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Proving the Soundness of a Proof System for Intuitionistic Hybrid Propositional Logic

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Abstract

At least two proof systems for modal hybrid intuitionistic logic have been shown to be complete, which is a promising direction of research. However, there has not yet been developed a complete proof system for *non-modal* hybrid intuitionistic logic. In this thesis I prove the soundness of a proof system for such a propositional hybrid intuitionistic logic, which provides the necessary axioms and proof rules for proving completeness of propositional hybrid intuitionistic logic. This is accomplished by first introducing modal logic, intuitionistic logic, classical hybrid logic, and the Kripke semantics for each of them. Some derivable formulas and generally interesting results about HIpL are noted and discussed throughout the latter half of the text.

Contents

1 Introduction: Scope and Aim of Research

The aim of this thesis is to prove the soundness of a proof for system hybrid intuitionistic propositional logic (HIpL) by considering the axioms and proof rules identified by professor Renardel de Lavalette as being necessary for completeness. I assume familiarity with classical logic.

A preliminary overview of the research in this article is as follows. First, I will give a brief introduction of modal logic, specializing that to the modal logic S4, and introducing the Kripke semantics for it. Then, I will give a brief introduction to intuitionistic logic and note the differences in Kripke semantics that accord with this change. I will then present the classic translation between intuitionistic logic and S4, and explain the significance in terms of Kripke semantics. The last major survey point will be the introduction of hybrid modal logic. This having been done, I will overview IHpL and prove the soundness of the proof system presented for it.

The value of this research is potentially twofold. Firstly, it attempts to further a line of research discussed in the literature but not yet pursued (to the author's knowledge): the development of a truly non-modal intuitionistic hybrid logic. The prevalence of intuitionistic hybrid *modal* logics is very helpful for the present research. However, the construction of these logics largely proceeds by substituting an intuitionistic basis for the classical basis in ordinary hybrid modal logics. This necessitates reaxiomatization, as will be seen later, but typically retains the modal operators $\Box$ and $\Diamond$. This retention of modal operators is generally sensible: not only is their use intuitive in the Kripke Semantics that will be introduced shortly, but their loss could at best leave the expressivity of a logic unchanged, and at worst severely weakened, as seen by comparing the proof-theoretic properties of classical logic with classical modal logic. Nevertheless, the second potential point of worth lies in that deriving an IHL in this way may lead to a significantly different logical system ([8]), and this new logical system seems plausible given the compatibility of hybrid operators with intuitionistic logic regardless of the presence of modal operators ([7]). As will become presently clear, for such a new system to be investigated as to its properties, it would first have to be axiomatized within the context of a specified semantics. In this thesis, the necessary Kripke semantics and a proof system for which completeness was derived during the course of my research by professor Renardel de Lavalette, relying heavily on work already present in the literature, followed my proof of the soundness lemma for this proof system.

It should be noted that important results such as completeness will not be investigated in this text, since that would be beyond the scope of this paper. Moreover, first-order intuitionistic hybrid logic will not be treated, only propositional. As such, the goal of this paper is laying some groundwork on the already established foundations.

2 Preliminary Notation

Throughout this text the elements of the (potentially infinite) set of propositional variables, which we denote PROP, will be denoted by letters p, q, r. In the case of $\top$, $\bot$ and the usual binary connectives $\to$, $\leftrightarrow$, $\wedge$, $\vee$, and $\neg$ the meanings are as usually defined in classical logic, except when in the intuitionistic context. This will be explained in section 5.

Formulae will be denoted by capital letters close to the start of the alphabet: A, B, C, and so on.

3 Modal Logic: The Basics

In [11] the definition of the modal logic language is given as

$$A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid A \leftrightarrow A \mid \Box A \mid \Diamond A$$

The only operators that are new here, relative to non-modal propositional logic, are the modal operators $\Box$ and $\Diamond$. Interpretations of what these operators mean abound, which in turn enables modal logic to be used in many contexts, both mathematical and philosophical (see for instance [12]).

For the purposes of mathematical logic as dealt with here, the meaning imposed on $\Box$ is 'necessary,' and that on $\Diamond$ is 'possible.' Naturally, these meanings must be in turn given logical translations for them to be applied usefully, which we shall define hereafter in section 4. However, the meanings may intuitively be understood as follows. $\Box A$, pronounced 'box A,' means that A is necessarily true everywhere in the logical context where A is being considered, no matter what rules may be applied. How this is distinct from an axiom will be examined by the end of section 4. On the other hand, $\Diamond A$, pronounced 'diamond A,' means that somewhere in the context being considered A is true, though it may also be the case that somewhere in the given context A is false.

It is useful to note for comparison to the intuitionistic case, where this does not hold, in classical logic $\Box$ and $\Diamond$ are dual. Consequently, $\Box A \iff \neg\Diamond\neg A$. This makes intuitive sense: if A is necessarily true, then it is not possible that A is not true somewhere. Similar reasoning applies when considering duality from the perspective of $\Diamond A$.

In order to make precise sense of the above, and of a logic in general, semantics are needed. We present and justify the use of Kripke semantics in the next section.

4 Kripke Semantics for Modal Logic

Below follows a brief overview of the thinking behind Kripke semantics, the formal inductive definition of the semantics, and then an explanation of each of the components thereof.

4.1 The Idea and Definitions of Kripke Semantics

It is first important to understand the distinction between a logical language and a semantics for it. What we have without a semantics is solely the ability to express formulae. Semantics give meaning to formulae by specifying when a statement is valid and when it is true, terms that will be defined shortly though they are fairly intuitively understandable.

Such a semantics may be constructed in myriad ways, but the one that has dominated the field of modal logic for the past decades has been Kripke's 'possible worlds' semantics, which built on some earlier work by, especially, Hintikka and Prior (see [14]).

The following are the key ideas of Kripke semantics. Introduce a set S of *states* $s \in S$, also known as *possible worlds* and *points*, which makes the 'somewhere' and 'everywhere' referred to in the previous section quantifiable. The set of states becomes the context, and the individual states become the places to which the assertions of the $\Box$ and $\Diamond$ operators refer. Kripke's great breakthrough was formalizing the idea of context using this concept of states. This is achieved via the *accessibility relation* R, in some cases denoted $\leq$, where if $s, t \in S$, then sRt means t is accessible from s. This naturally leads to the informal definition that if $\Box A$ holds at s, and sRt, then A is true at t. On the other hand, if $\Diamond A$ holds at s, then there is some state accessible via R such that A holds at that state.

This enables the defining of a *frame* $\mathcal{F} \equiv \langle S, R\rangle$, where we require that S be non-empty and R be an accessibility relation on S^2. Clearly, frames provide the context within we might say *e.g.* 'A is everywhere true.'

However, such an assertion also requires a means of determining the validity of an assertion such as $\Diamond p$. As it is always possible to reduce the process of checking whether an arbitrary sentence holds at a particular state to that of checking whether the individual propositions in the sentence hold at that state in such a manner as would make the sentence true, it is only necessary to be able to evaluate the truth values at particular states (this will be formalized shortly, after also introducing the concept of models). This leads us to the *valuation* V, which, for any $p \in$ PROP, gives the set of states in which p holds.

This enables the definition of a *model* $\mathcal{M} \equiv \langle \mathcal{F}, V \rangle$, which gives us both the context in which assertions can be made, the frame, and the means of evaluating the truth of those assertions at various points in this context.

The above gives meaning to any atom at a particular state. To be able to have the same capacity for parsing formulae and operators, and in order to be able to precisely denote what we described above, we define double turnstile $\models$, which is the *validity operator*, where $\mathcal{M}, s \models A$ means A *is true at state s in model M*, and $\mathcal{M} \models A$ means A *is valid in* $\mathcal{M}$. We define the meaning of this inductively as follows (I follow both [11] and [5] closely).

$$
\begin{array}{lcl}
\mathcal{M}, s \models p & \iff & s \in V(p) \\
\mathcal{M}, s \models \top & & \\
\mathcal{M}, s \not\models \bot & & \\
\mathcal{M}, s \models \neg A & \iff & \text{not } \mathcal{M}, s \models A\text{, denoted } \mathcal{M}, s \not\models A \\
\mathcal{M}, s \models A \wedge B & \iff & \mathcal{M}, s \models A \text{ and } \mathcal{M}, s \models B \\
\mathcal{M}, s \models A \vee B & \iff & \mathcal{M}, s \models A \text{ or } \mathcal{M}, s \models B \\
\mathcal{M}, s \models A \to B & \iff & \text{if } \mathcal{M}, s \models A \text{ then } \mathcal{M}, s \models B \\
\mathcal{M}, s \models A \leftrightarrow B & \iff & \mathcal{M}, s \models A \text{ if and only if } \mathcal{M}, s \models B \\
\mathcal{M}, s \models \Box A & \iff & \text{for all } t \in S\text{, if } sRt \text{ then } \mathcal{M}, t \models A \\
\mathcal{M}, s \models \Diamond A & \iff & \text{for some } t \in S\text{, } sRt \text{ and } \mathcal{M}, t \models A
\end{array}
$$

This brings the semantics to the point where it is possible to speak of the truth of formulas at specific, some, or all states. Moreover, restrictions can be imposed on where a given formula will be true through the valuation V, and on which states are accessible from given states through the accessibility relation R. Some last definitions are then in order, at which point a suitable level of generality is reached for this text. Firstly, validity in a model and validity:

$$
\begin{array}{ll}
\mathcal{M} \models A \iff \text{for all } s \in S,\ \mathcal{M}, s \models A & (A \textit{ is } \textbf{\textit{valid in model M}}) \\
\models A \iff \text{for all models } \mathcal{M},\ \mathcal{M} \models A & (A \textit{ is } \textbf{\textit{valid}})
\end{array}
$$

Secondly, validity of a set in a model and a set holding in a state in a model is defined as

$$
\begin{array}{ll}
\mathcal{M}, s \models \Gamma \iff \mathcal{M}, s \models A \text{ for all } A \in \Gamma & (\Gamma \textit{ holds at } s \textit{ in } \mathcal{M}) \\
\mathcal{M} \models \Gamma \iff \text{for all } s \in S,\ \mathcal{M}, s \models \Gamma & (\Gamma \textit{ is valid in } \mathcal{M})
\end{array}
$$

Lastly, it is further useful to define validity relative to a set of formulae Γ, where Γ *validates* A is defined by

$$\Gamma \models A \iff \text{for all models } \mathcal{M} \text{ and all states } s \in S\text{, if } \mathcal{M}, s \models \Gamma\text{, then } \mathcal{M}, s \models A$$

The above gives a sufficiently detailed overview for the purposes of this text. What remains are some remarks on the relation between frame properties and accessibility relations and introducing the modal logics K and S4.

