ": Let X be irreducible. Let $f_1, f_2 \in K[x_1, \ldots, x_n]$ such that $fg \in \mathcal{I}(X)$, then by the inclusion reversing property of $\mathcal{V}$, we have that $X \subseteq (\mathcal{V})(f_1) \cup \mathcal{V}(f_2)$. Then since $\mathcal{V}(f_i)$ is closed, and since X is irreducible, we must have that $\mathcal{V}(f_i) \subseteq X$ for $i \in \{1, 2\}$. Then this $f_i \in \mathcal{I}(X)$, hence $I(X)$ is a prime ideal, and $A(X)$ is a coordinate ring.

" $\impliedby$ ": By way of contradiction assume X is reducible, then $X = X_1 \cup X_2$ for $X_i \subsetneq X$, then $\mathcal{I}(X_1) \cdot \mathcal{I}(X_2) \subseteq \mathcal{I}(S)$. Then since $\mathcal{I}$ is inclusion reversing we have that $\mathcal{I}(X_i) \supsetneq \mathcal{I}(X)$. Then $\mathcal{I}(X)$ is not a prime ideal, thus $A(X)$ is not an integral domain. ■

The above proposition shows that $\mathbb{A}^n$ is irreducible, as $A(\mathbb{A}^n) = K[x_1, \ldots, x_n]/\langle 0 \rangle$, is a domain.

Corollary 1.20.

The bijection from Corollary 1.15 restricts to

$$\{\text{closed irreducible subsets of } \mathbb{A}^n\} \leftrightarrow \{I;\, I \trianglelefteq K[x_1, \ldots, x_n] \text{ is a prime ideal}\}.$$

Proof. Follows from noting that R/I is an integral domain $\iff$ I is a prime ideal. ■

Definition 1.21 (Noetherian topological space).

A topological space X is called *Noetherian* if any descending chain of closed subsets stabilises, said differently

$$\forall\, V_1, \supseteq V_2 \supseteq \cdots$$

$\exists n \in \mathbb{N}$ such that $V_n = V_{n+1} = \cdots$.

Example 1.22.

(i) $\mathbb{R}$ with the Euclidean topology is not a Noetherian topological space as one can easily make a descending chain of closed balls which does not stabilise.

(ii) $\mathbb{A}^n$ is a Noetherian topological space. This follows from Corollary 1.15, the inclusion reversing property of $\mathcal{I}$ and the fact that $K[x_1, \ldots, x_n]$ is a Noetherian ring.

(iii) Any subset of a Noetherian space with the inherited subset-topology, is a Noetherian topological space.

Proposition 1.23.

Any variety is a finite union of irreducible sub-varieties.

Proof. By way of contradiction assume that a variety X cannot be written as a finite union of proper sub-varieties. Then we can write $X = \cup_{i \in \mathbb{N}} V_i$, for V_i proper sub-varieties of X. Then we can construct the following descending chain of closed subsets, which does not stabilise:

$$X \supseteq X \setminus X_1 \supseteq X \setminus (X_1 \cup X_2) \supseteq \cdots$$

This contradicts the fact that X is Noetherian $\lightning$. Thus X can be written as a finite union of irreducible closed subsets. ■

Proposition 1.24.

Let X, Y be topological spaces, then if $V \subseteq X$ an irreducible set and $f : X \to Y$ is continuous, then $f(V)$ is irreducible.

Proof. Let $V \subseteq X$ be irreducible. By way of contradiction assume that $f(V)$ is reducible, then $f(V) = U_1 \cap U_2$, for U_i proper irreducible subsets of $f(V)$. Then $f^{-1}(U_1 \cap U_2) = f^{-1}(U_1) \cap$

$f^{-1}(U_2)$, and since f is continuous, f^{-1} sends closed sets to closed sets, thus $V = f^{-1}(U_1) \cap f^{-1}(U_2)$ $\lightning$. ■

Proposition 1.25.

