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Four-dimensional regular polytopes

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Student: S.H.E. Kamps

First supervisor: Prof.dr. J. Top

Second assessor: P. Kiliçer, PhD

Abstract

Since Ancient times, Mathematicians have been interested in the study of convex, regular polyhedra and their beautiful symmetries. These five polyhedra are also known as the Platonic Solids. In the 19th century, the four-dimensional analogues of the Platonic solids were described mathematically, adding one regular polytope to the collection with no analogue regular polyhedron. This thesis describes the six convex, regular polytopes in four-dimensional Euclidean space. The focus lies on deriving information about their cells, faces, edges and vertices. Besides that, the symmetry groups of the polytopes are touched upon. To better understand the notions of regularity and symmetry in four dimensions, our journey begins in three-dimensional space. In this light, the thesis also works out the details of a proof of prof. dr. J. Top, showing there exist exactly five convex, regular polyhedra in three-dimensional space.

Keywords: Regular convex 4-polytopes, Platonic solids, symmetry groups

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Contents

1 | Introduction

"He who in action sees inaction
and in inaction sees action
is wise among men."

— Chapter 4, verse 18, Bhagavad Gita, translated by Maharishi Mahesh Yogi [Yog90]

The above excerpt from the Hindu scripture *Bhagavad Gita* can be of help to understand the mathematical notion of *symmetry*. In Humanistic Mathematics Network Journal, professor of Mathematics Catherine Gorini [Gor96] illustrates it as follows. For an object as a set of points in $\mathbb{R}^n$, a symmetry of this object is a transformation (action) that leaves the object invariant (inaction). By definition, a transformation in $\mathbb{R}^n$ is a map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with the property $\|\varphi(v) - \varphi(w)\| = \|v - w\|$ for all $v, w \in \mathbb{R}^n$. Such maps are also called *isometries*.

The *convex, regular polyhedra* are an example of objects with an exceptionally high degree of symmetry. By the words 'a high degree of symmetry', we mean there exist a high (finite) amount of transformations that leave these objects invariant. This set of polyhedra is also called the *Platonic solids*.

Simply put, a Platonic solid is a three-dimensional object, a *polyhedron*, with each flat face being an equilateral polygon. There exist exactly five such shapes, called the *tetrahedron*, the *cube*, the *octahedron*, the *dodecahedron*, and the *icosahedron*.

Throughout the history of mathematics, these objects have been studied extensively. They earned a place in Plato's dialogue *Timaeus* in 360 B.C., and Euclid gave a mathematical description and exploration of their properties in *The Elements* in 300 B.C. Even from at least 1000 years before Plato's Timaeus, carved models were found of the solids. Whether these were studied in a similar way to Euclid's studies and his mathematical descriptions is rather questionable, but it affirms that these objects have been capturing human's eye throughout history [DeH16, p. 3].

Around the mid-19th century, an interest in the fourth dimension began to rise. Swiss mathematician L. Schläfli described an analogue version of the Platonic solids in four-dimensional space in his work *Theorie der vielfachen Kontinuität* [SG01]. The term *polytope* is used to generalise the notion of a polygon or polyhedron. Schläfli discovered that there exist exactly 6 regular, convex polytopes in four-dimensional space. These are called the *convex regular 4-polytopes* or sometimes *polychora*.

In this thesis, four-dimensional space denotes four-dimensional Euclidean space. This space is not to be confused with the notion of *spacetime*. The latter is topologically different [Zee67]. Just as one can imagine going from two-dimensional planes to three-dimensional space by adding one axis orthogonal to all others, the same reasoning can be applied to going from three- to four-dimensional space. Visualising such a space can be considered nearly impossible, although there are records of humans who were able to do so. One extraordinary example is mathematician Alicia Boole Stott. Without a formal education, she managed to create models of sections of 4-polytopes, through visualisation and self-study. Her works are on display in the University Museum of the University of Groningen [Bla02, p. 29].

1.1 Aim of this thesis

This thesis aims to describe the geometry of the six convex, regular 4-polytopes and their symmetry groups. The vertices of these figures are described, and their symmetry groups are touched upon. First, however, the Platonic solids are discussed. This somewhat inductive approach aims to help the reader familiarise with the mathematical concepts of regular polytopes, before moving into unimaginable territory.