4.2 R and $\mathcal{F}$: The Accessibility Relation and Defining Frame Properties

It is possible to induce various different (combinations of) properties on a frame. These properties characterize how the states belonging to the frame relate to each other, and this can only affected by restrictions imposed, in turn, on the accessibility relation. Two frame properties that will be important in the sense of being properties of the frames eventually studied in this paper are reflexivity and transitivity.

If a frame is reflexive, then for any state $s \in S$, s is accessible from s. This principally effects the meaning of $\Box A$: considering the semantics given in subsection 4.1, we see that such a frame relation requires that if $\Box A$ hold at state s, then A holds at s.

On the other hand, if a frame is transitive, then if for $s, t, u \in S$ we have both sRt and tRu, then sRu. Again, $\Box A$ is key to understanding the effect of this property. For, if $\Box A$ holds at state s, then A not only holds at all the states A would have held at without transitivity, but also at every state in the whole network

Axioms

(All propositional tautologies)	$\vdash A$
(Identity)	$A \vdash A$
(Distribution)	$\vdash \Box(A \to B) \to (\Box A \to \Box B)$

Rules

(Necessitation)	$\vdash A \implies \vdash \Box A$
(Weakening)	$\Gamma \vdash A \implies \Gamma, \Delta \vdash A$
(Cut)	$(\Gamma \vdash A \;\&\; \Gamma \vdash A \to B) \implies \Gamma \vdash B$
(Deduction)	$\Gamma, A \vdash B \implies \Gamma \vdash A \to B$

Figure 1: A proof system for modal logic K

of states related to each other without transitivity. At every state in this network it becomes possible to assert the validity of $\Box A$, since A will hold at every accessible state. This then also justifies the assertion that $\Box\Box A$ holds at all accessible states, since it follows that $\Box A$ will hold at every accessible state.

As such, frame properties affect the validity of assertions made at individual states. At the same time, it is possible to examine when requiring that certain forms of assertions hold, like $\Box A \to A$, forces certain frame properties to be part of the semantics in order for it to accurately render the meaning of the logic in which such assertions hold. These assertions then form part of the axiomatization of the logic (see [13] and [7]). This will be elaborated in the next section, where the modal logics K and S4 are defined, with the axiom restricting S4.

4.3 The Most Basic Modal Logic: K

The minimal definition of a modal logic is known as K, which has the operators introduced in the above paragraphs, but is further restricted by one rule and one axiom (see [13]).

The *necessitation rule*, if A is valid then $\Box A$ is valid, states that if formula A is true in all states in all models, then it must also be true that $\Box A$ is valid in all states in all models.

The *distribution axiom*, $\Box(A \to B) \to (\Box A \to \Box B)$, states that if it is true that in all accessible states A implies B, then if A is true at all accessible states, B must be also be true at all accessible states.

A proof system for K is presented in figure 1.

Many modal logics are extensions of K (see [13]), where the logic is extended through the addition of axioms and/or proof rules, as seen in the next section.

4.4 The Modal Logic S4

The modal logic S4 results from adding two axioms to K. We present these alongside the proof rules and axioms of K in the proof system of S4. Note that the frames in the semantics of S4 are reflexive, such that sRt holds if and only if tRs holds, and transitive, such that if sRt and tRu hold then sRu holds.

In order to construct a proof system, it is first necessary to have a notion of derivability, because the proof system will determine what derivation steps are possible. In this text we make use of sequent calculus, following [11], with $\Gamma \vdash A$ pronounced 'gamma proves A' and meaning that a formula A is derivable from the set of formulae Γ.

We further note that $\Gamma, \Delta \vdash A$ means that A is provable from the union of the set of Γ and Δ. Additionally, $\Gamma, A \vdash B$ means the union of Γ and the singleton set containing A proves B. Furthermore, $\vdash A$ means that A may be derived from the empty set and is therefore a tautology.

A last point to note is that when sequents appear in logical reasoning they are syntactic units such that the necessitation rule in sequent notation, $\Gamma \vdash A \implies \Gamma \vdash \Box A$, is composed of $\Gamma \vdash A$, $\implies$, and $\Gamma \vdash \Box A$.

Axioms

(All propositional tautologies)	$\vdash A$
(Identity)	$A \vdash A$
(Distribution)	$\vdash \Box(A \to B) \to (\Box A \to \Box B)$
(Axiom S)	$\vdash \Box A \to A$
(Axiom 4)	$\vdash \Box A \to \Box\Box A$

Rules

(Necessitation)	$\vdash A \implies \vdash \Box A$
(Strong Necessitation)	$\Gamma \vdash A \implies \Box\Gamma \vdash \Box A$
(Weakening)	$\Gamma \vdash A \implies \Gamma, \Delta \vdash A$
(Cut)	$(\Gamma \vdash A \;\&\; \Gamma \vdash A \to B) \implies \Gamma \vdash B$
(Deduction)	$\Gamma, A \vdash B \implies \Gamma \vdash A \to B$

Figure 2: A proof system for modal logic S4

This avoids potential confusion, since *e.g.* $\Gamma \vdash (A \implies \Gamma) \vdash \Box A$ would be notationally erroneous (and meaningless). Additionally, we define $\Box\Gamma \equiv \{\Box A | A \in \Gamma\}$.

A proof system for S4 is presented in figure 2. Note that contrary to typical best practice in axiomatization, there is significant redundancy included in this proof system. Specifically, the proof system in [11] is already complete, making use of *e.g.* the fact that the necessitation rule is a special case of the strong necessitation rule (see [17]). As such, the particular format of this proof system serves to highlight the lineage of S4 by including the axioms and rules of K and the axioms specific to S4 rather than aiming for conciseness.

Now, as anticipated earlier, the presence of $\Box A \to A$ and $\Box A \to \Box\Box A$ means that the semantics for S4 requires reflexive and transitive frames. Two things of note then follow.

Firstly, the widely known fact of S4's completeness depends on the proper correspondence between the frame properties and these axioms ([13]). It is therefore of paramount importance when axiomatizing a logic that the frame properties and axioms concur.

Secondly, these frame properties reflect important properties of the semantics of intuitionistic logic, which we will discuss in the next section. This fact enables a fascinating correspondence between S4 and intuitionistic logic first noted by Gödel in 1933. We shall review this in section 5.2.

5 Intuitionistic Logic: The Basics

5.1 An Overview of the Distinctive of Intuitionistic Logic

Intuitionism, or intuitionistic style logics under the cover term constructivism, is a philosophical approach to mathematics that is fundamentally different from the one reflected by classical logic. At its core, the logical consequences of a intuitionistic framework come down to what the statement 'A is true' is understood to mean. In illustrating this and further points in this section, I will rely heavily on the work of Troelstra and van Dalen in [21]. They specify that

> A statement is true if we have proof of it, and false if we can show that the assumption that there is a proof for the statement leads to a contradiction.

As a consequence, proof by contradiction is not considered a proof of any sentence, for reasons that will become clear shortly. Such a definition of truth and falsity has immediate consequences in what may and may not be asserted as valid.

A first consequence of this definition is that $A \vee \neg A$, which is valid for any formula A in classical logic, may be impossible to state at a given moment: $A \vee \neg A$ means that it is case that either a proof of A exists or the assumption that a proof of A exists leads to a contradiction (see *e.g.* [18]). Consequently, the validity of $A \vee \neg A$ cannot be asserted for any theorem that has not been either proven or disproven.

In fact, as is well-known, the law of the excluded middle (represented by the axiom $A \vee \neg A$) may be seen as the fundamental difference between classical and intuitionistic logic, since adding this axiom to intuitionistic logic will give you back classical logic, as the other rules such as $\neg\neg A \to A$ would be derivable from a complete proof system (see, for instance, [16]).

This shift in understanding is so fundamental it necessitates a redefinition of the logical operators $\to$, $\leftrightarrow$, $\wedge$, $\vee$, and $\neg$ (since we limit ourselves to propositional logic), with all logical operators taken as primitives. The operators must be taken as primitives since there is no reason think that a sentence such as '$A \wedge B$' could be intuitionistically expressed using $\to$ and $\neg$.

The definition of intuitionistic logical operators has some variation across the literature (see *e.g.* [18], [21], [2]), but all boil down to which proofs should be known in order for an assertion to be made, and how they relate to the sentences involved. They are as follows (with $\neg A$ repeated for convenience):

- $A \wedge B$ means that both a proof of A and a proof of B is known
- $A \vee B$ means that either a proof A or a proof of B is known
- $A \to B$ means that a method is known that turns any proof of A into a proof of B
- $A \leftrightarrow B$ means that a method is known that turns any proof of A into a proof of B, and *vice versa*.
- $\neg A$ means that any proof of A will lead to a contradiction

Naturally, the above makes any proof rule of the form $\neg\neg A \to A$ highly unlikely, since proving that it is not the case that any proof of A leads to a contradiction is not the same as having a proof for A, since $A \vee \neg A$ is not assumed in intuitionistic logic. Therefore, all that might be said for a sentence A of which it is only known that $\neg\neg A$ is valid in a model (or true at a state) is that in at least one of the states of the model (or in at least one of the states reachable from the state where $\neg\neg A$ holds) it is also the case that A holds.

5.2 Translating intuitionistic Propositional Logic to the Modal Logic S4: When, How, and then Why

This background material having been covered, we note before going on to present the semantics for intuitionistic logic a very useful means of motivating the use of Kripke semantics for intuitionistic logic: the translation that exists between classical modal and intuitionistic propositional logic.

The idea of a translation from intuitionistic (for interest's sake: specifically Brouwerian intuitionistic) logic to modal logic was first proffered by Gödel in 1933. The translation centers around the fact that to assert the validity of p in a intuitionistic context has an effectively identical meaning to asserting the validity of $\Box p$ in the context of modal S4: everywhere in the given context, p holds.

The following translation relies heavily on Goldblatt's presentation in [14]. It is worth noting that he also shortly discusses proofs of its correctness and the impact that the translation had. Notably, it inspired Kripke to formulate Kripke semantics for intuitionistic logic, which is explored in the next section.

This translation T from intuitionistic logic to modal S4 logic of an arbitrary atom p and formulae A, B is defined as

$$\begin{aligned} T(p) &= \Box p \\ T(\neg A) &= \Box \neg T(A) \\ T(A \to B) &= \Box(T(A) \to T(B)) \\ T(A \wedge B) &= T(A) \wedge T(B) \\ T(A \vee B) &= T(A) \vee T(B) \end{aligned}$$

with the co-domain of map T being equipped with Kripke semantics for classical modal logic rather than for intuitionistic logic. What makes this translation a good one is that it manages to preserve validity while reducing intuitionistic logic to S4. Consequently, this translation inspired a Kripke semantics for intuitionistic logic by demonstrating the compatibility of intuitionistic validity and Kripke states since intuitionistic validity can be reduced to S4 validity. Moreover, it highlights that there is a sense in which asserting A in the intuitionistic logic context is equivalent to both asserting that no proof of $\neg A$ is possible and that you have a proof of A. Therefore, no matter how much information may be additionally considered or proof rules applied, A will always hold. This constitutes the implicit assumption that contradictory information would never be found in such an expansion of available information. This naturally leads to considering accessible states with as much or more information than is available at the current state, which is what is considered in the next section.