Let $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ be varieties, then X, Y are irreducible $\implies X \times Y \subseteq \mathbb{A}^{m+n}$ is irreducible.

Proof. We want to show that $\mathcal{I}(X \times Y)$ is a prime ideal, as then by Corollary 1.20, $X \times Y$ is irreducible. Let $f_1, f_2 \in K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ and let $f_1 f_2 \in \mathcal{I}(X \times Y)$, then for $y_0 \in Y$ define $X_i(y_0) = \{x \in X;\ f_i(x, y_0) = 0\}$. $X_i \subseteq X$ is clearly closed. Now note that $X_1 \cup X_2 = X$, thus since X is irreducible we must have that $X_i(y_0) = X$ for $i = 1$ or $i = 2$. Now define $Y_i \coloneqq \{y \in Y;\ X_i(y) = X\} = \{y \in Y;\ (\forall x \in X) f_i(x, y) = 0\}$, thus Y_i is also closed, again we have that $Y_i = Y$ for $i = 1$ or $i = 2$. Then $(\forall y \in Y)(\forall v \in V)\ f_i(x, y) = 0 \implies f_i \in \mathcal{I}(X \times Y)$. Thus $\mathcal{I}(X \times Y)$ is prime and we are done. ■

Definition 1.26 ((Quasi) affine-variety).

Let $V \subseteq \mathbb{A}^n$ be a closed irreducible subset, then V is called a *affine variety*. Let V be an affine variety, then if $U \subseteq V$ is an open subset, then U is called a *quasi-affine variety.*

Definition 1.27 (Polynomial / rational functions).

Let $X \subseteq \mathbb{A}^n$, then $\phi : X \to K$ is called a

(i) *polynomial map* if

$$\exists f \in K[x_1, \ldots, x_n] \text{ such that } \phi = f.$$

(ii) *rational function at a point P* if

$$\exists f, g \in K[x_1, \ldots, x_n] \text{ such that } g(P) \neq 0 \text{ and } \phi|_U = f/g.$$

Definition 1.28 (Regular functions / Morphisms).

(i) Let $X \subseteq \mathbb{A}^n$ be a quasi-affine variety, and let $\phi : X \to K$, then ϕ *is regular at a point* P, if in some open neighbourhood U of P, $\phi|_U$ is a rational function, with no poles in U. Let $\mathcal{O}_X[U] \coloneqq \{\phi|_U\,;\, \phi \in K[x_1, \ldots, x_n]$ is regular.$\}$ denote the algebra of regular functions (where addition and (scalar)multiplication are performed point-wise). Sometimes we might not want to specify U and instead use $\mathcal{O}_{X,P}$ for the algebra of regular functions at a point P. Additionally let $\mathcal{O}_X$ be algebra of regular functions $X \to K$.

(ii) Let X, Y be varieties, for $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$, then $\varphi \coloneqq (\phi_1, \ldots, \phi_m) : X \to Y$ is a *morphism of varieties* if each ϕ_i is regular.

Remark 1.29.

We shall show below that for X a closed set we have that $\phi : X \to K$ is a regular function if $(\forall P \in X)\ \phi$ is a polynomial function.

Proposition 1.30.

(i) The ring $\mathcal{O}_X$ is isomorphic to the coordinate ring $A(X)$.

(ii) The ring $\mathcal{O}_{X,P}$ is isomorphic to $[A(X) \setminus \mathfrak{M}_P]^{-1}A(X)$, the localisation of the complement of the maximal ideal (without 0) associated to P by the Weak Nullstellensatz.

Proof.

(i) The homomorphism $F : K[x_1, \ldots, x_n] \to \mathcal{O}(X) : f \mapsto [x \mapsto f(x)]$ is clearly surjective, thus the first isomorphism theorem gives that $K[x_1, \ldots, x_n]/\ker(F) \cong \mathcal{O}(X)$, but $\ker(F) = \mathcal{I}(X)$, thus $K[x_1, \ldots, x_n]/\mathcal{I}(X) = A(X) \cong \mathcal{O}(X)$.

