University of Groningen

Bachelor's Thesis

Mathematics

Vassiliev knot invariants and the Conway weight system

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Abstract

Vassiliev invariant are conjectured to form a complete knot invariant making them one of the most powerful types of knot invariants. First we will discuss the basic theory of knots and knot invariants, after which we will develop the theory of Vassiliev invariants. In order to study Vassiliev invariants effectively we will explain chord diagrams, weight systems and the Fundamental theorem of finite-type invariants. The fundamental theorem of finite-type invariants states that each Vassiliev invariant corresponds to a weight system, which means we can generate Vassiliev invariants from given weight system. In this thesis we will discuss a prove of this fundamental theorem. We will also solve the problem of the dimension of the vector space of Vassiliev invariants of a given order. In the end we define the Conway weight system, give its corresponding Vassiliev invariant and show a connection with the adjacency matrix of the intersection graph of the knot.

Foreword

First and foremost, I would like to thank my supervisor Roland van der Veen for his guidance during this project. He allowed me to study interesting new mathematics and offered help and support whenever needed. I would also like to thank Daniel Valesin for agreeing to be my second supervisor.

Secondly, I would like to thank Jorge Becerra for co-organizing the knot theory seminar and many helpful comments and tips for learning knot theory. The knot theory seminar has been a valuable part of my studies as a place to learn more knot theory and topology, and be able to talk with others about my thesis.

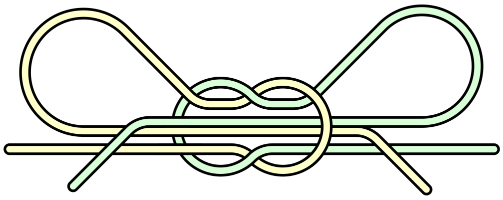
Lastly, the idea for this thesis came from a talk given by Dror Bar-Natan who visited Groningen in February 2020 and gave a talk in the Groningen knot theory seminar. This talk forms the basis for the ideas presented. A recording of this talk can be found here: [1, 2]. I want to thank professor Bar-Natan for giving this talk and uploading it afterwards, because without this talk my thesis would have been a lot harder to write.

Groningen, July 22, 2020

Contents

1 Introduction

Intuitively, a knot is nothing more than a piece of string which is tied up in some way and crosses itself at finitely many places. Knots appear in a variety of places, from maritime transport to mountain climbers and from your shoelaces to very complex knots that can be found in biochemistry. Knots are objects that have been studied by humankind for ages, either for applications or as a symbol in mythology [1]. However, it was not until the 19th century that mathematicians began to study these objects. When starting these studies, two questions immediately arose. First, can we untie any given knot with some algorithm? And second, given we have a way to untie a given knot, which other knots can we untie? Many other problems followed, but both questions are still being discussed until this day. In this thesis we will mainly be studying the second question, and ask ourselves which knots are the same and which are different.



(a) Shoelace knot [3].

(b) Gordian knot in mythology, painting by Jean-Simon Berthélemy [4].

[1]For example: the story goes that there was a knot, called the Gordian knot, which was impossible to untie. An oracle had predicted that whoever untied the knot would rule over Asia. After many people tried and failed, Alexander the Great is said to have untied the knot with his sword after which he ruled over Asia.

It was Gauss who started a tabulation of knots. In notes from 1794 he drew the first knots, and started checking whether they were the same ([5]). His study of knots did not prove to be very fruitful, until much later when he used knots in order to work on a theory in electrodynamics. He wanted to know how much energy it took to move a magnetic pole around a closed loop in the presence of a current. In the process he derived what is now known as the *Gauss linking integral* which will be discussed in an example in section 2. After Gauss many physicists came up with theories using knots. Most well-known is the theory by Lord Kelvin, who stated that particles must be made out of three-dimensional knotted tubes of ether (which was proven not to exist by Einstein). These kind of theories in physics led to more and more people thinking about knots, and the research into knot theory continues until this day with many applications in DNA research and theoretical physics. In this thesis we will see some of this modern research into knot theory, but before we can do that we will start in section 2 by defining what a knot is and develop a few tools to study them effectively.

The study of equivalence of knots is a topological question. If we can transform a knot into another knot by stretching, squeezing and moving it through space we can say these knots are indeed equivalent. However, it turned out topology is not enough to classify knots properly. We will turn to many different areas of mathematics like algebra, combinatorics and geometry to develop tools to study knots. One of the new tools is the knot invariant. Very simplistically, a knot invariant is a function which assigns the same number (or polynomial, or an other object like colour or shape) to all topologically equivalent knots. So if we calculate the number on knot A to be 42 and on knot B to be pi, we must have that these knots are not equivalent. It however turns out that it is very hard to make an invariant which actually distinguishes each pair of knots. In section 2 we will see many of these invariants. Especially the Alexander-Conway polynomial will be of great importance and in section 2.5, we will spend a great deal of time on showing this is actually a well-defined invariant.

A special class of knot invariants are the *Vassiliev invariants*. These invariants are special because the set of all Vassiliev invariants is conjectured to be 'powerful' enough to distinguish any pair of knots. Moreover, any invariant which is not Vassiliev can be approximated by Vassiliev invariants. These Vassiliev invariants were first created (or discovered, depending on your philosophy) by Victor Vassiliev in 1989. However, the definition as given by Vassiliev is a bit different from the definition which we will present, which was independently created by Joan Birman and Xiao-Song Lin [6], and Mikhail Goussarov [7]. The Vassiliev invariant will be the main topic of study for this thesis and will be defined in the beginning of section 3. In this section we will also give more precise motivation for studying the Vassiliev invariant.

Given that all Vassiliev invariants together might distinguish any pair of knots, two questions need answering. First of all, how many Vassiliev invariants are there? If we need to apply all of them it would be nice if there is a finite amount and if we know whether we applied them all. Secondly, how do we make new Vassiliev invariants? To answer both these questions we will need the concept of a *chord diagram* and a *weight system*. More details will follow in section 3, but to give you an idea a chord diagram is an object which stores the combinatorial information of a knot (i.e. the order in which the knot crosses itself when walking around the knot), and a weight system will be a specific function which attaches a number or a polynomial to this combinatorial information. These weight systems turn out be connected to Vassiliev invariants. This result is known as the Fundamental theorem of finite-type invariants (or Kontsevich theorem, after the person who proved the theorem). It states that each weight system corresponds to a Vassiliev invariant and vice versa. Therefore, if we can make a new weight system we make a new Vassiliev invariants. Similarly, if we know how many weight system there are, we also know how many Vassiliev invariants there are. The theorem will be of great importance and we will be proven (at least part of it) in section 3.

In section 4 we will see the fundamental theorem in action on an explicit example of a weight system called the *Conway weight system*. We will use this weight system to find an invariant, which turns out to be the Alexander-Conway polynomial. This weight system will also give us an interesting connection with the intersection graph of a knot which will be seen in section 4.5. For the convenience of the reader there is also an appendix with a few relevant results from linear algebra and module theory.

In summary, section 2 will be about knots and knot invariants. A few basic examples will be discussed. Section 3 will introduce the concepts of a Vassiliev invariant, chord diagrams and weight systems. Here, we also present a proof of the fundamental theorem of finite-type invariants. In section 4 we discuss an example of a weight system: the Conway weight system. We will also derive the invariant related invariant.

For those interested, most of the figures and uncommon LaTeX symbols are made by myself (unless stated otherwise) using Inkscape and my own knot theory LaTeX package. In case someone wants to have this package, they can contact me at o.l.kosterATstudent.rug.nl.

2 Introduction to knots and invariants

Knots are important objects in our daily lives, they keep our shoes in place, sailing boats going and DNA working. But there are many different flavours of knots, for example some knots feel impossible to untie and some knots fall apart just by looking at them. Some knots are simple (the knot in your shoelaces, at least after some practice) and some knots are more complex (the knots in proteins or DNA for example). All these knots have to be fitted under one definition. We will start by giving a mathematical definition of a knot in this section after which we will develop some tools to study these knots effectively.

2.1 Definition of a knot

Intuitively a knot is a knotted loop of string in space. There are several ways to define a knot, the simplest definition is given as follows:

Definition 2.1. A **parametrised knot** is an embedding of the circle $\mathbb{S}^1$ into $\mathbb{R}^3$.

Recall that for smooth manifolds M and N an *embedding* $f : M \to N$ is a smooth injective map, such that the image $f(N)$ with the subspace topology is homeomorphic to N under f. In other words, it is an injective immersion. More intuitively, a knot is a piece of string, in which we make loops in and move through space, and then tie the ends together. This might not be like the knots we see in daily life[2], as there the ends are usually not joint together.

There are many more possible definitions of a knot, for example a *topological knot* K^{Top} is a subset of $\mathbb{R}^3$ which is homeomorphic to the circle $\mathbb{S}^1$. An example which will come back a lot in this thesis is so called the *trefoil knot*, which is depicted in figure 1.

[2]This more intuitive knot will be called a *tangle*, we will give a better definition of a tangle in section 4.2.

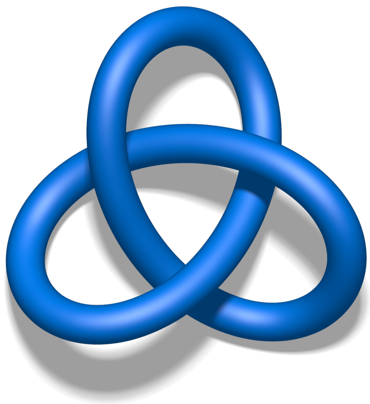


Figure 1: A 3D depiction of the trefoil knot [8].

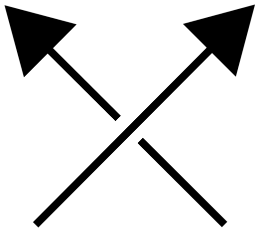
In this thesis we will denote knots by capital letters (e.g. $K, K_1, \tilde{K}$), while the 'corresponding' embedding is given by a lower case letter (e.g. $f, g : S^1 \to \mathbb{R}^3$).

A circle S^1 has two choices of an orientation. This means when walking over the knot, we can go in two directions. If we pick one, we give an orientation to all knots coming from this circle. A way to state this more formally is as follows:

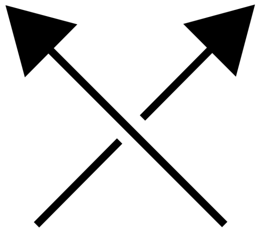
Definition 2.2. An **oriented knot** is an embedding of an oriented circle in $\mathbb{R}^3$. A link is called oriented if each of its components has an orientation. An **unoriented** knot is an embedding of a circle without a choice of orientation.

We will assume a knot is oriented counter-clockwise unless stated otherwise.

It is usually quite hard to draw knots in three dimensions. Therefore, we use *knot diagrams*. A knot diagram is a planar curve whose only singularities are transversal double points. These double points will be called crossings. We can make a choice for which branch we would would like to have on top of the other one, we will call one choice a *positive crossing* and the other one will be called a *negative crossing*. The choices are depicted in the following figure.



(a) Positive crossing



(b) Negative crossing

In other words, a knot diagram is a projection of a knot on the plane. An orientation of a knot will be denoted by arrows. When speaking about knots, we will mostly be referring to the corresponding knot diagram. A few examples of knots (and their knot diagrams) are:

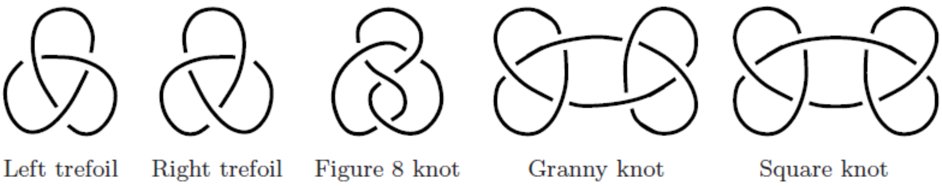


Figure 2: A few example of knots. [8]

There is also the case when the knot is the embedded circle itself. In this case we speak of the *unknot* or the *trivial knot*. In some sense this is the most basis knot there is. It will be denoted by ◯.

Another useful definition is the one of a link. A link is a combination of cut open knots, which are glued together.

Definition 2.3. A **link** L is a disjoint embedding of a finite number of copies of $\mathbb{S}^1$ into $\mathbb{R}^3$. Each copy of $\mathbb{S}^1$ is called a **component** of the link.

One can say that each knot is a link, when it is not connected to any other knot. We again use diagrams to depict links in an easier way. Some examples of links are:

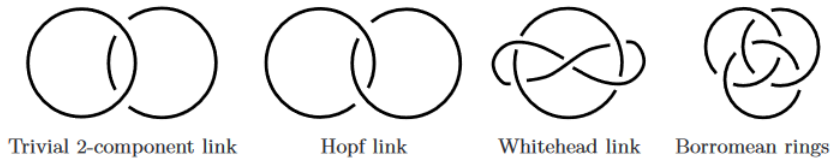


Figure 3: A few examples of links. [8]

Especially, the example of the Hopf link will come back many times in this thesis. In many cases, a result proven for knots will also hold for links. For most theorems in this thesis we will hence use knots where it could say links. Usually, the proof is the same for links.

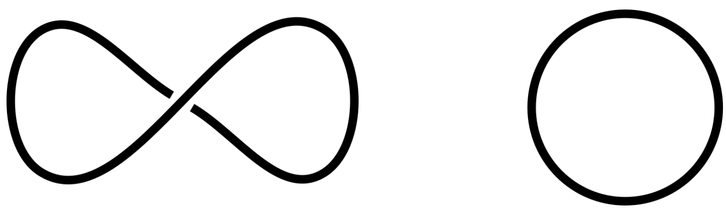
2.2 Isotopy

Using definition 2.1 there are infinitely many possible knots. To make the study of knots a bit more feasible we want to consider knots only up to a suitable notion of equivalence. As was mentioned in the introduction the equivalence of knots is a topological problem. Intuitively this means, we want to say two knots are equal when we can deform the first knot into the second knot and vice versa by a continuous deformation. This is made formal by a so called isotopy.

Definition 2.4. Two parametrised (smooth) knots $f_0, f_1 : S^1 \to \mathbb{R}^3$ are called **isotopic** if there is a smooth map $F : S^1 \times I \to \mathbb{R}^3$ such that $F(-, t)$ is an embedding for all t and $F(-, 0) = f_0$ and $F(-, 1) = f_1$

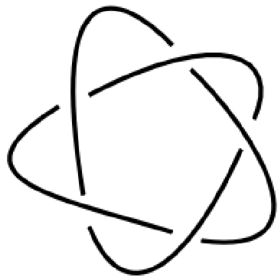
Using this definition we say that two knots are *equivalent* if they are isotopic.

Example 2.5. An example of a two isotopic knots are the following two knots.

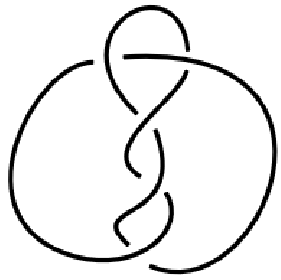


These knots are isotopic because the knot with a crossing (l) can be twisted and deformed to become the unknot (r).

An example of two knots which are not isotopic are the following two knots.



(a) Knot 5_1



(b) Knot 5_2

It is a bit harder to see that these two are not isotopic. In order to check this we will need new tools, which we will develop in section 2.4.

The notion of isotopy gives us a more precise definition of the unknot, as we can now say a knot isotopic to $\mathbb{S}^1$ is called the trivial knot or unknot.

In the definition of isotopy only the embedding of the circle itself gets deformed. In some cases it might be more useful to consider the case where also the space around the embedding is deformed. To work in this cases we define an ambient isotopy.

Definition 2.6. Two parametrised knots, f and g are **ambient isotopic** if there is a smooth map $\Psi : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ with the property that $\Psi(-,t)$ is a diffeomorphism for all t with $\Psi(\theta,t) = \psi_t(\theta)$ such that $\psi_0 = \mathrm{id}$ and $\psi_1 \circ f = g$.

We will not use ambient isotopy often in this thesis, since in [9, Chapter 1] is is shown that ambient isotopy and regular isotopy are equivalent definitions. However, we do need the definition for the following theorem to make sense. This theorem, called the Reidemeister theorem, will give us a simple way of checking whether two knots are isotopic.

Theorem 2.7 (**Reidemeister's theorem**)**.** Two unoriented knots K_1 and K_2 are isotopic if and only if a diagram of K_1 can be transformed into a diagram of K_2 by a sequence of ambient isotopies of the plane and one of the local moves R_1, R_2 and R_3 given in the following diagrams:

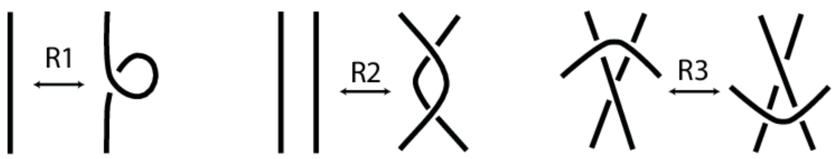


Figure 4: Unoriented Reidemeister moves

In other words, the equivalence class of knots under isotopy, is the same as the equivalence class of knots under planar isotopy and Reidemeister moves. The proof is too long to present here, but can be found in [10, Chapter 4].

We need to make two remarks about the Reidemeister theorem. First of all, the theorem is useful when we want to check two knots are indeed isotopic, but it is more complicated to show two knots are not isotopic using this theorem. Moreover, it is also not always easy to see that two knots are related by a finite number of Reidemeister moves. There are some famous examples of knots where one would need to do counter-intuitive Reidemeister moves to obtain an isotopic knot. Secondly, in the case for oriented knots, we have to equip the Reidemeister moves with all possible orientations for the theorem to work. For example, for oriented knots, the Reidemeister II moves are given in figure 5.

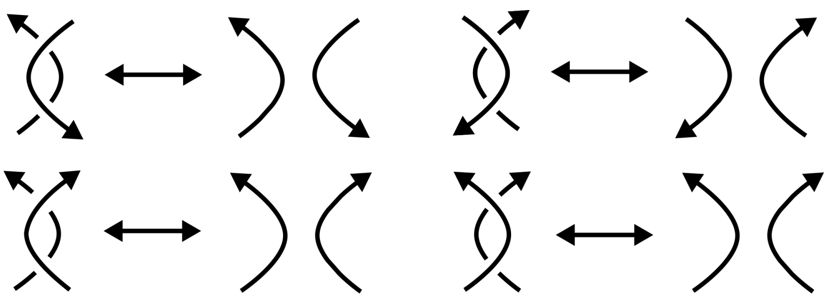


Figure 5: Oriented Reidemeister II moves.

Now that we have defined equivalence of knots we can start checking which knots are equivalent. To try this we use by the following lemma.

Lemma 2.8. Every pair of knots K_1 and K_2 are equivalent under Reidemeister moves and changing crossings from a positive crossing to a negative crossing or vice versa.

Proof. The idea of the proof is to show that using crossing changes and Reidemeister moves all knots can be made into the unknot. If this is the case we can change each knot K_1 into the unknot and then change the unknot into K_2.

Consider a knot diagram K_1 with $n \in \mathbb{Z}_{\geq 0}$ crossings in a plane with basis vectors X and Y. Pick a base point on the knot and pick an orientation, let us pick the counter-clockwise orientation. Take the knot diagram of K_1 in the XY-plane and parametrize the knot and walk around the knot increasing the height (Z-direction) proportionally to the length travelled over the knot. Walking around the knot we change each crossing such that we always first pass the crossing as the under strand, and the second time as the over strand. This means that each time we encounter a positive crossing we change it to a negative crossing. After we encountered the last crossing before getting back to the base point we decrease the height continuously along the Z-axis until we are back at the base point in the XY-plane. An example is for the trefoil is shown in figure 6.

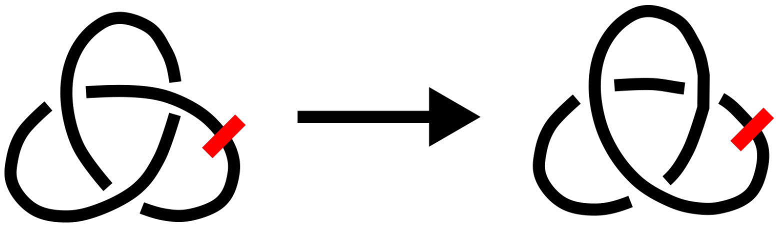


Figure 6: Change a crossing from the trefoil to get an ascending diagram. Notice that we could lift up the diagram from the plane forming a spiral like structure.

Notice that the path we created is an increasing loop until the n-th crossing, after which is decreases without crossings. This is isotopic to the unknot by applying Reidemeister move R_1 multiple times. Therefore, we can make the unknot from any given knot K_1 by changing crossings and Reidemeister moves, which proves the lemma. $\square$

2.3 Invariants

One of most fundamental problems in knot theory is deciding whether two knots are indeed equivalent. Take for example the Haken's Gordian knot [3] (figure 7).

[3]This knot is named after the knot cut by Alexander the Great mentioned in the introduction.

This knot looks complicated, but it is actually the unknot under a finite number of Reidemeister moves.

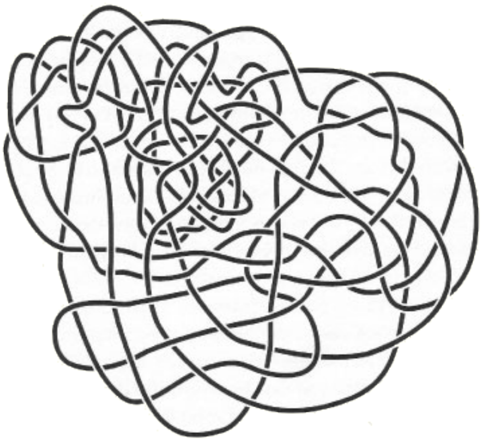


Figure 7: Haken's Gordian unknot[8].

The Reidemeister theorem is a good first step for distinguishing knots, but as was mentioned before it is not always easy to use. To solve this problem, we instead look for other properties of the knot which could distinguish two knots. This leads to the notion of a knot invariant.

Definition 2.9. A **knot invariant** is a map $f : \{\text{Knots}\} \to S$, where S is a set such that if two knots K_1 and K_2 are equivalent then $f(K_1) = f(K_2)$. Where $\{\text{Knots}\}$ denotes the set of all knots.

A knot invariant is hence a function which 'recognizes' whether two knots are inequivalent. However, notice that it does not have to be the case that if $f(K_1) = f(K_2)$ then K_1 and K_2 are equivalent. A function which has this property is called a complete knot invariant.

Definition 2.10. A knot invariant g is called **complete** if g has the property that for every pair of knots K_1 and K_2, $g(K_1) = g(K_2)$ if and only if K_1 and K_2 are equivalent knots.

Up until today there is only one known complete knot invariant. This is the fundamental group of the knot complement $\mathbb{R}^3 \setminus K$ plus some extra properties. The details and definitions are quite technical, so they will not be discussed here. For more more information about this one could also look at [11, Chapter 11], [12] or [13, Chapter 5].

Often an invariant can be easily extended to links. We will now give two examples of knot invariants.

Example 2.11 (**Crossing number**)**.** One of the most trivial things to do when you want to determine if two knots are isotopic is to check whether the two knots have the same number of crossings. Of course, using Reidemeister moves one could create more crossings by moving two strands of the knots over each other, to avoid this we will use the minimum number of crossings in a knot.

Definition 2.12. The **crossing number** of a knot K is the minimum number of crossings in a diagram of K.

The fact that we do have to make sure the knot diagrams have the minimum number of crossings makes this invariant hard to compute in some cases. It is also not a complete knot invariant. This can be seen when we consider the knots of figure 4a and 4b, we notice that both these diagrams have the minimal number of crossings, hence the crossing number is five. But as was mentioned these knots are not isotopic, hence the invariant is not complete for all knots.

Example 2.13 (**Linking number**)**.** We will first consider the sign of a crossing. The sign of a crossing is $+1$ for a positive crossing and -1 for a negative crossing. In figure 8 we see an example for a link. The linking number is only defined on links with two components.

