university of groningen

faculty of science and engineering

Master Project Mathematics

# Uniqueness of the infinite open cluster on the stationary distributions of interacting particle systems


*T.M. van Belle*

*s2766582*



**Abstract.** In classical percolation models the sites (or bonds) are declared open independently with a parameter $p$. The stationary distributions of interacting particle systems give a correlated field of 0's and 1's on $\mathbb{Z}^d$, which can be used to determine open and closed sites. The resulting percolation models have a long-range dependence. In this thesis we consider the stationary distributions of the voter model and the contact process. For both percolation models, it will be shown that the infinite open connected component is almost surely unique.



supervised by

prof. dr. D. Rodrigues Valesin

prof. dr. T. Müller

# Contents

# 1 Introduction

In statistical physics one often deals with questions like: "Assume that some liquid is poured on top of some porous material. Will the liquid be able to make its way from hole to hole and reach the bottom?". A commonly used strategy to solve such problems, is to introduce mathematical models that describe the physical problem in a useful yet simple way. In this case the problem can be modelled mathematically as a three-dimensional network of $n \times n \times n$ vertices, usually called sites. The edges, often referred to as bonds, between two neighbouring sites may be open or closed. The open bonds let the liquid flow through, whereas the closed bonds block the path of the liquid. Consider a system where each edge is declared open with probability $p$, and closed with probability $1 - p$. A natural question then is, for a given $p$, what is the probability that there exists an open path from the top to the bottom? This problem is referred to as bond percolation, and was introduced in mathematical literature by Broadbent and Hammersley [2] in 1957. The problem has been studied intensively by many mathematicians and physicists since then. In a slightly different model, known as site percolation, sites are occupied or empty. If a site is empty, all edges incident to the site are closed. Now consider a system where sites are occupied with probability $p$, and empty with probability $1 - p$. In this case, one is still interested in the probability that there exists an open path from the top to the bottom.

These percolation models can be extended to any lattice, or in fact any graph. Instead of considering a three-dimensional network of $n \times n \times n$ vertices, mathematicians often study the $d$-dimensional lattice $\mathbb{Z}^d$. In this case, one is interested in the existence of an infinite open cluster, so that there is a path of connected points of infinite length in the network. Another modification that can be made lies in the way sites (edges for bond percolation) are declared occupied (closed). Classically the sites are declared occupied independently, all with probability $p$. For the classical percolation models it has been shown that there exists a critical value $p_c$ so that for $p < p_c$, there almost surely does not exist an infinite open cluster, and for $p > p_c$, there almost surely exists an infinite open cluster [2]. Furthermore, it has been shown that in the supercritical case, that is $p > p_c$, the infinite open cluster is almost surely unique [3].

Interacting particle systems are Markov-processes describing the behaviour of stochastically interacting components. Examples of interacting particle systems are the voter model, the contact process, the asymmetric simple exclusion process (ASEP), the Glauber dynamics and the stochastic Ising model. This paper considers a percolation procedure that is not produced by making independent and identically distributed decisions on what to declare occupied. Rather, it is made from a highly correlated field on $\mathbb{Z}^d$, which is given by stationary distributions of the voter model and the contact process.

The voter model is an interacting particle system that serves as a rough model for changes of opinions among social agents. One can imagine that there is a "voter" at each vertex of a connected graph, where the connections indicate that some form of interaction between a pair of voters. The opinions of any given voter changes at random times under the influence of opinions of their neighbours. A voter's opinion at any given time can take one of two values, labelled 0 and 1. At random times, a random individual is selected and that voter's opinion is changed according to a stochastic rule. In case the connected graph is the usual nearest-neighbour lattice on $\mathbb{Z}^d$, it has been shown that the set of extremal stationary distributions of the process is given by $\{\mu_\alpha : \alpha \in [0,1]\}$ [8] . Here the measures $\mu_\alpha$ are defined as the distributional limit as time is taken to infinity of the voter model with the random initial configuration in which the states of all sites are independent and Bernoulli($\alpha$). Problems involving the voter model are often reformulated in terms of the dual system of coalescing random walks. The measures $\mu_\alpha$ be expressed in terms of the law of such a system as well.

The contact process is an interacting particle system that serves as a rough model for the spread of a infectious disease among social agents. One can imagine that there is an agent at each vertex of a connected graph, where the connections indicate that some form of interaction between a pair of agents. The healthy agents become infected at a rate proportional to the number of infected neighbours, while infected agents become healthy at a constant rate. Therefore, if we denote by $\lambda$ the proportionality constant, each agent remains infected for a random time period which is exponentially distributed parameter 1 and infects neighbouring agents at times of events of a Poisson process parameter $\lambda$ during this period. All processes are independent of one another and of the random period of time sites remains occupied. One can define the upper invariant measure $\mu_\lambda$ as the distributional limit of the contact process started from an initial configuration in which every agent is infected. The set of measures $\{\mu_\lambda : \lambda \in (0,\infty)\}$ can more easily be found using the self-duality of the contact process, which relates the process started from an initial configuration in which every agent is infected to a contact process started from an initial configuration in which only the agent at the origin is infected.

One can wonder whether vital results for independent percolation still hold in case we consider the measures $\mu_\alpha$ or $\mu_\lambda$ instead of the measures associated to a product Bernoulli distribution on $\mathbb{Z}^d$. In 1985 it was already known that the contact process exhibits a non-trivial phase-transition [8]. That is, there exists a critical value $\lambda_c$ so that for $\lambda < \lambda_c$, there almost surely does not exist an infinite open cluster, and for $\lambda > \lambda_c$, there almost surely exists an infinite open cluster. Recently, Valesin and Ráth have shown that for $d \geq 5$ there exists a phase transition for percolation on the stationary distributions of the voter model [11]. That is there exists a critical value $\alpha_c$ so that for $\alpha < \alpha_c$, there almost surely does not exist an infinite open cluster, and for $\alpha > \alpha_c$, there almost surely exists an infinite open cluster. In this paper, we will study percolation on the stationary distributions of the voter model and the contact process. In particular it will be shown that in the supercritical phase, the infinite open cluster is almost surely unique.

## 1.1 Overview of Markov processes and transition kernels

In this section, a brief overview of Markov processes and transition kernels will be given. A complete overview can be found in [12].

**Definition 1.1.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \mathcal{E})$ a measurable space. A stochastic process is a family $X = (X_t)_{t \in I}$ of random elements $X_t : \Omega \to E$, for $t$ in some set $I$.

In case $I \subseteq \mathbb{N}$ the family $X$ is called a discrete-time process, in case $I$ is an interval in $\mathbb{R}$, $X$ is called a continuous-time process. If the set $I$ consists of the non-negative real numbers, it is customary to write $(X_t)_{t \geq 0}$ for such a process. It will be useful to have a notion of the distribution of a process.

**Definition 1.2.** Let $X = (X_t)_{t \in I}$ be a stochastic process taking values in some measurable space $(E, \mathcal{E})$. For each $n \in \mathbb{N}$ and each $n$-tuple $(t_1, \ldots, t_n)$ of distinct elements of $I$, the distribution of $(X_{t_1}, \ldots, X_{t_n})$ is a probability measure on $(E^n, \mathcal{E}^n)$, given by:

$$\mu_{(X_{t_1}, \ldots, X_{t_n})}(A) = \mathbb{P}\left((X_{t_1}, \ldots, X_{t_n}) \in A\right) \text{ where } A \in \mathcal{E}^n.$$

The set of all measures of this form is called the set of finite-dimensional distributions of the process X.

An interesting property is the evolution of the process through time, also known as the trajectories of the process.

**Definition 1.3.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where we have a stochastic process $X = (X_t)_{t \geq 0}$ taking values in some measurable set $(E, \mathcal{E})$. The trajectories of the process are the functions

$$t \to X_t(\omega) \text{ for } \omega \in \Omega$$

Often-times Markov processes are defined in terms of their transition kernels.

**Definition 1.4.** Let $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ be measurable spaces. A probability kernel from the first space to the second is a function $K : E_1 \times \mathcal{E}_2 \to [0, 1]$ such that

- for each $x \in E_1$, the function $A \to K(x, A)$ is a probability measure on $\mathcal{E}_2$.
- for each $A \in \mathcal{E}_2$, the function $x \to K(x, A)$ is measurable function with respect to $E_1$.

In case $(E_1, \mathcal{E}_1) = (E_2, \mathcal{E}_2) = (E, \mathcal{E})$, we simply say that $K$ is a probability kernel on $(E, \mathcal{E})$.

Given a probability kernel $K$ and a bounded and measurable function $f : E_2 \to \mathbb{R}$, define $Kf : E_1 \to \mathbb{R}$ by

$$(Kf)(x) = \int_{E_2} f(y) K(x, \mathrm{d}y)$$

It can be shown that $Kf$ is measurable with respect to $\mathcal{E}_1$. Additionally, if $\mu$ is a probability measure on $\mathcal{E}_1$, define $\mu K : \mathcal{E}_2 \to [0,1]$ by

$$(\mu K)(A) = \int_{E_1} K(x, A)\mu(\mathrm{d}x)$$

Note that this gives a probability measure on $\mathcal{E}_2$. Finally, given a probability kernel $K_1$ from $(E_1, \mathcal{E}_1)$ to $(E_2, \mathcal{E}_2)$ and a probability kernel $K_2$ from $(E_2, \mathcal{E}_2)$ to $(E_3, \mathcal{E}_3)$, define $K_1 K_2 : E_1 \times \mathcal{E}_3 \to [0,1]$ by

$$K_1 K_2(x, A) = \int_{E_2} K_2(y, A) K_1(x, \mathrm{d}y).$$

It is then easy to check that $K_1 K_2$ defines a probability kernel.

**Definition 1.5.** A Markov semi-group on a measurable space $(E, \mathcal{E})$ is a family $(P_t)_{t \geq 0}$ of probability kernels on $(E, \mathcal{E})$ satisfying

$$P_t P_s = P_{t+s} \text{ for all } s, t \geq 0. \tag{1}$$

and

$$\lim_{t \downarrow 0} P_t = P_0 = 1.$$

Equation (1) is often referred to as the Chapman-Kolmogorov equation.

**Definition 1.6.** A filtration on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a family of $\sigma$-algebras $(\mathcal{F}_t)_{t \geq 0}$ such that

$$\mathcal{F}_t \subseteq \mathcal{F} \text{ and } \mathcal{F}_s \subseteq \mathcal{F}_t \text{ for all } 0 \leq s \leq t.$$

The structure $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ is called a filtered probability space. A process $(X_t)_{t \geq 0}$ is said to be adapted to the filtration if

$$X_t \text{ is } \mathcal{F}_t - \text{measurable for all } t \geq 0.$$

Intuitively, a process $X = (X_t)_{t \geq 0}$ is a Markov process if, to make a prediction at time $s$ on what is going to happen to the process in the future, it is useless to know anything more about the whole past up to time $s$ than the present state $X_s$. Mathematically this is represented as follows.

**Definition 1.7.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space, $X = (X_t)_{t \geq 0}$ an adapted process taking values in $(E, \mathcal{E})$, and $(P_t)_{t \geq 0}$ a Markov semi-group in $(E, \mathcal{E})$. The process $X$ is called a Markov process with respect to $(\mathcal{F}_t)_{t \geq 0}$ with semi-group $(P_t)_{t \geq 0}$ if, for any bounded and measurable function $f : E \to \mathbb{R}$ and any $s < t$,

$$\mathbb{E}(f(X_t) | \mathcal{F}_s) = P_{t-s}(X_s) \quad \mathbb{P} - \text{almost surely}$$

The probability measure given by the distribution of $X_0$, that is,

$$\mu(A) = \mathbb{P}(X_0 \in A), \text{ for } A \in \mathcal{B}(\mathbb{R})$$

is called the initial distribution of X.

### 1.1.1 Examples of Markov processes

The definitions in section 1.1 allow for the formal construction of some important examples of Markov processes.

