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# Summation of divergent series

Bachelor's Project Mathematics

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# 1 Introduction

Addition is perhaps the most natural and fundamental of all mathematics. To take two numbers and combine them to create a new number is the a very basic operation. From this, much of mathematics is born; other operations like subtraction and multiplication follow naturally. The process of addition can also be repeated to form a sum of multiple numbers. Expand this to infinitely many numbers and we are adding sequences. Take a sequence for which a sum cannot be trivially assigned and it will have brought to the subject of this thesis.

## 1.1 Prior knowledge and aim

This thesis will be centered around sequences, series, and convergence. As a brief reminder, a series is the sum of a sequence.

**Definition 1.1.** Let $(a_i)$ be a sequence with $a_i \in \mathbb{R}$ for all $i \in \mathbb{N}$. Then, the series over this sum, which we will call $A$ here, is given by

$$A = \sum_{i=1}^{\infty} a_i.$$

In this thesis, we will only be concerned with real-valued sequences and series and so we will always have that $a_i \in \mathbb{R}$. When assigning a value as "sum" to a series, we consider the $n$-th partial sum and its limit as $n$ tends to infinity (Abbott, 2016; Davis, 1962).

**Definition 1.2.** For any natural number $n$, the $n$-th partial sum of a series, denoted $s_n$, is given by

$$s_n = \sum_{i=1}^{n} a_i.$$

When considering the limit of these partial sums to assign a value to the series, we normally require this limit exist and be finite. If this is the case for a given series, then we call such a series convergent.

**Definition 1.3** (Convergent series)**.** Let $s_n$ denote the $n$-th partial sum of a series $A$. Then, if $\lim_{n\to\infty}$ exists *or* is finite, we call the series $A$ convergent. Moreover, we assign the value of this limit as the sum of series $A$.

Every sequence which does fall in this category is automatically divergent, meaning the partial sums do not converge to a finite limit.

**Definition 1.4** (Divergent series)**.** Let $s_n$ denote the $n$-th partial sum of a series $A$. Then, if $\lim_{n\to\infty}$ does not exist *or* is not finite, we call the series $A$ convergent. Alternatively, all series that not convergent are divergent.

For a more extensive explanation of sequences, series, and convergence, (re-)read "Understanding Analysis" by Stephen Abott.

The thesis is divided in three parts. At the base of each are Cesàro summation, defined in 'Summation methods', and the Grandi series, given by

$$G = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots.$$

Each section will expand, generalize, or otherwise build upon these in some way to further the reader's knowledge about this subject.

In section 2, 'Summation methods', we will define Cesàro summation amd give some examples. We will also generalize and expand this method to Hölder summation and prove a number of theorems about it. Last of the methods, we will briefly mention Voronoi summation. This will serve mostly as example for the final subsection about the matrix form of summation methods. Here, we will look at summation methods from a different angle and prove some properties using this notation.

Moving on to section 3, 'Adding infinitely many zeroes', we will make some observation about how injecting zeroes in a divergent series can change the outcome when using summation methods. We will then generalize this and prove a statement hypothesized by Daniel Bernoulli in the 18th century.

Lastly, in section 4, 'Visual approach' we will take a step back from our methods and instead consider things visually and suggest, without proof, an approach for assigning values to a series by considering the intersection of subseries of a specific type of series.

## 1.2 Notation and conventions

Throughout this thesis, we will try to maintain consistent notation as much as possible, some of the concepts described below will be propperly introduced later:

- The natural numbers, $\mathbb{N}$, start at 1.
- *i,j,k,m,n* for **indices** starting from 1. When summing, we will use $i$. For general indices, we will use $n$ and later also $m$. We will only use $k$ for Hölder summation;
- $a_1, a_2, a_3, \ldots, a_n$ will denote the **terms** of a series;
- $s_1, s_2, s_3, \ldots, s_n$ will denote the $n$-th **partial sum** of a series;
- $c_1, c_2, c_3, \ldots, c_n$ will denote the $n$-th **Cesàro sum** of a series;
- For each summation method, we indicate that a limit is assigned to a series using a certain method with an indication between brackets. For example,

$$\sum_{i=1}^{\infty} \frac{1}{i^2 \cdot (i+1)} = \frac{\pi^2 - 6}{6}(C)$$

  (Series No. 272, Jolley, 1961)
  means we obtained this limit using the Cesàro method of summation. When using classical summation, we will use no symbol.

We will also refer to summation in the usual sense as "classical summation" to avoid confusion with "regular summation", which will be defined later.

# 2 Summation methods

In this section, we will introduce and discuss Cesàro summation, Hölder summation, Voronoi summation, as well as some properties that come with these method and how they are related.

## 2.1 First definitions

We are interested in finding methods that allows us to map sequences to the real numbers in a meaningful way, we can describe such a map as follows,

$$(a_n) \mapsto \Sigma a_i \in \mathbb{R}.$$

For sequences that converge in the usual sense, it suffices to define $\Sigma a_i$ as the limit of the partial sums of $a_i$,

$$\Sigma a_i = \lim_{n\to\infty} s_n.$$

We seek to extend this notion to divergent series. We will still call this the "sum" of a series and keep using sigma notation to indicate we have mapped the elements from a sequence to a value in $\mathbb{R}$ although this is of course not the limit of the partial sums as it usually is. To keep our methods of assigning values to these sums meaningful, we will look at methods that satisfy Hardy's three axioms as described in "Divergent Series" (see Hardy, 1949, page 6).

**Definition 2.1** (Hardy's Axioms)**.** Let $(a_n)$ be a series, then we define the following properties:
**Scalability:** if $\sum_{i=1}^{\infty} a_i = s$ then $\sum_{i=1}^{\infty} k \cdot a_n = k \cdot s$ for any $k \in \mathbb{R}$
**Additivity:** if $\sum_{i=1}^{\infty} a_i = s$ and if $\sum_{i=1}^{\infty} b_n = t$ then if $\sum_{i=1}^{\infty}(a_i + b_i) = s + t$
**Stability:** if $a_0 + a_1 + a_2 + \cdots = s$ then $a_1 + a_2 + a_3 \cdots = s - a_0$ as well as the converse.

These properties by Hardy are satisfied by all methods that will be discussed, and occasionally used to support proofs. One property that arises naturally is regularity. We would like for our methods to be regular so we can consider them as extensions to conventional convergence, rather than alternatives.

**Definition 2.2.** A summation method is called **regular** if it sums every convergent series (in the normal sense) to its ordinary sum.

We also have a stronger version of this property, namely absolute regularity.

**Definition 2.3.** A summation method, S, is called **absolutely regular** if it is regular and, moreover, if for all sequences $(a_n)$ with partial sums $s_n$ for which $\lim_{n\to\infty} s_n = \infty$ we also have $\sum\limits_{i=1}^{\infty} a_i = \infty(S)$.

## 2.2 Cesàro summation

One method of summation is called **Cesàro summation** named after Ernesto Cesàro (1859 - 1906) and is defined as follows:

**Definition 2.4** (Cesàro summation)**.** Let $(a_n)$ be a sequence and let $s_k$ denote the $k$-th partial sum. Then, define $(c_n)$ by $c_n = \frac{1}{n}\sum_{k=1}^{n} s_k$. If the sequence $(c_n)$ converges to a limit $S$ as $n$ tends to infinity, then we say the sequence $(a_n)$ is Cesàro summable and has Cesàro sum $S$, which we will denote as $\sum_{n=1}^{\infty} a_n = S(C)$.

We would like that the series that are normally summable also are Cesàro summable and, moreover, that we can assign the same value to the series regardless of which summation method we pick. If this is the case then we see Cesàro summability is an extension of classical summability.

