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A characterization of teaching mathematical limits at university level

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Abstract

The purpose of this study is to characterize the teaching of the concept of limits to first-year bachelor degree students at a Dutch university. In order to achieve this purpose, two video-recorded lectures were analyzed through the analytical framework by Mali and Petropoulou (2017). Categories of teaching were used to identify explaining and extending actions within the lectures. This study suggests the DTP format was used in the lectures. Graphical, as well as symbolic representations were used in mathematical explanations. The analyzed lectures were mostly based on explaining definitions and theorems by proving them and showing its applications, rather than detailing the reasoning behind them.

Contents

1 Introduction

The concept of limits is a part of calculus that is used with continuity, differentiability and integration. The classical definition used in most calculus and analysis classes is the Weierstrass definition, also referred to as the (ϵ,δ)-definition (Bokhari & Yushau, 2006). The definition reads as follows (Stewart, 2010, p. 110):

Definition 1.1 ((ϵ,δ)-definition)**.** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the limit of $f(x)$ as x approaches a is L, and we write,

$$\lim_{x\to a} f(x) = L$$

if for every number $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon.$$

More generally, this definition is rewritten as

"$\lim_{x\to a} f(x) = L$ means that the distance between $f(x)$ and L can be made arbitrarily small by taking the distance from x to a sufficiently small (but not 0)".

Alternatively,

"$\lim_{x\to a} f(x) = L$ means that the values of $f(x)$ can be made as close as we please to L by taking x close enough to a (but not equal to a)".

Despite the concept of limits being a part of calculus that is used in other concepts, there seems to be a lack of understanding of this concept. For university students, limits is a rather difficult concept within calculus. Students tend to have problems with this concept when it comes to dealing with non-routine limit problems (Hardy, 2009). Non-routine problems refer to problems that are not solved algebraically, but by inspection. There seems to be reasons for this misconception. The misconception by students is caused by a difference between the everyday use of limits and the mathematical language (Liang, 2016). Furthermore, the difficulty of teaching this concept is strongly related to lecturers' attitude, content knowledge, and everyday instructions.

Limits are an important concept within calculus, and are used to define continuity, derivatives and integrals. It is important to study the teaching of calculus because it has been concluded that the quality of teaching is essential in encouraging students to continue with their university studies in mathematics (Bressoud, Carlson, Mesa, & Rasmussen, 2013). How any concept within mathematics is taught turned out very essential. Also Bezuidenhout (2001), who examined the understanding of limits in first-year students, concluded that the way mathematical concepts are taught is very crucial. According to these studies, there tended to exist a deficient conceptual understanding of the relationships between the concept of limits, continuity, and differentiability. Personally, I see great potential in explicitly focusing on the conceptual groundwork of these concepts in order to facilitate students in solving problems related to mathematical limits. Simply because there is not just one clear theorem around the concept of limits, choosing the right definition can be very challenging. From my own experience, I can tell it is not always clear-cut which definition is best to select in a particular situation and how to apply it.

Research on teaching the concept of limits is relatively under-investigated. This kind of research is not as broad as research on students learning the concept of limits (Güçler, 2013). How the lecturer explains different definitions and theorems while providing examples or interactions with students can be studied in order to analyze approaches of teaching.

This information can shed light on the effective teaching components. Simultaneously, the qualitative analysis on teaching the concept of limits can provide insights into the identification of approaches used in teaching the limit concept, and produces a categorization of those approaches. This knowledge may be useful to lecturers who intend to search for different approaches to do their teaching.

2 Literature Review

The research on teaching the concept of limit is not as extensive as other research on student learning on this concept (Güçler, 2013). Within this literature review, the literature found and analyzed included research done in several countries outside the Netherlands. A total of nine articles on the current topic were found, which indicates the limited research on this topic. The peculiar fact that no research has been conducted in the Netherlands thus far, motivated me to dive into the matter. Hence, with this thesis, I aim to contribute to this gap of missing data in the Netherlands. The methodology of the literature review, describing the outline of the literature search can be found in appendix A.

2.1 Weierstrass Definition

The formal definition of the concept of limit that is stated in the introduction and used in most countries, is the Weierstrass definition. This classical definition of the limit is conceived as the most problematic part of calculus for students (Bokhari & Yushau, 2006). Due to that, this definition is often skipped by students, because they both find it difficult to understand and do not consider it as useful. It is also said by the researchers that lecturers put little emphasis on this definition while teaching or even skip it. This is seen as a consequence of the shift of this definition in Calculus textbooks from the beginning to the end of the chapter. In some books the whole definition is surprisingly ignored at all. The role of this definition is however identified as crucial in the justification of certain statements and therefore cannot be ignored. Bokhari and Yushau (2006) investigated the reformulation of this definition.

The researchers concluded that the terminology used to define the Weierstrass definition influenced the insufficiency of the students' understanding of the definition. It is stated that an alternative definition, the local (L,ϵ)-approximation, presented by those researchers, will be interesting for lecturers as well as students. Furthermore, mathematical software used for showing examples can contribute to motivate students at the start of their calculus courses. The local (L,ϵ)-approximation is defined as follows: first, the researchers introduced a list of notations ($I_a, P_a, R_a, L_a, I_{-\infty}, I_\infty$) that appear in the definition. An open interval ($I_a, R_a, L_a, I_{-\infty}, I_\infty$) is identified as an interval that does not include its endpoints. A punctured interval (P_a) is identified as a neighborhood around a point a, excluding point a.

I_a : open interval containing a given point$a$.
P_a : punctured interval at a given by$I_a \backslash \{a\}$.
R_a : open interval on the right of a given by$I_a \cap (a, \infty)$.
L_a : open interval on the left of a given by$(-\infty, a) \cap I_a$.
$I_{-\infty}$: open interval of the form $(-\infty, M)$ where M is a real number.
I_∞ : open interval of the form (M, ∞) where M is a real number.

Definition 2.1 (Local (L,ϵ)-approximation)**.** For a given $\epsilon > 0$, we say that $y = L$ is a local

(i) (L,ϵ)-approximation of a function f at a point a if there is a punctured interval P_a at a in which the deviation between $f(x)$ and L is at most ϵ, i.e. $x \in P_a \implies |f(x) - L| < \epsilon$;
(ii) left (L,ϵ)-approximation of a function f at a point a if there is an interval L_a in which the deviation between $f(x)$ and L is at most ϵ;
(iii) right (L,ϵ)-approximation of a function f at a point a if there is an interval R_a in which the deviation between $f(x)$ and L is at most ϵ;
(iv) (L,ϵ)-approximation of a function f at ∞ if there is an interval I_∞ in which the deviation between $f(x)$ and L is at most ϵ;
(v) (L,ϵ)-approximation of a function f at $-\infty$ if there is an interval $I_{-\infty}$ in which the deviation between $f(x)$ and L is at most ϵ.

I intend to analyze my data with the awareness that the (ϵ, δ)-definition is difficult to grasp for students. Secondly, I intend to analyze if this definition is used by the lecturer, and if so, how it is explained.

2.2 Communication Between Lecturers and Students

Limit is a fundamental concept of calculus; however, it involves a lot of difficulties related to the underlying concept that causes major challenges for students. Güçler (2013) examined the discourse on the limit concept using a communicational approach based on the notion that learning is a "change in one's discourse through becoming a participant in a discourse community" (Güçler, 2013, p. 440). That is, the lecturer's discourse is seen as a central unit where the language, rules, and objects play a huge role in the different parts of the communication within the lecture. In order to provide further insight about the issues that arise when it comes to teaching the limit concept, Güçler identified the lecturer's discourse and his patterns of communicational links and failures during the lectures. This is done with a combination of eight video-taped classroom observations, a survey given to 23 students who were present during the classroom observations, task-based interviews with four students and students' written work. This all took place in a beginning-level undergraduate calculus classroom at an American university. The discourses of both the lecturer and the students were analyzed with respect to the following: word use, routines, and endorsed narratives.

The classroom observations showed that there were few instances of lecturer-student interactions. The lecturer-student interactions is an element that will matter in the data analysis of the current thesis, and whether this occurs or not. The analysis also showed that the most computation problems regarding the limit concept during the classroom observations were solved using only an algebraic approach (43 out of 64); a few of the problems were solved using a graphical approach (16 out of 64); the remaining problems were solved using both approaches. The lecturer used different underlying rules to identify specific approaches in his teaching, called metarules. He mostly used symbolic representation, but graphing and algebraic approaches were present as well. Additionally, the lecturer was very consistent in his terminology and routines. According to the researcher, this implied that he was well aware of different realizations and uses of the limit concept corresponding to them. Unfortunately, this case study does not address typical patterns of teaching calculus, because the metarules highlighted do not cover all of these. Despite that, this study did denote the essence of lecturers' coherent terminology to have students understand the subject correctly.

At college level, the limit concept itself is not treated as it should be. That is why Hardy (2009) wanted to examine the different interpretations of students and lecturers regarding the limit concept. The calculus course that was investigated had 14 different lecturers, with 25-35 enrolled students in each class. The research was based on task-based interviews with students, using tasks created to resemble the typical final examination tasks used the past few years. Additionally, lecturers' solutions and examples in the textbook were analyzed. A North American college institution was used for this research.

The results yielded that the knowledge to be learned differed from the knowledge to be taught. As stated, "students appear to classify limits of rational expressions into different types of tasks according to their algebraic appearance, instead of using some calculus criteria" (Hardy, 2009, p. 354). Specifically, students' way of tackling a limit problem, is based on the recall of a set of steps, given by the lecturer or the textbook, instead of mathematical thinking. Another observation found suggested that certain topics mentioned in the textbook and by the lecturer were not tested during the final examination. Obviously, not all theory can be tested during a single final examination. However, as it turned out that other subjects were repeated multiple times it was easy for students to simply recognize test questions and copy their previous answers. Due to this pitfall, students' own mathematical reasoning was not challenged during later examinations.

With these results in mind, I intend to emphasize my analysis on what methods are used and if links are made between different definitions and techniques for tackling limit problems. I intend to emphasize whether mathematical thinking is encouraged during lectures rather than the solely focus on a set of steps the students have to follow.

2.3 Teaching with Technological Tools

As reported by Kidron and Tall (2015), there has been extensive research on students' knowledge with respect to the concept of limit (Bezuidenhout, 2001; Fernández-Plaza & Simpson, 2016; Przenioslo, 2004; Szydlik, 2000; Williams, 1991). According to their findings, students encountered many difficulties in understanding limits. Their misconceptions affected students' mathematical core knowledge. Especially when mathematical matters became more complex, students experienced hard times understanding the matter.