5.3 The Semantics of Intuitionistic Logic

Again defining a model $\mathcal{M}$ as a set of states S with an accessibility relation R and a valuation V, but impose the following two specifications.

Firstly, the following is a condition imposed on the accessibility relation.

$$(\mathcal{M}, s \models p \ \& \ sRt) \Rightarrow \mathcal{M}, t \models p$$

where p is an atomic proposition. This condition is known as *monotonicity of atoms*. A direct consequence is an epistemic ordering imposed on the frame $\langle S, R\rangle$, since it requires that each successive state has at least enough information, in the sense of atoms and sentences, to validate all the same atoms as in the previous state.

Monotonicity of atoms and the consequent epistemic ordering also require an adjustment to the valuation V. To reflect the monotonicity of atoms (which is extended to formulae at the end of this section) in the valuation, we require that if $s \in V(p)$ and sRt hold, then $t \in V(p)$ ([19]). Formally, following [10], let $V : P \to \wp^R(S)$, where

$$\wp^R(S) = \{V \subseteq S \mid \forall s \in V \forall t \in S(sRt \Rightarrow t \in V)\}$$

meaning $\wp^R(S)$ is the collection of all R-upwards closed sets of accessible states, such that R is upwards closed.

We are now equipped to paraphrase [21] in giving the inductive definition of validity for intuitionistic propositional logic as follows.

$$\begin{array}{lll}
\mathcal{M}, s \models \top & & \\
\mathcal{M}, s \not\models \bot & & \\
\mathcal{M}, s \models p & \iff & s \in V(p) \\
\mathcal{M}, s \models \neg A & \iff & \text{for all } t \in S \text{ with } sRt,\ \mathcal{M}, t \not\models A \\
\mathcal{M}, s \models A \wedge B & \iff & \mathcal{M}, s \models A \text{ and } \mathcal{M}, s \models B \\
\mathcal{M}, s \models A \vee B & \iff & \mathcal{M}, s \models A \text{ or } \mathcal{M}, s \models B \\
\mathcal{M}, s \models A \to B & \iff & \text{for all } t \in S \text{ with } sRt,\ \mathcal{M}, t \models A \Rightarrow \mathcal{M}, t \models B \\
\mathcal{M}, s \models A \leftrightarrow B & \iff & \text{for all } t \in S \text{ with } sRt,\ \mathcal{M}, t \models A \Leftrightarrow \mathcal{M}, t \models B
\end{array}$$

Additionally, a derivable and useful rule is

$$\mathcal{M}, s \models \neg\neg A \iff \text{for all } t \in S \text{ with } sRt,\ \text{there exists a } u \in S \text{ such that } tRu \text{ and } \mathcal{M}, u \models A$$

Note that the frames are again reflexive and transitive. This was anticipated by the translation to S4, since the semantics of S4 is obtained by restricting to reflexive, transitive frames.

A last point to note is that of the monotonicity result, which extends the monotonicity of atoms to formulas, and which is also a direct consequence of assuming monotonicity of atoms. The proof is a simple formula induction (see for instance [21]).

For any formula A, and all states $s, t \in S$:

$\mathcal{M}, s \models A$ and $sRt \Rightarrow \mathcal{M}, t \models A$

Axioms

$$A \vdash A$$
$$\vdash A \to (B \to A)$$
$$\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$$
$$\vdash (A \wedge B) \to A$$
$$\vdash (A \wedge B) \to B$$
$$\vdash A \to (A \vee B)$$
$$\vdash B \to (A \vee B)$$
$$\vdash (A \to B) \to ((C \to B) \to ((A \vee C) \to B))$$
$$\vdash \bot \to A$$

Rules

(Cut) $(\Gamma \vdash A \wedge \Gamma \vdash A \to B) \implies \Gamma \vdash B$

(Deduction) $\Gamma, A \vdash B \implies \Gamma \vdash A \to B$

(Weakening) $\Gamma \vdash A \implies \Gamma, \Delta \vdash A$

Figure 3: A proof system for intuitionistic logic

5.4 Proof System for Intuitionistic Logic

We present in figure 3 the proof system for propositional logic that appears in [3], which is in turn a fairly well-known proof system for Hilbert type systems of intuitionistic propositional logics. Its usefulness and the body of research on it make it fairly representative choice, though there are of course other proof systems available (see [3] again). Note that, unlike in the modal case, a sequent calculus such as Gentzen's is noticeably less discussed in the literature, and the presentation of this axiomatization is a bit more to the point for the purposes of this text. We also include the provable deduction theorem as a proof rule for simplicity.

6 Going Further: Hybrid Modal Logic

Hybridizing a logic involves the addition of some concepts. Firstly, the set of *nominals* NOM with elements denoted i, j, k, and so on. A nominal names a state as follows: $\mathcal{M}, s \models i$ holds for only the state s named by i and $\mathcal{M}, t \not\models i$ for all states $t \neq s$. In other words, $V(i) = s$ is a singleton set. Furthermore, the *satisfaction operator* @ and also the down arrow *binder* $\downarrow$ are both typically added, though one may restrict to adding only one (see [4]). $@_i A$ means that A is true at the state denoted by i, and that $\downarrow i A$ as A is true when the state i is taken to denote the current state, with all occurrences of i in A bound. Note that we particularly have the treatment in mind presented in [15] in the above and in what follows, since this deviates in a manner that enables a fair bit of detail unimportant to the purposes of this text to be elided. This includes multiple accessibility relations with corresponding levels of states of knowledge (see *e.g.* [1] and [9]).

In this understanding of nominals, they are variable. This means that we may 'manually' name an arbitrary state t such that it is identifiable after the fact. In this case we will speak of a nominal being *bound* to a state to name it, and this will be done using the $\downarrow$ operator. This requires defining a *nominal assignment* g: NOM $\to S$ that maps a nominal to a state, such that for any $i \in$ NOM that has been bound to a state s to name it, we have $g(i) = s$, and so state s is named i. We also define *locally modified* variable assignments $g[i := s]$ which behave firstly such that for all $j \in$ NOM such that $i \neq j$, we have $g[i := s](j) = g(j)$, and secondly require that $g[i := s](i) = s$. The locally modified nominal assignment is therefore used to implement the meaning of nominal binding at the semantic level, while it is implented at the syntactic level by $\downarrow$ ([4]). A model $\mathcal{M}$ remains defined as before, meaning that when defining validity the assignment variable is an additional element of the definition and enables the same models with different assignment variables to be compared.

Additionally, let nom(A) be the set of nominals occurring in A, and fnom(A) the set of free nominals in A, with defining cases

$$\text{nom}(\downarrow iA) = \text{nom}(A) \cup \text{nom}(i)$$
$$\text{fnom}(\downarrow iA) = \text{fnom}(A) - \{i\}$$

A substitution is then defined to be a function σ: NOM $\rightarrow$ NOM, which may be redefined locally such that the substitution $\sigma[i := j]$ is identical to σ on NOM except at i, where $\sigma[i := j](i) = j$. The application of a substitution σ to a formula is denoted $A\sigma$, and the definition of such an application is straightforward except for the following case:

$$(\downarrow iA)\sigma = \downarrow j(A\sigma[i := j])$$

where $j \notin \sigma[\text{nom}(A)] = \{\sigma(k) | k \in \text{nom}(A)\}$, such that j is not one of the variables substituted by σ. Here, equivalence results from replacing every occurrence of i in A by j, such that the effect of binding i is identical to that of binding j. We then denote the application of a substitution that maps all nominals to themselves except for mapping a nominal i to j in a formula A by $A[i := j]$. Furthermore, the *substitution property* given in [15] states that

$$\mathcal{M}, g, s \models A\sigma \iff \mathcal{M}, g \circ \sigma, s \models A$$

Validity for hybrid logic in a Kripke semantics is then defined by

$$\begin{array}{lll}
\mathcal{M}, g, s \not\models \bot & & \\
\mathcal{M}, g, s \models p & \iff & s \in V(p) \\
\mathcal{M}, g, s \models i & \iff & g(i) = s \\
\mathcal{M}, g, s \models \neg A & \iff & \mathcal{M}, g, s \not\models A \\
\mathcal{M}, g, s \models A \wedge B & \iff & \mathcal{M}, g, s \models A \text{ and } \mathcal{M}, g, s \models B \\
\mathcal{M}, g, s \models \Box A & \iff & \text{for all } t \in S, \text{ if } sRt, \text{ then } \mathcal{M}, g, t \models A \\
\mathcal{M}, g, s \models @_i A & \iff & \mathcal{M}, g, g(i) \models A \\
\mathcal{M}, g, s \models \downarrow iA & \iff & \mathcal{M}, g[i := s], s \models A
\end{array}$$

which enables the definition of the proof system in figure 4 given in [15] to be repeated here, though without the infinitary proof rule that is beyond the scope of this text and with some notational changes. Note that the axioms and proof rules have both names and proof rules repeated as presented in [15] for the sake of completeness.

A last important result to repeat here is that one regarding the *J-equivalence of nominal assignments*, given as lemma 2.5 in [15], which shall be referred to here as *free nominal equivalence*. Defining J-equivalence first: two sets of models, $\mathcal{M} = \langle S, R, V \rangle$ and $\mathcal{M}' = \langle S', R', V' \rangle$, with nominal assignments g and g' respectively, are *J-equivalent* on a set of nominals J when $S = S'$, $R = R'$, $V = V'$, and $g = g'$ for all $j \in J$. The free nominal equivalence result then states that if $\langle \mathcal{M}, g \rangle$ and $\langle \mathcal{M}', g' \rangle$ are J-equivalent on fnom(A), then

$$\mathcal{M}, g, s \models A \iff \mathcal{M}', g', s \models A$$

which will be of use in investigating the validity of axioms and proof rules.

The benefit of hybridizing a logic is that greater expressivity and proof-theoretic behaviour is obtained without going all the way to the realm of first-order logic. For instance, hybrid modal logic is for instance better behaved proof-theoretically than modal logic, meaning that the added expressivity enables more properties of first-order logic to be enjoyed in the modal propositional context without losing important properties like *e.g.* completeness ([20]). Moreover, it enables further exploitation of Kripke semantics by providing more tools with which to work with in the context of states in models. It is for instance shown in [4] that a version of the above hybrid logic is as expressive as the bounded fragment of first-order logic.

