Relativity meets Ultra-Relativity

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Abstract

In this thesis we examine Carroll physics, taking the speed of light to 0 in a limit. The motivation for this is that in recent years, several applications of Carroll physics have come to light, such as in the study of black holes, gravitational waves and holography. We will examine what Carroll geometry is, and how particles and strings travel through a curved Carroll spacetime. In order to do this, we will compare Carroll physics with Galilean physics, which is in some sense dual to Carroll physics. The similarity between the Carroll algebra and the Galilean algebra is especially clear when considering only 2 spacetime dimensions. More generally, we may consider generalised 'Galilean' algebras which are important when describing strings and p-branes with low energies. These are very similar to generalised Carroll algebras. Given a 'Galilean' gravity theory adapted to p-branes, where the corresponding symmetry group is given by an extension of a p-brane Galilei group, we may use this similarity to immediately write down a similarly generalised Carrollian gravity theory. We will investigate this similarity at the level of the classical particle and string σ-models. When interpreting Carroll spacetime as a limit of a Lorentzian spacetime, in some cases the resulting σ-models describe superluminal objects. These cases include low energy p-branes coupling to p-brane Galilean gravity theories. We will however also encounter examples where this is not the case.

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1 Introduction

In this thesis we will be discussing non- and ultra-relativistic limits of general relativity, and in particular limits of particles, strings and p-branes coupling to the Riemannian metric.

A non-relativistic limit of general relativity is called Newton-Cartan geometry. This Newton-Cartan geometry can be used to describe Newtonian gravity in a formalism that is covariant under general coordinate transformations. In this way, Newtonian gravity can be understood in a geometric manner.

In modern theoretical physics, symmetries are often considered to be fundamental to theories. Famously, the standard model of particle physics is a gauge theory of a Lie group, and each particle corresponds to an irreducible representation of this group. A similar statement can be made for general relativity. The theory of general relativity may be obtained via a gauging of the Poincaré group. The non-relativistic limit of the Poincaré group is given by the Galilei group. However, a gauging of the Galilei group does not lead to Newton-Cartan geometry. Instead, Newton-Cartan geometry may be obtained via a gauging of the Bargmann group, an extension of the Galilei group [1].

The non-relativistic limit of general relativity corresponding to Newtonian gravity is particularly interesting when describing the dynamics of particles. However, when considering relativistic models describing the dynamics of extended objects such as strings, this limit leads to results that are in some way trivial, or unsatisfactory. Instead, in order to describe strings moving with low velocities, it is necessary to consider a different limit of general relativity [2]. The Galilei group, and any of its extensions are unrelated to this limit. Instead the group corresponding to this is a 'stringy' version of the Galilei group. We may further generalise this to p-brane Galilei groups, when considering objects extended in p spatial directions.

The opposite of a non-relativistic limit is the less famous ultra-relativistic limit. The ultra-relativistic limit of general relativity is called Carroll-geometry. It similarly describes a generally covariant geometry through which particles may propagate. There is a reason why this limit is not very well known. Single particles in Carroll geometry can not move. This can be understood by noting that in the ultra-relativistic limit we take the speed of light to 0, together with the fact that nothing can move faster than the speed of light. Following this reasoning, information can not move through space in a Carroll geometry.

Carroll geometry can be obtained by gauging the Carroll algebra [3]. This algebra may be obtained via a contraction of the Poincaré algebra, and consists of rotations, boosts and translations, similarly to the Poincaré and Galilei algebras. However, it differs from these two algebras in the way that the boosts work. Poincaré boosts leave only the spacetime distance between two points invariant, Galilean boosts leave time distances invariant, and Carroll boosts leave spatial distances invariant. Carroll geometry is different from Newton-Cartan geometry in that there is no analogue of the Bargmann algebra.

Since Carroll geometry does not allow any movement of particles, it is not straightforward to see why studying the geometry should be physically interesting. Indeed, when it was first discovered by Lévy Leblond, it was remarked that at that moment, the practical use of such a group was quite problematic [4]. However, in recent times it has come up frequently in very interesting physical subjects such as gravitational waves, dynamics of black hole horizons and holography. As such, it is more interesting than previously thought to study Carroll geometry.

The Carroll algebra can be generalised similarly to the Galilei algebra. However, whereas the p-brane Galilean algebra is particularly interesting when studying p-branes, it is not immediately clear that the same is true for a specific generalised Carroll algebra. Indeed, two different ultra-relativistic limits of the relativistic string have similar dynamics [5].

There are multiple ways in which the Carroll and Galilei geometries are similar to each other. In this thesis we will be mainly interested in a similarity that exists at the level of the algebra. Namely, there exists an invertible map between generalised versions of the Carroll and Galilei algebras [6] with interesting properties and consequences. Specifically, the map relates p-brane Galilean algebras with similarly generalised Carroll algebras. Although this map is not an isomorphism (except when considering only 2 spacetime dimensions), we may use this map to relate results obtained for the p-brane Galilei algebras to generalised Carroll algebras. Since these algebras and their extensions can be gauged to describe Galilean and Carroll geometries, we will also be able to use this map to relate Galilean and Carroll geometries. Given the large amount of research that has gone into non-relativistic geometry, it is an interesting question whether we may directly apply the obtained results to the less studied ultra-relativistic limit.

The main question we will be answering in this thesis is

"How may the similarity of generalised Galilean and Carroll algebras be used to obtain new results for ultra-relativistic geometry?"

In order to do this, we will find it convenient to mostly focus on the limits of models describing the movement of (extended) objects in a background of general relativity. We will find that these limits will allow us to deduce a lot about the related geometry. After all, an important part of general relativity is to describe how particles are affected by gravity. As we will see, in this context we may indeed use this similarity to relate models describing Galilean and Carroll particles to each other.

However the physical interpretation of these models can be more challenging. Namely, some of the models obtained by this similarity can be interpreted as describing objects moving faster than light. On the other hand, there are also exist models where this does not occur. Furthermore, the analysis of Carroll gravity considered in this thesis should not be considered complete, since we do not in all cases give the evolution equations of the background fields, or the matter coupling of the background fields to particles.

We will start by discussing the history and the applications of the Carroll group in section 2. After that, we will look at the Carroll algebra and Galilei algebra as limits of the Poincaré algebra in section 3 using the Inönü-Wigner contraction. We will also introduce the corresponding flat spacetimes as limits of Minkowski space. Furthermore, we consider possible extensions of the Galilei algebra and discuss stringy variants of non-relativistic and ultra-relativistic algebras. In section 4 we will discuss how general relativity can be formulated in terms of vielbein fields and the spin connection, and we will see how gauging the Poincaré algebra leads to general relativity. In section 5 we discuss how we can generalise the Inönü Wigner contraction in cases where it does not produce an invariant limit, using the Lie algebra expansion. In section 6 we will discuss non- and ultra-relativistic limits of the Einstein Hilbert action in both a first order and second order formulation, and discuss which limits are useful to understand the backgrounds of particles and strings. In section 7 we will discuss the non-relativistic limits of test particles and strings, using the Lie algebra expansion. In order to do this, we consider a 'stringy' version of the non-relativistic limit. We will furthermore pay special attention to the role played by torsion in this limit. In section 8 we will consider conventional ultra-relativistic limits of the particle and string, and look at the Polyakov and configuration space version of these actions. We compare the analogue of the 'stringy' non-relativistic limit with conventional Carroll limits. In section 9 we will look at similarities between the brane versions of the Carroll and Galilei algebras, and exploit these relations to write down new gravity and p-brane actions. We will comment on the properties of the obtained p-brane actions. In the conclusion 10 we will answer our main question by discussing the relations between the generalised Galilei and Carroll algebras described in the previous sections.

In the appendix we will discuss several technical details related to the limits we take in this thesis. In appendix A we will discuss the symmetry properties of a specific action that has an important role in this thesis. In appendix B we will discuss the occurrence of Lagrange multipliers when taking limits. In appendix C we will discuss the occurrence of extensions using the Lie algebra extension. In appendix D we will discuss two matrix identities that are useful when taking limits.

2 History and applications of the Carroll group

The Carroll group was first described as an ultra-relativistic limit in [4], where it is named after the writer and mathematician Lewis Carroll:

"The behaviour of a possible Universe which would be governed by the group invariance here is reminiscent of that of "Wonderland" [7]. The absence of causation is particularly clear in Alice's adventures as well as the arbitrary value of the time intervals (cf. in particular Chapter 7, "A mad tea-party")."

Another property of the Carroll group is captured in the sequel to "Alice in Wonderland", "Through the looking glass, and what Alice found there" [8]:

"Well, in our country," said Alice, still panting a little, "you'd generally get to somewhere else if you run very fast for a long time, as we've been doing." "A slow sort of country!" said the Queen. "Now, here, you see, it takes all the running you can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that!"

This reflects that particles can not move in a Carrollian space-time. The Carroll group is somehow the opposite of Galilei group, as in Newton-Cartan space-time any two non-simultaneous events can be causally related, time is universal and velocity is unrestricted.
The Carroll group is one of three possible contractions of the Poincaré group, the other being the Galilei group. The contraction to the Galilei group corresponds to taking $c \to \infty$, whereas the contraction to the Carroll group corresponds to taking $c \to 0$. The third possible contraction is the static group, which is a contraction of both the Carroll and Galilei group [9]. We will not discuss the static group further in this text.
Carroll symmetries occur in the strong coupling limit of general relativity, which can be used to calculate the quantum propagation amplitude between two gravitational field configurations [10]. Furthermore Carroll symmetries occur near spacetime singularities [11]. Here, spacetime points

spatially decouple. In particular, these two limits have in common that the spacetime points separate causally, i.e. the light cone collapses to a line. In particular, this means that the classical equations of motion, that determine the evolution of the fields, are ordinary differential equations, depending only on time. However, we do not have to go all the way to the singularity to obtain Carrollian physics. Any black hole horizon can be described as a Carrollian geometry emerging from an ultra-relativistic limit where the near-horizon radial coordinate plays the role of a virtual speed of light [12].
The Carroll group also appears as a subgroup of a Bargmann group in one dimension higher. These groups are the isometries of respectively Carroll space and Bargmann space, which have a specific metric structure. One such a Bargmann space is the plane gravitational wave, in which the Carroll space can be embedded. It follows that the group of isometries of the plane gravitational wave is the Carroll group [13].
The Carroll group also has possible applications to holography. According to [14], Warped Conformal field theories may be the simplest of field theories without Lorentz invariance that can be described holographically. The warped geometries that correspond to these field theories have global symmetries which correspond to Carroll symmetries, with added dilatations.

3 Introduction to the Carroll algebra

Here we will introduce the Carroll algebra as an Inönü-Wigner contraction of the Poincaré algebra. In order to this, we will first introduce the Poincaré group and the Galilei group. We will examine the (flat) spacetimes related to these algebras. We will also examine how the Carroll and Galilei algebra are related to each other via the Bargmann algebra, which gives us one way in which the Carroll and Galilei algebra are dual to each other. We will also consider generalised versions of these algebras, where we do not split the scaled translations in a time and space part, but differently. We will note that these generalised versions of the Carroll and Galilei algebras are similar to each other. This is the relation that we will exploit later in section 9.

3.1 The Poincaré group

The Poincaré group is usually defined as the Lorentz group with added translations. Perhaps the most well known application of the Poincaré group is as the isometry group of Minkowski spacetime. Minkowski spacetime is characterised as a manifold $\mathbb{M}^D$. This manifold is similar to $\mathbb{R}^D$, but has no canonical basis or an origin. We know only the spacetime interval between two points $\Delta s^2(p_1, p_2)$, and a set of invertible parametrizations $\phi : \mathbb{M}^D \to \mathbb{R}^D, p \to \vec{x}$ for which the distance function is given by

$$\Delta s^2(p_1, p_2) = (\vec{x}_1 - \vec{x}_2)^T \eta (\vec{x}_1 - \vec{x}_2), \tag{3.1}$$

where $\eta = diag(-1, 1..., 1)$ is the Minkowski metric. These parametrizations correspond to inertial observers. We will in practice only use these parametrizations. The Poincaré group is the global isometry group of Minkowski space. That is, the group of maps $f : \mathbb{M}^D \to \mathbb{M}^D$, satisfying $\Delta s^2(p_1, p_2) = \Delta s^2(f(p_1), f(p_2))$ is the Poincaré group. This is most easily seen using a parametrization ϕ with origin $\phi(p_0) = 0$. Let us denote $\vec{f}(\vec{x}) = \phi(f(p))$. Then $\vec{f} : \mathbb{R}^D \to \mathbb{R}^D$

satisfies the property

$$(\vec{f}(\vec{x}) - \vec{f}(0))^T \eta(\vec{f}(\vec{x}) - \vec{f}(0)) = \Delta s^2(f(p), f(p_0)) = \Delta s^2(p, p_0) = x^T \eta x. \tag{3.2}$$

Thus, $\vec{f}$ consists of a Lorentz transformation $L\vec{x} = \vec{f}(\vec{x}) - \vec{f}(0)$ plus a translation $\vec{f}(0)$. The generators of the D dimensional Poincaré algebra are translations $P_{\hat{A}}$ and Lorentz transformations $J_{\hat{A}\hat{B}}$, with $\hat{A} = 0, 1, ..., D-1$. The commutation relations are given by

$$\begin{aligned} [P_{\hat{A}}, P_{\hat{B}}] &= 0 \\ [J_{\hat{A}\hat{B}}, P_{\hat{C}}] &= 2\eta_{\hat{C}[\hat{B}} P_{\hat{A}]} \\ [J_{\hat{A}\hat{B}}, J_{\hat{C}\hat{D}}] &= 4\eta_{[\hat{A}[\hat{C}} J_{\hat{D}]\hat{B}]}. \end{aligned} \tag{3.3}$$

We can separate translations into time translations $P_0 = H$ and space translations P_a, $a = 1, ..., D-1$. We can separate the Lorentz transformation into the boosts $J_{0b} = G_b$ and rotations J_{ab}. It will be convenient to consider the algebra rewritten using these generators. The nonzero commutation relations are then given by

$$[G_b, P_c] = \delta_{cb} P_0 \tag{3.4a}$$

$$[J_{ab}, P_c] = 2\delta_{c[b} P_{a]} \tag{3.4b}$$

$$[G_b, H_0] = P_b \tag{3.4c}$$

$$[J_{ab}, J_{cd}] = 4\delta_{[a[c} J_{d]b]} \tag{3.4d}$$

$$[G_b, G_d] = -J_{db} \tag{3.4e}$$

$$[G_b, J_{cd}] = 2\delta_{b[c} G_{d]}. \tag{3.4f}$$

3.2 Inönü-Wigner Contraction

The Inönü-Wigner group contraction is a way to contract a group. The group contraction was first introduced in 1953 as a way to describe how groups can be limits of other groups. In particular, since classical mechanics is a limiting case of relativistic mechanics, the Galilei group should be a limit of the Poincaré group. The Inönü Wigner contraction formalises the sense in which this is true. We will limit our description of the group contraction to the contraction of Lie algebras.

To define an Inönü-Wigner contraction of a Lie algebra $\mathfrak{g}$ with bracket $[,]$, we use a linear mapping $\phi_\lambda : \mathfrak{g} \to \mathfrak{g}$ depending continuously on a dimensionless parameter $\lambda \in [0, 1]$. ϕ_λ is invertible at all λ except at $\lambda = 0$. We define a new bracket $[,]'$ contracted from $[,]$

$$[a, b]' := \lim_{\lambda \to 0} [a, b]'_\lambda := \lim_{\lambda \to 0} \phi_\lambda^{-1} [\phi_\lambda(a), \phi_\lambda(b)], \tag{3.5}$$

given that this limit is well defined for any elements $a, b \in \mathfrak{g}$. This is called an Inönü-Wigner contraction. In particular, we are interested in Inönü-Wigner contractions of the following form: Let $\mathfrak{g} = V_0 \oplus V_1$. We define ϕ_λ by setting $\phi_\lambda(v_0) = v_0$, $\phi_\lambda(v_1) = \lambda v_1$ for any $v_0 \in V_0$, $v_1 \in V_1$.

If $[V_0, V_0] \not\subset V_0$, the limit of $[a, b]'_\lambda$ is divergent and we do not obtain a new algebra. This sets constraints upon which subspaces of $\mathfrak{g}$ we can choose. Given this constraint, the contracted Lie algebra is given by the commutation relations

$$\begin{aligned} [v_0, w_0]' &= [v_0, w_0] \\ [v_0, v_1]' &= [v_0, v_1] \\ [v_1, w_1]' &= 0, \end{aligned} \tag{3.6}$$

for $v_0, w_0 \in V_0$, $v_1, w_1 \in V_1$.

The contracted Lie algebra has an invariant commutative subalgebra, given by V_1. To understand this physically, let us look at the contracted subgroup given by $exp(av_1)$ where a is a finite parameter. In taking the limit, v_1 becomes very small, and thus we effectively restrict ourselves to group elements infinitesimally close to the origin. We replace the finite parameter a with the infinitesimal parameter ϵ satisfying $\epsilon^2 = 0$. It is a simple calculation to show that any two such elements commute. The parameters of the subgroup $exp(av_0)$ remain finite. The commutation of an element with a finite parameter and an infinitesimal parameter remains infinitesimal, and hence the contracted group is invariant.

Since the algebra of the bracket $[,]'_\lambda$ is equivalent to the algebra $[,]$, the Inönü-Wigner contraction can equivalently be viewed as being defined by a continuously changing λ dependent basis on $\mathfrak{g}$. This can be seen as follows. The coefficients of $[,]'_\lambda$ given the basis (g_i) can be calculated in terms of the coefficients of $[,]$. In this basis, the matrix coefficients $A_{ij} = A_{ij}(\lambda)$ are given by $\phi_\lambda(g_i) = A_{ij} g_j$. Then

$$\begin{aligned} [g_i, g_j]'_\lambda &= \phi_\lambda^{-1}[A_{ik} g_k, A_{jm} g_m] \\ &= \phi_\lambda^{-1}(A_{ik} A_{jm} c^n_{km} g_n) \\ &= A_{ik} A_{jm} C^n_{km} A^{-1}_{np} g_p \\ &=: C'^p_{ij}(\lambda) g_p \end{aligned} \tag{3.7}$$

Thus the new coefficients are given by

$$C'^p_{ij}(\lambda) = A_{ik} A_{jm} C^n_{km} A^{-1}_{np}. \tag{3.8}$$

Instead, the same coefficients can also be obtained by using a change of basis. Given the generators g_i, we can define the new generators $g'_i = \phi_\lambda(g_i)$. We then have

$$\begin{aligned} [g'_i, g'_j] &= [A_{ik} g_k, A_{jm} g_m] \\ &= A_{ik} A_{jm} c^n_{km} g_n \\ &= A_{ik} A_{jm} c^n_{km} A^{-1}_{np} g'_p \\ &= C'^p_{ij} g'_p. \end{aligned} \tag{3.9}$$

This shows that the algebra defined by $[.]'_\lambda$ and $[,]$ are equivalent. For any basis of $\mathfrak{g}$, the coefficients of the new algebra $\mathfrak{g}$ are the same as the coefficients obtained by applying a basis transformation to the algebra. It also shows that the bracket $[g_i, g_j]'$ is isomorphic to the bracket defined by $[g'_i, g'_j] = \lim_{\lambda \to 0} C'^k_{ij}(\lambda) g'_k$. The coefficients of the algebra $[,]'$ can be calculated by using $C'^k_{ij} = \lim_{\lambda \to 0} C'^k_{ij}(\lambda)$.

3.3 The Galilei Algebra

Choosing $\mathfrak{g}$ to be the Poincaré algebra and

$$V_0 = \{J_{ab}, H\}, V_1 = \{G_a, P_a\}. \tag{3.10}$$

The corresponding Inönü-Wigner contraction is the Galilei algebra. Thus, the subgroup of the Poincaré algebra generated by the boosts and the spatial translations is contracted to the identity. Physically, this implies that we only consider small velocity differences and small spatial translations. We obtain the following nonzero commutation relations:

$$\begin{aligned}
[G_b, H] &= P_b \\
[J_{ab}, P_c] &= 2\delta_{c[b}P_{a]} \\
[J_{ab}, J_{cd}] &= 4\delta_{[a[c}J_{d]b]} \\
[G_b, J_{cd}] &= 2\delta_{b[c}G_{d]}.
\end{aligned} \tag{3.11}$$

The effect of the Inönü Wigner contraction on the Poincaré algebra (3.4) is to set $[G_b, G_d]$ (3.4e) and $[G_b, H]$ (3.4c) to zero. As the Galilean limit corresponds to taking the speed of light to ∞, we may consider the scaling factor λ to be proportional to the inverse speed of light, $\lambda = \frac{c_0}{c}$. In our discussion, the dimensionfull constant c_0 exists only to make the scaling factor dimensionless, should not occur in any final answer and has no physical interpretation. If we would attempt to make approximations for a small but finite $\frac{c_0}{c}$, c_0 should be considered as the characteristic velocity for which the approximation holds. However, since we will be taking the limit $c \to \infty$, any finite characteristic velocity will be in some sense equivalent. We will avoid explicitly discussing it by expressing anything in units where $c_0 = 1$, so that $\lambda = \frac{1}{c}$. The Galilei algebra is also the algebra of symmetry transformations of the action $S = \int d\tau \frac{1}{2} m\dot{x}^2$, which describes freely moving particles. Related to this is the fact that the Galilei group is part of the global symmetry group of flat Newton-Cartan geometry.
We can see this in the following way. The following maps on $\mathbb{R}^D$ are a faithful representation of the Galilei group.

$$\begin{aligned}
x'^a(t, x) &= \eta^a + R^a_b x^b + \lambda^a t, \\
t'(t, x) &= t + \eta^0.
\end{aligned} \tag{3.12}$$

Here η^a parametrizes the spatial translations, η^0 the timelike translations, λ^a the boosts and R^a_b an orthogonal matrix parametrizing the rotations. In order to obtain a good grasp of the flat Newton-Cartan structure it is helpful to switch to a "local" perspective. First we note that the time difference $\Delta t(p_1, p_2) = t(p_1) - t(p_2)$ is invariant. This implies that the one-form $\tau = dt$ is invariant under the Galilean transformations (3.12). The one-form τ determines surfaces of constant time, $t =$ constant. The squared spatial distance $d^2_S(p_1, p_2) = ||x^a(p_1) - x^a(p_2)||^2$ is preserved under Galilean boosts only if p_1 and p_2 occur at the same time, while all other transformations leave it invariant for any p_1 and p_2. Thus, we have a spatial metric on the surfaces of constant time. We cannot use this spatial metric to define a unique metric on the Galilean spacetime. Instead, the inverse metric may be pushed forward along the embedding to Galilean spacetime. The tensor $\delta^{ab}\frac{\partial}{\partial x^a}\frac{\partial}{\partial x^b}$ is invariant under the Galilean boosts. More specifically, it is invariant under the Lie derivative of the vector field $V_{Boost} = t\ \lambda^a\partial_a$. Thus, the flat Galilean structure is given by a one form $\tau = dt$ and a tensor $\gamma = \delta^{ab}\partial_a\partial_b$ on a D dimensional manifold.

These structure can be seen as the non-relativistic $c \to \infty$ limit of the Minkowski metric $\eta(c) = -c^2dt^2 + \delta_{ab}dx^a dx^b$ and the inverse metric $\eta^{-1}(c) = -c^{-2}(\partial_t)^2 + \delta_{ab}\partial_a\partial_b$. This can be seen as

$$\lim_{c\to\infty} \frac{\eta}{c^2} = -dt^2, \qquad\qquad \lim_{c\to\infty} c^2\eta^{-1} = \delta^{ab}\partial_a\partial_b. \tag{3.13}$$

A key property here is that dt spans the kernel of γ, when γ is viewed as a map from the co-tangent space to the tangent space. Interestingly, these structures are not only invariant under the group action (3.12), but also under time dependent space translations and rotations, i.e. the constants η^a are replaced by time dependent functions $\eta^a(t), R^a_{\ b}(t)$. Instead, the Galilei group is obtained by demanding that straight lines remain straight under the Galilean transformations. In other words, the Galilei group are those transformations that additionally leave the connection $\Gamma^\mu_{\nu\rho} = 0$ invariant (acting via the Lie derivative on the connection).

However, as the real world we experience in our daily lives is approximately a flat Newton-Cartan spacetime, we should expect that at constant time t, there is a notion of spatial distance. On the hyperplanes of constant t, the spatial distance function is invariant. To interpret this in a local sense we introduce the projective inverses of $\delta^{ab}\partial_a\partial_b$, dt^2. These projective inverses are given by a vector field $v = v^\mu\partial_\mu$ and a spatial metric $g = g_{\mu\nu}dx^\mu dx^\nu$ satisfying the boost-independent equations

$$(-\tau^2 + g)^{-1} = -v^2 + \gamma, \qquad v^\mu g_{\mu\nu} = 0, \qquad v^0 > 0. \tag{3.14}$$

The last equation determines the sign of v. The first equation can be rewritten as

$$(-\tau_\mu\tau_\nu + g_{\mu\nu})(-v^\nu v^\rho + \gamma^{\nu\rho}) = \delta^\rho_\mu. \tag{3.15}$$

We cannot use these equations to solve for all components. Leaving v^a independent, we obtain

$$v^0 = 1, \qquad g_{0a} = -v^b\delta_{ba}, \qquad g_{ab} = \delta_{ab}. \tag{3.16}$$

Thus, on any submanifold given by $t = \textit{constant}$, the metric induced by the pullback along the embedding is uniquely determined to be δ_{ab}, giving a notion of spatial distance between events occurring at the same time.

Since v^μ transforms under boosts, there is no unique symmetric affine connection connection that can be introduced on this space. For a discussion on the properties of the possible connections, see [15].

3.4 The Carroll algebra

Another way to contract the Poincaré algebra is to choose $V_0 = \{J_{ab}, P_a\}$, $V_1 = \{G_b, H\}$. This implies that in this limit we only consider small time differences and velocity differences. The Inönü-Wigner contraction then gives the Carroll algebra, with the following nonzero commutation relations:

$$\begin{aligned}
[G_b, P_c] &= \delta_{cb}H \\
[J_{ab}, P_c] &= 2\delta_{c[b}P_{a]} \\
[J_{ab}, J_{cd}] &= 4\delta_{[a[c}J_{d]b]} \\
[G_b, J_{cd}] &= 2\delta_{b[c}G_{d]}.
\end{aligned} \tag{3.17}$$

This algebra is obtained from the Poincaré algebra (3.4) by setting $[G_b, G_d]$ (3.4e) and $[G_b, P_c]$ (3.4a) to zero.
We may immediately notice that commutation of boosts and time translations do not produce a

translation. Since this commutation results in the Galilean algebra in a difference in velocity, we may expect that the boost must leave any velocity invariant. A group action of the Carroll group acting on the coordinate space of spacetime can be parametrized as

$$\begin{aligned} x'^a(t,x) &= x^a + \eta^a + R^a_b x^b, \\ t'(t,x) &= t + \eta^0 + \lambda_a R^a_b x^b. \end{aligned} \tag{3.18}$$

Here η^a parametrizes the spatial translations, η^0 the timelike translations, b^a the boosts, and R^a_b is an orthogonal matrix parametrizing the rotations. We will, similarly to the Poincaré and Galilei case introduce structures on a manifold, and define the Carroll group as the group leaving those structures invariant. To motivate the structures introduced we observe that as $c \to 0$, the Minkowski metric $\eta(c) = -c^2 dt^2 + \delta_{ab} dx^a dx^b$ and its inverse have the following limits:

$$g = \lim_{c\to 0} \eta = \delta_{ab} dx^a dx^b \tag{3.19}$$

$$-v^2 = \lim_{c\to 0} c^2 \eta^{-1} = -\partial_t^2. \tag{3.20}$$

Thus, we define a flat Carroll spacetime as a manifold equipped with the vector field $v = \partial_t$ and degenerate metric $g = \delta_{ab} dx^a dx^b$. The transformations given in (3.18) leave these structures invariant. Additionally, the structures are invariant under space-dependent time translations, i.e. $\eta^0(x^a)$ becomes dependent on the spatial coordinates x^a. The Carroll group is obtained by demanding that straight lines remain straight under the Carroll transformations. In other words, the Carroll group are those transformations that additionally leave the 'flat' connection $\Gamma^\mu_{\nu\rho} = 0$ invariant.