According to Liang (2016), who analyzed the techniques of teaching the limit concept with the aid of an online Desmos graphing calculator, the problem of the deficient explanation of the limit concept started within many textbooks. Textbooks focused on showing mathematical procedures rather than paying attention to its fundamental aspects. Textbooks treat this subject by showing mathematical procedures rather than explaining the theoretical understanding. This particular method is used to give students a good sense of the specific terminology used for mathematical matters. Namely, mathematical expressions significantly differ from day-to-day language. Apparently, it occurred that textbooks pay little attention to the fundamental aspects that underlie the definition of the limit concept, which resulted in lecturers treating the topic not as rigorous as it is needed to obtain the best understanding of this concept. This again influenced students' ability of studying more advanced mathematics courses, as calculus is required as a prerequisite for most advanced analysis courses. This domino effect is caused by the lack of awareness of the misconceptions of the limit concept.

In this study, the focus was on creating awareness on misconceptions of the limit concept among students first. Then, the software was used to provide the students with dynamic visualization of functions with a limit. Lastly, in combination with the software, the reasoning and techniques of proving limits was strengthened by understanding the concept rather than computations. It should be noted that the use of technology is not meant as a replacement of the teaching methods used to this day, instead it is meant as a complementary feature.

What stood out most in the results of the concerning research is that the underlying logic and the reasoning of the limit concept is not taught by lecturers, which is also one of the main reasons behind the current thesis. It turned out that the main focus of those lecturers is showing procedures and calculations rather than the conceptual understanding behind those procedures and calculations. In my data analysis, I plan to investigate the level of depth of the limit concept in the lectures. In other words, how the lecture goes beyond calculations and procedures.

In line with the issue mentioned above, there is touched upon the difficulties lecturers encounter due to the time restrictions of lectures. It is undesired to elaborate on particular subjects (e.g., the concept of limit) when this means that there is insufficient time left to discuss the rest of the topics. Therefore, a balance between teaching the set of subjects is key.

Liang (2016) analyzed the teaching of the limit concept with the use of a Desmos graphing calculator, whereas Kidron and Tall (2015) used Mathematica, a symbolic mathematical computation program. The latter analyzed how students proceeded from visual and symbolic approaches to the formal definition of the limit. This was done by using these approaches in Mathematica and follow the shift of the students to the formal definition. This study resulted in the conclusion that using a computer program using dynamic and symbolic visualizations can be seen as an alternative approach that facilitates the mathematical material. However, comparable to the findings of Liang (2016), it is not seen as a replacement of the used teaching methods, but as an addition to enhance students' understanding.

2.4 Approaches of Teaching

As mentioned earlier in the current thesis, there has been little research on the teaching of advanced mathematical courses (Weber, 2004). Nevertheless, there are different methods dominantly used in teaching. In the following section, two of those teaching methods and the examination of the lecture format are discussed in more detail.

The first mode that is used dominantly in the teaching of mathematics, is called the definition-theorem-proof format, abbreviated as the DTP format. The DTP format is known as a single teaching paradigm. Lectures mostly consist of the iteration of definition, theorem and proof. Within this iteration, examples and properties are treated. Nonetheless, according to this research, the DTP format appeared to be much more than that. The DTP format was seen as a diverse collection of techniques, including the focus on logical, procedural and semantic aspects, that all shared the same core feature: improving students' understanding of the subject. Research showed that this mode has been widely maligned yet is still applied by lecturers extensively. It was studied that to motivate meaningful learning while using the DTP format, it was not sufficient to tell students about mathematical procedures (Leron & Dubinsky, 1995). Therefore, Weber (2004) particularly described the teaching used in an advanced mathematical course at a university. It is required to understand the lecturer's thoughts behind their actions in order to understand why certain teaching methods are used in the classroom. This was done by weekly meetings with the lecturer and observations and recordings of the concerning lectures to create a more in-depth analysis.

Also, the lecturer in the research of Weber (2004) was a dedicated and respected lecturer, which led to his instructions being based on a solid belief structure. Even though I lack detailed knowledge on the lecturer in my own research, I do know that this person is a lecturer who has been teaching the course for several years now. Hence, I can assume that the instructions used in the lectures of my own research are based on the same, solid belief structure.

With the DTP format came a primary teaching method that is used at universities for all sorts of mathematics instructions, namely lecturing. Yet the amount of research done on this topic is insignificant. Weinberg, Wiesner, and Fukawa-Connelly (2016) addressed this gap and investigated how mathematical events in a lecture are ordered and connected. The researchers presented a framework to describe the structure and organization of a mathematics lecture, and analyzed the connections between different components of lectures. The study was done at a university in the United States, with a tenured professor whose research was focused on algebra. Lectures throughout the semester were video-recorded and analyzed. Those lectures were examined with respect to embedded narratives. That is, descriptions of the application of a mathematical process was given and connections between narratives were made in order to identify structures. This study suggested that paying attention to the narratives within lectures while analyzing mathematics lectures is an important component.

Even though the DTP format is preferred by many, some mathematics educators question this (Lew, Fukawa-Connelly, Mejía-Ramos, & Weber, 2016). The researchers stated that lectures were ineffective for developing students' understanding of the subject despite the lecturer being known as an excellent lecturer. In the following paragraphs, this paradox is investigated.

Lew et al. (2016) explained how students can fail to understand the mathematics taught during lectures. They analyzed a particular proof given in the lecture, and interviewed six pairs of students as well as the lecturer to examine their perceptions of this proof. This research was conducted at a university in the United States. The lecturer was known as an excellent teacher with three decades of teaching experience. As students were allowed to use previous notes during their analyses, these were reviewed by the researcher prior to the examination.

The results showed that the students who participated in the interviews failed to grasp many points the lecturer had emphasized during his interview. Understandably, it made sense students did not recall every detail mentioned during previous lectures by heart. However, it was quite surprising that most of them failed to reconstruct the key steps of the proof, as they could have written down the relevant information, highlighted by the lecturer as important in the interview, in their notes. Perhaps this could be assigned as a consequence of the lecturer denoting the relevant information orally instead of writing it on the blackboard. The limited research about the approaches of teaching inspired Bergsten (2007) as well to investigate the factors that account for quality of such lectures. He performed a case study at a Swedish university to try understanding the connection between teaching of mathematical matters and the learning in students and to highlight the main factors that contribute to the quality of mathematics lectures. A well-experienced lecturer taught the lecture in question in the research of Bergsten (2007). This lecture was observed and analyzed, together with an interview with the lecturer to ask questions in connection with the lecture.

According to this research, it turned out that the format mostly used was not the well-known DTP format, but the theorem-proof-application format, further referred to as the TPA format. The application part was only displaying the mathematical theory, leaving out the everyday use of mathematics. Within this format, the researcher identified the lecture as content-driven and rich in formation, meaning the aim of the lecture was to cover a large amount of information in the span of a lecture (Saroyan & Snell, 1997). Likewise, teacher immediacy emerged as a critical additional factor for the quality of the lecture in this research. To be more specific, gestures made by the lecturer, as well as personalization turned out to contribute positively to the lecture's quality.

3 Purpose and Research Question

The purpose of this study is to characterize the teaching of limits. In this thesis, I shed light on the teaching of mathematical limits to first-year bachelor degree students of a Mathematics program, at a Dutch university. As such, I analyzed video-recorded lectures and made use of the analytical framework by Mali and Petropoulou (2017). In this thesis, I aim to answer the following research question:

How is the concept of limits taught at university level in the Netherlands?

Accordingly, I answer this question by investigating the subsequent two subquestions: 1) *what kind of explaining actions are used in the teaching of the limit concept*, and 2) *what kind of extending actions are used in the teaching of the limit concept*? By analyzing university lectures and relevant literature with the focus on teaching the concept of limits, I aim to collect sufficient information which support me in answering these research questions. This is done using qualitative methods described in the book *Educational research: planning, conducting, and evaluating quantitative and qualitative research* by Creswell (2008), specifically chapters 7 and 8, about Collecting qualitative data and Analyzing and interpreting qualitative data, respectively.

4 Methodology

In this section, the methodology of the data collection and data analysis is described. Starting with the description of the confidentiality and research ethics, followed by the context of the study, data collection, and the analytical framework used in this thesis.

4.1 Confidentiality and Research Ethics

As this research took place at a university in the Netherlands, European Union's code of conduct is applied together with the code of conduct of personal data of the Association of Universities in the Netherlands (VSNU). Therefore, details that are not considered to contribute to the current research should be left out or anonymized. As such, the university name and personal information is left out from the data set. The video-recorded lectures used in this thesis is secondary data, which indicates the data was collected prior to this study. According to the code of conduct, the data must be anonymous. If possible, anonymous data is used exclusively. The publication of data is done only in a way that it is not traceable to the people involved, unless there is a mutual agreement of using personal data. The latter is not applicable in this research, hence all the data in this research will be anonymized.

4.2 Context of the Study

In this thesis, I used data that was previously collected. The data consisted of two lectures, each lecture had a duration of 90 minutes with a 15-minute break after 45 minutes. The lectures considered in this thesis were given for the course Calculus at a Dutch university and took place in a lecture hall. This course is offered to students of the following programs: astronomy, (applied) mathematics, and (applied) physics. This was a compulsory course in the appointed study programs. The particular lectures analyzed in this thesis, that consisted of the concept of limit, were the fourth and fifth lecture given in this course.

There were 350 student enrolled in this course, approximately 250 students were present each lecture. The students were first-year students, 40-50% international students, meaning 50-60% of the students were Dutch. The gender of the students broke down in 30-40% female students over 60-70% male students.

The lecturer who gave this course was a respected lecturer who has a PhD in Mathematics. He taught the course several years in advance of this research and had experience as a university lecturer.

4.3 Data Collection

The lectures considered in this thesis were video-recorded. The camera was positioned in order to capture the movements of the lecturer and the blackboard. Disruptive room sounds were minimized by using a wireless microphone the lecturer placed in his collar. However, in the first lecture, this microphone was not working. This led to a less accurate audio-recording where a back up microphone was used. In this audio-recording, the disruptive room sounds such as students coughing was not minimized. Nonetheless this audio-recording was accurate enough and still used in this thesis. Besides the audio- and video-recordings, during the lectures observation notes were made by an observer. These contained all the information that was written on the blackboard, as the visualizations of the blackboard in the video-recordings were not very evident at all times. Figure 1 shows a sample of the observation notes. The data was collected with the help of Creswell (2008). First, I transcribed the lectures by hand, capturing all that was said, by the lecturer as well as questions from students. I also captured what was written on the blackboard and given orally, simultaneously. Additionally, I captured when there was a mistake, and the mentioning of the textbook. Figure 2 shows a sample of the written transcript. The left column was used to capture time, while the right column was used for notes during the transcriptions, for example if the textbook was used or when the lecturer made a mistake in his discourse. The middle column captured all what was said by the lecturer, the blue writing corresponded to that. Black writing was used to capture when there appeared a pause in the lecturer's discourse (*pause*). Green writing was used when the lecturer wrote something on the blackboard (**bb**), the quotations in green marked what the lecturer wrote on the blackboard simultaneously while giving that information orally. Lastly, the red dots captured the lecturer pointing to something at the blackboard. This last notion was not used in the data analysis. After I finished the transcriptions by hand, I decided to copy it and write it in a digital text file in order to make use of *ATLAS.ti*, a computer program used for qualitative data analysis. This program helps to arrange and manage the data in systematic ways. By inserting actions as codes to phrases, I was able to classify my data in an efficient manner. Consequently, the program produced tables and overviews of what actions were used, particularly the total number, and combinations of actions.