At minimum, if hybrid modal logic has its classical base switched for an intuitionistic one, with all the requisite changes in semantics and axiomatization, then a complete proof system may be constructed (see,

Axioms		
Taut	$\vdash A$	(all propositional tautologies)
MP	$A, A \to B \vdash B$	(modus ponens)
$\mathbf{K}_{\Box}$	$\vdash \Box(A \to B) \to (\Box A \to \Box B)$	(distribution)
$\mathbf{K}_{@}$	$\vdash @_{\mathsf{i}}(A \to B) \to (@_{\mathsf{i}}A \to @_{\mathsf{i}}B)$	(distribution)
$\mathbf{SD}_{@}$	$\vdash @_{\mathsf{i}}A \to \neg @_{\mathsf{i}}\neg A$	(self-dual)
Intr	$\vdash \mathsf{i} \wedge A \to @_{\mathsf{i}}A$	(introduction)
$\mathbf{T}_{@}$	$\vdash @_{\mathsf{i}}\mathsf{i}$	(reflexivity)
Agree	$\vdash @_{\mathsf{i}}@_{\mathsf{j}}A \leftrightarrow @_{\mathsf{j}}A$	(agree)
Back	$\vdash \Diamond @_{\mathsf{i}}A \to @_{\mathsf{i}}A$	(back)
DA	$\vdash @_{i}(\downarrow \mathsf{j}A \leftrightarrow A[\mathsf{j} := \mathsf{i}])$	(downarrow)
Name	$\vdash \downarrow \mathsf{i} @_{\mathsf{i}}A \to A$, provided i is not free in A	(name)
BG	$\vdash @_{i}\Box \downarrow \mathsf{j} @_{\mathsf{i}}\Diamond \mathsf{j}$ provided $\mathsf{i} \neq \mathsf{j}$	(bounded generalization)
Proof Rules		
$\mathbf{SNec}_{\Box}$	if $\Gamma \vdash A$, then $\Box\Gamma \vdash \Box A$	(strong necessitation)
$\mathbf{SNec}_{@}$	if $\Gamma \vdash A$ then $@_{\mathsf{i}}\Gamma \vdash @_{\mathsf{i}}A$	(strong necessitation)
$\mathbf{SNec}_{\downarrow}$	if $\Gamma \vdash A$ then $\downarrow \mathsf{i}\Gamma \vdash \downarrow \mathsf{i}A$	(strong necessitation)
Ded	if $\Gamma, A \vdash B$ then $\Gamma \vdash A \to B$	(deduction)
W	if $\Gamma \vdash A$ then $\Gamma, \Delta \vdash A$	(weakening)

Figure 4: A proof system for classical hybrid modal logic

for instance, [6]). As such, at least some of the benefits of increased expressivity may be expected to be enjoyed by hybridizing intuitionistic logic, as is further suggested by the increase in expressivity arising from hybridizing intuitionistic modal logic. A proof system for hybrid intuitionistic modal logic in the next section, and then research into propositional IHL will be presented.

7 Hybrid Intuitionistic Propositional Logic

7.1 Semantics, Proof System, and Some Notes

The following relies heavily on [10], with the axiomatization and list of derivable rules repeated by the kind permission of Professor Renardel de Lavalette.

The language of HIpL is simply that of intuitionistic propositional logic with the hybrid operators and set of nominals added and defined as before. As such, the formulae of HIpL are defined by

$$A ::= \bot \mid p \mid A \wedge A \mid A \vee A \mid A \to A \mid i \mid @_i A \mid {\downarrow} i A \mid \Diamond i$$

with all operators defined as before, and the only innovation being $\Diamond i$, which means that the state named by nominal i is accessible from the current state.

However, differences in semantics and results such as monotonicity exist and are explained below. Firstly, the interpretation $M, g, s \models A$ of HIpL formula A in state s of Kripke model $\mathcal{M}$ with nominal assignment g is defined as follows.

$\mathcal{M}, g, s \not\models \bot$		
$\mathcal{M}, g, s \models p$	iff	$s \in V(p)$
$\mathcal{M}, g, s \models \neg A$	iff	for all $t \in S$ with sRt, $\mathcal{M}, g, t \not\models A$
$\mathcal{M}, g, s \models A \wedge B$	iff	$\mathcal{M}, g, s \models A$ and $\mathcal{M}, g, s \models B$
$\mathcal{M}, g, s \models A \vee B$	iff	$\mathcal{M}, g, s \models A$ or $\mathcal{M}, g, s \models B$
$\mathcal{M}, g, s \models A \to B$	iff	$\forall t \in S(sRt \;\&\; \mathcal{M}, g, t \models A \Rightarrow \mathcal{M}, g, t \models B)$
$\mathcal{M}, g, s \models A \leftrightarrow B$	iff	$\forall t \in S$ with sRt, $\mathcal{M}, g, t \models A \Leftrightarrow \mathcal{M}, g, t \models B$
$\mathcal{M}, g, s \models i$	iff	$g(i) = s$
$\mathcal{M}, g, s \models @_i A$	iff	$\mathcal{M}, g, g(i) \models A$
$\mathcal{M}, g, s \models {\downarrow} i A$	iff	$\mathcal{M}, g[i := s], s \models A$
$\mathcal{M}, g, s \models \Diamond i$	iff	$sRg(i)$ holds

Note that in binding a nominal j, the specific choice of j is irrelevant, since we have, for all $j_1, j_2 \notin \sigma[\mathsf{nom}(A)]$ that ${\downarrow} j_1(A[i := j_1])$ and ${\downarrow} j_2(A[i := j_2])$ are semantically equal: for any $\mathcal{M}, g, s$ we have

$$\mathcal{M}, g, s \models {\downarrow} j_1(A[i := j_1]) \text{ iff } \mathcal{M}, g, s \models {\downarrow} j_2(A[i := j_2]).$$

Let some sound and complete proof system for IpL be given, formulated with sequents $\Gamma \vdash A$ where Γ is a collection of formulae. It can then be extended to an axiomatization for HIpL by adding the following axioms and proof rules in figure 5.

It is worth noting that in section 7.5 that I show that the version of Nom I show to be derivable, Nom denotes $\vdash (@_i j \wedge @_i A) \leftrightarrow @_j A$, which has both the forward and the reverse direction as opposed to only the forward as in the case of the Nom in the list of derivable formulas from [10]. Inspiration for the reverse direction came from [6].

Further, note that monotonicity breaks down in this semantics. As most other logicians that have studied modal hybrid intuitionistic logic, Braüner and de Paiva impose various further restrictions, orderings, and (see [8], and also [9]) as compared with the semantics given here. As such, they are able to retain monotonicity, which is a feature lost in this IHpL due to its comparative simplicity. On the other hand, this simplicity greatly helps the comprehensibility of the semantics of IHpL as compared to these other logics. An example of where monotonicity doesn't hold is $\mathcal{M}, g, s \models i$, since we have

$$\mathcal{M}, g, s \models i \iff g(i) = s$$

so assuming $g(i) = s$, we have that $\forall t$ such that sRt and $s \neq t$ holds, $\mathcal{M}, g, t \not\models i$. This means that monotonicity of atoms fails, which will naturally also eliminate monotonicity of formulae.

Before verifying validity of axioms and proof rules, first some preliminary ground will be covered in the following subsections.

refl	$A \vdash A$	(reflexivity)
$\bot$ **L**	$\bot \vdash A$	($\bot$ left)
$\wedge$**R**	$A, B \vdash A \wedge B$	($\wedge$ right)
$\wedge$**L**	$A \wedge B \vdash A$ and $A \wedge B \vdash B$	($\wedge$ left)
$\vee$**R**	$A \vdash A \vee B$ and $B \vdash A \vee B$	($\vee$ right)
$\vee$**L**	if $\Gamma, A \vdash C$ and $\Gamma, B \vdash C$ then $\Gamma, A \vee B \vdash C$	($\vee$ left)
$\rightarrow$ **R**	if $A \vdash B$ then $\vdash A \rightarrow B$	($\rightarrow$ right)
$\rightarrow$ **L**	$A, A \rightarrow B \vdash B$	($\rightarrow$ left)
$\bot_{@}$	$@_i \bot \vdash \bot$	(falsity)
Dec$_{@}$	$\vdash @_i A \vee \neg @_i A$	(decidability)
$\vee_{@}$	$@_i(A \vee B) \vdash @_i A \vee @_i B$	(@-distributivity)
Intr	$i, A \vdash @_i A$ and $i, @_i A \vdash A$	(introduction)
T$_{@}$	$\vdash @_i i$	(reflexivity)
Agree	$\vdash @_i @_j A \leftrightarrow @_j A$	
refl$_{\Diamond}$	$\vdash @_i \Diamond i$	(reflexivity)
trans$_{\Diamond}$	$\Diamond i, @_i \Diamond j \vdash \Diamond j$	(transitivity)
mon$_{\Diamond}$	$p, \Diamond i \vdash @_i p$ and $A \rightarrow B, \Diamond i \vdash @_i(A \rightarrow B)$	(monotonicity)
DA	$i, \downarrow j A \vdash A[j := i]$ and $i, A[j := i] \vdash \downarrow j A$	(downarrow)
bindAt	$\vdash \downarrow i @_i A \leftrightarrow \downarrow i A$	(bind at)
vacBind	$\vdash \downarrow i A \leftrightarrow A$, provided $i \notin \mathsf{fnom}(A)$	(vacuous bind)
PR	if $\Gamma, @_i \Diamond j, @_j A \vdash @_j B$ and $j \notin \mathsf{fnom}(\Gamma, A, B)$ then $\Gamma \vdash @_i(A \rightarrow B)$	(paste rule)
SNec$_{@}$	if $\Gamma \vdash A$ then $@_i \Gamma \vdash @_i A$	(strong necessitation)
SNec$_{\downarrow}$	if $\Gamma \vdash A$ then $\downarrow i \Gamma \vdash \downarrow i A$	(strong necessitation)
W	if $\Gamma \vdash A$ then $\Gamma, \Delta \vdash A$	(weakening)
Cut	if $\Gamma \vdash A$ and $\Delta, A \vdash B$ then $\Gamma, \Delta \vdash B$	(cut)

Figure 5: A proof system for HIpL

7.2 Defining Validity of a Sequent

An additional definition must be given for $\mathcal{M}, g, s \models (\Gamma \vdash A)$, since this is necessitated by the use of a sequent calculus and will enable us to define *soundness* shortly.

I define *validity of sequents* by

$$\models (\Gamma \vdash A) \iff \forall \mathcal{M}, g, s(\mathcal{M}, g, s \models \Gamma \Rightarrow \mathcal{M}, g, s \models A)$$

with $\mathcal{M}, g, t \models \Gamma$ as defined before in subsection 4.1, and I claim

$$\models (\Gamma \vdash A) \iff \forall \mathcal{M}, g, s \forall t \in S(sRt \;\&\; \mathcal{M}, g, t \models \Gamma \Rightarrow \mathcal{M}, g, t \models A)$$

which is another definition of validity of sequents that occasionally occurs in the literature. Consequently, we can use the simpler and more concise definition without any loss of generality.