The necessary preliminaries are given in Chapter 2. The mathematical definitions of symmetries and *symmetry groups* are presented. The algebra of quaternions is introduced, as we will use this in Chapter 4 to describe the vertices and symmetries of the 24-cell and 600-cell.

The third chapter describes the three-dimensional analogue of these objects, the Platonic solids. The reader becomes familiarised with *symmetry groups* and *regularity*. Through a proof of the existence of exactly five such polyhedra, the objects are constructed.

The notions of regularity and symmetry will be looked into again in Chapter 4, but this time concerning four-dimensional polytopes. The six regular, polytopes in four-dimensional space are described one by one. For the four-dimensional analogues of the tetrahedron, cube and octahedron we derive their vertices, edges, faces and cells. For the analogues of the icosahedron and dodecahedron, as well as the 24-cell, we use quaternions to represent the vertices in terms of certain sets of quaternions.

2 Regular polytopes & Symmetry Groups

"POLYTOPE is the general term of the sequence
point, segment, polygon, polyhedron, . . .
— H.S.M. Coxeter [Cox48], p. 118

This section provides preliminary knowledge regarding *convex, regular polytopes* and *symmetry groups*. It serves as a support to understand the thesis and is based on [Cox48], [MT18] and [Arm88].

2.1 Regular polytopes

This thesis concerns only *convex* polytopes. We define convexity shortly as follows. A subset P of $\mathbb{R}^n$ is called *convex* if any segment between points in the set, is contained in the set. An example of a non-convex set is a *regular* star pentagon.

The following definition of a *polytope* is used, which is a generalisation to n dimensions of polygons (two-dimensional) and polyhedra (three-dimensional).

Definition 2.1.1 (Polytope)**.** An n dimensional *polytope* is a closed, bounded subset of $\mathbb{R}^n$ that is bounded by $n-1$ dimensional hyperplanes and has non-empty interior.

The *regular polytopes* form a specific collection of highly symmetrical polytopes. In two-dimensional space, for example, the regular polytopes are equilateral and equiangular polygons. In three dimensions, the regular polytopes are polyhedra with equal, regular faces, where an equal number of edges meets at each vertex [Bla02, p. 4]. As stated before, these regular polyhedra are the five Platonic solids.

A general definition of an n dimensional *regular polytope* can be given as follows.

Definition 2.1.2 (Regular polytopes [MS02])**.** An n *dimensional polytope* is called *regular* if, for each $j = 0, \ldots, n-1$ its symmetry group is transitive on the j-faces of it.

This thesis mainly concerns polytopes of dimension four, called *4-polytopes*. For 4-polytopes, where $j = 0, 1, 2, 3$, the j-faces are called respectively vertices, edges, faces and cells. Cells are three-dimensional faces that bound the four-dimensional objects.

A way to represent (regular) polytopes is by their so-called *Schläfli symbol*. These symbols are of the form $\{p, q, r, \ldots\}$, where $p, q, r, \ldots$ are natural numbers if the described polytope is convex and regular. For three-dimensional bodies, the symbol is of the form $\{p, q\}$. The first input p stands for p-gonal faces, the second input q stands for the number of 2-faces meeting in each vertex. For four-dimensions, the symbol is of the form $\{p, q, r\}$. The additional r represents the number of three-dimensional cells with symbol $\{p, q\}$ meeting in each edge [Bla02]. This notation will be used in Chapter 4 to denote different regular polytopes.

A notion that will show itself to be very useful in, for example, researching symmetry groups is that of a *dual* of a polytope. The *dual* of a regular polytope is defined as follows.

Definition 2.1.3. The *dual* of a regular polytope P in n dimensions is another polytope having one vertex in the centre of each $n-1$ face of the polytope P.

Remark. It turns out that the dual of a polytope with Schläfli symbol $\{p, q, r\}$ is a polytope with symbol $\{r, q, p\}$.

Later we will see that a polytope and its dual share the same symmetry group. An elaboration on symmetry groups is now given.

2.2 Symmetry groups

A *symmetry group* can be described informally as the group of transformations that leave an object 'unchanged' when applied to it. The group law of such a group is composition. Examples of a *symmetry* are rotations or reflections.

Symmetries are by definition distance preserving maps. Such maps are generally called *isometries*. For an isometry to be a symmetry, it is also required that the map sends an object to itself. In [MT18, p. 44], a definition for isometries is given.