**Example 1.8** (Discrete-time simple random walk on $\mathbb{Z}^d$)**.** Let $e_i$ denote the unit vector in the $i$-th direction, and define a random element $W : \Omega \to \mathbb{R}^d$ by

$$\mathbb{P}\left(W = e_i\right) = \mathbb{P}\left(W = -e_i\right) = \frac{1}{2d} \text{ for all } i \in \{1, \dots, d\}. \tag{2}$$

Let $W_1, W_2, \dots$ be an independent and identically distributed sequence of random elements, all with distribution as specified in equation (2). Finally define a stochastic process $Z = (Z_n)_{n\in\mathbb{N}}$ by

$$Z_0 = \vec{0} \text{ and for each } n \in \mathbb{N},\ Z_n = \sum_{i=1}^{n} W_i.$$

The process $Z$ is called a discrete-time simple random walk on $\mathbb{Z}^d$. It can be seen that $Z$ is a Markov process with semi-group $(P_n)_{n\in\mathbb{N}}$ defined by

$$P_1(x, y) = \frac{1}{2d} \text{ for all } x, y \in \mathbb{Z}^d \text{ such that } ||x - y||_1 = 1. \tag{3}$$

To find $P_n$ one can iteratively use

$$P_n(x, y) = \sum_{z\in\mathbb{Z}^d} P_{n-1}(x, z)P_1(z, y)$$

With a discrete-time process in mind, one can define a continuous-time analogue. In this case, the discrete-time process will be used as sequence of states to visit, and another sequence $U_1, U_2, \dots$ will be used as holding times. Let $c : \mathbb{Z}^d \to [0, \infty)$ be a function and $U_i \sim \mathrm{Exp}(c(Z_i))$ (with the convention that $\mathrm{Exp}(0)$ represents the distribution of a random variable that is identically equal to $\infty$). Given a discrete-time process $(Z_n)_{n\in\mathbb{N}}$, the continuous-time analogue $(X_t)_{t\geq 0}$ is defined by

$$X_t = Z_i \text{ for } i \in [T_i, T_{i+1}) \text{ where } T_i = \sum_{j=1}^{n} U_j \text{ for each } i \in \{1, 2, \dots\}. \tag{4}$$

Note that in order for the process $X$ to be a Markov process, the random variables $U_1, U_2, \dots$ need to be independent and their distributions should have the memoryless property, whence the exponential distribution. Note that the holding times may depend on the state of the process through the function $c$. Formally there are some requirements to ensure the continuous-time process is properly defined, however these conditions will not be discussed here. A complete overview can be found in [12].

**Example 1.9.** [Continuous-time simple random walk on $\mathbb{Z}^d$] Let $(Z_n)_{n\in\mathbb{N}}$ be a discrete-time simple random walk as defined in example 1.8. Let $c(x) = 1$ for all $x \in \mathbb{Z}^d$, that is independently of the state, $U_1, U_2, \cdots \sim \text{Exp}(1)$. Define the process $(X_t)_{t\geq 0}$ as in equation (4). This process is referred to as a continuous-time simple random walk on $\mathbb{Z}^d$. Since the exponential distribution is memoryless, the process $X$ is a Markov process. It can be shown that the Markov semi-group $(P_t)_{t\geq 0}$ associated to $X$ is defined by

$$P_t = e^{tQ}, \text{ where } Q(x,y) = P_1(x,y) - \mathbb{1}_{\{x=y\}}(x,y).$$

Here $P_1$ is the one-step transition kernel of the discrete-time process as given in equation (3).

# 2 Interacting particle systems

Let $S$ be a finite set, $\Lambda$ be a countable set, and $E \subset \Lambda \times \Lambda$ be the unordered edge set of the graph $(\Lambda, E)$. As usual, $S^\Lambda$ denotes the Cartesian product of $\Lambda$ copies of $S$, i.e. all elements $\eta \in S^\Lambda$ are of the form

$$\eta = (\eta(x))_{x\in\Lambda} \text{ where } \eta(x) \in S \text{ for all } x \in \Lambda.$$

Interacting particle systems are continuous-time Markov processes $H = (H_t)_{t\geq 0}$ with a state space of the form $S^\Lambda$ endowed with the Borel-$\sigma$-field on the product topology of $S$, that are defined in terms of maps from $\Lambda$ to $S$. That is, $(H_t)_{t\geq 0}$ is a Markov process such that at each time $t \geq 0$,

$$H_t = (H_t(x))_{x\in\Lambda} \text{ where } H_t(x) \in S \text{ for all } x \in \Lambda.$$

$H_t(x)$ is often referred to as the local state of $H$ at time $t$ and position $x$. The set $S$ is called the local state space, and $\Lambda$ is called the lattice. The evolution of a continuous-time Markov process is often characterised by its generator $G$. A generator is an operator acting on functions from the state space $S^\Lambda$ to the real line. If the lattice $\Lambda$ is finite, the generator is given by,

$$Gf(\eta) = \lim_{t\to 0} \frac{P_t f(\eta) - f(\eta)}{t} \text{ where } P_t f(\eta) = \sum_{\xi\in S^\Lambda} P_t(\eta,\xi) f(\eta).$$

Here $(P_t)_{t\geq 0}$ denotes the Markov semi-group of the process. For interacting particle systems the generator takes the form

$$Gf(\eta) = \sum_{m\in\mathcal{G}} r_m \left(f(m(\eta)) - f(\eta)\right),$$

where $\mathcal{G}$ is a set of local maps $m : S^\Lambda \to S^\Lambda$ that affect finitely many coordinates, and $(r_m)_{m\in\mathcal{G}}$ is a collection of non-negative constants called rates, that describe with which Poisson intensity the local map $m$ should be applied to $H_t$.

In the following subsections, formal constructions of interacting particle systems will be given using generators as well as Poisson point processes. Sections 2.1, 2.2, and 2.3 are based on Chapter 2 from "A course in interacting particle systems" by J.M. Swart [13].

## 2.1 Examples of interacting particle systems

### 2.1.1 The voter model

For each $x, y \in \Lambda$ the voter model map $\mathrm{vot}_{xy} : S^\Lambda \to S^\Lambda$ is defined as

$$\mathrm{vot}_{xy}(\eta) = \eta' \text{ where } \eta'(z) = \begin{cases} \eta(x) & \text{if } z = y \\ \eta(z) & \text{if } z \neq y \end{cases}$$

Applying $\text{vot}_{xy}$ to a configuration $\eta$ has as result that the local state of site $x$ is copied onto site $y$. The generator for the nearest neighbour voter model on $\Lambda$ is given by

$$G_{\text{vot}} f(\eta) = \sum_{(x,y)\in\mathcal{E}} \frac{1}{|\mathcal{N}_y|} \left[ f(\text{vot}_{xy}(\eta)) - f(\eta) \right]. \tag{5}$$

Here $\mathcal{E}$ denotes the set of all ordered pairs $(x, y)$ that correspond to an edge. That is

$$\mathcal{E} := \{(x, y) : \{x, y\} \in E\},$$

where $E$ is the edge set corresponding to the lattice $\Lambda$. The set $\mathcal{N}_y$ denotes the neighbourhood of the vertex $y$, and is defined by

$$\mathcal{N}_y := \{x \in \Lambda : \{x, y\} \in E\}.$$

The interpretation that has given the voter model its name describes the spread of opinions among a population. In this case, with rate one, an individual (site) becomes unsure what their opinion is, and asks for a randomly chosen neighbour to copy their opinion.

### 2.1.2 The contact process

The contact process is an interacting particle system with local state space $S = \{0, 1\}$. The process can be seen as a model for the spread of an infection. In this interpretation sites with local state 1 are said to be infected, whereas sites with local state 0 are said to be healthy. The dynamics of the contact process can be defined using two collections of maps. Firstly, for each $x, y \in \Lambda$ define an transmission map $\text{tra}_{xy} : \{0, 1\}^\Lambda \to \{0, 1\}^\Lambda$ as

$$\text{tra}_{xy}(\eta) = \eta' \text{ where } \eta'(z) = \begin{cases} \max\{\eta(x), \eta(y)\} & \text{if } z = y \\ \eta(z) & \text{if } z \neq y \end{cases}$$

For each $x \in \Lambda$ define a healing map $\text{hea}_x : \{0, 1\}^\Lambda \to \{0, 1\}^\Lambda$ as

$$\text{hea}_x(\eta) = \eta' \text{ where } \eta'(z) = \begin{cases} 0 & \text{if } z = x \\ \eta(z) & \text{if } z \neq x \end{cases}$$

It can be seen that if site $x$ is infected prior to the application of the map $\text{tra}_{xy}$, the site $y$ will also be infected after applying $\text{tra}_{xy}$. The map $\text{hea}_x$ can be seen as a healing potion for site $x$. That is, independent of the local state at $x$ prior to applying $\text{hea}_x$, after the application of $\text{hea}_x$, the local state at $x$ will be healthy. The generator for the nearest neighbour contact process on $\Lambda$ with infection rate $\lambda \geq 0$ and healing rate $\delta \geq 0$ is given by

$$Gf(\eta) = \lambda \sum_{(x,y)\in\mathcal{E}} \left[ f(\text{tra}_{xy}(\eta)) - f(\eta) \right] + \delta \sum_{x\in\Lambda} \left[ f(\text{hea}_x(\eta)) - f(\eta) \right].$$

Note that the ratio between $\lambda$ and $\delta$ determines the behaviour of the process. Therefore the generator in the equation above is often reformulated with $\delta = 1$. That is,

$$G_{\text{cont}} f(\eta) = \lambda \sum_{(x,y)\in\mathcal{E}} \left[f(\text{tra}_{xy}(\eta)) - f(\eta)\right] + \sum_{x\in\Lambda} \left[f(\text{hea}_x(\eta)) - f(\eta)\right]. \tag{6}$$

The formal construction of interacting particle systems as given above is not easy to deal with mathematically. Fortunately both systems can equivalently be defined in terms of Poisson point processes, which are better known objects.

### 2.2 Poisson point sets

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $S = \mathbb{R} \times \Lambda$ be a state space. Furthermore let $\mathcal{S}$ denote the Borel-$\sigma$-field on the product topology of $S$. A locally finite measure on $(S, \mathcal{S})$ is a measure $\mu$ such that $\mu(C) < \infty$ for all compact sets $C \in \mathcal{S}$. A random measure on $S$ is a function $\xi : \Omega \times \mathcal{S} \to [0, \infty]$ such that for fixed $\omega \in \Omega$, the function $\xi(\omega, \cdot)$ is a locally finite measure on $(S, \mathcal{S})$, and for fixed $A \in \mathcal{S}$, the function $\xi(\cdot, A)$ is measurable. Note that for all measurable functions $f : S \to [0, \infty]$, the integral $\int f \, \mathrm{d}\xi$ defines a random variable. It can be seen that there exists a unique measure $\nu$ so that

$$\int_S f(s) \, \mathrm{d}\nu(s) = \mathbb{E}\left(\int f \, \mathrm{d}\xi\right) = \int_\Omega \int_S f(s) \, \mathrm{d}\xi(\omega, s) \, \mathrm{d}\mathbb{P}(\omega).$$

The measure $\nu$ is often denoted by $\mathbb{E}(\xi)$ and is called the intensity of $\xi$. Let $\hat{\mathcal{S}}$ denote the set of measurable sets $A$ such that the closure of $A$ is compact.

**Proposition 2.1.** *Let $\mu$ be a locally finite measure on $(S, \mathcal{S})$. Then there exists a random measure $\xi$, unique in distribution, such that for any disjoint $A_1, \dots, A_n \in \hat{\mathcal{S}}$, the random variables $\xi(A_1), \dots, \xi(A_n)$ are independent and $\xi(A_i)$ is Poisson distributed with mean $\mu(A_i)$.*

*Proof.* The result follows from combining Lemma 10.1 and Proposition 10.4 in "Foundations of modern probability" by Olav Kallenberg [6]. $\square$

The random measure $\xi$ as in proposition 2.1 is called a Poisson point measure with intensity $\mu$. Note that for each $\omega \in \Omega$, $\xi(\omega, A) \in \mathbb{N}$ for each $A \in \hat{\mathcal{S}}$. Therefore for each $\omega \in \Omega$, $\xi(\omega, )$ is a locally finite counting measure. Note that each locally finite counting measure is of the form

$$\nu = \sum_{x\in\text{supp}(\nu)} n_x \delta_x. \tag{7}$$

Here $\text{supp}(\nu)$ denotes the support of the measure $\nu$, which is a locally finite subset of $S$. $n_x$ are positive integers and $\delta_x$ denotes the measure giving density one to $x$.

**Definition 2.2.** Let $\nu$ be a counting measure. Then $\nu$ is said to be simple if $n_x = 1$ for all $x \in \text{supp}(\nu)$ in equation (7).

A measure $\mu$ has an atom at $x$ if $\mu(\{x\}) > 0$. The measure is called atomless if it has no atoms. The aforementioned Proposition 10.4 in [6] shows the following.