The proof is as follows:

$$\begin{aligned}
&\forall \mathcal{M}, g, s \forall t \in S(sRt \;\&\; \mathcal{M}, g, t \models \Gamma \Rightarrow \mathcal{M}, g, t \models A)\\
\iff &\text{ (use that } (\forall x(A(x) \to B(x)) \iff (\forall x \forall y(xRy \wedge A(y) \to B(y))) \text{ as proven in subsection 7.4)}\\
&\forall \mathcal{M}, g, s(\mathcal{M}, g, s \models \Gamma \Rightarrow \mathcal{M}, g, s \models A)\\
\iff &\text{ (Definition of validity of a sequent)}\\
&\models (\Gamma \vdash A)
\end{aligned}$$

We now arrive at the central claim of this thesis.

7.3 The Soundness Lemma to be Proven

The rest of this thesis is aimed at developing the tools needed to prove the following lemma, proving the lemma in subsections 7.6 and 7.7, and showing why this proof does not accommodate certain classical axioms.

First, we must extend the definition of validity of a formula A in a set of formulae Γ to the hybrid context. We do this by defining

$$\Gamma \models A \iff \forall \mathcal{M}, g, s(\mathcal{M}, g, s \models \Gamma \implies \mathcal{M}, g, s \models A)$$

This enables us to define the notion central to this thesis: *soundness*.

Definition (Soundness) A proof system is *sound* if and only if it holds that if $\Gamma \vdash A$ then $\Gamma \models A$

This enables the central claim of this thesis to be stated as follows.

Lemma (Soundness) *The proof system in figure 5 is sound.*

Two initial results are immediately available to us. Firstly, I claim

$$\models A \to B \iff \forall \mathcal{M}, g, s(\mathcal{M}, g, s \models A \implies \mathcal{M}, g, s \models B) \tag{1}$$

The proof is achieved by noting

$$\begin{aligned}
&\models A \to B\\
\iff &\text{ (Definition of } \models)\\
&\forall \mathcal{M}, g, s[\forall t \in S(sRt \;\&\; \mathcal{M}, g, t \models A \Rightarrow \mathcal{M}, g, t \models B)]\\
\iff &\text{ (result in subsection 7.2)}\\
&\forall \mathcal{M}, g, s(\mathcal{M}, g, s \models A \Rightarrow \mathcal{M}, g, s \models B)
\end{aligned}$$

which proves the claim. Note that this means $A \models B \iff \models A \to B$, by definition of validity of a formula in a set of formulae (in this case the singleton set $\{A\}$).

Secondly, let $\bigwedge \Gamma$ be the conjunction of all formulas in Γ. Then the claim of (1) can be extended to the following.

$$\Gamma \models A \iff \models (\bigwedge \Gamma \to A) \tag{2}$$

The proof requires extending the definition of the truth of a set at a state, $\mathcal{M}, s \models \Gamma$ in subsection 4.1, to the hybrid context, which is done trivially as by defining:

$$\mathcal{M}, g, s \models \Gamma \iff \mathcal{M}, g, s \models A, \text{for all } A \in \Gamma$$

with the other definitions such as validity in a set being extended in the same way: simply adding the nominal assignment g as above.

The proof is as follows.

$$\begin{aligned}
&\models (\bigwedge \Gamma \to A) \\
\iff &\text{(Definition of } \models) \\
&\forall \mathcal{M}, g, s[\forall t \in S(sRt \,\&\, \mathcal{M}, g, t \models \bigwedge \Gamma \Rightarrow \mathcal{M}, g, t \models A)] \\
\iff &\text{(Result in subsection 7.2)} \\
&\forall \mathcal{M}, g, s(\mathcal{M}, g, s \models \bigwedge \Gamma \implies \mathcal{M}, g, s \models A) \\
\iff &\text{(Definition of } \models\text{, denoting by } B, \dots, C \text{ all the sets in } \Gamma) \\
&\forall \mathcal{M}, g, s([\mathcal{M}, g, s \models B \,\&\, \dots \,\&\, \mathcal{M}, g, s \models C] \implies \mathcal{M}, g, s \models A) \\
\iff &\text{(Definition of } \mathcal{M}, g, s \models \Gamma) \\
&\forall \mathcal{M}, g, s(\mathcal{M}, g, s \models \Gamma \implies \mathcal{M}, g, s \models A) \\
\iff &\text{(Definition of } \Gamma \models A) \\
&\Gamma \models A
\end{aligned}$$

which proves the claim. This also enables us to simplify our reasoning regarding axioms and proof rules. Consider for example the axiom $mon_\diamond$ which states in part that $p, \diamond i \vdash @_i p$. In order to take this as an axiom, it must not corrupt the (potential) soundness of the proof system. Therefore, we must prove that $p, \diamond i \models @_i p$ which is equivalent by claim (2) to $\models (p \wedge \diamond i) \to @_i p$, which will be used in subsubsection 7.6.7 to show the validity of this axiom.

This lays the groundwork for the general reasoning to be conducted. Some rules of predicate logic will now be overviewed to enable careful logical reasoning, and the use of these will then be demonstrated in subsection 7.5.

7.4 Some Rules of Inference in Predicate Logic

Some rules of inference from predicate logic will be needed in the process of proving validity of proof rules and axioms. Some of these are simply noted and others are proven.

Note first the classical rule of inference, universal instantiation:

$$\forall x A(x) \implies A(t/x)$$

where $A(t/x)$ has all occurrences of x in A replaced by t, and where t is a term.

Another proof rule that will be useful is what will be referred to as the *universal-existential rule* in this text:

$$\forall x(A(x) \to B) \iff \exists x A(x) \to B$$

of which there is the following proof:

$$\begin{aligned}
&\forall x(A(x) \to B) \\
\iff& \\
&\forall x(\neg A(x) \vee B) \\
\iff& \\
&\forall x(\neg(A(x) \wedge \neg B)) \\
\iff& \\
&\neg\exists x(A(x) \wedge \neg B) \\
\iff& \\
&\neg(\exists x A(x) \wedge \exists x \neg B) \\
\iff& \text{ (since } x \text{ not free in } B\text{)} \\
&\neg(\exists x A(x) \wedge \neg B) \\
\iff& \\
&\neg\neg(\neg\exists x A(x) \vee B)) \\
\iff& \\
&\exists x A(x) \to B
\end{aligned}$$

Note additionally that, if $\exists x A(x)$ is assumed, we have

$$\forall x(A(x) \to B) \iff \exists x A(x) \to B \iff B$$

meaning that this is also valid predicate logic rule of inference, provided $\exists x A(x)$ is assumed.

To simplify justifying several reasoning steps, I will refer to the following simple rule as $\wedge$-*instantiation/ elimination*: if A and B are both assumed to hold, then

$$A \iff A \wedge B$$

and

$$B \iff A \wedge B$$

Note also the following proof rule:

$$\forall x(A(x) \to B(x)) \iff \forall x \forall y(xRy \wedge A(y) \to B(y))$$

The $\implies$ direction is proven as follows. Assume $\forall x(A(x) \to B(x))$ and let x, y be arbitrary such that both

xRy and $A(y)$ hold. Then

$$
\begin{aligned}
& xRy \;\&\; A(y) \\
\Longrightarrow\; & (\forall x(A(x) \to B(x)) \text{ holds, so } \wedge\text{-instantiation/elimination permitted}) \\
& xRy \;\&\; A(y) \;\&\; \forall x(A(x) \to B(x)) \\
\Longrightarrow\; & (\text{universal instantiation}) \\
& xRy \;\&\; A(y) \;\&\; (A(y) \to B(y)) \\
\Longrightarrow\; & (\text{modus ponens}) \\
& xRy \;\&\; B(y) \\
\Longrightarrow\; & \\
& B(y)
\end{aligned}
$$

Since x, y were arbitrary, we conclude that the $\Longrightarrow$ direction of the rule holds as indicated. The $\Longleftarrow$ direction is proven as follows. Assume $\forall x \forall y(xRy \;\&\; A(y) \to B(y))$ and let y be arbitrary such that $A(y)$ holds. Then,

$$
\begin{aligned}
& A(y) \\
\Longrightarrow\; & (\text{Assumption is assumed to hold, so } \wedge\text{-instantiation/elimination permitted}) \\
& A(y) \;\&\; [\forall x \forall y(xRy \;\&\; A(y) \to B(y))] \\
\Longrightarrow\; & (\text{using universal instantiation on assumption twice}) \\
& A(y) \;\&\; [yRy \;\&\; A(y) \to B(y)] \\
\Longrightarrow\; & (\text{modus ponens}) \\
& yRy \;\&\; B(y) \\
\Longrightarrow\; & (yRy \text{ always holds, so } \wedge\text{-elimination permitted}) \\
& B(y)
\end{aligned}
$$

Since y was arbitrary, we conclude that $\forall x(A(x) \to B(x)) \Longleftarrow \forall x \forall y(xRy \wedge A(y) \to B(y))$ holds as well, and so the rule holds as given before.

7.5 Derivable formulas

I use the following derivation of the derivable formulas denoted $K_@$ and Nom as examples of precise reasoning in predicate logic, which I will often use in the following sections. Note, however, that more informal reasoning is sufficient for the purposes of this text in mathematical logic, and so will also be deployed.

By derivable formulas I mean formulas that are valid and that may be derived using the proof system given.

Nota bene: throughout this text, if the justification of a reasoning step is simply the application of the definition of validity, then I will not provide the justification for that step. As such, please take blank spaces besides the $\Longleftrightarrow$ symbol to indicate that the definition of validity is being applied. Additionally, note that in all proofs of axiom, proof rule, and formula validity, $\mathcal{M}, g, s$ is taken to be arbitrary.