In a Carroll spacetime, the spatial distance between points is boost-invariant and can be calculated with the metric g. Meanwhile, the time difference between two events taking place at different spatial locations is boost-dependent. We may similarly to the Galilean case introduce a boost-dependent metric by introducing a degenerate metric γ and a one-form τ, satisfying

$$(-v^2 + \gamma)^{-1} = -\tau^2 + g \qquad \gamma(\tau) = 0 \qquad \tau_0 > 0, \tag{3.21}$$

where the bivectors v^2, γ are seen as maps from the co-tangent space to the tangent space of $\mathbb{A}^D$. Leaving τ_a undetermined, we may solve for γ and τ_0 as

$$\gamma^{00} = \delta^{ab}\tau_a\tau_b, \qquad \gamma^{0a} = -\delta^{ab}\tau_b, \qquad \gamma^{ab} = \delta^{ab}, \qquad \tau_0 = 1. \tag{3.22}$$

Along lines of constant x^a, we may therefore measure the time in a Carroll invariant way using the pullback of τ_μ along the embedding of the line.

3.5 The Bargmann algebra

Another interesting algebra is the Poincaré algebra extended by a generator M, Poincaré $\otimes\, U(1)$. When we do an Inönü-Wigner contraction with the linear map

$$\begin{aligned} &H \to H + Z, \qquad Z \to -\frac{1}{c^2} Z, \qquad G_b \to \frac{1}{c} G_b \\ &J_{ab} \to J_{ab} \qquad P_a \to \frac{1}{c} P_a, \end{aligned} \tag{3.23}$$

we obtain the Bargmann algebra, which has the same commutation relations as the Galilei algebra, except

$$[P_a, G_b] = \delta_{ab} M. \tag{3.24}$$

Since M commutes with every element of the Bargmann group, i.e. it is in the center of the group, we say that the Bargmann group is a central extension of the Galilei group. Another way to obtain this algebra is from the conserved charges of the classical nonrelativistic point particle action

$$S = \int dt \frac{1}{2} m \dot{x}^2. \tag{3.25}$$

They are given by

$$\begin{aligned}
H &= \frac{1}{2} m \dot{x}^2 \\
P_a &= m \dot{x}_a \\
J_{ab} &= m \dot{x}_{[a} x_{b]} \\
G_b &= m \dot{x}_b t - m x_b
\end{aligned} \tag{3.26}$$

Using Poisson brackets they generate the Bargmann algebra. Specifically, $\{P_a, G_b\}_{PB} = \delta_{ab} m$ gives the central charge.
Related to this is that under Galilean boosts the Lagrangian (3.25) transforms as a total derivative,

$$\delta(v^b G_b) S = \int dt \; m v^a \delta_{ab} \dot{x}^b. \tag{3.27}$$

The Bargmann group was first discussed by Bargmann in [16]. It was encountered as the group under which quantum mechanical states of the non-relativistic Schrödinger equation transform, instead of the Galilei group. It was later recognised that, in order to properly describe the non-relativistic limit of a particle, the Bargmann group should be used instead of the Galilei group.

Similarly to the Galilei group, there is such a thing as a Bargmann spacetime. Bargmann spacetime is a semi-Riemannian manifold, equipped with a null vector field, which generates an isometry of the metric. We call a vector field that generates an isometry a Killing vector field. In other words, we have a non-degenerate metric $g_{\mu\nu}$ together with a vector field ξ^μ satisfying

$$g_{\mu\nu} \xi^\mu \xi^\nu = 0 \qquad\qquad L_\xi g_{\mu\nu} = 0. \tag{3.28}$$

A flat Bargmann space B is given by $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, ..., 1)$ and $\xi = (1, 0, ..., 0, -1)$. We can alternatively write this using light-cone coordinates $\eta_{+-} = 1$, $\eta_{++} = \eta_{--} = 0$, $\eta_{ab} = \delta_{ab}$, $\xi = \partial_-$. The Killing vector field is then given by

$$X = \eta^a \partial_a + \eta^+ \partial_+ + \eta^- \partial_- + \eta^a \partial_a + \lambda^a_{\ b} x^b \partial_a + \lambda^a (x^a \partial_- - x^+ \partial_a). \tag{3.29}$$

In terms of the Minkowski space, the boosts generated by λ^a correspond to Λ^{+a}, generating a Lorentz transformation. The Lorentz transformations generated by Λ^{-a} do not occur due to the invariance requirement of ξ. If we identify points of the Bargmann space related by a η^- translation we obtain a flat Galilean space, where x^+ has the role of time. Using this identification Newton-Cartan geometry, the non-relativistic limit of a semi-Riemannian geometry can be described in a generally covariant manner [17]. Furthermore, the unique connection of a Bargmann space uniquely defines a connection on Newton-Cartan geometry. Additionally, gauging the Bargmann algebra leads to Newtonian gravity, under an additional assumption on the curvature of spacetime [1].

The Bargmann space has another surprising property. Specific null surfaces, in particular $D-1$ dimensional hypersurfaces of dimension D that contain the null Killing vector field ξ, are Carroll manifolds [18]. Thus, Bargmann spaces contain in some sense both Carroll and Galilean manifolds. This also induces a unique connection on the Carroll manifold. This idea has led to some interesting applications of the Carroll algebra. For example, the relation between the BMS algebra and conformal Carroll algebras discussed in [19].

3.6 Newton-Cartan and Carroll algebras for extended objects

The Carroll and Galilei algebra do not exhaust all possible contractions of the Poincaré algebra. When discussing the non-relativistic particle the contraction to the Galilei algebra is used in order to take the limit. However, when discussing limits of extended objects such as strings and p-branes it is, in particular for the non-relativistic limit, much more interesting to use versions of these contractions that are adapted to these extended objects. In the Galilean and Carroll algebra we distinguish between time and space directions. Time is special first of all because timelike vectors have negative length squared. However, when considering particles in particular it is special because particles always travel in a timelike direction. When taking the non-relativistic limit of the particle, the parametric velocity of the worldline in the time direction is assumed to be very large compared to the parametric velocity in the spacelike direction. In other words, the time direction is longitudinal to the worldline. On the other hand, the space-like directions are called transverse. On the other hand, when discussing strings we have 2 spacetime directions in which the string is extended, and $D-2$ transverse directions in which the string is free to move. Therefore, the low velocity limit corresponds to letting the longitudinal directions become very large compared to the transverse directions. In order to distinguish the transverse directions from the longitudinal directions we split the indices labelling spacetime coordinates $\hat{A}=0,....,D-1$ into the longitudinal directions labelled by $A=0,1,...p$ and the transverse directions $a=p+1,...,D-1$. In particular, particles correspond to $p=0$ and strings correspond to $p=1$. In terms of these indices the Poincaré algebra is given by

$$[G_{Ab},P_c]=\delta_{\hat{c}\hat{b}}P_A \tag{3.30a}$$
$$[J_{ab},P_c]=2\delta_{c[b}P_{a]} \tag{3.30b}$$
$$[G_{Ab},P_C]=-\eta_{CA}P_b \tag{3.30c}$$
$$[J_{ab},J_{cd}]=4\delta_{[a[c}J_{d]b]} \tag{3.30d}$$
$$[J_{AB},J_{CD}]=4\eta_{[A[C}J_{D]B]} \tag{3.30e}$$
$$[G_{Ab},G_{Cd}]=\eta_{AC}J_{db}+\delta_{db}J_{AC} \tag{3.30f}$$
$$[G_{Ab},J_{cd}]=2\delta_{[b[c}G_{Ad]} \tag{3.30g}$$

The p-brane Galilean contraction of the Poincaré algebra then corresponds to setting

$$V_0=\{P_A,J_{AB},J_{ab}\}, \qquad\qquad V_1=\{J_{Ab},P_a\}, \tag{3.31}$$

contracting the transverse translations and J_{Ab}. J_{Ab} correspond to both boosts in the transverse directions ($A=0$), and rotations mixing the longitudinal and transverse direction ($A=1,...,p$).

Similarly J_{AB} correspond to both boosts in the longitudinal directions ($A = 0,\ B = 1, ..., p$) and rotations in the longitudinal directions ($A = 1, ..., p,\ B = 1, ..., p$). We will refer to J_{AB}, J_{ab} and J_{Ab} as longitudinal rotations, transverse rotations and boosts respectively.

On the other hand, the Carroll p-brane contraction of the Poincaré algebra corresponds to setting

$$V_0 = \{P_a, J_{AB}, J_{ab}\}, \qquad\qquad V_1 = \{J_{Ab}, P_A\}, \tag{3.32}$$

which corresponds essentially to exchanging $a, b \leftrightarrow A, B$. Throughout the thesis we will not exclusively associate p-brane actions with p-brane Galilei and Carroll algebras. In order to keep the names of the algebra consistent with their applications we will therefore in the remaining part of the thesis refer to these contractions as $p + 1$ Galilei contractions and $p + 1$ Carroll contractions, following [20].

4 General relativity as a gauge theory

In order to take non-relativistic limits, we must first be clear what we are actually taking limits of. Thus, in this section we will introduce general relativity. We will introduce the Einstein-Hilbert action, the frame field, the spin connection, and first and second order formalism. After that, we will rewrite the Einstein Hilbert action in terms of gauge fields. Thereafter we will discuss the actions governing the motion of relativistic particles, strings and branes and their symmetries.

4.1 The Einstein Hilbert action in terms of the vierbein and the spin-connection

The Einstein Hilbert action is given by

$$S_{EH} = G \int \sqrt{-g} R d^D x, \tag{4.1}$$

where G is the gravitational constant, R the Ricci scalar, a contraction of the Riemann curvature tensor

$$R^{\rho}_{\sigma\mu\nu} = 2\partial_{[\mu}\Gamma^{\rho}{}_{\nu]\sigma} + 2\Gamma^{\rho}{}_{\lambda[\mu}\Gamma^{\lambda}{}_{\nu]\sigma}, \tag{4.2}$$

and where $\Gamma^{\rho}_{\mu\nu}$ is the Levi-Civita connection, the torsionless connection for which the covariant derivative of the metric is zero,

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^{\sigma}{}_{\mu\nu} g_{\sigma\rho} - \Gamma^{\sigma}{}_{\mu\rho} g_{\sigma\nu} = 0 \qquad\qquad \Gamma^{\sigma}{}_{[\mu\nu]} = 0. \tag{4.3}$$

The contraction is given by $R = \delta^{\mu}_{\rho} g^{\sigma\nu} R^{\rho}{}_{\sigma\mu\nu}$. The action is invariant under gct's. The equations of motion determine the evolution of the metric.
An equivalent definition of the Riemann curvature tensor is that for any vector field v^ρ it satisfies

$$2\nabla_{[\mu}\nabla_{\nu]} v^\rho = \nabla_\mu \nabla_\nu v^\rho - \nabla_\nu \nabla_\mu v^\rho = R^{\rho}{}_{\sigma\mu\nu} v^\sigma. \tag{4.4}$$

To write the Einstein Hilbert action in terms of gauge fields, we will first rewrite it in terms of the frame field $E_\mu^{\hat{A}}$ and the spin connection $\omega_\mu^{\hat{A}\hat{B}}$. The frame field also goes under different names.

In one spacetime dimension, it is referred to as the einbein, in two spacetime dimensions as the zweibein and in four spacetime dimensions it is referred to as the vierbein. In unspecified spatial dimensions it is often called the vielbein.

The frame field consists of D orthonormal differential forms, one timelike one-form $E_\mu^{\ 0}$ of length -1, and $D-1$ spacelike one-forms $E_\mu^{\ a}$ of length 1. In other words, we have

$$E_\mu^{\ \hat{A}} g^{\mu\nu} E_\nu^{\ \hat{B}} = \eta^{\hat{A}\hat{B}}. \tag{4.5}$$

Since length is invariant under a Lorentz transformation, the frame fields are only defined up to local (spacetime coordinate dependent) Lorentz transformations, with infinitesimal transformations given by

$$\delta E_\mu^{\ \hat{A}} = \Lambda^{\hat{A}}_{\ \hat{B}} E_\mu^{\ \hat{B}}, \tag{4.6}$$

where $\Lambda^{\hat{A}\hat{B}}$ is antisymmetric in $\hat{A}$ and $\hat{B}$, and dependent on spacetime coordinates. It should not be a surprise that using a basis of the tangent space in which the metric is diagonal can simplify matters a lot. In generally it is not possible to choose a coordinate system of spacetime in which the metric is locally diagonal. Therefore we need the frame field.

The inverse vielbein is written as $E^\mu_{\ \hat{A}}$, and similarly consists of a timelike vector field of length -1, and $D-1$ spacelike vector fields of length $+1$, i.e.

$$E^\mu_{\ \hat{A}} g_{\mu\nu} E^\nu_{\ \hat{B}} = \eta_{\hat{A}\hat{B}}. \tag{4.7}$$

The one-forms $E^{\hat{A}} = E_\mu^{\ \hat{A}} dx^\mu$ form an orthonormal basis of the cotangent space at each point. In other words, they form a moving frame of the cotangent space. Similarly, the inverse frame field $E_{\hat{A}} = E^\mu_{\ \hat{A}} \partial_\mu$ form a moving frame of the tangent space. We may use the frame field to express tensors in terms of the corresponding moving frame. For example, for a vector field v^μ we have

$$v^\mu \partial_\mu = v^\mu E_\mu^{\ \hat{A}} E_{\hat{A}} =: v^{\hat{A}} E_{\hat{A}}, \tag{4.8}$$

where $v^{\hat{A}} = v^\mu E_\mu^{\ \hat{A}}$ denote the coefficients of the vector field in the moving frame. We note that these coefficients are scalars when considering general coordinate transformations. On the other hand, they do transform under Lorentz transformations whenever we transform the vielbein field. Thus, they may be called Lorentz vectors. These properties generalise to Lorentz one-forms, and more generally to Lorentz tensors.

We will sometimes refer to the indices $\hat{A}$ as flat indices, and to the indices μ as curved indices. Flat indices always correspond to Lorentz tensors, and curved indices to 'normal' tensors. We will also discuss objects that have both flat and curved indices. These are both Lorentz tensors and 'normal' tensors. For example $E_\mu^{\ \hat{A}}$ is both a one-form and a Lorentz vector.

The Einstein-Hilbert action can simply be rewritten in terms of the frame field by using these expressions for the metric and the inverse metric. This gives us the so called second order formalism of general relativity. We also want to discuss the first order formalism. In order to do so, we will first introduce the spin connection. The spin connection is closely related to the Levi-Civita connection. It is naturally encountered when we discuss the action of the covariant derivative on Lorentz tensors. Since the covariant derivative is an important operation in general relativity, it is useful to know how it acts on Lorentz tensors. To find out how this works, it is sufficient to add one property to the connection. Whereas the Levi-Civita connection is defined using the metric postulate (4.3), demanding the invariance of the metric under the covariant derivative, we define the spin connection $\omega_\mu{}^{\hat{A}\hat{B}}$ by demanding the vielbein postulate

$$D_\mu E_\nu{}^{\hat{A}} = \partial_\mu E_\nu{}^{\hat{A}} - \Gamma^\rho{}_{\mu\nu} E_\rho{}^{\hat{A}} - \omega_\mu{}^{\hat{A}}{}_{\hat{B}} E_\nu{}^{\hat{B}} = 0. \tag{4.9}$$

Here, $\Gamma^\rho{}_{\mu\nu}$ is the usual Levi-Civita connection. The covariant derivative is defined such that the covariant derivative of Lorentz tensors remain tensors. Since it is a derivative, this also implies the invariance of the metric, implying that the usual definition of the Levi-Civita connection is implied by the vielbein postulate. The spin connection is required in order to make this identity locally Lorentz invariant. The spin connection can be uniquely solved from these equations. Since Γ has no torsion, we have $\Gamma^\rho{}_{[\mu\nu]} = 0$. Therefore the unique torsionless spin-connection satisfies

$$2D_{[\mu} E_{\nu]}^{\hat{A}} = 2\partial_{[\mu} E_{\nu]}^{\hat{A}} - 2\omega_{[\mu}{}^{\hat{A}}{}_{\hat{B}} E_{\nu]}^{\hat{B}} = 0. \tag{4.10}$$

From this equation we may solve for the spin connection. It is given by

$$\omega_\mu{}^{\hat{A}\hat{B}} = -2E^{\nu[\hat{A}} \partial_{[\mu} E_{\nu]}^{\hat{B}]} + E^{\nu\hat{A}} E^{\rho\hat{B}} E_{\mu\hat{C}} \partial_{[\nu} E_{\rho]}^{\hat{C}} \tag{4.11}$$

The Levi-Civita connection is not a tensor. Similarly, the spin connection is not a Lorentz tensor. Instead, the transformation rule of the spin connection under an infinitesimal local Lorentz transformation is given by

$$\delta\omega_\mu{}^{\hat{A}\hat{B}} = -\partial_\mu \Lambda^{\hat{A}\hat{B}} + \Lambda^{\hat{A}}{}_{\hat{C}} \omega_\mu{}^{\hat{C}\hat{B}} + \Lambda^{\hat{B}}{}_{\hat{C}} \omega_\mu{}^{\hat{A}\hat{C}} \tag{4.12}$$

The Lorentz curvature can then be defined by

$$2D_{[\mu} D_{\nu]} v^{\hat{A}} = R_{\mu\nu}{}^{\hat{A}}{}_{\hat{B}} v^{\hat{B}}. \tag{4.13}$$

Since the covariant derivative of the vielbein is 0, they commute. In other words, first expressing a vector in the frame field and then taking the covariant derivative, is the same as taking the covariant derivative in a coordinate basis and then expressing it in the frame field. Therefore we have

$$R_{\mu\nu}{}^{\hat{A}}{}_{\hat{B}} v^{\hat{B}} = 2D_{[\mu} D_{\nu]} v^{\hat{A}} = 2E_\rho^{\hat{A}} D_{[\mu} D_{\nu]} v^\rho = 2E_\rho^{\hat{A}} R^\rho{}_{\sigma\mu\nu} v^\sigma = 2E_\rho^{\hat{A}} R^\rho{}_{\sigma\mu\nu} E^\sigma{}_{\hat{B}} v^{\hat{A}} \tag{4.14}$$

From this, we may conclude that $R_{\mu\nu}{}^{\hat{A}}{}_{\hat{B}} = E_\rho^{\hat{A}} E^\sigma{}_{\hat{B}} R^\rho{}_{\sigma\mu\nu}$. The Lorentz curvature and the Riemann curvature tensor are the same tensor expressed in a different basis. The curvature can be expressed in terms of the spin connection (and hence in terms of the frame field) as

$$R_{\mu\nu}{}^{\hat{A}\hat{B}} = 2\partial_{[\mu} \omega_{\nu]}^{\hat{A}\hat{B}} - 2\omega_{[\mu}{}^{\hat{A}}{}_{\hat{C}} \omega_{\nu]}^{\hat{C}\hat{B}}. \tag{4.15}$$

This expression allows us to write the Einstein-Hilbert action in terms of the spin connection and the vierbein field in first order formalism. The Lagrangian is then given by

$$\mathcal{L} = E E^\mu{}_{\hat{A}} E^\nu{}_{\hat{B}} R_{\mu\nu}{}^{\hat{A}\hat{B}}, \tag{4.16}$$

where E is the determinant $E = \epsilon_{\hat{A}\hat{B}\hat{C}\hat{D}}\epsilon^{\mu\nu\rho\sigma}E_\mu{}^{\hat{A}}E_\nu{}^{\hat{B}}E_\rho{}^{\hat{C}}E_\sigma{}^{\hat{D}}$. The Lagrangian can also be written as

$$\mathcal{L} = \epsilon^{\mu\nu\rho\sigma}\epsilon_{\hat{A}\hat{B}\hat{C}\hat{D}}E_\mu{}^{\hat{A}}E_\nu{}^{\hat{B}}R_{\rho\sigma}{}^{\hat{C}\hat{D}}. \tag{4.17}$$

The equations of motion (in a vacuum) are given by

$$R_{\hat{A}\hat{C}}{}^{\hat{B}\hat{C}} - \frac{1}{2}\delta_{\hat{A}}^{\hat{B}}R_{\hat{C}\hat{D}}{}^{\hat{C}\hat{D}} = 0. \tag{4.18}$$

The spin connection can also be considered as independent variables. This is called first order formalism, whereas a dependent spin connection is called second order formalism. In first order formalism, the Einstein-Hilbert action is given by the same expression, except that the spin connection is now independent. From the equation of motion of the spin connection follows the vielbein postulate (4.10), and hence the equations of motion of the frame fields are given by (4.18), as in the second order formalism.

In first order formalism, the Einstein-Hilbert action only contains first order derivatives, acting on $\omega_\mu{}^{\hat{A}\hat{B}}$. On the other hand, in second order formalism the Einstein Hilbert action contains second order derivatives of the frame field (or the metric). Thus the name.

4.2 The Einstein Hilbert action as the action of gauge fields

Now that we have written the Einstein Hilbert action in terms of the vielbein and spin connection, we will relate these fields to the gauge fields of the Poincaré algebra. Let us write down the gauge fields corresponding to $P_{\hat{A}}$ as $E_\mu{}^{\hat{A}}$, and the gauge fields corresponding to $\Lambda^{\hat{A}\hat{B}}$ as $\omega_\mu{}^{\hat{A}\hat{B}}$.

In particular, the transformation of $E_\mu{}^{\hat{A}}$and $\omega_\mu{}^{\hat{A}\hat{B}}$ under $\theta = \Gamma^{\hat{A}}P_{\hat{A}} + \frac{1}{2}\Lambda^{\hat{A}\hat{B}}M_{\hat{A}\hat{B}}$ are given by

$$\begin{aligned}\delta E_\mu{}^{\hat{A}} &= \partial_\mu\Gamma^{\hat{A}} - \omega_\mu{}^{\hat{A}}{}_{\hat{B}}\Gamma^{\hat{B}} + \Lambda^{\hat{A}}{}_{\hat{B}}E_\mu{}^{\hat{B}}\\ \delta\omega_\mu{}^{\hat{A}\hat{B}} &= \partial_\mu\Lambda^{\hat{A}\hat{B}} - 2\omega_\mu{}^{\hat{C}[\hat{A}}\Lambda^{\hat{B}]}{}_{\hat{C}}\end{aligned} \tag{4.19}$$

The fields transform like differential forms under gct's, with parameter Γ^μ

$$\begin{aligned}\delta E_\mu{}^{\hat{A}} &= \Gamma^\nu\partial_\nu E_\mu{}^{\hat{A}} + E_\nu{}^{\hat{A}}\partial_\mu\Gamma^\nu\\ \delta\omega_\mu{}^{\hat{A}\hat{B}} &= \Gamma^\nu\partial_\nu\omega_\mu{}^{\hat{A}\hat{B}} + \omega_\nu{}^{\hat{A}\hat{B}}\partial_\mu\Gamma^\nu.\end{aligned} \tag{4.20}$$

We would like to let the gauge field of translations "become" the frame field, and the gauge field of rotations the spin connection. However, this is not something that directly follows. After all, in GR we do not have a thing such as a local translation, and the vielbein is determined by the frame field. These 'issues' are resolved when we examine the equations of motion of the first order Einstein Hilbert Lagrangian, written in terms of both the Lorentz and translational gauge fields,

$$\begin{aligned}\mathcal{L} &= EE^\mu{}_{\hat{A}}E^\nu{}_{\hat{B}}R_{\mu\nu}{}^{AB},\\ R_{\mu\nu}{}^{\hat{A}\hat{B}} &= 2\partial_{[\mu}\omega_{\nu]}{}^{\hat{A}\hat{B}} - 2\omega_{[\mu}{}^{\hat{A}}{}_{\hat{C}}\omega_{\nu]}{}^{\hat{B}\hat{C}}.\end{aligned} \tag{4.21}$$

The equations of $\omega_\mu{}^{\hat{A}\hat{B}}$ are given by

$$R_{\mu\nu}{}^{\hat{A}} = 2\partial_{[\mu}E_{\nu]}{}^{\hat{A}} - 2\omega_{[\mu}{}^{\hat{A}}{}_{\hat{B}}E_{\nu]}{}^{\hat{B}} = 0. \tag{4.22}$$

From this equation, two results follow. First of all, we can solve this equation to express $\omega_\mu{}^{\hat{A}\hat{B}}$ in terms of the gauge field of local translations (similarly to how we can express the Levi-Civita connection in terms of the metric). By the elimination lemma we may substitute this into the action without changing the equations of motion. This lemma is stated as follows :

Let the action $I[\psi,\phi]$ depend on two sets of variables ψ and ϕ, such that the equation $\frac{\delta I}{\delta\phi} = 0$ can be solved algebraically for the variables ϕ as functions of the variables ψ, i.e. $\phi = \phi(\psi)$. In this case

$$\left.\frac{\delta I}{\delta\phi}\right|_{\phi=\phi(\psi)} \equiv 0. \tag{4.23}$$

The remaining equations of motion for are then equivalent to those obtained by variation of the new action $\hat{I}[\psi] = I[\psi;\phi(\psi)]$, i.e. that obtained by back-substitution.

Secondly, the transformation of the translational gauge field becomes equal to a gct plus a Lorentz transformation. This clarifies the role of the local translations in the second order formalism, namely they correspond to gct's. The correspondence between local translations and gct's is a bit more subtle then it seems, as we will see in the next section.