(ii) Note that $\mathfrak{m}_P$ is a maximal ideal of $K[x_1, \ldots, x_n]$, a principal ideal domain, thus $\mathfrak{m}_P$ is a prime ideal, and the complement is a multiplicative set. Now the localisation yields the set of rational functions f/g s.t. $f \in A(X)$ and $g \in A(X) \setminus \mathfrak{m}_P$, where we used the identification of elements of $A(X)$ and polynomial functions, from (i). Thus we obviously have an isomorphism.

■

Proposition 1.31.

Let $X \subseteq \mathbb{A}^n$ be Zariski closed. A function $\phi : X \to K$ is regular $\iff$ $\phi \in A(X)$.

Proof. Consider the ideal $I_\phi := \{g \in A(X);\ g\phi \in A(X)\}$, i.e. the ideal of denominators of ϕ. Then $\mathcal{V}(I_\phi) = \emptyset$ (as the denominators may not have any roots), $\iff 1 \in I_\phi \iff \phi \in A(X)$. ■

The above result shows that regular functions on $X \subseteq \mathbb{A}^n$ are polynomial functions.

Now we validate our definition of a morphism of varieties, by showing that regular maps are Zariski-continuous.

Proposition 1.32.

(i) A regular function $\phi : Y \to K$ is continuous when K is interpreted as $\mathbb{A}^1$.

(ii) A morphism of varieties is continuous.

Proof.

(i) It is enough to show local continuity. Thus let $U \subseteq Y$ be some open subset in which ϕ is represented by f/g. We shall check that ϕ^{-1} sends closed sets to closed sets. A closed set in $K = \mathbb{A}^1$ is a finite union of points, thus it is enough to show that the pre-image of a point is a closed set.

$$\phi^{-1}(a) \cap U = \{P \in U;\ \phi(P) = a\} = \{P \in U;\ f(P)/g(P) = a\} = \{P \in U;\ (ag - f)(P) = 0\}.$$

Thus $\phi^{-1}(a) \cup U = \mathcal{V}(f - ag) \cap U$, which is closed, thus ϕ is continuous.

(ii) Let $\varphi = (\phi_1, \ldots, \phi_m) : X \to Y$ be a morphism of varieties for $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$. Let $V \subseteq Y$ be a closed subset, then V is an ideal and thus by Hilbert's basis theorem we can write $V = \langle f_1, \ldots, f_r \rangle$. We know then that each ϕ_i is a polynomial function, then $\varphi \circ f_i$ is a polynomial on X, hence $\varphi^{-1}[\mathcal{V}(f_i)] = \mathcal{V}(\varphi \circ f_i)$, which is closed. Then $\varphi^{-1}[\mathcal{V}(\{f_i\}_{i \leq r})] = \cap_i \varphi^{-1}\mathcal{V}(f_i)$, which is again closed. Thus φ is continuous.

■

Appendix B Algebraic Groups

We shall now discuss algebraic groups. This section is based on [48, 49, 46]

Definition 2.1 (Algebraic group).

Let K be an algebraically closed field with characteristic 0. Let G be an algebraic variety over K, with group law $\mu : G \times G \to G : (x, y) \mapsto xy$, inverse $\iota : G \to G : x \mapsto x^{-1}$ and a unit element e such that the regular group axioms hold, furthermore let the operations, μ, ι be morphisms of varieties. Then the triple (G, μ, ι) is an *algebraic group*.
A Zariski closed subgroup of $\mathrm{GL}_n(K)$ is called a *linear algebraic group*, where an $n \times n$ matrix with coefficients in K can then be identified with a subset of $\mathbb{A}^{n^2}$.

Remark 2.2.

We note the following about algebraic groups:

(i) $G \times G$ is equipped with the Zariski topology and not the product topology, thus an algebraic group is not a topological group.

(ii) A closed subgroup of an algebraic group is again an algebraic group.