**Theorem 2.5.** *If* $\sum_{n=1}^{\infty} a_n = S$ *then* $\sum_{n=1}^{\infty} a_n = S(C)$.

*Proof.* Suppose a series $(a_n)$ has sum $S$, then $\lim_{n\to\infty} s_n = S$ so there exists an $N \in \mathbb{N}$ and an $\epsilon > 0$ such that for $n > N$ we have $|s_n - S| < \epsilon$.
Now,

$$\begin{aligned}
|c_n - S| =& \left|\frac{s_1 + s_2 + s_3 + \cdots + s_n}{n} - S\right| \\
=& \left|\frac{s_1 - S + s_2 - S + s_3 - S + \cdots + s_n - S}{n}\right| \\
\leq& \frac{|s_1 - S| + \cdots + |s_{N-1} - S|}{n} + \frac{|s_N - S| + \cdots + |s_n - S|}{n} \\
<& \frac{|s_1 - S| + \cdots + |s_{N-1} - S|}{n} + \left|\frac{n - N + 1}{n}\right| \epsilon.
\end{aligned}$$

We see that, as $n$ tends to infinity, the left term on the right hand side tends to 0 and the right term on the right hand side tends to $\epsilon$, so $|c_n - S| < \epsilon$ and thus $\lim_{n\to\infty} c_n = S$ which shows that $\sum_{n=1}^{\infty} a_n = S(C)$ so Cesàro is regular. □

**Example 2.6.** Let $(a_n) = \frac{1}{2^n}$, we know that that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

and that the partial sums $s_k$ are given by $s_k = 1 - \frac{1}{2^k}$. let us look at the Cesàro sum of this sequence. The $c_n$ terms are given by

$$c_n = \frac{1}{n}\sum_{k=1}^{n} s_k = \frac{(1 - \frac{1}{2^1}) + (1 - \frac{1}{2^2}) + \cdots + (1 - \frac{1}{2^n})}{n}.$$

So we now have

$$n \cdot c_n = \underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ times}} - \left(\frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}\right).$$

The expression between brackets can be recognised to be the $n$-th partial sum, for which we have a closed form. So we have

$$n \cdot c_n = \; n - (1 - \frac{1}{2^n}).$$

We divide by $n$ and obtain the following expression for $c_n$:

$$c_n = \; 1 - \frac{1}{n} + \frac{1}{n2^n}.$$

So we find that $\lim\limits_{n\to\infty} c_n = 1$.

Now that we have seen an example of how Cesàro summation is used, we want to see something more exciting. Consider for this the Grandi series.

The terms of the Grandi series are given by $g_n = (-1)^n$ so the sum would be given by $G = \sum_{n=0}^{\infty} g_n = 1 - 1 + 1 - 1 + \dots$.
One could argue that

$$G = (1-1) + (1-1) + (1-1) + \dots = 0 + 0 + 0 + \dots = 0$$

while at the same time their contrarian could argue that

$$G = 1 + (-1+1) + (-1+1) + \dots = 1 + 0 + 0 + \dots = 1.$$

So which one is it?

Intuitively, if we keep alternating between 0 and 1 then could we end up in the middle of those two? It could be tempting to argue that $1 - G = 1 - 1 + 1 - 1 + \dots = G$ and conclude that $1 - G = G$ so $2G = 1$ so $G = \frac{1}{2}$. While this reasoning is normally out of the question for divergent series, let us see how applying Cesàro summation can help us assign a value to this series.

**Example 2.7.** The Grandi series, $G = \sum_{i=1}^{\infty} g_i = 1 - 1 + 1 - 1 + \dots$ as described above, has partial sums $s_n$ given by

$$s_n = \sum_{i=0}^{n} (-1)^i = \frac{(-1)^n + 1}{2} = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Applying the Cesàro method gives

$$\begin{aligned} c_n = \frac{1}{n} \sum_{i=0}^{n} s_i &= \frac{1}{n} \sum_{i=0}^{n} \frac{(-1)^i + 1}{2} \\ &= \frac{1}{2n} \left( n + \sum_{i=0}^{n} (-1)^i \right) \\ &= \frac{1}{2} + \frac{1}{2n} \cdot s_n. \end{aligned}$$

So we find that $c_n = \frac{1}{2} + \frac{s_n}{2n}$ and as $|s_n| \leq 1$ we can see that $c_n$ tends to $\frac{1}{2}$ as $n$ tends to infinity. And so we conclude that $\sum_{n=0}^{\infty} g_n = \frac{1}{2}(C)$.

Interestingly, this can be seen as a continuation of the geometric series. In classical summation, we have, for $a, r \in \mathbb{R}$ and $|r| < 1$, that

$$a + ar + ar^2 + \dots = \frac{a}{1-r}.$$

(Series No. 3 Jolley, 1961)

Here we require that $r \in (-1, 1)$ in order for the series to converge. Setting $a$ equal to 1 for now, we can see that if we allow $r$ to be 1, this sum would tend to infinity. This makes sense as for $r = 1$ we are simply adding 1 (or $a$) infinitely many times. On the other end, if we let $r$ be -1, we see that the right hand side becomes a half, nothing strange, and on the left hand side the series becomes the Grandi series. So using Cesàro we have extended the range of convergence for the geometric series from $(-1, 1)$ to $[-1, 1)$.

### 2.3 Hölder summation

When we apply Cesàro summation, defined $c_n$ as the average of $s_1$ through $s_n$. We repeat this process, taking the average of $c_1$ through $c_n$ and call this quantity $H_n^2$. If we relabel $c_n$ to $H_n^1$, we find a general form for the quantity $H_n^k$, which we will need for the summation method known as Hölder summation.

**Definition 2.8.** Let $(a_n)$ be a series and call the $n$-th partial sum $H_n^0$. We inductively, in $k$, define $H_n^k$ as

$$H_n^{k+1} = \frac{H_1^k + H_2^k + \cdots + H_n^k}{n}.$$

If the limit $\lim_{n\to\infty} H_n^k$ exists, let call it $S$, for some *finite* $k$, then this limit is called the Hölder sum or $(H,k)$ sum of the sequence $(a_n)$ and we say $\sum_{i=1}^{\infty} a_i = S(H,k)$ (Hölder, 1882; Volkov, 2011).

To see why we require $k$ to remain finite, we will take a closer look at what $H_n^{k+1}$ does to $H_n^k$. Recall that

$$H_n^0 = a_1 + a_2 + \cdots + a_n$$

and take a closer look at $H_n^{\cdot}$

$$\begin{aligned} n \cdot H_n^1 =& H_1^0 + H_2^0 + H_3^0 \cdots + H_n^0 \\ =& a_1 + \\ & a_1 + a_2 + \\ & a_1 + a_2 + a_3 + \\ & \qquad \vdots \\ & a_1 + a_2 + a_3 + \cdots + a_n. \end{aligned}$$

So we end up with $n$ copies of $a_1$, $n-1$ copies of $a_2$ and so on with one copy of $a_n$. So we can rewrite $H_n^1$ as the following:

$$H_n^1 = a_1 + \frac{n-1}{n}a_2 + \frac{n-2}{n}a_3 + \cdots + \frac{1}{n}a_n.$$

If we now look for $H_n^2$ we will find something similar, namely

$$H_n^2 = a_1 + \left(\frac{n-1}{n}\right)^2 a_2 + \cdots + \frac{1}{n^2}a_n.$$

Moreover, we can write

$$\begin{aligned} H_n^k &= a_1 + \left(\frac{n-1}{n}\right)^k a_2 + \cdots + \frac{1}{n^k}a_n \\ &= \sum_{i=1}^{n} \left(\frac{n-i+1}{n}\right)^k a_i. \end{aligned}$$

And from this it is clear to see that, if we allow $k$ go to infinity, while keeping $n$ finite, this sum will reduce to be only $a_1$, which is not a very insightful or meaningful way of assigning a sum to a series. This would mean that classically summable series now have different results so this method would not be regular and while this method would be scalable and additive (also referred to as linear), it would not be stable.

So we cannot allow $k$ to be infinite, but it is advantageous to pick $k$ large. This is because, in essence, $k$ is the amount of times we successively apply Cesàro's method of summation. Since Cesàro doesn't decrease the amount of series we can sum, we have the following theorem.

**Theorem 2.9.** *If* $\sum_{i=1}^{\infty} a_n = S(H,k)$*, then* $\sum_{i=1}^{\infty} a_n = S(H,k')$ *for all* $k' > k$*.*

In other words, the series we can sum with a low value of $k$ can also be summed with a high value of $k$. The converse does not hold, as we will see later, a higher value of $k$ is 'stronger' than a lower one.