**Lemma 2.3.** *Let $\xi$ be a Poisson point measure with locally finite intensity $\mu$. Then $\xi$ is almost surely simple if and only if $\mu$ is atomless.*

Lemma 2.3 implies that if $\mu$ is atomless, the Poisson point measure $\xi$ with intensity $\mu$ is completely characterised by its support $D = \mathrm{supp}(\xi)$. The set $D$ is a random set and is called a Poisson point set with intensity $\mu$. The following lemma shows that Poisson point sets on the half-line can be constructed using exponentially distributed random variables.

**Lemma 2.4.** *Let $(\tau_k)_{k\in\mathbb{N}_0}$ be real-valued random variables such that $\tau_0 = 0$ and $\sigma_k = \tau_k - \tau_{k-1} > 0$ for $k \geq 1$. Then $D = \{\tau_k : k \geq 1\}$ is a Poisson point set on $[0, \infty)$ with intensity $c\ell$ if and only if the random variables $(\sigma_k)_{k\in\mathbb{N}}$ are i.i.d. exponentially distributed with mean $c^{-1}$.*

*Proof.* See Lemma 10.17 from [6]. □

Note that Lemma 2.4 provides a connection between Poisson point sets and the continuous-time simple random walk as defined in example 1.9.

### 2.3 Poisson construction of interacting particle systems

Recall that interacting particle systems are Markov processes on a state space of the form $S^\Lambda$ where $S$ is a finite set. In this subsection a Poisson construction of interacting particle systems will be given in case $\Lambda$ is finite too. In this case, it is known that $S^\Lambda$ is a finite state space. Conditions under which interacting particle systems with countable possibly infinite lattices can be defined in terms of Poisson point sets will also be given.

Assume $\Lambda$ is finite, denote by $\mathcal{G}$ a set whose elements are maps $m : S^\Lambda \to S^\Lambda$, and let $(r_m)_{m\in\mathcal{G}}$ be non-negative constants. In addition, consider the measurable space $(\mathcal{G} \times \mathbb{R}, \sigma(\{\{m\} : m \in \mathcal{G}\}) \otimes \mathcal{B}(\mathbb{R}))$, and equip it with the measure defined by

$$\rho(\{m\} \times A) = r_m \ell(A) \text{ for } A \in \mathcal{B}(\mathbb{R}).$$

Let $D$ be a Poisson point set with intensity measure $\rho$. It can be seen that $\nu = \sum_{(m,t)\in D} \delta_t$ is a Poisson point measure on $\mathbb{R}$ with intensity $r\ell$, where $r = \sum_{m\in\mathcal{G}} r_m$. Since the Lebesgue measure $\ell$ is atomless, the Poisson point measure $\nu$ is simple by lemma 2.3. That is, for each $t \in \mathbb{R}$ there exists at most one map $m$ such that $(m, t) \in D$. If $r < \infty$, the Poisson point measure $\nu$ is locally finite as well. Then, the set

$$D_{s,u} = \{(m, t) \in D : t \in (s, u]\}$$

can be ordered as

$$D_{s,u} = \{(m_1, t_1), \dots, (m_n, t_n)\} \text{ with } t_1 < \dots < t_n.$$

The ordering of maps can be used to define a collection of random maps $(X_{s,u})_{s\leq u}$ by

$$X_{s,u} = m_n \circ \cdots \circ m_1.$$

In case the set $D_{s,u}$ is empty, define $X_{s,u}$ to be the identity map. It can be seen that

$$\lim_{t\downarrow s} X_{s,t} = X_{s,s} \overset{\text{a.s.}}{=} \text{id}.$$

Furthermore, for any real numbers $s \leq t \leq u$,

$$X_{t,u} \circ X_{s,t} = X_{s,u}.$$

That is, the collection of maps $(X_{s,t})_{s\leq t}$ is a stochastic flow. It can be seen that by definition, $X_{s,t}$ is right-continuous in both $s$ and $t$. Furthermore, $(X_{s,t})_{s\leq t}$ has independent increments.

**Proposition 2.5.** *Let $(X_{s,t})_{s\leq t}$ be a stochastic flow associated to a Poisson point set $D$. Let $H_0$ be an $S^\Lambda$-valued random variable, independent of $D$. Then,*

$$H_t = X_{0,t}(H_0) \text{ for } t \geq 0 \tag{8}$$

*defines a Markov process $H = (H_t)_{t\geq 0}$ with generator*

$$Gf(\eta) = \sum_{m\in\mathcal{G}} r_m \left[ f(m(\eta)) - f(\eta) \right].$$

*Proof.* Since $S^\Lambda$ is finite one can define,

$$P_t(\eta, \eta') = \mathbb{P}\left( X_{s,s+t}(\eta) = \eta' \right) \text{ for } t \geq 0.$$

Since the law of the Poisson point set $D$ is invariant under translations in time, the definition of $P_t$ does not depend on $s \in \mathbb{R}$. Using that $(X_{s,t})_{s\leq t}$ has independent increments and $H_0$ is independent of $D$, it can be seen that the finite-dimensional distributions of $H$ satisfy, for each sequence $t_0 = 0 < t_1 < \cdots < t_n$,

$$\mathbb{P}\left(H_0 = \eta_0, \ldots, H_{t_n} = \eta_n\right) = \mathbb{P}\left(H_0 = \eta_0\right) P_{t_1}(\eta_0, \eta_1) \cdots P_{t_n - t_{n-1}}(\eta_{n-1}, \eta_n)$$

In other words, the stochastic process $(H_t)_{t\geq 0}$ is a Markov process with semi-group $(P_t)_{t\geq 0}$. By the properties of the Poisson point set,

$$\mathbb{P}\left(|D_{0,t}| \geq 2\right) = O(t^2) \text{ as } t \downarrow 0$$

Furthermore,

$$\mathbb{P}\left(D_{0,t} = \{(m, s)\} \text{ for some } s \in (0, t]\right) = r_m t + O(t^2) \text{ as } t \downarrow 0.$$

Combining these observations, it follows that for any $f : S^\Lambda \to \mathbb{R}$, as $t \downarrow 0$,

$$P_t f(\eta) = \mathbb{E}\left(f(X_{0,t}(\eta))\right) = f(\eta) + t \sum_{m\in\mathcal{G}} r_m \left(f(m(\eta)) - f(\eta)\right) + O(t^2).$$

This shows that,

$$\sum_{m\in\mathcal{G}} r_m \left(f(m(\eta)) - f(\eta)\right) = \lim_{t\to 0} \frac{P_t f(\eta) - f(\eta)}{t} = Gf(\eta).$$

Therefore, the Markov process defined by equation (8) has the generator,

$$Gf(\eta) = \sum_{m\in\mathcal{G}} r_m \left(f(m(\eta)) - f(\eta)\right).$$

□

Note that the theory provided relies on the finiteness of the lattice $\Lambda$. Under some conditions, the theory can be extended to infinite lattices. A brief summary will be given here. For full details please consult section 4.3 of "A course in interacting particle systems" by J. M. Swart [13]. Consider processes whose generator can be represented in terms of local maps, i.e., maps that change the local state of finitely many sites only, using only information about finitely many sites. For any map $m : S^\Lambda \to S^\Lambda$ let

$$\mathcal{D}(m) := \{x \in \Lambda : \text{There exists } \eta \in S^\Lambda \text{ s.t. } m(\eta)(x) \neq \eta(x)\}$$

denote the set of lattice points whose values can possibly be changed by $m$. A point $y \in \Lambda$ is said to be $m$-relevant for some $x \in \Lambda$ if there exist $\eta, \eta' \in S^\Lambda$ such that

$$m(\eta)(x) \neq m(\eta')(y) \text{ and } \eta(z) = \eta'(z) \text{ for all } z \neq y,$$

i.e., changing the value of $\eta$ in $y$ may change the value of $m(\eta)$ in $x$. For $x \in \Lambda$, define

$$\mathcal{R}_x(m) := \{y \in \Lambda : y \text{ is } m - \text{relevant for } x\}.$$

Observe that if $x \notin \mathcal{D}(m)$, then $m(\eta)(x) = \eta(x)$ for all $x \in \Lambda$, and hence

$$\mathcal{R}_x(m) = \{x\} \text{ if } x \notin \mathcal{D}(m).$$

**Definition 2.6.** A map $m : S^\Lambda \to S^\Lambda$ is said to be local if it satisfies the following three conditions.

1. $\mathcal{D}(m)$ is finite
2. $\mathcal{R}_x(m)$ is finite for all $x \in \Lambda$
3. For each $x \in \Lambda$, if $\eta(y) = \eta'(y)$ for all $y \in \mathcal{R}_x(m)$, then $m(\eta)(x) = m(\eta')(y)$.

Note that it is possible that $\mathcal{D}(m)$ is non-empty but $\mathcal{R}_x(m) = \emptyset$ for all $x \in \mathcal{D}(m)$. The following theorem gives sufficient conditions under which the Poisson point construction of interacting particle systems holds for countably infinite lattices.

**Theorem 2.7.** *Let $\mathcal{G}$ be a countable set whose elements are local maps $m : S^\Lambda \to S^\Lambda$, let $(r_m)_{m\in\mathcal{G}}$ be non-negative constants satisfying*

$$\sup_{x\in\Lambda} \sum_{m\in\mathcal{G}} r_m \left(|\mathcal{R}_x(m)| + 1\right) \cdot \mathbb{1}_{\mathcal{D}(m)}(x) < \infty,$$

*and let $D$ be a Poisson point set on $\mathcal{G} \times \mathbb{R}$ with intensity $r_m dt$. Then, for each $\eta \in S^\Lambda$ and $s \le u$, the pointwise limit*

$$X_{s,u}(\eta) := \lim_{\bar{D}_n \uparrow D_{s,u}} X_{s,u}^{\bar{D}_n}(\eta)$$

*exists a.s. and does not depend on the choice of the finite sets $\bar{D}_n \uparrow D_{s,u}$. If $H_0$ is an $S^\Lambda$-valued random variable, independent of $D$, then*

$$H_t := X_{0,t}(H_0) \text{ for } t \ge 0$$

*defines a Feller process with semi-group $(P_t)_{t\ge 0}$ given by*

$$P_t(\eta, \cdot) := \mathbb{P}\left(X_{0,t}(\eta) \in \cdot\right) \text{ for } \eta \in S^\Lambda \text{ and } t \ge 0.$$

*Proof.* See Theorem 4.14 from "A course in interacting particle systems" by J.M. Swart [13]. □

### 2.3.1 Poisson construction of the voter model

Recall that the generator of the voter model is given by equation (5). That is, each map $\mathrm{vot}_{xy}$ is applied with Poisson intensity $\frac{1}{|\mathcal{N}_y|}$. Note that the maps $\mathrm{vot}_{xy}$ are local as

$$\mathcal{D}(\mathrm{vot}_{xy}) = \{y\} \text{ and } \mathcal{R}_y(\mathrm{vot}_{xy}) = \{x\},$$

since only the local state at $y$ changes, and it suffices to know the type at $x$ to predict the new local state of $y$. For each $(x, y) \in \mathcal{E}$, let $D^{x,y}$ be a Poisson point set on $\{\mathrm{vot}_{xy}\} \times \mathbb{R}$ with intensity $\rho$ given by

$$\rho\left(\{\mathrm{vot}_{xy}\} \times A\right) = \frac{1}{|\mathcal{N}_y|}\ell(A) \text{ for all } A \in \mathcal{B}(\mathbb{R}).$$

Note that $D^{x,y}$ contains all the times that the local state of $x$ is copied onto $y$. Let $\Lambda_1 \subset \Lambda_2 \subset \dots$ be an increasing sequence of sub-lattices of $\Lambda$ such that $\Lambda_n \uparrow \Lambda$. Then, let

$$\bar{D}_n = \bigcup_{(x,y)\in\mathcal{E}_n} D^{x,y},$$

where $\mathcal{E}_n$ denotes the ordered edge set induced by the sub-lattice $\Lambda_n$. It can be seen that $\bar{D}_n \uparrow D = \cup_{(x,y)\in\mathcal{E}} D^{x,y}$. Theorem 2.7 guarantees that the point-wise limit of the stochastic flow associated to $\bar{D}_n$ exists and defines a Feller process. A visual representation of the Poisson construction of the voter model is given in figure 1. In this figure, an

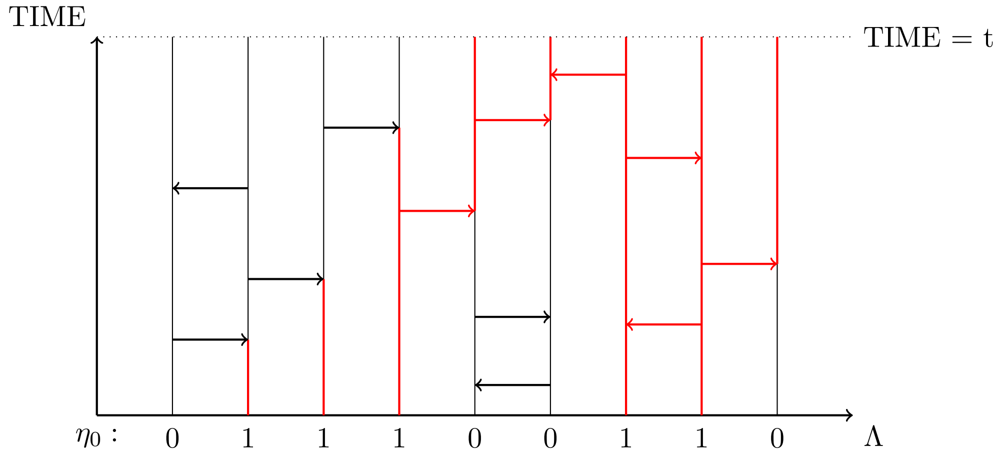


Figure 1: Poisson point set construction of the voter model on $\{0,1\}^\Lambda$ given initial configuration $\eta_0$. Local state 0 is represented in black and local state 1 in red.

arrow from $x$ to $y$ should be interpreted as the local state of $x$ is copied onto $y$, or equivalently applying the map $\mathrm{vot}_{xy}$. Given an initial configuration $\eta_0$, one can determine the configuration for any time $t$ by following the arrows; i.e. the process is completely determined by the collection of Poisson point sets $(D^{x,y})_{(x,y)\in\mathcal{E}}$.