Definition 2.14. Let L be an oriented link with components $L_1, \ldots, L_n$ and let D be the link diagram corresponding to L with components $D_1, \ldots, D_n$. The **linking number** $\operatorname{lk}(i, j)$ is defined to be the sum of all the signs of crossings between the components D_i and D_j where $i, j \in \{1, \ldots, n\}$ and $i \neq j$.

For example, consider the following link diagram with components a, b and c.

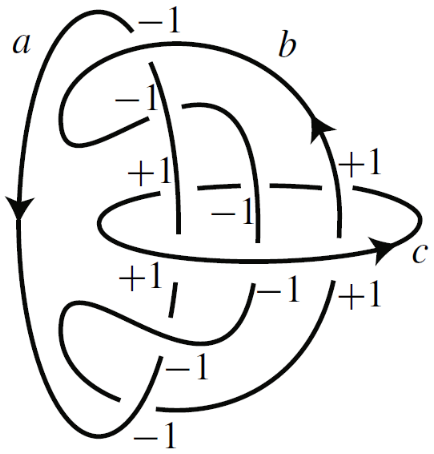


Figure 8: Link with components a, b and c [14].

In this case $\mathrm{lk}(a, b) = -4$, $\mathrm{lk}(a, c) = +2$ and $\mathrm{lk}(b, c) = 0$.

Notice that the linking number is not complete, simply because it cannot be computed on each knot.

The linking number is a nice invariant because there are several ways to express it as an integral. Most notably the following theorem by Gauss.

Theorem 2.15 (**Gauss' linking integral**, [9])**.** Let γ_1 and γ_2 be two non-intersecting differentiable curves in $\mathbb{R}^3$, the Gauss map $\Gamma : \mathbb{S}^1 \times \mathbb{S}_1 \to \mathbb{S}^1$ is given by:

$$\Gamma(s, t) = \frac{\gamma_1(s) - \gamma_2(t)}{|\gamma_1(s) - \gamma_2(t)|}.$$

The degree of the Gauss map $\deg(\Gamma(s, t))$ is given by the number of times the normalized vector connecting a point on $\gamma_1(s)$ to a point $\gamma_2(t)$ goes around the sphere $\mathbb{S}^2$.

Then the linking number is given by:

$$\mathrm{lk}(\gamma_1, \gamma_2) = \deg(\Gamma(s, t)) = \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{\det(\dot{\gamma_1}(s), \dot{\gamma_2}(t), \gamma_1(s) - \gamma_2(t))}{|\gamma_1(s) - \gamma_s(t)|^3} dsdt.$$

This gives an easy way to compute the linking number and has many applications electromagnetism (as was shown in the introduction) and in quantum field theory.

2.4 The Alexander-Conway polynomial

We have seen two invariants which take only integer values. We shall now show an example where the invariant takes a polynomial value.

The Alexander-Conway polynomial is defined recursively over the crossings of the knot using a so called *skein relation*.

Definition 2.16. The **Alexander-Conway polynomial** [4] $C(K) : \{\text{Knots}\} \to \mathbb{Z}[t^{\pm 1}]$, of a knot K, is defined by the skein relation:

1. $C(\bigcirc) = 1$
2. $C(\ ) - C(\ ) = tC(\ )$

Here $\bigcirc$ denotes the unknot. The term means we annul the given crossing. The way to do this is by making a straight line through a crossing, everything on a given side of the crossing becomes one connected component. In case the orientations of both strands are the same we make a vertical line, in case the orientations of the strands are reversed we make a horizontal line. This is illustrated figure 9.

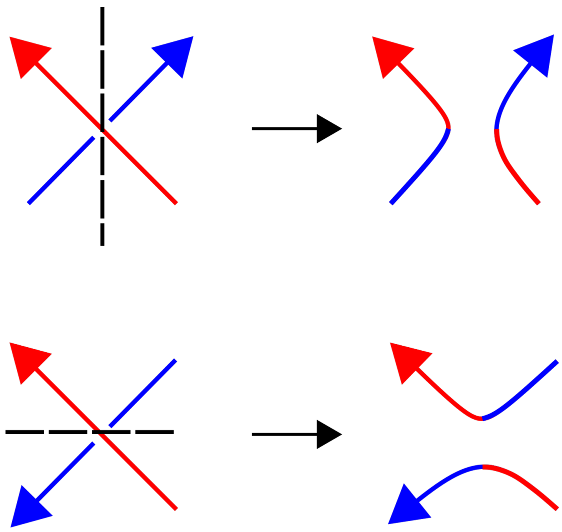


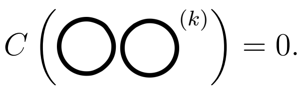
Figure 9: Smoothing of a crossing.

[4]A polynomial in x and x^{-1} is a Laurent polynomial

From the definition is is not clear whether this gives a well-defined knot invariant, in theorem 2.21 it wil be shown this is indeed the case.

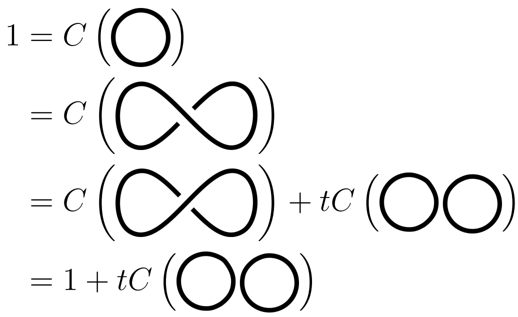
The Alexander polynomial was originally defined by J.W. Alexander in 1928 without using the skein relation but using algebraic topology (More details can be found in [11, Chapter 6]). Only later, the skein relation was found by J.H. Conway in 1967 which is why the skein relation is called the **Conway skein relation** and the associated polynomial is called the Alexander-Conway polynomial. We will stick to using the Conway skein relation as it is much more convenient to use for our purposes.

This invariant can be extended to links, but to do so we need to compute C on k disconnected links. This extension for links is necessary, because the term might give us terms which are no longer homeomorphic to one connected circle. Let us denote k disconnected unknots by $^{(k)}$.

Lemma 2.17. 

Proof. The proof will be given for $k = 2$ and can be extended to arbitrary k.

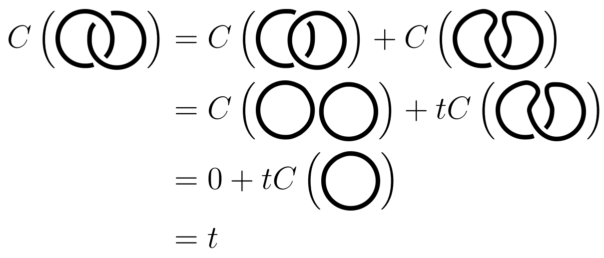
Notice that the value of C on and must be the same as they are isotopic. And that the value of can be computed using the skein relation. We hence get:



This can only be the case if $C\,(\ ) = 0$. To do the same proof for $^{(k)}$ we make k crossings in the unknot and repeat the same steps.

□

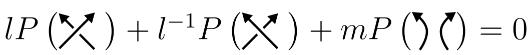
Example 2.18. We can now for example compute the Alexander-Conway polynomial of the Hopf link by rewriting the second term of the skein relation.



Example 2.19. Using the Conway polynomial, we can also finally see that the knots in figure 4a and 4b of example 2.5 are indeed not isotopic. Let us denote the knot in figure 4a by K_a and the knot in figure 4b by K_b. Then we compute $C(K_a) = t^4 + 3t^2 + 1$ and $C(K_b) = 2t^2 + 1$ [5]. Notice that $C(K_a) \neq C(K_b)$ and hence K_a and K_b are not isotopic.

The Alexander-Conway polynomial is part of a larger family of knot invariants called the *HOMFLYPT polynomial*. This is defined as follows.

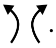
Definition 2.20. The **HOMFLYPT polynomial** $P : \{\text{Links}\} \to \mathbb{Z}[l^{\pm}, m^{\pm}]$ given by the following skein relation for a link L:

1. $P(\bigcirc) = 1$
2. 

Here {Links} denotes the set of all links. When we take the variables $m = -t$ and $l = 1$ we get back the Alexander-Conway polynomial, showing the HOMFLYPT polynomial is indeed a generalization of the Alexander-Conway polynomial. It turns out there are infinitely many pairs of distinct knots with the same HOMFLYPT polynomial [14]. This means the HOMFLYPT polynomial is not a complete invariant. As it is a generalization of the Alexander-Conway polynomial, we also know this is not a complete invariant.

We have seen how the Alexander-Conway polynomial can be useful when we want to distinguish knots. However, we did not actually show it is a unique and well-defined invariant. For generality, we will prove the HOMFLYPT polynomial is indeed a well-defined link invariant. The proof is quite long and technical, but the result will be useful in section 4.4.

[5]These values were computed using Mathematica which has packages to compute these polynomials.

Theorem 2.21. Let D_+ be the diagram , D_- be the diagram  and D_0 be the diagram . There is a unique function $P : \{\text{Links}\} \to \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ such that P takes the value 1 on the unknot and if L_+, L_- and L_0 are links which are the same except for a single point where they have diagrams D_+, D_- and D_0, then

$$lP(L_+) + l^{-1}P(L_-) + mP(L_0) = 0.$$

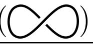
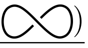
The proof of this theorem can be found in [11, Chapter 15], but I have chosen to present the proof here because there are a few unclarities and gaps to be filled in by the reader in the proof given in Lickorish. We will however closely follow the proof presented there.

Before we can prove the theorem we first need to prove two lemma's and give a three definitions which will be useful in the proof.

Lemma 2.22. Given that we denote k disconnected unknots by $\bigcirc\bigcirc^{(k)}$ we have that:

$$P\left(\bigcirc\bigcirc^{(k)}\right) = \left(-\frac{l+l^{-1}}{m}\right)^{k-1}.$$

Proof. The proof is similar to the proof of lemma 2.17. The unknot has only one component, so the value on the unknot is $P(\bigcirc) = 1 = \left(-\frac{l+l^{-1}}{m}\right)^{1-1}$. Rewriting the skein relation we can write:

$P(\bigcirc\bigcirc) = \big(-lP($$) - l^{-1}P($$)\big)/m = \left(-\frac{l+l^{-1}}{m}\right).$

When we use induction on the number of components k and apply the observation of the value on the unknot and the disconnected unlinks above, the result follows. □

The second lemma we need is a result from planar geometry.

Lemma 2.23. Suppose p and q are two arcs in $\mathbb{R}^2$ intersecting each other only at their endpoints A and B. Let R be the compact region bounded by p and q. Let $t_1, t_2, \ldots, t_n$ be arcs traversing R such that each t_i with i in $\{1, 2, \ldots, n\}$ intersects p once in one of its endpoints and intersects q once in the other endpoint. Suppose the intersections of all arcs are transverse [6] and there is no point where more than two arcs intersect. If any pair of traversing arcs t_i and t_j intersect in at most one point then the graph where the vertices are given by the intersections and the edges are given by the arcs t_i, p and q separates the region R into a collection of v-gons [7]. Moreover, if the set of traversing arcs is not empty, it is the case that in this collection of v-gons there is a 3-gon with an edge in p and a 3-gon with an edge in q [11, Lemma 15.1].

Proof. This lemma is proven by induction. Let n be the number of arcs traversing R. Suppose $n = 1$, in this case there is only one arc t_1 traversing R. Call the endpoints of t_1 X and Y respectively, then there are two 3-gons in R namely XYB and XYA. As the edges XA and XB are in q and YA and YB are in p, there exist a 3-gon with an edge in p and there exist a 3-gon with an edge in q. This is depicted in figure 10.

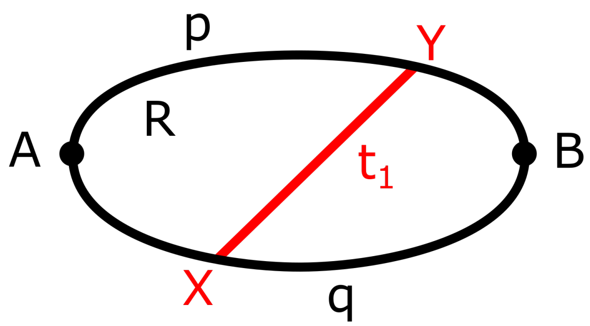


Figure 10: Base case of the lemma.

For the induction hypothesis, assume that for a region R with less than or equal to $n-1$ arcs crossing it, the statement holds. Now let R be a region between arcs p and q with $t_1, \dots, t_n$ traversing the region such that each pair of arcs intersects at most one time. The following algorithm gives use the desired 3-gons with an edge in q. The proof is also shown in figure 11.

1. Given the set of arcs $\{t_i : 1 \leq i \leq n\}$ pick the arc t_j with the endpoint in p which is closest to A. Call this endpoint in $p :$ X, and the corresponding endpoint in q: B.

2. If possible select a t_k from $\{t_i : i \neq j\}$ which intersects t_j in a point which

[6] An intersection of curves is called transverse if at every point of the intersection the tangent spaces at the given points generate the tangent space of the ambient space.

[7] A v-gon is a polygon with v vertices.

we call X' and intersects q in a point we will call A' such that t_k does not intersect any arc between A' and X'. If there are multiple options for the arc t_k, pick the one which gives X' a close as possible to B'.

3. In case there exist no t_k as described in step (2) select the arc p instead taking $A' = A$ and $X' = X$.

4. Now one can form an arc p' by following the lines $A'X'$ and $X'B'$. Moreover, label the segment $A'B'$ in AB by q'. Notice that p' and q' enclose a region which we call $R' \subseteq R$.

5. In case R' in not traversed by any t_i in $\{t_i : 1 \leq i \leq n, i \neq j, i \neq k\}$ then R' is our desired 3-gon with an edge in q.

6. In case R' is traversed by some arcs t_i in $\{t_i : 1 \leq i \leq n, i \neq j, i \neq k\}$ then there are at most $n-2$ arcs traversing R', so by the induction hypothesis there must exist a 3-gon in R' with an edge in q'. As q' is a segment of q, we have found the desired 3-gon.

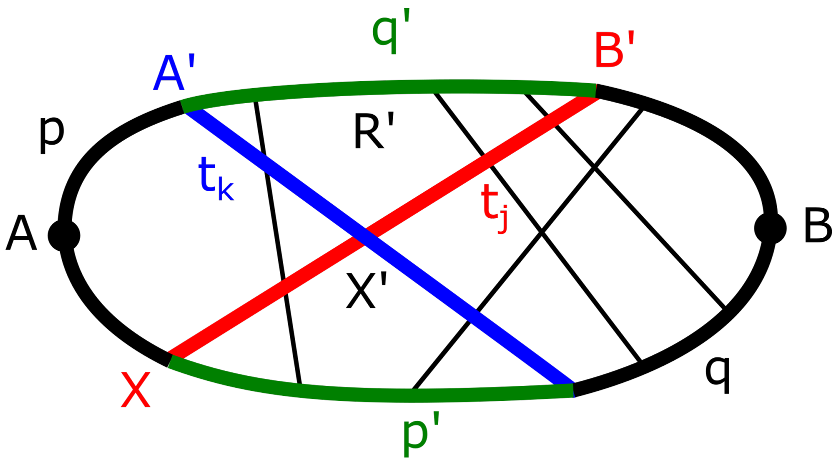


Figure 11: Induction step of the lemma.

A 3-gon with an edge in p can be found with a similar procedure. $\square$

The proof of the theorem will use a strategy similar to the proof of lemma 2.8. To do this let us define some terminology.

Definition 2.24. A diagram D of an oriented link with n-components is called **ordered** if an ordering is chosen for the link components $L_1, L_2, \ldots, L_n$. The diagram D is called **based** if a base point is selected in each link component from where a walk around that component starts.

This definition intuitively means, when walking over the link we have a certain order in which we need to go through the link components and we need to start the walk around each component at a specific point.

Definition 2.25. An ordered based diagram D of an oriented link is called **ascending** if in a walk around the link in the given order, each walk around a components starts at the base point and each crossing is first encountered as an under pass.

Given a diagram D we can make the corresponding ascending diagram αD by changing crossings from positive to negative. αD is the link above the diagram D in which each link component is entirely above the link components preceding it in the ordering. Now we can finally start the proof of theorem 2.21.

Before we start proving the theorem let us first give a sketch of the proof. In order to show P is a well-defined invariant (independent of the chosen ordering of link components, base points of link components and independent under Reidemeister moves) we will need to:

1. Define P on link diagrams. We will use induction on the number of crossings.
2. Show that:
$$lP(D_+) + l^{-1}P(D_-) + mP(D_0) = 0. \qquad (\diamondsuit)$$
holds for diagrams D_0, D_- and D_+ corresponding to the links L_0, L_- and L_+ respectively.
3. Verify that P is invariant under the Reidemeister moves, otherwise we could have two equivalent links K_1 and K_2 with values $P(K_1) \neq P(K_2)$.

The first two points will follow from a given inductive definition for which we need to show it is well-defined. The last point will be shown using the skein relation in ($\diamondsuit$). We will show that the value of P does not change under Reidemeister moves. Notice that in the skein relation ($\diamondsuit$) any of the terms $P\left(D_{+,-,0}\right)$ is uniquely determined by the knowledge of the two other terms. Moreover, one can check a general solution to ($\diamondsuit$) is given by:

$$\left(P\left(D_+\right), P\left(D_-\right), P\left(D_0\right)\right) = \left(x, x, \mu x\right),$$

where $\mu = -m^{-1}(l + l^{-1})$.

We can now start the proof of theorem 2.21.

Proof. Let us denote the set of oriented link diagrams with n crossings by $\mathcal{D}_n$. Using induction on the number of crossings we define the function $P : \mathcal{D}_n \to \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ on $\mathcal{D}_n$ as follows:

i ($\diamondsuit$) holds for any three diagrams in $\mathcal{D}_n$, related such as D_0, D_- and D_+, .

ii $P(D)$ does not change when we apply a Reidemeister move on D, which while applying the Reidemeister, move never involves more than n crossings.

iii If D is an ascending diagram of a link diagram in $\mathcal{D}_n$ with $\#D$ link components, then $P(D) = \mu^{\#D-1}$.

To prove the above we first show the properties hold on $\mathcal{D}_0$. Notice that any link without any crossing is an unlink with k components. Using lemma 2.22 we know that it must have the value μ^{k-1}. Because the k component is trivially ascending this means property (iii) is satisfied. Using Reidemeister moves does not change the number of components because we are still not allowed to have more than zero crossings. This means that again by lemma 2.22 we find $P(D) = \mu^{k-1}$, and the value is unchanged by Reidemeister moves. Therefore, property (ii) is also satisfied on $\mathcal{D}_0$. In order to show property (i) holds we first need to make sense of what D_+, D_- and D_0 are in this case. We know that D_0 is a version of $D_{+,-}$ ($D_{+,-}$ denotes the diagram with either a positive crossing or a negative crossing) where the crossing is left out. In this case let D_0 be the k-component unlink $(\bigcirc\bigcirc)^k$. Now adding a crossing between two components means we get that two of the components reduce to one component of the form  or . But these are both isotopic to the original unknot. This means that D_+ and D_- are $k-1$ component unlinks. Now let $x := (\mu)^{k-2}$. If we plug the above in the skein relation ($\diamondsuit$) and apply lemma 2.22 to find the value of P, we obtain:

$$
\begin{aligned}
lP(D_+) + l^{-1}P(D_-) + mP(D_0) &= l\,(\mu)^{k-2} + l^{-1}\,(\mu)^{k-2} + m\,(\mu)^{k-1} \\
&= lx + l^{-1}x + m\mu x \\
&= 0.
\end{aligned}
$$

This is zero, because $(x, x, \mu x)$ is a solution to ($\diamondsuit$). This shows property (i) and the base case of our induction.

Now assume that P is well-defined and property (i), (ii) and (iii) hold on diagrams with $n-1$ crossings $\mathcal{D}_{n-1}$. We can extend $\mathcal{D}_{n-1}$ to $\mathcal{D}_n$ using the following procedure. Pick a diagram D with n crossings, pick an ordering and a base point on each component, and let αD be the associated ascending diagram. Notice that in an ascending knot diagram each component is equivalent to the unknot in the same way as in lemma 2.8, this means that the value of $P(\alpha D) = \mu^{\#D-1}$. Notice that we can now walk through each component and change the crossings from negative to positive to obtain D. The value before and after the crossing change will be $P(D_-)$ and $P(D_+)$. This means the value $P(D_-)$ is known by recursion. The value of $P(D_0)$ will also be known, since by definition D_0 has an annulled crossing, and hence at least one crossing less than D. Using the induction hypothesis the value of D_0 will also be known. From the values $P(D_-)$ and $P(D_0)$, $P(D_+)$ is uniquely determined. We define the value $P(D)$ to be the value of $P(\alpha D)$ with the necessary crossing changes.

As an example of changing an ascending diagram into a knot consider the figure eight knot K and its ascending diagram αK in figure 12. We starting at the base point b, we have to change crossings 1 and 3 to change αK into K.

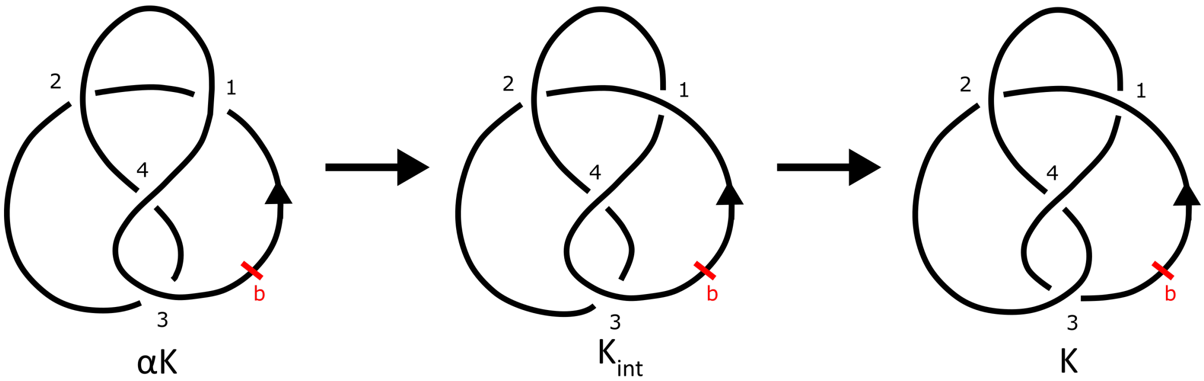


Figure 12: Figure eight knot K and its ascending diagram αK.

We need to show this is indeed well-defined. To do so we will prove three claims.

Claim A: The value $P(D)$ does not depend on the order in which we change the crossings of $P(\alpha D)$.

Claim B: The value $P(D)$ does not depend on the chosen base points.

Claim C: The value $P(D)$ does not depend on the order chosen for the link components.

Together, these claims will prove $P(D)$ is well-defined.

Proof of claim A. Take two crossings and label them 1 and 2 . Let us denote the crossing changes by C_1 and C_2, where C_r is the change of crossing r. Also let $D_{i,j}$ denote the diagram of the knot, where i, j are the signs of crossing 1 and 2 respectively. Then:

$$C_1 \circ C_2 (D_{-,-}) = C_1 (D_{-,+}) = D_{+,+} = C_2 (D_{+,-}) = C_2 \circ C_1 (D_{-,-}) .$$

Which proves the order in which the crossings are changed does not matter for the diagram D, so also not for the value $P(D)$.

Proof of claim B. Keep the order of the components fixed and pick one component. Let b be the base point of the chosen component. Let us consider what happens when we move the base point b from before a crossing to a base point b'after a crossing (Here we stay on the same strand of the knot). Let αD be the ascending diagram created using base point b, and βD the ascending diagram created using b'.

We want to show that $P(\alpha D) = P(\beta D)$, since then $P(D)$ would be the same in both cases. Notice there are two cases. In the first case b and b' are in segments which belong to different link components. If we start at b, we change the crossing and go to the next component. If there is more than one crossing in the component of b, we will return at some point in the link order to change the rest of the crossings of the component. In case we start at b' we start at a new component and get to the component of b in some other point in the order, and change all crossings then. Because it does not matter in which order we change crossings, we notice $\alpha D = \beta D$. Hence, $P(\alpha D) = P(\beta D)$.