4.3 The difference between local translation and gct's in the second order formalism

The Einstein-Hilbert action is manifestly invariant under gct's and local Lorentz transformations. However, the action is not invariant under P-transformations

$$\begin{aligned} \delta E_\mu{}^{\hat{A}} &= \partial_\mu\Gamma^{\hat{A}} - \omega_\mu{}^{\hat{A}}{}_{\hat{B}}\Gamma^{\hat{B}} \\ \delta\omega_\mu{}^{\hat{A}\hat{B}} &= 0 \end{aligned} \tag{4.24}$$

The usual transformation of the spin connection under P-transformations is zero. However, we can modify the transformation of the spin connection so that the P-transformation is a symmetry of the action. Furthermore, this modification disappears on the equations of motion, i.e. it is a Zilch symmetry. The difference between a local P-transformations with parameter $\Gamma^{\hat{A}}$ and gct's with parameter $\Gamma^\mu = E^\mu{}_{\hat{A}}\Gamma^{\hat{A}}$ acting on the gauge field of local translations and can be calculated using:

$$\delta(\Gamma^{\hat{A}}P_{\hat{A}})E_\mu{}^{\hat{A}} = \partial_\mu(\Gamma^\lambda)E_\lambda{}^{\hat{A}} + \Gamma^\lambda\partial_\mu(E_\lambda{}^{\hat{A}}) - \omega_\mu{}^{\hat{A}}{}_{\hat{B}}E_\lambda{}^{\hat{B}}\Gamma^\lambda \tag{4.25}$$

$$R_{\mu\nu}{}^{\hat{A}} = 2\partial_{[\mu}E_{\nu]}{}^{\hat{A}} - 2\omega_{[\mu}{}^{\hat{A}}{}_{\hat{B}}E_{\nu]}{}^{\hat{B}} \tag{4.26}$$

$$\delta(\Gamma^\nu\partial_\nu)E_\mu{}^{\hat{A}} = \partial_\mu(\Gamma^\lambda)E_\lambda{}^{\hat{A}} + \Gamma^\lambda\partial_\lambda E_\mu{}^{\hat{A}} \tag{4.27}$$

$$\delta(\Lambda^{\hat{C}\hat{B}}J_{\hat{C}\hat{B}})E_\mu{}^{\hat{A}} = \Lambda^{\hat{A}}{}_{\hat{B}}E_\mu{}^{\hat{B}}. \tag{4.28}$$

It can then be seen that

$$\delta(\Gamma^{\hat{B}}P_{\hat{B}})E_\mu^{\ \hat{A}} = \delta(\Gamma^\nu\partial_\nu)E_\mu^{\ \hat{A}} - \delta(\Gamma^\lambda\omega_\lambda^{\ \hat{C}\hat{B}}J_{\hat{C}\hat{B}})E_\mu^{\ \hat{A}} - \Gamma^\lambda R_{\lambda\mu}^{\ \ \hat{A}}. \tag{4.29}$$

Note that we have

$$0 = \delta(\Gamma^{\hat{C}}P_{\hat{C}})\omega_{\mu\ \hat{B}}^{\ \hat{A}} \neq \delta(\Gamma^\nu\partial_\nu)\omega_{\mu\ \hat{B}}^{\ \hat{A}} - \delta(\Gamma^\lambda\omega_\lambda^{\ \hat{C}\hat{D}}J_{\hat{C}\hat{D}})\omega_{\mu\ \hat{B}}^{\ \hat{A}}. \tag{4.30}$$

Since the Einstein-Hilbert action is invariant under general coordinate transformations and local Lorentz transformations, but not under

$$\delta E_\mu^{\ \hat{A}} = \Gamma^\lambda R_{\lambda\mu}^{\ \ \hat{A}}, \quad \delta\omega_{\mu\ \hat{B}}^{\ \hat{A}} = 0. \tag{4.31}$$

We will redefine $\delta(\Gamma^{\hat{A}}P_{\hat{A}})$ such that $\delta(\Gamma^{\hat{A}}P_{\hat{A}})S = 0$, by adding a term to the above $\delta\omega$ such that the transformation is a symmetry of the EH action.

We can do this in the following way. We consider an ansatz for $\delta\omega$ of the form

$$\delta\omega_\mu^{\ \hat{A}\hat{B}} = aR_\mu^{[\hat{A}}\Gamma^{\hat{B}]} + bE_\mu^{[\hat{A}}R_{\hat{C}}^{\ \hat{B}]}\Gamma^{\hat{C}} + cE_\mu^{[\hat{A}}R\Gamma^{\hat{B}]}, \tag{4.32}$$

where $R_\mu^{\hat{A}} = R_{\mu\hat{B}}^{\ \ \hat{A}\hat{B}}$. Using

$$\begin{aligned} \frac{\partial E}{\partial E_\rho^{\hat{C}}} &= EE_{\hat{C}}^{\ \rho} \\ \frac{\partial E_{\hat{A}}^{\ \mu}}{\partial E_\rho^{\hat{C}}} &= -E_{\hat{A}}^{\ \rho}E_{\hat{C}}^{\ \mu} \end{aligned} \tag{4.33}$$

and the Lagrangian density

$$\mathcal{L} = EE_{\hat{A}}^{\ \mu}E_{\hat{B}}^{\ \nu}R_{\mu\nu}^{\ \ \hat{A}\hat{B}} \tag{4.34}$$

we obtain

$$\frac{\delta\mathcal{L}}{\delta E_\rho^{\hat{C}}} = EE_{\hat{C}}^{\ \rho}E_{\hat{A}}^{\ \mu}E_{\hat{B}}^{\ \nu}R_{\mu\nu}^{\ \ \hat{A}\hat{B}} - 2EE_{\hat{A}}^{\ \rho}E_{\hat{C}}^{\ \mu}E_{\hat{B}}^{\ \nu}R_{\mu\nu}^{\ \ \hat{A}\hat{B}} \tag{4.35}$$

Furthermore

$$\frac{\delta\mathcal{L}}{\delta\omega_\rho^{\ \hat{E}\hat{F}}} = 2E[E_{\hat{A}}^{\ \alpha}E_{\hat{E}}^{\ \rho}E_{\hat{F}}^{\ \sigma} + E_{\hat{A}}^{\ \sigma}E_{\hat{E}}^{\ \alpha}E_{\hat{F}}^{\ \rho} + E_{\hat{A}}^{\ \rho}E_{\hat{E}}^{\ \sigma}E_{\hat{F}}^{\ \alpha}]R_{\sigma\alpha}^{\ \ \hat{A}} \tag{4.36}$$

So that, up to partial integration, we have

$$\begin{aligned} \delta\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta E_\rho^{\ \hat{C}}}\delta E_\rho^{\ \hat{C}} + \frac{\delta\mathcal{L}}{\delta\omega_\mu^{\ \hat{A}\hat{B}}}\delta\omega_\mu^{\ \hat{A}\hat{B}} = E\Gamma^{\hat{F}}(-R_{\hat{F}}R - 2R_{\hat{A}}^{\ \hat{E}}R_{\hat{E}\hat{F}}^{\ \ \hat{A}}) \\ &+ E\Gamma^{\hat{F}}(R_{\hat{F}}R(2a+4c) + R_{\hat{E}}R_{\hat{F}}^{\ \hat{E}}(-2a+4b) + R_{\hat{A}}^{\ \hat{E}}R_{\hat{E}\hat{F}}^{\ \ \hat{A}}(2a) \\ &= E\Gamma^{\hat{F}}((2a+4c-1)R_{\hat{F}}R + (2a-2)R_{\hat{A}}^{\ \hat{E}}R_{\hat{E}\hat{F}}^{\ \ \hat{A}} + R_{\hat{E}}R_{\hat{F}}^{\ \hat{E}}(-2a+4b)) = 0 \end{aligned} \tag{4.37}$$

Thus we obtain $a = 1, b = \frac{1}{2}, c = -\frac{1}{4}$. Modifying the transformation rule of the spin connection with (4.32), we obtain a symmetry of the Einstein-Hilbert Lagrangian.

4.4 Relativistic particles, string and branes

In order to discuss the non- and ultra-relativistic limits of particles, strings and branes, it will be useful to first introduce the actions governing the motion of these objects, and discuss their symmetries and other properties.

To describe a relativistic point particle, we need a spacetime manifold M with a metric $G_{\mu\nu}$. Furthermore we need a worldline I, which is embedded into M via a function $X : I \to M$. We denote the coordinates of the worldline by τ, and the coordinates of the point $X(\tau)$ as $X^\mu(\tau)$. Hence, we differentiate between the coordinates of the embedding map X^μ, and coordinates of the target space x^μ. A relativistic point particle without spin is governed by the action

$$S_{NG} = -m \int d\tau \sqrt{-\dot{X}^2}, \tag{4.38}$$

where $\dot{X}^2 = \dot{X}^\mu \dot{X}^\nu G_{\mu\nu}$. The action can alternatively be written in the Hamiltonian formalism. However, the canonical Hamiltonian, $H = p_\mu \dot{X}^\mu - L = 0$. Instead the Hamiltonian form of the action is

$$S_H = \int d\tau \; p_\mu \dot{X}^\mu - \frac{e}{2}(p^2 + m^2), \tag{4.39}$$

where p_μ are the momenta conjugate to X^μ, and the einbein e is a Lagrange multiplier. Its equation of motion results in the 'on shell' condition $p^2 + m^2 = 0$, which is the relativistic energy-momentum relation. The Hamiltonian is simply given by $H = \frac{e}{2}(p^2 + m^2)$. This is simply a Lagrange multiplier multiplied by the on-shell condition.

We can solve for the momenta of the action using its own equation of motion as

$$p_\mu = \frac{\dot{X}^\nu G_{\nu\mu}}{e} \tag{4.40}$$

to obtain a different form of the action, which we will call 'Polyakov-type'.

$$S_{Pol} = \int d\tau \frac{1}{2} \frac{\dot{X}^2}{e} - \frac{m^2 e}{2}. \tag{4.41}$$

We can than in turn eliminate e once again obtain the action (4.38). e is then given by

$$e = \frac{\sqrt{-\dot{X}^2}}{m}. \tag{4.42}$$

Form this solution it is clear that $-e^2$ should be interpreted as a one-dimensional metric of signature -1 on the worldline. e is precisely the single vielbein field of this 1-dimensional metric, or the einbein. If we substitute the solution (4.42 back into the action (4.41), we once again obtain the 'Nambu-Goto type' action (4.38). We can summarise this in the diagram

$$S_{NG} \xrightarrow{\text{on shell conditions}} S_{phase} \xrightarrow{\text{elimination of momenta}} S_{Pol} \xrightarrow{\text{elimination of worldvolume metric}} S_{NG} \tag{4.43}$$

As we will take limits, these three actions will give different but essentially equivalent perspectives of what is happening. Taking the ultra and non-relativisitic limits of these 3 actions will result in 3 actions that are related in a similar manner, cf. section 8 for the Carroll analogue, as well as the non-relativistic limit [21]. This action has various types of symmetries. The first one to discuss is reparametrization invariance of the worldline, in other words the general coordinate transformation of the worldline. An infinitesimal transformation is given by

$$\delta(\xi)X^\mu = \xi\dot{X}^\mu \qquad \delta(\xi)p_\mu = \xi\dot{p}_\mu \qquad \delta(\xi)e = \frac{d}{d\tau}(\xi e), \tag{4.44}$$

where ξ is a worldline-dependent parameter. In other words, this is a gauge symmetry. Under this transformation, the action (4.38) transforms only up to a boundary term

$$\delta S = -m\int d\tau \frac{d}{d\tau}(\xi\sqrt{-\dot{X}^2}). \tag{4.45}$$

The next type of symmetry are so called σ-model symmetries, corresponding to reparametrizations of the target space, given by

$$\delta X^\mu = \Gamma^\mu, \qquad \delta G_{\mu\nu} = L_\Gamma G_{\mu\nu}. \tag{4.46}$$

These 'symmetries' are in general not actual symmetries, as we transform the background fields. Thus, in general, Noethers' theorem does not apply to them. Only if the vector field is a killing vector, i.e. if it satisfies $L_\Gamma G_{\mu\nu} = 0$, does Γ generate an actual symmetry. In that case the transformation generates an isometry of the metric. In the flat case, where $G_{\mu\nu} = \eta_{\mu\nu}$ is the Minkowski metric, the isometries are given by the Poincaré group. Thus, the symmetry group of the particle in Minkowski spacetime is the Poincaré group, together with worldline reparametrizations.

If we write the action in terms of the vielbein fields, using $G_{\mu\nu} = E_\mu{}^{\hat{A}} E_\nu{}^{\hat{B}} \eta_{\hat{A}\hat{B}}$, we obtain additional σ-model symmetries given by the local Lorentz transformations (4.6). Thus, in the vielbein formalism the σ model symmetries of this action are given by the local Poincaré transformations, consisting of Lorentz transformations and general coordinate transformations.

A relativistic particle with charge q couples to an electromagnetic potential A_μ as

$$S = S_0 + q\int d\tau A_\mu \dot{X}^\mu, \tag{4.47}$$

where S_0 is the free point particle action. Transforming the electromagnetic potential by a total derivative, $\delta(a)A_\mu = \partial_\mu a$, changes the action by a total derivative

$$\delta S = q\int d\tau\ \dot{X}^\mu \partial_\mu a = q\int d\tau\ \dot{a}. \tag{4.48}$$

We will now discuss a string action along similar lines. A relativistic string is described by an embedding from a worldsheet into a semi-Riemannian spacetime manifold with the embedding $X : (\tau, \sigma) \to X(\tau, \sigma)$. Here, τ and σ are coordinates of the worldsheet. We similarly denote the coordinates of the embedding function by X^μ. The pullback of the metric to the worldsheet along the embedding induces a metric on the worldsheet, given by

$$G_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} \tag{4.49}$$

Here, α takes the values $\sigma^0 = \tau$, $\sigma^1 = \sigma$. The string action then takes the form

$$S = -T\int d\tau d\sigma \sqrt{-\det G_{\alpha\beta}}, \tag{4.50}$$

where T is the tension of the string. $d\tau d\sigma\sqrt{-\det G_{\alpha\beta}}$ is the area of a small piece of the worldsheet, and the action is proportional to the area of the worldsheet. This is the Nambu-Goto action, describing the string using only the embedding coordinates, i.e. in configuration space. Alternatively, using the notation

$$\partial_\tau X^\mu = \dot{X}^\mu, \qquad \partial_\sigma X^\mu = X'^\mu \tag{4.51}$$

for the worldsheet derivatives, we may write

$$-T\int d\tau d\sigma\sqrt{(\dot{X}^\mu G_{\mu\nu}X'^\nu)^2 - (\dot{X}^\mu G_{\mu\nu}\dot{X}^\nu)(X'^\rho G_{\rho\sigma}X'^\sigma)} \tag{4.52}$$

Similarly to the particle case, the canonical Hamiltonian, $H = p_\mu \dot{X}^\mu - L$ equals zero. Instead, the Hamiltonian form of the action is given by

$$S = \int d\tau\ P_\mu \dot{X}^\mu - \frac{e}{2}(P^2 + (TX')^2) - u(P_\mu X'^\mu). \tag{4.53}$$

Here, P_μ are the momentum densities, and e and u are Lagrange multipliers. The on shell conditions of the string are given by

$$P^2 + (TX')^2 = 0 \qquad P_\mu X'^\mu = 0. \tag{4.54}$$

These are the stringy analogue of the on-shell condition of the particle, $p^2 + m^2 = 0$. We can eliminate the momenta of the action by using their own equations of motion, obtaining

$$P_\mu = \frac{1}{e}G_{\mu\nu}\dot{X}^\nu - uG_{\mu\nu}X'^\nu. \tag{4.55}$$

We then obtain the Polyakov action, given by

$$S = \int d\tau d\sigma\ (\frac{1}{2e}\dot{X}^\mu - uX'^\mu)^2 - e(TX')^2. \tag{4.56}$$

More conventionally, the Polyakov action is written as

$$S = -\frac{T}{2}\int d\tau d\sigma\sqrt{-h}h^{\alpha\beta}G_{\alpha\beta}, \tag{4.57}$$

where $h_{\alpha\beta}$ is an independent worldsheet metric and $h = \det h_{\alpha\beta}$. We will call $h_{\alpha\beta}$ the conformal worldsheet metric. The reason for this is that the action (4.57) is invariant under a conformal transformation

$$h'_{\alpha\beta} = \Omega^2(\tau,\sigma)h_{\alpha\beta}. \tag{4.58}$$

This implies that $h_{\alpha\beta}$ has effectively only 2 degrees of freedom, instead of the 3 degrees of freedom of a metric on a two dimensional space. The relation between (4.57) and (4.56) is given by the coordinate transformation

$$[h_{\alpha\beta}] = \begin{bmatrix} u^2 - T^2e^2 & u \\ u & 1 \end{bmatrix}. \tag{4.59}$$

Thus, u and e correspond to the 2 independent components of the conformal metric. Eliminating $h_{\alpha\beta}$ using its equations of motion returns the Nambu-Goto action (4.50). For $h_{\alpha\beta}$ we obtain

$$h_{\alpha\beta} = \Omega^2(\tau,\sigma)G_{\alpha\beta}. \tag{4.60}$$

For later use we will also give the solutions of u and e in terms of X^μ.

$$u = \frac{G_{\tau\sigma}}{G_{\sigma\sigma}} \qquad\qquad e = \frac{\sqrt{-\det G_{\alpha\beta}}}{G_{\sigma\sigma}}. \tag{4.61}$$

Thus, the actions are related to each other as described in the diagram (4.43). As we take the ultra or non-relativistic limit of these three actions, we similarly to the particle case obtain three actions which are related in the same manner, cf. section 8 for the ultra-relativistic limit, and [22] for the non-relativistic limit.

We may quantize the string using lightcone quantization. Incredibly, requiring the Beta-functions of the string to be 0, implies that $G_{\mu\nu}$ satisfies the equations of motion of the Einstein-Hilbert action! Thus, the conformal invariance of the string implies general relativity.

The symmetries of the Nambu-Goto action are similar to the symmetries of the particle action (4.38). The action is invariant under general coordinate transformations of the worldsheet, up to a boundary term. Furthermore, the action has σ-model symmetries corresponding to general coordinate transformations of the target space, which reduce to symmetries whenever the metric is invariant under the corresponding transformation. Additionally, writing the metric as a product of the vielbein fields induces additional σ-model symmetries corresponding to local Lorentz transformations.

The string naturally couples to a generalisation of the electromagnetic potential. Instead of coupling to a one-form $A_\mu dx^\mu$, as the one-dimensional particle does, the string naturally couples to a two form $B_{\mu\nu}\, dx^\mu \wedge dx^\nu$. This two form is called a Kalb Ramond field. We may add the Kalb-Ramond field to any of the previous actions. The 3 actions are changed by adding an extra term

$$S = S_0 + \frac{1}{2}\int d\tau d\sigma\, \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}. \tag{4.62}$$

However, the addition of the Kalb-Ramond field changes the Hamiltonian action in such a way that it is no longer of Hamiltonian form. Indeed, we may calculate the momenta

$$\begin{aligned} P_\mu &= \frac{\partial}{\partial \dot{X}}(S_{Ham}(x,p') + \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}) \\ &= P'_\mu + B_{\mu\sigma}, \end{aligned} \tag{4.63}$$

where P' denotes the old momenta, and P_μ the new momenta. In order to put it back into Hamiltonian form we use this equation to define the new momenta. The action then becomes

$$S_{Ham}(X,P') + B_{\tau\sigma} = S_{Ham}(X, P_\mu - B_{\mu\sigma}). \tag{4.64}$$

In words, the Kalb-Ramond induces a shift of $B_{\mu\sigma}$ in the momenta, similarly to the particle case.

We will not extensively discuss p-brane actions. Nonetheless we will state the Nambu-Goto type p-brane action we will refer to. It is given by

$$S_{NG} = T\int d^{p+1}\sigma\sqrt{-\det G_{\alpha\beta}} + \epsilon^{\alpha_0\ldots\alpha_p}B_{\alpha_0,\ldots\alpha_p}, \tag{4.65}$$

where σ^α are the coordinate with $\alpha = 0, 1, ..., p$, and $B_{\mu_0,\ldots,\mu_p}$ is a generalisation of the electric field. For $B_{\alpha_0,\ldots B_{\alpha_p}} = 0$, this action corresponds to the volume of the worldsheet as measured by the target space metric.

5 The Lie algebra expansion

Algebras related to the Poincaré algebra can be found using the Lie algebra expansion. In this subsection we will discuss the Lie algebra expansion in an abstract manner, and discuss how we can use it to find limits of gauge theories. After that we will discuss how the Lie algebra expansion applies to the Poincaré algebra. We will also examine why this expansion appears of fundamental importance to the non-relativistic limit, and how it may be of importance for the ultra-relativistic limit. However for the most conventional ultra-relativistic limit it does not produce results different from the Inönü-Wigner contraction.

The Lie algebra expansion is a generalisation of the Inönü-Wigner contraction. In certain cases the Inönü-Wigner contraction can be used to take the limit of an action describing a gravity theory, a string or more generally a p-brane. However, in some cases the limit turns out to be somehow trivial. Trivial in this case usually means an action consisting only of a total derivative term. In such cases, rather then a contraction, an expansion can be used to produce a non-trivial limit, as was first done for a flat limit of the AdS superstring [23]. This Lie algebra expansion was also used to obtain actions for non-relativistic gravity theories [24], [25] and particle, string and p-brane actions, as well as equations of motion for the corresponding gravity theory [26], although the strategy had to be adapted here beyond simply applying the Lie algebra expansion to the action.

As the name suggests, rather than scaling certain generators of the algebra in terms of a scaling parameter, we expand objects using a scaling parameter. Specifically, the Lie algebra expansion is done by expanding the Maurer-Cartan one-forms of the algebra. Whereas an infinite expansion does not change the physics of the action, truncating this infinite expansion does. The Inönü-Wigner contractions corresponds to a specific truncation of the Lie algebra expansion. This means that the algebras generated by the Lie algebra expansion contain in general more generators than the Lie algebra from which they are generated.

We will first look at the definition of a Maurer Cartan form and examine one of its key properties, which we shall use to expand the Lie algebra. The Maurer-Cartan one-form ω is a specific Lie algebra valued one-form on the Lie group $\mathcal{G}$. Letting $h, g \in G$ we define the group action of left multiplication by $L_h : G \to G$, $g \to hg$. This allows us to relate a tangent vector at h, $v \in T_g G$ to its push forward $L_{h*}(v) \in T_{hg}(G)$, a tangent vector at hg. In particular, we may choose $h = g^{-1}$, so that $L_{g^{-1}*}(v) \in T_e(G) = \mathfrak{g}$, where we identify the tangent space at the identity e with the Lie algebra $\mathfrak{g}$. This pushforward to the identity is the Maurer-Cartan one-form, $\omega : T_g G \to \mathfrak{g}, v \to L_{g^{-1}*}(v)$.

The Maurer-Cartan one-form can be shown to satisfy a zero curvature condition

$$d\omega(v_1, v_2) + \frac{1}{2}[\omega(v_1), \omega(v_2)] = 0, \tag{5.1}$$

where the bracket simply denotes the Lie bracket. This equation is referred to as the Maurer-Cartan equation. In particular, we may use a basis of the Lie algebra a_i to write, $\omega = \omega^i a_i$

$$d\omega^k(v_1, v_2)a_k + \frac{1}{2}\omega^i(v_1)\omega^j(v_2)C^k_{ij}a_k = 0 \tag{5.2}$$

Thus, the Maurer-Cartan one-form determines the Lie bracket via the Maurer-Cartan equation.

The Lie algebra expansion is done by expanding the Maurer-Cartan one-form in a power series using a parameter λ. Each coefficient of a power of λ becomes a new one-form. This expansion will then be substituted into the Maurer-Cartan equation, and we demand that the equation holds for every power of λ. This results into a infinite set of Maurer-Cartan equations, that determine the commutation relations of an infinite dimensional Lie algebra. The Cartan one-forms of this algebra can in a precise sense be evaluated to give the original algebra. Truncating this algebra corresponds to approximating the original Maurer-Cartan one-form for small values of λ, up to a certain power of λ.

We will consider the Lie algebra expansion only in a specific case. A more general treatment can be found in [27]. More precisely, the Lie algebra expansion is applied to a Lie algebra $\mathfrak{g}$ by writing it as a direct sum $\mathfrak{g} = V_0 \oplus V_1$ with the following properties

$$[V_0, V_0] \subset V_0 \tag{5.3a}$$
$$[V_0, V_1] \subset V_1 \tag{5.3b}$$
$$[V_1, V_1] \subset V_0. \tag{5.3c}$$

We choose a basis (a_{i_0}) of V_0 labelled by indices i_0 and a basis (a_{i_1}) of V_1 labelled by indices i_1, so that we may write the Maurer-Cartan form of $\mathfrak{g}$ as $\omega = \omega^{i_0} a_{i_0} + \omega^{i_1} a_{i_1}$. We formally expand the one-forms ω^{i_0}, ω^{i_1} as

$$\begin{aligned} \omega^{i_0} a_{i_0} &= \overset{(0)}{\omega}{}^{i_0} a_{i_0} + \lambda^2 \overset{(2)}{\omega}{}^{i_0} a_{i_0} + ... \\ \omega^{i_1} a_{i_1} &= \lambda \overset{(1)}{\omega}{}^{i_1} a_{i_1} + \lambda^3 \overset{(3)}{\omega}{}^{i_1} a_{i_1} + ... \end{aligned} \tag{5.4}$$

We may write this expansion a bit more abbreviated as follows:

$$\omega(\lambda) = \sum_{p=0}^{\infty} \sum_{i_{\bar{p}}} \lambda^p \overset{(p)}{\omega}{}^{i_{\bar{p}}} a_{i_{\bar{p}}}, \tag{5.5}$$

where we denote $\bar{p} = p \mod 2$.

We will use this expansion to define a new Lie algebra. This Lie algebra is constructed by stating its Maurer-Cartan equations. We derive these equations by plugging the expansion of the Maurer-Cartan form into the Maurer-Cartan equation (5.2) and assume that it holds for any $\lambda \leq 1$. This implies that it holds for every term in the λ-expansion of the equation. At the order λ^p we obtain

$$d\overset{(p)}{\omega}{}^k(v_1, v_2) + \frac{1}{2} \sum_{m+n=p} \overset{(m)}{\omega}{}^{i_{\bar{m}}}(v_1) \overset{(n)}{\omega}{}^{j_{\bar{n}}}(v_2) C^{k_{\bar{p}}}_{i_{\bar{m}} j_{\bar{n}}} = 0, \tag{5.6}$$

where we purposefully leave out the basis of the Lie algebra a_p, setting the components to 0, We may now observe that these are the Maurer-Cartan equations of an infinite dimensional Lie algebra. Let us denote the basis of this Lie algebra by $\overset{(p)}{a}_{k_{\bar{p}}}$, so that rewriting to this new Lie algebra corresponds to replacing $\lambda^p a_{k_{\bar{p}}}$ by $\overset{(p)}{a}_{k_{\bar{p}}}$. We may explicitly write the Maurer-Cartan equations of the new Lie algebra as

$$d\overset{(p)}{\omega}{}^k(v_1, v_2) \overset{(p)}{a}_{k_{\bar{p}}} + \frac{1}{2} \sum_{m+n=p} \overset{(m)}{\omega}{}^{i_{\bar{m}}}(v_1) \overset{(n)}{\omega}{}^{j_{\bar{n}}}(v_2) C^{k_{\bar{p}}}_{i_{\bar{m}} j_{\bar{n}}} \overset{(p)}{a}_{k_{\bar{p}}} = 0, \tag{5.7}$$

Note that setting $\overset{(p)}{a}_{k_{\bar{p}}} = \lambda^p a_{k_{\bar{p}}}$ returns the usual Maurer-Cartan equations of the fully relativistic Lie algebra.

The vector space of the new Lie algebra's simply consists of copies of the original Lie algebra labelled by the corresponding power of λ. More explicitly let $\overset{(2k)}{V_0}$ and $\overset{(2k+1)}{V_1}$ be copies of the vector space V_0, V_1. $\overset{(2k)}{a}_{i_0}$, ranging over i_0 denotes the basis of $\overset{(2k)}{V_0}$, and similarly $\overset{(2k+1)}{a}_{i_1}$ denotes the basis of $\overset{(2k+1)}{V_1}$.