(iii) We may identify an element from $\mathrm{M}_n(K)$ with an element from $K^{n \times n} = K^{n^2}$. Let μ be matrix multiplication, then for $A = (a_{ij})_{i,j \leq n}, B = (b_{ij})_{i,j \leq n} \in \mathrm{M}_n(K)$, $\mu(A, B) = (c_{ij})_{i,j \leq n} = \sum_l^n a_{il} b_{lj}$. Thus matrix multiplication is a regular map.
If we restrict ourselves to $\mathrm{GL}_n(K)$, we can see that the standard matrix inverse also yields a regular map, using the adjoint, as $A^{-1} = \det(A)^{-1}\mathrm{adj}(A)$, is regular, since A is nonsingular. Thus $\mathrm{GL}_n(K)$ is a linear algebraic group with the identity matrix as a unit.

(iv) Let V be a K-vector-space, and let $\mathrm{GL}(V)$ be the group of K-linear automorphisms of V, then if $\dim(V) = n$, $\mathrm{GL}(V) \cong \mathrm{GL}_n(K)$.

Example 2.3.

(i) Consider $\mathbb{G}_a := \mathbb{A}^1$, $\iota(x) = -x$ and $\mu(x, y) = x + y$. This is an algebraic group, with unit 0. We shall call this the additive group.

(ii) Consider the algebraic group with underlying set $\mathcal{V}(xy - 1 = 0) \subseteq \mathbb{A}^2$, with component wise multiplication and and inverting. This group is isomorphic to $(K^\times, \cdot, 1)$. We shall call this group the multiplicative group and denote it by $\mathbb{G}_m$.

(iii) $\mathrm{GL}_1(\mathbb{K})$ is equal to the above multiplicative group, thus this is a linear algebraic group.

(iv) $\mathrm{T}_n(K)$, the set of upper triangular matrices is a linear algebraic group, as it is the set of zeros polynomials $\{X_{ij}\}_{j<i\leq n}$, and thus closed.

(v) $\mathrm{D}_n(K)$, the set of diagonal matrices is a linear algebraic group, as it is the set of zeros polynomials $\{X_{ij}\}_{i\neq j\leq n}$, and thus closed.

(vi) $\mathrm{U}_n(K)$, the set of upper triangular matrices with 1's on the diagonal is a linear algebraic group as it is the set of zero's for polynomials $\{X_{ij} - 1 = 0;\ i = j \leq n\} \cup \{X_{ij};\ j < i < n \text{ and } i \neq j\}$ and thus closed.

(vii) $\mathrm{SL}_n(K)$, the set of matrices with determinant 1, is a linear algebraic group, as it is the set of zeros polynomials $\det(X_{ij}) - 1$, and thus closed.

(viii) Similarly, representations of finite groups are linear algebraic groups, in fact any finite group is a linear algebraic group, as the permutation matrices of dimension $n \times n$ form matrix groups isomorphic to S_n, clearly these groups and subgroups are closed, thus by Cayley's theorem any finite group can be expressed as such a subgroup of a permutation matrix group, and is thus a linear algebraic group.

Remark 2.4.

We will talk about irreducible and connected components of algebraic groups. These will then refer to the subsets of the underlying variety of the group, and be meant in an topological sense.

Definition 2.5 (Morphism of algebraic groups).

Let G, H be algebraic groups then $\phi : G \to H$ is a *morphism of algebraic groups* if ϕ is a group homomorphism and a morphism of algebraic varieties.

Example 2.6.

(i) Let (G, μ, ι) be a commutative algebraic group then ι is a morphism of algebraic groups. Additionally since $\iota \circ \iota = \mathrm{id}_G$ it is an automorphism.

(ii) Conjugation, i.e. $\mu[g,\, \mu(-,\, g^{-1})]$ is an automorphism of algebraic groups.

Proposition 2.7.

(i) Let G be an algebraic group, then there is a unique irreducible component, denoted G^0, containing the identity element.

(ii) For algebraic groups a component is irreducible $\iff$ a component is connected.