Definition 2.2.1 (Isometry)**.** An *isometry* on $\mathbb{R}^n$ is a map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with the property $d(v, w) = d(\varphi(v), \varphi(w))$ for all $v, w \in \mathbb{R}^n$. Here $d(\cdot, \cdot)$ is the Euclidean distance.

In [MT18, p. 44] it is also proved that isometries mapping 0 to 0 are linear, and that the linear isometries on $\mathbb{R}^n$ are exactly the elements of the *orthogonal group* $O(n)$. For $n \in \mathbb{Z}_{>0}$ it is given by

$$O(n) = \{A \in GL_n(\mathbb{R}) : A^T A = I\}, \text{ where } A^T \text{ is the transpose of } A. \tag{2.1}$$

Elements of $O(n)$ have determinant equal to 1 or -1. This is easily seen from the fact that $\det(A^T A) = 1$ and $\det(A^T A) = \det(A^T)\det(A) = \det(A)^2$. A subgroup of $O(n)$ whose elements have determinant equal to 1 is the *special orthogonal group* and is given by

$$SO(n) = \{\{A \in GL_n(\mathbb{R}) : A^T A = I, \det(A) = 1\}. \tag{2.2}$$

A formal definition of a *symmetry group* of a set is given in [MT18] as follows.

Definition 2.2.2 (Symmetry group)**.** A *symmetry group* of a subset $P \subset \mathbb{R}^n$ is defined as the group of all isometries on $\mathbb{R}^n$ with the property that P is mapped to P.

From this it can be inferred that any object centred at the origin has a symmetry group that is a subgroup of $O(n)$.

We now describe basic knowledge of quaternions. These will be used to make representing points and rotations in $\mathbb{R}^4$ easier in Chapter 4.

2.3 Quaternions

In 1843, Hamilton invented the *algebra of quaternions* $\mathbb{H}(\mathbb{R})$ [Sti00, p. 18], or shortly $\mathbb{H}$. It is a normed algebra of quadruples, a two-dimensional extension of the complex numbers. A *quaternion* is of the form

$$q = q_0 + q_1 i + q_2 j + q_3 k,$$

with $q_0, \ldots, q_3 \in \mathbb{R}$ and where the imaginary units i, j, k are such that

$$i^2 = j^2 = k^2 = ijk = -1.$$

The conjugate of an element of $\mathbb{H}$ is defined as

$$\bar{q} = q_0 - q_1 i - q_2 j - q_3 k.$$

This can be used to define the norm map N of a quaternion. The norm map is defined as the square of the *norm* $\|q\| := \sqrt{q\bar{q}}$.

$$N(q) := \|q\|^2 = q\bar{q}$$

This norm has the property that $N(xy) = N(x)N(y)$ for all $x, y \in \mathbb{H}$. This follows from writing out $N(xy)$.

$$\begin{aligned} N(xy) &= xy\overline{xy} \\ &= xy\bar{y}\bar{x} \\ &= xN(y)\bar{x} \\ &= N(x)N(y). \end{aligned} \tag{2.3}$$

The second equality requires a small proof, as it uses the following equality.

$$\overline{xy} = \bar{y}\bar{x}, \text{ for all } x, y \in \mathbb{H}$$

It is not true in general that $\overline{xy} = \bar{x}\bar{y}$. To make notation easier, collect the quaternion coefficients in a vector. Hence, for $x = a + bi + cj + dk \in \mathbb{H}$, we write $\{a, b, c, d\}$ and for $y = e + fi + gj + hk \in \mathbb{H}$, we write $\{e, f, g, h\}$. To see that $\overline{xy}$ equals $\bar{y}\bar{x}$, but does not generally equal $\bar{x}\bar{y}$, let us list the three quaternions using the vector notation introduced above.

$$\begin{aligned} \overline{xy} &= \{ae - bf - cg - dh, -be - af + dg - ch, -ce - df - ag + bh, -de + cf - bg - ah\} \\ \bar{x}\bar{y} &= \{ae - bf - cg - dh, -be - af + dg - ch, -ce - df - ag + bh, -de + cf - bg - ah\} \\ \bar{y}\bar{x} &= \{ae - bf - cg - dh, -be - af - dg + ch, -ce + df - ag - bh, -de - cf + bg - ah\} \end{aligned}$$

Observe that for $\overline{xy} = \bar{x}\bar{y}$ to be true, it must hold that $dg = df = bg = 0$. On the other hand $\overline{xy} = \bar{y}\bar{x}$ for all $x, y \in R$. Looking at the second coefficient already reveals it, as for $\overline{xy}$ this is $-be - af + dg - ch$, and for $\bar{x}\bar{y}$ it is $-be - af - dg + ch$.