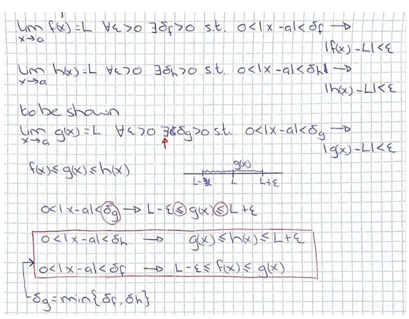


Figure 1: Sample of the observation notes.

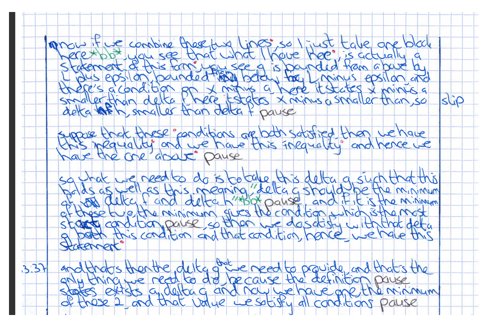


Figure 2: Sample of the written transcript.

4.4 Analytical Framework

I have used the analytical framework by Mali and Petropoulou (2017). The researchers defined actions and tools to identify teaching. Actions described what the teacher said and did, while tools specified what the teacher used to perform those actions. In my thesis, I will focus solely on the actions performed. Those actions are again divided into four categories: selecting, explaining, extending, and evaluating. Since selecting actions considered students in the lecture, this category was out of scope for this thesis. Lecturing is a form of teaching where the lecturer speaks and does not interact much with students, hence this choice is justified. Additionally, evaluating actions considered students' questions, students' work, and students' gestures and facial expressions during lectures. Students were not involved in this analysis, hence I will not use this action. The actions I did use in my analysis, were explaining and extending. Explaining actions acknowledged the mathematical content in the lecture by identifying verbal and mathematical representations, rhetorical questions, and demonstrating steps. Extending actions acknowledged the initiation of the lecturer to encourage the students' mathematical reflection. Particularly, providing reasoning to get students to think more about why something is done, connecting mathematical explanations to examples, and simplifying.

In the follow-up of the current thesis, the assigning of actions to phrases is defined as coding. To achieve an accurate analysis, I chose to code each phrase separately rather than combining phrases and observing that as a passage. While coding the data, as many actions as possible were attributed per phrase to create the most accurate analysis as possible. In appendix B, a sample of the coding can be found to observe that every phrase had as many actions assigned as possible to characterize that phrase.

Explaining actions
The category explaining actions contained eleven actions. *Concluding* considered phrases that are meant to conclude within a discourse, such phrases usually started with "so", or "hence". Particularly, phrases that consisted of explicit statements within a discourse are marked *concluding*. *Inferring* referred to concluding statements at the end of a discourse. Specifically, *inferring* is related to concluding after reasoning. *Summarising*, a different action within this category, referred to phrases that summarized what was done previously. *Demonstrating steps* considered phrases that presented steps in a mathematical procedure, it involved phrases such as "so the first, uhm, part of this is I consider the fraction in here, and write is as the ratio of two limits". In combination with this, we identified the action *describing*. This action referred to phrases that included a description of what is being done. For example, "now let's consider a *delta* which is less or equal than 1". The next action is *developing representational tools*. This action referred to phrases that included graphical, tabular, or symbolic mathematical representations. *Highlighting* referred to rhetorical questions. The lecturer asked those questions without expecting an answer from the students and continued with explaining without waiting for a response. *Parallel modelling* is another action within the explaining category, but not used in this thesis and therefore not explained. Additionally, *refining* referred to phrases where something that has been said earlier is being rephrased and put in other words. For instance, the lecturer first said the following: "well, I have already simplified this as I did before [the lecturer points to the blackboard]". After this, the following was said: "that's the simplification in there [the lecturer points to the blackboard]". This last phrase is marked as *refining*, as it is rephrased what is said previously. Consequently, *reminding* referred to phrases that referred to earlier mentioned statements. In the beginning of the second lecture, the lecturer referred to the first lecture and reminded students of the fact that had occurred: "last time I was almost finished with a, I almost finished an example". The action *repeating* is widely used simultaneously with *refining* as well as *reminding*, since it is in most cases repeated what has been said earlier and the action *repeating* is assigned to those phrases.

Extending actions
The category extending actions consisted of eight actions. *Connecting* referred to phrases that connected parts within the lecture and connected mathematical representations. For instance, a phrase is marked as *connecting* when a connection is made between a definition and an example: "okay, there the next subjects, directly related to the, uh, idea of the limit, is that of continuity". *Formulating* referred to phrases that considered the mathematical part of a definition, theorem, or example. "Well, if x minus 3 is smaller than 1, then the distance to 3 is smaller than 1, so x can be at most 4, and has to be at least 2" is marked with the action *formulating*. The following action is the action *providing reasoning*. This action marked phrases that consisted of reasoning why a computation is done, for example, "I am going to distinguish this *delta* [the lecturer points to the blackboard] from this *delta* [the lecturer points to the blackboard], because they are not the same". *Simplifying* is the last used action in this category. When a phrase consisted of

a simplification of wordings, this action was used. For example, "it says, in a sloppy way, that all values between f a $[f(a)]$ and f b $[f(b)]$ on this axis are function values for x values between a and b $[x \in [a,b]]$". Actions that also appeared in this category, but were not used in this thesis, were *analysing*, *generalising*, *interpreting*, and *synthesising*.

Other
This category is added, as in my opinion it is important to characterize a few other instances that might have appeared during the lectures. The acknowledgement of when the textbook was used (*textbook*), as well as characterizing questions asked by students (*question from student*) were added. Lastly, the action *previous lecture* was added. This action referred to phrases that mentioned the previous lecture.

4.5 Episodes

The collected data looks as one big lump without any structure. However, a structure can be identified when the data is being cut into episodes. These episodes are characterized as themes, to understand what is done inside that episode. Themes such as definitions, theorems, and proofs are used. After this, a structure can be identified and a teaching format can be derived. The theme *definition* described an episode where a definition was given and explained. *Theorems* described statements of theorems, and *proofs* described episodes where those theorems were proven. Additionally, *properties* described episodes that consisted of explaining properties related to the subject, for instance the limit laws were assigned as properties related to the limit concept. Lastly, the theme *examples* described episodes where examples were given.

5 Findings

Next, results are shown of the data analysis, supplied by the computer program *ATLAS.ti*. I start with analyzing the collected data in a general sense and describing the approaches of teaching that can be derived from the analysis. Then, I become more specific about the data, by exemplifications of data. Consequently, descriptions of certain episodes are given to identify a pattern of actions if such a pattern arose. Lastly, frequently used actions are featured, and combinations of certain actions.

5.1 Approaches of Teaching

The approaches of teaching that were used in the two lectures seemed to be diverse. Below, a description is given of both lectures.

Lecture 1
The lecturer started with explaining the limit laws. He followed with examples to explain those limit laws and its applications. After this introduction, the lecturer started with the explanation of the Weierstrass definition, when x approaches *infinity*. He started with a visualization of the definition, described in excerpt 1, table 2. Then, he stated the actual definition, and followed with an example of the application of this definition. After the break, the lecturer began with another example. Then he followed with the explanation of the Weierstrass definition, when x approaches a number a. He continued with examples. The lecturer was not able to finish the last example, but resumed with it the second lecture.

Lecture 2
In this lecture he started with recalling the Weierstrass definition. He continued with the example of the first lecture, followed with a few other examples and then explained the squeeze theorem. He first proved this theorem and then showed its application in an example, which is seen in the second excerpt, table 3. After this, the lecturer explained the concept of continuity, related to the limit concept. He analyzed this concept by showing an example and stating a theorem connected to this definition, the intermediate value theorem. Additionally, he did the same regarding the concept of differentiability, another concept that is related to the limit concept. After describing the definition and an example, the relation between differentiability and continuity was described in a theorem, which was then proven. This was the end of the second lecture and therefore the end of the explanations of the limit concept in this course.

Episodes
Table 1 shows what themes are used in which lecture, and the amount of those themes that appeared per lecture.

Table 1: *Episodes per lecture.*

	Lecture 1	*Lecture 2*	**Totals**
Definition	2	3	5
Example	10	8	18
Proof	0	2	2
Property	2	0	2
Theorem	0	3	3
Totals	14	16	30

In the order of appearance, the given definitions were the Weierstrass definition when x approaches *infinity*, the Weierstrass definition when x approaches a number a, a repetition of the Weierstrass definition when x approaches a, continuity, and differentiability. The properties considered in the first lecture were the limit laws. The discussed theorems in the second lecture were the squeeze theorem, the intermediate value theorem, and a theorem that connects continuity with differentiability.

The first lecture mostly consisted of examples that showed applications of the definitions given in the lecture, while the second lecture was more diverse. Seven examples were taken from the textbook. The first lecture started with the properties of limits. Then the lecturer followed with definitions and examples. On the other hand, in the second lecture, theorems were given with their proofs, as well as definitions and the applications of the definitions and theorems in the form of examples.

When zooming out of the two described lectures above, it was seen that the lecturer used visualizations and examples to explain a certain definition or theorem. When looking at the first lecture, no structure of episodes can be identified, so no format can be linked to this lecture. The lecturer started with explaining properties and definitions. He showed its applications in the form of examples, but no theorems or proofs were given. The second lecture on the contrary, was identified as the DTP format. The lecturer started with a definition, then moved to the explanation of theorems, and proved those consequently. Within this format, another format can be identified, namely the TPA format. After the stating of a theorem and the proof of that theorem, the lecturer explained its application in the form of examples. Figure 3 shows a snapshot of the observation notes of the second lecture where the structure of the DTP format can be identified. Note that after the recall of the definition, the lecturer gave two examples of its application before proceeding to the explanation of the squeeze theorem. In the figure, the identified episodes are marked.

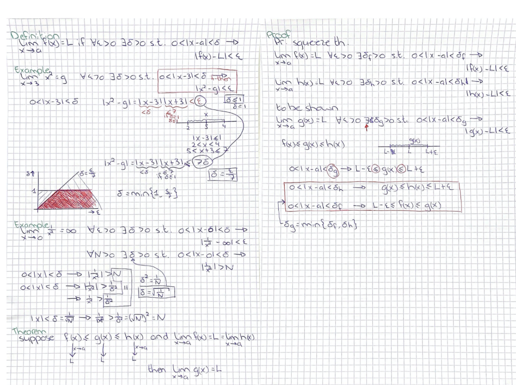


Figure 3: Sample of the structure of a DTP format.