*Proof.* Let $(a_n)$ be $(H,k)$ summable. Then, the limit $\lim_{n\to\infty} H_n^k$ exists. let us call this limit $S$. If we then have that for any $\epsilon$ larger than zero and indices $n$ larger than some natural $N$ the following inequality holds.

$$\left|H_n^k - S\right| \leq \epsilon.$$

We know that

$$H_n^{k+1} = \frac{1}{n}\sum_{i=1}^{n} H_i^k,$$

so we then consider the difference between $H_n^{k+1}$ and $S$.

$$\begin{aligned}
\left|H_n^{k+1} - S\right| &= \left|\frac{1}{n}\sum_{i=1}^{n} H_i^k - S\right| \\
&= \frac{1}{n}\left|\sum_{i=1}^{n}\left(H_i^k - S\right)\right| \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\left|H_i^k - S\right| \\
&= \frac{1}{n}\sum_{i=1}^{N}\left|H_i^k - S\right| + \frac{1}{n}\sum_{i=N+1}^{n}\left|H_i^k - S\right|
\end{aligned}$$

We can see that in this sum, the left term on the right hand side tends to 0 as $n$ tends to infinity because $H_i^k - S$ is finite for each $i$ and $\frac{1}{n}$ tends to 0. For the right term on the right hand side we use the fact that, since all $i$ are larger than $N$, we have $\left|H_i^k - S\right| \leq \epsilon$ for each $i$. So we find that

$$\left|H_n^{k+1} - S\right| \leq \frac{n-N-2}{n}\epsilon.$$

This tends to $\epsilon$ as $n$ tends to infinity. Since we can choose $\epsilon$ arbitrarily small, we can conclude that $\lim_{n\to\infty} H_n^{k+1} = S$.
This shows that if a sequence $(a_n)$ is $(H,k)$ summable, then $(a_n)$ is also $(H,k+1)$ summable and so, inductively, we can see that $(a_n)$ is $(H,k')$ summable for $k' > k$. □

We have now seen that a higher value of $k$ allows for a 'stronger' sense of summation in the sense that all summable series for a lower value of $k$ are also summable with a higher value of $k$ but the converse is not necessarily true. Can we then sum any series if we increase $k$ sufficiently while keeping it finite? Unfortunately, this is not the case, there are still limits to this method. To see this, consider the following example.

**Example 2.10.** Consider the sequence $(a_n)$ defined by $a_i = 1$ for all $i$. Clearly, the series $\sum_{i=1}^{\infty} a_i$ is divergent. We apply Hölder's method to get

$$\begin{aligned}
H_n^0 =& n \text{ (the } n\text{-th partial sum)} \\
H_n^1 =& \frac{1}{n}\sum_{i=1}^{n} H_i^0 = \frac{1}{n}\sum_{i=1}^{n} i \\
=& \frac{1}{n}\cdot\frac{n(n+1)}{2} = \frac{n+1}{2} \\
H_n^2 =& \frac{1}{n}\sum_{i=1}^{n} H_i^1 = \frac{1}{n}\cdot\sum_{i=1}^{n}\frac{i+1}{2} \\
=& \frac{1}{n}\sum\left(\frac{i}{2}+\frac{1}{2}\right) = \frac{1}{2n}\sum_{i=1}^{n}(i+1) \\
=& \frac{1}{2n}\sum_{i=1}^{n} i + \frac{1}{2n}\sum_{i=1}^{n} 1 = \frac{n+1}{4}+\frac{1}{2} \\
H_n^3 =& \frac{1}{n}\sum_{i=1}^{n} H_i^2 = \frac{1}{n}\sum_{i=1}^{n}\frac{i+1}{4}+\frac{1}{2} \\
=& \frac{1}{2n}\left(\sum_{i=1}^{n}\frac{i}{2}+\frac{1}{2}+1\right) \\
=& \frac{n+1}{8}+\frac{1}{4}+\frac{1}{2}
\end{aligned}$$

It seems we find a general form for $H_n^k$, that being

$$\begin{aligned}
H_n^k =& \frac{n+1}{2^k}+\sum\frac{1}{2^{k-1}} \\
=& \frac{n+1}{2^k}+\left(1-\frac{1}{2^{k-1}}\right) \\
=& \frac{n+1}{2^k}-\frac{2}{2^k}+1 \\
=& \frac{n-1}{2^k}+1.
\end{aligned}$$

We check to see if this pattern holds inductively.

*Proof.* Assume that for some $k$, $H_n^k$ is given by $H_n^k = \frac{n-1}{2^k}+1$. Then, we apply the definition of $H_n^k$ to find $H_n^{k+1}$.

$$\begin{aligned}
H_n^{k+1} = \frac{1}{n}\sum_{i=1}^{n} H_i^k =& \frac{1}{n}\left[\sum_{i=1}^{n}\left(\frac{i-1}{2^k}+1\right)\right] \\
=& \frac{1}{n}\left[\sum_{i=1}^{n}\frac{i-1}{2^k}+\sum_{i=1}^{n} 1\right] \\
=& \frac{1}{n}\cdot\frac{1}{2^k}\cdot\sum_{i=1}^{n}(i-1)+1 \\
=& \frac{1}{n}\cdot\frac{1}{2^k}\cdot\frac{n(n-1)}{2}+1 \\
=& \frac{n-1}{2^{k+1}}+1.
\end{aligned}$$

This only converges if we allow $k$ to tend to infinity, and this is not allowed. So this series is not $(H, k)$ summable for any $k$. □

The keen eye will spot that applying Hölder summation with $k = 0$ is the same as classical summation and, moreover, that applying Hölder with $k = 1$ is the same as Cesàro summation. We will use this to show the regularity of Hölder for any value of $k$.

**Theorem 2.11.** *Hölder summation is regular for any value of k.*

*Proof.* This follows immediately from Theorem 2.5, Theorem 2.9, and the observation that $(H, 1)$ summability is the same as Cesàro summability. □

We have now seen Hölder as an extension of Cesàro summation but have yet to see an example of it being necessary to use Hölder because Cesàro fails. For this, consider the following example.

**Example 2.12.** Let $(a_n) = (-1)^{n+1} \cdot n$. Then, the sequence of partial sums is given by

$$(s_n) = 1, -1, 2, -2, 3, \ldots$$

In general, we have that $s_n = (-1)^{n+1}\lceil \frac{n}{2} \rceil$. We can then compute the sequence $(c_n) = \frac{1}{n}\sum s_n$ for which the first few values are given by

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $c_n$ | 1 | 0 | $2/3$ | 0 | $3/5$ | 0 |

and deduce the general formula

$$c_n = \begin{cases} \frac{n+1}{2n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

We can see that, as $n$ tends to infinity, the sequence $(c_n)$ tends to an alternating sequence between 0 and $\frac{1}{2}$ so it does not converge if we have $k = 1$ or use Cesàro's method.
If we now consider $H_n^2$, we find that

$$\begin{aligned} H_n^2 =& \frac{1}{n}\sum_{i=1}^{n} H_i^1 \\ =& \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{j}\sum_{j=1}^{i} H_j^0\right) \\ =& \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{j}\sum_{j=1}^{i} s_j\right) \\ =& \frac{1}{n}\sum_{i=1}^{n} c_i. \end{aligned}$$

So we see that finding $H_n^2$ is the same as applying Cesàro's method on $c_i$ in the same way Cesàro is applied to $s_i$. This further illustrates that Hölder is $k$ repetitions of Cesàro. When we further work this out, we find that

$$\lim_{n\to\infty} H_n^2 = \frac{1}{4}.$$

We write this as

$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4}(H, 2),$$

or alternatively we can write this as

$$\sum_{i=1}^{\infty}(-1)^{i+1} \cdot i = \frac{1}{4}(H, 2).$$

Where the notation $(H, 2)$ indicated we used Hölder summation with $k = 2$.

We can only apply Cesàro to alternating sequences, because it is an not only a regular method as we saw in Theorem 2.5, it is also an absolutely regular method (see Hardy, 1949, page 10). Meaning that if a series' partial sums tend to positive *or* negative infinity, then the sum we assign using the summation will also tend to positive or negative infinity. As such, we have that the only alternating sequences and sequences that converge in the usual sense can converge using the Cesàro method. As a consequence of this, Hölder summation is also only applicable to alternating series and yet, it is stronger than Cesàro summation because it allows for the partial sums to grow faster as $k$ increases.