Using quaternions can make notation simpler, as we will see in Chapter 4 when describing vertices of the 24-cell and 600-cell. Quaternions can also be used to represent rotations in three as well as four-dimensional space [Zho91, p. 17 – 30].

To construct elements of $O(4)$ expressed as quaternions, take $q \in \mathbb{H}$ with $N(q) = 1$. Define $\lambda_q \colon \mathbb{H} \to \mathbb{H}$ by $\lambda_q(x) = qx$.

Take $x, y \in \mathbb{H}$ and $\alpha \in \mathbb{R}$. We will show that the map $\lambda_q(x)$ is linear, by showing that $\lambda_q(x+y) = \lambda_q(x) + \lambda_q(y)$ and $\lambda_q(\alpha x) = \alpha\lambda_q(x)$.

$$\begin{aligned} \lambda_q(x+y) &= q(x+y) \\ &= qx + qy \\ &= \lambda_q(x) + \lambda_q(y) \end{aligned}$$

Distributive properties of $\mathbb{H}$ are used here.

$$\begin{aligned} \lambda_q(\alpha x) &= q\alpha x \\ &= \alpha q x \\ &= \alpha\lambda_q(x) \end{aligned}$$

Associative properties of $\mathbb{H}$ are used here as well as the fact that for $z \in \mathbb{H}$ and $\alpha \in \mathbb{R}$ one has $\alpha z = z\alpha$. Because we assume $N(q) = 1$, the map $\lambda_q(x)$ is also distance-preserving. In other words, $\|x - y\| = \|\lambda_q(x) - \lambda_q(y)\|$.

After all, we can write out $\|\lambda_q(x) - \lambda_q(y)\|^2$ as follows.

$$\begin{aligned} \|\lambda_q(x) - \lambda_q(y)\|^2 &= N(qx - qy) \\ &= N(q)N(x-y) \\ &= \|x - y\|^2 \end{aligned}$$

A map $\lambda_p(x) : x \mapsto xp$ with $p \in \mathbb{H}, N(p) = 1$ is also linear and distance preserving by similar arguments. Any composition of the form $\varphi : x \mapsto qxp$ has the same properties.

Earlier it was stated that linear isometries on $\mathbb{R}^4$ are exactly the elements of $O(4)$. These kinds of maps φ will show to be useful in determining the symmetries of some four-dimensional regular polytopes in Chapter 4.

3 | The Platonic Solids

"Though analogy is often misleading, it is the least misleading thing we have."

— Samuel Butler (1835 – 1902) (Music, Pictures and Books) [Cox61, p. 401]

This chapter presents a description of the Platonic solids and their symmetry groups. Not only listing them, but constructing them through a proof inspired by [Top96, p. 66 – 69] that exactly five of them exist. This chapter can be seen as supporting the reader to read the subsequent chapter about their four-dimensional analogues. Besides that, it should provide a more complete story to understand regular polytopes. We will now describe useful excerpts from group theory that will be used in the proof.

3.1 Action, orbit and stabiliser

The lecture notes of prof. dr. J. Top and dr. J.S. Müller [MT18, p. 60 – 62] provide theory on the notions of *action, orbit,* and *stabiliser.* The original proof of the existence of five regular polyhedra in Top's Dutch version of the notes [Top96] does not mention their use explicitly. The extension of this proof given in the thesis will make use of theory on actions and orbits.

Definition 3.1.1 (Group action, from [MT18]). Let G be a group and X be a nonempty set. A *group action* of G on X is a map $G \times X \to X$ which we write as $(g, x) \mapsto gx$, satisfying

A1. $ex = x$ for every $x \in X$

A2. $(gh)x = g(hx)$ for all $g, h \in G$ and all $x \in X$

It is common to refer to a group action on X as 'G acts on X'.

Definition 3.1.2 (From [MT18]). Let the group G act on the set X and take $x \in X$.