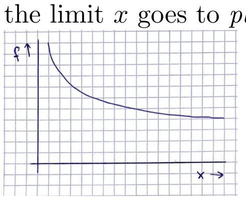
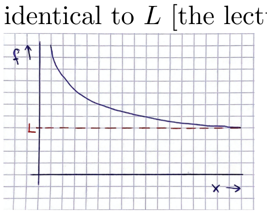
5.2 Exemplifications

In this section, exemplifications are shown of the coding of the lectures, complemented with observation notes where necessary and the assigned actions. The parts of episodes shown in the exemplifications were selected as they showed various combinations of actions (*refining* and *repeating*, *formulating* and *describing*) and represented approaches that appeared regularly during the lectures (visualizations of definitions, examples). Actions that appeared often, as well as combinations of actions were criteria for choosing these exemplifications. The graphics shown in the tables correspond to the sketching on the blackboard (graphical and symbolic representations).

Example 1
This segment of an episode came from the first lecture and consisted of the explanation of the Weierstrass definition graphically. This sample is chosen because the lecturer used a graphical approach quite often to supply the symbolic representation of the explanation of the definitions (nine combinations of graphical and symbolic approaches in the two lectures). This example showed the lecturer used many phrases to conclude within his explanation (*concluding*, *inferring*), as well as several combinations of the actions *refining* and *repeating*.

The lecturer started with an introduction of what he was about to do. “Uhm so far, for calculations. So we have some idea to calculate limits, but we do not have a definition yet. So in the next part, I’ll focus on the definition of the limit”. Next, the lecturer explained that he will be considering the limit where x goes to infinity, and the main question where this part was about, was what this actually meant: “question, not how to compute this limit, but what does this mean. That’s the question now. It’s a different question. How can we define meaning? We have wrote down some symbols there, what does it mean?”. The excerpt is shown below in table 2. It shows a two-minute explanation of the visualization of the Weierstrass definition of the limit concept.

Table 2: *Visualization of the definition of the limit concept.*

Excerpt 1	*Actions*
1. Well, let’s consider this first graphically, so here I’m going to draw f, of its x [the lecturer starts sketching the graph] and in the limit x goes to *plus infinity*.  (a) Reproduction of lecturer’s graph.	Connecting, Describing, Developing Representational Tools, Formulating.
2. So somewhere over there, it becomes interesting to consider function values [the lecturer points to the end of the function].	Concluding, Describing.
3. And the claim is, if you go very far in that direction [end of the function; large value for x], the function values become almost identical to L [the lecturer draws a line which corresponds to L].  (b) Reproduction of lecturer’s graph.	Describing, Formulating.
4. That’s what it really means.	Concluding, Describing, Refining.

5. In other words, if x approaches *infinity*, the function values will approach the limiting value L.	Formulating, Refining, Repeating.
6. That's what this means in words.	Concluding, Describing, Refining.
7. Now let's try to formalize that. This is my function f [the lecturer points to the function f]. Now let's first take L.	Describing, Demonstrating Steps, Formulating.
8. So this is our L. So that's a level. And that's the limiting value.	Concluding, Describing, Refining, Repeating.
9. Now the function approaches that if x becomes very, very large, so the function could be like this.	Describing, Formulating.
10. This is our f and if x becomes very large, the function values approach the limiting value L [the lecturer points to the function and limiting value].	Formulating, Refining, Repeating.
11. That's what you see here, more or less.	Describing, Inferring, Refining.

Below, a description is given to explain the assigned actions regarding the visualization of the definition. The numbers correspond to the numbers in table 2.

1. Representational tools are developed using a graphical approach. This graphical representation is connected to the definition. What was being done is described and the mathematical part of that (x goes to *infinity*) is formulated.
2. It is described what was being done and when it becomes interesting in a concluding sense ("so, ...").
3. A claim is described and formulated: when x takes very large values, the function values become almost identical to the limiting value L.
4. The lecturer put emphasis on this claim, by refining and concluding the claim.
5. Again, the claim is emphasized by repeating and refining what was said.
6. The claim is again emphasized by concluding and refining the claim.
7. Steps are demonstrated of what was to be executed (formalizing the visualization). It is formulated what is known (the function f and the limiting value L).
8. It is refined and repeated what the limiting value L is in a concluding sense ("so, ...").
9. It is described and formulated mathematically what happens when x becomes very large.
10. The formulation of x becoming very large is refined and repeated.
11. An inferring description is given to conclude the explanation of the visualization of the definition of the limit concept.

It showed that within his discourse, the lecturer put emphasis on the notion that when x approaches *infinity*, the function values become very large and become almost identical to the limiting value L, by refining and repeating this multiple times. He demonstrated what he was about to do once, and developed representational tools once as well. This indicates that no emphasis was put on demonstrating steps and developing representational tools, but on refining and repeating the limiting value L when x approaches *infinity.*

Example 2
The second segment of an episode came from the second lecture in the series of teaching the limit concept and considered an example of the squeeze theorem. This excerpt is chosen, because it consisted of actions that were not yet seen in the first excerpt (*textbook*, *highlighting*), but were as important as the actions that already appeared. Also, the lecturer made use of rhetorical questions in the explanation of this example (*highlighting*). Additionally, along every definition, theorem, or property, the lecturer showed its application with an example, in some cases even with multiple examples. Thus, in my opinion, it is key to show a sample of the transcript and coding of such an example and question if a pattern can be deduced. The example described in this excerpt was one from the textbook, section 2.3, example 11 (Stewart, 2010, p. 105).

Before the explanation of this example, the lecturer started with stating the theorem. The following statement of this theorem was used, and came from Stewart (2010) (p. 105):

Theorem 5.1 (Squeeze Theorem). If $f(x) \leq g(x) \leq h(x)$ when x is near to a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

The lecturer continued: "well this theorem is known. Perhaps you do know it, it's called the squeeze theorem, meaning you have a lower bound, you have an upper bound. We have the same limiting value, so what is squeezed between has the same limiting value". After this, he started proving the theorem and followed with an example. Table 3 shows the coding of a four-minute discourse of the lecturer explaining the example.

Table 3: *Example of an application of the squeeze theorem.*

Excerpt 2	*Actions*
1. Uhm, the squeeze theorem is actually a powerful way to compute a limit. And I'll give an example of this theorem, where you can use this theorem.	Connecting, Describing, Inferring.
2. Uhm, and actually this is an example from our book, section 2 point 3, example 11.	Describing, Refining, Textbook.
3. So this is the limit that will, is to be evaluated [the lecturer writes down the limit]. $\lim_{x \to 0} x^2 \sin \frac{1}{x}$ (a) Reproduction of lecturer's reasoning.	Demonstrating Steps, Describing.
4. Uhm, let me do it first in a wrong way, uhm, in order to show, uh, that it's, you cannot simply use the product rule for the limit because if you do so, you would write down this [the lecturer writes down the reasoning]. $\lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{1}{x} = 0$ (b) Reproduction of lecturer's reasoning.	Demonstrating Steps, Describing, Providing Reasoning, Reminding.
5. So this is a product of two functions. Now we take the limit of the product as the product of the limits.	Concluding, Formulating, Refining, Reminding, Repeating.
6. And then you could reason, well, this becomes 0, if x goes to 0, x to the power 2 becomes 0 times something is 0.	Formulating.
7. Hence the result is 0. That's wrong.	Concluding, Describing, Formulating.
8. The reasoning is wrong because this equality sign here [the lecturer points to the first equality sign in figure (b)], holds only if both limits in here exist. That's the condition. You can write the limit of a product of functions as the product of the limits only if both components in that product of limits exists. And that's not the case here. Because the limit of the sine of 1 over x if x goes to 0 does not exist.	Describing, Formulating, Providing Reasoning, Refining, Repeating.
9. And hence we cannot have this and we cannot have this. Okay? Well, how to proceed then? Well.	Describing, Highlighting, Inferring.

10. The function *sine* of 1 over x, is still bounded like any *sine*; at most of this plus 1 and at least it is minus 1. If x goes to 0 this takes, this goes to plus *infinity* or minus *infinity*. So that argument takes very, very large values. It is $2pi$ [2π] periodic, so this takes actually any value between minus 1 and plus 1. But it's still bounded by minus and plus 1. Yeah?	Concluding, Describing, Developing Representational Tools, Formulating, Highlighting, Refining, Reminding, Repeating.
11. So it can make use of that, uhm, that means that the function that we consider here, has an upper bound and a lower bound.	Concluding, Formulating, Refining, Repeating.
12. So let's focus on the function, x power 2 times sine of 1 over x. Well, it's bounded from above by x to the power 2 and from below by minus x to the power 2 [the lecturer writes down the bounds]. $-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$ (c) Reproduction of lecturer's reasoning.	Demonstrating Steps, Describing, Developing Representational Tools, Formulating, Refining, Repeating.
13. Yeah? I have an upper bound, I have a lower bound. Now I take the limit where x goes to 0.	Describing, Developing Representational Tools, Formulating, Highlighting, Reminding, Repeating.
14. Well, if x goes to 0, then x to the power 2 also approaches 0, and minus x to the power 2, minus 0, but that's 0.	Formulating.
15. So I have a function squeezed in between two other functions. The lower one has a limit 0. The upper bound has the limit 0. So the squeeze theorem will tell us that this has the limit 0 too. $\begin{array}{ccccc} -x^2 & \leq & x^2 \sin\frac{1}{x} & \leq & x^2 \\ \downarrow{\scriptstyle x\to 0} & & \downarrow & & \downarrow{\scriptstyle x\to 0} \\ 0 & & 0 & & 0 \end{array}$ (d) Reproduction of lecturer's reasoning.	Concluding, Describing, Developing Representational Tools, Formulating, Summarising.
16. And that's the only way in which you can reason here. That's the correct way of getting the 0 out of this. Okay, so, this is an application of the squeeze theorem.	Connecting, Describing, Inferring.

Below, a description is given to explain the assigned actions regarding the explanation of the example. The numbers correspond to the numbers in table 3.

1. The beginning of this episode started with describing that the squeeze theorem is a powerful way to compute a limit. Then, the theorem was connected to the example.
2. It was specified what example was being done by stating that it was an example from the textbook.
3. It was described what limit was to be evaluated, together with demonstrating what needed to be done next.
4. Steps were demonstrated and reasoning was provided. Students were reminded about the product rule of the limit: the limit of a product of two functions is equal to the product of the limits of the two functions.
5. It was formulated and concluded that the example contained a product of two functions. Additionally, students were reminded of the product rule of the limit, and this limit law was refined and repeated.
6. The mathematical reasoning of applying this limit law was formulated.
7. First, it was formulated what the result was. Then, it was described and concluded that this reasoning was wrong.
8. Reasoning was provided why this reasoning was wrong. The limit law was refined and repeated by specifying that this limit law can only be applied if both components in that product of limits exists. This was not the case in this example, so reasoning was provided why this was not the case and it is refined and repeated why this specification applied to this example.