In the second case b and b' are in the same component. This means βD is created from αD by changing one crossing. Given that αD is an ascending diagram with $\#D$ components, we get $P(\alpha D) = \mu^{\#D-1}$. If the crossing we change is annulled, we get an ascending diagram D_0 which has $\#D+1$ components and $n-1$ crossings. This means $P(D_0) = \mu^{\#D}$. Using the skein relation ($\diamondsuit$) we get the following:

$$lP(\beta D) + l^{-1}P(\alpha D) + mP(D_0) = 0.$$

Rewriting this yields:

$$
\begin{aligned}
P(\beta D) &= l^{-1}\left(-l^{-1}P(\alpha D) - mP(D_0)\right) \\
&= l^{-1}\left(-l^{-1}\mu^{\#D-1} - m\mu^{\#D}\right) \\
&= l^{-1}\left(-l^{-1} - m\mu\right)\mu^{\#D-1} \\
&= l^{-1}\left(-l^{-1} + l + l^{-1}\right)\mu^{\#D-1} \\
&= \mu^{\#D-1}.
\end{aligned}
$$

This shows, $P(\alpha D) = P(\beta D)$, so $P(D)$ is the same whether we start at b or b'.

The proof of claim C uses the invariance of $P(D)$ under Reidemeister moves. We will hence show property (ii) of the definition of $P(D)$ first. First, we remark that if a link diagram has an ordering before a Reidemeister move then there is an ordering on the link diagram after the move as well. We will show invariance one move at the time.

Type I. Suppose we want to change a crossing by means of a Type I move. Consider without loss of generality we use the move from left to right in the figure below. We have shown in claim B that $P(D)$ is invariant of the place of the base point. If we put the base point at the position of b_1 before the move and we put it at b_2 after the move, we get an ascending diagram before and after the move. Because the computation of $P(D)$ does not depend on where the base point is, both the ascending diagram using b_1 and the ascending diagram using b_2 will give the same value $P(D)$. Therefore, computation of $P(D)$ is invariant under Reidemeister I moves.

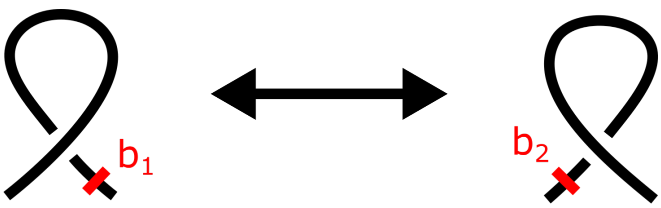


Figure 13: Reidemeister I move.

Type II. Suppose change a knot using a Reidemeister II move. First consider the case where both strands of the crossing have the same direction.

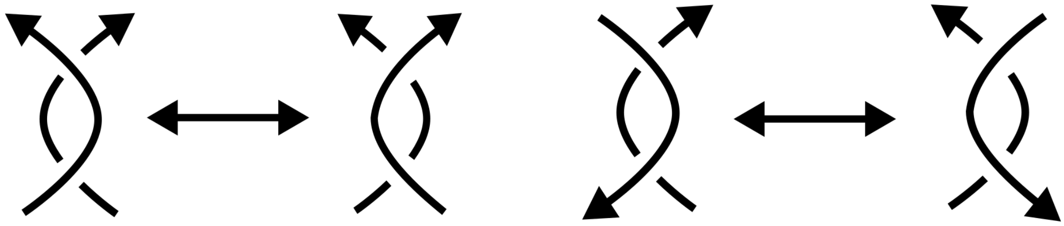


Figure 14: Reidemeister II moves with different orientations.

In figure 15 both sides of the Reidemeister moves are denoted $D_{+,1}$ and $D_{+,2}$ respectively. We want to show that $P(D_{+,1}) = P(D_{+,2})$. To use the skein relation we need $D_{0,1}, D_{0,2}, D_{-,1}$ and $D_{-,2}$. These diagrams are created by changing either the upper crossing (diagrams with subscript 1) or the lower crossing (diagrams with subscript 2). As can be seen in figure 15 we find that $D_{-,1}$ is the same as $D_{-,2}$. Similarly, $D_{0,1}$ and $D_{0,2}$ are the same up to isotopy (in this case just a rotation). Therefore, by the skein relation ($\diamondsuit$) we find that $P(D_{+,1}) = P(D_{+,2})$.

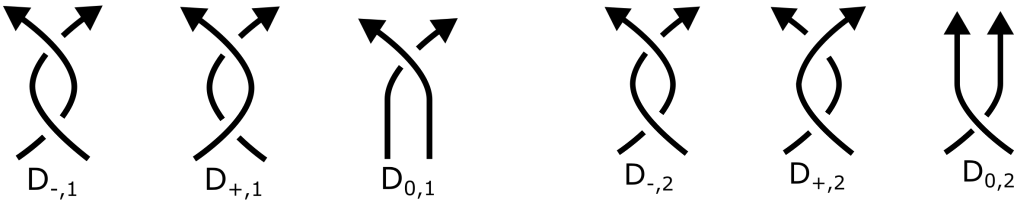


Figure 15: Same orientation.

In the case the directions of the strands are reversed we can use the same trick but instead we use the diagrams such as in 16.

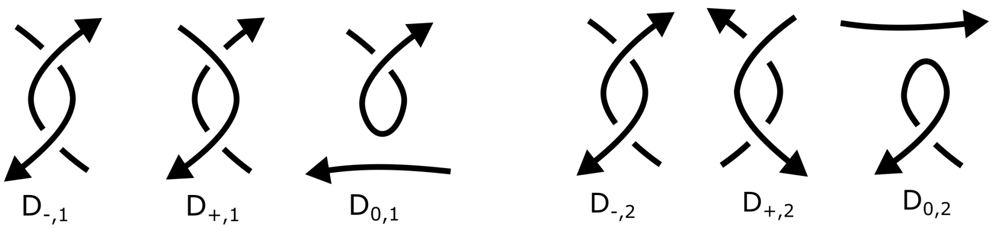


Figure 16: Different orientation.

Together this shows $P(D)$ is invariant under Reidemeister II moves.

Type III. The same strategy for showing $P(D)$ is invariant under type III moves is the same as for type II moves.

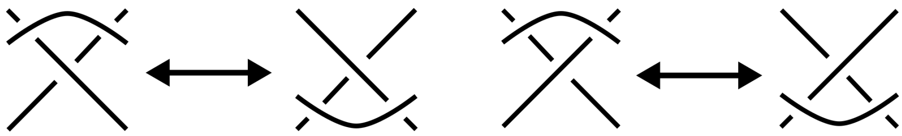


Figure 17: Reidemeister III moves.

In figure 18 the possible crossings are given. The Reidemeister III move transforms D_1 into D_1' or D_2 into D_2'. The difference between D_1 and D_2 is the sign of the bottom crossing. Now notice that using the skein relation ($\diamondsuit$) we can relate D_1 and D_2 with either D_3 or D_4, depending on the orientation of the strand [8]. We can now use that D_3 and D_4 are related to D_3' and D_4' respectively using a type II move. Moreover, because the difference between D_1 and D_2 is only one crossing, either one of them will be part of the ascending diagram and hence either $P(D_1)$ and $P(D_1')$ or $P(D_2)$ and $P(D_2')$ are known. This means that either the remaining value is determined uniquely as before, proving that $P(D_1) = P(D_1')$ or $P(D_2) = P(D_2')$.

For example, the skein relation gives an argument of the form:

$$lP(D_1) = -l^{-1}P(D_2) - mP(D_3) = -l^{-1}P(D_2') - mP(D_3') = lP(D_1).$$

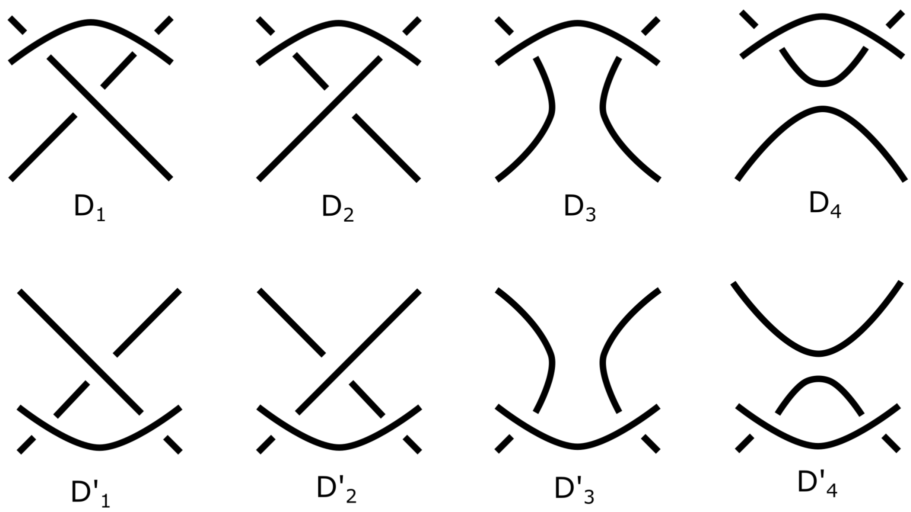


Figure 18: Reidemeister III with crossing changes.

We have now shown that $P(D)$ is invariant under each Reidemeister move. Which

[8]One would need to work this out for all possible orientations on all three strands, this would be 2^3 different cases, so the pictures do not show the orientations explicitly.

proves property (ii) for from the inductive definition of $P(D)$. We can now prove claim C.

Proof of claim C. Suppose D is any link diagram in $\mathcal{D}_n$ with an ordering for its components. Let αD be the associated ascending diagram. Suppose βD is the ascending diagram with respect to a different ordering. We now want to give the components of βD the ordering of αD. In this way we compute βD from αD. We can show this by showing that $P(\beta D) = \mu^{\#D-1}$. This means we can start computing $P(D)$ from αD, as well as from βD.

Consider an innermost loop of the diagram βD. A loop is a sub-arc of a diagram which begins and ends at the same crossing. The loop is said to be innermost if no link component is completely within the area bounded by the loop. If necessary, any component of the diagram with no crossings that bounds a disc whose interior is disjoint can be moved away from the diagram.

If the loop does not contain crossings (except for the begin- and endpoints), the loop can be removed using a Reidemeister I move. In this case the crossing vanishes and we get that βD is ascending with $n-1$ crossings. Therefore, $P(\beta D) = \mu^{\#D-1} = P(\alpha D)$ by the induction hypothesis.

If the loop does contain crossings this means other arc traverse the loop. Using the fact that the loop is innermost and isotopy one can make sure the traversing arc meets the loop at only two points. One transversal arc and (part of) the loop bound a 2-gon and also pairs of transversal arc bound 2-gons. Choose an innermost 2-gon inside the loop. Here innermost means there is no connected component within the area covered by the 2-gon. Call the two arcs involved p and q, ending at points A and B, and bounding a region R. Notice that any remaining arc traversing R must intersect both the arcs p and q. Moreover, two arcs traversing R can only intersect once, since if they would intersect more times this would form a connected component, which would then be the innermost 2-gon.

Now we can use lemma 2.23 to see there must also be a 3-gon T with an edge in p (and one with an edge in q). Assume all base points are outside the region R (in case they are not they can be moved there using claim B). If βD is indeed ascending, it must be the case that each vertex of our 3-gon T is both a positive and a negative crossing. This means that we can move a part of the arc p between two vertices of T by means of a Reidemeister III move. Move the arc p across the 3-gon in such a way that we have less lines intersecting R. In this way, we create a new region R'. We can repeat the process until there are not more lines intersecting the region R' or no more 3-gons. In the first case, one can remove the region R' by means of a Reidemeister II move. Removing the region R' means that we remove the crossings at the points endpoints of p: A and B. This means that using Reidemeister moves we have changed βD in an ascending diagram with $n-2$ crossings. Then by the induction hypothesis we have that $P(\beta D) = \mu^{\#D-1} = P(\alpha D)$. Where we notice that we can not have deleted a whole link component using this process , because such a components would have been moved out. This proves claim B, because using both orders αD and βD we can compute the same value for $P(D)$.

This means we have shown property (i), (ii) and (iii) for $P(D)$ with D in $\mathcal{D}_n$. The corresponding link invariant $P(L)$ for an oriented link L with diagram D can be defined by defining $P(L) \coloneqq P(D)$. All that is left to show is that P is unique. This follows from the fact that $P(L)$ can always be computed from an ascending diagram which as said before is equivalent to a k-component unlink. This means the resulting function P will be fully determined by the value on the k-components unlink. Lemma 2.22 uniquely fixes the value on this k-component unlink, and therefore P is unique. This proves theorem 2.21. $\square$

3 Vassiliev invariants

In the previous section we have seen several examples of knot invariants. In this section we will build a large class of invariants in order to study all these invariants at once. This class of invariants will be called the Vassiliev invariants. Before we can define a Vassiliev invariant, we will need to look at knots in a different way.

3.1 The Vassiliev extension

The idea of a knot can be generalised to a so called *singular knot*. Intuitively, a singular knot is a knot that is allowed to intersect itself a finite number of times, which was not allowed in a regular knot. In order to define these knots formally we need the definition of an embedding.

Definition 3.1. Let N and M be two topological manifolds. A smooth function $f : N \to M$ is called an **embedding** if f is an injective immersion [9]

Definition 3.2. A **singular knot** is a smooth map $S^1 \to \mathbb{R}^3$ that fails to be an embedding.

We will denote singular crossings by . A singular knot is a knot that is for instance not injective i.e. multiple points of the circle are mapped to the same point in the knot, which is singular crossing. For simplicity we will only consider the case where two points of the circle map to the same point, so called *ordinary double points* ([15].

Definition 3.3. Given a map $f : S^1 \to \mathbb{R}^3$ a point $p \in \mathrm{im}(f) \subset \mathbb{R}^3$ is a *ordinary double point* of f if $f^{-1}(p)$ consists of two points t_1 and t_2 and the two tangent vectors $f'(t_1)$ and $f'(t_2)$ are linearly independent. Geometrically a double points means that the tangent vectors in a neighbourhood of the point are non-collinear.

It would of course be possible to define more complicated singular knots such as points where three or more lines intersect. The ideas presented here can be extended to these more complicated cases.

In order to discuss knots more effectively we can define a vector space of singular knots. This vector space is defined as follows:

Definition 3.4. The **vector space of knots** $\mathcal{K}$ is the vector space of finite formal linear combinations of isotopy classes of oriented knots over the field $\mathbb{F}$.

[9]A continuous function $f : N \to M$ is an immersion if for every point $x \in N$ there exists an open neighbourhood B_x such that $f : B_x \to M$ is a homeomorphism.

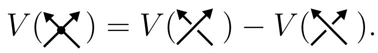
This means that $\mathcal{K}$ is the vector space over a field $\mathbb{F}$ with a basis consisting of the isotopy classes of knots. In appendix A the formal linear combination will be discussed in more detail. The subspace spanned by the set of singular knots with at least m singular crossings will be denoted by $\mathcal{K}_m$. Observe that if we have a knot with at $m+1$ crossings, this would also be in $\mathcal{K}_m$. This observation gives us a *filtration* of spaces of knots:

$$\mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \dots .$$

This filtration enables us to make the quotient space $\mathcal{K}_m/\mathcal{K}_{m+1}$, which is the space of knots with exactly m singular crossings.

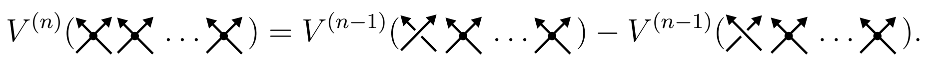
The knot invariants as defined in section 2 are not well-defined for singular knots. In order to extend the notion of an invariant to singular knots we need the following definition.

Definition 3.5. Let $\mathbb{F}$ be any field of characteristic zero. Any $\mathbb{F}$-valued invariant V can be extended to a singular knot using the following rule:

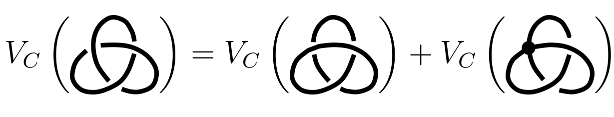


This relation is called the **Vassiliev skein relation**.

Likewise, given n in $\mathbb{Z}_{\geq 0}$ singular crossings we can define the knot invariant $V^{(n)}$ on a knot with n-singularities by applying this definition 3.5 recursively. This gives the formula:



To base the recursion we take $V^0 = V(K)$. The process of recursively applying the Vassiliev skein relation is called *resolving a double point*. As an example one could a knot invariant V on the singular trefoil knot, this would yield:



Remark 3.6. In this thesis we will assume the invariants take values in any field with characteristic 0 (e.g. $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}[X]$ etc.), while in general this could be done

over any Abelian group. This, however, would complicate the dimension of the spaces we work with as we will see later.

The process of resolving double points is well defined as the order in which we apply the Vassiliev skein relation does not matter. This can be shown using the following proposition.

Proposition 3.7. Let K be a knot with m-singular points. Let us denote the set of singular points by S_K. Given a subset $A \subseteq S_K$, we denote by K_A the knot where all singular points in A are given by a negative crossing and all singular points which are not in A are resolved by a positive crossing. Then:

$$V^m(K) = \sum_{A \subseteq S_K} (-1)^{|A|} K_A.$$

Where $|A|$ denotes the number of singular points in the set A.

Proof. The proof of this proposition can be done by induction. Notice that by definition we have $V^{(0)}(K) = V(K)$, which can be used to base the induction. Let us assume that we have:

$$V^m(K) = \sum_{A \subseteq S_K} (-1)^{|A|} K_A.$$

We want to show that given we have a set of $m+1$ singular points $\tilde{S_K}$:

$$V^{m+1}(K) = \sum_{A \subseteq \tilde{S_K}} (-1)^{|A|} K_A$$

.

Let us pick one singular crossing r in $\tilde{S_K}$. Then by the Vassiliev skein relation we get:

$$
\begin{aligned}
V^{m+1}(K) &= V^m\left(\times\,\underbrace{\times\cdots\times}_{m}\right) - V^m\left(\times\,\underbrace{\times\cdots\times}_{m}\right) \qquad (\clubsuit)\\
&= \sum_{A\subseteq S_K, r\notin\tilde{A}} (-1)^{|A|}\, K_A - \sum_{A\subseteq S_K, r\in\tilde{A}} (-1)^{|A|}\, K_A.
\end{aligned}
$$

Notice there are two options for the crossing r. It can either be in $\tilde{A}$ or it is not in $\tilde{A}$. If it is in $\tilde{A}$, we have one more crossing, which means $\left|\tilde{A}\right| = |A| + 1$, this means we get that the terms in which r is in $\tilde{A}$ are of the form $-(-1)^{|A|}V(K_A)$. This means that if we work out the resolution in ($\clubsuit$) we get exactly:

$$
V^{m+1}(K) = \sum_{\tilde{A}\subseteq\tilde{S_K}} (-1)^{\left|\tilde{A}\right|}\, K_{\tilde{A}}.
$$

Which proves the claim. $\square$

The resolution map above will be called the *Vassiliev resolution*. Notice that by the formula in proposition 3.7 it does not matter in which order we resolve the singular points, this makes the Vassiliev resolution a singular knot well-defined.

When working with Vassiliev invariants we will identify elements of the vector space $\mathcal{K}$ with their Vassiliev resolution. This means that a singular knot K is the same as sum of the resolved crossings. For example, a singular trefoil knot is the same as the formal linear combination of the trefoil knots constructed by replacing the singular crossing by a positive and a negative crossing.

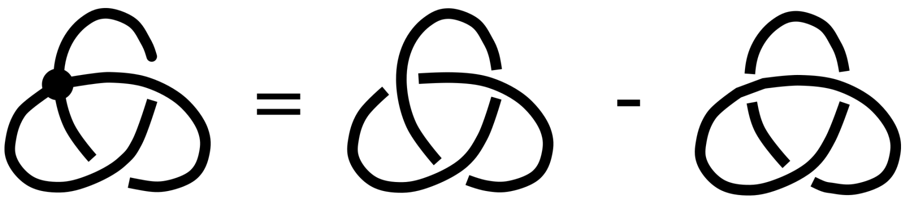


Figure 19: Trefoil knot and its Vassiliev resolution.

One of the most important definition of this thesis is the definition of a Vassiliev invariant.

Definition 3.8. A knot invariant V is said to be a **Vassiliev invariants of degree/type** n if $V^{(n+1)} = 0$. A Vassiliev invariant is said to be of **order** n if it is of type n but not of type less than or equal to $n-1$.[10]

This means V is a Vassiliev knot invariant of type n if $V(\underbrace{\times, \ldots \times}_{n+1}) = 0$. We will denote the set of Vassiliev invariants of type n by $\mathcal{V}_n$.

Analogously, one could think of a Vassiliev invariant as a higher-order derivative applied to a polynomial. For example applying the $n+1$th order derivative to a polynomial of degree n gives us zero, but the nth order derivative will give us a constant.

Example 3.9. An example of a Vassiliev invariant is the n-th coefficient of the Alexander-Conway polynomial which was defined in definition 2.16. The Alexander-Conway polynomial C can be defined by the Conway skein relation:

$$
\begin{aligned}
&C(O) = 1 \\
&C(\nwarrow\!\!\!\!\!\nearrow) - C(\nwarrow\!\!\!\!\!\nearrow) = tC(\curvearrowright\,\curvearrowleft)
\end{aligned}
$$

where O is the unknot. Including the Vassiliev skein relation for a n-singular knot, we obtain:

$$
C\left(\times \ldots \times\right) = t^n C\left(\curvearrowright\,\curvearrowleft \ldots \curvearrowright\,\curvearrowleft\right).
$$

This means that if a knot K has more than n double points then $C(K)$ must be divisible by at least t^{n+1}. But the n-th coefficient is of the form at^n, where a is some coefficient. A term like is only divisible by t^{n+1} if it is zero. Therefore, it must be the case that the n-th coefficient of the Conway polynomial is zero. This means that the n-th coefficient of the Conway polynomial is a Vassiliev invariant of type n, because it is zero for any knot with more than n singular points.

In a very similar way one could proof that the n-th coefficient of the HOMFLYPT polynomial is a Vassiliev invariant of order n. Many more knot invariants are Vassiliev invariants. For example the HOMFLYPT polynomial as was shown in 2.4 and the linking number in definition 2.14 are examples of Vassiliev invariants. Many more examples can be found in [16].

[10] In literature this is sometimes called a finite-type invariant, because the type n is an integer.

3.2 Motivation for studying Vassiliev invariants

In the last section we defined what a Vassiliev invariant is, in the remaining sections of the thesis we will be studying Vassiliev invariants in several ways. But before we can do that, we first need to answer the question why we should study this specific type of invariants. But before we can do that, let us consider how people started thinking about Vassiliev invariants in the first place.

Originally the definition given by V.A. Vassiliev was not meant to deal with knot theory at all. The definition was given to study the complements of discriminants in spaces of maps, in branch of mathematics called *catastrophe theory*. This is a branch of dynamical system theory which studies maps with singularities of some kind and calls the subspace of maps which fail to be an embedding the discriminant. This is why we started working on singular knots. It turned out that the complement of this discriminant can be considered as *the space of knots*. This turned out to be very useful in studying knots, and hence more and more definitions for the Vassiliev invariant specifically for knots where given. But why are these Vassiliev invariants so useful?

First of all, many well known invariants are indeed Vassiliev invariants. For example, each coefficient of the Alexander-Conway polynomial is Vassiliev, as well as the more general HOMFLYPT polynomial. The linking number, which was discussed in example 2.13 is also a Vassiliev invariant. Other well known Vassiliev invariants such as the Jones polynomial (another specialization of the HOMFLYPT polynomial) and the Kauffman polynomial are discussed in [9].

There are also many invariants which are not Vassiliev. However, there is a conjecture that each invariant which is not Vassiliev can be approximated by Vassiliev invariants in some way. This is analogous to a theorem such as the Stone-Weierstrass theorem for continuous functions and polynomials. An example of this conjecture is the Alexander-Conway polynomial. The whole polynomial is not a Vassiliev invariant, but each term is. In this way, there is a countable number of Vassiliev invariants which approximates the Alexander-Conway polynomial.