**Theorem 2.13.** *Let $(a_n)$ be a sequence defined for which $(s_n) = (-1)^{n+1} \cdot t_n^q$, where $t_n^q = \sum_{j=0}^{q} h_j n^j$ for some constant $q \in \mathbb{N}$ and $h_j \geq 0$. Then, $(a_n)$ is Hölder summable for $k > q$.*

*Proof.* Consider a sequence as described above. We have

$$s_n = (-1)^{n+1} \cdot t_n^q.$$

Here,

$$t_n^q = \sum_{j=0}^{q} h_j n^j = h_0 + h_1 n + h_2 n^2 + \cdots + h_q n^q.$$

We apply Cesàro's method of summation to find

$$\begin{aligned}
c_n =& \frac{1}{n}\sum_{i=1}^{n} s_i \\
=& \frac{1}{n}\sum_{i=1}^{n}(-1)^{i+1} \cdot t_i^q \\
=& \frac{1}{n}\sum_{i=1}^{n}(-1)^{i+1}\left[h_0 + h_1 i + h_2 i^2 + \cdots + h_q i^q\right] \\
=& \frac{1}{n}\sum_{i \text{ odd}}^{n}\left[h_0 + h_1 i + h_2 i^2 + \cdots + h_q i^q\right] - \\
& \frac{1}{n}\sum_{i \text{ even}}^{n}\left[h_0 + h_1 i + h_2 i^2 + \cdots + h_q i^q\right] \\
=& \frac{1}{n}\left[h_0\left(\sum_{i \text{ odd}}^{n} 1\right) + h_1\left(\sum_{i \text{ odd}}^{n} i\right) + h_2\left(\sum_{i \text{ odd}}^{n} i^2\right) + \cdots + h_q\left(\sum_{i \text{ odd}}^{n} i^q\right)\right] - \\
& \frac{1}{n}\left[h_0\left(\sum_{i \text{ even}}^{n} 1\right) + h_1\left(\sum_{i \text{ even}}^{n} i\right) + h_2\left(\sum_{i \text{ even}}^{n} i^2\right) + \cdots + h_q\left(\sum_{i \text{ even}}^{n} i^q\right)\right]
\end{aligned}$$

We know that $\sum_{i \text{ odd}}^{n} i = n^2$, $\sum_{i \text{ odd}}^{n} i^2 = \frac{1}{3}(4n^3 + n,)$ and that $\sum_{i \text{ odd}}^{n} i^3 = 2n^4 - n^2$ (Series No. 25 & 26, Jolley, 1961).

Note that summing $i^q$ results in a polynomial of degree $q+1$. This is similarly true for polynomials if we sum over even indices $i$. Take for example

$$\sum_{i \text{ even}}^{n} i^2 = \frac{2n(n+1)(2n+1)}{3} = \frac{4n^3+6n^2+2n^2}{3}$$

(Series No. 28, Jolley, 1961).

Moreover, the coefficient of the higher power term is independent of whether we sum over the even or odd indices $i$ as can be seen above when summing $i^2$. To see this, we introduce the following notation:

$$\begin{aligned} I(n) &= \sum_{i=1}^{n} i^q, \\ I_e(n) &= \sum_{i \text{ even}}^{n} i^q = u_0 + u_1 n + u_2 n^2 + \cdots + u_q n^q, \\ I_o(n) &= \sum_{i \text{ odd}}^{n} i^q = v_0 + v_1 n + v_2 n^2 + \ldots v_q n^q. \end{aligned}$$

Here, $u_i, v_i \in \mathbb{R}\ \forall i \in \mathbb{N} \cup \{0\}$. We claim that $u_q = v_q$. First, we'll show that $I_e(n)$ and $I_o(n)$ have the same limit as $n$ tends to infinity. Then, we will show that this implies $u_q = v_q$ as these are the terms that dominate the value of $I_e(n)$ and $I_o(n)$ for large values of $n$.

To see that $I_e(n)$ and $I_o(n)$ have the same limit, we explicitly write out the sums they represent. We will write $2k+1$ and $2k$ instead of $n$ to indicate odd/even numbers, so $I_e(2k) = I_e(n)$ and $I_e(2k+2) = I_e(n+1)$ for $n$ even.

$$\begin{aligned} I_e(2k) &= 2^q + 4^q + \cdots + (2k)^q \\ < I_o(2k+1) &= 1^q + 3^q + 5^q + \cdots + (2k+1)^q \\ < I_e(2k+2) &= 2^q + 4^q + 6^q + \cdots + (2k+2)^q \end{aligned}$$

The inequalities can easily be confirmed by noticing that the terms are, vertically, in increasing order and by remembering that $q \in \mathbb{N}$. Since $\lim_{n\to\infty} I_e(n) = \lim_{n\to\infty} I_e(n+1)$, it follows that $\lim_{n\to\infty} I_o(n) = \lim_{n\to\infty} I_e(n)$ since the limit of $I_o(n)$ bounds it from below and above. Similarly, the limit of $I_e$ is bounded by the limit of $I_o$ in the same way.

We now know that $\lim_{n\to\infty} I_e(n) = \lim_{n\to\infty} I_o(n)$. Writing them out as polynomials of $n$ again, we get

$$\lim_{n\to\infty} u_0 + u_1 n + u_2 n^2 + \cdots + u_q n^q = \lim_{n\to\infty} v_0 + v_1 n + v_2 n^2 + \ldots v_q n^q.$$

Since the limits of $I_o(n)$ and $I_e(n)$ are equal, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{I_o(n)}{I_e(n)} = 1 &= \lim_{n\to\infty} \frac{u_0 + u_1 n + u_2 n^2 + \cdots + u_q n^q}{v_0 + v_1 n + v_2 n^2 + \ldots v_q n^q} \\ &= \lim_{n\to\infty} \frac{u_0 + u_1 n + u_2 n^2 + \cdots + u_q n^q}{v_0 + v_1 n + v_2 n^2 + \ldots v_q n^q} \cdot \frac{n^{q-1}}{n^{q-1}} \\ &= \lim_{n\to\infty} \frac{u_q n}{v_q n} \\ &= \frac{u_q}{v_q} \end{aligned}$$

So we see that the final coefficients, the ones for the highest power of $n$ have to be equal in the even and odd versions of $I$. When we look for the highest power of $n$ in our expression for $c_n$, we find it in the terms with the highest value of $q$, namely

$$h_q\left(\sum_{i \text{ odd}}^{n} i^q\right) = \tilde{h_q} n^{q+1} + \mathcal{O}(n^q) \text{ and } h_q\left(\sum_{i \text{ even}}^{n} i^q\right) = \tilde{h_q} n^{q+1} + \mathcal{O}(n^q).$$

Here, $\tilde{h_q}$ is the coefficient in front of the highest order term. We use a tilde because the sum of $i^q$ over the odd or even integers $i$ results in a polynomial and we multiply by $h_q$ afterward. Since we subtract these two polynomials, we are left with terms of order $n^q$ and lower. Diving by $n$ after summing all $s_i$'s lowers this to $n^{q-1}$. In conclusion, when applying Cesàro's method, if $s_n$ was an alternating polynomial of order $q$, then $c_n$ will be an alternating polynomial of order $q-1$. And since Hölder $k$ is $k$ repetitions of Cesàro. Remember that we have already seen that the sequence

$$(a_n) = (-1)^{n+1} \cdot 1$$

is Hölder summable with $k = 1$ and that the sequence

$$(a_n) = (-1)^{n+1} \cdot n$$

is Hölder summable with $k = 2$.

Using these examples as our base case, we can conclude by induction, that if $s_n$ is an alternating polynomial of order $q$, then it is is Hölder $q + 1$ summable.

□

## 2.4 Voronoi summation

We have seen one of way of generalizing the Cesàro method of summation. Now let us look at another. Named after at Georgy Feodosevich Voronoy (1868-1908), this method takes an auxilary sequence $(p_n)$ and uses it to assign weights to the terms of a sequence. It should be noted that this method was later rediscovered by Niels Erik Nørlund (1885-1981) so this method may also be referred to as the Nørlund method, Nørlund means, Nörlund method, or Nörlund means. We will refer to this method as Voronoi Summation.

**Definition 2.14** (Voronoi summation)**.** For a given sequence $(a_n)$, the Voronoi method considers the limit of the expression

$$\frac{a_1p_1 + a_2p_2 + \cdots + a_np_n}{p_1 + p_2 + \cdots + p_n}$$

as $n$ tends to infinity. If this limit exists, then the sequence $(a_n)$ is called $V(p_n)$ summable (Voronoi, 1932) and we assign the value of the limit as the sum of the series.

Note that we can very easily confirm that this is indeed a generalization of Cesàro summation by picking $p_i = 1$ for each $i$ (Kharshiladze, 2011).

One reason to study Voronoi summation is that, unlike Cesàro and Hölder, the method is not regular at all times. Nor is it always non-regular. It, similarly to Hölder, has a second parameter. For Hölder this was $k$ and for Voronoi it is the sequence $(p_n)$. The reguliarity of Voronoi is dependant on the choice of this sequence $(p_n)$.