The vector space of this new infinite dimensional Lie algebra thus consists of the direct sum

$$\bigoplus_{k=0}^{\infty} \overset{(2k)}{V_0} \oplus \overset{(2k+1)}{V_1}. \tag{5.8}$$

The commutators corresponding to the Maurer-Cartan equation (5.7) are given by

$$[\overset{(n)}{a}_{i_{\bar{n}}}, \overset{(m)}{a}_{j_{\bar{m}}}] = C_{i_{\bar{m}}\ j_{\bar{n}}}^{k_{\overline{m+n}}} \overset{(m+n)}{a}_{k_{\overline{m+n}}}. \tag{5.9}$$

We note here that the commutation of a n-order term and an m-order term only gives $m+n$ order terms. In general, the coefficients of the infinite dimensional algebra can be denoted by $C_{i_{\bar{m}}m\ j_{\bar{n}}n}^{k_{\bar{p}}p}$. They are given by

$$C_{i_{\bar{m}}m\ j_{\bar{n}}n}^{k_{\bar{p}}p} = C_{i_{\bar{m}}j_{\bar{n}}}^{k_{\bar{p}}} \delta_{m+n}^{p}. \tag{5.10}$$

The Lie algebra expansion is then obtained by truncating the power series at some power k. That is, all terms of order $p > k$ are ignored. This is consistent since the sub-algebra consisting of terms of order $p > k$ is an ideal. If we truncate the series at $k = 1$ then we obtain an Inönü-Wigner contraction with some extra conditions given by (5.3b) and (5.3c). In particular, the Inönü-Wigner contraction from the Poincaré to the Galilei and Carroll algebra satisfy these conditions. Thus, the Lie algebra expansion can be considered as a generalisation of the Inönü-Wigner contraction.

5.1 Gauge fields and the Lie algebra expansion

As we expand the Maurer-Cartan forms of the Lie algebra, so too can we expand the gauge fields. We will first introduce gauge fields and their curvatures, and then explain how we can expand these in order to obtain limits of actions.
Given a Lie algebra $\mathfrak{g}$ and a manifold M, we will consider gauge fields as Lie-algebra valued one forms on the manifold. Thus, a gauge field ω is given by

$$\omega = \omega_{\mu}^{\ i} a_i dx^{\mu}. \tag{5.11}$$

Gauge fields transform under local actions of the group. We will only consider infinitesimal actions of the group generated by the Lie algebra. Under a local transformation $\theta = \theta^i a_i$of the group a gauge field transforms as

$$\delta(\theta)\omega = d\theta - \theta^j \omega^k C_{jk}^{\ i} a_i \tag{5.12}$$

$$\tag{5.13}$$

The curvature of the gauge field is then given by

$$R = d\omega + \frac{1}{2}\omega^i \wedge \omega^j C_{ij}^k a_k \tag{5.14}$$

The curvatures transform covariantly under gauge transformations. That is, no derivatives occur in the transformation rules assigned to them. The curvatures transform as

$$\delta(R) = R^i \theta^j C_{ij}{}^k a_k \tag{5.15}$$

The gauge fields corresponding to the Lie algebras can be expanded in a very similar way. Given a gauge field

$$\omega_\mu = \omega_\mu{}^i a_i = \omega_\mu{}^{i_0} a_{i_0} + \omega_\mu{}^{i_1} a_{i_1}, \tag{5.16}$$

where we once again consider the Lie algebra to be a direct sum of two vector spaces $\mathfrak{g} = V_0 \oplus V_1$. We will consider this to be a Lie algebra valued one-form on a manifold. We may be given a way to expand it in a way similar to the Lie algebra expansion

$$\omega_\mu(\lambda) = \lambda^p \overset{(p)}{\omega}{}_\mu{}^{i_0} a_{i_0} + \lambda^p \overset{(p)}{\omega}{}_\mu{}^{i_1} a_{i_1}. \tag{5.17}$$

It is these expansions that motivate the Lie algebra expansion in this thesis. We may replace, similarly to the Maurer-Cartan forms the $\lambda^p a_{i_{\bar{p}}}$ by $\overset{(p)}{a}_{i_{\bar{p}}}$. Thus we obtain a gauge field of the infinite dimensional Lie algebra.

$$\omega_\mu = \overset{(p)}{\omega}{}_\mu{}^{i_0} a_{i_0} + \lambda^p \overset{(p)}{\omega}{}_\mu{}^{i_1} a_{i_1}. \tag{5.18}$$

The expansion of the covariant curvatures of the gauge field is given by

$$R = R^{i_0} a_{i_0} + R^{i_1} a_{i_1} = \lambda^p \overset{(p)}{R}{}^{i_{\bar{p}}} a_{i_{\bar{p}}}, \tag{5.19}$$

where $\overset{(p)}{R} a_{i_{\bar{p}}}$ are the curvatures of the infinite dimensional Lie algebra given by

$$\overset{(p)}{R}{}^{k_{\bar{p}}} = d\omega^k + \sum_{m+n=p} \frac{1}{2} C_{i_{\bar{m}} j_{\bar{n}}}^{k_{\bar{p}}} \overset{(m)}{\omega}{}^{i_{\bar{m}}} \wedge \overset{(n)}{\omega}{}^{j_{\bar{n}}}. \tag{5.20}$$

Thus, the curvatures transform covariantly under transformations generated by the infinite dimensional Lie algebra. When we cutoff the expansion at order k, the expression for the curvatures up to $\overset{(k)}{R}$ remain unchanged, as higher order terms do not occur in the curvatures.

The infinite dimensional algebra can be extended with a dilatation in the following way. We define the commutator of the dilatation D, with a homogeneous element $\overset{(p)}{a}$ of the infinitely expanded algebra with grade p, as

$$[\overset{(p)}{a}, D] = p\overset{(p)}{a}. \tag{5.21}$$

This also applies to truncated versions of the algebra. We will denote the parameter of the dilatation by Λ_D. The interpretation of this parameter acting on the gauge fields of the infinite dimensional

Lie algebra is as a small change in the parameter λ. This can be seen be using the evaluation $\overset{(p)}{a}_{i_{\bar{p}}} = \lambda^p a_{i_{\bar{p}}}$. We then have

$$\begin{aligned}
\delta(\Lambda_D D)(\omega_\mu) &= \Lambda_D \overset{(p)}{\omega}{}_\mu{}^{i_{\bar{p}}} [\overset{(p)}{a}_{i_{\bar{p}}}, D] \\
&= p\Lambda_D \overset{(p)}{\omega}{}_\mu{}^{i_{\bar{p}}} \overset{(p)}{a}_{i_{\bar{p}}} \\
&= p\lambda^p \Lambda_D \overset{(p)}{\omega}{}_\mu{}^{i_{\bar{p}}} a_{i_{\bar{p}}} \\
&= \Lambda_D \lambda \frac{\partial}{\partial\lambda} \omega_\mu(\lambda) \\
&= \delta\lambda \frac{\partial}{\partial\lambda} \omega_\mu(\lambda),
\end{aligned} \tag{5.22}$$

where in the last line we identified $\Lambda_D = \frac{\delta\lambda}{\lambda}$. Thus, a constant dilatation can be seen as a small change in the parameter λ. On the other hand, a local dilatation could be seen as promoting the parameter λ from a constant to a function. These dilatations will later be interpreted as conformal symmetries which occur in non-relativistic holography [28] and the BMS algebra [19].

5.2 Invariant actions and the Lie algebra expansion

To examine how the Lie algebra expansion can be used to obtain non-relativistic actions for particle, string, p-brane and gravity actions, we first examine how the Lie algebra expansion can be used to write down new actions, obtained from previous ones. We will assume that we have a Lie algebra $\mathfrak{g}$ with generators g_i. Furthermore, we have a (spacetime) manifold M, on which we have gauge fields $\omega_\mu^i g_i$. We furthermore assume that we are given an action S, depending on the gauge fields. There are 2 separate ways on how this action may depend on the gauge fields. First of all, the gauge fields may be the dynamic variables of the action. Secondly, the gauge fields may occur as background fields of the action, in which case they have already been determined and play the role of parameters. In this second case, transformations of the gauge fields that leave the actions invariant are not true symmetries, as we transform the background fields of the actions. Rather, they are known as σ-model symmetries. In any case, we will assume the action to be invariant under gauge transformations

$$\delta(\theta^i g_i)\hat{S}(\hat{\omega}^i g_i) = 0. \tag{5.23}$$

Here, the θ^i are functions of spacetime. We may then be given a way to expand the gauge fields with a parameter λ, so that we obtain a new action

$$S(\omega) = S(\omega(\lambda)) = \lambda^k S^{(k)} + \lambda^{k+2} S^{(k+2)} + ... \tag{5.24}$$

Every action $S^{(n)}$ will be invariant under the expanded transformations $\delta(\overset{(n)}{\theta^i}\overset{(n)}{g_i})$ [24]. Thus, we have obtained new actions which are invariant under the gauge transformations. After having obtained an interest in one of the actions in this expansion, we may note that only gauge fields of order p or lower occur in the action. We may then truncate the Lie algebra expansion at order p, without changing the action.

5.3 The Lie algebra expansion of the Poincaré algebra

We will now apply the Lie algebra expansion method to the Poincaré algebra. We will slightly adapt the Lie algebra expansion method in order to obtain an expansion that is useful when taking limits. We will denote the gauge fields ω_μ of the Poincaré algebra as

$$\omega_\mu = \hat{E}_\mu{}^{\hat{A}} \hat{P}_{\hat{A}} + \hat{\Omega}_\mu{}^{\hat{A}\hat{B}} \hat{J}_{\hat{A}\hat{B}}. \tag{5.25}$$

For the non-relativistic Lie algebra expansion the Lie algebra is written as a direct sum with the same subspaces as in the Galilean Inönü-Wigner contraction in (3.10.) We then expand the gauge fields corresponding to the translations in the following way:

$$\begin{aligned}
\hat{E}_\mu{}^0 &= c^z \sum_{p\geq 0} \frac{1}{c^{2p}} \overset{(2p)}{E}{}_\mu{}^0 \\
&= c^z(\tau_\mu + \frac{1}{c^2} m_\mu + ...) \\
\hat{E}_\mu{}^a(c) &= c^z \sum_{p\geq 0} \frac{1}{c^{2p+1}} \overset{(2p)}{E}{}_\mu{}^0 \\
&= c^{z-1}(E_\mu{}^a + ...).
\end{aligned} \tag{5.26}$$

The parameter c^z does not change any of the arguments made in section 5.1 about the Lie algebra expansion, as the spacetime translations form an invariant commutative subgroup of the Poincaré group. This implies that it also does not change anything for the Inönü-Wigner contraction. When we use the Lie algebra expansion to take the limit of an action, cf. equation (5.24), the parameter z can be selected to be valued such that we obtain a finite action. The gauge fields corresponding to the boosts and translations are expanded as prescribed by the Lie algebra expansion

$$\hat{\Omega}_\mu{}^{ab} = \sum_{p\geq 0} \frac{1}{c^{2p}} \overset{(2p)}{\Omega}{}_\mu{}^{ab} \tag{5.27}$$

$$\hat{\Omega}_\mu{}^{ab} = \sum_{p\geq 0} \frac{1}{c^{2p+1}} \overset{(2p+1)}{\Omega}{}_\mu{}^{ab}. \tag{5.28}$$

For the Carroll algebra we obtain a similar expansion. The gauge field corresponding to translations is expanded as

$$\hat{E}_\mu{}^0 = c^{z+1} \sum_{p\geq 0} c^{2p} \overset{(2p)}{E}{}_\mu{}^0 \tag{5.29a}$$

$$= c^{z+1}(\tau_\mu + ...) \tag{5.29b}$$

$$\hat{E}_\mu{}^a(c) = c^z \sum_{p\geq 0} c^{2p} \overset{(2p)}{E}{}_\mu{}^0 \tag{5.29c}$$

$$= c^z(E_\mu{}^a + c^2 n_\mu{}^a + ...). \tag{5.29d}$$

In principle, when taking either the ultra or non-relativistic limit of a curved spacetime, we should always include an infinite expansion of the gauge fields. However, in the limit only some of the highest order terms are relevant. As we will see, in the Carroll limit only the highest order term is relevant. This means that in this specific case the Lie algebra expansion does not give additional results when compared with the Inönü-Wigner contraction. Of course, if we want a better approximation when $c << 1$, we should consider the additional terms in the expansion. For example, post-Newtonian physics can be discussed using the non-relativistic expansion [29].

6 Non- and ultra-relativistic gravity actions

In this section we will discuss the limits of the Einstein Hilbert action using the Lie algebra expansion. Specifically, we will consider the limit in an empty space, without matter. Our motivation for doing this is to understand Carrollian geometry as a limit of general relativity. We will restrict our discussion to the conventional Carroll limit, i.e. A only takes the value 0.

6.1 The Lie algebra expansion: Gravity actions

The Lie algebra expansion can be used to find gravity actions, derived from the Einstein-Hilbert action, that are invariant under the algebras generated by the Lie algebra expansion. We will first consider the Lie algebra expansion with subspaces generated by $V_0 = \{J_{ab}, H\}, V_1 = \{G_{0b}, P_a\}$, in other words corresponding to the Galilei algebra. We will denote the gauge field of $G_{0b} = J_{0b}$ by $\omega_\mu^{\ 0b}$, and the gauge field of H by $\tau_\mu = E_\mu^{\ 0}$. The expansions of the gauge fields of the Lie algebra expansion are then given by

$$\begin{aligned}
\hat{\tau}_\mu &= \overset{(0)}{\tau}_\mu + c^{-2}\overset{(2)}{\tau}_\mu + ... \\
\hat{E}_\mu^{\ a} &= c^{-1}\overset{(1)}{E}{}_\mu^{\ a} + c^{-3}\overset{(3)}{E}{}_\mu^{\ a} + ... \\
\omega_\mu^{\ 0a} &= c^{-1}\overset{(1)}{\omega}{}_\mu^{\ 0a} + c^{-3}\overset{(3)}{\omega}{}_\mu^{\ 0a} + ... \\
\omega_\mu^{\ ab} &= \overset{(0)}{\omega}{}_\mu^{\ ab} + c^{-2}\overset{(2)}{\omega}{}_\mu^{\ ab} + ...
\end{aligned} \tag{6.1}$$

Assuming the gauge field $E_\mu^{\ \hat{A}}$ to be invertible, we also obtain an expansion for the inverse gauge fields

$$\begin{aligned}
\hat{\tau}^\mu &= \overset{(0)}{\tau}{}^\mu + c^{-2}\overset{(2)}{\tau}{}^\mu + ... \\
\hat{E}^\mu_{\ a} &= c^1\overset{(-1)}{E}{}^{\ \mu}_{\ a} + c^{-1}\overset{(1)}{e}{}^\mu_{\ a} + ...
\end{aligned} \tag{6.2}$$

Where each coefficient in the expansion is defined such that the matrix $(\tau^\mu, E^\mu_{\ a}) = (E^\mu_{\ \hat{A}})$ is the inverse of $(\tau_\mu, E_\mu^{\ a}) = (E_\mu^{\ \hat{A}})$ (for any value of c). The starting point here is the first order Einstein-Hilbert action (4.1), which is invariant under gct's and local Lorentz transformations. We can then substitute the expansion of these gauge fields into the action

$$
\begin{aligned}
\mathcal{S} &= \int d^4x \; \epsilon^{\mu\nu\rho\sigma} \epsilon_{\hat{A}\hat{B}\hat{C}\hat{D}} E_\mu^{\ \hat{A}} E_\nu^{\ \hat{B}} R_{\rho\sigma}^{\ \ \hat{C}\hat{D}} \\
&= \int d^4x \; \frac{2}{c} \epsilon^{\mu\nu\rho\sigma} \epsilon_{a0cd} \overset{(1)}{E}{}_\mu^{\ a} \overset{(0)}{\tau}_\nu \overset{(0)}{R}{}_{\mu\nu}^{\ \ cd} + ... \\
&= \frac{1}{c} S^{(1)} + \frac{1}{c^3} S^{(3)} + ...
\end{aligned} \tag{6.3}
$$

Every action in this expansion will be invariant under the full infinite dimensional algebra generated by the Lie algebra expansion. The first action corresponds to Galilei gravity. Note that Galilei gravity is entirely different from Newtonian gravity. For one thing, it is invariant under the Galilei algebra without central charge, whereas Newtonian gravity requires a central charge [1]. Alternatively this expansion can be written as

$$
\begin{aligned}
\mathcal{S} &= G \int d^4x \; E E^\mu_{\hat{A}} E^\nu_{\hat{B}} R_{\mu\nu}^{\ \ \hat{A}\hat{B}} \\
&= G \int d^4x \; c^{-1} \overset{(3)}{E} \overset{(-1)}{E}{}^{\ \mu}_{a} \overset{(-1)}{E}{}^{\ \nu}_{b} \overset{(0)}{R}{}_{\mu\nu}^{\ \ ab} + c^{-3} ...
\end{aligned} \tag{6.4}
$$

We can do a similar thing for the Carroll gravity. The Lie algebra expansion then works with subspaces with generators $V_0 = \{J_{ab}, P_a\}, V_1 = \{G_b, H\}$. The expansion of the gauge fields is then given by

$$
\begin{aligned}
\tau_\mu &= c \overset{(1)}{\tau}_\mu + c^3 \overset{(3)}{\tau}_\mu + ... \\
E_\mu^{\ a} &= \overset{(0)}{E}{}_\mu^{\ a} + c^2 \overset{(2)}{E}{}_\mu^{\ a} + ... \\
\omega_\mu^{\ a} &= c^1 \overset{(1)}{\omega}{}_\mu^{\ a} + c^3 \overset{(3)}{\omega}{}_\mu^{\ a} + ... \\
\omega_\mu^{ab} &= \overset{(0)}{\omega}{}_\mu^{\ ab} + c^2 \overset{(2)}{\omega}{}_\mu^{\ ab} + ...
\end{aligned} \tag{6.5}
$$

Assuming the gauge field $E_\mu^{\hat{A}}$ to be invertible, we also obtain an expansion for the inverse gauge fields

$$
\begin{aligned}
\tau^\mu &= c^{-1} \overset{(-1)}{\tau}{}^\mu + c^1 \overset{(1)}{\tau}{}^\mu + ... \\
E^\mu_a &= \overset{(0)}{E}{}^{\ \mu}_{a} + c^2 \overset{(2)}{E}{}^{\ \mu}_{a} + ...
\end{aligned} \tag{6.6}
$$

This produces the action

$$
\begin{aligned}
\mathcal{S} &= G \int d^4x \; \hat{E} \hat{E}^\mu_{\hat{A}} \hat{E}^\nu_{\hat{B}} \hat{R}_{\mu\nu}^{\ \ \hat{A}\hat{B}} \\
&= G \int d^4x \; c \overset{(1)}{E} \overset{(0)}{E}{}^{\ \mu}_{a} \overset{(0)}{E}{}^{\ \nu}_{b} \overset{(0)}{R}{}_{\mu\nu}^{\ \ ab} + c \overset{(1)}{E} \overset{(0)}{E}{}^{\ \mu}_{a} \overset{(-1)}{\tau}{}^\nu \overset{(1)}{R}{}_{\mu\nu}^{\ \ a} + O(c^3)
\end{aligned} \tag{6.7}
$$

The $O(c)$ terms give the action for Carroll gravity.

6.2 Carroll gravity as described by the first order action

The equations of motion of the second order action, obtained by varying the action 6.7 are given by

$$\begin{aligned} R_{\mu\nu}(H) &= 0 \\ R_{\mu\nu}{}^{a}(P) &= 0 \\ R_{\mu a}{}^{a}(G) &= 0 \\ R_{0b}{}^{ab}(J) &= 0 \\ R_{ac}{}^{bc}(J) + R_{0a}{}^{b}(G) &= 0 \end{aligned} \tag{6.8}$$

This completely determines the geometry of Carroll gravity. The first two equations determine that the geometry is torsionless, and can be used to solve for the spin connection in terms of the vielbein fields.

$$\begin{aligned} \omega_{\mu}^{\ a} &= \tau_{\mu}\tau^{\nu}E^{\rho a}\partial_{[\nu}\tau_{\rho]} + E^{\nu a}\partial_{[\mu}\tau_{\nu]} + S^{ab}E_{\mu}^{b} \\ \omega_{\mu}^{\ ab} &= -2E^{\rho[a}\partial_{[\mu}E_{\rho]}^{b]} + E_{\mu c}E^{\rho a}E^{\nu b}\partial_{[\rho}E_{\nu]}^{c} \end{aligned} \tag{6.9}$$

Note that we have an undetermined symmetric component S^{ab}, which shows that these equations do not uniquely determine ω_{μ}^{a}. We can then define the degenerate metric

$$h_{\mu\nu} = E_{\mu}^{a}E_{\nu a}. \tag{6.10}$$

The extrinsic curvature is then given by the Lie derivative of the metric:

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_{\tau}(h_{\mu\nu}) = \tau^{\rho}\partial_{\rho}h_{\mu\nu} + h_{\rho\nu}\partial_{\mu}\tau^{\rho} + h_{\mu\rho}\partial_{\nu}\tau^{\rho}. \tag{6.11}$$

From the 2nd equation of motion it follows that that $K_{ab} = 0$. We can furthermore define the affine connection $\Gamma_{\mu\nu}^{\rho}$ by setting

$$\begin{aligned} D_{\mu}\tau_{\nu} &= \partial_{\mu}\tau_{\nu} - \Gamma_{\mu\nu}^{\rho}\tau_{\rho} - \omega_{\mu\ a}^{\ 0}E_{\nu}^{\ a} = 0, \\ D_{\mu}E_{\nu}^{a} &= \partial_{\mu}E_{\nu}^{\ a} - \Gamma_{\mu\nu}^{\rho}E_{\rho}^{\ a} - \omega_{\mu\ b}^{\ a}E_{\nu}^{\ b} = 0. \end{aligned} \tag{6.12}$$

We have now introduced the main objects describing the geometry of Carroll gravity. Thereby we note that from the first two equations of motion follows that the geometry is torsionless. This means that $\Gamma_{[\mu\nu]}^{\rho} = 0$.

We can pass from this first order approach to a second order approach by substituting the solution of the first two equations back into the action, thus obtaining a second order action. We can separate the $S^{(ab)}$ term in the second order action to obtain

$$S = \int E\left(2\tau^{\mu}E_{a}^{\nu}R(G)_{\mu\nu}^{a}\big|_{S^{ab}=0} + E_{a}^{\mu}E_{b}^{\nu}R(J)_{\mu\nu}^{ab} + 2K_{ab}S^{ab} - 2\delta^{ab}\delta_{cd}K_{ab}S^{cd}\right). \tag{6.13}$$

The equation $K_{ab} = 0$ is only fulfilled once we vary with respect to S^{ab}, which acts as a Lagrange multiplier.

6.3 Obtaining the Galilean and Carroll gravity from the second order EH action

We have already performed an Inönü Wigner contraction of the Poincaré algebra to either the Galilei algebra or the Carroll algebra (which can be viewed as the first step of a Lie algebra expansion). We obtain a gravity theory with a corresponding action in both cases. We can solve for the spin connections, which are however only determined up to certain classes, where two spin connections correspond to the same class if they differ by a tensor with a specific structure. Similarly, the corresponding affine connection is also dependent on boosts.In the second order action of these theories these undetermined tensors act as Lagrange multipliers. Variation of the action with respect to these tensors provide a constraint on the geometry. On the other hand we can first write down the relativistic spin connection for general relativity as a function of E^a_μ, τ_μ and their inverses, in second order formulation. We now expand E and τ. In this way we obtain an expansion of the spin connections and the Einstein Hilbert action. By rewriting the expanded Einstein Hilbert action, introducing an extra field in the process, we can obtain the Carroll gravity action as a limit of this action. Also, given the constraints on the geometry for Carroll and Galilei gravity, we check whether these expressions for the spin connection correspond to the spin connection in the respective gravity theory. For Galilei gravity we have not yet obtained such a result, but a similar statement should hold.

We expand the vielbeins corresponding to the Inönü-Wigner contraction

$$\hat{E}_\mu^{\ a} = c^{k-1} E_\mu^{\ a} \qquad\qquad \hat{\tau}_\mu = c^k \tau_\mu, \tag{6.14}$$

We note that the contraction corresponds to $c \to 0$ in the Carroll case. In the following discussion we will not label the vielbein fields with their corresponding power of c, since no confusion is possible. When solving for the vielbein fields using equation (4.22), we note that the equation is linear in $\hat{E}_\mu^{\hat{A}}$. Therefore the overall factor c^k in (6.14) can be divided out of the equation. As a consequence, the factor c^k only affects the Einstein-Hilbert action by an overall factor $c^{(D-2)k}$. We will now explicitly do this expansion, and show that the $c \to 0$ limit of the action produces the action of second order Carrol gravity (6.13).

6.3.1 The expansion of the action

We will first write down an explicit c-dependent form of the action. In order to do this, we will first discuss the boost and rotational spin connection, and the curvatures. Then we will substitute these into the action. The c-dependent spin connections are given by

$$\begin{aligned}
\omega_\mu^{\ ab} &= -c^2 E^{\nu a} E^{\rho b} \tau_\mu \partial_{[\nu} \tau_{\rho]} \\
&\quad - 2E^{\nu[a} \partial_{[\mu} E_{\nu]}^{\ b]} + E^{\nu a} E^{\rho b} E_{\mu c} \partial_{[\nu} E_{\rho]}^c \\
&= \overset{(-2)}{\omega}{}_\mu^{\ ab} c^2 + \overset{(0)}{\omega}{}_\mu^{\ ab}
\end{aligned} \tag{6.15}$$

$$\begin{aligned}\omega_\mu{}^{0b} &= c(\tau^\nu E^{\rho b}\tau_\mu\partial_{[\nu}\tau_{\rho]} + E^{\nu b}\partial_{[\mu}\tau_{\nu]}) \\ &+ \frac{1}{c}(dE^{(a\,b)}_{\;\;0}E_{\mu a}) \\ &= c\overset{(-1)}{\omega_\mu}{}^{0b} + c^{-1}\overset{(1)}{\omega}{}_\mu^{0b}\end{aligned} \tag{6.16}$$

One thing we may note is that from there exist terms $\overset{(-2)}{\omega_\mu}{}^{ab}$ and $\overset{(-1)}{\omega_\mu}{}^{0b}$, which do not appear in the Galilean Lie algebra expansion. However, these terms are all proportional to the torsion constraint $\partial_{[\mu}\tau_{\nu]} = 0$. Only if this constraint holds, do the expansions of the spin connections (6.15) and (6.16) match the 'non-relativistic' Lie algebra expansion of the Poincaré gauge fields (6.1). Similarly, in the Carroll Lie algebra expansion we do not have a term $\overset{(1)}{\omega}{}_\mu^{0b}$. This term however is proportional to the torsionlessness constraint

$$K_{ab} = dE_0^{(ab)} = 0 \tag{6.17}$$

Another thing to note is that the full Lie algebra expansion of $E_\mu^{\;a}$ and τ_μ, with higher (Carroll)/lower (Galilei) order c terms, leads to different spin connections and hence to different limits. We will not extend our discussion to the full expansion. From this expansion of the spin connection follows the following expansion of the curvatures:

$$\begin{aligned}R^{ab}(J) =& c^4[-\overset{(-2)}{\omega}{}^{ac}\wedge\overset{(-2)}{\omega}{}^{cb}] \\ &+c^2[d\overset{(-2)}{\omega}{}^{ab} - \overset{(-2)}{\omega}{}^{ac}\wedge\overset{(0)}{\omega}{}^{cb} - \overset{(0)}{\omega}{}^{ac}\wedge\overset{(-2)}{\omega}{}^{cb} - \overset{(-1)}{\omega}{}^{0a}\wedge\overset{(-1)}{\omega}{}^{0b}] \\ &+c^0\;[d\overset{(0)}{\omega}{}^{ab} - \overset{(0)}{\omega}{}^{ac}\wedge\overset{(0)}{\omega}{}^{cb} - 2\overset{(1)}{\omega}{}^{[a}\wedge\overset{(-1)}{\omega}{}^{b]}] \\ &+c^{-2}\;[\overset{(1)}{\omega}{}^{0a}\wedge\overset{(1)}{\omega}{}^{0b}] \\ =& c^4\overset{(-4)}{R}{}^{ab}(J) + c^2\overset{(-2)}{R}{}^{ab}(J) + c^0\overset{(0)}{R}{}^{ab}(J) + c^{-2}\overset{(2)}{R}{}^{ab}(J)\end{aligned} \tag{6.18}$$

$$\begin{aligned}R^{0b}(J) =& c^3[-\overset{(-2)}{\omega}{}^{bc}\wedge\overset{(-1)}{\omega}{}^{0c}] \\ &+c^1[d\overset{(-1)}{\omega}{}^{0b} - \overset{(-2)}{\omega}{}^{bc}\wedge\overset{(1)}{\omega}{}^{0c} - \overset{(0)}{\omega}{}^{bc}\wedge\overset{(-1)}{\omega}{}^{0c}] \\ &+c^{-1}\;[d\overset{(1)}{\omega}{}^{0b} - \overset{(0)}{\omega}{}^{bc}\wedge\overset{(1)}{\omega}{}^{0c}] \\ =& c^3\overset{(-3)}{R}{}^{0b}(G) + c^{+1}\overset{(-1)}{R}{}^{0b}(G) + c^{-1}\overset{(1)}{R}{}^{0b}(G)\end{aligned} \tag{6.19}$$

We can then expand the action as follows:

$$\begin{aligned}\mathcal{S} = G\int c^{(k-1)(D-2)+1}e\big(&+c^0[2\tau^\mu E_b^\nu R_{\mu\nu}^{(1)b} + E_a^\mu E_b^\nu R_{\mu\nu}^{(2)ab}] \\ &+c^2[2\tau^\mu E_b^\nu R_{\mu\nu}^{(-1)b} + E_a^\mu E_b^\nu R_{\mu\nu}^{(0)ab}] \\ &+c^4[2\tau^\mu E_b^\nu R_{\mu\nu}^{(-3)b} + E_a^\mu E_b^\nu R_{\mu\nu}^{(-2)ab}]\big)\end{aligned} \tag{6.20}$$

We have $R_{ab}^{(-4)ab} = 0$, and hence it does not occur in the action

6.3.2 The Carroll limit of the second order Hilbert action

In this section we take the $c \to 0$ limit of the Einstein-Hilbert action. The conclusion is that we obtain the same limit as obtained for the first order action [30].