The local state of a vertex $x$ at time $t$ can be traced back to the initial configuration by following the process in the reverse-time direction. That is, one follows the arrows in the opposite direction whenever possible. This creates paths in $\Lambda \times [0,\infty)$ space, which will be referred to as trajectories. Graphical examples are highlighted in Figure 2. These trajectories are local state preserving in a sense that for any two points $(y,s),(y',s')$ contained in the trajectory, the local state of $y$ at time $s$ equals the local state of $y'$ at time $s'$. The local state of all the points in a trajectory is determined by $\eta_0(y)$. Note that the rate at which a vertex $y$ loses its opinion equals one, as each of the $|\mathcal{N}_y|$ neighbours imposes its opinion to $y$ at rate $\frac{1}{|\mathcal{N}_y|}$. Lemma 2.4 guarantees that trajectories spend an exponentially distributed (with mean one) amount of time at each vertex they visit. Therefore, they are trajectories of a continuous-time simple random walk on $\Lambda$.

In Figure 2, it can be seen that the configuration at time $t$, $H_t$, can be determined by the initial configuration of the three vertices $x, y$, and $z$. In Figure 1, an initial configuration with $\eta_0(y) = \eta_0(z) = 1$ and $\eta_0(x) = 0$ was used to determine the evolution of the local states 0 and 1. It can be seen that the vertices that connect to either $y$ or $z$ (in Figure 2) indeed have local state 1 at time $t$, whereas vertices that connect to $x$ have local state 0 at time $t$. Trajectories as in Figure 2 can be generated by a system of coalescing random walks. That is, a continuous-time simple random walk is started at each vertex. Whenever two of these walks collide, i.e. they are in the same place at the same time, they will join and continue as one.

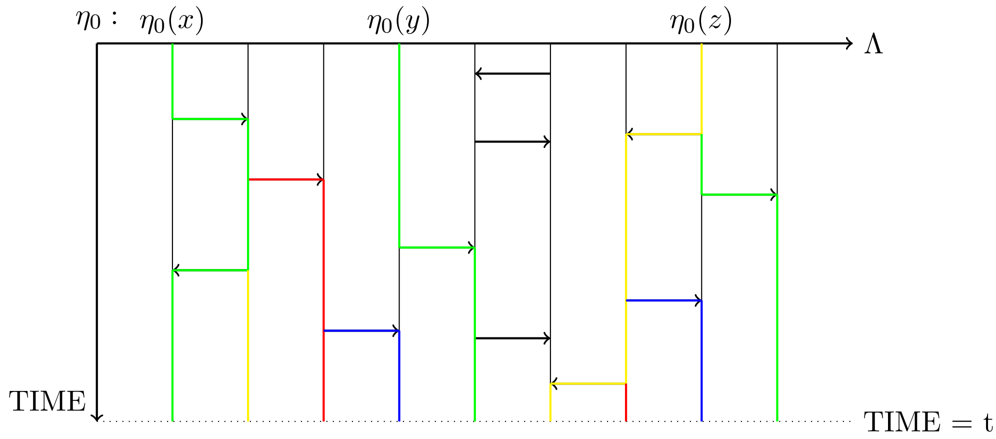


Figure 2: Reversed-time trajectories for the realisation of the voter model given in Figure 1.

**Definition 2.8.** A system of coalescing random walks on a lattice $\Lambda$ is a collection of stochastic processes

$$\{(Y_t^x)_{t\geq 0} : x \in \Lambda\}$$

That satisfy

1. For each $x \in \Lambda$, $(Y_t^x)_{t\geq 0}$ is a continuous-time simple random walk on $\Lambda$ started in $x$.
2. If there exist a $t \geq 0$ such that $Y_t^x = Y_t^y$ for some $x, y \in \Lambda$, then $Y_{t'}^x = Y_{t'}^y$ for all $t' \geq t$.

The voter model process $(H_t)_{t\geq 0}$ can equivalently be defined in terms of a system of coalescing random walks by

$$H_t(x) = \eta_0(Y_t^x) \text{ for all } x \in \Lambda \text{ and } t \in [0, \infty).$$

This relation is known as the coalescing duality. Another way to construct a system of coalescing random walks on $\mathbb{Z}^d$ will be given in section 2.4.1.

### 2.3.2 Poisson construction of the contact process

Recall that the generator of the contact process is given by equation (6). That is, each map $\text{tra}_{xy}$ is applied with Poisson intensity $\lambda$, and each map $\text{hea}_x$ is applied with Poisson intensity one. Note that the maps $\text{tra}_{xy}$ are local as

$$\mathcal{D}(\text{tra}_{xy}) = \{y\} \text{ and } \mathcal{R}_y(\text{tra}_{xy}) = \{x, y\},$$

since only the local state at $y$ changes, but we need to know both the local states of $x$ and $y$ to predict the new local state of $y$. Furthermore, the maps $\text{hea}_x$ are local as

$$\mathcal{D}(\text{hea}_x) = \{x\} \text{ and } \mathcal{R}_x(\text{hea}_x) = \emptyset,$$

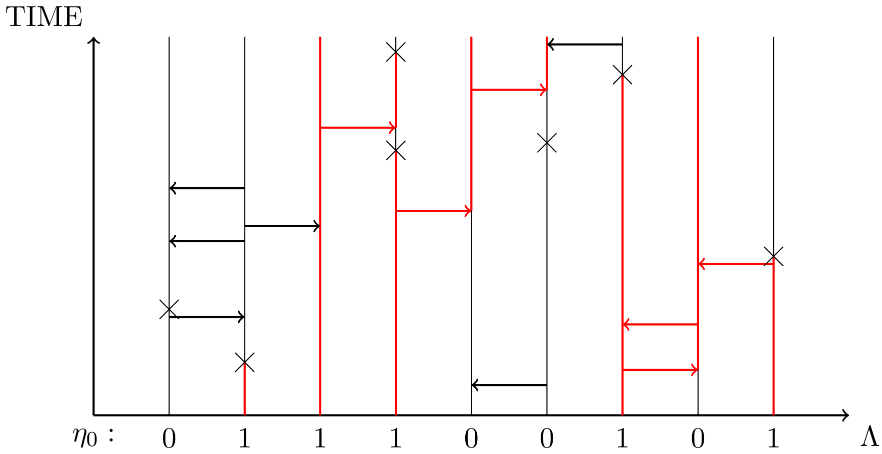


Figure 3: Poisson point set construction of the contact process given initial configuration $\eta_0$. Infected sites are represented in red and healthy sites are represented in black.

since only the local state at $x$ changes, and the new local state of $x$ is 0 regardless of $\eta$. For each $(x,y) \in \mathcal{E}$, let $D^{x,y}$ be a Poisson point set on $\{\mathrm{tra}_{xy}\} \times \mathbb{R}$ with intensity $\rho$ given by

$$\rho\left(\{\mathrm{tra}_{xy}\} \times A\right) = \lambda\ell(A) \text{ for all } A \in \mathcal{B}\left(\mathbb{R}\right).$$

For each $x \in \Lambda$, let $D^x$ be a Poisson point set on $\{\mathrm{hea}_x\} \times \mathbb{R}$ with intensity $\rho$ given by

$$\rho\left(\{\mathrm{hea}_x\} \times A\right) = \ell(A) \text{ for all } A \in \mathcal{B}\left(\mathbb{R}\right).$$

Note that $D^{x,y}$ contains all the times that $x$ is infects vertex $y$, and $D^x$ contains all the times that $x$ is healed. Let $\Lambda_1 \subset \Lambda_2 \subset \dots$ be an increasing sequence of sub-lattices of $\Lambda$ such that $\Lambda_n \uparrow \Lambda$. Then, let

$$\bar{D}_n = \left(\bigcup_{(x,y)\in\mathcal{E}_n} D^{x,y}\right) \bigcup \left(\bigcup_{x\in\Lambda_n} D^x\right).$$

It can be seen that $\bar{D}_n \uparrow D = \left(\cup_{(x,y)\in\mathcal{E}} D^{x,y}\right) \cup \left(\cup_{x\in\Lambda} D^x\right)$. Theorem 2.7 guarantees that the point-wise limit of the stochastic flow associated to $\bar{D}_n$ exists and defines a Feller process. A visual representation of the Poisson construction is given in figure 3. In this figure, an arrow from $x$ to $y$ should be interpreted as $x$ infects $y$, and a cross at $x$ corresponds to $x$ is healed. Similar to the voter model, one can determine the configuration for any time $t$ by following the arrows and crosses; i.e. the process is completely determined by the collections of Poisson point sets $(D^{x,y})_{(x,y)\in\mathcal{E}}$ and $(D^x)_{x\in\Lambda}$. An infection path in $\Lambda \times [0,\infty)$ is a connected oriented path which moves along the time lines in the increasing $t$ direction without passing through a recovery symbol, and along infection arrows in the direction of the arrow. If $x, y \in \Lambda$ and $0 \le s \le t$ then denote by $(x,s) \rightsquigarrow (y,t)$ the event that there is an infection path connecting $(x,s)$ to $(y,t)$.

**Definition 2.9.** A path in $\Lambda \times [0, \infty)$ is said to be an infection path if there exist

$$s = s_0 < s_1 < \cdots < s_{n+1} = t \text{ and } z = x_0, x_1, x_2, \ldots, x_n = x \in \Lambda$$

such that the following two conditions hold:

1. for $i = 1, 2, ..., n$, there is an arrow $x_{i-1} \to x_i$ at time $s_i$ and
2. for $i = 0, 1, ..., n$, there is no cross on the segment $\{x_i\} \times (s_i, s_{i+1})$.

If $x, y \in \Lambda$ and $0 \leq s \leq t$ then denote by $(x, s) \rightsquigarrow (y, t)$ the event that there is an infection path connecting $(x, s)$ to $(y, t)$. If this holds, we also say that there is a dual path $(y, t) \rightsquigarrow (x, s)$. In other words, dual paths are defined as paths except that they evolve backward in time and follow the arrows of the graphical representation in the opposite direction.

**Definition 2.10.** Define the $\{0, 1\}$-valued random variables, called infection path indicators

$$\Xi_t(x, y) =: \mathbb{1}_{[(x,0) \rightsquigarrow (y,t)]}, \text{ where } x, y \in \Lambda \text{ and } t \geq 0.$$

For any $\eta_0 \in \{0, 1\}^\Lambda$, the contact process $(H_t)_{t \geq 0}$ with infection rate $\lambda$ and initial state $\eta_0$ can be constructed by letting

$$H_t(y) = \max_{x \in \Lambda} \{\eta_0(x) \cdot \Xi_t(x, y)\}, \text{ for } y \in \Lambda \text{ and } t \geq 0.$$

Note that this definition is equivalent to the definition in terms of stochastic flows.