**Theorem 2.15.** *Let* $\sum_{i=1}^{n} p_i = P_n$. *Then, Voronoi's summation method is regular if and only if*

$$\lim_{n\to\infty} \frac{p_n}{P_n} = 0.$$

(see Hardy, 1949, Theorem 16, page 64)

So we need $p_n$ to not grow too fast. If $p_n$ is much larger than $p_i$ for all $i < n$ then we this method will not be regular.

## 2.5 Matrix form

When looking at a summation method, we can view the transformation as a lower triangular $n$ by $n$ matrix that multiplies with the sequence $(a_n)$. Doing this gives us a new sequence that takes the form of a column vector. Here, taking the limit as $n$ tends to infinity means to look at the bottom of the infinite column. This will be illustrated with an example.

**Example 2.16** (Classical summation)**.** For normal summation, the matrix is given by only 1's.

$$\begin{bmatrix} 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & \dots \\ 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} a_1 \\ a_1 + a_2 \\ a_1 + a_2 + a_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \end{bmatrix},$$

Where $s_n$ is the $n$-th partial sum. So if we want to take the limit of $s_n$ for $n$ tending to infinity, then we have to keep looking lower and lower in this resulting column vector.

Now that we had a refresher on matrix multiplication, let us look at some examples of matrices related to methods we have discussed earlier.

**Example 2.17** (Cesàro summation)**.** For Cesàro, the non-zero entries in the $n$-th row are given by $\frac{1}{n}$ (Enyeart, 2010; Szilágyiová, 2016).

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 & \dots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \\ \vdots \end{bmatrix} = \begin{bmatrix} a_1 \\ \frac{a_1+a_2}{2} \\ \frac{a_1+a_2+a_3}{3} \\ \vdots \\ \frac{a_1+a_2+\dots+a_n}{n} \\ \vdots \end{bmatrix} = \begin{bmatrix} s_1 \\ \frac{1}{2}s_2 \\ \frac{1}{3}s_3 \\ \vdots \\ \frac{1}{n}s_n \\ \vdots \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \\ \vdots \end{bmatrix}$$

More information can be found in Hardy, including definitions, theorems, and additional lemmas (Hardy, 1949) which views a summation method as a transformation of a sequence $(a_n)$ in the form $w_m = \sum v_{m,n}a_n$ and then subsequently defines the summation matrix $T$ by $w_{m,n} = v_{m,n}$. For now, it suffices to see there exists a relationship between summation methods and matrices so we can look at some examples instead and get some intuition for how they are used.

We have already seen two examples, classical summation and Cesàro summation. As we have seen before, applying Cesàro's summation method multiple times gives rise to Hölder's summation method and the matrices follow a similar patern.

**Definition 2.18** (Hölder summation matrices)**.** Let $C \in \mathbb{R}^{n\times n}$ be the matrix associated with Cesàro summation. Then, the matrix associated with Hölder $k$ summation, $H_k$, is given by $H_k = C^k$ (Enyeart, 2010).

We saw earlier that for $p_i = 1$ for all $i$, Voronoi's method of summation reduces to Cesàro summation. This, however, does not extend to Hölder summation for any value of $k$ other than 1. Not even for $k = 0$ which means we can't find a sequence $(p_n)$ such that Voronoi summation reduces to classical summation.

**Theorem 2.19.** *For $k \neq 1$ there does not exist a sequence $(p_n)$ such that Voronoi summation reduces to Hölder summation.*

Before we will look at the proof of this for $k = 0$, let us first look at the summation matrix for Voronoi, which will be key in the proof and serve to further illustrate these matrices.

**Example 2.20** (Voronoi)**.** The entries in the Voronoi matrix are given by $v_{m,n}(V) = \frac{p_m}{P_n}$, where $P_n = \sum_{i=1}^{n} p_i$, and the Voronoi matrix $V$ is given by

$$V = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & \dots \\ \frac{p_1}{p_1+p_2} & \frac{p_2}{p_1+p_2} & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ \frac{p_1}{\sum_{i=1}^{n} p_i} & \dots & \frac{p_2}{\sum_{i=1}^{n} p_i} & \dots & \frac{p_n}{\sum_{i=1}^{n} p_i} & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \ddots \end{bmatrix}$$

*Proof of Theorem 2.19 for $k = 0$.* First, we will show this is true for $k = 0$. Consider again the matrix given in example 3.1. If we equate these two matrices we get that $\frac{p_1}{p_1+p_2} = 1 = \frac{p_2}{p_1+p_2}$. From this, we get that $p_1 = p_2$ which implies that $\frac{p_1}{p_1+p_2} = \frac{p_1}{2p_1} = \frac{1}{2} \neq 1$. This shows that classical summation is not a specific case of Voronoi summation. □

# 3 Adding infinitely many zeroes

In this section, we will look at what happens when we inject additional zeroes in the Grandi series and then consider the Cesàro sum. Summation method wise, this will not be very interesting as we will not go into more properties or new methods. However, we will be able to prove a result, described in theorem 3.2, that was speculated by Daniel Bernoulli and going through these observations and the proof will help better understand the underlying theory.

## 3.1 Introduction and observations

While we can use Stability to add zeroes to our sum, we can, interestingly, only do so finitely many times. let us look at the Grandi series and a modified version of the Grandi series as an example. We have already established that

$$\sum_{i=1}^{\infty}(-1)^i = 1-1+1-1+\cdots = \frac{1}{2}(C)$$

and since the Cesàro method is stable (see Hardy, 1949, Theorem 40, page 95), we also have the following equations by applying the stability axiom repeatedly,

$$\begin{aligned}
1-1+1-1+\cdots =& \frac{1}{2}(C)\\
-1+1-1+\cdots =& \frac{1}{2}-1(C) = -\frac{1}{2}(C)\\
1+1+\cdots =& -\frac{1}{2}+1(C) = \frac{1}{2}(C)\\
0+1+1+\cdots =& \frac{1}{2}+0(C) = \frac{1}{2}(C)\\
-1+0+1-1+\cdots =& \frac{1}{2}(C)\\
&\vdots\\
1-1+0+1-1+\cdots =& \frac{1}{2}(C).
\end{aligned}$$

By repeating this process $N$ many times, we get that

$$\underbrace{1-1+0+1-1+0+1-1+0+\dots}_{N \text{ times}} +1-1+1-1+\cdots = \frac{1}{2}(C).$$

This is not yet very exciting but watch what happens if we consider the modified Grandi series

$$G' = 1-1+0+1-1+0+\dots.$$

We check the first few partial sums and find they are $s_1 = 1$, $s_2 = 0$, $s_3 = 0$, $s_4 = 1$ and so on with every $(3n+1)$-th partial sum being 1 and the others being 0. Explicitely, we now have

$$s_n = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 3,\\ 0 & \text{if } n \equiv 2 \bmod 3,\\ 0 & \text{if } n \equiv 3 \bmod 3.\end{cases}$$

We want to apply Cesàro to evaluate the sum, so we first find a general form for the sum of $s_n$ and by computing the first few sums of partial sums we find

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ... |
|---|---|---|---|---|---|---|---|---|---|---|
| $\sum_{i=1}^{n} s_i$ | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | ... |

and from this we find the general formula

$$\sum_{i=1}^{n} s_i = \left\lceil \frac{n}{3} \right\rceil,$$

which means we can write $c_n$ as

$$c_n = \frac{1}{n} \sum_{i=1}^{n} s_n = \frac{\lceil \frac{n}{3} \rceil}{n}.$$

Notice that $\frac{n}{3} \leq \lceil \frac{n}{3} \rceil \leq \frac{n}{3} + 1$, both $\frac{1}{n} \cdot \frac{n}{3}$ and $\frac{1}{n}\left(\frac{n}{3}+1\right)$ tend to $\frac{1}{3}$ as $n$ tends to infinity and so we find that

$$\lim_{n\to\infty} c_n = \frac{1}{3} \neq \frac{1}{2}.$$

This might seem weird, since all we did was add some zeroes. Infinitely many, but still just zeroes. To reassure our intuition, recall how, with the normal Grandi series, we considered the outcome logical since it was the average of the two possible outcomes for the partial sums. We can do that here as well by remarking that $s_n$ is equal to 1 'a third of the time'.
If you are not yet convinced, or want to see more odd behaviour related to adding (infinitely many) zeroes, consider the next example.