(iii) $G^0 \trianglelefteq G$.

(iv) G^0 has finite index in G.

(v) Let $H \subseteq G$ be a closed subgroup of finite index, then $H \subseteq G^0$.

Proof.

(i) By Proposition 1.23 we may write $G = \bigcup_{i \leq m} V_i$ for V_i irreducible components. Let $\{V_j\}_{j \leq r}$ for $r \leq m$ be the subset of components containing the unit element $e \in G$. Now we take the multiplication map $M : G^r \to G$, defined recursively as $\mu\left(\mu(x_1,\, x_2),\, x_3\right)$ etc. This map is continuous, and $\bigtimes_{i \leq r} G_i$ is irreducible by Proposition 1.25, thus by Proposition 1.24 $M(\bigtimes_{i \leq r} V_i) \coloneqq X$ is irreducible. Since e is contained in all the subsets, we have that $V_i \subseteq X$, but we must also have that $V_{i_0} = X$ for some i_0, this can only be true if $r = 1$.

(ii) By (i), the irreducible component containing e is unique. Then, since $g \mapsto hg$ is a homeomorphism, there is also a unique irreducible component containing h. This is true for all $h \in G$, thus G is partitioned into irreducible components, thus these are the connected components.

(iii) Since taking the conjugate of G^0, maps it to an irreducible subset containing e, is must be an automorphism, thus G^0 is normal.

(iv) Since $\mu(g, -)$ is continuous, $\mu(g, G^0)$ is irreducible, but by Proposition 1.23 the number of irreducible subsets must be finite, thus the index of G^0 in G is finite.

(v) Let $\{H_i\}_{i \leq n}$ be the set of cossets of H. These partition G and are all closed, since multiplication is a homeomorphism. Then $\bigcup_{i \leq n} G^0 \cap H_i = G^0$, means that there is only one i, for which the intersection is non-empty (otherwise G^0 would not be irreducible). Now the union of the cosets not equal to H is closed. The complement of this union is H, thus H is also open. Then if $H \subseteq G^0$, H would partition G^0 in the subspace topology, thus $H \supseteq G^0$. The finiteness of the index is required for the union to remain closed. ■

Definition 2.8 (Identity component / connected).

(i) Let G be an algebraic group, then the irreducible subset of G containing the unit of G is called the *identity component*, and denoted G^0.

(ii) Let $G = G^0$, then we call G *connected*.

Proposition 2.9.

(i) Let G be an algebraic group, and H a closed subgroup, which is a connected component, then $[H, H]$ is connected.

(ii) Let G be an algebraic group, with H a closed normal subgroup, such that G/H is abelian and H^0 is solvable, then G^0 is solvable.

Proof.

(i) H a connected component, then H is irreducible, thus $H \times H \to H : (h_1, h_2) \mapsto h_1 h_2 h_1^{-1} h_2^{-1}$ is a morphism of algebraic groups and thus $[H, H]$ is connected.

(ii) G/H is abelian, thus $[G, G] \subseteq H$. Then $[G^0, G^0] \subseteq H$, and then by (i), $[G^0, G^0]$ is connected. As H^0 is a connected component and H^0 and $[G^0, G^0]$ contains the unit, $[G^0, G^0] \subseteq H^0$. H^0 is solvable, hence $[G^0, G^0]$ is solvable and then G^0 is solvable. ■

Remark 2.10.

Let $\phi : G \to H$ be a morphism of algebraic groups, then $\phi(G^0) = \phi(G)^0$. This follows from the fact that $e \in \phi(G^0)$, as ϕ is a group homomorphism, and must be connected as ϕ is continuous.

Definition 2.11 (Unipotent group / element).

A unipotent group is closed subgroup of U_n. We will denote such a group by $\mathbb{U}_n$. Each element of a unipotent group is unipotent meaning $\forall u \in \mathbb{U}\ \exists n \in \mathbb{N}$ s.t. $(u - I)^n = 0$, i.e. $u - I$ is nilpotent.