In Carroll gravity the spin connection $\omega_\mu^{\ ab}$ is uniquely determined, but $\omega_\mu^{\ a}$ is not. It is only determined up to $S^{ab}E_\mu^{\ b}$, where S^{ab} is a symmetric tensor [30]. $\omega_\mu^{(1)} = K^{ab}E_\mu^{\ b}$ can be written in this way, using the symmetric tensor $K^{ab} = 2E^{\rho(a}\tau^\nu\partial_{[\nu}E_{\rho]}^{\ b)}$.This relation can be exploited to write the action of second order Carroll gravity as a limit of the expanded Einstein Hilbert action. Upon identifying the expanded gauge fields with the gauge fields of Carrol gravity , we obtain

$$\begin{aligned}\omega^{(0)ab} &= \omega_C^{(0)ab} \\ \omega^b &= \lambda^{-1}\omega^{(-1)b} + \lambda\omega^{(1)b} = \lambda^{-1}\omega_C^b|_{S^{ab}=\lambda^2K^{ab}}\end{aligned} \tag{6.21}$$

Where spin connections labelled with C denote the Carroll spin connections in terms of the vierbein fields. We would like to take the limit $\lambda \to \infty$ of the action, however the highest order term is not at all similar to the second order Carroll gravity action. There are however strong similarities between the second highest order term and Carroll gravity. We use this as a motivation to drop all lower order terms and continue only with the two highest order terms. The lower order terms will tend to 0 as $\lambda \to \infty$.

$$\begin{aligned}\mathcal{S} = \int E\big(&+ \lambda^2[2\tau^\mu E_b^\nu R_{\mu\nu}^{(1)b} + E_a^\mu E_b^\nu R_{\mu\nu}^{(2)ab}] \\ &+ [2\tau^\mu E_b^\nu R_{\mu\nu}^{(-1)b} + E_a^\mu E_b^\nu R_{\mu\nu}^{(0)ab}]\big)\end{aligned} \tag{6.22}$$

By expressing the spin connections in terms of the vielbein fields and the curvature K^{ab} we find that

$$\begin{aligned}\mathcal{S} &= \int E\big(2\tau^\mu E_b^\nu R_{C\mu\nu}^b|_{S^{ab}=c^{-2}K^{ab}} + E_a^\mu E_b^\nu R_{C\mu\nu}^{ab} + c^{-2}E_a^\mu E_b^\nu R_{\mu\nu}^{(2)ab}\big) \\ &= \int E\big(2\tau^\mu E_b^\nu R_{C\mu\nu}^b|_{S^{ab}=0} + E_a^\mu E_b^\nu R_{C\mu\nu}^{ab} + c^{-2}(K^{ab}K^{ab} - K^{aa}K^{bb})\big),\end{aligned} \tag{6.23}$$

where $R_{C\mu\nu}^{\ \ b}, R_{C\mu\nu}^{\ \ ab}$ denote the curvature tensors of boosts and rotations, where the spin connections are evaluated as in 2nd order Carroll gravity. The first two terms correspond directly to terms in the Carroll action, and we will denote them with $\mathcal{L}_0$. The last term is a divergent quadratic form, and has to be rewritten before we can take a finite limit. The action

$$\mathcal{S}' = \int \mathcal{L}_0 + 2K^{ab}S^{ab} - 2K^{aa}S^{bb} - c^2(S^{ab}S^{ab} - S^{aa}S^{bb}) \tag{6.24}$$

has the same equations of motion, while it does not have any divergent terms. As we take the limit $c \to 0$ we obtain the second order action of Carroll gravity (6.13). We conclude that this action is simply the $c \to 0$ limit of the second order Einstein Hilbert action.

An interesting question is whether the same trick can be done for Galilei gravity.

7 Nonrelativistic particles and strings

In previous sections we have discussed the Lie algebra expansion as a method of taking limits of actions, the particle actions in general relativity and the Galilei algebra. In this section we will

use these tools to describe the non-relativistic particle and string actions. We apply these tools to these cases first of all because we already now much about the non-relativistic limit, and we can do a sanity check of our methods by applying it to a particle travelling around a Schwarzschild black hole. More importantly however, the non-relativistic is physically very interesting, and there are many articles on the subject and applications. Therefore, taking into account the similarity between the generalised (p-brane) Galilei and Carroll algebras, we may use this to learn interesting things about Carroll geometry.
It will be our point of view that in curved space, it does not matter which particle or string action we use to obtain the limit, whether we use the configuration space (i.e. Nambu-Goto type), Hamiltonian, or the Polyakov action. As we will see, all limits of these actions can be related to each other by either introducing new coordinates or eliminating coordinates. This only becomes unclear if we fix a certain gauge. In the current section we will use Polyakov type actions.
We show that the limit of the relativistic particle in curved space has σ-model symmetries which satisfy the Bargmann algebra without local translations. As discussed previously, the local translations correspond to general coordinate transformations. That remains true in the limit if we keep the curvature of the translations equal to 0. We will then discuss applications of the non-relativistic string and take the non-relativistic limit of the relativistic string. We will explain that the appropriate non-relativistic limit requires a different c-dependence of the vielbein fields, as mentioned in section 3.6.
In this section we will focus our discussion on the dynamics of the particles and strings, and keep the comments about the background fields to a minimum. In particular we will not discuss the dynamics of the background fields. However, when taking the limit the torsion plays an important role. This is because in the torsionlessness condition puts certain conditions on the vielbein fields. We will first deal with the torsionless case, and then investigate for which types of torsion the limit we take is legitimate.
While we focus in this section on Newton-Cartan geometry as a limit of semi-Riemannian geometry, there are physical applications for Newton-Cartan geometry with any torsion, such as Horava-Lifshitz gravity [31]. We are only interested in distinguishing the types of Newton-Cartan geometry which can be interpreted as a limit, and which types can not.

7.1 The non-relativistic particle in a torsionless background

In this section we will take the non-relativisitic limit of the 'Polyakov'-type relativistic particle action 7.1, interacting with an 'electric field' A_μ, given by

$$S = -\int d\tau \frac{1}{2}(-\frac{\hat{G}}{\hat{e}} + m^2\hat{e}) + \hat{A}_\mu \dot{x}^\mu, \tag{7.1}$$

where we use the notation $\hat{G} = \dot{X}^\mu \dot{X}^\nu \hat{G}_{\mu\nu}$, and m denotes the mass of the particle. The configuration space version of the action (4.38) can be obtained by eliminating e using its equation of motion. We obtain

$$S = -m\int d\tau \sqrt{-\hat{G}} \tag{7.2}$$

is obtained from the Polyakov action by solving the equation of motion of $\hat{e}$, obtaining $\hat{e} = \frac{\sqrt{-\hat{G}}}{m}$. The non-relativistic limit of this action is taken by first scaling the background fields in a 'non-relativistic particle expansion'

$$\begin{aligned}\hat{\tau}_\mu &= c^2\tau_\mu + m_\mu + O(c^{-2})\\ \hat{E}^a_\mu &= c^1 E_\mu^{\ a} + O(c^{-1})\end{aligned} \tag{7.3}$$

We then expand the einbein as $\hat{e} = c^2 e + O(c^0)$, as the expansion of its solution would suggest. We then choose a particular dependence for the electric field, $\hat{A}_\mu = -c^2\tau_\mu + A_\mu$, which is chosen such that, if we solve for e, it precisely cancels the leading order term of the action, which is given by

$$S_{leading} = -m\int c^2\dot{x}^\mu\tau_\mu. \tag{7.4}$$

We also note that, if and only if $2\partial_{[\mu}\tau_{\nu]} = 0$, the divergent term of the field $\hat{A}_\mu$ is locally a total derivative. We will call a field τ_μ satisfying this condition torsionless, because it is implied by taking the limit of the vielbein postulate with zero torsion (4.10). In particular, using the Lie algebra expansion we may expand the relativistic curvature of time translations as

$$\begin{aligned}\lim_{c\to\infty}\frac{1}{c^2}R_{\mu\nu}^{\ \ 0}(\hat{P}) &= \lim_{c\to\infty}R_{\mu\nu}^{\ \ 0}(H) + \frac{1}{c^2}R_{\mu\nu}^{\ \ 0}(Z)\\ &= R_{\mu\nu}^{\ \ 0}(H)\\ &= 2\partial_{[\mu}\tau_{\nu]} = 0.\end{aligned} \tag{7.5}$$

Thus, torsionlessness of the relativistic spin connection implies torsionlessness in this limit. It is interesting to note that, unlike in the relativistic case, the torsionlessness condition in the non-relativistic limit gives rise to a constraint on the geometry τ_μ. We noted in section 3.3 that a Galilean manifold consists of surfaces of constant time. We can interpret this in two different ways. Firstly, we may assume that there exists a clock function τ, satisfying $\tau_\mu = \partial_\mu\tau$. This functions serves as a global clock on which every observer agrees. If we only assume this function to exist locally, this is equivalent to the torsionlessness constraint (7.5). On the other hand, we may only assume the less stringent condition of existence of a foliation of spacetime consisting of spacelike hypersurfaces, orthogonal to τ_μ. By this we mean that any tangent vector contained in the hypersurface has (timelike) length 0. In other words, the subbundle of the tangent space spanned by E^μ_a is integrable. The existence condition of such a foliation is given by the Frobenius theorem. This tells us that such a foliation exists if and only if E^μ_a equipped with the Lie bracket spans a Lie algebra, i.e.

$$2E^\mu_{[a}\partial_\mu E^\nu_{\ b]} = C_{ab}^{\ \ c}E^\nu_{\ c}, \tag{7.6}$$

where we antisymmetrize a and b. Equivalently, it exists if and only if

$$\tau_{[\mu}\partial_\nu\tau_{\rho]} = 0, \tag{7.7}$$

where we antisymmetrize μ, ν and ρ. In words, this condition tells us that the exterior derivative of τ is proportional to τ. A geometry satisfying this condition has twistless torsion.

In this derivation we will continue with a torsionless geometry. Furthermore, we will set the field $A_\mu = 0$, since we will not use it in our further analysis.

In the Polyakov formalism a divergent leading order term remains however, since we do not solve for e. The action is given by

$$S = \int d\tau\frac{1}{2e}(c^2(\tau_\tau^2 - 2\tau_\tau me + m^2e^2) + E^a_\tau E^b_\tau\delta_{ab} - 2m_\tau\tau_\tau) + O(c^{-2}), \tag{7.8}$$

where we use the τ subscript to denote the pullback to the worldline. The divergent term may be cancelled by realising that the highest order term is a quadratic form, which implies that we can rewrite the action by introducing a single new variable λ, in the following way,

$$S = -\int d\tau \frac{1}{2e}(-2\lambda(\tau_\tau - me) + E_\tau^a E_\tau^b \delta_{ab} - 2m_\tau \tau_\tau) + \frac{\lambda^2}{e} c^{-2} + O(c^{-2}). \tag{7.9}$$

In this action, the $O(c^{-2})$ term is taken to be independent of λ. We can explicitly solve for λ using its equation of motion to find the previous action. Now we may take the limit without any divergent terms, obtaining

$$S = -\int d\tau \frac{1}{2e}(-2\lambda(\tau_\tau - me) + E_\tau^a E_\tau^b \delta_{ab} + 2m_\tau \tau_\tau) + A_\tau. \tag{7.10}$$

Now λ has become a Lagrange multiplier. This is a 'Polyakov type' non-relativistic particle action. We note that both e and λ are only a single variable. We may solve for both e and λ in terms of the other variables using both equations of motion. Since it is a Lagrange multiplier, the term containing λ cancels and we obtain only

$$S = -\int d\tau \frac{m E_\tau^a E_\tau^b \delta_{ab}}{2\tau_\tau} - m m_\tau + A_\tau, \tag{7.11}$$

This action is invariant under gct's of the worldline. We may observe that the fields A_μ and m_μ, have the same 'role' in this action, the action only depends on $A_\mu - m_\mu$. We will use this to redefine m_μ to absorb the term A_μ, so that the action only depends on m_μ. If we use this gauge freedom to set $X^0 = \tau$ and additionally set $E_\mu{}^a = \delta_\mu^a$, $\tau_\mu = \tau_\mu = \delta_\mu^0$ and $m_i = 0$ and $A_\mu = 0$, we obtain

$$S = -\int dX^0 \frac{m \dot{x}^a \dot{x}^b \delta_{ab}}{2} - m m_0, \tag{7.12}$$

where the 0 denotes the zeroth component of a one-form or vector. This action has a clear interpretation in classical mechanics. The first term is the kinetic term of a particle moving in a space with metric δ_{ab}, with a potential $V = m m_0$. The potential m_0 is the gravitational potential of Newton's gravity.

This interpretation is particularly clear when we look at the Schwarzschild black hole. A metric describing the geometry of the Schwarzschild black hole is given by

$$ds^2 = -(1 - \frac{2GM}{c^2 r})c^4 dt^2 + (1 - \frac{2GM}{rc^2})^{-1} c^2 dr^2 + r^2 c^2 d\theta + c^2 r^2 \sin^2\theta d\phi^2. \tag{7.13}$$

Where we have chosen the c-dependence in such a way that $S = -m\int ds$. We can then choose the nonzero components of the vielbein fields to be

$$\begin{aligned}
\hat{E}_t{}^0 &= c^2\sqrt{1 - \frac{2GM}{c^2 r}} \\
\hat{E}_r{}^1 &= c\sqrt{1 - \frac{2GM}{c^2 r}}^{-1} \\
\hat{E}_\theta{}^2 &= cr \\
\hat{E}_\phi{}^3 &= cr\sin\theta.
\end{aligned} \tag{7.14}$$

This choice of vielbeins corresponds to the expansion (7.3) We then obtain, when expanding in c the nonzero components

$$\begin{aligned}
\tau_t &= 1 \\
m_t &= -\frac{GM}{r} \\
E_r^{\ 1} &= 1 \\
E_\theta^{\ 2} &= r \\
E_\phi^{\ 3} &= r \sin\theta.
\end{aligned} \tag{7.15}$$

In $E_\mu^{\ a} E_\nu^{\ b} \delta_{ab}$ we recognise the spatial euclidean metric, whereas τ simply measures the time difference t between any two events and $m m_t$ is the gravitational potential. Filling in the values of these fields into the action (7.12), without an electric field, we obtain

$$S = \int dt \frac{\dot{X}^a \dot{X}^b \delta_{ab}}{2} + \frac{GMm}{r}. \tag{7.16}$$

This is just the classical action for a non-relativistic particle in a gravitational field. The action (7.10) has σ-model symmetries, which are not actually symmetries of the action as they act on the background fields, instead of on the variables. The σ-model symmetries of this field are given by the Bargmann group in addition to a scaling symmetry. The Bargmann group consists of the Galilean symmetries with a central extension. The action of this central extension is given by $\delta(\xi) m_\mu = \partial_\mu \xi$. The action of the scaling symmetry is given by

$$\tau_\mu \to c^2 \tau_\mu, \qquad e_\mu^{\ a} \to c e_\mu^{\ a}, \qquad m_\mu \to m_\mu \tag{7.17}$$

Thus, this scaling symmetry corresponds to a local rescaling of scaling parameter c in the expansion (7.3). Essentially this means that for this particular model, the exact value of c does not matter and may vary. The torsionlessness $\delta R_{\mu\nu}(H) = 0$ implies that $e^\mu_{\ a} \partial_\mu c = 0$, indicating that c only depends on the time $\int \dot{x}^\mu \tau_\mu d\tau$ if we want to keep the background field torsionless.

We re-iterate that given a set of equations, or an action, determining the evolution of the vielbein fields, these will generally not be invariant under the local scaling transformation, or even a constant scaling transformation. It is when studying only the σ-models that it is necessarily gauge symmetry.

7.2 The torsionless non-relativistic limit of the string, and the string Newton Cartan algebra

The non-relativistic limit of the Polyakov string action is very similar to the non-relativistic limit of the 'Polyakov' particle action. There is much interest in non-relativistic string theory because it promises to open a window into non-relativistic quantum gravity. Another point of interest is that non-relativistic string theory is related to a limit of AdS/CFT correspondence, as shown in [32]. There are two related non-relativistic string actions, defined on somewhat different geometries. The first one is the Newton-Cartan string of Gomis and Ooguri [2]. This string is constructed by using a torsionless limit of the relativistic string action. The related geometry is called string Newton-Cartan geometry [26], [33]. Secondly we have a string embedded in a torsional Newton-Cartan

(TNC) geometry [32]. These two types of string Newton-Cartan geometry are related to each other [34].

We will now discuss the non-relativistic limit of the Polyakov string action. We will keep our comments about the background fields to the minimum required for taking this limit, only discussing the background fields that occur in the action and the constraints we put on them. We will take the limit similarly to [26], where the only differences are that we do not write down the field $C_\mu^{\ A}$, which is related to symmetries of the background fields in the non-relativistic limit. Since this field does not occur in the final limit, we will not include it here. If one wishes to include this gauge field, one can simply replace every $\tau_\mu^{\ A}$ by $X_\mu^{\ A} = \tau_\mu^{\ A} + \frac{1}{c^2}C_\mu^{\ A}$, where $C_\mu^{\ A}$ satisfies the same curvature conditions as $\tau_\mu^{\ A}$. Another difference is that we keep the action classical and do not use the effective action used when studying the path integral of this string theory.

The limit we are going to take is analogous to the particle limit. The limit starts at the relativistic string Polyakov action, interacting with a Kalb-Ramond field. We introduced this action in equation (4.57). Together with the Kalb-Ramond field, the action is given by

$$\hat{S} = -T\int d^2\sigma\sqrt{-\hat{h}}\hat{h}^{\alpha\beta}\hat{G}_{\alpha\beta} + \epsilon^{\alpha\beta}\hat{B}_{\alpha\beta} \tag{7.18}$$

Objects labelled by the world-sheet indices $\alpha, \beta, ... = 0, 1$ refer to coordinates on the worldsheet, and often are used to denote pullbacks so that for example $\hat{G}_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \hat{G}_{\mu\nu}$. We define $\hat{h} = \det \hat{h}_{\alpha\beta}$.

The metric $\hat{h}_{\alpha\beta}$ has conformal invariance. That is, $\hat{h}_{\alpha\beta}(t,\sigma)$ and $\gamma(t,\sigma)\hat{h}_{\alpha\beta}(t,\sigma)$ lead to the same actions if γ is an everywhere nonzero scalar. Thus only two degrees of freedom of the three that $h_{\alpha\beta}$ possesses actually occur in the action.

We proceed similarly to the particle case. We start by splitting up to coordinates labelled by $\hat{A} = 0, ..., D$ into longitudinal coordinates labelled by $A = 0, 1$ and transverse coordinates labelled by $a = 2, ..., D$. We will then expand the vierbein fields in a 'non-relativistic manner', where c can be assumed to be large.

$$\begin{aligned}\hat{E}_\mu^{\ A} &= c\tau_\mu^{\ A} + c^{-1}m_\mu^{\ A} + O(c^{-3})\\ \hat{E}_\mu^{\ a} &= E_\mu^{\ a} + O(c^{-2}).\end{aligned} \tag{7.19}$$

We have chosen this particular scaling of the vielbein fields in order to obtain a finite answer when taking the limit without requiring a rescaling of the mass tension. Using this expansion we expand the metric $\hat{G}_{\mu\nu}$ as

$$\begin{aligned}\hat{G}_{\mu\nu} &= c^2\tau_\mu^{\ A}\tau_\nu^{\ B}\eta_{AB} + E_\mu^{\ a}E_\nu^{\ b}\delta_{ab} + 2\tau_{(\mu}^{\ A}m_{\nu)}^{\ B}\eta_{AB}\\ &= c^2\tau_\mu^{\ A}\tau_\nu^{\ B}\eta_{AB} + G_{\mu\nu}.\end{aligned} \tag{7.20}$$

When entering this expansion into the Nambu-Goto string action (4.50), the divergent leading order term is given by

$$S_{lead} = -T\int d\tau c^2|\tau|, \tag{7.21}$$

where $\tau = \det\tau_\alpha^{\ A} = \epsilon^{\alpha\beta}\tau_\alpha^{\ A}\tau_\beta^{\ B}\epsilon_{AB}$. We will assume τ to be non-zero, as implied by the name longitudinal. We consider the field strength $R_{\mu\nu}^{\ \ A} = 2\partial_{[\mu}\tau_{\nu]}^{\ A} - \Omega_{[\mu\ \ B}^{\ \ A}\tau_{\nu]}^{\ B}$. If $\tau_{[\mu}R_{\nu\rho]}^{\ \ A} = 0$ then the leading order term is locally a total derivative.

More generally, we recognise that this divergent term can be cancelled with a Kalb-Ramond field given by

$$\hat{B}_{\mu\nu} = c^2 \tau_\mu{}^A \tau_\nu{}^B \epsilon_{AB} + B_{\mu\nu} \tag{7.22}$$

and hence we will use this Kalb-Ramond field in the Polyakov action. Of course, if the leading order term of the action is locally a total derivative, adding this Kalb-Ramond field to the action will not change the equation of motion of the action.

Upon elimination of $h_{\alpha\beta}$ we obtain $h_{\alpha\beta}(\vec{\sigma}) = \gamma(\vec{\sigma}) c^2 \tau_\alpha{}^A \tau_\beta{}^B \eta_{AB} + O(c^0)$, suggests that the leading order term of $h_{\alpha\beta}$ is invertible, since $\tau_\alpha{}^A$ is invertible. Hence any lower order terms of $\hat{h}_{\alpha\beta}$ will not contribute to the action. We can use the conformal invariance to cancel any scaling factor in front of $\hat{h}_{\alpha\beta}$, and hence we use the expansion $\hat{h}_{\alpha\beta} = h_{\alpha\beta} + O(\frac{1}{c^2})$. We enter this into the Polyakov action to obtain

$$S = -T \int d^2\sigma (\sqrt{h} h^{\alpha\beta} (c^2 \tau_\alpha{}^A \tau_\beta{}^B \eta_{AB} + G_{\alpha\beta}) + \epsilon^{\alpha\beta} (\tau_\alpha{}^A \tau_\beta{}^B \epsilon_{AB} + B_{\alpha\beta}) + O(\frac{1}{c^2}). \tag{7.23}$$

Since $h^{\alpha\beta}$ is symmetric and $\epsilon^{\alpha\beta}$ we may simply add them together in the following way if we ensure that the tensors they originally contract with are symmetric resp. anti-symmetric.

$$S = -T \int d^2\sigma ((\sqrt{h} h^{\alpha\beta} + \epsilon^{\alpha\beta})(c^2 \tau_\alpha{}^A \tau_\beta{}^B (\eta_{AB} - \epsilon_{AB})) + \sqrt{h} h^{\alpha\beta} G_{\alpha\beta} + \epsilon^{\alpha\beta} B_{\alpha\beta}) + O(\frac{1}{c^2}). \tag{7.24}$$

We may also rewrite $h^{\alpha\beta} = e^{(\alpha}_C e^{\beta)}_D \eta^{CD}$, and $\frac{\epsilon^{\alpha\beta}}{\sqrt{h}} = -e^{[\alpha}_C e^{\beta]}_D \epsilon^{CD}$. Thus we may write

$$S = -T \int d^2\sigma \sqrt{h} (c^2 e^\alpha_C \tau_\alpha{}^A e^\beta_D \tau_\beta{}^B (\eta_{AB} - \epsilon_{AB})(\eta^{CD} - \epsilon^{CD})) + \sqrt{h} h^{\alpha\beta} G_{\alpha\beta} + \epsilon^{\alpha\beta} B_{\alpha\beta} + O(\frac{1}{c^2}. \tag{7.25}$$

We recognise here that the highest order term is a quadratic form in $e^{\alpha A} \tau_\alpha{}^B$. However, the matrix of $\eta_{AB} - \epsilon_{AB}$ is degenerate. We may simplify our expression by using light cone coordinates. We have

$$\begin{aligned} (\eta^{CD} - \epsilon^{CD}) e^\alpha_C e^\beta_D &= -2 \bar{e}^\alpha e^\beta \\ (\eta_{AB} - \epsilon_{AB}) \tau_\alpha{}^A \tau_\beta{}^B &= -2 \tau_\alpha \bar{\tau}_\beta, \end{aligned} \tag{7.26}$$

where we define the light cone components of $\tau_\alpha{}^A, e^\alpha_A$ as

$$e^\alpha = \frac{1}{\sqrt{2}}(e^\alpha_0 + e^\alpha_1), \qquad \bar{e}^\alpha = \frac{1}{\sqrt{2}}(e^\alpha_0 - e^\alpha_1) \tag{7.27}$$

$$\tau_\alpha = \frac{1}{\sqrt{2}}(\tau_\alpha{}^0 + \tau_\alpha{}^1) \qquad \bar{\tau}_\alpha = \frac{1}{\sqrt{2}}(\tau_\alpha{}^0 - \tau_\alpha{}^1) \tag{7.28}$$

We thus obtain the action

$$S = -T \int d^2\sigma (\sqrt{h} (4c^2 \bar{e}^\alpha \tau_\alpha e^\beta \bar{\tau}_\beta + G_{\alpha\beta} + B_{\alpha\beta}). \tag{7.29}$$

We now introduce the parameters λ $\bar{\lambda}$ in an equivalent rewriting of the action.

$$S = -T \int d^2\sigma(\sqrt{h}(4\lambda e^{\beta}\bar{\tau}_{\beta} + 4\bar{\lambda}\bar{e}^{\alpha}\tau_{\alpha} + G_{\alpha\beta} + B_{\alpha\beta}) - \frac{4\lambda\bar{\lambda}}{c^2}) + O(\frac{1}{c^2}). \tag{7.30}$$

Where the $\lambda, \bar{\lambda}$ dependence of the $O(c^{-2})$ term is explicitly written out. Solving for $\lambda, \bar{\lambda}$ using their equations of motion we obtain

$$\lambda = c^2\bar{e}^{\alpha}\tau_{\alpha} \qquad \bar{\lambda} = c^2 e^{\alpha}\bar{\tau}_{\alpha}. \tag{7.31}$$

Substituting these values for λ into the action (7.30), we obtain (7.29), thus demonstrating the equivalence of the action.
We may take the limit $c \to \infty$ to obtain

$$S_{\text{nonrel}} = -T \int d^2\sigma(\sqrt{h}(4\lambda e^{\beta}\bar{\tau}_{\beta} + 4\bar{\lambda}\bar{e}^{\alpha}\tau_{\alpha} + G_{\alpha\beta} + B_{\alpha\beta}). \tag{7.32}$$

In this action λ and $\bar{\lambda}$ are Lagrange multipliers. The restrictions associated with these Lagrange multipliers indicate that the lightcones of $h_{\alpha\beta}$ and $\tau_{\alpha}{}^{A}\tau_{\beta}{}^{B}\eta_{AB}$ point in the same direction, i.e. $\bar{e}_{\alpha}\epsilon^{\alpha\beta}\bar{\tau}_{\beta} = e_{\alpha}\epsilon^{\alpha\beta}\tau_{\beta} = 0$

Solving for both the equations of λ, $\bar{\lambda}$, e_{α}, $\bar{e}_{\beta}$, we obtain the action

$$S_{\text{nonrel}} = -T \int d^2\sigma|\tau|\eta^{AB}\tau^{\alpha}_{A}\tau^{\beta}_{B}G_{\alpha\beta} \tag{7.33}$$

This action equals the non-relativistic string action given in [6]. We note that all m_{μ} dependence can be absorbed into the Kalb Ramond field using a redefinition of $B_{\mu\nu}$. This is especially clear in the Nambu-Goto type action. However, in the remainder of this thesis we will choose not to do this and essentially set $B_{\mu\nu} = 0$.