## 2.4 Stationary distribution of interacting particle systems

Let $H$ be an interacting particle system with transition kernel $(P_t)_{t \geq 0}$ given by

$$P_t(\eta, \cdot) = \mathbb{P}(H_t \in \cdot | H_0 = \eta) = \mathbb{P}^\eta(H_t \in \cdot)$$

A measure $\mu$ on $S^\Lambda$ is said to be stationary if

$$\mu P_t(\cdot) := \int_{S^\Lambda} P_t(\eta, \cdot) \mathrm{d}\mu(\eta) = \mu(\cdot) \text{ for all } t \geq 0.$$

Denote by $\mathbb{Z}^d$ the usual nearest-neighbor lattice on $\mathbb{Z}^d$, i.e., two points of $\mathbb{Z}^d$ are adjacent if they differ only in one coordinate, by 1. In the following sections, we are particularly interested in the stationary distributions of the given interacting particle systems on the state space $\{0, 1\}^{\mathbb{Z}^d}$.

### 2.4.1 System of coalescing random walks

To construct a system of coalescing walks, the set of vertices needs to be ordered. Note that there are many ways to orderer the vertices of $\mathbb{Z}^d$, however the chosen ordering does not affect the law of the system. For each vertex $x \in \mathbb{Z}^d$ process of coalescing random walks $(Y_t^x)_{t\geq 0}$ should comply with the following rules.

1. For each $x \in \mathbb{Z}^d$, $Y_0^x = x$
2. If there exist a $t \geq 0$ such that $Y_t^x = Y_t^y$ for some $x, y \in \mathbb{Z}^d$, then $Y_{t'}^x = Y_{t'}^y$ for all $t' \geq t$.

A way to construct this process $(Y_t)_{t\geq 0}$ is to use independent random walks started from each vertex, and revealing the paths of the random walks following the order of the vertices. Then, in each iteration, a path is revealed until it has met with one of the already known random walks. From this point on, it will simply follow (the already known) path of this random walk. Algorithm 1 gives a more mathematical definition of the process $Y$. In this algorithm the independent random walks started from each vertex $x$ are denoted by $X^x$.

**Algorithm 1** Coalescing Random walks

**Input**: $(X^x)_{x\in\mathbb{Z}^d}$ with ordering $\{x_1, x_2, \dots\}$
Set $Y_t^{x_1} = X_t^{x_1}$.
**for** $i$ in $\{1, 2, \dots\}$ **do**
  Find $\sigma = \min\{t \geq 0 : X_t^{x_i} = Y_t^x \text{ for some } x \in \{x_1, \dots, x_{i-1}\}\}$
  Find $y = \text{argmin}_{x\in\{x_1,\dots,x_{i-1}\}}\{t \geq 0 : X_t^{x_i} = Y_t^x\}$
  Let $Y_t^{x_i} = \begin{cases} X_t^{x_i} & \text{if } t < \sigma \\ Y_t^y & \text{if } t \geq \sigma \end{cases}$
**end for**

### 2.4.2 Stationary distribution of the voter model

For fixed $d \geq 1$ and $\alpha \in [0, 1]$, one defines a probability measure $\mu_\alpha$ on $\{0, 1\}^{\mathbb{Z}^d}$ as the distributional limit as time is taken to infinity, of the voter model with the random initial configuration in which the states of all sites are independent and Bernoulli($\alpha$). It has been shown that this distributional limit exists, see e.g. Lemma 1.15 in Chapter V from "Interacting particle systems" by T.M. Liggett [8]. The measure $\mu_\alpha$ is then stationary for the voter model dynamics. It can be shown that each measure $\mu_\alpha$ is invariant and ergodic with respect to translations of $\mathbb{Z}^d$, see Theorem 2.5 of Chapter V and Corollary 4.14 of Chapter I from [8].

**Definition 2.11.** Definition 4.10 Chapter I [8]. Let $\tau_x$ denote a shift in $\mathbb{Z}^d$. A translation invariant measure $\mu$ is called ergodic if $\tau_x f = f$ for all $x \in \mathbb{Z}^d$ and $f$ measurable implies that $f$ is almost surely constant with respect to $\mu$.

Note that the measures $\delta_{\underline{0}}$ and $\delta_{\underline{1}}$, i.e. the measures giving density one to the configuration zero (respectively one) everywhere, are stationary measures. It can also be seen that any convex combination of stationary measures defines another stationary measure. One is often interested in stationary distributions that are extremal, i.e. that cannot be expressed as non-trivial convex combinations of other stationary distributions. It has been shown that the set of extremal stationary distributions is exactly the family

$$\{\mu_\alpha : \alpha \in [0,1]\}.$$

Note that in case $d = 1$ or 2, this set consists only of $\delta_{\underline{0}}$ and $\delta_{\underline{1}}$, as the simple random walks on $\mathbb{Z}$ and $\mathbb{Z}^2$ are recurrent. Furthermore,

$$\mu_\alpha\left(\{\eta \in \{0,1\}^{\mathbb{Z}^d} : \eta(0) = 1\}\right) = \alpha,$$

so that $\alpha$ is a density parameter. The duality between the voter model and continuous-time simple random walks can be used to construct the measures $\mu_\alpha$. Given a realisation of a system of coalescing random walks $\{(Y_t^x)_{t\geq 0} : x \in \mathbb{Z}^d\}$, define the equivalence relation

$$x \sim y \iff \text{There exists } t > 0 \text{ such that } Y_t^x = Y_t^y.$$

An equivalence relation induces a partition of $\mathbb{Z}^d$ into equivalence classes. Instead of equivalence classes, they will be referred to as coalescence classes. For each coalescence class $[x] \in \mathbb{Z}^d/\sim$, let $\beta^{[x]}$ be a Bernoulli trial with parameter $\alpha$.

**Claim 2.12.** *Define*

$$\eta(x) := \beta^{[x]} \text{ for } x \in \mathbb{Z}^d.$$

*Then $(\eta(x))_{x\in\mathbb{Z}^d}$ has the law $\mu_\alpha$.*

*Proof.* Note that

$$\lim_{t\to\infty} H_t(x) = \lim_{t\to\infty} \eta_0\left(Y_t^x\right) = \beta^{[x]}$$

□

A way to perform Bernoulli trials per coalescence class uses of the order of the vertices. Let $(\beta^x)_{x\in\mathbb{Z}^d}$ denote a collection of independent Bernoulli trials with parameter $\alpha$. For each coalescence class $[x]$, let $x^* \in [x]$ denote the element of highest order in the equivalence class $[x]$, then let

$$\eta(x) = \beta^{x^*} \text{ for all } x \in [x^*] \tag{9}$$

so that the Bernoulli trial of the highest ordered element acts as the Bernoulli trial for the whole equivalence class, $\beta^{[x]}$.

### 2.4.3 Stationary distribution of the contact process

The upper invariant measure of the contact process, denoted $\mu_\lambda$, is defined as

$$\mu_\lambda := \lim_{t\to\infty} H_t^{\underline{1}}$$

where $\left(H_t^{\underline{1}}\right)_{t\geq 0}$ is the contact process started from the identically-one configuration, and the limit in distribution can be shown to exist using Theorem 2.3 from Chapter III in [8]. This distribution is invariant and ergodic with respect to translations of $\mathbb{Z}^d$, see Theorem 1.5 in Chapter VI in [8]. Note that the identically-zero element of $\{0,1\}^{\mathbb{Z}^d}$, denoted $\underline{0}$, is an absorbing state for the contact process. The following claim from [9] is useful to construct the stationary distribution of the contact process.

**Claim 2.13.** *Define*

$$\eta(x) =: \lim_{t\to\infty} \max_{y\in\mathbb{Z}^d} \Xi_t(x,y), \text{ for } x\in\mathbb{Z}^d.$$

*Then $(\eta(x))_{x\in\mathbb{Z}^d}$ has the law $\mu_\lambda$ of the upper invariant measure of the process with infection rate $\lambda$.*

*Proof of Claim 2.13.* The claim can be proven via the dual process.

**Definition 2.14.** The dual process starting at $(y,t)$ is defined in terms of dual paths as

$$\hat{H}_s(x,t) = \max_{y\in\Lambda} \mathbb{1}_{[(y,t)\rightsquigarrow(x,t-s)]} \text{ for all } 0\leq s\leq t.$$

It follows from the definition of paths and dual paths and from the construction of the contact process from the graphical representation that we have the following important property called duality relationship:

$$\begin{aligned} H_t^{\underline{1}}(x) = 1 &\iff H_0^{\underline{1}}(y)\cdot \max_{y\in\Lambda} \mathbb{1}_{[(y,0)\rightsquigarrow(x,t)]} = 1 \\ &\iff H_0^{\underline{1}}(y)\cdot \max_{y\in\Lambda} \mathbb{1}_{[(x,t)\rightsquigarrow(y,0)]} = 1 \\ &\iff \hat{H}_t(x,t) = 1 \end{aligned}$$

Therefore, $H_t^{\underline{1}} = \max_{y\in\Lambda} \Xi_t(x,y)$, and hence $\eta$ defined by

$$\eta(x) := \lim_{t\to\infty} \max_{y\in\mathbb{Z}^d} \Xi_t(x,y), \text{ for } x\in\mathbb{Z}^d.$$

has the law $\mu_\lambda$ of the upper invariant measure of the process with infection rate $\lambda$. □

# 3 Percolation theory

Percolation theory is a branch of probability theory that studies the process of removing edges, known as bond percolation, or vertices, known as site percolation, at random from a given graph, and studying the connected components that remain. In this thesis the focus will be on site percolation. Classically, this process is carried out on the infinite lattice $\mathbb{Z}^d$ and the vertices are declared open independently, all with probability $p \in [0, 1]$. Define the event Perc $\subset \{0,1\}^{\mathbb{Z}^d}$ which consists of those configurations $\eta \in \{0,1\}^{\mathbb{Z}^d}$ for which the sub-graph of the lattice $\mathbb{Z}^d$ spanned by the set of open sites $\{x : \eta(x) = 1\}$ has an infinite connected component. A principle quantity of interest is the percolation probability, being the probability that a given vertex belongs to an infinite open cluster [4]. For percolation on $\mathbb{Z}^d$ where the vertices are declared open independently with probability $p$, the percolation probability $\theta(p)$ can be defined by

$$\theta(p) = \mathbb{P}_p(|C(0)| = \infty),$$

where $C(x)$ denotes the open cluster containing the vertex $x$, and $\mathbb{P}_p$ is the measure associated to a product Bernoulli distribution with parameter $p$. It is fundamental to percolation theory that there exists a critical value $p_c = p_c(d)$ of $p$ such that

$$\theta(p) \begin{cases} = 0 & \text{if } p < p_c \\ > 0 & \text{if } p > p_c. \end{cases}$$

$p_c$ is called the critical probability and is formally defined by

$$p_c = \sup\{p : \theta(p) = 0\}.$$

Note that for $d = 1$, the critical probability $p_c(1) = 1$. This can be seen by assuming $p < 1$, and noting that this implies that there exist infinitely many closed vertices to the left and to the right of the origin almost surely, implying that $\theta(p) = 0$. For dimension two and higher, the situation is quite different.

**Theorem 3.1.** *If $d \geq 2$ then $p_c(d) \in (0, 1)$.*

This theorem shows that in two or more dimensions, there are two phases of the process. In the sub-critical phase where $p < p_c(d)$, every vertex is almost surely in a finite open cluster, hence all open clusters are almost surely finite. In the super-critical phase where $p > p_c(d)$, each vertex has a strictly positive probability of being in an infinite open cluster, hence there exists almost surely at least one open cluster.

**Theorem 3.2.** *The probability $\mathbb{P}_p(Perc)$ that there exists an infinite open cluster satisfies*

$$\mathbb{P}_p(Perc) = \begin{cases} 0 & \text{if } \theta(p) = 0, \\ 1 & \text{if } \theta(p) > 0. \end{cases}$$

*Proof.* Note that the event Perc does not depend on the states of any finite collection of vertices. By the usual zero-one law, $\mathbb{P}_p(\text{Perc})$ takes the values 0 and 1 only. If $\theta(p) = 0$ then

$$\mathbb{P}_p(\text{Perc}) \leq \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(|C(x)| = \infty) = 0$$

□

In case there exists an infinite component, that is, $\mathbb{P}_p(\text{Perc}) = 1$, it can also be shown to be unique. Let $N : \{0,1\}^{\mathbb{Z}^d} \to \{0, 1, 2, \ldots, \infty\}$ be a random variable denoting the number of infinite open clusters.

**Theorem 3.3.** *If $p$ is such that $\theta(p) > 0$, then*

$$\mathbb{P}_p(\{\eta \in \{0,1\}^{\mathbb{Z}^d} : N(\eta) = 1\}) = 1$$

This theorem is a vital result in percolation theory and was shown by Burton and Keane [3].