7.5.1 Derivable Formula $K_@$

The formula named $K_@$ is $\vdash @_i(A \to B) \to (@_i A \to @_i B)$. Validity is confirmed as follows. Take $\mathcal{M}, g, s$ to be arbitrary.

$$\models @_i(A \to B) \vdash (@_i A \to @_i B)$$

$\iff$ (Result in section 7.3)

$$\forall \mathcal{M}, g, s(\mathcal{M}, g, s \models @_i(A \to B) \implies \mathcal{M}, g, s \models (@_i A \to @_i B))$$

$\iff$

$$\forall \mathcal{M}, g, s(\mathcal{M}, g, g(i) \models (A \to B) \implies \mathcal{M}, g, s \models (@_i A \to @_i B))$$

$\iff$

$$\forall \mathcal{M}, g, s[\forall u \in S(g(i)Ru \;\&\; \mathcal{M}, g, u \models A \Rightarrow \mathcal{M}, g, u \models B) \implies \mathcal{M}, g, s \models (@_i A \to @_i B)]$$

$\iff$ $(\forall x \in S[\forall y \in S[A(y)]] \iff \forall x \in S[A(x)])$

$$\forall \mathcal{M}, g, s[(g(i)Rs \;\&\; \mathcal{M}, g, s \models A \Rightarrow \mathcal{M}, g, s \models B) \implies \mathcal{M}, g, s \models (@_i A \to @_i B)]$$

Considering $\mathcal{M}, g, t \models @_i A \to @_i B$ separately to help clarity:

$$\mathcal{M}, g, t \models @_i A \to @_i B$$

$\iff$

$$\forall u \in S(tRu \;\&\; \mathcal{M}, g, u \models @_i A \Rightarrow \mathcal{M}, g, u \models @_i B)$$

$\iff$

$$\forall u \in S(tRu \;\&\; \mathcal{M}, g, g(i) \models A \Rightarrow \mathcal{M}, g, g(i) \models B)$$

$\iff$ (since u no longer free in consequent, use $A(x) \to B \iff \exists x A(x) \to B$)

$$\exists u \in S(tRu \;\&\; \mathcal{M}, g, g(i) \models A) \Rightarrow \mathcal{M}, g, g(i) \models B$$

$\iff$ (use the logical law $[\exists x(A(x) \wedge B) \iff \exists x A(x) \wedge B]$)

$$[\exists u \in S(tRu) \;\&\; \mathcal{M}, g, g(i) \models A] \Rightarrow \mathcal{M}, g, g(i) \models B$$

$\iff$ (since tRt, $\exists u \in S(tRu)$ holds so $\wedge$-instantiation/elimination permitted)

$$\mathcal{M}, g, g(i) \models A \Rightarrow \mathcal{M}, g, g(i) \models B$$

This simplifies the last line of the previous block of reasoning to

$$\forall \mathcal{M}, g, s[(g(i)Rs \;\&\; \mathcal{M}, g, s \models A \Rightarrow \mathcal{M}, g, s \models B) \implies (\mathcal{M}, g, g(i) \models A \Rightarrow \mathcal{M}, g, g(i) \models B)]$$

$\iff$ $(\forall x(A(x) \to B) \iff \exists x A(x) \to B)$

$$\exists \mathcal{M}, g, s[g(i)Rs \;\&\; \mathcal{M}, g, s \models A \Rightarrow \mathcal{M}, g, s \models B] \implies (\mathcal{M}, g, g(i) \models A \Rightarrow \mathcal{M}, g, g(i) \models B)$$

$\iff$ ($g(i)Rg(i)$ and $\mathcal{M}, g, g(i) \models A \Rightarrow \mathcal{M}, g, g(i) \models B$ from 2nd block: use $\exists x A(x) \to B \iff B$)

$$\mathcal{M}, g, g(i) \models A \Rightarrow \mathcal{M}, g, g(i) \models B$$

which is clearly valid, so $K_@$ is a valid formula.

7.5.2 *Nom*

Nom denotes the potential axiom $\vdash (@_i j \wedge @_i A) \leftrightarrow @_j A$, which was identified as a derivable formula in [10]. The derivation is as follows.

$$\mathcal{M}, g, s \models (@_i j \wedge @_i A) \leftrightarrow @_j A$$

$$\iff$$

$$\forall t \in S[sRt \implies (\mathcal{M}, g, t \models (@_i j \wedge @_i A) \Leftrightarrow \mathcal{M}, g, t \models @_j A)]$$

$$\iff$$

$$\forall t \in S[sRt \implies (\mathcal{M}, g, s \models (@_i j \wedge @_i A) \Leftrightarrow \mathcal{M}, g, g(j) \models A)]$$

$$\iff$$

$$\forall t \in S[sRt \implies ((\mathcal{M}, g, s \models @_i j \text{ and } \mathcal{M}, g, s \models @_i A) \Leftrightarrow \mathcal{M}, g, g(j) \models A)]$$

$$\iff$$

$$\forall t \in S[sRt \implies ((\mathcal{M}, g, g(i) \models j \text{ and } \mathcal{M}, g, g(i) \models A) \Leftrightarrow \mathcal{M}, g, g(j) \models A)]$$

$$\iff \text{(since sRs holds, use } \forall x(A(x) \to B) \iff \exists x A(x) \to B \iff B)$$

$$(\mathcal{M}, g, g(i) \models j \text{ and } \mathcal{M}, g, g(i) \models A) \Leftrightarrow \mathcal{M}, g, g(j) \models A$$

which is valid since $\mathcal{M}, g, g(i) \models j$ if and only if $g(i) = g(j)$ so using substitution we get both directions of the equivalence above.

Therefore, $\vdash (@_i j \wedge @_i A) \leftrightarrow @_j A$ is a valid formula.

7.5.3 Axiom *Name*

Another classical axiom that turns out to be a derivable formula is *Name*, which denotes '$\vdash \downarrow i @_i A \to A$, provided $i \notin \text{fnom}(A)$'. This turns out to be the combination of the *bindAt* and *vacBind* axioms, which are proven to be valid in the section below. It expresses that if the current state is named i and A is then shown to hold at i, then A holds in all accessible states. This is because the state named with $\downarrow i$ was arbitrary, so showing $@_i A$ is then showing A holds at an arbitrary state. The validity of this axioms is shown as follows.

$$\mathcal{M}, g, s \models \downarrow i @_i A \to A$$

$$\iff$$

$$\forall u \in S([sRu \;\&\; \mathcal{M}, g, u \models \downarrow i @_i A] \Rightarrow \mathcal{M}, g, u \models A)$$

$$\iff$$

$$\forall u \in S([sRu \;\&\; \mathcal{M}, g[i := u], u \models @_i A] \Rightarrow \mathcal{M}, g, u \models A)$$

$$\iff$$

$$\forall u \in S([sRu \;\&\; \mathcal{M}, g[i := u], g[i := u](i) \models A] \Rightarrow \mathcal{M}, g, u \models A)$$

$$\iff \text{(definition of locally changed nominal assignment)}$$

$$\forall u \in S([sRu \;\&\; \mathcal{M}, g[i := u], u \models A] \Rightarrow \mathcal{M}, g, u \models A)$$

Now, consider that i is not free in A means $g = g[i := u]$ on $\text{fnom}(A)$. It is then clear that for any $u \in S$ such sRu and $\mathcal{M}, g[i := u], u \models A$ both hold, it follows from J-equivalence that $\mathcal{M}, g, u \models A$. Therefore, the last line in the above block of reasoning is always valid and so *Name* may be taken as an axiom.

7.6 Validity of Axioms

I proceed by examining the axioms presented in professor Renardel de Lavalette in [10] and repeated in subsection 7.1, in no particular order.

7.6.1 Axiom $Dec_@$

The $Dec_@$ axiom, $\vdash @_i A \lor \neg @_i A$, which expresses the decidability of satisfaction operators, is shown to be valid as follows.

$$\begin{aligned}
&\mathcal{M}, g, s \models @_i A \lor \neg @_i A \\
\iff& \\
&\mathcal{M}, g, s \models @_i A \text{ or } \mathcal{M}, g, s \models \neg @_i A \\
\iff& \\
&\mathcal{M}, g, g(i) \models A \text{ or } \forall t \in S(sRt \Rightarrow \mathcal{M}, g, t \not\models @_i A) \\
\iff& \\
&\mathcal{M}, g, g(i) \models A \text{ or } \forall t \in S(sRt \Rightarrow \mathcal{M}, g, g(i) \not\models A) \\
\iff& \text{ (universal-existential rule)} \\
&\mathcal{M}, g, g(i) \models A \text{ or } ([\exists t \in S(sRt)] \Rightarrow \mathcal{M}, g, g(i) \not\models A) \\
\iff& \text{ (since } sRs\text{, so use } \exists x A(x) \to B \iff B) \\
&\mathcal{M}, g, g(i) \models A \text{ or } \mathcal{M}, g, g(i) \not\models A
\end{aligned}$$

which is a tautology and so $@_i A \lor \neg @_i A$ is valid and $Dec_@$ may be taken as an axiom.

7.6.2 Axiom $Intr$

The proposed axiom $Intr$ has the two parts of $\vdash (i \land A) \to @_i A$ and $\vdash (i \land @_i A) \to A$. The first is proven as follows.

$$\begin{aligned}
&\mathcal{M}, g, s \models i \land A \to @_i A \\
\iff& \\
&\forall t \in S([sRt \ \& \ \mathcal{M}, g, t \models i \land A] \Rightarrow \mathcal{M}, g, t \models @_i A) \\
\iff& \\
&\forall t \in S([sRt \ \& \ (\mathcal{M}, g, t \models i \ \& \ \mathcal{M}, g, t \models A)] \Rightarrow \mathcal{M}, g, g(i) \models A) \\
\iff& \\
&\forall t \in S([sRt \ \& \ g(i) = t \ \& \ \mathcal{M}, g, t \models A] \Rightarrow \mathcal{M}, g, g(i) \models A) \\
\iff& \text{ (using substitution)} \\
&\forall t \in S([sRt \ \& \ g(i) = t \ \& \ \mathcal{M}, g, g(i) \models A] \Rightarrow \mathcal{M}, g, g(i) \models A)
\end{aligned}$$

which is valid as may be checked by a simple truth table.

Likewise, the second is proven by the following reasoning. Firstly, by definition, for arbitrary $\mathcal{M}, g, s$ it is the case that

$$\models (i \wedge @_i A) \rightarrow A$$

$$\iff$$

$$\forall \mathcal{M}, g, s(\mathcal{M}, g, s \models (i \wedge @_i A) \implies \mathcal{M}, g, s \models A)$$

$$\iff$$

$$\forall \mathcal{M}, g, s([\mathcal{M}, g, s \models i \text{ and } \mathcal{M}, g, g(i) \models A] \implies \mathcal{M}, g, s \models A)$$

$$\iff$$

$$\forall \mathcal{M}, g, s([s = g(i) \text{ and } \mathcal{M}, g, g(i) \models A] \implies \mathcal{M}, g, s \models A)$$

which is again clearly valid.

As such, *Intr* may be taken as an axiom.

7.6.3 Axiom $T_@$

The reflexivity axiom $T_@, \vdash @_i i$, which expresses the reflexivity of frames in terms of the satisfaction operator @, is shown to be an axiom as follows.

$$\mathcal{M}, g, s \models @_i i$$

$$\iff$$

$$\mathcal{M}, g, g(i) \models i$$

$$\iff$$

$$g(i) = g(i)$$

which is valid, so $T_@$ may be taken as an axiom.