Another good reason to study Vassiliev invariants is the following conjecture.

Conjecture: The set of all rational valued Vassiliev invariants is a complete knot invariant.

It was already mentioned in section 2.3 that a complete invariant is an invariant which can distinguish any pair of knots up to isotopy. To this day only one such invariant is known: the fundamental group of the knot complement where we keep track of the meridian and the longitude when walking around the knot [11]. This result is known as Waldhausen's theorem [17]. However, computing the fundamental group is very complicated. It can be done by hand for knots with a low number of crossing but it gets increasingly difficult for knots with more crossings. There are computer programs like SnapPy [18] which you can use to do these computations but this is computationally intensive if you want to work with knots of very high order (such as the knots occuring when you want to distinguish two strands of DNA). This means it would a great result if we state that the set of rational Vassiliev invariants is a complete invariant. Sadly, it is not even known yet if this set of invariants can distinguish between the unknot and another knot ([9]), which means a lot of work has to be done before the conjecture will be proven.

In the remaining chapters we will try to answer a few questions which are relevant in case we would like to prove this conjecture. For example, if we want to see whether the set of Vassiliev invariants is complete, it would be nice if there is at least a finite amount of them per order. If this is not the case, it would be hard to apply each Vassiliev invariant to a knot. A second question is whether we can find a way of generating new Vassiliev invariants. We will see in section 3.5 there is a structure called the weight system, which can be used to make new Vassiliev invariants. In the following we will try to define this and see how it works.

3.3 Counting finite type invariants

Now that we know Vassiliev invariants exists, we wonder how many there are of a certain degree. In order to do this we first need to relate the invariants with a mathematical object that we can count. To find such an object we are going to reduce the problem to a linear algebra problem. In this section we will follow [14, Chapter 11] quite closely. The next definition will produce a vector space, in order to have a definition of dimension.

Definition 3.10. The set $\mathcal{V}_m$ equipped with addition and scalar multiplication defined by:

$$(v+w)(K) = v(K) + w(K), (\lambda v)(K) = \lambda v(K)$$

where $v, w \in \mathcal{V}_m$, $\lambda \in \mathbb{F}$ and K is any knot, is a vector space.

More generally, it can be shown that the set $\mathcal{V}_m$ form a module. As was mentioned

[11]For more information about fundamental group and the fundamental group as a knot invariant, one could look at [13], [12] or [11, Chapter 11]

in remark 3.6 we picked our invariants to be field valued. This means that $\mathcal{V}_m$ is actually a module over a field and hence a vector space. If we would have picked for example a ring or an abelian group for the values of our invariant the set $\mathcal{V}_m$ would have a module structure over a ring. The 'dimension' of a module is more complicated, therefore we have chosen to use field valued invariants. In most following cases we will use the field of complex number $\mathbb{C}$.

We note that each invariant of degree m is also an invariant of degree $m+1$, to make this formal we have the following proposition.

Proposition 3.11. Each Vassiliev invariant of order m is also a Vassiliev invariant of degree $m+1$ i.e. we have the following filtration on the space $\mathcal{V}$:

$$\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \dots.$$

Proof. Let K be a knot with at least $m+2$ singular crossings. Then K also has at least $m+1$ singular crossings. Which means we have $\mathcal{K}_{m+2} \subset \mathcal{K}_{m+1}$. Given a Vassiliev invariant of degree m, $v \in \mathcal{V}_m$ we have that $v(K) = 0$ because K is in $\mathcal{K}_{m+1}$. Because $\mathcal{K}_{m+2} \subset \mathcal{K}_{m+1}$, it must also be the case that $v(K) = 0$ for any knot $K \in \mathcal{K}_{m+2}$. Therefore, v is a Vassiliev invariant of degree $m+1$, i.e. $v \in \mathcal{V}_{m+1}$. Using induction, the proposition follows. □

In some sense an invariant is nothing more than a map that takes a given knot, or the properties of that knot, and maps it to an an element in a field $\mathbb{F}$. Meaning an invariant is an element of the dual space of the knot. Given a vector space W, we will denote the dual of this vector space by $W^\star$. Notice that for dual spaces we have a theorem which states that the dimension of a given space is equal to the dimension of the dual space. This will also be the case here, meaning we can make an invariant for each singular knot. We would like to find a specific Vassiliev invariant for knots of each order. To start this, we have the following lemma.

Lemma 3.12.

$$\mathcal{V} \cong (\mathcal{K}_m/\mathcal{K}_{m+1})^\star$$

Proof. Notice that we can rewrite $\mathcal{V}_m$ as the set of functionals from the space of knots that give zero for the knots with $m+1$ singular crossings. Symbolically, $\mathcal{V}_m = \{V \in \mathcal{K}^\star : V(\mathcal{K}_{m+1}) = 0\}$. Using the definition of the set of morphisms (definition A.1), we notice this is equivalent to writing:

$$\mathcal{V}_m = \{V \in \operatorname{Hom}(\mathcal{K}, \mathbb{C}) : \mathcal{K}_{m+1} \subset \operatorname{Ker} V\}.$$

Therefore, using lemma A.3 from the appendix we get:

$$\mathcal{V}_m = \{V \in \mathrm{Hom}(\mathcal{K}, \mathbb{C}) : \mathcal{K}_{m+1} \subset \mathrm{Ker}\, V\} \cong \mathrm{Hom}\left(\mathcal{K}/\mathcal{K}_{m+1}, \mathbb{C}\right) = \left(\mathcal{K}/\mathcal{K}_{m+1}\right).$$

Which proves the claim. $\square$

The previous lemma does show that for each singular knot with m crossings we have a Vassiliev invariant of some order. We would now like to show the found Vassiliev invariant would be of order m. This means we want to find a map which uniquely identifies a Vassiliev invariant of each order, with the space of knots of each order. A map which makes such a unique identification is an isomorphism.

Proposition 3.13. There exists an isomorphism between the dual space of knots of order n and the space of Vassiliev invariants of order n i.e.

$$\frac{\mathcal{V}_m}{\mathcal{V}_{m-1}} \cong \left(\frac{\mathcal{K}_m}{\mathcal{K}_{m+1}}\right)^\star$$

Proof. First, we need to describe the elements of $\frac{\mathcal{V}_m}{\mathcal{V}_{m-1}}$. Let $[f], [g]$ be elements in $\mathcal{V}_m/\mathcal{V}_{m-1}$ where $f, g \in \mathcal{V}_m$. Using the fact that $\mathcal{V}_m$ is a module for all $m \in \mathbb{Z}_{\geq 0}$, we have $\mathcal{V}_m/\mathcal{V}_{m-1}$ is a quotient module using definition A.2. We want to define $[f]$ and $[g]$ as equivalence classes of elements in $\mathcal{V}_m$ under the following equivalence relation: $f \sim g$ if and only if $f - g \in \mathcal{V}_{m-1}$. Using the definition of Vassiliev invariants, we also have that:

$$\mathcal{V}_{m-1} = \{v \in \mathcal{K}^\star : v(\mathcal{K}_m) = \{0\}\}.$$

Therefore, $f \sim g$ if and only if $(f - g)(\mathcal{K}_m) = \{0\}$. Which means, $f \sim g$ if and only if $f|_{\mathcal{K}_m} = g|_{\mathcal{K}_m}$ [12]. We define the equivalence class:

$$[f] = \{v \in \mathcal{V}_m : v|_{\mathcal{K}_m} = f|_{\mathcal{K}_m}\}.$$

We now want to show that there is a well-defined map between the given spaces. Using the construction above, we can identify elements in $\frac{\mathcal{V}_m}{\mathcal{V}_{m-1}}$ with elements in

[12] With $f|_{\mathcal{K}_m}$ is mean the map f with restricted domain $\mathcal{K}_m$.

$\left(\frac{\mathcal{K}_m}{\mathcal{K}_{m+1}}\right)^\star$. This correspondence between the two spaces is given by defining the action of $[f]$ on the knot as: $[f](K) \coloneqq f(K)$. This correspondence is well-defined (not dependent on the element chosen in the equivalence class) because of the definition of the equivalence class $[f]$. Using the definition of Vassiliev invariants of order m, we rewrite the right hand side and define the map:

$$\phi : \mathcal{V}_m/\mathcal{V}_{m-1} \to \{v \in \mathcal{K}_m^\star : v(\mathcal{K}_{m+1}) = \{0\}\}.$$

Given any m-singular knot K we map $\phi([f])(K) \coloneqq f(K)$.

It can be checked this map is indeed linear. We need to show that the map ϕ is a linear isomorphism. In order to do this, we first show that ϕ is surjective. We need to show that given any $g \in \{v \in \mathcal{K}_m^\star : v(\mathcal{K}_{m+1}) = \{0\}\}$ there exists a map $\tilde{g}$ such that $\phi([\tilde{g}]) = g$. Take any knot $K \in \mathcal{K}$, then define $\tilde{g} : \mathcal{K} \to \mathbb{C}$ that takes:

$$K \mapsto \begin{cases} g(K) & \text{if } K \in \mathcal{K}_m \\ 0 & \text{otherwise} \end{cases}.$$

Notice, that this map can be formed for any g in this way. It is follows from the construction that $\tilde{g} \in \mathcal{V}_m$. Take the equivalence class of $\tilde{g}$ given by $[\tilde{g}] \in \mathcal{V}_m/\mathcal{V}_{m-1}$. Given any $K \in \mathcal{K}_m$ we have:

$$\phi([\tilde{g}])(K) = \tilde{g}(K) = g(K).$$

Because the construction is done for an arbitrary knot $K \in \mathcal{K}_m$ we can drop the dependence on K. Therefore, we have proven the claim and we have shown that ϕ is surjective.

It remains to be shown that ϕ is also injective. We need to show that for $f, g \in \mathcal{K}^\star$ we have $\phi([f])(K) = \phi([g])$ implies $[f] = [g]$. Given an arbitrary $K \in \mathcal{K}_m$ we assume that $\phi([f])(K) = \phi([g])(K)$. Applying the maps we get $f(K) = g(K)$, so $(f-g)(K) = 0$. Using the definition of $\mathcal{V}_m$ we have $f - g \in \mathcal{V}_{m-1}$, which by the definition of the equivalence relation defined above implies that $[f] = [g]$, proving injectivity. Thus, together with what was shown before, we have proven that ϕ is an isomorphism between the dual space of knots of order n and the space of Vassiliev invariants of order n.

$\square$

This result establishes that if a knot with n singular crossings exists, a Vassiliev invariant of order n exists. Because knots with any finite integer number of singular crossings can be formed, we know Vassiliev invariants of every finite positive integer degree exist. Let us start counting the Vassiliev invariants of some low degrees and see how far we get or if we can design a method.

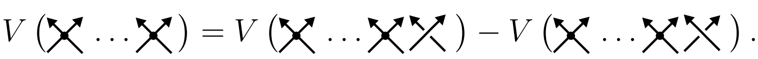
Proposition 3.14. $\mathcal{V}_0 = \{\text{const.}\}$ and hence $\dim \mathcal{V}_0 = 1$.

Proof. Let V be an invariant in $\mathcal{V}_0$. If we take any knot K with a singular points, we get that $V(\;) = 0$ because the invariant is of type zero. Expanding the left-hand side using the Vassiliev skein relation, we obtain $V(\;) = V(\;) - V(\;) = 0$ and hence, $V(\;) = V(\;)$.

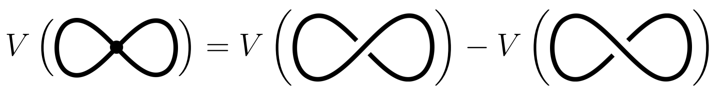
This means that V is invariant under changing crossings. As was shown in lemma 2.8 we have that any knot can be turned into the unknot using Reidemeister moves and changing crossings. Therefore, we have that $V(K) = V(O) = \text{const.}$. $\square$

Proposition 3.15. $\dim \mathcal{V}_1 = 1$.

Proof. Let $V \in \mathcal{V}_1$. Take any knot with more than one double point, then:



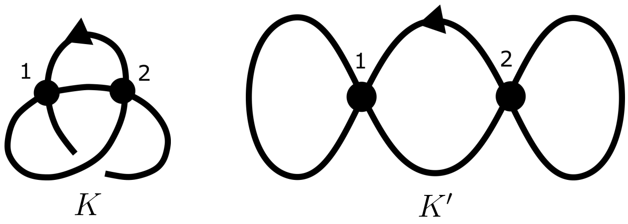
Therefore we have, similarly to the previous proof, that V is invariant under changing crossings except for one. This means that by means of changing crossings we can rewrite any knot K to . Using the fact that V is Vassiliev we obtain:



Since the positive crossing and the negative crossing are both isotopic to the unknot by Reidemeister moves, we note that $V(\;) = V(\bigcirc) - V(\bigcirc) = 0$. Therefore, the dimension of the space of finite type invariants of order 1 is one. $\square$

Proposition 3.16. $\dim \mathcal{V}_2 = 2$

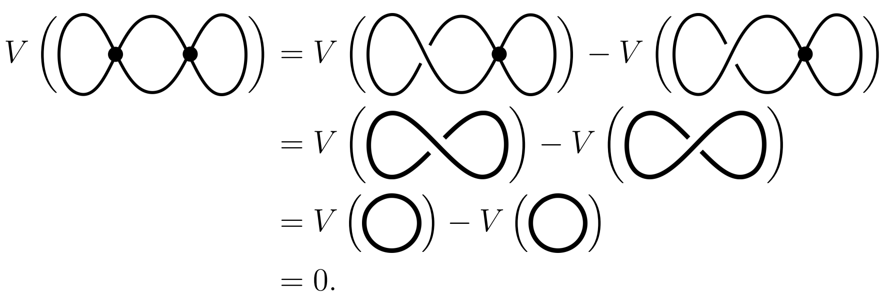
Proof. By similar reasoning as in the proofs of propositions 3.14 and 3.15 we get that the Vassiliev invariant of a knot are invariant under crossing changes, except for two arbitrary singular points. Using crossing changes and Reidemeister moves we always get either one of the following knots:



These knots differ because when walking around the knot in the specified direction the double points are encountered in a different order. Depending on the starting point for a walk around the knot different permutations of crossings will be obtained, but they will never be the same for K and K'. Any permutation of transversing 4 singular crossings will be either of the form $\sigma(1122)$ (which is equivalent to K') or of the form $\sigma(1212)$ (which is equivalent to K).

Therefore, given any knot $\tilde{K}$ with two singular crossings, we can use crossing changes and Reidemeister moves in order to change $\tilde{K}$ such that either $V(\tilde{K}) = V(K)$ or $V(\tilde{K}) = V(K')$, depending on the original order in which the crossings are encountered.

Computing $V(K')$ we obtain:

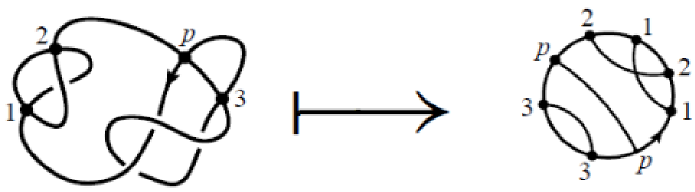


The first equality follows from resolving the knot using the Vassiliev skein relation. The second equality follows from using the Reidemeister III moves on both terms and again using the Vassiliev skein relation. This means the dimension of the space $\mathcal{V}_2$ is at most 2 as only $V(K)$ can have a different value. As was shown in example 3.9, it must be the case that every coefficient of the Conway polynomial is a Vassiliev invariant. Notice that the second term of the Conway polynomial does make a distinction between the unknot and the singular trefoil ($C\left(\bigcirc\right) = 1 + 0$ and $C\left(\right.$$\left.\right) = 1 + t^2$). Therefore, it must be the case that there are exactly two Vassiliev invariants of order 2, the trivial one and the second term of the Conway polynomial [13]. $\square$

The method as described in the last three proposition works, and would also work for higher degrees. But the process of counting knots and finding all its invariants gets more complicated for higher degrees and is already very hard to do for $\mathcal{V}_3$. Hence, a more efficient method is necessary. In the previous proposition we have seen that in order to see which knots have the same Vassiliev invariant the only relevant information is the cyclic order in which the crossings occur. This information about the knot can be stored in a *chord diagram*. A chord diagram of a singular knot is an oriented parametrized circle on which the double points of the knot are marked in order and chords are drawn between corresponding double points. A chord diagram is created by the following ‘recipe’:

- Step 1: Pick an orientation for the knot and walk around the knot in the direction of the orientation, parametrise this walk on a circle.
- Step 2: Record the crossings encountered, put labels of the corresponding crossings on the circle.
- Step 3: You always pass both a negative and a positive crossing. Connect the corresponding labels by a chord.

As an example of a knot and its corresponding chord diagram are given in figure 20. The *degree* of a chord diagram is the number of chords it contains.



K C

Figure 20: A knot K with its chord diagram C. [14]

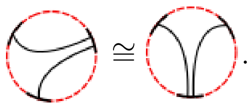
[13]The second term of the Conway polynomial is called the Casson invariant.

The idea is that if we can establish a one-to-one correspondence between Vassiliev invariants of a given order and chord diagrams of a certain degree, and give the number of possible chord diagrams of a degree we would be able to count the number of Vassiliev invariant of some order. Let us start by showing there are at least finitely many chord diagrams of a given degree.

Lemma 3.17. Let $\mathcal{C}_m$ be the set of chord diagrams of degree m, then $\mathcal{C}_m$ contains at most $2m!$ chord diagrams.

Proof. Notice that if we have m chords, there are m singular crossings in the corresponding singular knot, so there are $2m$ points we can connect with a chord. Now we can reduce the problem to the question: In how many ways can we connect $2m$ points with each other? This is an easy problem in combinatorics. Take one of the $2m$ points there are $2m$ options, we can connect this to $2m-1$ points. Then take a next point, we have $2m-2$ options and we can connect it to $2m-3$ options. This product continues like this, which means the total number of options is $2m!$. We now that as long as m is finite this $2m$ is a finite number. Which means that $|\mathcal{C}_m| \leq 2m!$.

The real number of chord diagrams is much harder to compute, because in the $2m!$ above we found there are many chord diagrams which are topologically the same. Take for instance the following case with just two chords:



Here $\cong$ denotes topological equivalence. This means $2m!$ is really an upper bound. A more specific bound, taking into account topological equivalence, is given in [19]. $\square$

In order to establish a more formal correspondence between Vassiliev invariants and chord diagrams, we need to show that the Vassiliev invariants do not depend on the structure of the non-singular crossings, and similarly that Vassiliev invariants do depend on the order in which the singular points occur in a walk around the knot.

Proposition 3.18. A Vassiliev invariant V of degree m does not depend on non-singular crossings. That is: given two knots K and K' in $\mathcal{K}_m$ such that K and K' differ by a finite number of crossing changes and isotopy, we then have $V(K) =$

$V(K')$.

Proof. Let K and K' be knots in $\mathcal{K}_m$ that differ by only one crossing change. Let $\tilde{K}$ be a singular knot with $m+1$ crossing, which has Vassiliev resolutions K and K'. Assume, without loss of generality, K has the positive crossing and K' has the negative crossing. Given that V is a degree m vassiliev invariant we compute:

$$\begin{aligned} 0 &= V\left(\tilde{K}\right) \\ &= V(K) - V(K') \end{aligned}$$

Therefore, $V(K) = V(K')$. In case K and K' differ by a finite number of crossing changes, we can apply the reasoning above recursively to prove the claim. ☐

The independence of the order in which the singular crossings occur, and a more formal way to see the difference between K and K' in 3.16, is given in the following proposition.

Proposition 3.19. Given a Vassiliev invariant V of degree m and a singular knot K in $\mathcal{K}_m$, then $v(K)$ only depends on the cyclic order in which singular points appear in a walk around the knot in the direction of the orientation of the knot.

Proof. We note that if we have a given knot, changing the order in which two crossings are traversed corresponds to changing a negative crossing into a positive crossing. As was shown before, a Vassiliev invariant is invariant under changing crossings. Therefore, we have shown the proposition. ☐

We need to define some notions around chord diagrams before we can work with them. First of all, similarly to what we did with knots and Vassiliev invariants we can make a vector space of the chord diagrams.

Definition 3.20. The set $\mathcal{C}$ is the vector space of all finite formal linear combinations of chord diagrams over $\mathbb{C}$. The subspace $\mathcal{C}_m$ is the subspace of $\mathcal{C}$ generated by all chord diagrams of degree m.

In appendix A we will give more details on formal linear combinations, and explain why the space of chord diagrams is indeed a vector space Another important point is to define what it means for two chord diagrams to be 'the same'.

Definition 3.21. Two chord diagrams C and C' are equivalent if, after relabelling the chords, the same circular sequence of crossings is obtained.

Now that we have given some definitions, we will try to make a one-to-one correspondence between the space of knots and the space of chord diagrams. The map between the space of chord diagrams and the space of knots can be constructed as follows. Take any oriented singular knot K in $\mathcal{K}_m$. Select any non-singular point on K as a base point and walk around the knot in the direction of its orientation. During this tour label the singular points with arbitrary labels $1, \ldots, m$. The same crossings get the same label when traversed for a second time. Notice that because crossings consist of a positive crossing and a negative crossing each integer in $\{1, \ldots, m\}$ is in the resulting sequence exactly twice. When this sequence is put around a circle and chords are drawn between the corresponding labels, we get a chord diagram. The following figure illustrates this process.

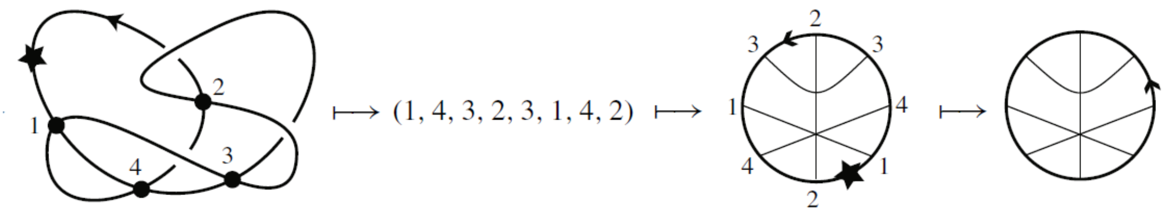


Figure 21: ϕ_4 on a knot [14]

The map $\phi_m : \mathcal{K}_m \to \mathcal{C}_m$ is defined by this construction. As an arbitrary base point was chosen it is clear the map does not depend on the choice of base point.

Similarly, a knot can be created from a chord diagram. This process is called *contracting the chords*. Any chord diagram can be made into a knot by replacing the chords as in figure 22 and lifting the resulting knot diagram from the paper.

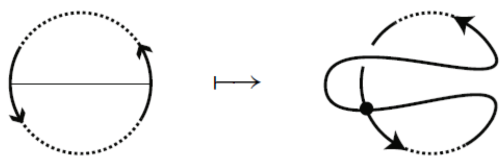


Figure 22: Contracting the chords [14]

This can be done for each chord. Now, we continue to show that that ϕ_m is surjective and not injective.

Proposition 3.22. $\phi_m : \mathcal{K}_m \to \mathcal{C}_m$ is surjective.

Proof. Let C be any chord diagram in $\mathcal{C}_m$. By contracting the chords, we form a knot K_C in $\mathcal{K}_m$. Applying the map ϕ_m, we get $\phi_m(K_C) = C$. As this can be done for each chord diagram and corresponding knot, we prove that ϕ_m is surjective. □

Proposition 3.23. $\phi_m : \mathcal{K}_m \to \mathcal{C}_m$ is not injective.

Proof. This is exercise 11.15 in [14] and can be proven by an example. Consider the singular knots K and K' in figure 23 with the same chord diagram D.