7.3 Symmetries of the non-relativistic particle and string actions

In order to fully understand all the σ-model symmetries of the Newton-Cartan particle it will be useful to first analyse the symmetries of the trivial highest order term. Although we cancel the term, it will give us some interpretation of the symmetries we encounter in the Newton-Cartan particle. The term is specifically given by

$$S = m \int d\tau\ \tau_{\mu}\dot{X}^{\mu}. \tag{7.34}$$

We are particularly interested in σ-model symmetries. Clearly, the action is invariant under the $U(1)$ gauge symmetry $\delta(\Gamma)\tau_{\mu} = \partial_{\mu}\Gamma$. This can be interpreted in the following way. Since the action is a total derivative, the action is, up to boundary terms, completely invariant of X^{μ}. We can use this freedom to set $\delta X^{\mu} = 0$ under general coordinate transformations. Thus, gct's act only on the field τ_{μ}, via equation (4.29). Since we assume $R^{\ 0}_{\mu\nu} = 0$ this implies that the action is invariant under local time translations, acting only on the coordinates. Equivalently, it means that the action is invariant under general coordinate transformations acting only on the fields. Thus, the 'triviality' of the action implies the symmetry under local time translations. All other local Galilean transformations, do not affect τ_{μ}, and are therefore not interesting.

As we explain in the appendix C, the existence of a σ-model symmetry acting on this action, implies the existence of a σ-model symmetry acting on the Newton-Cartan particle action, given by

$$\delta(\Gamma)m_\mu = \delta_{gct}(\Gamma)\tau_\mu. \tag{7.35}$$

This can be alternatively interpreted as a combination of boosts and local time translation of order $\frac{1}{c^2}$, as

$$\delta(\Gamma)m_\mu = \delta(\Gamma^0 H)\tau_\mu - \Gamma^\nu R_{\mu\nu}(H) = \partial_\mu\Gamma^0 - \Gamma^\nu R_{\mu\nu}(H). \tag{7.36}$$

Thus if $R_{\mu\nu}(H) = 0$, we may instead write $\partial(\Gamma^0)m_\mu = \partial_\mu\Gamma^0$. This shows that only one of the parameters of the gct paramater Γ^μ actually generates a new symmetry. This transformation corresponds precisely to the extension of the Galilei algebra that occurs in the Bargmann algebra. Furthermore, the conformal transformation (7.17) leaves the action invariant. This conformal symmetry, together with the Bargmann symmetries forms the full symmetry group of this action. They satisfy the Lie algebra

$$\begin{aligned}
&[G_b, H] = P_b && [J_{ab}, P_c] = 2\delta_{c[b}P_{a]} \\
&[J_{ab}, J_{cd}] = 4\delta_{[a[c}J_{d]b]} && [G_b, J_{cd}] = 2\delta_{b[c}G_{d]}. \\
&[G_b, P_c] = \delta_{bc}Z && [G_b, D] = -G_b \\
&[P_a, D] = P_a && [H, D] = 2H.
\end{aligned} \tag{7.37}$$

Usually, when discussing the conformal symmetries D, the special conformal transformation generated by C are also included in the algebra, satisfying the relations

$$\begin{aligned}
&[H, C] = D, && [C, D] = -2C \\
&[P_a, C] = -G_a
\end{aligned} \tag{7.38}$$

This algebra is also called the Schrödinger algebra, since it is the algebra of the free particle Schödinger equation, see for example [35]. The transformations generated by C do not transform the gauge fields $E_\mu{}^A, \tau_\mu$, which explains why we did not encounter the generator in our analysis. However, the Schrödinger algebra is also important for the non-relativistic free particle σ-model, see section 7.4.

Let us now consider the symmetries of the non-relativistic string. Of course, as can be seen from the Lie algebra expansion, the obtained action is invariant under the homogeneous Galilean group, consisting of the Galilean group without translations. Furthermore, by the Lie algebra expansion, it is also invariant under second order longitudinal rotations. The second order transverse rotations do not act on any of the fields contained in the action and therefore may be ignored.

Similarly to the particle there are also extra 'accidental' symmetries, which we would not a priori expect from the Lie algebra expansion. These can be most intuitively explained by considering the trivial first term in the expansion of the action (7.21), given by

$$S = \int d\tau d\sigma |\det \tau_\alpha{}^A|. \tag{7.39}$$

Since this is a total derivative, the general coordinate transformations of the target space coordinates only are a symmetry up to boundary terms, given by $\delta X^\mu = \Gamma^\mu$. Since general coordinate transformations are already symmetries of the action, this implies that general coordinate transformations of the background fields also contribute only boundary terms. Additional symmetries are given by

$$\delta(\bar{\Lambda})\tau_\mu{}^A = 2\bar{\Lambda}^A_B\tau_\mu{}^B, \qquad \bar{\Lambda}^A_A = 0 \tag{7.40}$$

where $\bar{\Lambda}^A_B$ is any traceless tensor. We use the bar to denote a traceless tensor. Given any tensor with two indices X^A_B that has a trace, we may decompose it in an isotropic part and a traceless part.

$$X^A_B = \frac{\delta^A_B X^D_D}{\delta^C_C} + \bar{X}^A_B. \tag{7.41}$$

The invariance of the action under this transformation can be seen using a short calculation, see appendix A. In particular, when writing out the action explicitly,

$$\det \tau_\alpha{}^A = \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \tau_\mu{}^A \tau_\nu{}^B \epsilon_{AB}, \tag{7.42}$$

we see that it depends on the two-form $\tau_{[\mu}{}^A\tau_{\nu]}{}^B\epsilon_{AB}$. Anything that leaves this two-form invariant leaves the area of the worldsheet invariant (at highest order). These transformations are precisely the transformations (7.40).

We may interpret these transformations generated by Λ^A_B at a single point in spacetime as the area preserving transformations of two dimensional Minkowski spacetime, or equivalently of $\mathbb{R}^2$. They act on coordinates X^A via matrix multiplication. The corresponding Lie group is given by $\mathrm{SL}(2,\mathbb{R})$. For $\mathbb{R}^2$, a basis of the Lie algebra is given by

$$a_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad a_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{7.43}$$

Here, a_1 generates rotations, a_2 generates boosts and a_3 generates squeezes, stretching one direction while compressing the other. However, we will use a different notation for the bases, denoting them by the traceless part of the matrices $(M_A{}^B)^C_D = 2\delta^C_A\delta^B_D$, given by

$$(\bar{M}_A{}^B)^C_D = 2\delta^C_A\delta^B_D - \delta^B_A\delta^D_C \tag{7.44}$$

The commutation relations of this basis can be computed using the matrix multiplication, and are given by

$$[\bar{M}_A^B, \bar{M}_C^D] = 2\delta^C_B M_A{}^D - 2\delta^D_A M_B{}^C = 2\delta^C_B \bar{M}_A{}^D - 2\delta^D_A \bar{M}_B{}^C. \tag{7.45}$$

Specifically, the Lie algebra consists of infinitesimal rotations $\frac{1}{2}(M_1{}^0 - M_0{}^1)$, boosts $\frac{1}{2}(M_0{}^1 + M_1{}^0)$ and squeezes $\frac{1}{2}(M_0{}^0 - M_1{}^1)$. Interpreted as actions on Minkowski spacetime, squeezes squeeze space and time in an area preserving manner, and rotations rotate space and time. These are considered to be accidental symmetries of the action, whereas the boosts follow from the boost invariance of the relativistic particle.

Following the discussion in the appendix C, the accidental symmetries in the trivial leading order term imply accidental symmetries in the non-relativsitic limit of the string action (7.33), corresponding to the transformations (7.40) at order $\frac{1}{\omega^2}$. These transformations can be viewed as the gauge transformations of the string Newton-Cartan algebra with generators.

H_A	Longitudinal translations
P_a	transverse translations
$M = \frac{1}{2}\epsilon^{AB}J_{AB}$	longitudinal string rotations
$J_{[ab]}$	transverse string rotations
J_{Aa}	string Galilean boosts
Z_A	2nd order local longitudinal translations
$\bar{Z}_{AB}$,	2nd order area preserving traceless linear maps

The Newton-Cartan algebra is an extension of the Galilean algebra. The string Newton-Cartan algebra is given by

$$[H_A, J_{BC}] = 2\eta_{A[B}H_{C]} \qquad [H_A, J_{Bc}] = \eta_{AB}P_c \tag{7.46}$$
$$[P_a, J_{bc}] = 2\delta_{a[b}P_{c]} \qquad [J_{Aa}, J_{BC}] = 2\eta_{A[B}J_{C]a} \tag{7.47}$$
$$[J_{Ab}, J_{cd}] = 2J_{A[d}\delta_{c]b} \qquad [J_{ab}, J_{cd}] = 4\delta_{[b[c}J_{a]d]} \tag{7.48}$$
$$[P_a, J_{Bc}] = -\delta_{ac}Z_B \qquad [Z_A, J_{BC}] = 2\eta_{A[B}Z_{C]} \tag{7.49}$$
$$[J_{Ab}, J_{Cd}] = \delta_{db}\bar{Z}_{[AC]}, \qquad [H_A, \bar{Z}_{BC}] = 2\eta_{AC}Z_B - \eta_{BC}Z_A \tag{7.50}$$
$$[J_{AB}, \bar{Z}_{CD}] = 2\eta_{C[B}\bar{Z}_{A]D} - 2\bar{Z}_{C[B}\eta_{A]D} \tag{7.51}$$

Furthermore, the action (7.33) is invariant under local conformal transformations

$$\tau_\mu{}^A = c\tau'_\mu{}^A \qquad m_\mu^A = \frac{1}{c}m'^A_\mu. \tag{7.52}$$

Similar to the particle case this transformation has the property that it does not leave the triviality of the divergent term invariant, i.e. $\det \tau_\alpha{}^A$ does not remain a total derivative. However, it does leave invariant the foliation constrain $d\tau_{ab}^A = 0$. We will discuss this further in section 7.4.

7.4 Torsion in non-relativistic geometry

In the previous sections we have discussed the non-relativistic particle and string particle actions in the torsionless case. In this section we will discuss how much torsion can be allowed in Newton Cartan and string Newton Cartan geometry, but still be considered a limit of general relativity. We also discuss how the dilatations and torsion are related.

Whereas in the non-relativistic particle limit we took we assumed torsionlessness, there are also formulations of Newton-Cartan theory including torsion. For example, gauging the Schrödinger algebra, which consists of the Bargmann algebra along with the previously discussed dilatations and special conformal transformations, leads to twistless torsion Newton-Cartan geometry [36]. Further considerations of this algebra, by introducing an additional Stückelberg scalar, can lead to torsional Newton-Cartan geometry. We note here that torsion is introduced only after taking the

non-relativistic limit. Torsional and twistless torsion Newton-Cartan geometry are closely related to Lifshitz holography [37] and the fractional quantum Hall effect [38].
In the method we discussed for taking the non-relativistic limit, the torsionless condition $R_{\mu\nu}(H) = 0$ is necessary in order to guarantee that the divergent term of the action (7.4) is a total derivative. This requirement implies that, classically, we do not modify the relativistic particle in order to take the limit. We only add a total derivative. This also implies that if we introduce torsion in general relativity, that does not satisfy $R_{\mu\nu}(H) = 0$, the divergent term is not a total derivative. In order to take the limit we then need to cancel the divergent term with an electric-magnetic potential that is not a total derivative.

However, after taking the limit, we have seen that additional σ-model symmetries occur, which do not preserve the 0 torsion condition. Indeed, the conformal transformation (7.17) changes the torsion

$$\partial_{[\mu}\tau_{\nu]} = c^2\partial_{[\mu}\tau'_{\nu]} + 2c\tau'_{[\nu}\partial_{\mu]}c = 0, \tag{7.53}$$

but leaves the equation

$$d\tau_{ab} = 0 \tag{7.54}$$

invariant. Whenever τ satisfies this equation, we refer to it as twistless torsion. This is equivalent to the hypersurface orthogonality condition of the Frobenius theorem (7.6). However, there exist limits leading to twistless torsion for which the divergent term is not a total derivative. In particular, a necessary condition for the term to be a total derivative is

$$D_{[a}d\tau_{b]0} = 0, \tag{7.55}$$

where D_μ is a connection satisfying $D_{[a}D_{b]} = 0$.

In the stringy non-relativistic limit, there is a bit more subtlety. As it turns out, the torsionlessness condition

$$R_{\mu\nu}{}^{A} = 2\partial_{[\mu}\tau_{\nu]}^{A} - 2\Omega_{[\mu}{}^{A}{}_{B}\tau_{\nu]}^{B} = 0 \tag{7.56}$$

is not necessary to guarantee that the divergent term (7.21) is a total derivative. Instead, a necessary condition is given by $R_{aB}{}^{B} = 0$, and $R_{ab}{}^{C} = 0$. We may distinguish two separate reasons for this. Firstly, $R_{\mu\nu}{}^{A}$ is dependent on the gauge field of longitudinal rotations $\Omega_\mu{}^{AB}$. Thus, various components of the curvature $R_{\mu\nu}{}^{A}$ can be used to solve for the spin connection. The spin connection does not occur in the particle action, and therefore, its value does not affect the action. In particular, the equations

$$R_{AB}{}^{C} = 0, R_{a}{}^{[BC]} = 0, \tag{7.57}$$

may be used to solve for the spin connection. Thus, only the remaining equations

$$R_{ab}{}^{C} = d\tau_{ab}{}^{C} = 0, \qquad R_{a}{}^{(BC)} = d\tau_{a}{}^{(BC)} = 0, \tag{7.58}$$

affect the value of the action, and only torsion in these terms affect whether the divergent term is a total derivative. The second reason is of a similar vein. As discussed previously, the divergent term (7.21) is invariant under the action of the $\mathrm{SL}(2,\mathbb{R})$ given by (7.40). This symmetry group is larger than the Lorentz group of longitudinal rotations. This implies that the curvature $R_{\mu\nu}{}^{A}(H)$ of the Lorentz group is not covariant under these transformations, and hence equation (7.56) is not invariant under these transformations. In order to obtain a covariant curvature, it is convenient to introduce a gauge field

$$\bar{\Omega}_{\mu}{}^{A}{}_{B}\bar{J}_{A}^{B}, \tag{7.59}$$

where $\bar{\Omega}_{\mu}{}^{A}{}_{A} = 0$. The gauge fields corresponds to the transformation $\bar{J}_{A}^{B}$ of $\mathrm{SL}(2,\mathbb{R})$. This gauge field will only be used as a convenient tool in this calculation. The covariant curvature of the time translations is then given by

$$R_{\mu\nu}^{\mathrm{SL}\ A} = 2\partial_{[\mu}\tau_{\nu]}^{A} - 2\bar{\Omega}_{[\mu}{}^{A}{}_{B}\tau_{\nu]}^{B} = 0, \tag{7.60}$$

which we may set to 0. The components

$$R_{AB}^{\mathrm{SL}\ C} = 0, \tag{7.61a}$$

$$\bar{R}_{aB}^{\mathrm{SL}\ C} = R_{aB}^{\mathrm{SL}\ C} - \frac{1}{2}R_{aD}^{\mathrm{SL}\ D}\delta_{B}^{C} = 0, \tag{7.61b}$$

may be used to non-uniquely solve for $\bar{\Omega}_{\mu}{}^{A}{}_{B}$. The remaining equations

$$\bar{R}_{aB}{}^{B} = d\tau_{aB}{}^{B} = 0 \tag{7.62a}$$

$$\bar{R}_{ab}{}^{C} = d\tau_{ab}{}^{C} = 0, \tag{7.62b}$$

form a necessary condition for the divergent term (7.21) to be a total derivative, as is shown in the appendix A. We note that these conditions are equivalent to setting $\partial_{[\mu}\tau_{\nu}{}^{A}\tau_{\rho]}{}^{B}\epsilon_{AB} = 0$.

It is interesting to note that the hypersurface orthogonality condition $d\tau_{ab}{}^{A} = 0$ is implied by the torsional constraints (7.56) as well as by (7.60), implying that in these limits there exists a foliation of transverse surfaces.

After taking the limit, the action is furthermore invariant under the conformal transformations (7.52). This transformation only leaves invariant the hypersurface orthogonality condition (7.62b), and do not leave invariant the restriction (7.62a). This is related to the fact that the action of $\mathrm{SL}(2,\mathbb{R})$ and the conformal transformations on $\tau_{\mu}{}^{A}$ form precisely the general linear group $\mathrm{GL}(2,\mathbb{R})$.

However, this does not complete the discussion on possible torsion in the non-relativistic action. One of the reasons is that the only geometries that can be considered limits of a relativistic geometry, have $R_{aC}{}^{C}$ which is purely gauge. This sets constraints on the possible values of τ. The exact formulation of these constraints is beyond the scope of this thesis.

Another reason is that it may be useful to also consider torsional stringy Newton Cartan geometry without any constraints, since this also has application with Newton Cartan geometry, i.e. in the particle case [37].

7.5 Massless particles and tensionless strings in the non-relativistic limit

In this section we have put considerable attention to the appearance of the extensions to the Galilei algebra that should be considered in the non-relativistic limit of massive particles. However, these do not appear in the massless limit of the non-relativistic particle. This can be clarified by noting that the divergent term is proportional to the mass. Hence, if we take this parameter to 0, there is no divergent term to be cancelled, and no extension to the algebra. The massless limit of the Galilean particle is very similar to the massless limit of the Carroll particle, as we will see in the next section. We will treat the massless limit in the Hamiltonian formalism. The non-relativistic Hamiltonian is given by

$$S = \int p_\mu \dot{X}^\mu - \frac{e}{2}(-2mp_0 + (p_a)^2), \tag{7.63}$$

where einbein e acts as a Lagrange multiplier setting the mass-shell condition $p_0 = \frac{1}{2m}(p_a)^2$. Furthermore, the Polyakov action (7.10) without A_μ-field can be obtained by eliminating p_a using its equation of motion. This action is also linear in the energy p_0, implying that it also is a Lagrange multiplier. The constant λ in (7.10) corresponds to the energy of the particle. The massless limit is taken by letting $m \to 0$ in expression (7.63). We obtain

$$S = \int p_\mu \dot{X}^\mu - \frac{e}{2}p_a^2. \tag{7.64}$$

The energy p_0 is now a Lagrange multiplier setting $\dot{X}^0 = 0$. Therefore, the particle is restricted to a surface of constant time. The remaining equation of motions are

$$\dot{p}_\mu = 0, p_a = \frac{1}{e}\dot{X}^a, \tag{7.65}$$

implying that the path the particle takes is straight. This limit matches the non-relativistic limit of the massless relativistic particle.

Since there is no divergent term when taking the limit, we may also consider a limit of the relativistic particle where we scale the vierbein in the way used to deal with the non-relativistic string, with $\hat{E}_\mu{}^A = \tau_\mu{}^A + O(\frac{1}{c^2})$ and $\hat{E}_\mu{}^A = E_\mu{}^A$, with $A = 0, 1$ and $a = 2, ..., D$. In general we may also take $A = 0, ..., p$, and $a = p+1, ..., D$, and obtain very similar limits. The actions corresponding to these limits are similarly given by equation (7.65), where only the range of a changes. In this limit, P_A act as Lagrange multipliers setting $\dot{X}^A = 0$.

8 Ultra-relativistic limits of the relativistic particle and string

Here we will consider certain ultra-relativistic limits of the Carroll particle and string actions, with single particles. Following previous articles on the dynamics of Carroll particles [39] and strings [5] we will begin with the Hamiltonian form of the action, where the Carroll limit has a very direct interpretation as a limit of the on-shell conditions. Furthermore, taking the limit will be straightforward compared to the other choice of action, as we will not need to introduce new variables which play the role of Lagrange multipliers. We will also comment on the dynamics of these objects, which turn out to be trivial. When we use Lie algebra expansions adapted to non-relativistic p-branes we obtain dynamics of objects which are very similar to the massless non-relativistic limits.

As noted in [5], there are (at least) two different limits which can be referred to as Carroll limits. The first one ('à la particle') is taken by splitting up the relativistic vielbein into τ_μ, $E_\mu{}^a$, and scaling these fields separately. The second one is taken by taking the 'stringy' limit, splitting up the variables in $A = 0, 1$, $a = 2, ..., D-1$. Unlike in the non-relativistic limit, the actions obtained have similar properties. A string in a Carroll background resembles an collection of particles in a Carroll background, and a 2-Carroll contracted string resembles a collection of 2-Carroll contracted particles. An exception to this statement holds for strings that are precisely longitudinal to the contracted directions.

8.1 Ultra-relativistic limit of the Hamiltonian particle action

The Hamiltonian version of the relativistic particle action is given in equation (4.39). For convenience we will restate it here,

$$S = \int d\tau p_\mu \dot{X}^\mu - \frac{e}{2}(p_\mu \hat{G}^{\mu\nu} p_\nu + m^2). \tag{8.1}$$

Here, e, assumed to be bigger than 0, is a Lagrangian multiplier fixing the mass-shell condition $p^2 + m^2 = 0$ and p_μ are the momenta associated with x^μ. We note that all independent components of the momenta occur in the mass-shell condition. We may solve for momenta as in equation (4.40),

$$p_\mu = \frac{\hat{G}_{\mu\nu}\dot{x}^\nu}{e} \tag{8.2}$$

We may enter this into the action to obtain the usual Polyakov-type action for the particle given in (7.1). We remind ourselves that solving for the einbein $\hat{e}$ results in

$$\hat{e} = \frac{\sqrt{-\dot{x}^2}}{m} \tag{8.3}$$

We use the following scaling of the vierbein fields:

$$\begin{aligned} \hat{E}_\mu^{\ a} &= \frac{1}{c} E_\mu^{\ a} + O(c) \\ \hat{\tau}_\mu &= \tau_\mu + O(\frac{1}{c^2}). \end{aligned} \tag{8.4}$$

This implies the following relations for the inverse vierbein:

$$\begin{aligned} \hat{E}^\mu_{\ a} &= c e^\mu_{\ a} + O(\frac{1}{c}) \\ \hat{\tau}^\mu_{\ A} &= \tau_\mu^{\ A} + O(\frac{1}{c^2}). \end{aligned} \tag{8.5}$$

We will assume that the worldsheet and target space coordinates are not expanded. We will rewrite e for reasons we will justify later as

$$\hat{e} = e + O(\frac{1}{c^2}). \tag{8.6}$$

This then leads to the action

$$S = \int d\tau p_\mu \dot{X}^\mu - \frac{e}{2}(-p_\mu p_\nu \tau^\mu \tau^\nu + m^2) + O(c^2) \tag{8.7}$$

We have chosen the expansion of $\hat{E}_\mu^{\ a}$ and $\hat{\tau}_\mu^{\ A}$ such that the action is convergent. There are no terms that require cancellation or other complication. We obtain in the limit

$$S = \int d\tau p_\mu \dot{X}^\mu + \frac{e}{2}((p_\mu \tau^\mu)^2 - m^2). \tag{8.8}$$

We may remark that the only background field occurring in this action is the boost invariant vector field τ^μ. The mass-shell constraint only puts a constraint on the energy given by $E^2 := (p_\mu \tau^\mu)^2 = m^2$. This implies that the energy becomes completely independent of the transverse momenta. The equations of motion are given by

$$\dot{x}^\mu = -e p_\nu \tau^\nu \tau^\mu \qquad \dot{p}_\mu = e \partial_\mu \tau^\rho p_\rho p_\nu \tau^\nu \qquad (p_\mu \tau^\mu)^2 = m^2. \tag{8.9}$$

The first equation of motion shows that $\dot{x}^\mu$ moves along the integral trajectories of τ^μ, independently of the transverse momenta. The second equation shows that in a non-trivial background the momenta have non-trivial equations of motion.

In order the inspect Carroll versions of the Polyakov and Nambu-Goto type particle actions, we may eliminate some of the momenta from the Hamiltonian actions. In order to do this, we will first switch coordinates

$$p_\mu, \quad x^\mu \quad \to \quad p_0 = \tau^\mu p_\mu, \quad p_a = E^\mu_{\ a} p_\mu, \quad x^\mu. \tag{8.10}$$

We may then eliminate p_0 from the action using its own equation of motion. We obtain

$$p_0 = -\frac{1}{e} \dot{x}^\mu \tau_\mu. \tag{8.11}$$

Using $p_\mu \dot{X}^\mu = p_0 \tau_\mu \dot{X}^\mu + p_a E_\mu^{\ a} \dot{X}^\mu$, we can use this to rewrite the action as a Polyakov type action. We obtain

$$S = \int d\tau \ p_a \dot{x}^a - \frac{(\dot{x}^\mu \tau_\mu)^2}{2e} - \frac{m^2 e}{2}, \tag{8.12}$$

where $p_a = E^\mu_{\ a} p_\mu$, $\dot{x}^a = E_\mu^{\ a} \dot{x}^\mu$. In this action, p_a occur as Lagrange multipliers, setting $\dot{x}^a = 0$.