Instead of considering the event Perc, one can also be interested in the infinite connected component of the vacant set. Define the event $\text{Perc}^* \subset \{0,1\}^{\mathbb{Z}^d}$ which consists of those configurations $\eta \in \{0,1\}^{\mathbb{Z}^d}$ for which the sub-graph of the lattice $\mathbb{Z}^d$ spanned by the set of closed sites $\{x : \eta(x) = 0\}$ has an infinite connected component. In case of i.i.d. percolation, it can be seen that there exists a critical probability $p_c^* = 1 - p_c$ so that

$$\mathbb{P}_p(\text{Perc}^*) = \begin{cases} 0 & \text{if } p > p_c^*, \\ 1 & \text{if } p < p_c^*. \end{cases}$$

Here we used the symmetric nature of the Bernoulli trials that decide whether or not a site is closed. Also, for $p < p_c^*$,

$$\mathbb{P}_p(\{\eta \in \{0,1\}^{\mathbb{Z}^d} : N^*(\eta) = 1\}) = 1$$

where $N^* : \{0,1\}^{\mathbb{Z}^d} \to \{0, 1, 2, \ldots, \infty\}$ is a random variable denoting the number of infinite closed clusters.

## 3.1 Percolation on stationary distributions of interacting particle systems

Rather than by making independent and identically distributed decisions on which vertices to declare closed, one can use other methods to distribute the open and closed vertices. One of these methods uses the stationary distributions of interacting particle systems, which gives a highly correlated field on $\mathbb{Z}^d$.

### 3.1.1 Percolation on the stationary distributions of the voter model

Let $\mu_\alpha$ denote the stationary distribution of the voter model with the random initial configuration in which the states of all sites are independent and Bernoulli($\alpha$) distributed. The value of interest is the probability that the event Perc occurs. Note that the family $\{\mu_\alpha\}$ is stochastically increasing: in the partial order on $\{0,1\}^{\mathbb{Z}^d}$ induced by the order $0 < 1$ on the coordinates, $\mu_\alpha$ is stochastically dominated by $\mu_{\alpha'}$ when $\alpha < \alpha'$. By this stochastic ordering, $\mu_\alpha(\text{Perc})$ is non-decreasing in $\alpha$. Define the critical probability $\alpha_c$ as the supremum of all the values of $\alpha$ for which $\mu_\alpha(\text{Perc}) = 0$. It has been shown that in dimension five or higher, the family of measures $\{\mu_\alpha : \alpha \in [0,1]\}$ exhibits a non-trivial percolation phase transition, see Theorem 1.1 in [11].

**Theorem 3.4.** *If $d \geq 5$, there exists $\alpha_c \in (0,1)$ such that for $\alpha < \alpha_c$, $\mu_\alpha(Perc) = 0$, and for $\alpha > \alpha_c$, $\mu_\alpha(Perc) = 1$.*

Due to the symmetric nature of the Bernoulli trials, it can be seen that if $d \geq 5$, there exists $\alpha_c^* = 1 - \alpha_c$ such that for $\alpha < \alpha_c^*$, $\mu_\alpha\left(\text{Perc}^*\right) = 1$ and for $\alpha > \alpha_c^*$, $\mu_\alpha\left(\text{Perc}^*\right) = 0$

### 3.1.2 Percolation on the stationary distributions of the contact process

Let $\mu_\lambda$ denote the stationary distribution of the contact process with infection rate $\lambda$ and initial configuration $\eta_0(x) = 1$ for all $x \in \mathbb{Z}^d$. The value of interest is again the probability that the event Perc occurs. The survival probability of the infection is

$$\rho(\lambda) =: \mathbb{P}_\lambda\left(H_t^{\{0\}} \neq \underline{0} \text{ for all } t \geq 0\right),$$

where $\mathbb{P}_\lambda$ is a probability measure under which the contact process on $\mathbb{Z}^d$ with infection rate $\lambda$ is defined and $\left(H_t^{\{0\}}\right)_{t\geq 0}$ is the contact process started with a single infected site at the origin. It can be seen that the function $\lambda \to \rho(\lambda)$ is non-decreasing. Using this observation, one then defines the critical infection rate $\lambda_c$ as

$$\lambda_c =: \sup\{\lambda > 0 : \rho(\lambda) = 0\}$$

It is also known that

$$\rho\left(\lambda_c\right) = 0 \text{ and } \lim_{\lambda \downarrow \lambda_c} \rho(\lambda) = 0$$

The first equality is the celebrated result by Bezuidenhout and Grimmett [1]. The second equality is a consequence of the first and the fact that $\lambda \to \rho(\lambda)$ is continuous on $[\lambda_c, \infty)$ [8]. Moreover,

$$\mu_\lambda(\{\eta \in \{0,1\}^{\mathbb{Z}^d} : \eta(0) = 1\}) = \rho(\lambda)$$

It has been shown (see Theorem 1.6 in Chapter VI of [8]) that the contact process exhibits a non-trivial phase transition.

**Theorem 3.5.** *There exists $\lambda_c \in (0,\infty)$ such that for $\lambda < \lambda_c$, $\mu_\lambda(Perc) = 0$, and for $\lambda > \lambda_c$, $\mu_\lambda(Perc) = 1$.*

For $\lambda \leq \lambda_c$, the measure $\mu_\lambda$ is the Dirac measure concentrated on the identically-zero configuration denoted by $\delta_{\underline{0}}$. For $\lambda > \lambda_c$, the measure $\mu_\lambda$ is a non-trivial measure supported on configurations with infinitely many infected sites.

The contact process does not have a symmetry like the voter model, therefore some work is required to show the vacant set (i.e. the closed sites) exhibits a non-trivial phase transition. Note that for $\lambda \leq \lambda_c$,

$$\mu_\lambda \left(\mathrm{Perc}^*\right) = \delta_{\underline{0}} \left(\mathrm{Perc}^*\right) = 1.$$

# 4 Uniqueness of the infinite open cluster

## 4.1 Uniqueness of the infinite open cluster for percolation on stationary distributions of the voter model

Let $\{\mu_\alpha : \alpha \in [0,1]\}$ denote the family of stationary distributions of the voter model. In this section it will be shown that for any $\alpha \in [0,1]$ there exists at most one infinite open cluster.

**Theorem 4.1.** *For any $\alpha \in [0,1]$ there exists $\mu_\alpha$-almost surely at most one infinite open component in $\mathbb{Z}^d$.*

By combining Theorem 4.1 with Theorem 3.4 it can be seen that

**Corollary 4.2.** *If $d \geq 5$, for any $\alpha > \alpha_c$, there exists $\mu_\alpha$-almost surely a unique infinite open component in $\mathbb{Z}^d$.*

Furthermore the symmetry in Bernoulli trials can be used to obtain

**Corollary 4.3.** *If $d \geq 5$, for any $\alpha < \alpha_c^*$, there exists $\mu_\alpha$-almost surely a unique infinite closed component in $\mathbb{Z}^d$.*

Let $N$ be the number of infinite open components in $\mathbb{Z}^d$. Since $\mu_\alpha$ is ergodic and invariant under translations of $\{0,1\}^{\mathbb{Z}^d}$, it can be seen that $N$ is $\mu_\alpha$-almost surely constant. That is

$$\mu_\alpha (N = k) = 1 \text{ for some } k \in \{0, 1, \ldots, \infty\}. \tag{10}$$

The value of $k$ is dependent on the choice of $\alpha$. This result allows us to rule out certain values of $k$. We will prove Theorem 4.1 with the aid of the following two lemmas

**Lemma 4.4.** *For any $\alpha \in (0,1)$,*

$$\mu_\alpha (N = k) = 0 \textit{ for all } 2 \leq k < \infty.$$

**Lemma 4.5.** *For any $\alpha \in (0,1)$,*

$$\mu_\alpha (N = \infty) = 0.$$

*Proof of Theorem 4.1.* If $\alpha = 0$ it can easily be deduced that

$$\mu_0 (N = 0) = \delta_{\underline{0}} (N = 0) = 1.$$

Similarly if $\alpha = 1$ it can be seen that

$$\mu_1 (N = 1) = \delta_{\underline{1}} (N = 1) = 1.$$

By Lemma 4.4, the value of $k$ in equation 10 satisfies $k \in \{0, 1, \infty\}$. By Lemma 4.5 we also have that $k \neq \infty$, hence for any $\alpha \in [0,1]$,

$$\mu_\alpha (N = k) = 1 \text{ for some } k \in \{0, 1\}.$$

That is there exists $\mu_\alpha$-almost surely at most one infinite open component in $\mathbb{Z}^d$. $\square$

*Proof of Lemma 4.4.* For any finite set of vertices let $N_B(0)$ denote the number of infinite open clusters if all vertices in $B$ are declared to be closed. Similarly let $N_B(1)$ denote the number of infinite open clusters if all vertices in $B$ are declared to be open.

**Claim 4.6.** *Let $B$ be a finite subset of $\mathbb{Z}^d$ and let $0 < \alpha < 1$. Then,*

$$\mu_\alpha \left( \{\eta \in \{0,1\}^{\mathbb{Z}^d} : \eta(x) = 0 \text{ for all } x \in B\} \right) > 0.$$

*Similarly,*

$$\mu_\alpha \left( \{\eta \in \{0,1\}^{\mathbb{Z}^d} : \eta(x) = 1 \text{ for all } x \in B\} \right) > 0.$$

*Proof of Claim 4.6.* This claim will be proven via the coalescence duality. Recall that for the system of coalescing random walks, an ordering of the vertices is needed, and the choice of ordering does not influence the distribution. Pick an ordering of vertices in $\mathbb{Z}^d$ such that any vertex in $B$ has higher order than any vertex in $\mathbb{Z}^d \setminus B$. Note that

$$\mathbb{P}_\alpha \left( \beta^x = 0 \text{ for all } x \in B \right) = (1-\alpha)^{|B|},$$

where $|B|$ denotes the cardinality of the set $B$, and $\mathbb{P}_\alpha$ denotes the measure associated to a product Bernoulli distribution with parameter $\alpha$. Let $x \in B$, as the vertices in $B$ have priority over vertices outside of $B$, the highest ordered element $x^*$ of the coalescence class $[x]$ is an element of $B$. Therefore,

$$\beta^x = 0 \text{ for all } x \in B \implies \beta^{x^*} = 0 \text{ for all } x \in B$$

By equation 9,

$$\beta^{x^*} = 0 \text{ for all } x \in B \iff \eta(x) = 0 \text{ for all } x \in B$$

Therefore for $0 < \alpha < 1$,

$$\mu_\alpha \left( \{\eta \in \{0,1\}^{\mathbb{Z}^d} : \eta(x) = 0 \text{ for all } x \in B\} \right) \geq (1-\alpha)^{|B|} > 0$$

By a similar argument for $0 < \alpha < 1$,

$$\mu_\alpha \left( \{\eta \in \{0,1\}^{\mathbb{Z}^d} : \eta(x) = 1 \text{ for all } x \in B\} \right) \geq \alpha^{|B|} > 0.$$

□

From Claim 4.6 and the almost sure constantness of $N$ (Equation (10)), it also follows that

$$\mu_\alpha \left( N_B(0) = N_B(1) = k \right) = 1 \text{ for some } k \in \{0, 1, \ldots, \infty\}.$$

Note that under the assumption that $k < \infty$, $N_B(0) = N_B(1) = k$ if and only if $B$ intersects at most one infinite open cluster. Therefore,

$$\mu_\alpha \left( M_B \geq 2 \right) = 0.$$

Here $M_B$ denotes the amount of clusters intersecting $B$. It can be seen that $M_B$ is non-decreasing in $B$, and $M_B$ grows to $N$ as $B$ grows towards $\mathbb{Z}^d$. By taking $B$ to be the diamond $S(n) = \{x \in \mathbb{Z}^d : |x| \leq n\}$ and taking the limit as $n$ tends to infinity, it can be seen that

$$0 = \mu_\alpha \left(M_{S(n)} \geq 2\right) \to \mu_\alpha \left(N \geq 2\right).$$

Therefore, $\mu_\alpha \left(N \geq 2\right) = 0$. $\square$

*Proof of Lemma 4.5.* Assume that $k$ in equation (10) satisfies $k = \infty$. That is

$$\mu_\alpha \left(N = \infty\right) = 1.$$

A contradiction will be derived using a geometrical argument. The following definition will be useful for the argument.