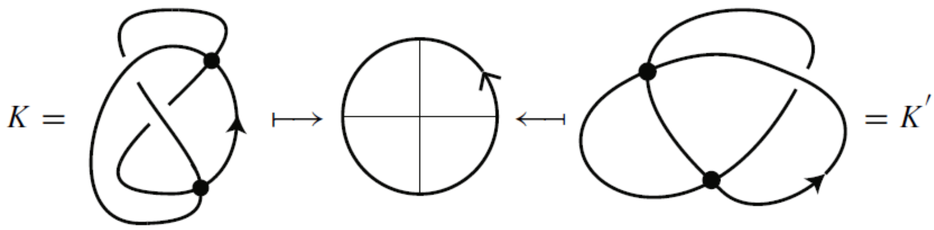


Figure 23: Two knots with the same chord diagram [14]

Notice that if we compute the Alexander-Conway polynomial of K and K' we get $C(K) = -2 - z^2$ and $C(K') = -2 + z^2$. This means that K and K' cannot be equivalent. Therefore, there are two different knots K and K' such that $\phi_2(K) = D$ and $\phi_2(K') = D$, but $K \neq K'$, so ϕ_m is not injective. □

3.4 Vassiliev invariants and Chord diagrams

In the previous sections we have shown there is a relation between chord diagrams and knots We have also shown there is a relation between knots and Vassiliev invariants. To make the circle round again, we will now show there is a relation between chord diagrams and Vassiliev invariants. It will turn out that in order to understand Vassiliev invariants, we only need to understand the dual space of the vector space of chord diagrams. We will again follow [14, Chapter 11] quite closely. We start by proving that the value a Vassiliev invariant on a given knot, only depends on the chord diagram of that knot.

Theorem 3.24. The value of a finite-type invariant of degree m on a knot K with m singularities depends only on the chord diagram of K.

Proof. Let $\sigma(K)$ denote the chord diagram of a knot K in $\mathcal{K}_m$. We want to show that if $\sigma(K_1) = \sigma(K_2)$ then $V(K_1) = V(K_2)$ for any finite-type invariant in $\mathcal{V}_m$ for K_1 and K_2 knots with m double points. Suppose $\sigma(K_1) = \sigma(K_2)$. This assumption tells us there is a one-to-one correspondence between the double points of the knot. If we place these double points above each other we can deform the rest of the chords around these points such that they coincide as well. In this process we might create new singularities, but these singularities can be resolved. An extra singularity would mean there are $m+1$ singularities, denote the knot created by $\tilde{K}$. We notice that:

$$V(\tilde{K}) = V(\underbrace{\times\times\dots\times}_{m+1}) = V(\nearrow\!\!\!\!\nwarrow\ \underbrace{\times\dots\times}_{m}) - V(\nwarrow\!\!\!\!\nearrow\ \underbrace{\times\dots\times}_{m}) = 0.$$

Hence, we get $V(\nearrow\!\!\!\!\nwarrow\ \underbrace{\times\dots\times}_{m}) = V(\nwarrow\!\!\!\!\nearrow\ \underbrace{\times\dots\times}_{m})$, which means that finite-type invariant is invariant under changing all but the m original crossings. Therefore, in deforming the knot by R, the value $V(K)$ does not change, therefore $V(K_1) = V(K_2)$. Proving the theorem. $\square$

Now that we have established that Vassiliev invariants only depend on chord diagrams, we want to show that every Vassiliev invariant is a linear map from the space of chord diagrams into a given field i.e. an element of the dual of the space of chord diagrams. This is done in the following theorem.

Theorem 3.25. Let v be an element of $\mathcal{V}_m$. The following properties hold.

1. There exists a unique map $\omega_v : \mathcal{C}_m \to \mathbb{C}$ such that $v = \omega_v \circ \phi_m$ i.e. the following commutative diagram commutes:

$$\begin{tikzcd} \mathcal{K}_m \arrow[r, "\phi_m"] \arrow[dr, "v"'] & \mathcal{C}_m \arrow[d, "\exists!\omega_\theta"] \\ & \mathbb{C} \end{tikzcd}$$

2. There exists a linear map $\alpha_m : \mathcal{V}_m \to \mathcal{C}_m^{\star}$.

3. The map $\omega_v : \mathcal{C}_m \to \mathbb{C}$ can be written as $\omega_v(C) = v(K)$ for any knot related to C, K_C, such that $\phi_m(K) = C$.

Proof. Let $v \in \mathcal{V}_m$.

1. We need to show that the map $\omega_v : \mathcal{C}_m \to \mathbb{C}$ exists, is well-defined and unique. First, we show existence. Using proposition 3.22, we know that given a chord diagram $C \in \mathcal{C}_m$ there must exist a related singular knot K_C such that $C = \phi_m(K_C)$. Let us define the map ω_v, from the chord diagram to the value of $v(K)$ i.e. $\omega_v(C) = v(K_C)$. Then it is clear that ϕ_m first takes the singular knot K_C to its related chord diagram C, which using ω_v is evaluated in C. This shows that $\omega_v \circ \phi_m = v$, proving the existence of the map ω_v.

   It is necessary to show this ω_v is indeed well-defined. Assume there exists an other knot K' in $\mathcal{K}_m$ such that K is not the same as K', but $\phi_m(K') = C$. It was shown in theorem 3.24 that if $\phi_m(K) = \phi_m(K')$, then $v(K) = v(K')$. This means that $\omega_v(C)$ does not depend on the choice of the given knot and hence the given chord diagram.

   The uniqueness is shown by supposing another map $\bar{\omega}_v : \mathcal{C}_m \to \mathbb{C}$ exists such that $\bar{\omega}_v \circ \phi_m = v$, and show that $\omega_v = \bar{\omega}_v$. Notice that $\bar{\omega}_v \circ \phi_m(K) = v(K)\omega_v \circ \phi_m(K)$. Notice that, $\phi_m(K) = C$, hence $\bar{\omega}_v(C) = \omega_v(C)$. Using that the map is well-defined, we know $\bar{\omega}_v = \omega_v$, proving that $\omega_v : \mathcal{C}_m \to \mathbb{C}$ is the unique map such that $v = \omega_v \circ \phi_m$.

2. We want to show there exists a linear map α_m from $\mathcal{V}_m$ to $\mathcal{C}_m^{\star}$. Define, $\alpha_m(v) = \omega_v$. We need to show this is indeed a linear map. Take two arbitrary Vassiliev invariants v_1, v_2 in $\mathcal{V}_m$ and a scalar $\lambda \in \mathbb{C}$. Moreover, take an arbitrary chord diagram C in $\mathcal{C}_m$, with a related singular knot K_C such that $\phi_m(K_C) = C$. Now we can expand,

$$
\begin{aligned}
\alpha_m(\lambda(v_1+v_2))(C) &= \omega_{\lambda(v_1+v_2)}(C) \\
&= \omega_{\lambda(v_1+v_2)} \circ \phi_m(K) \\
&= (\lambda(v_1+v_2))(K) \\
&= \lambda\left(v_1(K_C)+v_2(K_C)\right) \\
&= \lambda\omega_{v_1}(C)+\lambda\omega_{v_2}(C) \\
&= \lambda\alpha_m(v_1)(C)+\lambda\alpha_m(v_2)(C).
\end{aligned}
$$

As C was chosen arbitrarily, we drop the dependence on C, leaving us with $\alpha_m(\lambda(v_1+v_2)) = \lambda\alpha_m(v_1)+\lambda\alpha_m(v_2)$, which shows α_m is linear.

3. Using the fact that ω_v is defined to be a unique map taking C to $v(K)$, we have shown the third part of the theorem.

$\square$

This theorem states that given any Vassiliev invariant, we can view it as an element of the dual space of the chord diagram. In order to study Vassiliev invariants of a specific order we need to get a similar map from the quotient space $\mathcal{V}_m/\mathcal{V}_{m-1}$ to the dual space of the space of chord diagrams of degree m given by $\mathcal{C}_m$.

Lemma 3.26. Given $\alpha_m : \mathcal{V} \to \mathcal{C}_m^\star$ we have:

1. $\ker(\alpha_m) = \mathcal{V}_{m-1}$.

2. The map $\overline{\alpha}_m : \mathcal{V}_m/\mathcal{V}_{m-1} \to \mathcal{C}_m^\star$ is injective.

3. $\overline{\alpha}_m([v]) = \omega_v$.

Proof. Let v be an element in $\mathcal{V}_m$.

1. Let us first show that $\mathcal{V}_{m-1} \subseteq \ker(\alpha_m)$. Let v be an element in $\mathcal{V}_{m-1}$. Notice that for a given C in the space $\mathcal{C}$ we have $\alpha_m(v)(C) = \omega_v(C)$. The chord diagram can also be written as $\phi_m(K_C)$ for K_C in $\mathcal{K}_m$ by proposition 3.22. Using theorem 3.25 and the fact that K_C is in the the space of singular knots order order n, while v is a Vassiliev invariant of lower degree we obtain:

$$
\alpha_m(v)(C) = \omega_v(C) = \omega_v \circ \phi_m(K_C) = v(K_C) = 0.
$$

This shows that v must be a Vassiliev invariant in $\ker(\alpha_m)$, and hence $\mathcal{V}_{m-1} \subseteq \ker(\alpha_m)$.

Conversely, we need to show that $\ker(\alpha_m) \subseteq \mathcal{V}_{m-1}$. Let $v \in \ker(\alpha_m)$, then we know $\alpha_m(v) = \omega_v(C) = 0$ for all C in $\mathcal{C}_m$. This implies that $\omega_v(\phi_m(K)) = v(K) = 0$ for all $K \in cK_m$, which is equivalent to $v(\mathcal{K}_m) = \{0\}$. Therefore, it must be the case that $v \in \mathcal{V}_{m-1}$ and hence $\ker(\alpha_m) \subseteq \mathcal{V}_{m-1}$. Together with what was shown before we can conclude $\ker(\alpha_m) = \mathcal{V}_{m-1}$.

2. Notice that $\mathcal{V}_m$ and $\mathcal{C}_m^\star$ are both modules over a field $\mathbb{C}$. Therefore, using the first isomorphism theorem for modules (theorem A.5) we can conclude that $\ker(\alpha_m)$ is a submodule of $\mathcal{V}_m$ and the image of α_m ($\mathrm{Im}(\alpha_m)$) is isomorphic to the quotient module $\mathcal{V}_m/\ker(\alpha_m)$. Notice that $\mathrm{Im}(\alpha_m)$ is a subset of $\mathcal{C}_m^\star$. Because of the established isomorphism we know that $\overline{\alpha}_m$ will only map to elements in it's image, which means that map $\overline{\alpha}_m$ as a map to $\mathcal{C}_m^\star$ is still injective (but when $\mathcal{C}_m^\star \neq \mathcal{V}_m$ no longer surjective).

3. Notice that the map $\overline{\alpha}_m$ maps the equivalence class $[v]$ to $\alpha_m(v) = \omega_v$, proving the third part of this theorem.

$\square$

For now we leave it at the fact that $\overline{\alpha}_m$ is injective. But we will see in lemma 3.33 in section 3.5 that $\overline{\alpha}_m$ is not surjective. However, when we restrict ourselves to Vassiliev invariants of order zero, we do find an isomorphism.

Corollary 3.27. The map $\overline{\alpha}_0 : \mathcal{V}_0 \to \mathcal{C}_0^\star$ is an isomorphism given by $\overline{\alpha_0}(v) = \omega_v$.

The corollary follows straightforward from the fact that α_0 is linear and the dimensions of $\mathcal{V}_0$ and $\mathcal{C}_0^\star$ are the same and the kernel of $\overline{\alpha}_0$ must be trivial.

Lemma 3.26 finally gives us the possibility to achieve what we set out to do in the introduction and bound the number of Vassiliev invariants of a certain degree. Let us first show that there are at least finitely many invariants in $\mathcal{V}_m/\mathcal{V}_{m-1}$.

Proposition 3.28. The vector space $\mathcal{V}_m/\mathcal{V}_{m-1}$ is finite dimensional for each $m \geq 1$.

Proof. First, we notice that there are only a finite number of chord diagrams of degree m by lemma 3.17. Moreover, the dimension of a finite dimensional vector space and its dual are always the same. Hence, $\dim \mathcal{C}_m = \dim \mathcal{C}_m^\star$, so the dual space of $\mathcal{C}_m$ is also finite dimensional. Because for the map $\overline{\alpha}_m$ to be injective it must hold that each element v in $\mathcal{V}_m/\mathcal{V}_{m-1}$ only maps to one element it must be the case that there is a finite amount of elements v in $\mathcal{V}_m/\mathcal{V}_{m-1}$, which proves the claim. □

Now that we have shown that the space of Vassiliev invariants of each specific order is finite, we want to modify the proof and show the space of Vassiliev invariants of a certain degree m (i.e. order less than or equal to m) is finite dimensional. We define the vector space of chord diagrams of at most degree m as the subspace of $\mathcal{C}$ generated by all chord diagrams of degree at most m. Or more formally,

$$\mathbb{C}_{\leq m} \coloneqq \mathcal{C}_m \oplus \mathcal{C}_{m-1} \oplus \cdots \oplus \mathcal{C}_0.$$

We need to have a similar lemma as lemma 3.26 for this space.

Lemma 3.29. For $m \geq 0$ there exists a map $\beta_m : \mathcal{V}_m \to \mathcal{C}_m^\star$.

Proof. Notice that each $\mathcal{V}_m$ is a vector space and $\mathcal{V}_m/\mathcal{V}_{m-1}$ is a vector subspace of $\mathcal{V}_m$. We can use theorem A.4 to show that

$$\mathcal{V}_m \cong \frac{\mathcal{V}_m}{\mathcal{V}_{m-1}} \oplus \mathcal{V}_{m-1}.$$

Notice that the last term of this expression can again be expanded like this. Therefore, we can recursively write:

$$\mathcal{V}_m \cong \frac{\mathcal{V}_m}{\mathcal{V}_{m-1}} \oplus \frac{\mathcal{V}_{m-1}}{\mathcal{V}_{m-2}} \oplus \cdots \oplus \mathcal{V}_0.$$

Using lemma 3.26, we know that each of these space $\frac{\mathcal{V}_k}{\mathcal{V}_{k-1}}$ is isomorphic to $\mathcal{C}_k^\star$ for all $1 \leq k \leq m$ using the map $\overline{\alpha_k}$. By corollary 3.27 we know that $\mathcal{V}_0$ is isomorphic to $\mathcal{C}_0^\star$. Because the direct sum of $\mathcal{C}_0^\star \oplus \ldots \mathcal{C}_m^\star$ is isomorpic to $\mathcal{C}_{\leq m}^\star$, we must have there is also an injection between $\mathcal{C}_{\leq m}^\star$ and $\mathcal{V}_m$ since the direct sum of injective morphisms is again an injective morphism. This can be summarised in the following diagram.

$$\begin{tikzcd}
\mathcal{V}_m \arrow[d, dashed, "\beta_m"] & \cong & \frac{\mathcal{V}_m}{\mathcal{V}_{m-1}} \arrow[d, "\overline{\alpha_m}"] & \oplus & \frac{\mathcal{V}_{m-1}}{\mathcal{V}_{m-2}} \arrow[d, "\overline{\alpha_{m-1}}"] & \oplus \cdots \oplus & \mathcal{V}_0 \arrow[d, "\overline{\alpha_0}"] \\
\mathcal{C}_{\leq m}^\star & \cong & \mathcal{C}_m^\star & \oplus & \mathcal{C}_{m-1}^\star & \oplus \cdots \oplus & \mathcal{C}_0^\star
\end{tikzcd}$$

$\square$

Using this injection β_m we can find a bound on the number of Vassiliev invariants of a specific order.

Corollary 3.30. The vector space $\mathcal{V}_m$ of degree m Vassiliev invariants is finite dimensional for each m. And moreover,

$$\dim(\mathcal{V}_m) \leq \sum_{k=0}^{m} \dim \mathcal{C}_k.$$

Proof. Notice that by lemma 3.29 there exists an injective map from $\mathcal{V}_m$ to $\mathcal{C}^{\star}_{\leq m}$. Therefore, we have that the dimension of $\mathcal{V}_m$ is less than or equal to the dimension of $\mathcal{C}^{\star}_{\leq m}$. Using properties of the direct sum the dimension of $\mathcal{C}^{\star}_{\leq m}$ equals the sum of the dimensions of the dual space of chord diagrams of each specific dimension. When also noticing that a space and its dual have the same dimension we are done.

$$\begin{aligned}\dim(\mathcal{V}_m) &\leq \dim(\mathcal{C}^{\star}_{\leq m}) \\ &= \sum_{k=0}^{m} \dim(\mathcal{C})^{\star}_{k} \\ &= \sum_{k=0}^{m} \dim(\mathcal{C})_k\end{aligned}$$

$\square$

Using the bound on the number of chord diagrams of degree k that was given in lemma 3.17, we can say that:

$$\dim(\mathcal{V}_m) = \sum_{k=0}^{m} 2k! < \infty.$$

So there are at least finitely many Vassiliev invariants of each degree/type. Using more sophisticated methods the following table of number of Vassiliev invariants for a given degree was made. The precise dimensions of $\mathcal{V}_n$ are known up to $n = 12$ ([9]).

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\mathcal{V}_n$	1	1	2	3	6	10	19	33	60	104	184	316	548

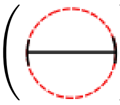
3.5 Weight systems

As was seen in the proof of lemma 3.26 we have an isomorphism between $\mathcal{V}_m/\mathcal{V}_{m-1}$ and the image of α_m. If we can describe the image of this map in a nicer way we will be able to give a better description of the space $\mathcal{V}_m/\mathcal{V}_{m-1}$. It turns out each element in this image satisfies two properties called *framing independence* and the *4T-relation*. In order to define these relations we first need to define the notion of an isolated chord.

Definition 3.31. Given a chord diagram in the space of chord diagrams $\mathcal{C}$, an *isolated chord* is a chord that does not intersect any other chord of the diagram.

We need this because each element in the image of $\overline{\alpha}_m$ evaluates a chord diagram to zero if the chord diagram has an isolated chord. This relation is called the *framing independence* or the *1T relation.*

Lemma 3.32. Every element W in the image of $\overline{\alpha}_m$ satisfies the 1T relation, i.e.

$W\Big($  $\Big) = 0.$

Proof. Let K_C be a singular knot in the space $\mathcal{K}_n$, whose related chord diagram C contains an isolated chord. For a given W in the image of $\overline{\alpha}_m$ we know we can write $W = \overline{\alpha}_m([v])$ for some equivalence class $[v]$ of $v \in \mathcal{V}_m$. Following lemma 3.26, we also have that $W(C) = \overline{\alpha}_m([v])(C) = \omega_v(C) = \omega_v \circ \phi_m(K) = v(K_C)$. Therefore, in order to proof this lemma, we need to show that for a given knot K_C related to a chord diagram C with an isolated chord, we have $v(K_C) = 0$.

Let us denote the singular crossing responsible for the isolated chord by p. Notice that p divides the chord diagram into two segments, which will be called S_1 and S_2. Then the following claim is true.

Claim: Given the knot K_C we can change crossings and use Reidemeister moves to show that $v(K_C) = v(K'_C)$ where K'_C has the same chord diagram C, but the segments S_1 and S_2 both lie on the opposite side of some plane in $\mathbb{R}^3$ passing through the double point p.

The situation is as depicted in figure 24

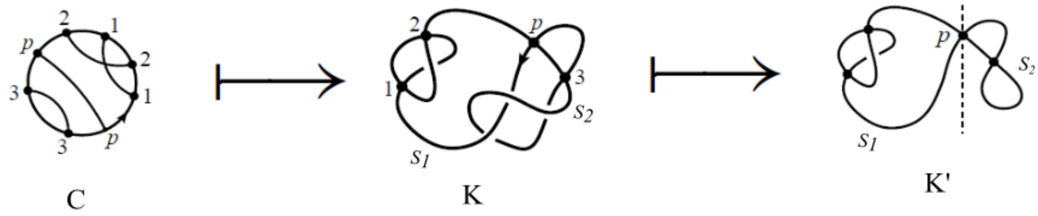
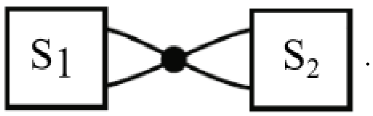


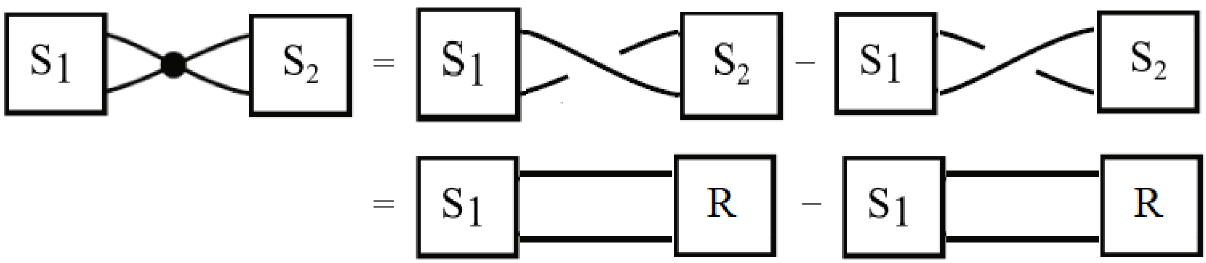
Figure 24: Crossing changes [9].

Because the chord is isolated, S_1 and S_2 do not have common double points. What might happen is that one branch of a non-singular crossing is in S_1, while the other is in S_2. Because the value $v(K)$ only depends on the singular crossings, we can safely change any negative crossing in a positive crossing. Using Reidemeister moves, we can create the situation in figure 24, proving the claim.

Now we notice that $v(K_C) = v(K'_C)$. Let us resolve the knot K_C at the point p in $v(K^+_{C,p})$ and $v(K^-_{C,p})$. The same can be done for K'_C. Let us depict the resulting value of the knot as:



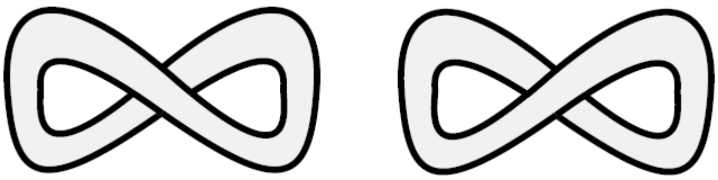
Here we have that both sides S_1 and S_2 can be seen as knots, linked by the point p. Then resolving the knot using the Vassiliev Skein relation gives us:



$= 0$

Where R denotes the reversed diagram of the knot S_2. This means we have shown that each element W in the image of $\overline{\alpha}_m$ satisfies the 1T relation. $\square$

The reason this relation is also called the framing independence relation is because of so called *framed knots*. A framed knot is not made from a circle, but from a annulus instead. This means our knot can be thought of as thickened or made of a band instead of a piece of string. Most of the theory described in this thesis could also have been presented in the framed case and there exist formula's for framing and de-framing certain invariants, chord diagrams and weight systems (see [9, Section 3.5])[14]. One important difference between framed knots and unframed knots is that the diagrams from [14]:



are not equivalent. This means the proof of lemma 3.32 does not hold for framed knots and hence not every function on a chord diagram chord diagram satisfies the 1T-relation. This means that unframed knots are the cases where the value of the weight system does not depend on the framing, hence framing independence.

We continue with a nice application of framing independence. In section 3.4 we already stated that $\overline{\alpha}_m : \mathcal{V}_m/\mathcal{V}_{m-1} \to \mathcal{C}_m^\star$ is not surjective. Using framing independence we can give a formal proof.

Lemma 3.33. The map $\overline{\alpha}_m : \mathcal{V}_m/\mathcal{V}_{m-1} \to \mathcal{C}_m^\star$ is not surjective.