7.6.4 Axiom *Agree*

The *Agree* axiom, $\vdash @_i @_j A \leftrightarrow @_j A$, expresses that if it can be proven that a formula holds at a particular state named j, then that formula can also be proven to hold at that state named j at another state named i. It's validity is demonstrated as follows.

$$\mathcal{M}, g, s \models @_i @_j A \leftrightarrow @_j A$$

$$\iff$$

$$\forall t \in S(sRt \,\&\, \mathcal{M}, g, s \models @_i @_j A \iff sRt \,\&\, \mathcal{M}, g, s \models @_j A)$$

$$\iff$$

$$\forall t \in S(sRt \,\&\, \mathcal{M}, g, g(i) \models @_j A \iff sRt \,\&\, \mathcal{M}, g, s \models @_j A)$$

$$\iff$$

$$\forall t \in S(sRt \,\&\, \mathcal{M}, g, g(j) \models A \iff sRt \,\&\, \mathcal{M}, g, g(j) \models A)$$

which is valid. Therefore, $@_i @_j A \iff @_j A$ is valid. and so *Agree* may be taken as an axiom

7.6.5 Axiom $\bot_@$

The potential axiom $\vdash @_i\bot \to \bot$, denoted by $\bot_@$, expresses that if falsity is derivable at any state, then it is derivable at all accessible states. This is consistent with the semantics of HIpL that permits no validation of $\bot$ at any state, which in turn overlaps with the intuitionistic background assumption mentioned by Troelstra and van Dale in [21], which states that it is assumed that no information leading to a contradiction will be found as one moves from state to state. Consequently, if a contradiction would hold at a particular state, it is indicative of a model in which a state without contradiction cannot be expected to exist.

$$\begin{aligned}
&\mathcal{M}, g, s \models @_i\bot \to \bot \\
\iff& \\
&\mathcal{M}, g, s \models @_i\bot \Rightarrow \mathcal{M}, g, s \models \bot \\
\iff& \\
&\mathcal{M}, g[i := s], s \models \bot \Rightarrow \mathcal{M}, g, s \models \bot
\end{aligned}$$

Now, since by the definition of semantics it is known that $\mathcal{M}, g[i := s], s \not\models \bot$, it follows that $\mathcal{M}, g[i := s], s \models \bot$ gives a contradiction, and since anything follows from a contradiction, $\mathcal{M}, g[i := s], s \models \bot \Rightarrow \mathcal{M}, g, s \models \bot$ is valid. This in turn means $@_i\bot \to \bot$ is valid, and so may be taken as an axiom.

7.6.6 Axiom $refl_\diamond$

The reflexivity axiom $\vdash @_i\diamond i$ expresses the reflexivity of frames in HIpL in terms of reachability of a state named i from that same state named i. It's validity is shown as follows.

First, for arbitrary $\mathcal{M}, g, s$ and by the definition of $\models$, we have that $\models @_i\diamond i$ if and only if $\mathcal{M}, g, g(i) \models \diamond i$, which by definition holds if and only if $g(i)Rg(i)$, which always holds by the reflexivity of HIpL frames. Therefore, $\models @_i\diamond i$ is implied by a valid expression, and is therefore valid.

As such, the $refl_\diamond$ may be taken as an axiom.

7.6.7 Axiom $mon_\diamond$

The monotonicity axiom $mon_\diamond$ has two parts. Firstly, $p, \diamond i \vdash @_ip$, and secondly, $A \to B, \diamond i \vdash @_i(A \to B)$. This expresses that HIpL retains partial monotonicity. The first part of the axiom expresses that if a propositional variable p holds at a state where the state named i is accessible from, then p holds at the state named i also, such that we have monotonicity of propositional variables in the set of accessible states. The second part expresses that this same result holds for implication with formulae, which is by no means full-fledged monotonicity of formulae, but does give monotonicity of formulae in this subcase. The validity of this axiom is shown as follows.

For the first part, recall that in subsection 7.3 it was noted that $p, \diamond i \models @_ip$ if and only if $\models (p\wedge i) \to @_ip$, which by the definition of $\models$ is equivalent to asserting that for arbitrary $\mathcal{M}, g, s$, we have

$$(\mathcal{M}, g, s \models p \ \&\ \mathcal{M}, g, s \models \diamond i) \text{ implies } \mathcal{M}, g, g(i) \models p \tag{3}$$

Now, the definition $\models$ gives that $\mathcal{M}, g, s \models p$ if and only $s \in V(p)$, which by the definition of the intuitionistic valuation function is the set $\{V \subseteq S \mid \forall s \in V \forall t \in S(sRt \Rightarrow t \in V)\}$. Note further that by definition of $\models$, $\mathcal{M}, g, s \models \diamond i$ if and only if $sRg(i)$ holds. Therefore, $g(i) \in V(p)$, meaning that if the antecedent of (3) holds, then $\mathcal{M}, g, g(i) \models p$. As such, $\models (p \wedge i) \to @_ip$ is valid.

For the second part, it is necessary to show validity of $\models ([A \to B] \wedge \diamond i) \to @_i(A \to B)$. As this is equivalent by the definition of $\models$ to asserting that for arbitrary $\mathcal{M}, g, s$ we have

$$([\mathcal{M}, g, s \models A \implies \mathcal{M}, g, s \models B] \ \&\ \mathcal{M}, g, s \models \diamond i) \text{ implies } \mathcal{M}, g, g(i) \models A \implies \mathcal{M}, g, g(i) \models B \tag{4}$$

By the definition of $\models$, if $\mathcal{M}, g, s \models A \implies \mathcal{M}, g, s \models B$ holds then $\forall t \in S(sRt \;\&\; \mathcal{M}, g, t \models A \Rightarrow \mathcal{M}, g, t \models B)$. Since $sRg(i)$ holds as in the reasoning for the first part of $mon_\diamond$, it immediately follows that the antecedent implies $\mathcal{M}, g, g(i) \models A \implies \mathcal{M}, g, g(i) \models B$ and so $\models ([A \to B] \wedge \diamond i) \to @_i(A \to B)$ is also valid.

Therefore, since both parts of $mon_\diamond$ is valid, it may be taken as an axiom.

7.6.8 Axiom *bindAt*

The *bindAt* axiom, $\vdash \downarrow i @_i A \leftrightarrow \downarrow i A$, expresses that if the current state is named i using $\downarrow i$, then to shift the point of evaluation to the state named i by using $@_i$ is vacuous: $@_i$ then shifts the point of evaluation from the current state to the current state. What is to be demonstrated is that $\models \downarrow i @_i A$ if and only if $\models \downarrow i A$. This is proven as follows. For arbitrary $\mathcal{M}, g, s$ we have

$$\begin{aligned} & \mathcal{M}, g, s \models \downarrow i @_i A \\ \iff & \\ & \mathcal{M}, g[i := s], g[i := s](i) \models A \\ \iff & \text{ (Using } g[i := s](i) = s) \\ & \mathcal{M}, g[i := s], s \models A \\ \iff & \\ & \mathcal{M}, g, s \models \downarrow i A \end{aligned}$$

meaning that the validity of one implies the validity of the other, and so we have that the *bindAt* may be taken as an axiom.

7.6.9 Axiom *vacBind*

The axiom *vacBind* denotes $\vdash \downarrow i A \leftrightarrow A$, provided $i \notin \text{fnom}(A)$. This expresses that binding a formula in which the nominal being used for binding is not free in, is equivalent to not binding it, since whether A holds at the state named i is not influenced by the occurence of i in A precisely because $i \notin \text{fnom}(A)$. Now, it is to be demonstrated that $\models \downarrow i A$ if and only if $\models A$, provided $i \notin \text{fnom}(A)$.

Assume $i \notin \text{fnom}(A)$, and note for arbitrary $\mathcal{M}, g, s$ we have that $\mathcal{M}, g, s \models \downarrow i A$ if and only if $\mathcal{M}, g[i := s], s \models A$ by definition. Now, consider that i is not free in A means $g = g[i := u]$ on $\text{fnom}(A)$. It then follows from J-equivalence that $\mathcal{M}, g[i := s], s \models A$ if and only if $\mathcal{M}, g, s \models A$. As such, the equivalence $\models \downarrow i A$ if and only if $\models A$ holds, provided $i \notin \text{fnom}(A)$.

Consequently, *vacBind* may be taken as an axiom.

7.7 Validity of Proof Rules

7.7.1 Proof Rule $\to$R

The proof rule denoted by $\to$**R** states that 'if $A \vdash B$ then $\vdash A \to B$', so we must demonstrate 'if $A \models B$ then $\models A \to B$.' Assume $A \models B$. Then, by the reasoning in subsection 7.3, $A \models B \iff \models A \to B$ and so the validity of $A \to B$ follows from the validity of proof rule holds.

7.7.2 Proof Rule *Cut*

The *Cut* rule states that 'if $\Gamma \vdash A$ and $\Delta, A \vdash B$ then $\Gamma, \Delta \vdash B$', so it must be demonstrated that if $\Gamma \models A$ and $\Delta, A \models B$, then $\Gamma, \Delta \models B$.

Assume $\Gamma \models A$ and $\Delta, A \models B$. By definition, this is equivalent to assuming that for arbitrary $\mathcal{M}, g, s$, we have

$$\mathcal{M}, g, s \models \Gamma \text{ implies } \mathcal{M}, g, s \models A \tag{5}$$

and

$$\mathcal{M}, g, s \models \Delta \wedge A \text{ implies } \mathcal{M}, g, s \models B \tag{6}$$

Now, $\Gamma, \Delta \models B$ if and only if $\mathcal{M}, g, s \models \Gamma \wedge \Delta$ implies $\mathcal{M}, g, s \models B$, by the definition of $\Gamma \models A$ and claim (2) in subsection 7.3. This enables us to reason as follows.

$$\begin{aligned}
&\mathcal{M}, g, s \models \Gamma \wedge \Delta \\
\Longleftrightarrow& \\
&\mathcal{M}, g, s \models \Gamma \;\&\; \mathcal{M}, g, s \models \Delta \\
\Longrightarrow& \text{ (Using (5))} \\
&\mathcal{M}, g, s \models A \;\&\; \mathcal{M}, g, s \models \Delta \\
\Longleftrightarrow& \\
&\mathcal{M}, g, s \models A \wedge \Delta \\
\Longrightarrow& \text{ (Using (6))} \\
&\mathcal{M}, g, s \models B
\end{aligned}$$

Therefore, the consequent of the proof rule has been shown to hold using the assumptions in the antecedent. I conclude that the cut rule is valid.

7.7.3 Proof Rule $SNec_{@}$

The satisfaction strong necessitation proof rule is stated as 'if $\Gamma \vdash A$ then $@_i\Gamma \vdash @_i A$.' As such, it must be demonstrated that 'if $\Gamma \models A$ then $@_i\Gamma \models @_i A$' is a valid inference.