1. The *stabiliser* of x in G, denoted G_x is defined as

$$G_x := \{g \in G : gx = x\} \subset G$$

2. The *orbit* of x under G, denoted by Gx, is defined as

$$Gx := \{gx : g \in G\} \subset X$$

If G is a group of permutations, the stabiliser may be described as the set of permutations such that an element x remains invariant. The orbit of an element x will be the set of elements that x is mapped to by elements of G.

3.2 Five regular, convex polyhedra

The only convex, regular polyhedra are the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron. Proofs of this fact using geometry (angle sums) and topology (Euler's polyhedral formula) are given in [DeH16, p. 11 – 18]. The proof included here is based on group theory and works out the details of [Top96, p. 66 – 69].

Theorem 3.2.1. There exist five convex, regular polyhedra.

Let us first set up some 'mise en place' for proving this theorem. We begin with a sphere in three-dimensional space, centred at the origin. One can construct a Platonic solid by taking the convex hull of n suitably-picked points on this sphere. Any isometry preserves the distance between the n points. The only point that has equal distance to all of them is the origin of the sphere. Therefore, this point gets mapped to itself by any isometry, and as stated in Section 2.2 any isometry fixing the origin is linear. This shows that symmetries of such polyhedra are in fact linear (also called *orthogonal*) isometries.

The following proposition from [Top96, p. 59] is an important result for proving the existence of exactly five regular polyhedra.

Proposition 3.2.2. If φ is an orthogonal map on $\mathbb{R}^3$ with $\det(\varphi) = \epsilon$, then $\epsilon = \pm 1$. There also exist a line L and a plane W orthogonal to L such that φ maps L and W to itself. Here φ acts on W as a rotation around L or as a reflection in a line $M \subset W$, and on L as multiplication with ϵ in the first case, resp. with $-\epsilon$ in the second.

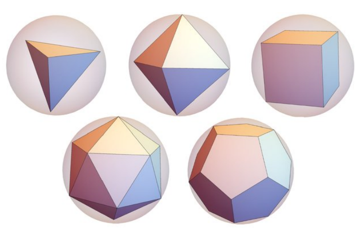

Figure 3.1: The Platonic solids as constructed by picking n points on a sphere, taken from [TG20].

The proof of this proposition uses the fact that the characteristic polynomial of the 3-by-3 matrix A in $O(3)$ corresponding to φ has at least one real root λ, with eigenvector v. By distance preservation of the map φ represented by A it follows that $\lambda = \pm 1$. Since $Av = \lambda v$ it follows that φ maps a line L through v and the origin to itself. By looking at a plane W through the origin orthogonal to v, it is shown that φ maps this plane to itself by showing that for any vector $w \in W$, $\varphi(w)$ is orthogonal to v and hence also in the plane W. The map φ is therefore either a rotation of W or a reflection in a line in the plane through the origin.

From this (one deduces, using that if λ is a non-real eigenvalue then so is $\bar{\lambda}$) that elements of $SO(3)$ (where $\det = 1$) can be seen as rotations around a line L through the origin.

It is known from Linear Algebra that linear maps are determined by their effect on a basis and specifically in this case by the n vertices of a Platonic solid. Without loss of generality we can assume that the origin is the centre of the solid so that the symmetry group consists of linear maps, and then three vertices can be picked as a basis for $\mathbb{R}^3$, since not all lie in the same plane. Therefore, the symmetries of Platonic solids are determined by their action on the vertices. The consequence is that the order of the symmetry group is at most equal to the number of permutations on the n vertices. The symmetry group is thus isomorphic to a subgroup of S_n.

By Definition 2.1.1 it is also known that the symmetry group of a Platonic solid acts transitively on its edges and vertices. This simply means that for each pair of vertices x, y there exists a symmetry that sends x to y, and similar for pairs of edges. In more mathematical terms, if $x, y \in \mathbb{R}^3$ are vertices of polytope P, there exists a φ in the symmetry group of P, such that $\varphi(x) = y$. Furthermore, we stated that symmetry groups are subgroups of $O(3)$, as $O(3)$ consists of all linear isometries.

Now the claim is that vertices of a Platonic solid can be mapped to each other using purely rotations (so, elements in $SO(3)$). This is supported by the following intuitive argument, which provides some additional explanation to the brief text given on p. 66 of [Top96]. Take an arbitrary Platonic solid P, centred at the origin, and a vertex n on the North Pole. Say v is another vertex of P. Symmetries that fix a vertex at the North Pole are either rotations around the z-axis, or reflections in a plane containing the z-axis. Such symmetries have the property of preserving the z-coordinates of all vertices. Firstly, we can map v to n by rotating the solid. Now in order to construct a symmetry that maps v to n, we need to ensure that P ends up in its original position.