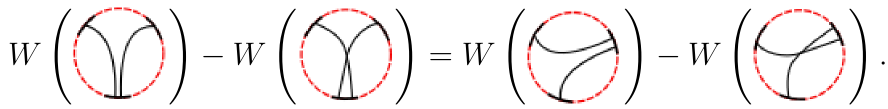
Proof. Let us define the map $f : \mathcal{C}_m \to \mathbb{C}$ by the following definition:

$$f(C) = \begin{cases} 1 & \text{if C has an isolated chord;} \\ 0 & \text{Otherwise.} \end{cases}$$

This map is well-defined, but does not satisfy the framing independence relation. Therefore, f cannot be in $\mathrm{Im}(\alpha_m)$, and hence cannot come from an invariant in $\mathcal{V}_m/\mathcal{V}_{m-1}$. Therefore, $\overline{\alpha}_m : \mathcal{V}_m/\mathcal{V}_{m-1} \to \mathcal{C}_m^\star$ is not surjective. □

[14] In the literature about weight systems framing independence is often left out as a condition because we could consider framed knots instead.

In a similar way, we find that every element in the image of α_m satisfies the *4T-relation*. This relation states the following:



Notice that this relation works on any two chords related in this way. This means that if any other chords are in the chord diagram these do not matter for the 4T-relation.

Lemma 3.34. Every element $W \in \mathrm{Im}(\alpha_m)$ satisfies the 4T-relation.

In order to proof this lemma, we first need to consider knots related to the chord diagrams in the 4T-relation. We can find many different knots with these chord diagrams, but under isotopy and non-singular crossing changes these are all equivalent. These knots can be given by represented by:

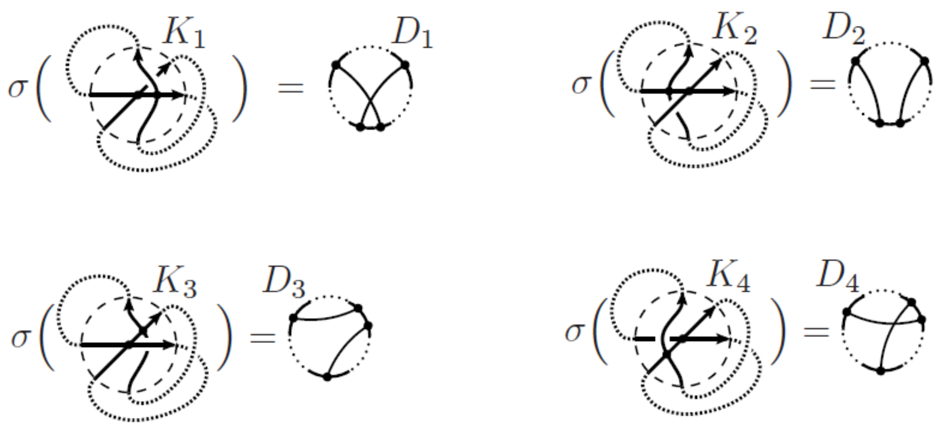
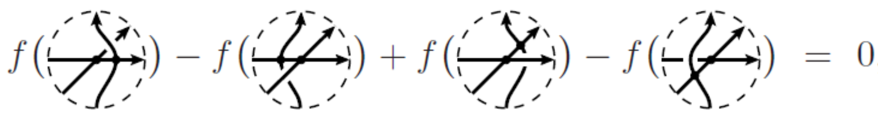


Figure 25: Knots related to the chord diagrams of the 4T relation [9].

Using this, we can prove the 4T-relation for Vassiliev invariants.

Lemma 3.35. Any Vassiliev invariant f satisfies the 4T-relation, i.e.



Proof. This lemma can be proven by resolving a singular points in each term. The resulting sum is an alternating sum, which will be zero. A detailed proof can be found in [9]. □

The proof of lemma 3.34 can now be done as follows.

Proof. (Lemma 3.34). We want to show that each W in the image of $\overline{\alpha}_m$ satisfies the $4T$-relation. Write $W = \overline{\alpha}_m[v]$ for some Vassiliev invariant v. Using the same trick as in the proof of lemma 3.32 we notice that for some chord diagram C in $\mathcal{C}$:

$$W(C) = \overline{\alpha}_m([v])(C) = \omega_v(C) = \omega_v \circ \phi_m(K) = v(K_C)$$

For a knot K_C related to the chord diagram. Using lemma 3.35 we already know that for the Vassiliev invariant v, $v(K_1) - v(K_2) + v(K_3) - v(K_4) = 0$. Notice K_i, for $i \in \{1,2,3,4\}$, are as in figure 25. Therefore, it must also hold that $W(D_1) - W(D_2) + W(D_3) - W(D_4) = 0$, where D_i, for $i \in \{1,2,3,4\}$, are as in figure 25. Which proves the lemma. □

We will now give the elements in $\mathrm{Im}(\alpha_m)$ a name.

Definition 3.36. An **Weight system** of degree m is an element $W \in \mathcal{C}_m^\star$ which satisfies both the framing independence and the $4T$-relation. The subspace $\mathcal{W}$ of $\mathcal{C}^\star$ is the vector space of all weight systems. Similarly, the subspace $\mathcal{W}_m$ of $\mathcal{C}_m^\star$ is the vector space of all weight systems of degree m.

Similarly to before with Vassiliev invariants, we have no filtration but a grading on the vector space of weight systems. This means that:

$$\mathcal{W} := \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \dots.$$

In a similar way as for chord diagrams, we want to establish a link between the space of Vassiliev invariants of a specific order and the the space of weight-systems. The following lemma states how to do this.

Theorem 3.37. The map $\overline{\alpha}_m : \mathcal{V}_m/\mathcal{V}_{m-1} \to \mathcal{C}_m^\star$ restricted to the image $\mathcal{W}_m$ is an isomorphism.

Proof. The surjectivity is very complicated and uses the so called Universal Kontsevich invariants. This proof is very interesting but too long and difficult to show here. For a detailed proof look at [9, 14, 20].

To show injectivity, first notice that $\overline{\alpha}_m : \mathcal{V}_m/\mathcal{V}_{m-1} \to \mathcal{C}_m^\star$ is injective by lemma 3.26. Any function W in the image of $\overline{\alpha}_m \subset \mathcal{C}_m^\star$ is a weight system of degree m by lemma 3.32 and lemma 3.34, therefore $\mathrm{Im}(\overline{\alpha}_m) \subseteq \mathcal{W}_m$. So $\overline{\alpha}_m$ is also injective when we restrict it to the image $\mathcal{W}_m$. Proving that $\overline{\alpha}_m$ restricted to the image $\mathcal{W}_m$ is an isomorphism. $\square$

It follows from theorem 3.37 that the equivalence class of Vassiliev invariants of a given order m is isomorphic to the vector space of weight systems of degree m. Using $\mathcal{V}_m \cong \frac{\mathcal{V}_m}{\mathcal{V}_{m-1}} \oplus \mathcal{V}_{m-1}$ recursively and the fact that $\mathcal{W}_{\leq m}$ is isomorphic to the direct sum of subspaces of weight systems of a given order, we can give a map ξ_m which is an isomorphism between $\mathcal{V}_m$ and $\mathcal{W}_{\leq m}$. This can also be seen as the following commutative diagram.

$$\begin{tikzcd}
\mathcal{V}_m \arrow[d, dashed, "\xi_m"] & \cong & \frac{\mathcal{V}_m}{\mathcal{V}_{m-1}} \arrow[d, "\overline{\alpha_m}"] & \oplus & \frac{\mathcal{V}_{m-1}}{\mathcal{V}_{m-2}} \arrow[d, "\overline{\alpha_{m-1}}"] & \oplus \cdots \oplus & \mathcal{V}_0 \arrow[d, "\overline{\alpha_0}"] \\
\mathcal{W}_{\leq m} & \cong & \mathcal{W}_m & \oplus & \mathcal{W}_{m-1} & \oplus \cdots \oplus & \mathcal{W}_0
\end{tikzcd}$$

This shows that $\mathcal{V}_m \cong \mathcal{W}_{\leq m}$, which means that we have proven that studying Vassiliev invariants is equivalent to studying weight systems.

Theorem 3.37 together with lemma 3.32 and lemma 3.34 can be summarised as the following theorem.

Theorem 3.38 (Fundamental theorem of finite-type invariants). For each weight system W there exists a unique corresponding Vassiliev invariant V and for each Vassiliev invariant V we can find a unique weight system W.

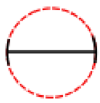
Theorem 3.38 is known as the Fundamental theorem of finite-type invariants or the Kontsevich theorem, which was proven by M.L.Kontsevich in 1993. He won the Fields medal for, amongst other things, this proof. As was mentioned before, the proof gets very complicated because of the surjectivity of $\overline{\alpha}_m$. This problem is resolved by finding a universal Vassiliev invariant which can be related to weight systems. In order to define such an universal Vassiliev invariant, we would need a lot more representation theory, algebra and geometry.

The goal for the next section we will be to make an example of a weight system and find its related Vassiliev invariant using the fundamental theorem.

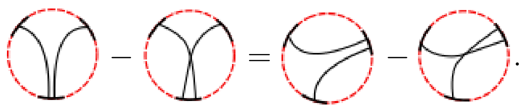
In this section we have shown that studying the space of Vassiliev invariants is equivalent to studying weight systems. But we have also shown that a weight system is just a linear functional from the space of chord diagrams to a field (in our case $\mathbb{C}$) which satisfies the 1T-relation and the 4T-relation. This means we could write:

$$\mathcal{W}_m \cong \left(\frac{\mathcal{C}_m}{1\mathrm{T}, 4T}\right)^{\star}.$$

Because of this duality it would be enough to just worry about the space $\frac{\mathcal{C}_m}{(1\mathrm{T}, 4T)}$. In all honesty, we should make a difference between the 4T-relation and the 1T-relation and the corresponding relations on the dual space. Formally, we can use lemma A.6 to states that for every equivalence relation on the dual, there must exist equivalence relations on the original space. In what follows we will try to not make the notation unnecessarily complicated. We will call both the relations the 4T-relation and the 1T-relation on the space of chord diagrams and on its dual space. The distinction will be clear from the context. This means we have the *1T-relation*:


$= 0$

And the *4T-relation* which is:



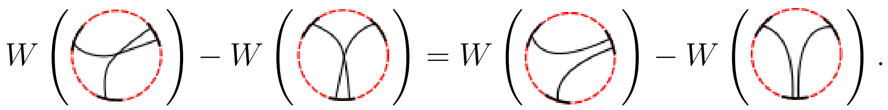
The space of chord diagrams which are equivalent under 4T and 1T, will be called $\mathcal{A}$. The subspace generated by degree m chord diagrams will be denoted by $\mathcal{A}_m$.

4 Example of a weight system

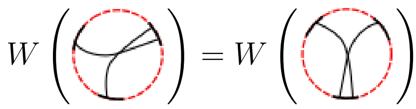
Now that we know what a weight system is we want to construct an example and use the fundamental theorem 3.38 to construct a related Vassiliev invariant. However, this turns out to be harder than one might expect. In order to make an example I will first define a stronger version of the 4T-relation, the so called 2T-relation. This relation will make it easier to create an example. I will give a different representation of chord diagrams, which is easier to work with in some cases.

4.1 The 2T-relation

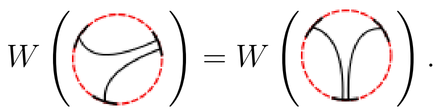
Let us first of all simplify the 4T-relation. If we recall the 4T-relation and move the terms around such that we get both terms where the chords intersect on the left-hand side and both terms where the chords do not intersect on the right-hand side, we get the following equation:



Here W is an element in the space of weight systems $\mathcal{W}$. If we now set both sides to zero, we obtain the following relations:



and,



We say W in $\mathcal{W}$ satisfies the **2T-relation** if both these relations are satisfied. Intuitively, these relations say that if we have a chord we can slide it along any other chord. In terms of knots, you can visualize this relation by saying we can vary the place of any singular crossing as long as we do not cross any other singular crossing. This is again defined on two chords, any other chords in de diagram do not matter for the 2T-relation.

Notice that if a functional on the space of chord diagrams satisfies the 2T-relation is also satisfies the 4T-relation. Notice that if we have a chord diagram which satisfies the 2T-relation, then both sides of the 4T-relation are zero, and hence the 4T-relation is satisfied. The other way around is not true. Therefore, we can say that the 2T-relation is a stronger version of the 4T-relation.

In general it is possible to make many more stronger versions of the 4T-relation by setting any combination of elements in the 4T-relation to zero. In the paper [21] is is shown that using these it is possible to make several weight systems. In this example we will use the 2T-relation as defined above.

4.2 Linear chord diagrams and long knots

Instead of working with round chord diagrams, we can build a 'linear' representation. Intuitively, this new representation is obtained by cutting open the skeleton of a chord diagram at an arbitrary point and put the chords on a line instead of a circle. This means we cut open the knot somewhere, walk along it, and mark the singular crossings on a straight line, and connect the two markings corresponding to the same singular crossing by a chord. In lemma 4.3 we will show that this operation is well-defined. We will refer to the original case of chord diagrams as *round chord diagrams*, and the case on the line will be a *linear chord diagram*.

This linear representation can be obtained by looking at so called *tangles* instead of knots. Tangles are a generalization of knots, but instead of being homeomorphic to a circle, a tangle is homeomorphic to a straight line with both its endpoints fixed. A knot is hence a tangle where both endpoints of the line are at the same position. An advantage in studying tangles is that they form a better representation for knots encountered in daily life, as most knotted chords are not circular, and hence might be more applicable. However, tangles are in some cases harder to work with, while often the theory for tangles can be derived from the theory of knots. For example, most of the theory discussed in this thesis could have been discussed on tangles instead, using the linear chord diagrams instead of using its round counterparts. It would, however, have been more complicated to decide which base point to pick when walking along the knot. Formally, we define [15] a tangle as follows.

Definition 4.1. A **tangle** is a smooth embedding of a one-dimensional compact oriented manifold T, possibly with boundary, into a box:

[15] There are several ways to define a tangle, instead of a box we can also use an embedding into a sphere.

$$B = \{(x, y, z) : a \leq x \leq b, -1 \leq y \leq 1, c \leq z \leq d\} \subset \mathbb{R}^3$$

where a, b, c, d are in $\mathbb{R}$ such that the boundary of T is mapped into the intersection of the plane given by $y = 0$ with either the upper or the lower face of the box. [9]

Notice that the only compact one-dimensional manifolds are closed intervals and circles. Together they form the *components* of the tangle. The boundary points are divided into two set, the set of boundary points in the upper face of B and the boundary points in the lower face of B. The manifold T together with the two sets of boundary points will be called the *skeleton* of the tangle. Similar to knots we use a projection of a tangle onto a plane to depict a tangle. In figure 26 a tangle with and a tangle without boundary are shown.

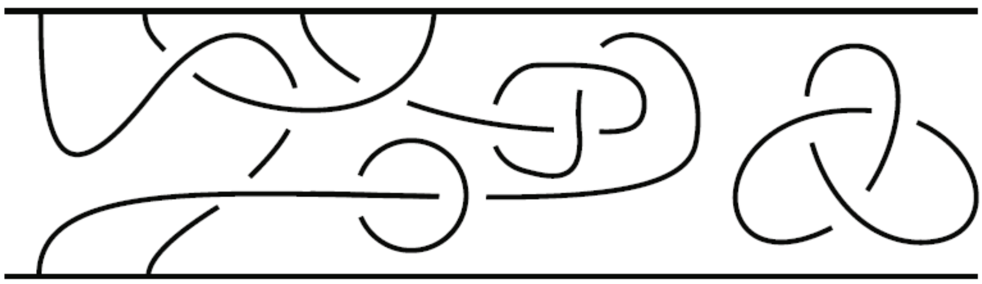


Figure 26: A tangle with and without boundary [14].

Notice that an n-component link is a tangle whose skeleton consists of a disjoint union of n-circles. Another special type of tangle is a *string link*.

Definition 4.2. Let us fix n distinct upper boundary points p_i on the plane given by $y = 0$. Given the projections q_i on to the bottom of the box we obtain n intervals $[p_i, q_i]$. A **string link** on n strings is a tangle whose skeleton consists of the intervals $[p_i, q_i]$. A string link with one string is called a **long knot**.

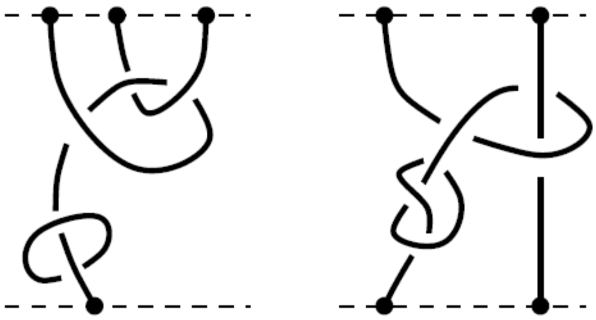


Figure 27: A tangle (l) and a string link with 2 components (r) [9].

In a similar way to knots we can define *singular long knots*. By applying the procedure of making a chord diagram as described in section 3.3 to a singular long knot, we obtain a *linear chord diagram*. Let us call the vector space of linear chord diagram $\mathcal{C}^l$, this vector space is formed similar to the vector space $\mathcal{C}$. We can form a round chord diagram from a linear chord diagram by glueing together the left and right end of the linear chord diagram. To show that a linear chord diagram is just a different (sometimes simpler) representation of the round chord diagram we need to show that the vector spaces $\mathcal{C}$ and $\mathcal{C}^l$ are isomorphic. This is not true in general, but it is true under the 4T-relation and framing independence.

Lemma 4.3. Glueing together the ends of a linear chord diagram gives rise to a vector space isomorphism $G : (\mathcal{C})^l \to \mathcal{C}$ modulo the 4T-relation.

Proof. It is clear G is a linear map between the vector spaces of formal linear combinations of round chord diagrams and linear chord diagrams. Let us first show that G is surjective. We need to show that for each round chord diagram rD there exists a linear chord diagram lD such that $G(lD) = rD$. This is trivial, because we can cut open any round chord diagram rD to obtain a linear chord diagram lD for which $G(lD) = rD$.

To show injectivity we need to show that if we have two round chord diagrams rD_1 and rD_2 for which the chord diagrams are equivalent, we get the equivalent linear chord diagram independent of where we cut open the round diagram. Suppose we cut open We use the 2T-relation instead of the 4T-relation. It is clear that if two diagrams are equivalent under 2T, then they must be equivalent under 4T. Notice that the 2T-relation on linear chord diagram states that the following diagrams are equivalent.

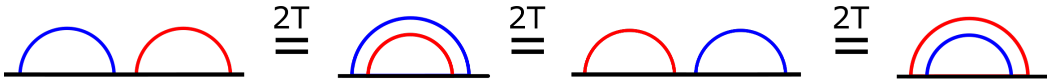


Figure 28: Linear 2T-relation for non-intersecting chords.

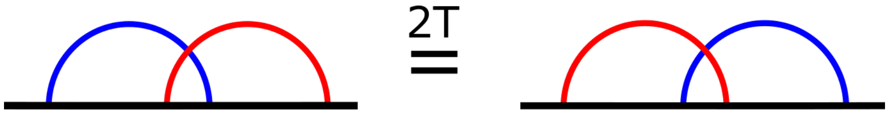


Figure 29: Linear 2T-relation for intersecting chords.

Let us assume, without loss of generality, that the orientation of the chord diagram is counter-clockwise. The idea of the proof is to show that if we change the place where we cut the round chord diagram, we obtain the same linear chord diagram. Suppose we cut open rD_1 at a point x, if we cut rD_2 at a position y between x and the next crossing, belonging to a chord c, we obtain the same linear chord diagram. Suppose we cut rD_2 at a point y after the next crossing looking from the perspective of x. In case it is the only chord in the diagram, we get trivially the same linear chord diagram. If there are other chords in the diagram $t_1, \dots, t_{n-1}$, then pick one of these chord t_i (depicted in black in figure 30), with $1 \leq i \leq n-1$ and notice that c can either intersect t_i or not intersect t_i. In case the two chord do not intersect, we get a nested case or a disjoint case. All cases are depicted in figure 30, together with the case where we cut at x or at y.

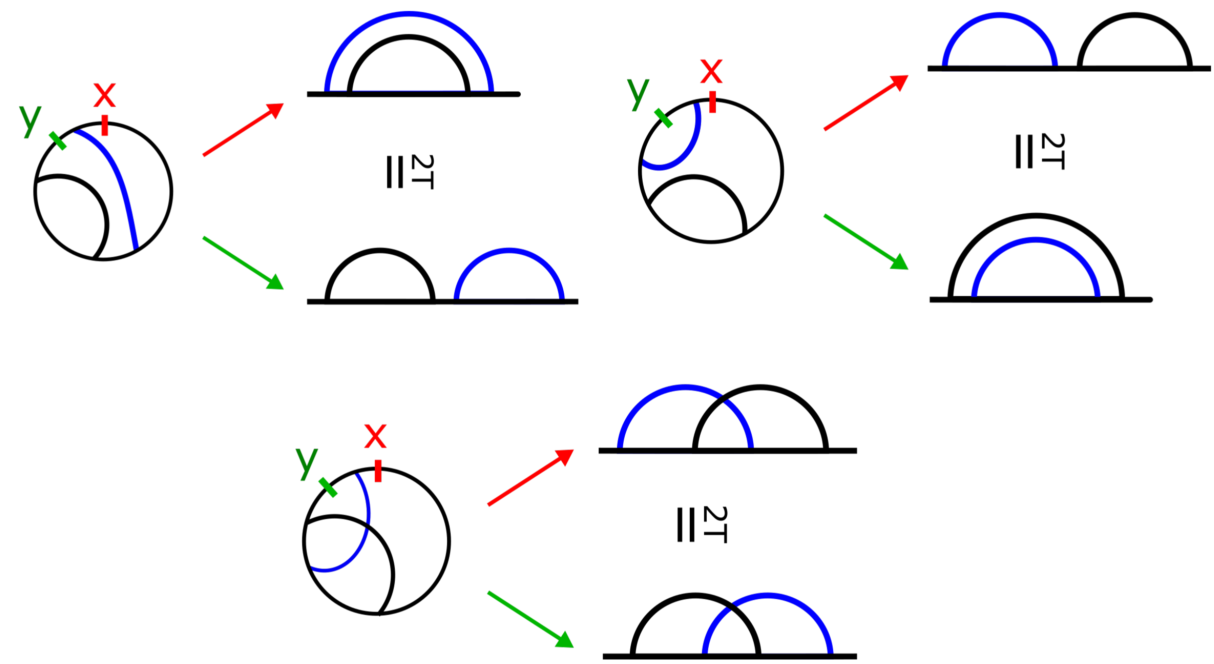


Figure 30: Different cases for cutting a round chord diagram. The red arrow depicts cutting at x, the green arrow depicts cutting at y.

Notice that under the 2T-relation, the diagram where we cut at x and where we cut at y are equivalent for each separate case. Moreover, the cases where the chords do not intersect are equivalent all together. This means that under the 2T-relation, it does not matter whether we cut at x or at y. Using induction, we can also move the cut of the round chord diagram to a place after the next crossing from the perspective of y: let us call this position z. Now, notice that the linear chord diagrams created from cutting at x and at z are also equivalent, because they can both be shown to be equivalent to y using the above reasoning. Using that the fact that if two diagram are equivalent under the 2T-relation they are also equivalent under the 4T-relation, we have shown that the map G is independent of the place where we cut open the round chord diagram under 4T. Therefore, G is injective and hence an isomorphism. $\square$

In the rest of the thesis we will, whenever convenient, freely change between linear chord diagrams and round chord diagrams, given that the 4T-relation and framing independence are satisfied.

It turns out that under the 2T-relation we can make a nice simplification. Namely if three chords are connected as in the figure 31, we can use the 2T-relation to simplify this to one single chord and only two connected chords.

In order to be able to prove this and use the result, the following terminology was invented.

Definition 4.4. A chord diagram with two connected chords is called a **2-humped camel**, a single (isolated) chord will be called a **1-humped camels** [16]. A sequence of g 2-humped camels and n 1-humped camels is called a **(g,n)-caravan**.

Now we can prove the following theorem.

Theorem 4.5. Any chord diagram with m chords is equivalent to a caravan under the 2T-relation.

Proof. First consider the case where we have three connected chords. Using the path drawn by the gray arrows, we can slide the blue chord out of the red chords. This sliding is allowed by the 2T-relation.