Assume $\Gamma \models A$. Observe the following equivalence:

$$\begin{aligned}
&@_i\Gamma \models @_i A \\
\Longleftrightarrow& \text{ (Definition of } \Gamma \models A) \\
&\forall \mathcal{M}, g, s(\mathcal{M}, g, s \models @_i\Gamma \Rightarrow \mathcal{M}, g, s \models @_i A) \\
\Longleftrightarrow& \\
&\forall \mathcal{M}, g, s(\mathcal{M}, g, g(i) \models \Gamma \Rightarrow \mathcal{M}, g, g(i) \models A) \\
\Longleftrightarrow& \text{ (since } s \text{ is not free in any term, it maybe eliminated and introduced freely)} \\
&\forall \mathcal{M}, g(\mathcal{M}, g, g(i) \models \Gamma \Rightarrow \mathcal{M}, g, g(i) \models A)
\end{aligned}$$

We now use the assumption as follows:

$$\begin{aligned}
&\Gamma \models A \\
\Longleftrightarrow\ & (\text{Definition of } \Gamma \models A) \\
&\forall \mathcal{M}, g, s(\mathcal{M}, g, s \models \Gamma \Rightarrow \mathcal{M}, g, s \models A) \\
\Longrightarrow\ & (\text{since } g(i) \in S \text{ we may instantiate}) \\
&\forall \mathcal{M}, g(\mathcal{M}, g, g(i) \models \Gamma \Rightarrow \mathcal{M}, g, g(i) \models A) \\
\Longleftrightarrow\ & (\text{by the first block of reasoning}) \\
&@_i\Gamma \models @_i A
\end{aligned}$$

Therefore, since the validity of $@_i\Gamma \models @_i A$ follows from the validity of $\Gamma \models A$, this is a valid proof rule.

7.7.4 Proof Rule $SNec_\downarrow$

The $SNec_\downarrow$proof rule asserts that if $\Gamma \vdash A$ then $\downarrow i\Gamma \vdash \downarrow iA$. What is to be demonstrated is that $\Gamma \models A$ implies $\downarrow i\Gamma \models \downarrow iA$. Assume $\Gamma \models A$. Note the following.

$$\begin{aligned}
&\downarrow i\Gamma \models \downarrow iA \\
\Longleftrightarrow\ & (\text{Definition of } \models) \\
&\forall \mathcal{M}, g, s(\mathcal{M}, g[i := s], s \models \Gamma \implies \mathcal{M}, g[i := s], s \models A)
\end{aligned}$$

The assumption can then be used as follows.

$$\begin{aligned}
&\Gamma \models A \\
\Longleftrightarrow\ & (\text{Definition of } \Gamma \models A) \\
&\forall \mathcal{M}, g, s(\mathcal{M}, g, s \models \Gamma \Rightarrow \mathcal{M}, g, s \models A)
\end{aligned}$$

This gives rise to a case distinction. Firstly, if $i \notin \text{fnom}(\Gamma, A)$ then, by the same reasoning as in the proof of validity of the *vacBind* axiom, it follows from J-equivalence that $\mathcal{M}, g[i := s], s \models A$ if and only if $\mathcal{M}, g, s \models A$ and the desired implication is therefore obtained directly.

Secondly, if $i \in \text{fnom}(\Gamma, A)$ then binding Γ and A simultaneously by $\downarrow i$ means the state mapped to by $g(i)$ is simply changed in both $\mathcal{M}, g[i := s], s \models A$ and $\mathcal{M}, g[i := s], s \models \Gamma$ to state s, which preserves the $\mathcal{M}, g, s \models \Gamma \Rightarrow \mathcal{M}, g, s \models A$ implication. As such, in this case $\mathcal{M}, g, s \models \Gamma \Rightarrow \mathcal{M}, g, s \models A$ implies $\mathcal{M}, g[i := s], s \models \Gamma \Rightarrow \mathcal{M}, g[i = s], s \models A$.

As such, we conclude

$$
\begin{aligned}
& \Gamma \models A \\
\iff & \text{ (Definition of } \Gamma \models A) \\
& \forall \mathcal{M}, g, s(\mathcal{M}, g, s \models \Gamma \Rightarrow \mathcal{M}, g, s \models A) \\
\implies & \text{ (Above reasoning)} \\
& \forall \mathcal{M}, g, s(\mathcal{M}, g[i := s], s \models \Gamma \Rightarrow \mathcal{M}, g[i := s], s \models A) \\
\iff & \\
& {\downarrow} i \Gamma \models {\downarrow} i A
\end{aligned}
$$

Therefore, the validity of $\Gamma \models A$ implies the validity of ${\downarrow} i \Gamma \models {\downarrow} i A$, and so the $SNec_{\downarrow}$is a valid proof rule.

7.7.5 Proof Rule PR

The paste rule PR for HIpL is as follows: 'if $\Gamma, @_i \Diamond j, @_j A \vdash @_j B$ and $j \notin \mathsf{fnom}(\Gamma, A, B)$ then $\Gamma \vdash @_i(A \to B)$.' As such, it must be demonstrated that if $\Gamma, @_i \Diamond j, @_j A \models @_j B$ then $\Gamma \models @_i(A \to B)$ with $j \notin \mathsf{fnom}(\Gamma, A, B)$. Firstly, assume $\Gamma, @_i \Diamond j, @_j A \models @_j B$. Note that j either does not occur in Γ, A, or B, or is bound by $\downarrow$ in them, since $j \notin \text{fnom}(\Gamma, A, B)$. We have by claim (2) that $\Gamma, @_i \Diamond j, @_j A \models @_j B \iff \models (\Gamma \wedge @_i \Diamond j \wedge @_j A) \to @_j B$.

Simple reasoning shows the following equivalence.

$$
\begin{aligned}
& \Gamma \models @_i(A \to B) \\
\iff & \text{ (Definition of } \Gamma \models A \text{ and } \models) \\
& \forall \mathcal{M}, g, s(\mathcal{M}, g, s \models \Gamma \implies \forall t \in S(g(i) R t \;\&\; \mathcal{M}, g, s \models A \Rightarrow \mathcal{M}, g, t \models B))
\end{aligned}
$$

We now use the assumption as follows.

$$
\begin{aligned}
& \models (\Gamma \wedge @_i \Diamond j \wedge @_j A) \to @_j B \\
\iff & \text{ (Definition of } \models) \\
& \forall \mathcal{M}, g, s([\mathcal{M}, g, s \models \Gamma \;\&\; \mathcal{M}, g, g(i) \models \Diamond j \;\&\; \mathcal{M}, g, g(j) \models A] \implies \mathcal{M}, g, g(j) \models B) \\
\iff & \\
& \forall \mathcal{M}, g, s([\mathcal{M}, g, s \models \Gamma \;\&\; g(i) R g(j) \;\&\; \mathcal{M}, g, g(j) \models A] \implies \mathcal{M}, g, g(j) \models B) \\
\iff & \text{ (Using } x \to y \iff \neg x \vee y \text{ twice with rearrangement)} \\
& \forall \mathcal{M}, g, s(\mathcal{M}, g, s \models \Gamma \implies (\neg g(i) R g(j) \text{ or } \neg \mathcal{M}, g, g(j) \models A \text{ or } \mathcal{M}, g, g(j) \models B) \\
\iff & \text{ (Using } x \to y \iff \neg x \vee y) \\
& \forall \mathcal{M}, g, s(\mathcal{M}, g, s \models \Gamma \implies (g(i) R g(j) \;\&\; \mathcal{M}, g, g(j) \models A \Rightarrow \mathcal{M}, g, g(j) \models B)
\end{aligned}
$$

$$\Longrightarrow \ (i \notin \text{fnom}(\Gamma, A, B))$$

$$\forall \mathcal{M}, g, s(\mathcal{M}, g, s \models \Gamma \implies \forall t \in S(g(i)Rt \ \& \ \mathcal{M}, g, t \models A \Rightarrow \mathcal{M}, g, t \models B)$$

$$\Longleftrightarrow \ (\text{Using first block of reasoning})$$

$$\Gamma \models @_i(A \to B)$$

where the implication justified by $i \notin \text{fnom}(\Gamma, A, B)$ is justified in this way since this means that j names an arbitrary *accessible* state (relative to $g(i)$) named j, and as such replacing j by another nominal naming another *accessible* state is permissible, but this is the same as ranging over all accessible states, as is done.

Consequently, the paste rule PR may be taken as a valid proof rule in the proof system.

7.8 Lemma Proven

Similar reasoning shows the soundness of the remaining axioms and proof rules as well. As such, this completes the proof of the soundness of the proof system presented in subsection 7.1.

8 Counterexamples

In this section I overview some formulas and rules which one might expect could be potential axioms and proof rules and yet fail to be such in the current semantics.

8.0.1 Classical DA

This version of the downarrow axiom came from [15] but does not survive the transition from a classical to intuitionistic context. Classical DA denotes $\vdash @_i(\downarrow jA \leftrightarrow A[j := i])$.

$$\mathcal{M}, g, s \models @_i(\downarrow jA \leftrightarrow A[j := i])$$

$$\Longleftrightarrow$$

$$\mathcal{M}, g, g(i) \models \downarrow jA \leftrightarrow A[j := i]$$

$$\Longleftrightarrow$$

$$\forall t \in S(g(i)Rt \ \& \ \mathcal{M}, g, t \models \downarrow jA \iff g(i)Rt \ \& \ \mathcal{M}, g, t \models A[j := i])$$

$$\Longleftrightarrow$$

$$\forall t \in S(g(i)Rt \ \& \ \mathcal{M}, g[j := t], t \models A \iff g(i)Rt \ \& \ \mathcal{M}, g, t \models A[j := i])$$

$$\Longleftrightarrow \ (\text{by substitution property})$$

$$\forall t \in S(g(i)Rt \ \& \ \mathcal{M}, g[j := t], t \models A \iff g(i)Rt \ \& \ \mathcal{M}, g \circ [j := i], t \models A)$$

Now, consider that if $A = j$, then

$$\mathcal{M}, g \circ [j := i], t \models j \iff g \circ [j := i](j) = g(i) = t$$

On the other hand,

$$\mathcal{M}, g[j := t], t \models j \iff g[j := t](j) = t$$

where the second always holds, and the first only holds if $g(i) = t$. As such, the equivalence in the last line of the above block of reasoning does not hold for all t in S.

As such, $\vdash @_i(\downarrow jA \leftrightarrow A[j := i])$ cannot be taken as an axiom.

9 Conclusion

Introductions to modal, intuitionistic, and hybrid modal logic were all presented, with necessary concepts and results such as Kripke semantics and complete proof systems included. The semantics of hybrid intuitionistic logic was then presented, followed by a sufficient number of proofs of validity of various axioms and proof rules in the axiomatization presented in subsection 7.1 to conclude soundness of this axiomatization. Various additional results regarding HIpL were also noted, as well as a discussion of potential axioms that were found to be derivable, and a discussion of classical DA which turned out to be invalid in the context of the semantics presented in this thesis. To sum up, soundness of a proof system for HIpL was proven and various aspects of the logic explored.

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