9. It was described that what was done could not happen. This was done in an inferring sense, meaning the lecturer meant this phrase as to conclude this part of his discourse. This part of the lecturer's discourse was highlighted by asking rhetorical questions. Asking rhetorical questions is meant as a reminder for students that what was said earlier was important.
10. This line contained phrases that were about explaining what happened to the *sine* function ($\sin(\frac{1}{x})$). This was done by formulating it mathematically, describing what was being done, refining, reminding, and repeating that the function stayed bounded, and developing representational tools by formulating the knowledge that the *sine* function was bounded. The first phrase formulated the course of the *sine* function: "the function *sine* of 1 over x, is still bounded like any *sine*; at most of this plus 1 and at least it is minus 1.". As this is knowledge students were aware of, this phrase was marked with the actions *refining*, *reminding* and *repeating*. Additionally, representational tools were developed by formulating this knowledge. It was then formulated mathematically what happened when x goes to 0. Then it was described and concluded that that argument then took very large values.
11. A conclusion was described and it was formulated mathematically that this function had an upper bound and a lower bound. This knowledge was already mentioned before, so the actions *refining* and *repeating* were assigned as well.
12. It was described what was going to be done by demonstrating steps, that the focus shifted from the *sine* function to the actual function in this example ($x^2 \sin(\frac{1}{x})$). Then, the lecturer developed representational tools by formulating what happened orally, as well as giving symbolic representation on the blackboard. With that, the bounds were refined and repeated by giving the actual bounds orally.
13. This knowledge was highlighted by a rhetorical question. Then, students were reminded by the fact that there was an upper bound and a lower bound, this was described and repeated by the lecturer. Lastly, it was formulated what was done, namely taking the limit where x goes to 0. The lecturer also gave a symbolic representation of this on the blackboard (*developing representational tools*).
14. The mathematical reasoning of what happened was formulated.
15. A description of a conclusion of what was achieved was given, specifically that the function ($x^2 \sin(\frac{1}{x})$) was squeezed between two other functions. Then, it was formulated in two phrases what the limits were from the functions. The last phrase on line 15 described, summarised and concluded that the function ($x^2 \sin(\frac{1}{x})$) had a limit 0 as well, and a symbolic representation was given on the blackboard.
16. All three phrases showed on this line, were marked with the actions *describing* and *inferring*. The lecturer described what was done, and concluded the explanation of this example. The last phrase was also marked with the action *connecting*, as this example is again connected to the squeeze theorem.

In this excerpt, it is shown that the lecturer developed representational tools, symbolic representations to be exact. The example was first treated with a wrong approach. Students were reminded of the limit laws mentioned in the first lecture, and this definition was refined (this limit law can only be applied if both components in the product of limits exist). Reasoning was provided why this approach was wrong. This was then followed by the right approach to tackle this example. The lecturer refined and repeated to emphasize the importance of the upper bound and the lower bound of the *sine* function. Students were reminded of the periodicity of the *sine* function. Again, the lecturer refined and repeated statements about the upper bound and the lower bound to emphasize its importance. Then, the reasoning is formulated for the application of the squeeze theorem, and it was concluded what the solution was of this limit. The lecturer ended his discourse of this example by inferring and connecting the example to the squeeze theorem.

5.3 Episodes

In this section, an attempt is made to identify patterns of actions within episodes. By identifying those patterns, a structure that may have occurred can be derived. For instance, if it appeared all proofs consisted of the same sequence of actions, a structure was used in the explanation of proofs. Here, definitions, theorems, proofs and examples are examined separately. Properties are excluded of this analysis.

Definitions
There appeared five definitions in the lectures. The first definition was the Weierstrass definition when x ap-

proaches *infinity*. This definition is explained using a graphical approach as well as a symbolic approach. The visualization of this is seen and analyzed in the first excerpt, table 2. The lecturer emphasized x approaching *infinity*, as well as emphasizing on the limiting value L and *epsilon*. Emphasizing is done by refining and repeating statements such that students identify these statements as essential. Lastly, the formal definition is given. The second episode that is identified as a definition is the Weierstrass definition when x approaches a finite number a. This definition is also explained using a graphical and symbolic approach. The lecturer broke this definition down into two pieces. First, he emphasized the bandwidth with respect to the limiting value L that is created by *epsilon*. Second, he emphasized on the interval around a, $a-delta$ and $a+delta$. He ended with the formal definition. This structure can be identified by phrases that were marked with the action *demonstrating steps*. The third definition was given at the start of the second lecture, and had a reminding function for the students, because this definition was a repetition of the Weierstrass definition when x approaches a. The lecturer put emphasis on *epsilon* being arbitrary, and on the existence of a *delta*. This was done by phrases that were marked with the actions *refining* and *repeating*. The fourth and fifth definitions were the definitions of continuity and differentiability, respectively. As those definitions were not about the actual Weierstrass definition, they are not included in the analysis of the definition.

When trying to find a pattern of actions, it is identified that many phrases were characterized as *refining* and *repeating*. This was done so that students knew that parts that were refined and repeated, may have an important function within that definition. The emphasis was put on x approaching *infinity* and x approaching a, respectively. Additionally, in both definitions the lecturer emphasized on the limiting value L and its bound with bandwidth *epsilon*. When considering x approaching a, the lecturer also emphasized on the interval around a. Both episodes concluded with the full definition given orally in words as well as its symbolic representations on the blackboard. The first two definitions followed the same pattern of actions within its explanation. By excluding the third definition also, as it was a repetition of an earlier explained definition, it can be concluded that the same structure occurred in the explanations of the two definitions. Both definitions began with a graphical representation (*developing representational tools*) which was connected to episodes prior to these definitions (*connecting*). Then, steps were demonstrated what was done afterwards (*demonstrating steps*). The second definition consisted of more steps (six) compared to the first definition (three), however the structure was still explicit. During the execution of these steps, the computations were refined and repeated regularly (*refining*, *repeating*). Reasoning was provided rarely (*providing reasoning*), three times in the first definition against once in the second definition. Lastly, both definitions ended with concluding the full definition in words (*concluding*, *inferring*).

Examples

While analyzing examples, the attention was focused on the following actions: *demonstrating steps*, *developing representational tools* and *providing reasoning*. This is done as a result of the following observation. When a relation of those actions can be characterized, possibly a pattern can be identified that might have been followed by the lecturer in explaining examples. Other actions were not considered, as they did not contribute to a certain structure within an explanation if such a structure appeared.

A total number of 18 examples appeared during the explanation of the limit concept. Seven of those examples were taken from the textbook, as the lecturer mentioned during the lectures. The origin of all other examples was unknown, this information was not specified during his discourse. Twelve examples were considered using a symbolic representation, one example was explained with a graphical representation. All other examples, five to be exact, were considered using a combination of a symbolic and graphical representation. No structure of actions can be identified when focusing on the representations used. Some examples started with the graphical representations and were followed by the symbolic representation, while others were structured the other way around. Additionally, there appeared to be no connection between the kind of representation and the duration of the explanations of the examples, as this differed too much.

The examples were divided into categories that were identified as “short”, “medium” and “long” examples. These names referred to the duration of the explanations of the examples: short examples lasted a minute, medium examples lasted approximately 2-4 minutes, while long examples lasted over 5 minutes. Long examples were mostly used (eight times), followed by medium examples (six times). Short examples were used the least (four times). Furthermore, short examples were identified as not including many statements that demonstrated steps, developed representational tools, or provided reasoning. Medium examples consisted of more detailed explanations. Steps were identified as well as representations were developed in all medium examples. Reasoning however was not provided consistently. Some medium examples had none, while others appeared to have multiple occurrences of this action. Lastly, considering long examples, there appeared to be no structure. Some long examples followed a detailed description which was characterized by steps demonstrated often, while analyzing other long examples, there appeared to be a lack of demonstrating steps. The same observation occurred considering the action *developing representational tools*. Despite the difference between representations, some consisted of many descriptions while others consisted of a few.

To summarize, no relation can be found by connecting the type of representation with the frequencies of the actions *demonstrating steps*, *developing representational tools* and *providing reasoning* when analyzing short and long examples. Medium examples on the other hand, appeared to have more structure, as steps were demonstrated consistently. Considering all examples, there appeared to be no pattern that was followed during explaining examples as many differences were found in duration, thoroughness and mathematical representation.

Theorems
There were three theorems given in the lectures, specifically in the second lecture. Those theorems were the squeeze theorem (page 16), the intermediate value theorem, and a theorem about the connection between differentiability and continuity. These theorems are stated as follows (Stewart, 2010, p. 125,p. 158):

Theorem 5.2 (Intermediate Value Theorem)**.** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in $[a, b]$ such that $f(c) = N$.

Theorem 5.3. If f is differentiable at a, then f is continuous at a.

Considering the intermediate value theorem and the theorem about the connection between differentiability and continuity, the theorems are named prior to defining the theorems. The squeeze theorem on the other hand was first defined, and later named as the squeeze theorem. When focusing on the stating of the squeeze theorem, all parts of this theorem (the ordering of the functions, the limiting value of the functions f and h), were refined and repeated in order to highlight these properties. The same was done with the consequence, the part after the "then"-statement. Lastly, the theorem was refined and repeated repeatedly to conclude the stating of the squeeze theorem. Considering the intermediate value theorem, a different approach was taken in the explanation. First, a graphical representation was executed to sketch the idea of the proof. The emphasis was put on the function being continuous by showing a counterexample where the function was discontinuous. Then, the sketch was formalized by stating the theorem. This theorem was lastly refined and repeated to reach the desired result, which was the detailed description of the intermediate value theorem. When looking at the theorem about the connection between differentiability and continuity, it was introduced that such a connection existed, and the students were reminded what differentiability was. Then, an example was given in order to show the students that continuity and differentiability do not mean the same; they have different definitions. With that example, the theorem was stated, refined and repeated multiple times. Lastly, an explanation was given why this theorem only holds in one way.

The stating of theorems mostly consisted of refining and repeating what was already said. Considering all theorems, representational tools were developed using a symbolic representation. In the intermediate value theorem, a graphical representation was used additionally. All theorems were given without demonstrating steps of those theorems. To make an attempt of identifying a pattern, such pattern cannot be found. The explanations of the three theorems differ to recognize one. The only observation that can be made, is that by refining and repeating, essential parts of a theorem were emphasized. Other than that, no conclusion can be drawn, as different representational tools were used, and no steps were demonstrated in the theorems.