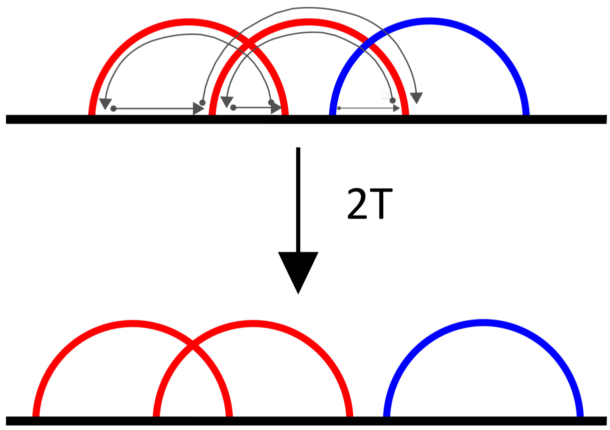
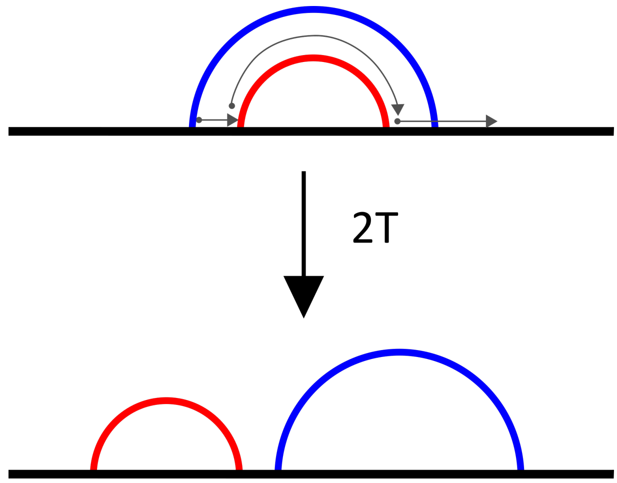


Figure 31: Three connected chords to a caravan.

When we have more than three chords, say k, this same method can be used to slide out chords one by one until we obtain one 2-humped camel and $k-2$ 1-humped camels.

Another possibility is that chords are nested. Again applying the 2T-relation we can slide out the blue chord following the gray arrows as drawn below and create two 1-humped camels.

[16] I would suggest to call this a dromedary, but this is sadly not the chosen name in the field.



Again when we have k nested chords we can apply this method several times to be left with k 1-humped camels. As all possible cases have been covered and resulted in only 2-humped camels and 1-humped camels, we have proven that each combination of chords is equivalent to a caravan under the 2T-relation. □

This is of course a nice proof of the theorem, but in my opinion a more elegant proof can be given. This proof has the added advantage of showing more of the connection between geometry, topology and chord diagrams, where up to now we have mainly been studying chord diagrams algebraically and using combinatorics. To give this proof we need the classification theorem for closed surfaces.

Theorem 4.6 (Classification theorem for closed surfaces). Any closed surface is homeomorphic either to:

1. the sphere;

2. the sphere with a finite number of handles added;

3. or the sphere with a finite number of discs removed and replaced by Möbius strips.

No two of the surfaces mentioned are homeomorphic. [13]

A surface is called *closed* if it is compact, connected and has no boundary. Adding a handle to the sphere means we remove the interior of a disc at two places in the sphere, and connecting the two discs by attaching a cylinder by glueing its boundary circles to the edges of the two holes in the sphere. 'Sewing' in a Möbius strip is done in a similar way. First, remove a single disc and add a Möbius strip in its place. This can be done because the Möbius strip has a single circle as its boundary. This is also known as a *cross-cap*. Both operations on the sphere are depicted in figure 32. Both operations on the sphere can be done multiple times at different places on the sphere. A proof of theorem 4.6 is given in [13, Chapter 7].

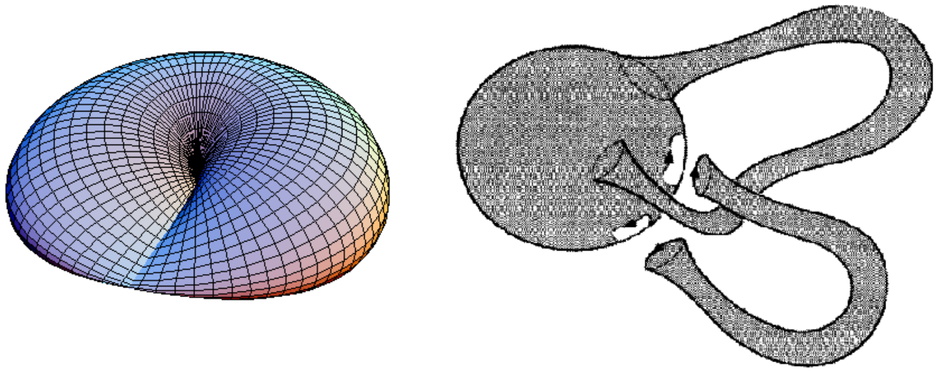


Figure 32: Sphere with an added Möbius strip (Left) and a sphere with a handle (Right). [13]

The *genus* is the number of cuttings in a surface along a closed curve. This means it is in some sense the number of holes in the surface. The sphere does not have any holes and hence has genus zero. But adding a handle or a cross-cap to the sphere changes the number of holes. Adding either a cross-cap or a handle adds one to the genus of the original surface. There is a difference in the definition of the genus for an orientable and a non-orientable surface. It turns out that the genus of an orientable surface equals the number of handles on sphere homeomorphic to the surface, while the genus of a non-orientable surface equals the number of cross-caps on the sphere homeomorphic to the surface. Using this, we can give an equivalent formulation of the classification theorem as follows.

Lemma 4.7. A closed surface is completely determined is determined by its genus and whether it is orientable or not.

But how does the classification theorem for surfaces relate to chord diagrams? We can thicken a chord diagram by replacing each chord by bridge as in the following figure:

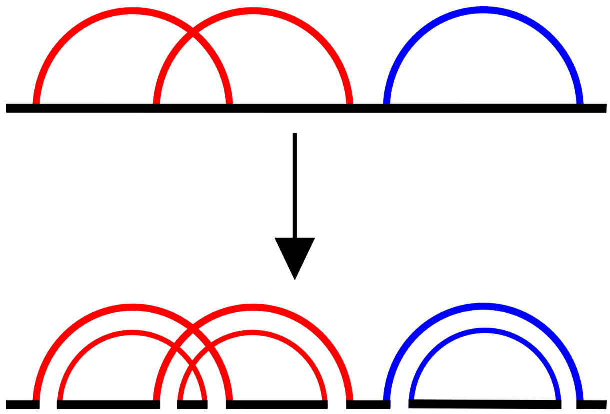


Figure 33: Bridged chord diagram

A thickened chord diagrams can be considered as a closed surface once we close and thicken the skeleton as in figure 34

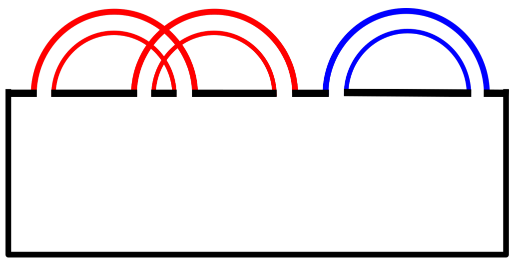


Figure 34: A chord diagram can be considered to be a box with handles.

P.M. Melvin commented that we hence consider chord diagrams and caravans as a 'box with handles' [22].

Every thickened chord, now will create a handle on the sphere homeomorphic to the surface of the thickened chord diagram. This means a chord diagram with m chords, will have genus m. Similarly, a (g,n)-caravan will have genus $2g + n = m$. Therefore, it must be the case that the space of (g,n)-caravans such that $2g+n = m$ is homeomorphic to the space of chord diagrams $\mathcal{C}_m$ by the classification theorem of surfaces. Here we have to notice that both a chord diagrams and a caravan can endowed with an orientation, and hence both the surface corresponding to the chord diagrams and the surface corresponding to the caravan will be orientable.

Now that we have shown that each chord diagram is equivalent to a caravan under the 2T-relation, we want to reduce the problem of counting the number of chord diagrams with m chords under the 2T relation to counting the number of (g,n)-caravans for which $2g + n = m$. The last condition on the number of chords m holds because each 2-humped camel consists of two chords and hence accounts for $2g$ chords in total, while the other n chords come from the 1-humped camels in the caravan.

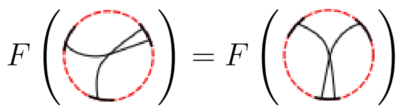
Theorem 4.8. *The dimension of the space of chord diagrams of m chords ($\mathcal{C}_m$) equals the number of (g, n)-caravans for which $2g + n = m$.*

Proof. From theorem 4.5 we already know that each chord diagram can be represented by a (g, n)-caravan, what is left to be shown is that each chord diagram can be represented by a unique caravan. The idea of the proof is to construct a functional $F : \mathcal{C}_m/2T \to \mathbb{Q}[N]$, where $\mathbb{Q}[N]$ are the polynomials in variable N with coefficients in $\mathbb{Q}$. Using this functional we will be able to say there are as many chord diagrams of m chords as there are (g, n)-caravans with $2g + n = m$.

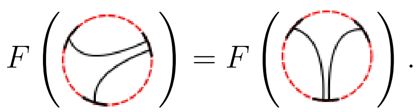
First we construct a functional $F : \mathcal{C}_m/2T \to \mathbb{Q}[N]$. To do this, first we replace each chord in a linear chord diagram by a bridge as in figure 33.

In this process we open the skeleton of the diagram at the 'feet' of the chords, and connect the chords by lines at their feet. The obtained diagram will be called a *bridged chord diagram* D'. Let us denote the number of connected components by C. For example the chord diagram above has two connected components which in the following picture have been depicted in blue and red. The functional F on a chord diagram D' is defined by $F(D') :- N^C$.

Notice that F is invariant under the 2T-relation. This is the case because the 2T relation does not change the number of chords or the number of connected components, hence:



and



The invariance under the 2T-relation shows that the functional F is well-defined.

We can now look at the effect of the functional F on a given caravan. We notice that all the 2-humped camels together form one connected component (this is the red line in figure 35), while each 1-humped camel adds one connected component (this is the blue line in figure 35) to the number of connected components.

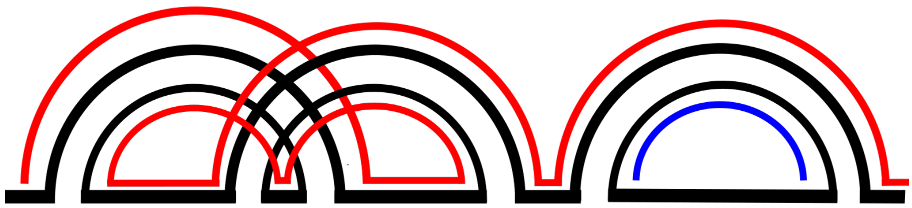


Figure 35: The red line connects all 2-humped camels, the blue lines is created by a 1-humped camel.

This means that for an (g, n)-caravan, there are n connected components from the n 1-humped camels, and one connected component from the 2-humped camels. So if we now compute F of a caravan we get:

$$F\left(\mathrm{Caravan(n, g)}\right) = N^{n+1}.$$

For each n the polynomials N^{n+1} are linearly independent in $\mathbb{Q}[N]$, and hence all caravans are linearly independent. As we have already proven that each chord diagrams can be expressed as a caravan, which means we have that the (n, g)-caravans span the space of chord diagrams under the 2T relation. Together, these two facts show that all (n, g)-caravans for which $n + 2g = m$ form a basis for the space $\mathcal{C}_m/2T$. This also proves that the dimensions of both sides agree, and hence:

$$\dim\left(\mathcal{C}_m/2T\right) = \#\left((g, n)\text{-caravans s.t. } 2g + n = m\right).$$

Proving the theorem. $\square$

In summary, this theorem shows us that when we want to find the dimension of the space of chord diagrams under the 2T-relation, all we need to do is look at the number of 1-humped camels and add one to the dimension for all the 2-humped camels.

4.3 The Conway weight system

One might wonder what happens when we also apply framing independence to framing independence to a chord diagrams in the space $\mathcal{C}_m/2T$. We notice that when we apply framing independence all isolated chords become irrelevant to the value of the weight system. It follows from the definition that the isolated chords on a chord diagram are equivalent to 1-humped camels. Thus the functional F applied to a chord diagram is invariant under the number of 1-humped camels n. Hence, we can set n to be equal to 0. This shows:

$$\dim(\mathcal{C}_m/2T) = N^{1+n} = N^1.$$

Therefore, we find only one basis vector, namely this monomial N. This means there is only one weight system which satisfies both the 2T-relation and the framing independence relation. This unique weight system is called the *Conway weight system.*

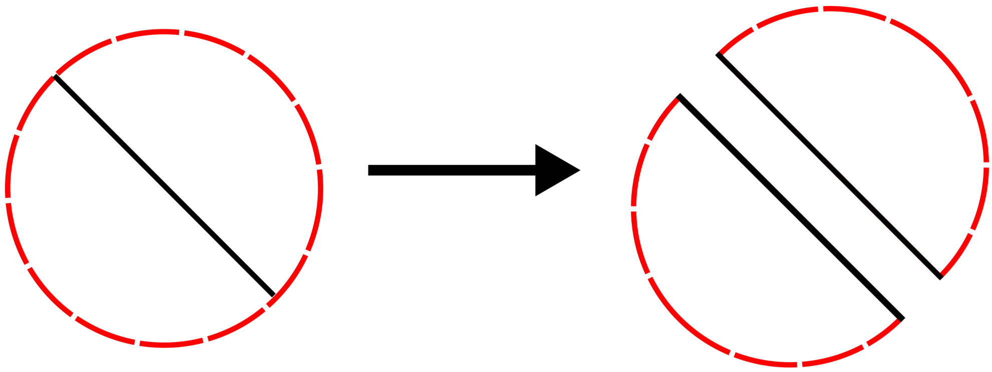
Definition 4.9. Given a chord diagram D in $\mathcal{C}_m$ and a bridged chord diagram D' the **Conway weight system** $W_{C,m}$ is an element in $(\mathcal{C}_m)^\star$ such that:

$$W_{C,m}(D') = \begin{cases} 1 & \text{if } D' \text{ has one connected component;} \\ 0 & \text{if } D' \text{ has more than one connected componenent.} \end{cases}$$

Actually, the Conway weight system is defined to be the the coefficient of N in $F(D)$ for a diagrams D in $\mathcal{C}_m$. This weight system in some way mimics the way our functional $F : (\mathcal{C}_m/2T) \to \mathbb{Q}[N]$ was built. Before we dive deeper into the details of this weight system, we first need to show that what we defined actually is a weight system.

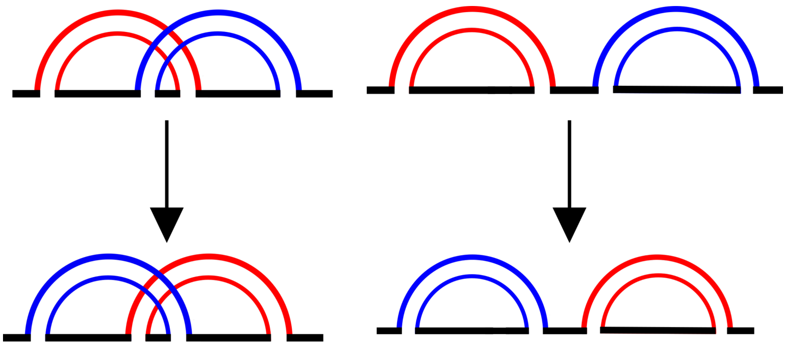
Theorem 4.10. The Conway weight system $W_{C,m}$ is a weight system.

Proof. To show that $W_{C,m}$ is a weight system we need to check it is invariant under the 4T-relation and under framing independence. Let us first show it is framing independent. Let us take a diagram D with m chords and at least one isolated chord. When now creating the bridged diagram D' we obtain the following picture:

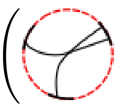
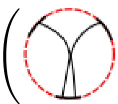


This means that for any isolated chord in D, we get two disjoint connected components in D', meaning that $W_{C,m} = 0$. Therefore, we have shown that $W_{C,m}$ is framing independent.

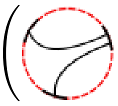
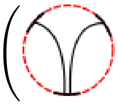
To check $W_{C,m}$ satisfies the 4T-relation, we can also check it satisfies the 2T-relation. This is because the 2T-relation implies the 4T-relation. Notice that the 2T-relation can only slide one chord along another chord. This means only situations as in the following figure can occur:



In both these cases the number of connected components does not change and hence:

$W_{C,m}$ () $= W_{C,m}$ ()

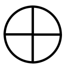
and

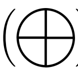
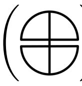
$W_{C,m}$ () $=W_{C,m}$ ().

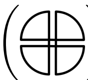
This proves that $W_{C,m}$ satisfies the 2T-relation, and hence also the 4T-relation. Therefore, together with the framing independence $W_{C,m}$ is a weight system. $\square$

Until this point the Conway weight system is defined on the spaces $\mathcal{C}_m$. We know $W_{C,m}$ is the unique weight system for a chord diagram of type m. However, there are still infinitely many types. In order to make a knot invariant on a knot with an arbitrary amount of crossings we can 'pack all weight systems together'. We do this by taking a direct sum of all the spaces of chord diagram modulo 4T and framing independence and mapping that to the polynomial ring over Z. So we obtain: $W_C : \mathcal{A} = \oplus \mathcal{A}_m \to \mathbb{Q}[Z]$ such that $W_C := \sum_{m=0}^{\infty} W_{C,m} Z^m$. Here $\mathcal{A}$ was the space of chord diagrams under 4T and framing independence.

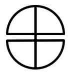
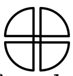
We notice that $W_{C,m}(D) = 0$ on a diagram D which is not in $\mathcal{C}_m$. Moreover, using $W_C(D)$ on a diagram D with m chords is just $W_{C,m}$. It might not look like we changed a lot, but one thing we can do now is talk about chord diagrams of links and define W_C recursively. The necessity for this becomes clear in the next example.

Example 4.11. Take chord diagram $D =$  in $\mathcal{C}_2$,

W_C () $= ZW_C$ ()

$= Z^2 W_C$ ()

$= Z^2$

Here we notice that we cannot bridge any more chords in , and we have one connected component. The main advantage of defining W_C is that we can now speak about chord diagrams such as  and  which consist of multiple parts and no longer just one circle. An example of a chord diagram of a links is given in figure 36.

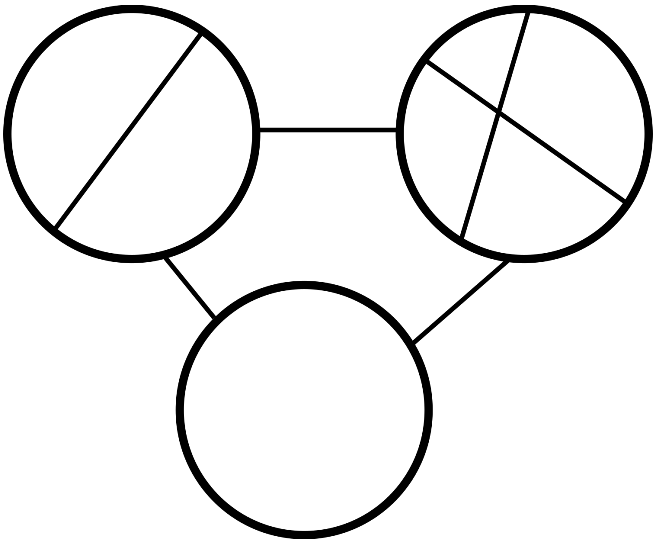
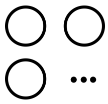
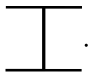


Figure 36: Chord diagram for a link.

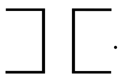
Using the above we can define W_C inductively as follows. let us denote k arbitrary not connected chord diagrams by:



A chord between two objects (either between two link component chord diagrams or a ‘normal chord’ inside a diagram) will be denoted by:

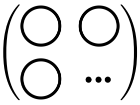


The bridge coming from this chord is denoted by:



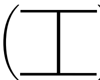
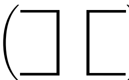
Definition 4.12. The Conway weight system $W_C : \mathcal{C} \to \mathbb{Q}[Z]$ is inductively defined as follows:

1. The weight system on k not connected chord diagrams is:

$W_C\Big($  $\Big) = \delta_{k1},$

where δ_{k1} is the Kronecker delta function.

2. The weight system on each chord is:

$W_C\Big($  $\Big) = Z \cdot W_C\Big($  $\Big).$

4.4 Invariant from the Conway weight system

From the Kontsevich theorem (theorem 3.38) we know that each weight system has a unique associated Vassiliev invariant. In this section we will try to find the invariant related to the Conway weight system. In other words, we are looking for $V_C : \{\text{Links}\} \to \mathbb{Q}[Z^{\pm 1}]$ such that the weight system associated to V_C is W_C. Again, we consider links for the case where we have multiple connected chord diagrams for connected knots. The invariant we get should in some way be equivalent to what is happening in the weight system. Therefore, we guess the following invariant V_C and then show it is the invariant related to the Conway weight system.

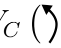
Definition 4.13. The Conway invariant is given by $V_C : \{\text{Links}\} \to \mathbb{Q}[Z^{\pm 1}]$ the following skein relation on singular crossings:

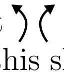
1. The value of V_C on k disconnected unknots is:

$$V_C\left(O^{(k)}\right) = \delta_{k1}.$$

Here $O^{(k)}$ denotes k unknots which are not linked. Again δ_{k1} is the Kronecker delta function.

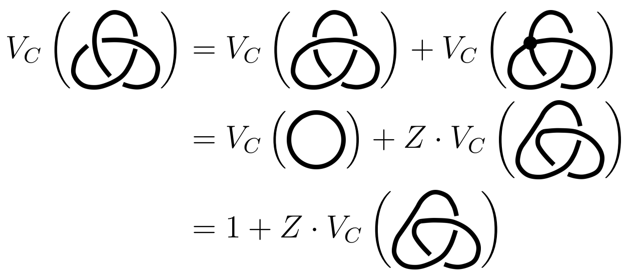
2. For a singular crossing  we have:

$V_C\big($  $\big) = V_C\big($  $\big) - V\big($  $\big) = Z \cdot V_C\big($  $\big).$

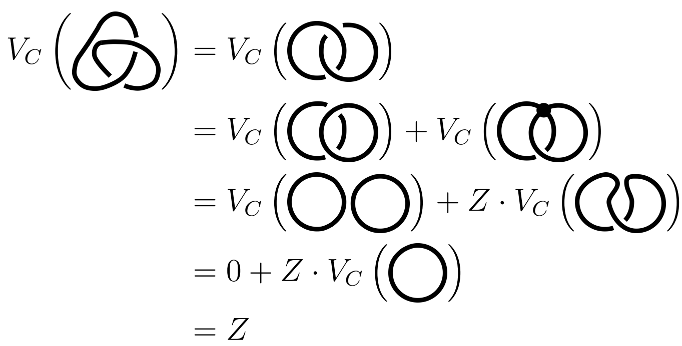
Here the main guess is that  is actually the knot analogue of making a bridge on a chord. Now notice that this skein relation is same as the skein relation given for the Alexander-Conway polynomial (definition 2.16), if we replace the base unlink case with lemma 2.22.

How to work with the Conway invariant will be illustrated in the following example.

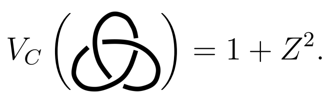
Example 4.14. The value of V_C on the trefoil knot can be computed as follows.



Where we use that  is isotopic to the unknot. Moreover, notice that  is actually equivalent to a Hopf link, so we get:



Here we notice that  is equivalent to the unknot. Taking the above together, we get:



Now there are several important things to prove about this invariant. First of all is the function we defined actually a unique and well-defined invariant?

Theorem 4.15. V_C is a unique and well-defined link invariant.