We may further reduce the amount of variables encountered in this classical action by eliminating e using its equation of motion. We then obtain

$$e = \frac{|\dot{x}^\mu \tau_\mu|}{m}, \tag{8.13}$$

using the assumption $e > 0$. This is consistent with the einbein expansion (8.6), in that e is invariant under a small change in c_0. We then obtain the action

$$S = \int d\tau \ p_a \dot{x}^a - m |\dot{x}^\mu \tau_\mu|. \tag{8.14}$$

The σ model symmetries of the Carroll action are given by homogeneous Carroll transformations of the background fields, along with a local scaling symmetry given by

$$E_\mu^{\ a}(x) \to c(x) E_\mu^{\ a}(x). \tag{8.15}$$

This scaling symmetry corresponds to a local rescaling of the parameter c_0 discussed in section (3.3). However, in general this local scaling transforms the torsionless field $E_\mu^{\ a}$ into a torsional field. If we want to keep working with torsionless Carroll geometry we have to require the constraint

$$R_{\mu\nu}^{\ \ a}(P) = 0 \tag{8.16}$$

to be invariant under these scaling symmetries. In particular, this equation implies a restriction on the geometry given by

$$2\tau^{\mu}E^{\nu(b}\partial_{[\mu}E_{\nu]}{}^{a)} = 0, \tag{8.17}$$

which must remain invariant. Assuming this restriction holds, the restricted term transforms as

$$0 = 2\tau^{\mu}e^{\nu(b}\partial_{[\mu}E_{\nu]}{}^{a)} \to \delta^{ab}\tau^{\mu}\partial_{\mu}c \tag{8.18}$$

Setting this to 0 to preserve this restriction implies $\tau^{\mu}\partial_{\mu}(c) = 0$. Thus, the torsionless Carroll geometry is preserved for scaling transformations of the transverse vierbein fields, where the scaling factor c is constant in the longitudinal direction.

However, the full σ-model symmetries of the Carroll particle action transform the background field into torsionfull background fields, which seems to indicate that certain components of the relativistic torsion are not relevant when describing the trajectories of Carroll particles.

In order to obtain a better understanding of what the expansion of the relativistic einbein $\hat{e}$ (8.6) implies, we apply this assumption to its classical solution (8.3). We obtain

$$e^2 = \frac{(\tau_{\mu}\dot{x}^{\mu})^2 - \frac{1}{c^2}\dot{x}^{\mu}\dot{x}^{\nu}E_{\mu}{}^{a}E_{\nu}{}^{b}\delta_{ab}}{m^2} \tag{8.19}$$

Thus, our rewriting $\hat{e} = e + O(\frac{1}{c^2})$, and the assumption that this remains finite implies that $\dot{x}^{\mu}E_{\mu}{}^{a} = O(c^2)$. This is precisely tells us that in the limit $c \to 0$, the particle does not move in the spatial directions. This clarifies what assumption has led to the restriction of the particle to a line in spacetime.

The Polyakov (8.12) and Nambu-Goto (8.14) type Carroll actions can also be derived by taking the limit of the respective relativistic particle actions. In these cases, a divergent quadratic form needs to be used to restore the momenta p_a as variables, with the method described in appendix B.

8.2 Carroll string action

The ultra-relativistic limit of the phase-space action is performed in [5]. We will in particular consider the 'stringy' limit, where both a timelike and a single spacelike vielbein field is scaled. These fields are usually referred to as longitudinal, but we will refrain from doing so since the string does not need to be longitudinal to the rescaled directions. In order to avoid confusion we will call the longitudinal directions the τ directions, and the transverse spacetime directions the E directions. The limit is taken in a way similar to the Carroll particle, but appears to describe a string that is not without dynamics.

On the other hand there is a different limit of the action in which the $\tau_{\mu}{}^{A}$ is assumed to be longitudinal to the string. In this limit, the dynamics are trivial.

$$S = \int d\tau d\sigma p_{\mu}\dot{X}^{\mu} - \frac{\hat{e}}{2}(P^2 + T^2\hat{G}_{\sigma\sigma}) - \hat{u}(P_{\mu}\partial_{\sigma}X^{\mu}) \tag{8.20}$$

We then apply the Inönü Wigner contraction to the Poincaré algebra, where we choose the following rescaling of the vielbein fields

$$\tau_\mu{}^A = c^{1/2}\tau_\mu{}^A$$
$$\hat{E}_\mu{}^a = c^{-1/2}E_\mu{}^a. \tag{8.21}$$

This re-scaling is chosen in such a way that the action we obtain in configuration space has no c-dependence, and we are therefore not required to rescale the tension. The corresponding c dependence of $\hat{e}$ is given by

$$\hat{e} = ce + O(c^3), \tag{8.22}$$

whereas $\hat{u} = u$ does not depend on c. We then obtain in the limit

$$S = \int d\tau d\sigma\ P_\mu \dot{X}^\mu - \frac{e}{2}(P_A\eta^{AB}P_B + T^2 E_\sigma{}^a E_\sigma{}^b \delta_{ab}) - u(P_\mu \partial_\sigma X^\mu). \tag{8.23}$$

Similarly to the Carroll particle, we are only taking a limit in the on-shell condition of e. $P^2 + T^2\hat{G}_{\sigma\sigma} = 0$ is replaced by $P_A\eta^{AB}P_B + T^2 G_{\sigma\sigma} = 0$, where G is the degenerate spatial metric.

In order to interpret the equations of motion we move to the Carroll equivalent of the conformal gauge, given by $e = 1, u = 0$, and consider a flat background. We then obtain the equations of motion of P_A, P_a, and X^μ respectively

$$\dot{X}^A = P_B\eta^{AB} \qquad \dot{X}^a = 0 \qquad \dot{P}_\mu = -\partial_\mu \tau^\nu{}_A \eta^{AB}\tau^\rho{}_B P_\nu P_\rho + T^2\partial_\mu(E_\sigma{}^a E_\sigma{}^b\delta_{ab})X'^\nu. \tag{8.24}$$

In a flat spacetime this reduces to

$$\dot{X}^A = P_B\eta^{AB} \qquad \dot{X}^a = 0 \qquad \dot{P}_\mu = 0. \tag{8.25}$$

We see that the E-coordinates of the string are constant in time, whereas the τ-coordinates of the string are not. This shows that the string can only move in two directions. This does not constrain the entire movement of the string. For example, consider the open string in a 3 dimensional flat spacetime at an instant $\tau = 0$ given by

$$X^0 = P_0(\sigma)\tau, \qquad X^1 = P_1(\sigma)\tau, \qquad X^2 = \sigma, \tag{8.26}$$

where

$$P_0^2 - P_1^2 = T^2. \tag{8.27}$$

P_0 and P_1 are constant in time, but not necessarily constant in σ. Then the string both moves and satisfies the equations of motion. This is a counter-intuitive example since for constant $P_A(\sigma) = P_A$, this string is spatially extended only in the E-direction, which, in the non-relativistic limit we called transverse. However, this string does minimize the action (8.23). In order to avoid a discussion of boundary conditions in this non-Lorentzian spacetime we can compactify the X^2 direction, i.e. identify $X^2 \sim X^2 + 2\pi R$, so that X^2 is a parameter along a circle. This implies that the string is closed, wound around a circle. The spacetime is given by $\mathbb{M}^2 \times S^1$, the product of two dimensional Minkowski spacetime and a circle. The string then traces out a flat cylinder $\mathbb{R} \times S^1$ embedded in spacetime, moving in a timelike direction. It seems thus that this string is only partially restricted in its motion.

In a certain sense however, the motion of the string does not have any dynamics. In particular, points having different values of x^a, do not affect each other. This holds for any solution of the string. Hence, the string resembles a collection of individual relativistic massive particles, labelled by the coordinate X^a, with independent momenta, each moving in a two-dimensional space, with the appropriate mass-shell condition (8.27). This two dimensional space is given in the flat case by $X^2 = const.$, and more generally by the manifold perpendicular to $E_\mu{}^2$, if it exists.

We may look at the implications of the e dependence (8.22). We have per equation (4.61), $\hat{e} = \frac{\sqrt{-\det \hat{G}_{\alpha\beta}}}{\hat{G}_{\sigma\sigma}}$. We may introduce the c dependence as

$$\hat{e} = \frac{\sqrt{-\det E_\alpha{}^a E_\beta{}^b \delta_{ab} + c^2 \tau_\alpha{}^A \tau_\beta{}^B \eta_{AB})}}{E_\sigma{}^a E_\sigma{}^b \delta_{ab} + c^2 \tau_\sigma{}^A \tau_\sigma{}^B \eta_{AB} + O(c^2)} = O(c). \tag{8.28}$$

We may conclude therefore that the c dependence of $\hat{e}$ implies that $\det E_\alpha{}^a E_\beta{}^b \delta_{ab} \to 0$. In particular this assumes that $E_\alpha{}^a$ has rank 1 in the limit. This condition implies that the string is spatially extended in the $E_\mu{}^a$ directions, but does not move in these directions.

Let us examine the Polyakov and Nambu-Goto versions of the Hamiltonian Carroll action. We may first obtain the Polyakov action by eliminating all momenta occurring quadratically in the action (8.23). We obtain

$$S = \int d\tau d\sigma \frac{(D_t X^A)^2}{2e} + P_a D_t X^a - \frac{e}{2}(X'^a)^2, \tag{8.29}$$

where $D_t = \partial_\tau - u\partial_\sigma$. The justification that this limit is of Polyakov type is given by the fact that the limit of the Polyakov action taken using (8.21) and (8.22) produces this action. The divergent quadratic term occurring in this limit is used to restore the variables P_a, as in the appendix B. Interestingly, the dependence of the worldsheet metric $h_{\alpha\beta}$ on c is of Carroll type. We may say that we take a Carrollian limit on the worldsheet. This can be seen from the dependence of $h_{\alpha\beta}$ on e and u, given in equation (4.59). In particular, the zweibeins satisfying $h_{\alpha\beta} = e_\alpha{}^A e_\beta{}^B \eta_{AB}$ are given by

$$e_0{}^0 = Tce, \qquad e_0{}^1 = 0, \qquad e_1{}^0 = u, \qquad e_1{}^1 = 1. \tag{8.30}$$

We may furthermore eliminate e, u and $P = \sqrt{P_a P^a}$ using their equations of motion to obtain an action of Nambu-Goto type. This shows that we retain $D - 3$ variables which play a role analogous to Lagrange multipliers, in addition to the D variables X^μ. This limit is particularly clear for $D = 3$, where all momenta are eliminated. In this case, we may denote $X^a = X^2 = X$. We obtain

$$S = -T \int d\tau d\sigma \sqrt{-(X'\dot{X}^A - \dot{X}X'^A)^2} \tag{8.31}$$

In light of a formal map between Carroll and Galilei actions discussed in section 9, it is interesting to note that the highest order term in the non-relativistic limit 'à la particle' of the relativistic string is given by [40],[41]

$$S = -T \int d\tau d\sigma \sqrt{(\dot{X}^0 X'^a - X'^0 \dot{X}^a)^2}. \tag{8.32}$$

The similarities are self evident.

The derivation of the Nambu-Goto version of the Carroll limit is most easily done by taking the limit of the relativistic Nambu-Goto action. This requires the restoration of the $D-3$ variables occurring in the action, obtained by using a quadratic form. The derivation of this limit makes use of the Weinstein-Aronszajn identity, and the Cauchy-Binet identity. These identities are used in the computation of matrix determinants and are discussed in the appendix D. We start with the relativistic Nambu-Goto action (4.50). We will first examine the $D=3$ case; we expect an answer matching (8.31). We substitute the vielbein expansions (8.21) into the action. We obtain

$$\begin{aligned} S &= -T\int d\tau d\sigma\sqrt{-\det(\frac{1}{c}E_\alpha E_\beta + c\tau_{\alpha\beta}) + O(c^4)} \\ &= -T\int d\tau d\sigma\frac{1}{c}\sqrt{-\det(\tau_{\alpha\beta})\det(\tau^{\gamma\alpha}E_\alpha E_\beta + c^2\delta^\gamma_\beta) + O(c^4))} \end{aligned} \tag{8.33}$$

where $\tau_{\alpha\beta} = \tau_\alpha{}^A\tau_\beta{}^B\eta_{AB}$. Note that the determinant of $\tau^{\gamma\alpha}E_\alpha E_\beta$ is 0, preventing us from directly taking a limit. Using the Aronszajn-Weinstein identity we obtain

$$\det\tau^{\gamma\alpha}E_\alpha E_\beta + c^2\delta^\gamma_\beta = c^2(E_\alpha\tau^{\alpha\beta}E_\beta + c^2). \tag{8.34}$$

Thus we obtain

$$\lim_{c\to 0} S = -T\int d\tau d\sigma\ \sqrt{-\det(\tau_{\alpha'\beta'})\tau^{\alpha\beta}E_\alpha E_\beta} + O(c^2). \tag{8.35}$$

We note that although in the calculation of the limit we used the inverse of $\tau_{\alpha\beta}$, both the Nambu-Goto action and the limit (8.35) are in general differentiable at $\det\tau_{\alpha\beta} = 0$. In particular the adjunct of $\tau_{\alpha\beta}$, given by $\mathrm{adj}(\tau)^{\alpha\beta} = \det(\tau_{\alpha'\beta'})\tau^{\alpha\beta}$ is differentiable. This shows that the action describes without issues states where the worldsheet is 'orthogonal' to $\tau_\mu{}^1$. Instead, the action is not differentiable if the worldsheet is 'orthogonal' to E_μ, i.e $E_\alpha = 0$.
In the more general case $D \geq 3$, we have some complications, as we obtain a quadratic divergent term. We once again start at the Nambu-Goto action, in $D \geq 3$.

$$S = -T\int d\tau d\sigma\frac{1}{c}\sqrt{-\det(E_\alpha{}^a E_\beta{}^b\delta_{ab} + c^2\tau_{\alpha\beta}) + O(c^4)} \tag{8.36}$$

Directly taking the limit $c \to 0$ has an issue, since the term in the square root becomes negative. This can be dealt with in two separate ways, where we obtain two separate results. We will first discuss the method that corresponds directly to the previous discussion. In this method, we expand the determinant in powers of c^2,

$$S = -T\int d\tau d\sigma\frac{1}{c}\sqrt{-\det(E_\alpha{}^a E_\beta{}^b\delta_{ab}) - c^2\overset{(2)}{\det}(E_\alpha{}^a E_\beta{}^b\delta_{ab} + c^2\tau_{\alpha\beta}) + O(c^4)}. \tag{8.37}$$

$\overset{(2)}{\det}$ gives the coefficient corresponding to the power c^2. At this point we may note that the first term under the square root is quadratic, using the Cauchy-Binet theorem, discussed in appendix D. Explicitly, we may write using equation (D.6)

$$\det E_\alpha{}^a E_\beta{}^b\delta_{ab} = \sum_{S\in\binom{[n]}{D-2}}(\det E_\alpha{}^{S(\beta)})^2, \tag{8.38}$$

Here $\binom{[n]}{k}$ is the set of increasing maps $S : \{0,1\} \to \{2, ..., D-1\}$. These maps match a value of the index $\alpha = 0, 1$ to a value of the index $a = 2, ..., D-1$. These maps label all 2×2 minors of the matrix $E_\alpha^{\ a}$. Clearly, equation (8.38) is quadratic, and hence we may introduce Lagrange multipliers

$$\frac{1}{c^2} \det E_\alpha^{\ a} E_\beta^{\ b} \delta_{ab} \to \sum_{S \in \binom{[n]}{D-2}} 2\lambda_S \det E_\alpha^{\ S(\beta)} - c^2 \lambda_S^2. \tag{8.39}$$

We may note that for $D > 4$, the number of possible maps S given by $(D-2)(D-3)/2$, is larger than $D-3$, which is the expected number of Lagrange multipliers as we deduced by eliminating variables from the Polyakov action. Nevertheless, these restrictions are equivalent to linear dependence of $\dot{X}^a$ and X'^a. The second order terms can be calculated using the Aronszajn-Weinstein identity, and are given by

$$\overset{(2)}{\det} \left(E_\alpha^{\ a} E_\beta^{\ b} \delta_{ab} + c^2 \tau_{\alpha\beta}\right) = \det\left(\tau_{\alpha'\beta'}\right)\left(\tau^{\alpha\beta} E_\alpha^{\ a} E_\beta^{\ b} \delta_{ab}\right). \tag{8.40}$$

The final limit of the action is therefore given by

$$\lim_{c\to 0} S = -T \int d\tau d\sigma \sqrt{-\det\left(\tau_{\alpha\beta}\right)\left(\tau^{\alpha\beta} E_\alpha^{\ a} E_\beta^{\ b} \delta_{ab}\right) + 2 \sum_{S \in \binom{[n]}{D-2}} \lambda_S \det E_\alpha^{\ S(\beta)}} \tag{8.41}$$

As shown in [5], there is nothing stopping us from discussing a Carroll limit of the string 'à la particle', i.e. rescaling only one time dimension separately from the spatial directions. In this limit we do in fact obtain a string without movement. Alternatively, we may also consider 'stringy' limits of the particle, where we rescale the time coordinate and an additional direction separately from the remaining directions. This leads to a particle that is restricted to only move in one direction, similar to the 'stringy' limit of the string.

This is in stark contrast with the non-relativistic limit, where the particle has an interesting physical in the limit 'à la particle', and the string has an interesting physical interpretation in the 'stringy' limit.

8.2.1 A completely static Carroll string

In this section we will discuss an alternative limit of the Carroll string, obtained by starting with (8.36). $E_\alpha^a E_\beta^b \delta_{ab}$ is quadratic in $E_\alpha^{\ a}$, and hence we may rewrite it to the equivalent action

$$S = -T \int d\tau d\sigma \frac{1}{c} \sqrt{-\det(2c^2 E_\alpha^{\ a} \lambda_\beta^{\ b} \delta_{ab} + c^2 \tau_{\alpha\beta} - c^4 \lambda_\alpha^{\ a} \lambda_\beta^{\ b} \delta_{ab} + O(c^4))} \tag{8.42}$$

For $\tau_{\alpha\beta}$ invertible, this limit converges to

$$S = -Tc \int d\tau d\sigma (\sqrt{-\det(2E_\alpha^{\ a} \lambda_\beta^{\ b} \delta_{ab} + \tau_{\alpha\beta})} + O(c^2)). \tag{8.43}$$

This is precisely an action that describes a string longitudinal to the directions $\tau_\mu^{\ A}$. This is because the Lagrange multipliers set the restrictions $E_\alpha^{\ a} = 0$. By Frobenius' theorem, this equation of

motion only has a solution if $dE_{AB}{}^{c} = 0$.
We note that the action obtained approaches 0 as $c \to 0$. As such, if we were to write the original action (8.36) as a combination of both (8.42) and (8.38), then this action would only contribute at $O(c)$. This shows that for this particular limit the scaling of the vielbein (8.21) is not appropriate, unless we re-scale the mass. Instead, we could use the c-dependence

$$\begin{aligned} \tau_\mu{}^A &= \tau_\mu{}^A \\ \hat{E}_\mu^{\ a} &= c^{-1} E_\mu^{\ a}. \end{aligned} \tag{8.44}$$

This action is, in a certain sense, much more similar to the ultra-relativistic particle. In both cases, the classical solutions to the equations of motion correspond to the sub-manifolds obtained by integrating τ^μ respectively τ^μ_A. However, if we are to view τ as a time parameter, for the particle any initial condition (i.e. any position of the particle) is allowed. However, in the string case, the constraints imposed by λ^b_β do not allow for all initial conditions of the string. The string must necessarily be transverse to the $E_\mu^{\ a}$, i.e. parallel to τ^μ_1.

9 Carroll and Galilei from a brane perspective

In this section we will carefully discuss the formal map between k-Galilei Lie algebras, with $A = 0, ..., k-1$, and $D-k$-Carroll Lie algebra's, with $A = 0, ..., D-k-1$, discussed in [6]. This duality also holds for the Lie-algebra expansion, or any other extensions of the Galilei and Carroll algebras. We will interpret this map as interchanging a space direction with a time direction. Using this formal map we can map Galilei and Carroll gravity actions to each other. We can also use this map to relate extended objects to each other. This is a subtle matter, since the map changes the physical meaning of the actions. We will look at the dynamics of these actions and interpret them as the limits of relativistic actions.

9.1 σ-models with a central charge symmetry

As noted in [6], the formal mapped can be used to obtain from a (homogeneous) Galilei invariant σ-model a (homogeneous) Carroll invariant σ-model. This includes any extensions of the Galilei and Carroll algebras. This is in particular applied to the non-relativistic p-brane actions when $D = 2(p+1)$, to obtain Carroll invariant σ-models. As a first application let us consider the Nambu-Goto action of the non-relativistic particle in $D = 2$, in a curved background. This action is given by

$$S = m \int d\tau \frac{(\dot{X}^1)^2}{\dot{X}^0} + m_\mu \dot{X}^\mu, \tag{9.1}$$

where 0 and 1 are flat indices. The map $A = 0 \leftrightarrow a = 1$ gives an action for a moving Carroll 'particle',

$$S = m \int d\tau \frac{(\dot{X}^0)^2}{\dot{X}^1} + n_\mu \dot{X}^\mu. \tag{9.2}$$

This action is completely analogous to the non-relativistic particle in 2 dimensions, having an analogous symmetry group and analogous dynamics. If we want to retain the interpretation of a Carroll space as having a 0 light speed, this implies that this particle is moving faster than the speed of light. Indeed, it can be interpreted as the limit of the 2-dimensional relativistic particle action

$$S = m \int d\tau \sqrt{\dot{X}^2}, \tag{9.3}$$

where $\dot{X}^2 > 0$, i.e. the particle is moving faster than the speed of light. Thus interchanging $a \leftrightarrow A$ in this case has led to a particle with a larger than light speed.

For $D = 4$, we can similarly compare the non-relativistic Nambu-Goto action (7.33), explicitly given by

$$\det(\tau_{\alpha'}{}^{A'})\eta^{AB}\tau_A^\alpha \tau_B^\beta (E_\alpha{}^a E_\beta{}^b \delta_{ab} + \tau_\alpha{}^C m_\beta{}^D \eta_{CD}), \tag{9.4}$$

with its ultra-relativistic counterpart

$$\det(E_{\alpha'}^{a'})\delta^{ab} E_a^\alpha E_b^\beta (\tau_\alpha{}^A \tau_\beta{}^B \eta_{AB} + E_\alpha{}^c n_\beta{}^d \delta_{cd}). \tag{9.5}$$

If we interpret the Carrollian spacetime as a limit of a relativistic spacetime, the condition $\det E_{\alpha'}^{a'} \neq 0$ is the condition for a string being spatially extended, i.e. moving faster than the speed of light. This action can indeed be obtained as the ultra-relativistic limit of a relativistic string moving faster than the speed of light, completely analogous to the limit used to obtain (7.33).

It is worth noting that this similarity does not only involve $D = 2(p+1)$. Indeed, the action (9.5) is invariant under the generalised homogeneous Carroll transformations with central extensions for $A = 0, ..., D-3$, $a = D-2, D-1$, i.e. the only condition required is that a ranges over 2 values. This, however, forces us to consider strings in an ultra-relativistic limit of the Poincaré algebra that is usually not considered to be 'stringy'. It is not difficult to show that these strings are related to each other via dimensional reduction, precisely analogous to the way that the non-relativistic strings are related to each other via dimensional reduction. An analogous statement can be made for the Carroll particle action (9.2), where we may replace $0 \to A = 0, ..., D-2$, $1 \to D-1$.

As shown in [6], it is possible to write down an action where a ranges over only 1 value, which does have a central extension. This action can be obtained from the action (9.5) by assuming constraints on the background fields similar to dimensional reduction of a string. To be more precise, we split the available spacetime indices into $\mu = 0, ..., D-1 \to \tilde{\mu} = 0, ..., D-2$, keeping $D-1$ separate. Similarly, a splits into $D-2$ and $D-1$. We assume all background fields to be independent of x^{D-1}, and we assume the conditions

$$E_{\tilde{\mu}}{}^{D-1} = 0 \quad E_{D-1}{}^{D-1} = 1 \qquad E_{D-1}{}^{D-2} = 0 \qquad \tau_{D-1}{}^{A} = 0 \qquad n_\mu{}^{D-1} = 0 \qquad n_{D-1}{}^{D-2} = 0 \tag{9.6}$$

The action obtained is given by

$$S = T \int d\tau d\sigma \epsilon^{\alpha\beta} \partial_\alpha X^{D-1} n_\beta - \frac{\epsilon^{\alpha_0\alpha_1}\epsilon^{\beta_0\beta_1}(-\tau_{\alpha_0}\tau_{\beta_0} + \partial_{\alpha_0} X^{D-1} \partial_{\alpha_0} X^{D-1}) h_{\alpha_1\beta_1}}{2\epsilon^{\alpha\beta}\partial_\alpha X^{D-1} E_\beta} \tag{9.7}$$

As it turns out, this action is the formal dual of the non-relativistic string action introduced in [32], which couples to torsional Newton-Cartan geometry in $D-1$ dimensions. This non-relativistic string action can be obtained from a non-relativistic string coupling to a stringy Newton-Cartan geometry (7.33) in a similar fashion.

9.2 σ-models without a central charge symmetry

For the σ-models considered in the previous section, the formal duality between the generalised Carroll algebra and the Galilei algebra relates an object moving slower than the speed of light with an object moving faster than the speed of light. There exist examples where this is does not occur. In this section we will discuss examples where the map relates 2 objects that either both move slower than the speed of light, or at the speed of light.
We have already two actions related in this fashion, namely actions (8.31), (8.32). However in a certain sense, both of these actions have trivial dynamics. Since, in a torsionless Carroll geometry (more precisely, if there exists a foliation causally separated surfaces), (8.31) all points on the string are causally separated from each other, it seems unlikely that this string as very interesting dynamics. On the other hand (8.32) is considered in [40], where it is noted that the action simply prescribes the string to be of minimal length, and does not describe any dynamics. However, in [41] a supersymmetric version of this Galilean string is considered, which has drawn some interest. Perhaps these ideas extend to the related Carroll string action.

This formal duality between these two actions also holds for any spacetime dimension $D \geq 2$. The corresponding $D-1$-Carroll algebra corresponds to letting $A = 0, ..., D-2$, $a = D-1$. Physically this string can be interpreted as being extended a large amount in the E_μ^{D-1} direction, and a small amount in the ${\tau_\mu}^A$ directions, including the time direction.

We may also consider σ-models in which the objects are moving at the speed of light. This is for example the case for the massless Galilean particle (7.64), given by

$$S = \int p_\mu \dot{X}^\mu - \frac{e}{2} p_a^2. \tag{9.8}$$

The map from a Galilean particle to a Carroll particle corresponds to sending the massless Galilean particle to the massless Carroll particle, obtained by letting $m \to 0$ in the Carroll particle action 8.8

$$S = \int p_\mu \dot{X}^\mu - \frac{e}{2} p_A^2. \tag{9.9}$$

Alternatively, these actions can be considered as an non- respectively ultra-relativistic limit of a massless relativistic point particle. These actions can be obtained for any choice of the generalised Galilean and Carroll algebras, i.e. for any $A = 0, ..., k-1$, $a = k, ..., D-1$. Similar actions exist for any tensionless p-brane.