**Example 3.1.** Let $G_1 = 1 - 1 + 0 + 0 + 1 - 1 + 0 + 0 + \ldots$ and $G_2 = 1 + 0 - 1 + 0 + 1 + 0 - 1 + 0 + \ldots$. Then we have

$$s_{1,n} = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 4, \\ 0 & \text{if } n \equiv 2 \bmod 4, \\ 0 & \text{if } n \equiv 3 \bmod 4, \\ 0 & \text{if } n \equiv 4 \bmod 4, \end{cases} \text{ and } s_{2,n} = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 4, \\ 1 & \text{if } n \equiv 2 \bmod 4, \\ 0 & \text{if } n \equiv 3 \bmod 4, \\ 0 & \text{if } n \equiv 4 \bmod 4. \end{cases}$$

Subsequently, we then find $G_1 = \frac{1}{4}(C)$ and $G_2 = \frac{1}{2}(C)$.

So not only does its it matter *that* we add infinitely many zeroes but it also matters *how* we add them.

## 3.2 Generalization

We have now seen that by tactically adding infinitely many zeroes, we can obtain the fractions $\frac{1}{4}$ and $\frac{1}{3}$ as well as the $\frac{1}{2}$ that we were already familiar with. Can we extend this to get the reciprocal of any natural number greater than 2? To any fraction in the interval $(0, 1)$? As it turns out, we can! By choosing the amount of zeroes and their place specifically, we can get any fraction we desire in this interval.

**Theorem 3.2.** *By adding zeroes to the terms in the Grandi series, any fraction in $\mathbb{Q} \cap (0, 1)$ can be attained as the Cesàro sum of the modified Grandi series.*

*Proof.* For the proof, we have to realize that this series is periodic in the sense that we have repeating terms in our sum. For example in $G_1$, the periodic term was $1-1+0+1$ and in $G_2$ it was $1+0-1+0$. With this in mind, note that we can only add zeroes in two places. Namely, after the 1 terms and after the $-1$ term.

$$G_{a,b} = 1 + \underbrace{0 + \cdots + 0}_{a-1 \text{ times}} - 1 + \underbrace{0 + \cdots + 0}_{b-1 \text{ times}} + \ldots.$$

This results in the corresponding partial sums being given by

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 \\ &\vdots \\ s_a &= 1 \\ s_{a+1} &= 0 \\ &\vdots \\ s_{a+b} &= 0 \end{aligned}$$

This results in

$$\sum_{i=1}^{n} s_i = \begin{cases} n & \text{if } n \leq a, \\ a & \text{if } a < n \leq a+b. \end{cases}$$

Next, let $n = \alpha_n \cdot (a+b) + \beta_n$ with $\alpha_n$ and $\beta_n$ both natural numbers depending on $n$ with $0 \leq \beta_n < a+b$. We will use this similarly to division with remainder in the sense that we consider $\alpha_n$ copies of $s_1$ through $s_{a+b}$ and $\beta_n$ remaining terms. So clearly $\beta_n$ is less than $a+b$. This is shown below to further illustrate.

$$\underbrace{\begin{array}{ccccc} s_1 & s_{a+b+1} & \vdots & s_{(\alpha_n-1)(a+b)+1} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ s_{a+b} & s_{2(a+b)} & \vdots & s_{\alpha_n(a+b)} \end{array}}_{\alpha_n \text{ many}} \quad \left.\begin{array}{l} s_{\alpha_n(a+b)+1} \\ \vdots \\ s_n \end{array}\right\} \beta_n \text{ many}$$

Figure 1: The values of $s_i$ arranged in columns of lenght $a+b$

Knowing this, we can see that summing $n$ many $s_i$'s, we sum $\alpha_n$ copies of $s_1$ through $s_{a+b}$ and the add the 'loose' $\beta_n$ terms that remain. When we take the limit as $n$ tends to infinity, we will see that these $\beta_n$ many terms become negligibly small, as $\beta_n$ is finite. Note also that $s_i = s_{i+a+b}$ so we can redraw the above diagram with only 1's and 0's.

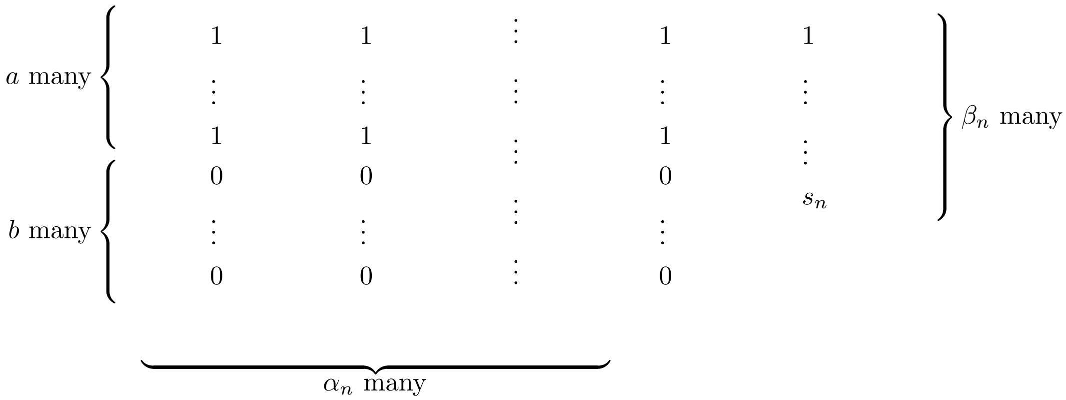


Figure 2: The values of $s_i$ arranged as in figure 3.2 with their values filled in

Here, we leave $s_n$ undetermined as it will depend on $n$ wether it will be 0 or 1. From this, it is clear to see that we have $\alpha_n a$ many 1's and then $\beta_n$ more *or* $a$ more, depending on whether or not $\beta_n$ is greater than $a$ or not. Mathematically, we say that

$$\sum_{i=1}^{n} s_i = \sum_{i=1}^{\alpha_n(a+b)+\beta_n} s_i = \alpha_n \sum_{i=1}^{a+b} s_i + \sum_{i=1}^{\beta_n} s_i = \alpha_n a + \begin{cases} \beta_n & \text{if } \beta_n < a, \\ a & \text{if } \beta_n \geq a. \end{cases}$$

And with this, paired with the knowledge that $s_i = s_{i+a+b}$, we can find an expression for $c_n$, namely

$$\begin{aligned} c_n = \frac{1}{n}\sum_{i=1}^{n} s_i =& \frac{1}{n}\begin{cases} a\alpha_n + \beta_n & \text{if } \beta_n < a, \\ a(\alpha_n + 1) & \text{if } \beta_n \geq a. \end{cases} \\ =& \begin{cases} \dfrac{a\alpha_n + \beta_n}{n} & \text{if } \beta_n < a, \\ \dfrac{a\alpha_n + a}{n} & \text{if } \beta_n \geq a. \end{cases} \\ =& \begin{cases} \dfrac{a\alpha_n}{n} + \dfrac{\beta_n}{n} & \text{if } \beta_n < a, \\ \dfrac{a\alpha_n}{n} + \dfrac{a}{n} & \text{if } \beta_n \geq a. \end{cases} \end{aligned}$$

Now, we have to consider the limit of $c_n$ as $n$ tends to infinity. To avoid having to work with a case distinction any more than necessary, note that $\beta_n$ and $a$ are both less than $a+b$ and so they are finite. Hence, we can see that

$$\lim_{n\to\infty} \frac{a}{n} = \lim_{n\to\infty} \frac{\beta_n}{n} = 0.$$

With this in mind, we can move on with only the first term in our sum when considering the limit of $c_n$.

$$
\begin{aligned}
\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} s_i &= \lim_{n\to\infty} \frac{a\alpha_n}{n} \\
&= \lim_{n\to\infty} \frac{a(n-\beta_n)}{n(a+b)} \\
&= \lim_{n\to\infty} \frac{na}{n(a+b)} - \lim_{n\to\infty} \frac{\beta_n a}{n(a+b)} \\
&= \frac{a}{a+b}
\end{aligned}
$$

Here, we can see that if we choose $a = 1$, we can indeed get $\frac{1}{n}$ as the Cesàro sum of our modified series for $n > 1$ as we picked $b \in \mathbb{N}$. Moreover, since we can also choose $a$ anywhere in the natural numbers, we can indeed get any fraction in $\mathbb{Q} \cap (0, 1)$ as we can write

$$\frac{a}{a+b} = \frac{p}{q}, \text{ with } q \geq p \text{ and } p, q \in \mathbb{N}.$$

Which is exactly the interval $\mathbb{Q} \cap (0, 1)$. With this, we conclude that

$$G_{a,b} = 1 + \underbrace{0 + \cdots + 0}_{a-1 \text{ times}} - 1 + \underbrace{0 + \cdots + 0}_{b-1 \text{ times}} + \cdots = \frac{a}{a+b}(C).$$

□

Interestingly, it can be observed that if we now, after having taken the limit in $n$, take a limit by letting either $a$ or $b$ tend to infinity, then we get $G_{\infty,b} = 1$ and $G_{a,\infty} = 0$ respectively. This makes sense seeing as when we extend the row of $a - 1$ zeroes in our expression of $G_{a,b}$ to be infinitely long, then we are left with a 1 followed by an infinite amount of 0's. And similarly, if we let $b$ go to infinity then we only have a 1 followed by an arbitrary (but finite!) amount of zeroes, then the $-1$ term, followed by infinitely many 0's, leaving us with 0.