Proof. Because V_C has the same skein relation as the Alexander-Conway polynomial, which is a HOMFLYPT polynomial for $m = -t$ and $l = 1$, the result follows directly from theorem 2.21. $\square$

Now that we know V_C is an invariant, it remains to be shown that it is actually the invariant which is related to the Conway weight system. We know that the full Alexander-Conway polynomial is not a Vassiliev invariant. But we do know something about each of its terms.

Lemma 4.16. The m-th term of the link invariant $V_C : \{\text{Links}\} \to \mathbb{Q}[Z^{\pm 1}]$ is a Vassiliev invariant of order m.

Proof. From example 3.9 we know that each term of this polynomial is a Vassiliev invariant. Moreover, we have shown there that the m-th coefficient of the Alexander-Conway polynomial is a Vassiliev invariant of order m. $\square$

Rewriting the value V_C on a link L we get $V_C(L) = \sum_{m\geq 0} V_{C,m}(L)Z^m$ where $V_{C,m}$ denotes the m-th coefficient of polynomial V_C. Here we notice that $V_{C,m}$ is a Vassiliev invariant of type m by the previous lemma. By theorem 3.38, there must be a weight system W_m of degree m such that $W_m(D) = V_{C,m}(L)$ of a diagram D such that $\phi_m(L) = D$. Here ϕ_m is the map which maps knots to their diagrams. In the case at hand it turns out that the Conway weight system $W_{C,m}$ is the same as the coefficient of the m-th term of the polynomial V_C.

Theorem 4.17. The weight system which corresponds to $V_{C,m}$ is $W_{C,m}$.

Proof. We will prove this theorem using the defined skein relations for V_C and W_C. Given a link with m crossings, we can annul all of them using the second relation of both skein relations. In the case of V_C we get:

$$V_C(\underbrace{\times \cdots \times}_{m}) = Z^m V_C\left(\underbrace{)\,(\cdots)\,(}_{m}\right)$$

This means $V_C\left(\underbrace{)\,(\cdots)\,(}_{m}\right)$ equals the m-th coefficient of the Alexander-Conway polynomial $V_{C,m}(L)$. But notice that annulling each singular crossing results in a diagram which can be changed into a k-component unlink by a finite amount of non-singular crossing changes. Here k is a number less than or equal to m. But as a Vassiliev invariant does not depend on a finite number of non-singular crossing changes, we can use the value on the k-component unlink. This means we that, by lemma 2.22, we have:

$$V_{C,m}(L) = \begin{cases} 1 & \text{if } k = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we take a diagram D with m chords such that $\phi_m(K) = D$. Let us make a bridge of each chord to obtain:

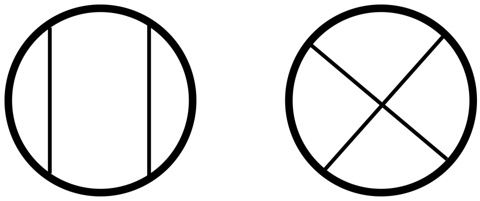
$$W_C\left(\underbrace{\perp \cdots \perp}_{m}\right) = Z^m W_C\left(\underbrace{\sqsupset\sqsubset \cdots \sqsupset\sqsubset}_{m}\right).$$

Because on a diagram with m chords we have that $W_C(D) = W_{C,m}(D)Z^m$, it must be the case that the value of $W_C\left(\underbrace{\sqsupset\sqsubset \cdots \sqsupset\sqsubset}_{m}\right) = W_{C,m}(D)$. But notice that if we bridge every chord in a diagram, all that remains are ℓ components without any chords. Here ℓ is a number less than or equal to m. Applying the first relation of the skein relation for W_C gives us the following value for $W_{C,m}$.

$$W_{C,m}(D) = \begin{cases} 1 & \text{if } \ell = 1; \\ 0 & \text{otherwise.} \end{cases}$$

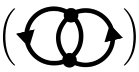
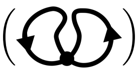
This means that $W_{C,m}(D) = V_{C,m}(L)$ for diagrams D such that $\phi_m(L) = D$ provided that k and ℓ are the same on links and diagrams related like this. In order to prove k equals l we bridge every chord in a diagram of degree m and see it leaves us with m link components. The link components are connected as long as the diagram components are connected. This means the chord diagram with k empty components has the k-unlink as its corresponding link. Similarly, if we create a k-unlink by smoothing crossings, the corresponding chord diagram will have k disconnected components. As an arbitrary number of singular crossings was chosen this shows that for any given link L and chord diagram D with the property that $D = \phi(L)$ we have $W_{C,m}(D) = V_{C,m}(L)$. Proving that the Conway weight system indeed corresponds to the m-th term coefficient of the Conway invariant. $\square$

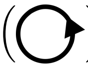
As an example of the proof above, consider the case $m = 2$. We know by proposition 3.16 there are two chord diagrams D_1 and D_2 with two chords. These are:



D1 D2

Notice that the value $W_C(D_1)$ is trivially zero, because of the framing independence relation. But the value of $W_C(D_2)$ equals Z^2 by example 4.11. We should now show that for a knot corresponding to the diagram D_i, with $i \in \{1, 2\}$, the m-th term of the Conway polynomial agree with $W_C(D_i)$. A knot corresponding to D_2 is given by the Hopf link with two singular crossings L_2. Keeping track of the orientation [17] carefully we compute the following.

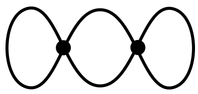
V_C () $= Z \cdot V_C$ ()

$= Z^2 \cdot V_C$ ()

$= Z^2.$

Which $V_{C,2}(L_2) = W_C(D_2)$

A knot corresponding to D_1 is given by:

$L_1 =$ 

The value here was already computed to be zero in proposition 3.16. This shows that in the case m equals 2, the m-th term of the Conway polynomial agrees with the value of W_C on diagram with two crossings.

4.5 Intersection graphs

The definition of the Conway weight system as described in the previous section is quite hard to compute. In this section we will give a different characterization

[17]Notice that when annulling the crossing the orientations make sure that one of the crossings is cut open 'horizontally' and the other 'vertically' as depicted in figure 9

of the Conway weight system, which should be easier to compute. This idea was first presented in [22], which we will follow closely in this section.

Definition 4.18. Let C be a linear chord diagram in $\mathcal{C}_m$. The **labeled intersection graph** $\Gamma(D)$ is the graph whose vertices correspond to the chords of D, numbered from 1 to m in the order in which they appear from left to right on the base line. Two vertices are connected by an edge if and only if the corresponding chords intersect.

A simple example is given in figure 37.

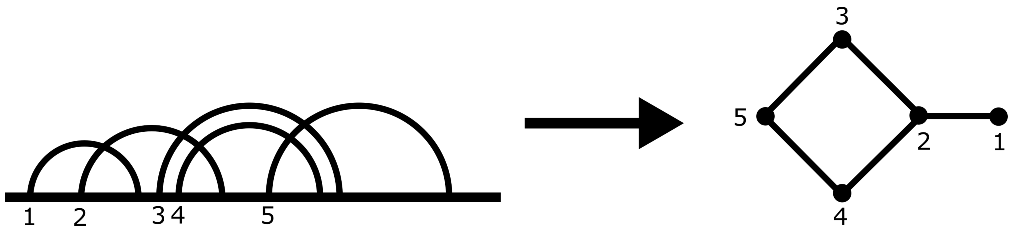


Figure 37: Example of an intersection graph $\Gamma(D)$.

Intersection graphs are useful because they give a simplified representation of a chord diagram, but still contain a great amount of information about the chord diagram. In some cases, it is even true that when the intersection graphs of two diagrams are equal then the diagrams must be equal (see [9, Section 4.8.4.]). On these intersection graphs graphs it sometimes turns out to be easier to define a weight system. In the case of the Conway weight system it will be the determinant of the so called *intersection matrix*.

Definition 4.19. The **intersection matrix** $\mathrm{IM}(D)$ of a labeled intersection graph $\Gamma(D)$ with D in $\mathcal{C}_m$ is given by an $m \times m$ matrix with the following entries:

$$\mathrm{IM}(\Gamma(D))_{ij} = \begin{cases} \mathrm{sign}(i-j) & \text{if vertices } i,j \text{ are connected in } \Gamma(D). \\ 0 & \text{otherwise} \end{cases}$$

Notice that we have chose the labels i and j to be applied from left to right. The intersection matrix is an anti-symmetric variation of the adjacency matrix. The diagonal will only contain the zero element, as a chord never intersects with itself. As an example, the intersection matrix of our example in figure 37 is the following 5×5 matrix:

$$\mathrm{IM}(\Gamma(D)) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

We can now show that the Conway weight system is given by the determinant of the intersection matrix.

Theorem 4.20. For any chord diagram D we have that $W_C(D) = \det(\mathrm{IM}(\Gamma(D)))$.

Proof. Let $W(D) := \det(\mathrm{IM}(\Gamma(D))$ and let D be a chord diagram in $\mathcal{C}_D$. Because W_C is the unique weight system satisfying the 2T-relation, all we need to do is show that $W(D)$ is indeed a weight system satisfying the 2T-relation. Let us start by showing $W(D)$ satisfies the 2T-relation.

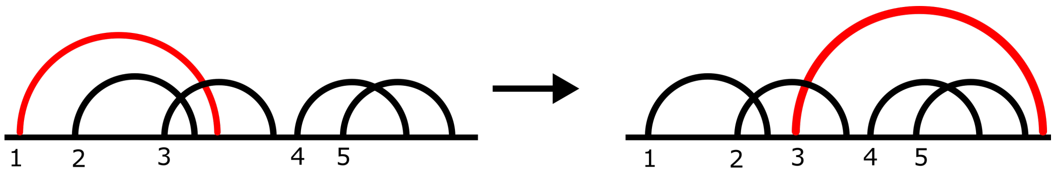


Figure 38: Independence of the base point.

First, we will need to show that $W(D)$ is independent of the base point of the chord diagram D. Suppose we have a linear chord diagram D_1 and a linear chord diagram D_2 obtained from D_1 by moving the left-most vertex to the right end (see figure 38). We now want to show that $W(D_1) = W(D_2)$. Notice that we do not add extra intersections in the graph because we picked the left-most vertex, hence we only change the labels of the vertices. In the intersection graph we only change the first row and column. Here the signs flip, because we now meet the original 'endpoint' of the first chord, before we meet it 'beginning'. Instead of the first chord, this chord is now the j-th chord, where $1 \le j \le m$, we change row 1 to row j and flip the sign. We do the same thing for the first column. Then the sign flips cancel each other, which means we obtain the following:

$$\det(\mathrm{IM}(D_2)) = (\pm 1)(\pm 1)\det(\mathrm{IM}(D_1)) = \det(\mathrm{IM}(D_1)).$$

We now want to show $W(D)$ satisfies the 2T-relation. Consider the 2T-relation, which moves b over a stationary chord s to the chord a as in figure 39. Let D_1 be the diagram before the 2T-move (i.e. with chords b and s) and let D_2 be the diagram after the 2T-move (i.e. with a and s). We now want to show $W(D_1) = W(D_2)$.

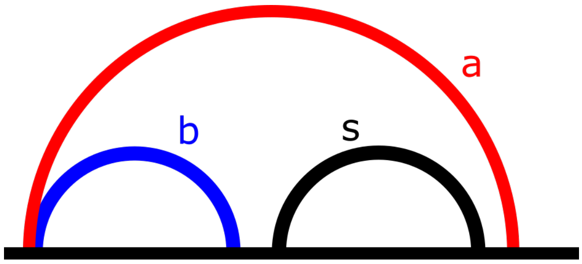


Figure 39: 2T relation for $W(D)$.

Using the independence of the base point we can make chords b and a to be the first chords of this linear chord diagram. We also state no other chord can start or end between the endpoint of b and the start of s, and between the endpoint of s and the endpoint of a. If any other chord o does end in one of these regions we would have that a and b would first have to move over o, before it could move over s. From this assumption is follows any other chord can either intersect none of the chords a, b and r or exactly two of them. In case no other chords intersect a, b and r, nothing changes in the intersection matrix if we go from D_1 to D_2, so $W(D_1) = W(D_2)$.

Suppose a chord does intersect exactly two of the chords a, b and s. Let us consider the case where other chords $o_1, \ldots, o_r$ cross a and s, but not b. Then when moving b to a, we would get extra intersections with a. These are exactly the the intersections with r, so in the intersection matrix we would need to add the rows corresponding to chords $o_1, \ldots, o_r$ to the first row. Similarly for the first column. Notice that the determinant of a matrix does not change when we add a row to another row multiple times, so $W(D_1) = W(D_2)$. Notice that if the chords $o_1, \ldots, o_r$ would intersect b and r, we would subtract the rows corresponding to the crossings, and the same reasoning holds. Similarly, when a chord would only intersect a and b, nothing changes to the intersection matrix, as the crossings stay the same. This shows $W(D)$ satisfies the 2T-relation, and hence also the 4T-relation.

We need to show that $W(D)$ is also framing independent. Notice that, using the 2T-relation, any chord diagram can be changed to a (g, n)-caravan with g 2-humped camels and n 1-humped camels. The intersection matrix G for a 2-humped camel is given by:

$$G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

And the intersection matrix N for a 1-humped camel is the 1×1 matrix $[0]$, as it does not intersect any chords by definition. This means the intersection matrix of a chord diagram can always be written as a block diagonal matrix of the form:

$$\mathrm{IM}(D) = \begin{pmatrix} G & 0 & & & \dots & & 0 \\ 0 & \ddots & & & & & \\ & & G & & & & \vdots \\ \vdots & & & & N & & \\ & & & & & \ddots & 0 \\ 0 & & \dots & & & 0 & N \end{pmatrix}$$

Notice that the determinant of a block diagonal matrix, is the product of the determinant of the blocks. Notice that $\det(G) = 1$ and $\det(N) = 0$, so:

$$W(D) = \det(G)^g \det(\mathrm{N})^n = 1^g 0^n.$$

This means that if we have one 1-humped camel, we get that $W(D) = 0$, so $W(D)$ is framing independent. Together with what we have shown before $W(D)$ is a weight system which satisfies the 2T-relation. By uniqueness of the Conway weight system we have found that $W_C(D) = W(D)$ for an arbitrary chord diagram D, proving the theorem. □

Under the 2T-relation the connection between the determinant of the intersection matrix and the Conway weight system can even be made more explicit. Given a diagram D, we have seen that under the 2T-relation, we get write:

$$\det(\mathrm{IM}(D)) = 0^n 1^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We have seen before that the only way there could only one connected components would be if $n = 0$. This means the determinant of the intersection matrix indeed gives the same formula for the Conway weight system.

5 Conclusion and outlook on further research

In this thesis we have seen what a Vassiliev invariant is and how it relates to chord diagrams and weight systems. We have proven part of the Fundamental theorem of finite-type invariants and we have seen that there are at most finitely many Vassiliev invariants of a given degree. In the end we have defined the Conway weight system and found its corresponding Vassiliev invariant. This however only covers a small portion of the theory of Vassiliev invariants. In [9] and [14] there are already several chapters which give more information about the surrounding theory of Vassiliev invariants. Here also the generalizations to invariant which are not field-valued can be found. These sources however assume more knowledge about representation theory and lie algebras. In this section I would like to give a few references for associated literature and some interesting applications of the presented theory which I did not have time for during this thesis.

5.1 Geometric view of Vassiliev invariants

In this thesis we have looked at knot invariants and Vassiliev invariants, which very generally said are topological invariants on a manifold. However, when working with these invariants we have taken a quite algebraic and combinatorial approach, using basic combinatorics, linear algebra and module theory to say something about these invariants. Of course topology and geometry are always in the background here, as was for example seen when working with chord diagrams as a closed surface, but from the example of the Linking integral (example 2.13) we know that some invariants can also be seen as an integral. An integral formula like the linking integral for a Vassiliev invariant would give a more geometric interpretation of what is going on when applying these invariants to a knot. It turns out that for the Casson knot invariant (the quadratic term of the Alexander-Conway polynomial) there is such an integral formula. This formula is presented in [23]. If it would be possible to extend the method in this paper to more general Vassiliev invariants, this would give a very nice geometric characterization of the Vassiliev invariant. In the future I would like to examine this paper more closely.

5.2 Tangles

In section 4.2 we defined a tangle. This object is very closely related to a knot. This means we might wonder if we can also define a Vassiliev ‘tangle invariant’ and a corresponding ‘tangle weight system’. The idea for a tangle invariant would be to cut up the tangle in smaller parts, compute the corresponding polynomials, and ‘glue’ these polynomials back together. Cutting open the tangle (or any knot for that matter) would give us ‘open’ strands (with open I mean not circular), but all our polynomials are defined on 'closed' strands. This means the classical Alexander-Conway polynomial cannot be computed. But luckily there exists a generalization of the Alexander-Conway polynomial which is defined on open strands and hence on tangles. The generalization of the polynomial is called the *multivariable Alexander polynomial* (MVA), and this is treated in detail in the thesis of Jana Archibald [24]. Under the right substitutions, and by setting all but one of the variables to a constant, one can obtain the Alexander-Conway polynomial as described in definition 2.16. In [24] it is, amongst other things, shown that the n-th term of the multivariable Alexander polynomial is always a Vassiliev invariant of order n. Moreover, a weight system is computed for this polynomial, given by $cMVA : \{\text{Links}\} \to \mathbb{Z}[t_1, \ldots, t_n]$

$$\mathrm{cMVA}(D) := \frac{\det\left(M^i(D)\right)}{t_i}$$

where $M^i(D)$ denotes the so called 'Alexander Matrix' with row and column i removed. This Alexander matrix is very closely related to the intersection matrix that was seen in section 4.5. For more details I will refer to [24], but from the above it can be seen that it should be possible to generalize the Conway weight system on knots to a Conway weight system on tangles, which is something to look into in further studies.

5.3 Making new Vassiliev invariants

One of the goals set in the introduction was to be able to make new Vassiliev invariants. As was shown in theorem 3.38 for every weight system there exists a related Vassiliev invariant. This means that by making new weight systems we directly make new Vassiliev invariants. It may be noted however that the way we have found the invariant from the Conway weight system involved a guess that annulling a crossing was equivalent to bridging a chord. This guess turned out to be true because of the uniqueness of the weight system invariant under 2T, which is not true in general. This means the method is hard to apply in general. It may however be noted that in [21] different types of 2T-relation are derived. This is done by setting different parts of the 4T-relation equal to zero. Using these different 2T-relations, more weight systems can be derived using similar techniques as described in 4.4. In this paper, also a weight system for the HOMFLYPT polynomial is given. This is a generalization of the Alexander-Conway polynomial, but also of many other polynomial invariants. These other polynomial invariants related to the HOMFLYPT polynomial will hence also be Vassiliev invariants in some way. It may also be noted that using the intersection graphs as described in section 4.5, it is also possible to derive many more weight systems and hence more Vassiliev invariants.

Appendices

A Vector spaces and modules

A.1 Relevant theorem and definitions

In this section some relevant theorems and definitions from module theory are given which are used in some proofs in Section 3. For more background information or the proofs one could look at [14] or [25].

Definition A.1. If V, W are modules over the same ring $\mathbb{K}$ then $f : V \to W$ is a *morphism* if for all $x, y \in V$ and $\lambda \in \mathbb{K}$ we have that:

1. $f(x + y) = f(x) + f(y)$
2. $f(\lambda x) = \lambda f(x)$.

The set of morphisms from V to W is denoted $\mathrm{Hom}(V, W)$.

It is clear that if the ring $\mathbb{K}$ is a field, the modules V and W become vector spaces over $\mathbb{K}$ and the definition of a morphism becomes the definition of a linear map.

Definition A.2. Given a module V over a field $\mathbb{K}$ and a subspace W of V we can define the quotient module V/W by using the equivalence relation $x \sim y$ if and only if $x - y \in W$. We get $V/W := \{[x] : x \in V\}$ is a module. Here $[x]$ denote the equivalence class $x/\sim$.

Lemma A.3. Let V, W, I be vector spaces and suppose that I is a subspace of V then:

$$\mathrm{Hom}(V/I, W) \cong \{f \in \mathrm{Hom}(V, W) : I \subseteq \mathrm{Ker}\, f\}.$$

Theorem A.4. Let V be a vector space and $U \subset V$ be a subspace then

$$V \cong (V/U) \oplus U.$$

In particular, if the dimension of V is finite then:

$$\dim V = \dim(V/U) + \dim U.$$

Theorem A.5 (**First isomorphism theorem for modules**)**.** Let $f \in \mathrm{Hom}\,(V, W)$ then $\mathrm{Im}\,(f)$ is a submodule of W and $\mathrm{Ker}(f)$ is a submodule of V and we have:

$$V/\,\mathrm{Ker}(f) \cong \mathrm{Im}(f).$$

If f is surjective, we find that:

$$V/\,\mathrm{Ker}(f) \cong W.$$

Theorem A.6. Let U and V be finite dimensional vector spaces, and $r_1, \ldots, r_n$ elements in U. Let F be the vector space of all linear maps from U to V that satisfy the relation that $f(r_i) = 0$ for $i \in \{1, \ldots, n\}$, then:

$$F \cong \mathrm{Hom}\left(\frac{U}{(r_1, \ldots, r_n)}, V\right) = \left(\frac{U}{(r_1, \ldots, r_n)}\right)^\star.$$

A.2 Formal linear combinations

In section 3 the concept of a formal linear combination is used to create vector spaces of knots and chord diagrams. In this section we will give a short overview of what a formal linear combinations is and how it forms a vector space. We follow [26].

First, let us construct an abelian group from a set.

Definition A.7. Let A be an additive abelian group and let M be a set. Then:

$$A[M] = \{f : A \to M \mid f^{-1}(A \setminus \{0\}) \text{ is finite}\}$$

is called the **A-linearization** of M.

Elements f of $A[M]$ are of the form $f = a_1x_1 + \cdots + a_nx_n$, where $x_1, \ldots, x_n$ are all elements of the set M, and $a_i = f(x)_i$ for $1 \leq i \leq n$ such that $f(x_i) \neq 0$. Elements of $A[M]$ are called *formal linear combinations*. This can be seen as taking the elements of some set as basis vectors, and going through all combinations of these basis vectors using the function f. The addition on $A[M]$ is defined to be same as on the additive abelian group A. This means that $(f + g)(x) = f(x) + g(x)$. But how do we make a vector space out of this? The way to do that is by taking the abelian group to be a field $\mathbb{F}$. In this case $\mathbb{F}[M]$ is a vector space over $\mathbb{F}$, with scalar multiplication $(\lambda f)(x) = \lambda f(x)$ for $\lambda \in \mathbb{F}$.

This is very abstract, so let us try to apply this to the set of chord diagrams.

Example A.8. Take A to be the field $\mathbb{C}$ and take the set of chord diagram $\mathcal{C}$. Then the A-linearization is given by:

$$\mathbb{C}[\mathcal{C}] = \{f : \mathcal{C} \to \mathbb{C} \mid f^{-1}(\mathbb{C} \setminus \{0\}) \text{ is finite}\}.$$

This means we get elements of the form:

$$f = z_1 C_1 + \cdots + z_n C_n$$

Where z_i is an element of $\mathbb{C}$ and C_i is an element of $\mathcal{C}$ for $1 \leq i \leq n$. A single chord diagram as an element of the space $\mathbb{C}[\mathcal{C}]$: C_k, can be obtained by taking $f = \delta_{ik}$ for $1 \leq i \leq n$.

Addition on chord diagrams is defined using the trick above. Let us say we have two chord diagrams C_1 and C_2. Then $C_1 + C_2 = 1 \cdot C_1 + 1 \cdot C_2 + 0 \cdot C_3 + \cdots + 0 \cdot C_n$. Notice that $f^{-1}(\mathbb{C} \setminus 0) = \{C_1 + C_2\}$ has one element, so it is finite. Multiplication, is defined similarly. We can for example say $(a + bi)C_1 = aC_1 + biC_1 = \underbrace{C_1 + \ldots C_1}_{a \text{ times}} + \underbrace{iC_1 + \cdots + iC_1}_{b \text{ times}}$.

The construction is still a bit abstract, but it is clear this forms a vector space.

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