**Definition 4.7.** A point $y \in \mathbb{Z}^d$ is said to be a trifurcation if it belongs to an infinite open component of $\{x \in \mathbb{Z}^d ; \eta(x) = 1\}$ which is split into three disjoint infinite components by the removal of $y$.

For $x \in \mathbb{Z}^d$, let $T_x$ denote the event that $x$ is a trifurcation. Note that $\mu_\alpha \left(T_x\right)$ is constant for all $x \in \mathbb{Z}^d$, and therefore

$$\frac{1}{|S(n)|} \mathbb{E}_\alpha \left( \sum_{x \in S(n)} \mathbb{1}_{T_x} \right) = \mu_\alpha \left(T_0\right), \tag{11}$$

where $T_0$ denotes the event that the origin is a trifurcation. It will be useful to show that the origin is a trifurcation with positive probability. Let $M_B$ be the number of infinite open clusters that intersect $B$, and let $M_B(0)$ denote the number of infinite open clusters that intersect $B$ if all vertices in $B$ are closed. Note that under the assumption that $k = \infty$,

$$\mu_\alpha \left(M_{S(n)}(0) \geq 3\right) \geq \mu_\alpha \left(M_{S(n)} \geq 3\right) \xrightarrow{n \to \infty} \mu_\alpha \left(N \geq 3\right) = 1.$$

Therefore we can fix $n \in \mathbb{N}$ such that

$$\mu_\alpha \left(M_{S(n)}(0) \geq 3\right) \geq \frac{1}{2}$$

The exterior boundary of $S(n)$ is defined by

$$\partial_{\text{ext}} S(n) = \{x \notin S(n) : x \in \mathcal{N}_y \text{ for some } y \in S(n)\}.$$

Note that if the event $\{M_{S(n)}(0) \geq 3\}$ occurs, there exist $z_1, z_2, z_3 \in \partial_{\text{ext}} S(n)$, lying in distinct open infinite clusters of $\mathbb{Z}^d$. It can be seen that there exist paths $\gamma_1, \gamma_2$ and $\gamma_3$ such that

$$\bigcup_{i \in \{1,2,3\}} \text{range } \gamma_i \subset S(n) \cup \{z_1, z_2, z_3\},$$

and for each $i \neq j$

$$\text{range } \gamma_i \bigcap \text{range } \gamma_j = \{0\},$$

and for all $i \in \{1, 2, 3\}$, $\gamma_i$ is a path starting at the origin and ending in $z_i$. Let $J_z$ denote the event that all the vertices in these paths are open, and that all other vertices in $S(n)$ are closed. Note that

$$\begin{aligned}
\mu_\alpha (T_0) &\geq \mu_\alpha \left(J_z | M_{S(n)}(0) \geq 3\right) \cdot \mu_\alpha \left(M_{S(n)}(0) \geq 3\right) \\
&\geq \frac{1}{2} \mu_\alpha \left(J_z | M_{S(n)}(0) \geq 3\right)
\end{aligned}$$

Therefore, we aim to show that $\mu_\alpha \left(J_z | M_{S(n)}(0) \geq 3\right)$ is positive. This will be done via the coalescence duality. Note that we can choose any order of the vertices of $\mathbb{Z}^d$. We choose an order $\succ$ on $\mathbb{Z}^d$ satisfying,

$$x \in \partial_{\text{ext}} S(n) \implies x \succ y \text{ for all } y \notin \partial_{\text{ext}} S(n)$$

and

$$x \notin \bar{S}(n) \implies x \succ y \text{ for all } y \in S(n)$$

here $\bar{S}(n) = S(n) \cup \partial_{\text{ext}} S(n)$. Note that by choosing this ordering, the event $\{M_{S(n)}(0) \geq 3\}$ can be decided by the collection of random walks $(X_t^x)_{x \notin S(n)}$, and the Bernoulli trials $(\beta^x)_{x \notin S(n)}$. For each $z \in \partial_{\text{ext}} S(n)$ define,

$$A_z^\delta = \{X_t^z = z \text{ for all } t \leq \delta\}$$

Let $\mathbb{P}$ denote the measure under which the random walks $(X_t^z)_{z \in \mathbb{Z}^d}$ are defined and note that for each $z \in \mathbb{Z}^d$, $\mathbb{P}(X_t^z = z) \xrightarrow{t \to 0} 1$. Therefore, there exists a $\delta > 0$ such that

$$\mathbb{P}\left( \bigcap_{z \in \partial_{\text{ext}} S(n)} A_z^\delta \right) > \frac{1}{2}.$$

Therefore, there exists a $\delta > 0$ so that the event $\{M_S(n)(0) \geq 3\}$ and $\cap_{z \in \partial_{\text{ext}} S(n)} A_z^\delta$ happen simultaneously with positive probability. Since $z_1, z_2, z_3$ is lie in distinct open clusters of $\mathbb{Z}^d$, there exists $x \in \partial_{\text{ext}} S(n)$ such that $x$ is closed. Let

$$\mathcal{H} = S(n) \setminus \left( \bigcup_{i \in \{1,2,3\}} \text{range}(\gamma_i) \right).$$

Then, for each $z \in S(n) \setminus \{0\}$ define

$$A_z^\delta = \begin{cases} \{X_t^z \in \text{range } \gamma_1 \text{ for all } t \leq \delta\} \cap \{\exists t \leq \delta : X_t^z = z_1\} & \text{if } z \in \text{range } \gamma_1 \\ \{X_t^z \in \text{range } \gamma_2 \text{ for all } t \leq \delta\} \cap \{\exists t \leq \delta : X_t^z = z_2\} & \text{if } z \in \text{range } \gamma_2 \\ \{X_t^z \in \text{range } \gamma_3 \text{ for all } t \leq \delta\} \cap \{\exists t \leq \delta : X_t^z = z_3\} & \text{if } z \in \text{range } \gamma_3 \\ \{X_t^z \in \mathcal{H} \cup \{x\} \text{ for all } t \leq \delta\} \cap \{\exists t \leq \delta : X_t^z = x\} & \text{if } z \in \mathcal{H} \end{cases}$$

For the origin, we define

$$A_0^\delta = \{X_t^0 \in \text{range } \gamma_1 \cup \{z_1\} \text{ for all } t \leq \delta\} \cap \{\exists t \leq \delta : X_t^0 = z_1\}.$$

The events $\left(A_z^\delta\right)_{z \in S(n)}$ are independent, as they can be decided using the independent continuous time random walks $(X_t^z)_{z \in S(n)}$. Therefore,

$$\mathbb{P}\left(\bigcap_{z \in S(n)} A_z^\delta\right) = \prod_{z \in S(n)} \mathbb{P}\left(A_z^\delta\right) > 0,$$

as for all $\delta > 0$ and any $z \in S(n)$, $\mathbb{P}\left(A_z^\delta\right) > 0$. Note that the events $\left(A_z^\delta\right)_{z \in S(n)}$ are also independent of $\left(A_z^\delta\right)_{z \in \partial_{\text{ext}} S(n)}$ and $\{M_{S(n)}(0) \geq 3\}$. Therefore,

$$\mu_\alpha\left(J_z | M_{S(n)}(0) \geq 3\right) > 0.$$

This implies that the origin is a trifurcation with positive probability. It follows from 11 that the expected number of trifurcations inside $S(n)$ grows similarly as $|S(n)|$ as $n \to \infty$. Select a trifurcation $t_1$ in $S(n)$, and choose a vertex $y_1 \in \partial_{\text{ext}} S(n)$ such that $t_1$ and $y_1$ in are in the same infinite open component. Then, select another trifurcation $t_2$ in $S(n)$. Using the definition of a trifurcation, it can be seen that there exists $y_2 \in \partial S(n)$ such that $y_2 \neq y_1$ and $t_2$ and $y_2$ are in the same infinite open component. Continuing this process, that is at each stage pick a new trifurcation $t_k \in S(n)$ and a new vertex $y_k \in \partial_{\text{ext}} S(n)$. If there exist $K$ trifurcations in $S(n)$, $K$ distinct vertices $y_k \in \partial_{\text{ext}} S(n)$ are obtained. Hence, $|\partial_{\text{ext}} S(n)| \geq K$. However, $\mathbb{E}_\alpha(K)$ is comparable to $|S(n)|$. Since $|\partial_{\text{ext}} S(n)|$ grows like $n^{d-1}$ and $|S(n)|$ grows like $n^d$, this is a contradiction. □

The result of Theorem 4.1 has some interesting consequences for the system of coalescing random walks.

**Corollary 4.8.** *Let $\mathcal{B}$ denote the set of infinite connected components of coalescence classes of the system of coalescing random walks. Then the following statements hold:*

- $\mathbb{P}(|\mathcal{B}| \geq 2) = 0$
- *For $d \geq 5$,* $\mathbb{P}(|\mathcal{B}| = 1) = 0$

*Proof.* Suppose $\mathbb{P}(|\mathcal{B}| \geq 2) > 0$, then it can be seen that

$$\mu_\alpha(N \geq 2) \geq \alpha^2 \cdot \mathbb{P}(|\mathcal{B}| \geq 2) > 0.$$

This contradicts Lemma 4.4, therefore $\mathbb{P}(|\mathcal{B}| \geq 2) = 0$. Now suppose $d \geq 5$ and $\mathbb{P}(|\mathcal{B}| = 1) > 0$, then, for any $\alpha \in (0, 1)$,

$$\mu_\alpha(N = 1) \geq \alpha \cdot \mathbb{P}(|\mathcal{B}| = 1) > 0.$$

Hence, there exists $0 < \alpha < \alpha_c$ such that,

$$\mu_\alpha(N = 1) > 0.$$

Since this contradicts Theorem 3.4, $\mathbb{P}(|\mathcal{B}| = 1) = 0$. □

From the corollary, one can conclude that in dimensions five and higher, there are almost surely no infinite connected components of coalescence classes. The following lemma will be proven as it is closely related to corollary 4.8, however the result is of independent interest.

**Lemma 4.9.** *All coalescence classes of the system of coalescing random walks are almost surely infinite.*

*Proof.* For each $t > 0$ define the equivalence relation $\sim_t$ by

$$x \sim_t y \iff \text{There exists } t' \leq t \text{ such that } Y_{t'}^x = Y_{t'}^y.$$

The equivalence classes related to the equivalence relation $\sim_t$ will be denoted by $[\cdot]_t$. Assume the coalescence class of the origin is finite with positive probability, i.e.

$$\mathbb{P}\left(|[0]| < \infty\right) > 0.$$

This implies that with positive probability, there exists some $t_0$ such that the coalescence class does not change after time $t_0$. That is,

$$\delta := \mathbb{P}(\exists t_0 \geq 0 : \ [0]_t = [0]_{t_0} \text{ for all } t \geq t_0) > 0;$$

we will derive a contradiction from this assumption. Note that

$$\delta = \lim_{t_0 \to \infty} \mathbb{P}([0]_t = [0]_{t_0} \text{ for all } t \geq t_0),$$

we can fix $t_0 > 0$ such that

$$\mathbb{P}([0]_t = [0]_{t_0} \text{ for all } t \geq t_0) > 0.$$

By the union bound, the probability on the left-hand side is bounded from above by

$$\sum_{y \in \mathbb{Z}^d} \mathbb{P}([0]_t = [0]_{t_0} \text{ for all } t \geq t_0, \ Y_{t_0}^0 = y),$$

so there exists a site $y$ for which the probability inside the above sum is strictly positive. That is,

$$\mathbb{P}([0]_t = [0]_{t_0} \text{ for all } t \geq t_0, \ Y_{t_0}^0 = y) > 0.$$

This also gives

$$\delta' := \mathbb{P}(\exists x : \ [x]_t = [x]_{t_0} \text{ for all } t \geq t_0, \ Y_{t_0}^x = y) > 0.$$

By translation invariance, the value of $\delta'$ does not depend on $y$, so

$$\mathbb{P}(\exists x : \ [x]_t = [x]_{t_0} \text{ for all } t \geq t_0, \ Y_{t_0}^x = 0) = \delta'. \tag{12}$$

Now, fix $t_1 > t_0$ and define the events

$$\begin{aligned} &A_{x,y,z} := \left\{ [x]_t = [x]_{t_0} \text{ for all } t \geq t_0,\ Y^x_{t_0} = y,\ Y^x_{t_1} = z \right\}, \quad x, y, z \in \mathbb{Z}^d, \\ &A'_{y,z} = \cup_x A_{x,y,z}, \quad y, z \in \mathbb{Z}^d. \end{aligned}$$

Note that we can now rewrite equation (12) as:

$$\mathbb{P}\left(\cup_{x,z} A_{x,0,z}\right) = \mathbb{P}(\cup_z A'0, z) = \delta'. \tag{13}$$

Moreover, it is easy to check that

$$y_1 \neq y_2 \quad \Longrightarrow \quad A'_{y_1,z} \cap A'_{y_2,z} = \varnothing, \tag{14}$$

$$z_1 \neq z_2 \quad \Longrightarrow \quad A'_{y,z_1} \cap A'_{y,z_2} = \varnothing. \tag{15}$$

We are now ready to estimate:

$$\rho_{t_1} = \mathbb{P}(\exists x \in \mathbb{Z}^d\ Y^x_{t_1} = 0) \geq \mathbb{P}(\cup_y A'(y, 0)).$$

Using equation (14), it can be seen that

$$\mathbb{P}(\cup_y A'(y, 0)) = \sum_{y \in \mathbb{Z}^d} \mathbb{P}(A'(y, 0)).$$

By translation invariance, the right-hand side equals

$$\sum_z \mathbb{P}(A'(0, z)) \overset{(15)}{=} \mathbb{P}(\cup_z A'(0, z)) = \mathbb{P}(\cup_{x,z} A(x, 0, z)) \overset{(13)}{=} \delta'.$$

This implies that $\rho_{t_1} > \delta' > 0$ for all $t_1 > 0$, which contradicts Van den Berg and Kesten [15] who have shown that $\rho_t \to 0$. □

### 4.2 Uniqueness of the infinite open cluster for percolation on stationary distributions of the contact process

In this section it will be shown that in the supercritical phase of percolation on the stationary distribution of the contact process there exists at most one infinite open cluster.