Since the solid is regular, it looks the same seen from all vertices when looking towards the origin. From this point of view, some vertices can be said to be 'underneath' the vertex from which we are looking. These 'underlying' vertices lie in one or more planes $P_i, i \in \mathbb{N}$ orthogonal to the line through the origin and our viewpoint. Drawing a line L through the vertex v and origin, we define the 'height' of the other vertices as the distance between vertex v and the point(s) $P_i \cap L$. From whichever vertex we are looking, the heights between the planes and the vertex will be equal because the solid is regular. It also holds that at each vertex, an equal number of equal angles meet. Therefore, it is possible to rotate P until it is in its original position.

It was seen in Proposition 3.2.2 elements of $SO(3)$ are such rotations. We would like therefore to consider the rotation group $SO(3)$, because the vertices of any Platonic solid can be mapped to each other by purely using such rotations. We recall some facts concerning rotations around the origin in the plane.

Using, e.g., [Leo15, p. 202] these rotations considered in the two-dimensional plane W can be represented by a matrix of the form

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

where θ is the angle of rotation about the origin in the counterclockwise direction. These matrices clearly have determinant equal to 1.

For any vertex, there exist rotations in the symmetry group that fix that vertex. More precisely, these are rotations of a certain angle around the line through the origin and the vertex itself. The rotations fix this line pointwise and send a plane through the origin and orthogonal to the line to itself. The following theorem from [Top96, p. 61] tells us something about this group of rotations.

Theorem 3.2.3. If G is a subgroup of $SO(2)$ existing of m elements, then G consists entirely of rotations about a multiple of $2\pi/m$. In particular $G \cong \mathbb{Z}/m\mathbb{Z}$.

Hence, the group of such rotations fixing a vertex is isomorphic to $\mathbb{Z}/m\mathbb{Z}$ for some $m \in \mathbb{N}$. The rotations in such a group map each face connected to the vertex cyclicly to one another. Because of the order of the group, it is clear that precisely m such faces exist.

From the above follows a sketch of the proof of Theorem 3.2.1.

Because the symmetry groups are finite and any vertex can be mapped to another using rotation, we first determine all *finite* subgroups of $SO(3)$. For all possible vertices of Platonic solids there exists at least one rotation that maps this vertex to itself, as we described above. We are interested in these vertices, and therefore determine all such points on the sphere. Then focusing on one point and looking at its orbits under an arbitrary finite subgroup of $SO(3)$, denoted as G, we can find the regular polyhedra. We now provide the proof in more detail.

Proof. (of Theorem 3.2.1) Let G be an arbitrary finite nontrivial subgroup of $SO(3)$. Denote its order by $N \geq 2$. Consider the boundary of the unit ball, so $B = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. Any element g of G is an orthogonal map with determinant equal to 1. By Proposition 3.2.2, if $g \neq id$ then g is a rotation around a unique line L through the origin. Any such line L intersects the sphere B in two antipodal points $\pm P$. The points $\pm P$ are the only points of B that are fixed by the linear map g, i.e. $g(P) = P$.

We now count the number of pairs (g, P) such that $g \in G$ is not the identity map and $P \in B$ satisfies $g(P) = P$. Since $\#G = N$, and for each $g \neq id$ in G we have two points $P \in B$ such that $g(P) = P$ we see that there are $2(N-1)$ such pairs. The set X consisting of those pairs is denoted as

$$X = \{(g, P) : g \in G,\ g \neq id,\ P \in B,\ g(P) = P\}.$$

Group G acts on X. The group action is defined as $hx := (hgh^{-1}, h(P))$ where h is an element of G. Note that hx is again an element of X, because $hgh^{-1}h(P) = hg(P) = h(P)$. Here we use the fact that $x = (g, P) \in X$ implies $g(P) = P$.

Consider now the set Y given by

$$Y = \{P \mid \exists\, g \in G : (g, P) \in X\}.$$

This set consists of all points on the sphere that are held fixed by some element $\neq id$ of G. In the sketch of the proof, we argued that those points are possible vertices of the regular polyhedra.