Proofs
Two theorems were proven, these episodes are marked as proofs. The first proof considered proving the

squeeze theorem (page 16), whereas the second proof considered proving the theorem about the connection between differentiability and continuity. Both proofs started with stating what was known, and what needed to be proven. The proof of the squeeze theorem broke down what was known. It started with working out the known limits of the functions f and h. Halfway through the proof, it was demonstrated what steps were needed to be executed in order to prove the theorem. The lecturer followed with the other part of what was known, namely the ordering of the functions. This was worked out and lastly those two components were combined by reminding and repeating what was accomplished earlier. The lecturer completed the proof by refining and repeating what was achieved, and inferring that what was done, needed to be done. The proof of the connection of differentiability and continuity started with rewriting the limit that described the derivative by introducing a new variable. By refining and repeating what was said, the lecturer emphasized what was to be done. Then, a phrase appeared with the action *demonstrating steps*, saying what needed to be shown afterwards and what needed to be done to complete the proof. Then, the concept continuity is rewritten in a way that it contributed to this proof. Lastly, the desired conclusion was achieved, this was described by refining, inferring, and repeating what was done. Additionally, students were reminded that this theorem only works in one way (this notion was discussed earlier in the lecture).

In both proofs, many phrases appeared to have the action *demonstrating steps* assigned to it (17 occurrences in the first proof, against 12 occurrences in the second proof). During the discourses of the proofs, students were told what was done and what needed to be shown in order to prove the theorem. Some steps were even refined and repeated, to emphasize these steps were of high importance or had a more difficult computation. Additionally, both proofs consisted of phrases marked with the action *providing reasoning.* Steps that were about to be executed, were justified by offering reasoning. By combining these observations, a pattern of actions can be identified within those proofs. Steps were demonstrated and emphasized by refining and repeating them when those steps were essential. Those steps were then justified by provided reasoning. Both proofs were completed by refining and inferring that was done, needed to be done to finish the proof. Lastly, both proofs were explained using a symbolic representation.

5.4 Frequently Assigned Actions

All the assigned actions were counted, frequencies were calculated. In the next table, table 4, the overall appearance of each action is seen. With that, percentages are calculated of the occurrence of each action in each lecture, and the occurrence in total.

Table 4: *Overview of occurrence of all actions per lecture.*

	Lecture 1(%)	*Lecture 2(%)*	**Totals(%)**
Concluding	137 (9%)	130 (8%)	267 (8%)
Connecting	29 (2%)	16 (1%)	45 (1%)
Demonstrating Steps	90 (6%)	106 (7%)	196 (6%)
Describing	316 (20%)	376 (23%)	692 (22%)
Developing Representational Tools	100 (6%)	80 (5%)	180 (6%)
Formulating	356 (22%)	284 (18%)	640 (20%)
Highlighting	54 (3%)	36 (2%)	90 (3%)
Inferring	20 (1%)	24 (2%)	44 (1%)
Providing Reasoning	34 (2%)	32 (2%)	66 (2%)
Refining	209 (13%)	192 (12%)	401 (13%)
Reminding	69 (4%)	91 (6%)	160 (5%)
Repeating	170 (11%)	216 (13%)	386 (12%)
Simplifying	2 (0%)	6 (0%)	8 (0%)
Summarising	13 (1%)	20 (1%)	33 (1%)
Totals	1599 (100%)	1609 (100%)	3208 (100%)

The actions with the highest percentages in both lectures were the actions *describing* (22%) and *formulating* (20%). Those actions referred to phrases with what is done and mathematical computations respectively. Other than that, the actions that appeared most according to the percentage, after *describing* and *formulating*, were *refining* and *repeating* (13% and 12%, respectively). This can perhaps be related to the fact that those actions, together with *reminding*, are often assigned to phrases simultaneously that explain the meaning of earlier mentioned statements.

Describing and formulating

Each phrase had either *describing* or *formulating* assigned to it, thus the high percentages that appeared seemed to be a logical consequence of that. This was the case, because in my opinion every phrase included either a description of what was done or a formulation of a mathematical explanation. Table 5 shows the co-occurrence of all the actions with respect to *describing* and *formulating*. In parenthesis, the number of appearances in total of that particular action is shown, along with percentages of the occurrences of each action with respect to *describing* and *formulating*. As seen in the excerpts in table 2 and 3, each phrase had as many actions assigned as possible to characterize that phrase. Particularly, the total number of assigned actions is 3209, while the total number of phrases that appeared in the lecturer's discourse and that were identified by assigning actions, is 663. Because of this, the numbers in the following table do not add up. For instance, when we look at table 2, line 4, the phrase had three assigned actions, *concluding*, *describing* and *refining*, hence the numbers can never add up to the number of appearances of each action.

Table 5: *Co-occurrence regarding* Describing *and* Formulating.

	Describing (692)	Formulating (640)
Concluding (267)	133 (47%)	150 (53%)
Connecting (45)	37 (76%)	12 (24%)
Demonstrating Steps (196)	146 (63%)	86 (37%)
Developing Representational Tools (180)	49 (24%)	152 (76%)
Highlighting (90)	12 (71%)	5 (29%)
Inferring (44)	33 (77%)	10 (23%)
Providing Reasoning (66)	46 (61%)	30 (39%)
Refining (401)	194 (47%)	223 (53%)
Reminding (160)	73 (43%)	97 (57%)
Repeating (386)	181 (44%)	227 (56%)
Simplifying (8)	3 (37%)	5 (63%)
Summarising (33)	20 (57%)	15 (43%)

For the analysis of these percentages, I adopted a criteria that the percentages must be far apart from each other. Specifically, distributions between 40-60% were perceived as the notion that those actions were distributed somewhat equally regarding *describing* and *formulating*. Such actions were *concluding*, *refining*, *reminding*, *repeating* and *summarising*. The other actions were distributed differently in the sense that actions were used more consistently with either *describing* or *formulating*. Actions that were mostly used simultaneously with *describing*, were *connecting* (76%), *demonstrating steps* (63%), *highlighting* (71%), *inferring* (77%) and *providing reasoning* (61%). Actions that were mostly used together with *formulating*, were *developing representational tools* (76%), as this action was about creating symbolic or graphical representations, which considered mathematical explanations, by all means.

Concluding and inferring

Concluding statements occurred frequently (9%). The difference in concluding statements was distinguished by identifying two different actions, *concluding* and *inferring*. *Concluding* was assigned to phrases that described conclusions made within the lecturer's discourse, whereas *inferring* was assigned to phrases that included a conclusion made at the end of the lecturer's discourse. Table 6 shows the percentages related to each action per lecture, together with the frequency of the actions per lecture.

Table 6: *Occurrence of* Concluding *and* Inferring.

	Lecture 1(%)	*Lecture 2(%)*	**Totals(%)**
Concluding	137 (87%)	130 (84%)	267 (86%)
Inferring	20 (13%)	24 (16%)	44 (14%)
Totals	157 (100%)	154 (100%)	311 (100%)

Concluding occurred 86% of the times, against 14% times of *inferring*. These overall percentages agree with the percentages per lecture, the distribution *concluding/inferring* in lecture 1 happened to be 87%/13%, against 84%/16% in lecture 2. According to these lines, the lecturer concluded repeatedly within his explanation. Examples of such phrases were "so both conditions are to be satisfied", and "so everything is allowed". Examples of phrases that had the action *inferring* assigned to it, were "so we have given the proof at this point", and "make use of what you know, that's essentially what we do here". There can be made a connection between this observation and the excerpts shown in section 5.2. The lecturer used concluding phrases quite often during his discourse, as well as statements at the end of his discourse.

Demonstrating steps
An action that occurred every now and then, was *demonstrating steps*. In both lectures, this action appeared in episodes, 6% in the first lecture, against 7% in the second lecture. This led to the belief that the lecturer steadily explained what he was going to do within his discourse. With explaining what he was about to do, he involved students in a way that they would get a feeling of what was going to be done, rather than being oblivious and just execute steps randomly and not telling students what was going to happen next. It was found that steps were demonstrated mostly in episodes that were marked as examples and proofs. This action also occurred in other episodes, however not as consistent as in examples and proofs, therefore nothing can be concluded from that observation. When considering proofs, it can be concluded that steps were demonstrated consistently while explaining them, so that students would get a feel of tackling such a problem themselves. On the other hand, some examples consisted of a consistent use of the code, while others had a lack thereof.

Developing representational tools
This action occurred 6% in total, steadily divided over both lectures. As seen in table 5, *developing representational tools* was mostly used in combination with *formulating* (76%), rather than *describing* (24%). This might be the consequence of the fact that this action referred to phrases that included mathematical representations, graphical or symbolic. Consequently, *formulating* was an action that referred to phrases that considered the mathematical part of a definition, theorem, or example. Hence, this action had a similar meaning, which contributed to this high percentage. All 30 episodes consisted of representational tools, either only symbolic representation, or a combination of symbolic and graphical representation. Symbolic representations arose in all episodes, whereas graphical representations arose only nine times within both lectures as an addition to the symbolic representation. The graphical representation was used on itself only once.

Highlighting
The action *highlighting* referred to phrases that included rhetorical questions to the students where the lecturer did not wait for the students to respond, the intention of those questions was not to be answered. This action is used 90 times in total. In 10% of the cases of the use of this action, it was assigned to phrases that did intend a student to answer. In these cases, a student wanted to ask a question to the lecturer, the student's hand was raised in order to get attention from the lecturer. The lecturer said "yeah?", and looked in the direction of the student. The following table, table 7, shows the frequencies regarding what kind of questions were asked by students. The categories were assigned as follows. *Fixing error* referred to a student acknowledging a mistake the lecturer made, whether this mistake was made intentionally or not. Additionally, as the microphone was not working in the first lecture, a student acknowledged this in the beginning of the lecture by raising his/her hand. The associated questions were marked as *irrelevant*, as those questions did not have a mathematical meaning. The questions that were related to mathematical explanations, or about the final examination, were marked as *meaningful*. An example that appeared during

the second lecture, was a student asking why the lecturer did not write down that something was positive. The lecturer answered with "yeah you can. I, I do not write that explicitly because I do not use it in the reasoning, but it still holds".

Table 7: *Questions from students.*

Content	Frequency
Meaningful	4
Fixing error	4
Irrelevant	1
Total	9

All other phrases marked as *highlighting* (90%), defined rhetorical questions that were not intended to be answered. A difference was characterized within those questions. Questions that were considered meaningful, like "what does that mean?", and "well, how to proceed then?" occurred 41% (37 times). On the other hand, questions without meaning, but meant as a check for the students, like "okay?" or "yeah?" occurred 49% of the times (44 times). The use of rhetorical questions may be justified by the intention to let students think for themselves, to highlight the importance of something that has been said, or to pause for a moment.

Refining, reminding and repeating
These explaining actions seemed to appear very often. Below, in table 8, the occurrence of the actions *refining*, *reminding* and *repeating* are shown together with multiple combinations of those actions. In the first column, the actions are stated that occurred. For example, the fourth row consists of the number of appearances of the combination of *refining* and *repeating*. In the second column, the frequency of each combination is shown with the percentage of this occurrence with respect to the total. Specifically, in the first row, the frequency is shown of the appearance of solely the action *refining*. Such a phrase can also have other assigned actions to it, like *concluding*, but not *reminding* or *repeating*. Those combinations are shown in row 5 and row 6 respectively.