At the start of this proof, we said there are only two places where zeroes could be added and you might argue that there is a third, namely in *front* of the first 1. This would add another lever of generalization and we would add a third variable to our (generalized) expression of $G$. We will show, however, that this is not the case and that adding any zeroes in this place can be described with the two existing variables.

*Proof.* Consider adding an amount of infinitely repeating zeroes at the start of our repeating term and call this $d$ for "don't add zeroes here", which of course has to be a natural number. We then have the general form

$$G_{a,b,d} = \underbrace{0 + \cdots + 0}_{d \text{ times}} + 1 + \underbrace{0 + \cdots + 0}_{a-1 \text{ times}} - 1 + \underbrace{0 + \cdots + 0}_{b-1 \text{ times}} + \ldots.$$

Nothing special so far but let us write out a few more terms and do some rewriting to illustrate the point.

$$
\begin{aligned}
G_{a,\tilde{b},d} = &\underbrace{0 + \cdots + 0}_{d \text{ times}} + 1 + \underbrace{0 + \cdots + 0}_{a-1 \text{ times}} - 1 + \underbrace{0 + \cdots + 0}_{\tilde{b}-1 \text{ times}} + \underbrace{0 + \cdots + 0}_{d \text{ times}} + 1 \\
&+ \underbrace{0 + \cdots + 0}_{a-1 \text{ times}} - 1 + \underbrace{0 + \cdots + 0}_{\tilde{b}-1 \text{ times}} + \underbrace{0 + \cdots + 0}_{d \text{ times}} + \underbrace{0 + \cdots + 0}_{a-1 \text{ times}} - 1 \\
&+ \underbrace{0 + \cdots + 0}_{\tilde{b}-1 \text{ times}} + \ldots
\end{aligned}
$$

First, we remove the first $d$ zeroes. Since we remove finitely many zeroes, this does not change the outcome by Hardy's first axiom.

$$\begin{aligned} G_{a,\tilde{b},d} =& 1 + \underbrace{0 + \dots + 0}_{a-1 \text{ times}} - 1 + \underbrace{0 + \dots + 0}_{\tilde{b}-1 \text{ times}} + \underbrace{0 + \dots + 0}_{d \text{ times}} + 1 + \underbrace{0 + \dots + 0}_{a-1 \text{ times}} \\ & - 1 + \underbrace{0 + \dots + 0}_{\tilde{b}-1 \text{ times}} + \underbrace{0 + \dots + 0}_{d \text{ times}} + \underbrace{0 + \dots + 0}_{a-1 \text{ times}} - 1 + \underbrace{0 + \dots + 0}_{\tilde{b}-1 \text{ times}} + \dots \end{aligned}$$

Next, we rewrite the two subsequent strings of zeroes as one, since they are immediately after each other.

$$\begin{aligned} G_{a,\tilde{b},d} =& 1 + \underbrace{0 + \dots + 0}_{a-1 \text{ times}} - 1 + \underbrace{0 + \dots + 0}_{\tilde{b}+d-1 \text{ times}} + 1 + \underbrace{0 + \dots + 0}_{a-1 \text{ times}} - 1 \\ & + \underbrace{0 + \dots + 0}_{\tilde{b}+d-1 \text{ times}} + \underbrace{0 + \dots + 0}_{a-1 \text{ times}} - 1 + \underbrace{0 + \dots + 0}_{\tilde{b}+c-1 \text{ times}} + \dots \end{aligned}$$

Now, since we have $\tilde{b} > 0$ and $d \geq 0$, we also have $\tilde{b} + d > 0$. All that is left is is to rename $\tilde{b} + d - 1$ to $b$ and we have the same general formula for $G_{a,b}$. □

### 3.3 Discussion

Returning to the sentiment that not only may adding infinitely many zeroes change things, and that the *way* we add them also matters, one might wonder if something similar would also be applicable to classical summation. That is, without using Cesàro or any other summation method and just considering the limit of the partial sums.

Imagine we add zeroes to a converging sequence. Consider any sequence $(a_n) = (a_1, a_2, a_3, \dots)$ and add a zero to it between each term, then we would have a sequence $(\tilde{a}_n) = (a_1, 0, a_2, 0, a_3, 0, \dots)$ for which we would have that

$$\begin{aligned} \tilde{s}_1 =& a_1 = s_1, \\ \tilde{s}_2 =& a_1 + 0 = a_1 = s_1, \\ \tilde{s}_3 =& a_1 + 0 + a_2 = a_1 + a_2 = s_2, \\ & \vdots \\ \tilde{s}_n =& \begin{cases} s_{\frac{n}{2}} & \text{if } n \text{ is even,} \\ s_{\frac{n+1}{2}} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

So here we can see that $\lim_{n\to\infty} \tilde{s}_n = \lim_{n\to\infty} s_n$, which means the same limit is achieved except it takes more iterations to get closer to it. When we have to pick an $N$ in the natural numbers sufficiently large such that we can get within a distance $\epsilon$ from our limit for indices larger than $N$, we would have to pick $N$ roughly twice as large. This holds in general as well. We can only "postpone" how long it takes to reach each $s_n$ but we can only extend it by finitely many steps as we add finitely many zeroes between any two terms.

A similar result was proposed by Daniel Bernoulli (1700 - 1782) (Sandifer, 2006a). However, this result lacked any rigorous proof as Bernoulli simply argued that for the Grandi series, the partial sums were equal to 1 half the time and equal to 0 half the time so therefore the sum should be equal to $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}$. Bernoulli noted that

$$1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 + \cdots = \frac{2}{3}$$

by the same argument. This isn't, in essence, too different from what we have done here, if not a bit less refined. We also started by looking at the Grandi series and observed how adding a zeroes in the repeating term could give us 1/3 as a result and noted that the partial sums were equal to $nicefrac13$ 'a third' of the time. For us, however, this was simply to illustrate and not to serve as a way of proving.

Bernoulli goes on to conclude that by inserting 0's in the right places, any value between 0 and 1 could be attained. It is unclear if "any value" was meant to describe fractions or irrational numbers as well. At the same time, it isn't clear whether "between 0 and 1" was meant to include or exclude the boundary. Furthermore, when Bernoulli made his claim roughly 250 years ago, the terms "progression" and "sum" were used interchangeably. As well as there being no distinction between a "series" and a "sequence". This is not too surprising, as this was roughly 100 years before the term "matrix" was first used (Sandifer, 2006b). Moreover, Cesàro's method of summation, which we used to prove our result, wasn't well documented yet. Mostly due to the fact that Ernesto Cesàro was not born yet when Daniel Bernoulli was alive. Altogether, this calls into question what exactly what was meant by Bernoulli and whether or not he had a rigorous proof for any of these claims.

Should it be possible, though, to attain any *irrational* value in the interval $(0, 1)$, it would require us to pick our $a$ and $b$ dynamically. Meaning we don't add the same amount of zeroes after each $-1$ but a varying amount and similarly a different amount before each $-1$. Unfortunately, we can't just change our earlier result to $\frac{a_m}{a_m+b_m}$ (we take $m$ instead of $n$ to avoid confusion) and take a limit as $m$ tends to infinity. This is because a dynamic $a_m$ and $b_m$ would change the way we add our values of $s_n$. It might be tempting to think that, as $s_i$ is a natural number for each $i$, we would get a natural number when summing $n$ many of them and that, as a result of this. We would be left with a fraction after dividing this sum by $n$ when we take the average. But it must be remembered that, when taking limits (especially to infinity), unexpected things might show up. Recall that the sum of all $\frac{1}{n^2}$ is irrational and that we just saw how adding infinite zeroes in a strategic way changes the outcome of our sum.