Proposition 2.12.

Let G be an unipotent group, then G is solvable, additionally there is a subnormal series $(G_i)_{i \leq n}$, where $G_i/G_{i+1} \cong \mathbb{G}_a$.

Proof. Let G be an n-dimensional unipotent group. Consider the chain.

$$\begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 1 \end{bmatrix} \subseteq \begin{bmatrix} 1 & 0 & * & \cdots & * \\ 0 & 1 & * & \cdots & * \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & * \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \subseteq \cdots \subseteq \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & * & \cdots & * \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & * \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \subseteq \cdots \mathrm{I}_{n\times n},$$

where we replace more and more off-diagonal elements by zero-entries. Consider the following map

$$\begin{bmatrix} 1 & a_{1,2} & \cdots & a_{1,n} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1,n} \\ 0 & \cdots & 0 & 1 \end{bmatrix} \mapsto \mathrm{I}_{n\times n} + a_{ij},$$

where a_{ij} is the matrix with only non-zero element a_{ij} at position (i,j). The image of this homomorphism is isomorphic to $\mathbb{G}_a$ and its kernel is of the from of a matrix in the above chain, thus from the first isomorphism theorem we see that the a chain consists of normal subgroups, and each quotient at each step of the chain is isomorphic to $\mathbb{G}_a$, which is abelian, thus G is solvable. ∎

Remark 2.13.

It is not hard to see that $\mathrm{D}_n \cong \mathbb{G}_m^n$, thus D_n is connected, as products of irreducible sets are irreducible. Let $\mathbb{D}_n$ denote a closed and connected subgroup of D_n, then $\mathbb{D}_n \cong \mathrm{D}_m$ for some $n \leq m$.

Theorem 2.14 (Lie-Kolchin, [50]).

Let V be a finitely generated vector-space. Let $G \subseteq \mathrm{GL}(V)$ be a connected solvable group, then there is a basis for V such that G is upper triangular.

Corollary 2.14.

Any connected solvable linear algebraic group G is isomorphic to $U \ltimes D$, for U a unipotent group and D a diagonal group.

Proof. By the Lie-Kolchin theorem we have that G is upper-triangular. The map which sends a matrix to it's diagonal is a homomorphism for upper triangular matrices. $\mathbb{U}_n$ is it's kernel, and thus a normal subgroup. Then clearly $\mathbb{D}_n\mathbb{U}_n = G$ and $\mathbb{D}_n \cap \mathbb{U}_n = I$, thus $\mathbb{D}_n \ltimes \mathbb{U}_n$. ∎

Definition 2.15 (Semi-simple group / element).

A semi-simple element of a linear algebraic group, is an element which is diagonalisable. A semi-simple group is a linear algebraic group in which each element is semi-simple.

Theorem 2.16 (Jordan-Chevally decomposition, [51]).

(i) Let G be an linear algebraic group, over a field with characteristic 0. Then every $g \in G$ can be decomposed as $g = g_s g_u = g_u g_s$, where g_s, $g_u \in G$ are respectively semi-simple and unipotent.

(ii) Let $\phi : G \to H$ be a morphism of algebraic groups, then $\phi(g_s) = \phi(g)_s$ and $\phi(g_u) = \phi(g)_u$.

Proposition 2.17.

Let K, H be unipotent groups and $e \longrightarrow K \xhookrightarrow{\phi} G \xrightarrow{\psi} H \longrightarrow e$ be a short exact sequence of algebraic groups. Then G is unipotent.

Proof. Since by Theorem 2.16, each element $g \in G$ is fo the form $g_s g_u$. Then $\psi(g) = \psi(g_s g_u) = \psi(g)_s \psi(g)_u$ is an element in H and thus unipotent, therefore $\psi(g)_s = e$. Thus the semi-simple elements of G are in the kernel of ψ and thus in the image of ϕ, and thus must be unipotent, hence g must be unipotent, thus G is unipotent. ■

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