Table 8: *Occurrence of* Refining, Reminding *and* Repeating.

	Frequency (%)
Refining	127 (24%)
Reminding	15 (3%)
Repeating	17 (3%)
Refining and Repeating	227 (43%)
Refining and Reminding	3 (1%)
Reminding and Repeating	98 (18%)
Refining, Reminding and Repeating	44 (8%)
Total	531 (100%)

The combination that occurred most often, 43%, was the combination of the actions *refining* and *repeating*. With that comes the notion that the action that appeared most often alone with respect to *refining*, *reminding* and *repeating*, was *refining*, namely 24%. The combination that appeared the least, 1%, was the combination of *refining* and *reminding*. The high percentages in the rows where the action *refining* appeared, agree with the high percentage of the total occurrence of this action shown in table 4. *Refining* and *repeating* appeared most frequently (13% and 12%, respectively), against *reminding* (5%). When considering the descriptions of the episodes in section 5.3, the conclusions made there about *refining* and *repeating* can be confirmed with the knowledge of the high percentages of those actions. Namely, it was noted that certain statements were refined and repeated often to highlight their importance.

Another observation was made regarding these actions. It emerged that phrases such as "that's what this implication means", and "well that means we can make f x $[f(x)]$, the function values, arbitrarily close to

L" had *refining* and *repeating* as assigned actions to it often. Such formalizing of phrases was perhaps used with a motive. Explicitly saying what meaning underlies a mathematical procedure, may enhance students' understanding of that procedure.

Providing reasoning
The action *providing reasoning* occurred rarely, namely 2%. Table 9 shows the co-occurrence of *providing reasoning* with all other actions. Actions that did not appear with *providing reasoning*, are not shown in the table. Note that the numbers in the table do not add up, for the reason that as many actions were assigned as possible to characterize each phrase.

Table 9: *Co-occurrence regarding* Providing Reasoning.

	Providing Reasoning (66)
Concluding (267)	7
Connecting (45)	4
Demonstrating Steps (271)	14
Describing (692)	46
Developing Representational Tools (180)	5
Formulating (640)	30
Refining (401)	13
Reminding (160)	10
Repeating (386)	15
Simplifying (8)	1
Summarising (33)	1

Apart from the high frequencies of the co-occurrence of *providing reasoning* with *describing* and *formulating*, *providing reasoning* was mostly assigned to phrases along with *demonstrating steps*, *refining*, *reminding* or *repeating*. This led to the belief that when the lecturer refined a phrase or told the students what to do, he did so by justifying their reasoning. Also, the notion seen in section 5.3, focusing on the episodes marked as proofs, supplied this observation. This observation was that often steps were demonstrated and followed by providing reasoning. Hence, the high frequency of the co-occurrence of *providing reasoning* and *demonstrating steps* seemed to be a reasonable effect of the observations made in section 5.3. On the contrary, reasoning was not provided consistently when focusing on definitions, examples and theorems. To finalize, reasoning was provided minimally, as it only occurred frequently in proofs.

Connecting, simplifying and summarising
These actions were rarely assigned to phrases. *Connecting* occurred 1%, *simplifying* 0%, and *summarising* 1%. Phrases that were marked *connecting* were used to make links between definitions and examples, for instance: "well, let's consider this first graphically, so here I'm going to draw f, of its x and in the limit x goes to *plus infinity*". A phrase that was used with *simplifying* and *summarising* assigned, was "it says, in a sloppy way, that all values between f a $[f(a)]$ and f b $[f(b)]$ on this axis are function values for x values between a and b $[x \in [a, b]]$".

Errors
The lecturer made a few errors in his discourse, eleven in total. 36% of those errors were fixed by students, meaning that students raised their hands to ask a question about it. In the other cases, the lecturer fixed his error right away, meaning he was aware of the error. Errors fixed by students, may be made intentionally by the lecturer. Lecturers view intentional errors as positive as it opens possibilities for discussion, and that it is okay for students to make mistakes (Matteucci, Corazza, & Santagata, 2015).

Textbook
The action *textbook* was assigned to 12 phrases. In 83% of the occurrences of this action, the phrase referred to an example from the textbook. The other cases of the occurrence of this action, considered phrases referring to certain definitions in the textbook. Specifically, in what section that definition could be found.

6 Discussion and Conclusion

The definition used in these lectures was the Weierstrass definition, stated in definition 1.1. This is done despite the fact that it was investigated that this definition is the most problematic part of calculus for students (Bokhari & Yushau, 2006). This however may be done since the Weierstrass definition is still the main definition that is used in textbooks for explaining the concept of limit, because it uses symbolization that is needed. As this definition cannot simply be removed from the curriculum of a course, it is suggested to pay attention to the difficult parts of the definition, or to make use of the alternative definition described in definition 2.1. This alternative definition is then not seen as a replacement of the Weierstrass definition, it is meant as a complementary definition. When considering the definitions explained in the lectures, a pattern is deduced regarding those episodes. In this pattern, it is seen that statements within the definitions were repeated and refined occasionally, to emphasize its importance. This might be done as a result of the definition being interpreted as difficult.

A structure can be recognized in the second lecture, namely the DTP format (definition-theorem-proof format). Within this DTP format, the TPA format (theorem-proof-application format) was also slightly derived. While there was no format deducted in the first lecture, the structure can be defined as a definition-application format. After every definition and property, the lecturer explained its application in the form of examples. The second lecture began with a definition, following some examples. Then, a theorem was given and proven, followed by some applications of this theorem in the form of examples. Again, the lecturer stated a theorem, gave examples, followed by another definition and its application. Subsequently, another theorem was given with an example, and proven additionally. Lastly, another example was given to end the lecture and the limit concept came to an end.

The highest percentages that occurred considering the total occurrences of each action in total, seen in table 4, where *describing* (22%) and *formulating* (20%). This can be explained by the fact that each phrase was coded either *describing*, *formulating*, or both. 48% of all phrases were coded with *describing*, 44% was coded with *formulating*, while the rest was coded with both actions. Furthermore, the lecturer used quite a lot of phrases within his discourse that repeated and refined what was said before. Apart from *describing* and *formulating*, the actions *refining* and *repeating* occurred a lot comparing to all other actions (13%, 12%, respectively). The argument of why these actions had such high percentages in occurrence, may be a consequence of the lecturer putting emphasis on the importance of these statements within his lecture. Combining the observations made in section 5.3 with the high percentages, obtained the same conclusion. *Highlighting* was an action that appeared regularly (3%), in the form of "yeah?" or an actual question ("what will happen?"). This might be done by the lecturer to check in with the students, and to let students think for themselves to reason what was done or why something was done. Furthermore, *concluding* was an action that appeared frequently (8%). It was seen in the excerpts shown in section 5.2 that phrases were used regularly that concluded within the explanation. Additionally, phrases were identified to make concluding statements at the end of explanations (*inferring*, 1%). The knowledge that the lecturer concluded within episodes along with concluding at the end of an episode regularly was established. Lastly, *demonstrating steps* and *developing representational tools* occurred steadily (6% and 6%, respectively). The action *demonstrating steps* mostly occurred in the episodes proofs and examples. This might be done to let students think for themselves of how to tackle such a mathematical problem. The action *providing reasoning* occurred minimally during the lectures. Specifically, this action occurred frequently in proofs, while in other episodes this action appeared exceptionally. Additionally, when this action appeared, it did in combination with *demonstrating steps*. Thus, if steps were demonstrated, this was justified by providing reasoning. Regarding the action *developing representational tools*, a symbolic approach was mostly used on its own (20 episodes). Consequently, the graphical approach was used in addition in nine episodes. One episode was treated solely with a graphical representation. This episode was marked as an example, and showed a sketch of a discontinuous function.

Furthermore, phrases that were formalized as "meaning that x is strictly positive", and "that means *delta* is *epsilon* over 7 [$\delta = \frac{\epsilon}{7}$]", were often used and were defined as *refining*. This may have had an underlying motive. That is, by teaching, the lecturer might aim for enhancing students' understanding, rather than aiming on students passing final examinations.

Additionally, students were not involved in the lectures. The only moments where students were involved, were when a student asked a question. Other than that, there appeared to be no interaction in the lectures between students and the lecturer. This observation did not come as a surprise, as it was studied by Güçler (2013) that lecturer-student interaction arose infrequently during lectures. This observation may also be a consequence of the fact that the average number of students present during these lectures was 250 students, which makes involving students rather impractical.

To answer the research question where this thesis is centered around, the subquestions are answered first. The kind of explaining actions that were used most often were *describing*, *refining*, and *repeating*. The kind of extending actions mostly used was *formulating*. When it comes to identifying a pattern of actions within episodes of the lectures, it differed per episode if such a pattern was identified. When considering definitions, a pattern was identified. Both analyzed definitions were explained using a symbolic and graphical representation. Many phrases were identified as *refining* and *repeating*. Emphasis was put on the limiting value L, *epsilon* being arbitrary, and the existence of *delta*. Finally, the analyzed definitions concluded with giving the definition in full detail. When considering theorems, no pattern of actions was identified. The explanations of the theorems differed. Different representations were used, steps were not demonstrated. The only common observation made was that essential parts of the theorems were emphasized by *refining* and *repeating*. When focusing on proofs, a pattern was identified. It was found that steps were demonstrated and students were told what needed to be done to complete the proofs. Essential steps were refined and repeated to emphasize them. Also, reasoning was provided to justify those steps. Both proofs ended with concluding what needed to be done, was done. Lastly, considering examples, no pattern of actions was identified as the examples differed. Some consisted of brief descriptions without reasoning and demonstrating steps, while others were described in much detail. Some consisted of demonstrating steps, while others consisted more of providing reasoning. With all the results described above, I can conclude that the concept of limits is mostly based on explaining definitions and theorems by proving them and showing its applications. It is about showing steps and mathematical procedures rather than the reasoning behind them. This may be related to the course being a first-year course in the beginning of a mathematics program. This course built on high school knowledge, and therefore cannot describe reasoning extensively.

While analyzing these lectures, it appeared the lecturer did not use technological tools in his discourse. Visualizations of definitions, examples, and theorems were done by sketching graphs on the blackboard instead. As visualizing such graphs contributes to a better understanding in students of that particular explanation (Kidron & Tall, 2015; Liang, 2016), I suggest in further research the application of technological tools is studied in order to question if this benefits students' understanding as well as the lecturer's discourse. Furthermore, lecturers have limited time to give their lectures (Liang, 2016). Offering visualizations through technological tools rather than sketches on the blackboard, may ensure that there is less time needed for the explanation of visualizations, hence time can be redistributed in such a way that it contributes to the explanation of certain topics that thereafter may enhance students' understanding. In order to achieve that, this must be investigated by executing such research.