As a final remark. We could use the scalability of Cesàro to introduce one more generalization. We will trade the ability to call our series the Grandi series for some additional results. If we rescale the Grandi series by a real constant $r$, we also multiply our result by the same constant as per scalability. This means we would have the rescaled generalized Grandi series as

$$G_{a,b,r} = r + \underbrace{0 + \cdots + 0}_{a-1 \text{ times}} - r + \underbrace{0 + \cdots + 0}_{b-1 \text{ times}} + \cdots = r \cdot \frac{a}{a+b} (C).$$

The range of solution is now not restricted to $(0, 1) \cap \mathbb{Q}$ but instead can take any value in $\mathbb{Q}$ by taking $r \in \mathbb{Q}$ and any value in $\mathbb{R}$ by taking $r \in \mathbb{R}$.

# 4 A visual approach

## 4.1 Introduction

When graphing a series that converges in the classical sense, we plot the individual points and can then see, by squeezing our eyes sufficiently, that the points don't seem to stop going up or down for $n$ large enough. This is in essence what it *normally* means for a series to converge. For divergent series, however, we can make some extra observations.

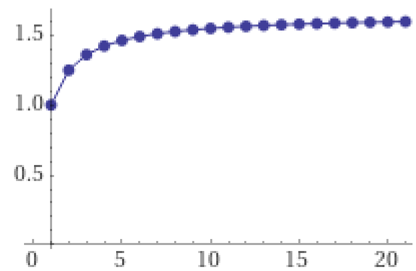


Figure 3: The first few partial sums of the series $\sum_{i=1}^{n} \frac{1}{i^2}$ from $n = 1$, a series which is known to converge. Clearly, the partial sums approach a fixed value which is the sum assigned to this series.

In figure 3, we see the partial sums $\sum_{i=1}^{n} \frac{1}{i^2}$, which we know converge nicely to $\frac{\pi^2}{6}$ (Jolley, 1961; Davis, 1962). In this example, we can clearly see the graph flattening, which is what we expect for converging series. For diverging series, however, it is not the case that the graph of the partial sums flatten an $n$ increases.
One of the examples we have seen before is $(a_n) = (-1)^{n+1} \cdot n$ which is an alternating sequence for which the graph of partial sums does not flatten but instead keeps going up and down more and more as $n$ increases.

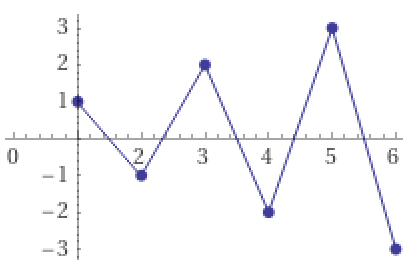

Figure 4: The first handful of partial sums of the alternating series.

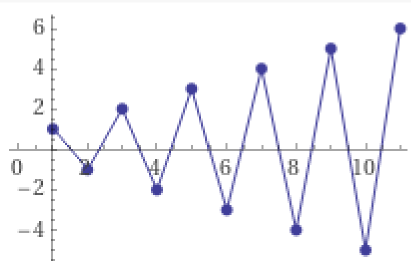

Figure 5: More partial sums of the same series. Here, the triangle shape is already more visible

When we plot enough points, we can see the graph starts to look a lot like a triangle. Using some simple math, we can find the height 'tip' of this triangle at the left side which, perhaps surprisingly, is exactly at $\frac{1}{4}$, which is value we assigned to this series using the methods we discussed.

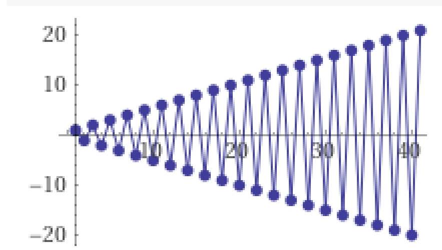


Figure 6: With many more partial sums plotted, the triangular shape is undeniable. We are interested in finding the location of the 'tip' of the triangle that we can extrapolate from these points.

### 4.2 Generalization

Now that we have gotten a taste for this, we can generalise this idea a bit more and make it more explicit what we are really doing. For this, consider the sequence $(a_n) = (-1)^{n+1} \cdot \frac{n(n+1)}{2}$. Here, we can't just draw a set of straight lines and call it a day. What we will do instead is consider the odd-indexed points and the even-indexed points as separate sequences and describe both of them with their own equation and then find their intersection.

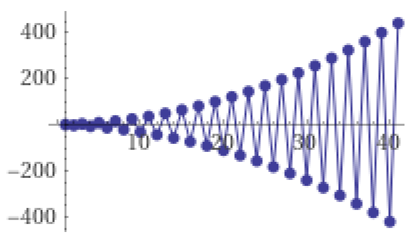


Figure 7: The partial sums of $(-1)^{n+1} \cdot \frac{n(n+1)}{2}$ form two clear sub-sequences, both non-linear.

Let the odd-indexed points of $(s_n)$ be denoted as $s_n^o$ and the even-indexed points as $s_n^e$. After some computation we find

$$\begin{aligned} s_n^o =& \frac{1}{8}\left(2n^2 + 4n + 2\right) \\ =& \frac{1}{4}\left(n^2 + 2n + 1\right). \end{aligned} \qquad \begin{aligned} s_n^e =& -\frac{1}{8}\left(2n^2 + 4n\right) \\ =& -\frac{1}{4}\left(n^2 + 2n\right). \end{aligned}$$

Equating these, we find $s_n^o = s_n^e = \frac{1}{8}$ which is exactly the $(H, 3)$ sum of this series.

Explicitely, we take the sequence of partial sums $s_n$, extract two subsequences $s_n^o$ and $s_n^e$, denoting the odd indexed partial sums and the even indexed partial sums respectively. Then, we view them not as a sequence but as a continuous function. Thereby allowing $n$ to be in $\mathbb{R}$, rather than just $\mathbb{N}$. Lastly, we equate the resulting continuous expressions and assign the resulting value for $s_n^o$ (or $s_n^e$, equivalently) as the sum of the series. In general, if we write $s_n^o$ as the opposite of $s_n^e$ plus some displacement $d$, where $d$ is in $\mathbb{R}$,

$$s_n^o = -s_n^e + d,$$

then we will find that the value that we assign to our sum by taking the intersection of the even and odd sequences, is given by

$$s_n^o = s_n^e = \frac{d}{2}.$$

Intuitively, this should make sense. As this gives the same result as Hölder which is known as an "averaging method" and taking the average of two series which are identical bar a displacement, results in the average being shifted by half that displacement.

This is under the observation (and the hope) that we can continue to easily recognise $s_n^o$ in $s_n^e$ or vice versa. If we can not, then we are left with an equation of two polynomials. We can then subtract these from each other and solve the resulting root-finding problem. It is important to note that, if we do this, we will get a value of $n$ which is not in $\mathbb{N}$ and so we should evaluate the continuous interpretation of $s_n^o$ in $s_n^e$ as this value of $n$ and **not** the discrete one. Doing this can result in non-real values. For example if $n = -1/2$ at the intersection, then $(-1)^{n+1} = (-1)^{0.5} = \hat{\imath}$, where $\hat{\imath}$ denotes the imaginary unit.

If it can be shown that this holds for all series with alternating polynomials as described in Theorem 2.13, then it would save considerable amounts of computation if the degree $q$ of a series' partial sums gets very large. This is because utilizing the approach described above would not become notably difficult. Given $s_n$ we just have to extract the two sub-sequences, equate them, and then compute the intersection. Whereas with Hölder, we essentially have to perform Cesàro's method $k$ many times. Not to mention the increasingly complex form of $H_n^k$ as $k$ increases. While a computer might be able to do so, this would be incredibly cumbersome to do by hand.

Should this method be applicable and consistently provide us with the Hölder sum of a series, then we have found a new summation method that is compatible (or equivalent) with Hölder summation. If this is the case, then properties like stability, scalability, and additivity would immidiately follow. As we have seen that the degree of the polynomial $q$ directly corresponds to a value $k > q$ in theorem 2.13, we can also find the value of $k$ very quickly by taking $k = q+1$. Overall, this method has certain potential but further research is needed.

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