10 Conclusion and discussion

In this thesis we started with the main question

"How may the similarity of generalised Galilean and Carroll algebras be used to obtain new results for ultra-relativistic geometry?"

In order to answer this question partially, we have studied Galilean and Carroll σ-models. These σ-models give an action principle describing how particles, strings and p-branes couple to types of Galilean and Carroll geometry. We found that every Galilean invariant σ-model has a formal

Carroll dual (and vice versa), which may be immediately written down and is guaranteed to be invariant under the transformations of the respective algebras. There are multiple particle and string actions which each shed a different light on this formal duality.

In order to understand the formal duality it is necessary to consider generalisations of the Carroll and Galilei algebra. These generalisations can, similarly to the Carroll and Galilei algebras, be obtained as limits of the Poincaré algebra via the Inönü-Wigner contraction. Whereas the Galilei algebra can be considered to be a limit of the Poincaré algebra where we only consider large time differences and small spatial differences, the generalisations of the Galilei algebra are limits of the Poincaré algebra where we consider large time differences, large spatial differences in p spatial directions, and $D-(p+1)$ spatial directions in which we only consider small differences. These generalised algebras are especially well suited to describe the low velocity limits of p-branes.

The generalisations of the Carroll algebra related to the p-brane Galilean algebras are instead obtained as a limit from the Poincaré algebra where only small time differences are considered. For the Carroll algebra specifically, all considered spatial differences are large. For the generalised Carroll algebra related to the p-brane Galilei algebra we consider small spatial differences in $D-(p+2)$ directions. In the $p+1$ remaining spatial directions, we consider only large differences. Thus, only when in $D=2$ spacetime dimensions is the Carroll algebra directly related in this way to the Galilei algebra. In all other spacetime dimensions this formal duality relates generalised versions of the algebra to each other.

While the Galilean limit of the relativistic particle can 'informally' be obtained by sending the speed of light c to infinity, this is not really a correct interpretation. Only dimensionless parameters have physical meaning. This means that in practice we do not take a limit of the speed of light, but instead the ratio of the speed of light with a velocity. The velocity can for example be the maximum velocity of an object that is considered. A particle can move in any direction, and therefore we have to assume that the velocity in each of those directions is small. Strings only move in directions transverse to the string. Therefore, in order to take a non-relativistic limit, we first have to determine directions in which the string is extended, and then assume that the velocity of the string in the transverse directions is small. This leads to a splitting of the algebra into two directions longitudinal to the string (one timelike and one spacelike) where we consider large distances, and we only consider short distances in the $D-2$ remaining transverse directions.

The non-relativistic particle in D spacetime dimensions is formally dual to an ultra-relativistic 'particle' that has a large velocity (faster than light) in a single longitudinal spatial direction, and a small velocity in the remaining directions (including the time dimension). This can be interpreted as an ultra-relativistic limit of a relativistic tachyon. This interpretation raises questions about the physical relevance of this model, but being the limit of a tachyon does not exclude physical applications. Since in many applications Carroll spacetime does not emerge directly as a limit of a relativistic spacetime, it is even unclear whether the σ-models considered actually describe faster than light objects in those applications. Furthermore, a massless particle with a property called colour can be obtained as the non-relativistic limit of a tachyon [42], providing an example where the limit of a tachyon leads to an interesting result.

Similarly, the non-relativistic string in D spacetime dimensions is formally dual to an ultra-relativistic surface that is longitudinal to 2 spatial directions. This surface can be thought of as a string moving faster than the speed of light.

We may also consider non-relativistic strings embedded in a $D-1$ dimensional Newton-Cartan geometry, instead of a D dimensional string Newton-Cartan geometry. This non-relativistic string can be interpreted instead as being embedded in a string Newton-Cartan geometry with a compact spatial longitudinal isometry [34]. As with any action invariant under an extension of the Galilean group, this string has a Carroll formal dual. The formal dual to a string in $D = 3$ Newton-Cartan geometry was discussed in [6] as a dimensional reduction of the $D = 4$ Carroll string model with non-central extensions.

There also exist 'non-relativistic' limits without (non)central extensions, which can not precisely be considered as low velocity limits. One example is the somewhat trivial limit of the relativistic string where we consider only small distances in all spatial directions, including the longitudinal direction. In other words, we take the limit corresponding to the Galilei algebra. One reason why this limit is particularly interesting is that the formal dual is apparently the most general ultra-relativistic limit of the relativistic string where the speed of the string does not exceed the speed of light. The Carrollian limit has one spatial direction where we consider large distances, which is longitudinal to the string. Since we only consider small time differences, this means that in general any two points on the string appear to be causally separated from each other. Due to the simplicity of this limit it is an interesting candidate to test whether the duality holds up in a more general context, such as in the supersymmetric version of the non-relativistic string discussed in [41].

We also have some additional remarks related to torsion. Torsion in Newton-Cartan geometry is closely related to the existence of surfaces of constant time, a characterising feature of Newton-Cartan geometry. From a symmetry point of view it is an important concept, as the invariance of the non-relativistic particle action under Bargmann and Schrödinger gauge transformations is directly related to the torsion. From the point of view of the Lie algebra expansion, the action is however always invariant under homogeneous Galilean transformations, and 0th order as well as 2nd order general coordinate transformations of the worldline, independent of the torsion. However, the algebra that these symmetries form is dependent on the torsion. This is most easily seen by noting the torsion dependent identity relating general coordinate transformations and local translations in general relativity.

From a different point of view the torsion is not a vital concept. This point of view can be held in p-brane non-relativistic limit. In particular in the string case, we may tune the Kalb-Ramond field in precisely such a way that the divergent term appearing in the non-relativistic limit of the string cancels. In doing this, the gauge fields corresponding to the extensions of the Galilei algebra can be absorbed into the Kalb-Ramond field. The Kalb-Ramond field should then be considered as an integral part of the string Newton-Cartan geometry, as its transformation rule allows for the invariance of the non-relativistic string action under the string Galilei algebra.

In this thesis we have also considered specific conformal transformations of the gauge fields occurring in σ-models, corresponding to variations of the expansion parameter. These transformations are local symmetries for any limit of a relativistic σ-model. However, this is not necessarily true for the corresponding limit of the Einstein-Hilbert action. Therefore, they can in general not truly be considered gauge symmetries of the introduced gauge fields.

In this thesis we have applied the formal duality between Carroll and Galilei only at the level of the σ-models, and the application has led to interesting particle and string models, generalisations and extensions of the Carroll algebra. It will be interesting to see whether these ideas will lead to new applications.

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A Symmetries of $\int d^{p+1}\sigma\ \tau$

Here we prove the claims of invariance made in section 7.3 about the action $S = \int d\sigma\tau$. First of all, we check that the action is invariant under the traceless tensor $\delta(\Lambda)\tau_\mu^A = \Lambda^A_B\tau_\mu{}^B$:

$$\begin{aligned}\delta(\Lambda)S &= 2T\int d^{p+1}\sigma\ \tau\ \tau^\alpha_A\Lambda^A_B\tau_\alpha{}^B \\ &= 2\int d\sigma\ \tau\Lambda^A_A = 0.\end{aligned}$$

The invariance of the action under local translations $\delta(\xi^B P_B)\tau_\mu{}^A = \partial_\mu\xi^A - \Omega_\mu{}^A{}_B\xi^{\ B}$ involves a bit more work, and is only a σ-model symmetry symmetry up to boundary terms. Furthermore, the invariance only holds if a constraint on $R_{\mu\nu}{}^A(P_A)$ is satisfied.

We will assume $\tau = \det\tau_\alpha{}^A > 0$.

$$\begin{aligned}\delta(\xi)S &= T\int d^{p+1}\sigma\ \delta(\xi)\tau \\ &= T\int d^{p+1}\sigma\ \tau\tau^\alpha_A D_\alpha\xi^A \\ &= T\int d^{p+1}\sigma\ \partial_\alpha(\tau\tau^\alpha_A\xi^A) - \xi^A D_\alpha(\tau\tau^\alpha_A) \\ &= S_{boundary} - T\int d^{p+1}\sigma\ 2\xi^A\tau\tau^\alpha_A\tau^\beta_B D_{[\alpha}\tau_{\beta]}{}^B + \\ &= S_{boundary} - T\int d^{p+1}\sigma\ \xi^A\tau\tau^\alpha_A\tau^\beta_B R_{\alpha\beta}{}^B(P)\end{aligned}$$

Thus, the local translation of the gauge fields is a σ-model symmetry of this term if and only if the background fields satisfy

$$\tau^\beta_B R_{\alpha\beta}{}^B(P_A) = 0 \tag{A.1}$$

for all X^μ. The invertibility conditions of the longitudinal and transverse vierbein imply

$$\tau^\beta_B\partial_\beta X^\nu = \tau^\nu_B + E^a_\beta E^\nu_a\tau^\beta_B. \tag{A.2}$$

Hence the action is invariant if $\tau^{\nu}_{B} R_{\alpha\nu}{}^{B}(P_A) + \tau^{\beta}_{B} E^{a}_{\beta} E^{\nu}_{a} R_{\alpha\nu}{}^{B}(P_A) = 0$ for all X^μ. A sufficient condition for this to hold is

$$R_{\mu B}{}^{B}(P) = 0 \qquad \text{and} \qquad R_{\mu a}{}^{B}(P) = 0. \tag{A.3}$$

However, for particles, $p = 0$, equation (A.1) is trivially satisfied. For strings, $p = 1$, (A.3) is equivalent to $R_{\mu\nu}{}^{A} = 0$.

B Divergent quadratic forms

When taking the non or ultra-relativistic limits of an action it occurs relatively often that one finds a divergent quadratic form, i.e. we have a Lagrangian of the form

$$\mathcal{L} = \mathcal{L}_0(\phi) + \omega^2 Q(K), \tag{B.1}$$

where the quadratic form $Q : V \to \mathbb{R}$ can be written as a symmetric bilinear form $Q(K) = B(K, K)$. For example, this occurs when taking the non-relativistic of the Polyakov string or particle action, cf. section 7, when taking the Polyakov type limit of the Carroll particle and string actions cf. section 8, and also when discussing gravity actions in second order formalism, cf. section 6. All these cases have in common that they can be derived by eliminating variables from actions with more variables. For example, the relativistic Polyakov actions can be obtained by eliminating the momenta from the Hamiltonian action. However, after taking the non-relativistic limit of the Hamiltonian action, not all momenta can be eliminated without also eliminating the Lagrange multipliers e, in the particle case. This is because the energy p_0 becomes a Lagrange multiplier. Thus, we should expect some complications when taking the non-relativistic limit of the Polyakov action. This complication occurs in the form of a divergent quadratic form. We can in general use this quadratic form to essentially restore the previously eliminated variables in the following way.

$\mathcal{L}$ has the same equations of motion as the Lagrangian

$$\mathcal{L}' = \mathcal{L}_0(\phi) + 2B(S, K) - \omega^{-2} B(S, S), \tag{B.2}$$

where $c \neq 0$, and $S \in V$. This can be seen by eliminating the variable S from this equation using its equation of motion,

$$\delta_S \int \mathcal{L}' = \int B(\delta_S S, K) - \omega^{-2} B(\delta_S S, S) = 0. \tag{B.3}$$

Note that equation (B.3) must hold for any $\delta S \in V$. Therefore this equation implies that $\int \omega^{-2} B(S, S) = \int B(K, S) = \int \omega^2 B(K, K)$. By substituting this into the action (B.2) we obtain the original action (B.1). This shows via the elimination lemma that $\mathcal{L}$ and $\mathcal{L}'$ have the same equations of motion.

Taking the limit $\omega \to \infty$ gives in the limit

$$\mathcal{L} = \mathcal{L}_0(\phi) + 2B(S, K). \tag{B.4}$$

Here, S has become a Lagrange multiplier.

We have formally only considered cases where K is a field on its own, however does this also hold if K is a function of fields ϕ and its derivatives? The answer is yes, since the above proof uses only the variation of S, which remains a field on its own. Furthermore, the bilinear form B is also allowed to depend on additional fields.

More generally, we can apply the same procedure to Lagrangians when the Lagrangian is a function of a quadratic form, under the condition this function is differentiable with respect to K for all possible values of the quadratic form. In particular, in the limit $\omega \to \infty$ it should be differentiable at $B(K(\phi), K(\phi)) = 0$. The variables S in a Lagrangian of the form

$$\mathcal{L}(\phi, 2B(S, K(\phi); \phi) - \omega^{-2} B(S, S; \phi)), \tag{B.5}$$

can be eliminated to obtain the action

$$\mathcal{L}(\phi, \omega^2 Q(K(\phi); \phi)). \tag{B.6}$$

In particular, the Carroll p-brane limit, including the particle and string limits configuration space are of this form cf. section 8.

Furthermore, there is a correspondence between the symmetries of $\mathcal{L}$ and $\mathcal{L}'$. In particular, if δ represents a symmetry of $\mathcal{L}$, then the difference in the variations $\int \delta\mathcal{L} - \delta\mathcal{L}' = \int \delta B(\omega K - \omega^{-1} S, \omega K - \omega^{-1} S)$. If the variation of S is defined such that this term is 0, δ is also a symmetry of $\mathcal{L}'$. In particular, if the quadratic form is invariant, i.e. $\int \delta B(K, K) = 0$, then $\delta S = \delta K$ has this property.

C The occurrence of gauge field extensions and accidental symmetries in brane actions

The following part is based on a comment in section 4.2.2. of [24], where it stated that, in a specific case, the higher order term actions in the Lie algebra expansion can be obtained by examining the lowest order term, and then replacing the gauge fields in them by gauge fields of higher order. We will be a bit more explicit. Let us consider a brane action $S = S[X](E_\mu^{\hat{A}})$, and let us consider the Carroll limit $c \to 0$. Using the scaling $\tau_\mu^A(c) = c\tau_\mu^A$, we obtain an expansion of the action

$$S[X](e_\mu{}^a, c\tau_\mu{}^A) = c^k S^{(k)} + c^{k+2} S^{(k+2)} + O(c^{k+4}) \tag{C.1}$$

for some integer k. If we expand the vielbein fields further, as in the Lie algebra expansion (5.17), we obtain for small c

$$S[X](e_\mu{}^a + c^2 n_\mu{}^a, c\tau_\mu{}^A + c^2 m_\mu{}^A +; c) = c^k S^{(k)} + c^{k+2}(S^{(k+2)} + \frac{\partial S^{(k)}}{\partial e_\mu{}^a} n_\mu{}^a + \frac{\partial S^{(k)}}{\partial \tau_\mu{}^A} m_\mu{}^A) + O(c^{k+4}). \tag{C.2}$$

From this, we can draw a few conclusion about the occurrence of central charges in the limit. First of all, the extensions of the gauge fields only occur in the limit if the first nonzero term $S^{(k)}$ of the expansion of the action is cancelled by another term, such as the near-critical electric field in the Galilean particle limit. Secondly, if the highest order term $S^{(k)}$ has symmetries which transform

$e_\mu{}^a$ and/or $\tau_\mu{}^A$, beyond the symmetries of S, then these symmetries will re-occur in the next term, as $\delta' n_\mu{}^a = \delta e_\mu{}^a$, $\delta' \tau_\mu{}^A = \delta m_\mu{}^a$.

$$\begin{aligned}\delta'(S^{(k+2)} + \frac{\partial S^{(k)}}{\partial e_\mu{}^a} n_\mu{}^a + \frac{\partial S^{(k)}}{\partial \tau_\mu{}^a} m_\mu{}^a) &= \frac{\partial S^{(k)}}{\partial e_\mu{}^a} \delta' n_\mu{}^a + \frac{\partial S^{(k)}}{\partial \tau_\mu{}^A} \delta' m_\mu{}^A \\ &= \frac{\partial S^{(k)}}{\partial e_\mu{}^a} \delta e_\mu{}^a + \frac{\partial S^{(k)}}{\partial \tau_\mu{}^A} \delta \tau_\mu{}^A \\ &= \delta S^{(k)} = 0\end{aligned} \tag{C.3}$$

This holds in particular also for symmetries of $S^{(k)}$ which do not correspond to symmetries of S. This also occurs in the Galilean/$2(p+1) = D$ Carroll limit, where the local translations paramatrized by ξ^A/ξ^a and local boosts, rotations and scalings parametrized by traceless $\Lambda^{AB}/\Lambda^{ab}$ discussed in section 7.3 re-occur as symmetries of the action acting on $m_\mu{}^A/n_\mu{}^a$.

D A short aside on the computation of matrix determinants

In this text we use two identities to computes matrix determinants. The first one is called the Weinstein-Aronszajn identity, or the Sylvester determinant identity, given by

$$\det 1 + AB = \det 1 + BA, \tag{D.1}$$

where A is a $n \times k$ matrix and B is an $k \times n$ matrix. This determinant relates an $n \times n$ matrix with a $k \times k$ matrix. This identity is easily derived using the $n + k \times n + k$ block matrices

$$M = \left[\begin{array}{c|c} 1 & A \\ \hline 0 & 1 \end{array}\right], \quad N = \left[\begin{array}{c|c} 1 & -A \\ \hline B & 1 \end{array}\right] \tag{D.2}$$

We have

$$\det 1 + AB = \left[\begin{array}{c|c} 1 + AB & 0 \\ \hline B & 1 \end{array}\right] = \det MN = \det NM = \left[\begin{array}{c|c} 1 & 0 \\ \hline B & 1 + BA \end{array}\right] = \det 1 + BA. \tag{D.3}$$

We use a re-scaled version of the identity given by

$$\det \lambda 1 + AB = \lambda^{n-k} \det \lambda 1 + BA \tag{D.4}$$

The second identity is the Cauchy-Binet formula. For $n > k$ we have

$$\det BA = \sum_{S \in \binom{[n]}{k}} \det A_{S\times[k]} \det B_{[k]\times S}, \tag{D.5}$$

where S ranges over all k-element ordered subsets of $[n] := (1, ..., n)$, $A_{S\times[k]}$ is the $k \times k$ minor consisting of the rows S, and similarly $B_{[k]\times S}$ is the $k \times k$ minor consisting of the columns S. The proof of the Cauchy-Binet formula happily consists of inspecting the $n - k$ order term of (D.4), see [43], page 297-298.
For $B = A^T$ we have

$$\det A^T A = \sum_{S \in \binom{[n]}{k}} (\det A_{S\times[k]})^2, \tag{D.6}$$

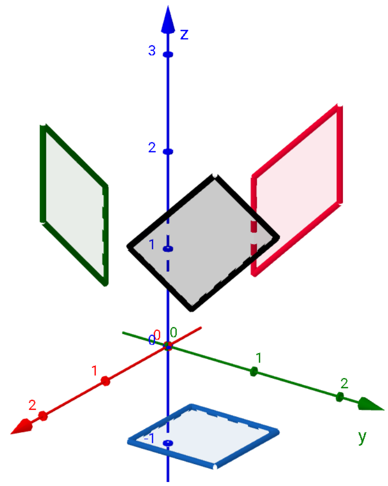

Figure 1: The squared area of the black parallelogram equals the sum of the squared areas of its orthogonal projections. In this case, the squared areas are given by $3 = 1 + 1 + 1$.

If A is a column vector, this is just the Pythagorean theorem. If $k > 1$ this generalises the Pythagorean theorem. This formula has a nice geometric interpretation. $\sqrt{\det A^T A}$ is the volume of a k-dimensional parallelepiped constructed by the k column vectors A_i in $\mathbb{R}^n$. $|\det A_{S\times[k]}|$ is the area of the orthogonal projection of the parallelepiped onto the subset spanned by the basis vectors $e_i, i \in S$. Thus, the volume squared of the parallelepiped equals the sum of squares of the orthogonal projections of the parallelepiped, see figure 1.

Specifically, the Cauchy-Binet theorem (D.5) tells us that $\det A^T A$ is a quadratic form of A, where the vector space the corresponding bi-linear form acts on consists of the determinants of $k \times k$ minors of A.

References

[1] Roel Andringa, Eric Bergshoeff, Sudhakar Panda, and Mees De Roo. Newtonian gravity and the bargmann algebra. *Classical and Quantum Gravity*, 28(10):105011, 2011.

[2] Jaume Gomis and Hirosi Ooguri. Nonrelativistic closed string theory. *Journal of Mathematical Physics*, 42(7):3127–3151, 2001.

[3] Jelle Hartong. Gauging the carroll algebra and ultra-relativistic gravity. *Journal of High Energy Physics*, 2015(8):69, 2015.

[4] Jean-Marc Lévy-Leblond. Une nouvelle limite non-relativiste du groupe de poincaré. In *Annales de l'IHP Physique théorique*, volume 3, pages 1–12, 1965.

[5] Biel Cardona, Joaquim Gomis, and Josep M Pons. Dynamics of carroll strings. *Journal of High Energy Physics*, 2016(7):50, 2016.

[6] Eric Bergshoeff, José Manuel Izquierdo, and Luca Romano. Carroll versus galilei from a brane perspective. *arXiv preprint arXiv:2003.03062*, 2020.

[7] Lewis Carroll. *Alice's Adventures in Wonderland*. Macmillan, 1865.

[8] Lewis Carroll. *Through the Looking-Glass, and What Alice Found There*. Macmillan, 1871.

[9] Henri Bacry and Jean-Marc Lévy-Leblond. Possible kinematics. *Journal of Mathematical Physics*, 9(10):1605–1614, 1968.

[10] Marc Henneaux, Martin Pilati, and Claudio Teitelboim. Explicit solution for the zero signature (strong-coupling) limit of the propagation amplitude in quantum gravity. *Physics Letters B*, 110(2):123–128, 1982.

[11] Thibault Damour, Marc Henneaux, and Hermann Nicolai. Cosmological billiards. *Classical and Quantum Gravity*, 20(9):R145, 2003.

[12] Laura Donnay and Charles Marteau. Carrollian physics at the black hole horizon. *Classical and Quantum Gravity*, 36(16):165002, 2019.

[13] Christian Duval, GW Gibbons, PA Horvathy, and PM Zhang. Carroll symmetry of plane gravitational waves. *Classical and Quantum Gravity*, 34(17):175003, 2017.

[14] Diego M Hofman and Blaise Rollier. Warped conformal field theory as lower spin gravity. *Nuclear Physics B*, 897:1–38, 2015.

[15] Hans-Peter Künzle. Galilei and lorentz structures on space-time: comparison of the corresponding geometry and physics. In *Annales de l'IHP Physique théorique*, volume 17, pages 337–362, 1972.

[16] Valentine Bargmann. On unitary ray representations of continuous groups. *Annals of Mathematics*, pages 1–46, 1954.

[17] C Duval and HP Kunzle. Newtonian connections and nontrivial extension of galilei group. *Comptes rendus herbomadaires des seances de l'academie des sciences serie A*, 285(12):813–816, 1977.

[18] Christian Duval, GW Gibbons, PA Horvathy, and PM Zhang. Carroll versus newton and galilei: two dual non-einsteinian concepts of time. *Classical and Quantum Gravity*, 31(8):085016, 2014.

[19] Christian Duval, GW Gibbons, and PA Horvathy. Conformal carroll groups and bms symmetry. *Classical and Quantum Gravity*, 31(9):092001, 2014.

[20] Andrea Barducci, Roberto Casalbuoni, and Joaquim Gomis. Confined dynamical systems with carroll and galilei symmetries. *Physical Review D*, 98(8):085018, 2018.

[21] Josef Klusoň. Canonical analysis of non-relativistic particle and superparticle. *The European Physical Journal C*, 78(2):1–9, 2018.

[22] Josef Klusoň. Nonrelativistic string theory sigma model and its canonical formulation. *The European Physical Journal C*, 79(2):108, 2019.

[23] Machiko Hatsuda and Makoto Sakaguchi. Wess-zumino term for the ads superstring and generalized inönü-wigner contraction. *Progress of theoretical physics*, 109(5):853–867, 2003.

[24] Eric Bergshoeff, José Manuel Izquierdo, Tomás Ortín, and Luca Romano. Lie algebra expansions and actions for non-relativistic gravity. *Journal of High Energy Physics*, 2019(8):48, 2019.

[25] Eric A Bergshoeff, Kevin T Grosvenor, Ceyda cSimcsek, and Ziqi Yan. An action for extended string newton-cartan gravity. *Journal of High Energy Physics*, 2019(1):178, 2019.

[26] Eric A Bergshoeff, Jaume Gomis, Jan Rosseel, Ceyda Simsek, and Ziqi Yan. String theory and string newton–cartan geometry. *Journal of Physics A: Mathematical and Theoretical*, 53(1):014001, 2019.

[27] J.A. de Azcrraga, J.M. Izquierdo, M. Picn, and O. Varela. Generating lie and gauge free differential (super)algebras by expanding maurercartan forms and chernsimons supergravity. *Nuclear Physics B*, 662(1-2):185219, Jul 2003.

[28] Marika Taylor. Non-relativistic holography. *arXiv preprint arXiv:0812.0530*, 2008.

[29] Joaquim Gomis, Axel Kleinschmidt, Jakob Palmkvist, and Patricio Salgado-Rebolledo. Symmetries of post-galilean expansions. *Physical Review Letters*, 124(8):081602, 2020.

[30] Eric Bergshoeff, Joaquim Gomis, Blaise Rollier, Jan Rosseel, and Tonnis Ter Veldhuis. Carroll versus galilei gravity. *Journal of High Energy Physics*, 2017(3):165, 2017.

[31] Jelle Hartong and Niels A Obers. Hořava-lifshitz gravity from dynamical newton-cartan geometry. *Journal of High Energy Physics*, 2015(7):155, 2015.

[32] Troels Harmark, Jelle Hartong, and Niels A Obers. Nonrelativistic strings and limits of the ads/cft correspondence. *Physical Review D*, 96(8):086019, 2017.

[33] Eric Bergshoeff, Jaume Gomis, and Ziqi Yan. Nonrelativistic string theory and t-duality. *Journal of High Energy Physics*, 2018(11):133, 2018.

[34] Troels Harmark, Jelle Hartong, Lorenzo Menculini, Niels A Obers, and Gerben Oling. Relating non-relativistic string theories. *Journal of High Energy Physics*, 2019(11):71, 2019.

[35] Yusuke Nishida and Dam T Son. Nonrelativistic conformal field theories. *Physical Review D*, 76(8):086004, 2007.

[36] Eric A Bergshoeff, Jelle Hartong, and Jan Rosseel. Torsional newton–cartan geometry and the schrödinger algebra. *Classical and Quantum Gravity*, 32(13):135017, 2015.

[37] Morten H Christensen, Jelle Hartong, Niels A Obers, and Blaise Rollier. Torsional newton-cartan geometry and lifshitz holography. *Physical Review D*, 89(6):061901, 2014.

[38] Dam Thanh Son. Newton-cartan geometry and the quantum hall effect. *arXiv preprint arXiv:1306.0638*, 2013.

[39] Eric Bergshoeff, Joaquim Gomis, and Giorgio Longhi. Dynamics of carroll particles. *Classical and Quantum Gravity*, 31(20):205009, 2014.

[40] CF Yastremiz. Galilean extended objects. *Classical and Quantum Gravity*, 9(11):2395, 1992.

[41] Joaquim Gomis and Paul K Townsend. The galilean superstring. *Journal of High Energy Physics*, 2017(2):105, 2017.

[42] Carles Batlle, Joaquim Gomis, Luca Mezincescu, and Paul K Townsend. Tachyons in the galilean limit. *Journal of High Energy Physics*, 2017(4):120, 2017.

[43] Terence Tao. *Topics in random matrix theory*, volume 132. American Mathematical Soc., 2012.