**Theorem 4.10.** *For any $\lambda > \lambda_c$ there exists $\mu_\lambda$-almost surely a unique infinite open component in $\mathbb{Z}^d$.*

Let $N$ be the number of infinite open components in $\mathbb{Z}^d$. Since $\mu_\lambda$ is ergodic and invariant under translations on $\{0, 1\}^{\mathbb{Z}^d}$, it can be seen that $N$ is $\mu_\lambda$-almost surely constant. That is

$$\mu_\lambda\left(N = k\right) = 1 \text{ for some } k \in \{0, 1, \ldots, \infty\}. \tag{16}$$

Naturally the value of $k$ is dependent on the choice of $\lambda$. Similar to the proof of Theorem 4.1, we will use the following to lemmas to proof Theorem 4.10.

**Lemma 4.11.** *For any $\lambda \in (\lambda_c, \infty)$,*

$$\mu_\lambda (N = k) = 0 \textit{ for all } k \in \{2, 3, \dots\}.$$

**Lemma 4.12.** *For any $\lambda \in (\lambda_c, \infty)$,*

$$\mu_\lambda (N = \infty) = 0.$$

*Proof of Theorem 4.10.* Suppose $\lambda > \lambda_c$. By Lemma 4.11, the value of $k$ in equation 16 satisfies $k \in \{1, \infty\}$. By Lemma 4.12 we also have that $k \neq \infty$, hence for any $\lambda \in (\lambda_c, \infty)$,

$$\mu_\lambda (N = 1) = 1,$$

that is, there exists $\mu_\lambda$-almost surely a unique infinite open component in $\mathbb{Z}^d$. □

The proof of lemma 4.11 is quite similar to the proof of lemma 4.4. The only part that needs revision, it the proof of the claim.

**Claim 4.13.** *Let $B$ be a finite subset of $\mathbb{Z}^d$. Then for any $\lambda \in [0, \infty)$,*

$$\mu_\lambda \left( \{\eta \in \{0,1\}^{\mathbb{Z}^d} : \eta(x) = 0 \textit{ for all } x \in B\} \right) > 0.$$

*For any $\lambda > \lambda_c$,*

$$\mu_\lambda \left( \{\eta \in \{0,1\}^{\mathbb{Z}^d} : \eta(x) = 1 \textit{ for all } x \in B\} \right) > 0.$$

*Proof of Claim 4.13.* Note that for $\lambda \leq \lambda_c$, the $\mu_\lambda$ equals $\delta_{\underline{0}}$, so that in this case

$$\mu_\lambda \left( \{\eta \in \{0,1\}^{\mathbb{Z}^d} : \eta(x) = 0 \text{ for all } x \in \mathbb{Z}^d\} \right) = 1$$

As $\{\eta : \eta(x) = 0 \text{ for all } x \in B\}$ is a subset of $\{\eta : \eta(x) = 0 \text{ for all } x \in \mathbb{Z}^d\}$, it is clear that

$$\mu_\lambda \left( \{\eta \in \{0,1\}^{\mathbb{Z}^d} : \eta(x) = 0 \text{ for all } x \in B\} \right) = 1 > 0$$

Recall that the survival probability $\rho(\lambda)$ gives the probability that the infection survives given it started in one vertex. Therefore,

$$\mu_\lambda \left( \{\eta \in \{0,1\}^{\mathbb{Z}^d} : \eta(x) = 0 \text{ for all } x \in B\} \right) \geq (1 - \rho(\lambda))^{|B|}$$

and

$$\mu_\lambda \left( \{\eta \in \{0,1\}^{\mathbb{Z}^d} : \eta(x) = 1 \text{ for all } x \in B\} \right) \geq \rho(\lambda)^{|B|}$$

Noting that for $\lambda > \lambda_c$, $\rho(\lambda) > 0$ gives the desired result. □

*Proof of Lemma 4.12.* Assume that $k$ in equation (16) satisfies $k = \infty$. That is

$$\mu_\lambda (N = \infty) = 1.$$

Similar to the proof of lemma 4.5, a geometrical argument will be used to derive a contradiction. Recall that $T_x$ denotes the event that $x$ is a trifurcation and note that $\mu_\lambda (T_x)$ is constant for all $x \in \mathbb{Z}^d$, and therefore

$$\frac{1}{|S(n)|}\mathbb{E}_\lambda \left( \sum_{x \in S(n)} \mathbb{1}_{T_x} \right) = \mu_\lambda (T_0), \tag{17}$$

This suggest we can use the same geometrical argument as for the voter model, if we manage to show that the origin is a trifurcation with positive probability. Note that also for the contact process we can pick $n \in \mathbb{N}$ such that

$$\mu_\lambda \left( M_{S(n)}(0) \geq 3 \right) \geq \frac{1}{2}$$

Furthermore,

$$\begin{aligned} \mu_\lambda (T_0) &\geq \mu_\lambda \left( J_z | M_{S(n)}(0) \geq 3 \right) \cdot \mu_\lambda \left( M_{S(n)}(0) \geq 3 \right) \\ &\geq \frac{1}{2} \mu_\lambda \left( J_z | M_{S(n)}(0) \geq 3 \right) \end{aligned}$$

Therefore, we aim to show that $\mu_\lambda \left( J_z | M_{S(n)}(0) \geq 3 \right)$ is positive. The self-duality of the contact process will be used to proof this. Recall the Poisson point constuction of the contact process, where the process is defined in terms of random sets $(D^x)_{x \in \Lambda}$ and $(D^{x,y})_{(x,y) \in \mathcal{E}}$. For $z \in \partial_{\text{ext}} S(n)$ define the events

$$A_z^\delta = \{ D_{0,\delta}^z = \emptyset \} \cap \{ D_{0,\delta}^{z,y} = \emptyset \text{ for all } y \in \mathcal{N}_z \}$$

Note that

$$\mathbb{P} \left( \bigcap_{z \in \partial_{\text{ext}} S(n)} A_z^\delta \right) = \prod_{z \in \partial_{\text{ext}} S(n)} \mathbb{P} \left( D_{0,\delta}^z = \emptyset \right) \cdot \mathbb{P} \left( D_{0,\delta}^{z,y} = \emptyset \text{ for all } y \in \mathcal{N}_z \right)$$

For all $z \in \partial_{\text{ext}} S(n)$,

$$\mathbb{P} \left( D_{0,\delta}^z = \emptyset \right) \xrightarrow{\delta \to 0} 1 \text{ and } \mathbb{P} \left( D_{0,\delta}^{z,y} = \emptyset \text{ for all } y \in \mathcal{N}_z \right) \xrightarrow{\delta \to 0} 1$$

Therefore, there exists a $\delta > 0$ such that

$$\mathbb{P} \left( \bigcap_{z \in \partial_{\text{ext}} S(n)} A_z^\delta \right) \geq \frac{1}{2},$$

So that the events $\cap_{z\in\partial_{\text{ext}}S(n)}A_z^\delta$ and $\{M_{S(n)(0)}\geq 3\}$ occur simultaneously with positive probability. For $z\in\mathcal{H}$ define the events

$$A_z^\delta=\{D_{0,\delta}^z\neq\emptyset\}\cap\{D_{0,\delta}^{z,y}=\emptyset \text{ for all } y\in\mathcal{N}_z\}$$

Since $S(n)$ is finite, $\gamma_1,\gamma_2$, and $\gamma_3$ are finite paths. For $i\in\{1,2,3\}$ let

$$L_i:=|\text{range}(\gamma_i)| \text{ and } L=\max_{i\in\{1,2,3\}}L_i$$

For $i\in\{1,2,3\}$ we write $\{0=x_1^i,x_2^i,\dots x_{L_i}^i=z_i\}$ for the collection of ordered vertices in the range of $\gamma_i$. For $x_j^i$, $i\in\{1,2,3\}$ and $j\in\{2,3,\dots,L_i-1\}$ define the events

$$A_{x_j^i}^\delta=\{D_{0,\delta}^{x_j^i}=\emptyset\}\cap\{D_{j\epsilon,(j+1)\epsilon}^{x_j^i,x_{j+1}^i}\}.$$

where $\epsilon=\delta/L$. For the origin we define the event

$$A_0^\delta=\{D_{0,\delta}^0=\emptyset\}\cap\{D_{0,\epsilon}^{0,x_2^1}\}\cap\{D_{0,\epsilon}^{0,x_2^2}\}\cap\{D_{0,\epsilon}^{0,x_2^3}\}$$

Note that the events $\left(A_z^\delta\right)_{z\in S(n)}$ are independent, as they are determined by independent Poisson point sets. By noting that all these events have positive probability as well we obtain that

$$\mathbb{P}\left(\bigcap_{z\in S(n)}A_z^\delta\right)=\prod_{z\in S(n)}\mathbb{P}\left(A_z^\delta\right)>0.$$

Therefore,

$$\mu_\lambda\left(J_z|M_{S(n)}(0)\geq 3\right)>0.$$

The proof can now be completed using the same geometrical argument as in the proof of Lemma 4.5. $\square$

## References


[1] Carol Bezuidenhout and Geoffrey Grimmett. The critical contact process dies out. *The Annals of Probability*, pages 1462–1482, 1990.

[2] Simon R Broadbent and John M Hammersley. Percolation processes: I. crystals and mazes. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 53, pages 629–641. Cambridge University Press, 1957.

[3] Robert M Burton and Michael Keane. Density and uniqueness in percolation. *Communications in mathematical physics*, 121(3):501–505, 1989.

[4] Geoffrey Grimmett. *Percolation.* Springer, 1989.

[5] Richard A Holley and Thomas M Liggett. Ergodic theorems for weakly interacting infinite systems and the voter model. *The annals of probability*, pages 643–663, 1975.

[6] Olav Kallenberg. *Foundations of modern probability.* Springer, 1997.

[7] Nicolas Lanchier. *Stochastic modeling.* Springer, 2017.

[8] T. Liggett. *Interacting particle systems.* Springer, 1985.

[9] Balázs Ráth and Daniel Valesin. On the threshold of spread-out contact process percolation. *arXiv preprint arXiv:1912.09825*, 2019.

[10] Balázs Ráth, Daniel Valesin, et al. On the threshold of spread-out voter model percolation. *Electronic Communications in Probability*, 22, 2017.

[11] Balázs Ráth, Daniel Valesin, et al. Percolation on the stationary distributions of the voter model. *The Annals of Probability*, 45(3):1899–1951, 2017.

[12] D. Rodrigues Valesin. Lecture notes stochastic processes, 2019.

[13] Jan M Swart. A course in interacting particle systems. *arXiv preprint arXiv:1703.10007*, 2017.

[14] Augusto Teixeira et al. On the uniqueness of the infinite cluster of the vacant set of random interlacements. *The Annals of Applied Probability*, 19(1):454–466, 2009.

[15] J Van den Berg and Harry Kesten. Asymptotic density in a coalescing random walk model. *Annals of Probability*, pages 303–352, 2000.