Group G acts on this set Y, with group action defined as $hP := h(P), h \in G, P \in Y$. To see that $hP \in Y$, note that by definition there exists g in G such that $(g, P) \in X$. We saw earlier that $(hgh^{-1}, h(P))$ is in X as well. Hence, $hP \in Y$.

For $P \in Y$ we write $G_P \subset G$ for its stabiliser under the action of G. Looking at the proof of Lagrange's theorem given in [MT18], we see that G is the disjoint union of cosets $g_i G_P$. This can be expressed as follows.

$$G = g_1 G_P \cup g_2 G_P \cup \cdots \cup g_{n_p} G_P$$

with $n_P = [G : G_P]$.

By writing any element of a coset as $g_i h \in g_i G_P$, observe that $g_i h(P) = g_i(P)$. Also, $g_i(P) \neq g_j(P)$ for any $i \neq j$, because otherwise it would imply that $g_i G_P$ and $g_j G_P$ are not disjoint, leading to a contradiction. A more detailed explanation of this fact can be found in the proof of Lagrange's theorem on p. 27 of [MT18]. It follows that the orbit $GP \subset Y$ consists of n_P points.

We continue to look at an orbit $GP \subset Y$. As stated above, the order of an orbit is n_P. Given $Q \in GP$, its stabiliser G_Q has the same order as G_P. Indeed, taking $g \in G$ with $gQ = P$ and $\tau \in G_P$, one finds $g\tau g^{-1} \in G_Q$. The same holds the other way around, showing that G_P and G_Q are conjugate within G. Hence, their order is equal.

For any point $P \in Y$, we have now discussed the number of points in its orbit: $\#GP = n_P$, and the number of elements of G in the stabiliser of any point in this orbit: for $Q \in GP$ one has

$\#G_Q = \#G_P = N/n_P$; this integer we will denote by $m_Q = m_P$. The orbit GP therefore yields $n_P(\frac{N}{n_P} - 1)$ pairs in X, namely

$$\{(h, Q) \ : \ Q \in GP, h \in G, h \neq \mathit{id}, h(Q) = Q)\} \subset X.$$

Note that this number is independent of P: it depends solely on the orbit GP.

The cardinality of the set X was counted before as being equal to $2(N-1)$. One can now infer a second way of counting by taking the sum over each orbit $GP = C \subset Y$; for all $Q \in C$ the number n_Q is the same, and the same holds for m_Q. Hence we write these numbers as n_C and m_C, respectively. This yields

$$\#X = \sum_C n_C(m_C - 1),$$

the sum taken over the different orbits $C \subset Y$.

Combining the two expressions for $\#X$ one finds

$$2(N-1) = \sum_C n_C(m_C - 1) = \sum_C (N - n_C). \tag{3.1}$$

Equation 3.1 can be manipulated by dividing by N and using that $n_C m_C = N$. This leads to

$$2 - \frac{2}{N} = \sum_C (1 - \frac{1}{m_C}). \tag{3.2}$$

Since $N \geq 2$, the left hand side of Equation 3.2 has value at least 1 and less than 2. Note that every $P \in Y$ satisfies $m_P = \#G_P \geq 2$, hence each term of the sum in the right hand side has value at least $1/2$ and less than 1. Therefore, this sum must consist of two or of three terms.

The proof in [Top96] concludes that there must be three terms, or the equation is concluded with one vertex, which will not lead us to a regular polyhedron. The order of the remaining orbits denoted C_i is chosen such that $m_1 \leq m_2 \leq m_3$, where m_{C_i} is denoted by m_i. The original proof excludes the case where all m_i with $i = 1, 2, 3$ are at least 3 and sets $m_1 = 2$. The reasoning used is similar to that in the previous paragraph. Similarly, it is excluded that the remaining m_2 and m_3 are both at least 4. A case distinction is made for m_2 and it is concluded that its value is not equal to two. This case would yield regular n-gons.

Another case distinction is made for m_3, with $m_1 = 2$ and $m_2 = 3$. Using the same argument as before for obtaining a case distinction for m_2, it is found that m_3 is at most 5. The remaining options for the value of m_3 are $3, 4$ and 5.