The research done in this thesis contributes to existing literature in a way that it fills a gap in the research done in teaching the limit concept at university level in the Netherlands. However, it is a case study, so generalizations can not be made regarding the results achieved in this thesis. It may occur that, for instance, different approaches of teaching can be derived when researching the teaching of the limit concept at different universities. Therefore, I suggest further research that considers the analysis of lectures, is done on more than one university, perhaps even consider universities in several countries. Moreover, the lecturer was not interviewed, so the reasoning of the lecturer behind the mathematical explanations is not examined.

Likewise, it was studied errors are sometimes made intentionally (Matteucci et al., 2015). However, the lecturer's reasoning behind making mistakes was not analyzed, thus no conclusion could be made regarding errors. Moreover, as the first lecture was analyzed using an audio-recording, the use of gestures could not be identified. The video-recording of the second lecture showed the lecturer made use of gestures such as walking around and looking into the classroom. To conclude anything regarding gestures made, it is necessary no data is missing and to analyze the lecturer's reasoning. In my opinion, interviews will therefore benefit further research in a way that it contributes to the understanding of the lecturer's reasoning. Interviews with students can also contribute to the understanding of the misconceptions of the limit concept, and to check whether the mathematical material given in lectures is understood. Lew et al. (2016) studied that students failed to grasp the key ideas of an explanation, thus I suggest for further research, this concept is taken into account and added to the analysis of the teaching of the limit concept.

In further research, I suggest investigating the approach of teaching by questioning the lecture style and examine if more lecturer-student interaction will benefit students' understanding of the concept of limits.

Despite being a case study, this study contributes to previous literature in a way that it agrees with existing literature. The identified format was the widely used DTP format, little lecturer-student interaction appeared and the used definition was indeed the Weierstrass definition. Results of existing literature described in section 2 have been confirmed by the findings of the analysis done in this thesis. When lecturers seek to analyze their approach of teaching and perhaps want to change that approach, this thesis will function as a description of the DTP format used considering the concept of limits. On account of this thesis, hope is pinned upon further research following the recommendations suggested, focusing on the teaching of the mathematical limit at university level.

References

Bergsten, C. (2007). Investigating quality of undergraduate mathematics lectures. *Mathematics Education Research Journal*, *19*(3), 48–72.

Bezuidenhout, J. (2001). Limits and continuity: some conceptions of first-year students. *International journal of mathematical education in science and technology*, *32*(4), 487–500.

Bokhari, M., & Yushau, B. (2006). Local (l, ε)-approximation of a function of single variable: an alternative way to define limit. *International Journal of Mathematical Education in Science and Technology*, *37*(5), 515–526.

Bressoud, D. M., Carlson, M. P., Mesa, V., & Rasmussen, C. (2013). The calculus student: insights from the mathematical association of america national study. *International Journal of Mathematical Education in Science and Technology*, *44*(5), 685–698.

Creswell, J. W. (2008). *Educational Research: Planning, Conducting, and Evaluating Quantitative and Qualitative Research* (4th ed.). Upper Saddle River, N.J.: Pearson/Merrill Prentice Hall.

Fernández-Plaza, J. A., & Simpson, A. (2016). Three concepts or one? students' understanding of basic limit concepts. *Educational Studies in Mathematics*, *93*(3), 315–332.

Güçler, B. (2013). Examining the discourse on the limit concept in a beginning-level calculus classroom. *Educational Studies in Mathematics*, *82*(3), 439–453.

Hardy, N. (2009). Students' perceptions of institutional practices: the case of limits of functions in college level calculus courses. *Educational Studies in Mathematics*, *72*(3), 341.

Kidron, I., & Tall, D. (2015). The roles of visualization and symbolism in the potential and actual infinity of the limit process. *Educational Studies in Mathematics*, *88*(2), 183–199.

Leron, U., & Dubinsky, E. (1995). An abstract algebra story. *The American Mathematical Monthly*, *102*(3), 227–242.

Lew, K., Fukawa-Connelly, T. P., Mejía-Ramos, J. P., & Weber, K. (2016). Lectures in advanced mathematics: Why students might not understand what the mathematics professor is trying to convey. *Journal for Research in Mathematics Education*, *47*(2), 162–198.

Liang, S. (2016). Teaching the concept of limit by using conceptual conflict strategy and desmos graphing calculator. *International Journal of Research in Education and Science*, *2*(1), 35–48.

Mali, A., & Petropoulou, G. (2017). Characterising undergraduate mathematics teaching across settings and countries: an analytical framework. *Nordic Studies in Mathematics Education*, *22*(4), 23–42.

Matteucci, M. C., Corazza, M., & Santagata, R. (2015). Learning from errors, or not. an analysis of teachers' beliefs about errors and error-handling strategies through questionnaire and video. *Progress in Education*, *37*, 33–54.

Przenioslo, M. (2004). Images of the limit of function formed in the course of mathematical studies at the university. *Educational Studies in Mathematics*, *55*(1-3), 103–132.

Saroyan, A., & Snell, L. S. (1997). Variations in lecturing styles. *Higher Education*, *33*(1), 85–104.

Stewart, J. (2010). *Calculus: early transcendentals*. Cengage Learning.

Szydlik, J. E. (2000). Mathematical beliefs and conceptual understanding of the limit of a function. *Journal for Research in Mathematics Education*, 258–276.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *The Journal of Mathematical Behavior*, *23*(2), 115–133.

Weinberg, A., Wiesner, E., & Fukawa-Connelly, T. (2016). Mathematics lectures as narratives: Insights from network graph methodology. *Educational Studies in Mathematics*, *91*(2), 203–226.

Williams, S. R. (1991). Models of limit held by college calculus students. *Journal for research in Mathematics Education*, 219–236.

A Methodology Literature Review

This thesis showed a literature review of a selection of relevant articles found with a literature search related to the teaching of the concept of limit. I used a time scale of approximately 15 years for my literature search, because however the concept of limit has not changed over the years, teaching methods change and articles written in the nineties are therefore outdated. For the literature review, it is needed to be as accurate as possible, hence the use of only recent articles is justified. To ensure the quality of the articles, I narrowed down my literature search to a selection of international journals in mathematics and science education. Those international journals were peer-reviewed, guaranteeing the quality of the articles. The articles in this literature review were from the following journals: *Educational Studies in Mathematics*, *International Journal of Mathematical Education in Science and Technology*, *International Journal of Research in Education and Science*, *Journal for Research in Mathematics Education*, *Mathematics Education Research Journal*, and *The Journal of Mathematical Behavior*.

Firstly, key words were identified to initiate the literature search. Key words such as "concept of limits", "teaching", "university level", "tertiary education", and "calculus" were used. Together with the key words, the other criteria mentioned above were used to narrow down the search. Secondly, the literature was located. This was done with the use of multiple databases. The databases used in this literature search were ERIC, Browzine Library, Google Scholar, and SmartCat. Articles were scanned to find out whether they were relevant for this literature review. By scanning I imply reading the title and the abstract. The third step consisted of critically analyzing the found articles for relevant information. This started with reading the abstract and introduction to acquire a general idea of the focus of the article. This then established a more compact and accurate selection of the found articles. Likewise, double articles found in the different databases were sorted. Fourth, the found literature is organized. That is, related articles are clustered to create a literature map in order to clarify all the information found. This then will benefit with the start of the analysis of the literature (Creswell, 2008). Below, the literature map that is used in this thesis is shown in figure 4.

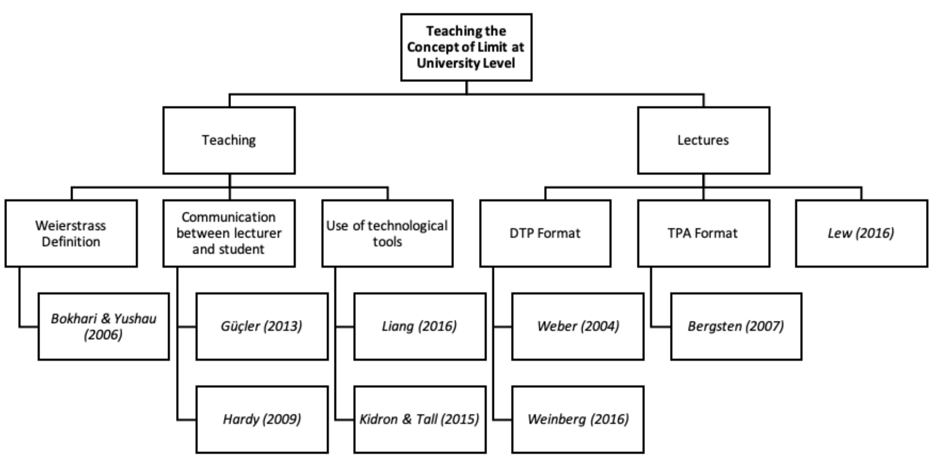


Figure 4: Literature Map

B Sample of Coding

2:562 ¶ 274 in 16 September 2019.docx

Content:

> This is what we can reason directly, assuming that, we know that the square root is an increasing function.

2 Coding:

- **Describing**
- **Formulating**

2:563 ¶ 274 in 16 September 2019.docx

Content:

> Hence its maximum is over there.

2 Coding:

- **Concluding**
- **Describing**

2:564 ¶ 274 in 16 September 2019.docx

Content:

> This is what we need to prove, and this shows the relation between delta and epsilon bb pause.

4 Coding:

- **Demonstrating Steps**
- **Describing**
- **Reminding**
- **Repeating**

2:565 ¶ 274 in 16 September 2019.docx

Content:

> And you can actually read it off here pause.

2 Coding:

- **Demonstrating Steps**
- **Describing**

2:566 ¶ 274 in 16 September 2019.docx

Content:

> And from that, I can also use the value of delta, “delta is then the square root out of epsilon’ bb pause.

3 Coding:

- **Concluding**
- **Demonstrating Steps**
- **Formulating**

1:10 ¶ 15 in 18 September 2019.docx

Content:
> So in order to show, that the only thing we have to do is to provide that delta.

2 Coding:
- **Concluding**
- **Describing**

1:11 ¶ 15 in 18 September 2019.docx

Content:
> So that's mainly what we'll do uhm, pause.

5 Coding:
- **Concluding**
- **Demonstrating Steps**
- **Describing**
- **Refining**
- **Repeating**

1:12 ¶ 18 in 18 September 2019.docx

Content:
> Last time I was almost finished with a, I almost finished an example.

3 Coding:
- **Describing**
- **Previous Lecture**
- **Reminding**

1:13 ¶ 18 in 18 September 2019.docx

Content:
> I'll do it again today.

2 Coding:
- **Demonstrating Steps**
- **Describing**

1:14 ¶ 18 in 18 September 2019.docx

Content:
> It's the most difficult example, uhm, because it's not straightforward in there.

2 Coding:
- **Describing**
- **Refining**