Inserting the obtained values for m_1 and m_2, we write Equation 3.2 as

$$\frac{1}{6} + \frac{2}{N} = \frac{1}{m_3} \tag{3.3}$$

If $m_3 = 3$, $N = 12$. The order of C_2 and C_3 is equal to $N/m_2 = 12/3 = 4$. The order of the stabiliser of this set of points is 3. These are rotations of degree $2\pi/3$. The regular polyhedron consists of 4 vertices where 3 faces meet at each vertex. This is the tetrahedron with Schläfli symbol {3,3}.

If $m_3 = 4$, $N = 24$. By the same reasoning we find $|C_2| = 24/3 = 8$ and $|C_3| = 24/4 = 6$. These span respectively the cube and octahedron with Schläfli symbols $\{4, 3\}$ and $\{3, 4\}$.

If $m_3 = 5$, $N = 60$. Again we find $|C_2| = 60/3 = 20$ and $|C_3| = 60/5 = 30$. These span respectively the dodecahedron and the icosahedron, with Schläfli symbols $\{5, 3\}$ and $\{3, 5\}$. □

What follows is a description of the symmetry groups that belong to these regular polyhedra.

3.3 Symmetry groups

In the previous section, we considered rotations as the symmetries of our polyhedra. These symmetries were described as orthogonal maps with determinant 1.

The full symmetry group G of a regular polyhedron is, however, a subgroup of $O(3)$ and therefore also contains maps τ with determinant -1. These maps are point reflections in the origin, as stated in Proposition 3.2.2. For such maps τ, we know that $-\tau$ is a rotation. Therefore, the full symmetry group consists of rotations τ and maps $-\tau$. Hence $G \cap SO(3)$ has index 2 in G.

For completeness, the full symmetry groups of the Platonic solids will now be repeated from [Top96].

Let us now examine the symmetry group of the tetrahedron. It was stated before that the symmetry group of a polyhedron with n vertices is isomorphic to a subgroup of S_n. The order of the group S_4 is equal to $4! = 24$. This number is also the amount of permutations of 4 vertices. These permutations determine entirely the symmetry group of the tetrahedron. Hence, the number of elements in the subgroup is equal to the number of elements of its group. Therefore, the symmetry group is equal to the group S_4.

When we now explore the symmetries of the cube, we know the cube shares its symmetries with its dual, the octahedron. In the above proof we found the order of the rotational symmetry group to be 24. The full symmetry group has order $2 \cdot 24 = 48$. It is worked out in [Top96, p. 69 – 70] that the symmetry group is isomorphic to $S_4 \times \{\pm 1\}$. The author makes use of the fact that any symmetry of the cube maps the four diagonals from vertex to vertex to each other, and that each symmetry has determinant 1 or -1.

Recall that the last two solids are also each other's dual, the dodecahedron and icosahedron. In [Top96, p. 70] it is worked out that their symmetry group is isomorphic to $A_5 \times \{\pm 1\}$. Here, A_5 denotes the alternating group consisting of even permutations in S_5. The reasoning used splits the edges of the dodecahedron up into five collections of edges that are either parallel or perpendicular in direction. The symmetry group then works on the set of these five collections are permutations. After all, it sends angles to angles of equal measure and the collections were picked such that the angles between the directions of the edges in one collection are $\pi/2$ or 0.

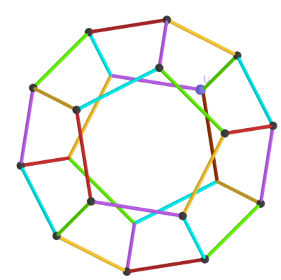

Figure 3.2: The six collections of edges used in [Top96, p. 70]

In the above proof we saw the order of the rotational group G is 60. The symmetry group as well as $A_5 \times \{\pm 1\}$ both have order $2 \cdot 60 = 120$.

The next chapter will discuss the four-dimensional analogues of the Platonic solids. One of them has no analogue in three-dimensional space. This is the 24-cell.

4 | Regular polytopes in four-dimensional space

This chapter describes the regular polytopes in four dimensions, specifically their vertices. It also derives their symmetries from the geometrical description. The information about the polytopes is based on [Cox61] and [Cox48].

In Coxeter's *Introduction to Geometry* [Cox61] it is shown that the only possible finite regular polytopes in four dimensions are the hypertetrahedron, the hypercube, the hyperoctahedron, the 24-cell, the 120-cell and the 600-cell.

In general we can describe a polytope as the convex hull of its vertex points. The edges, faces and cells can then be explored. This puzzle, as one could call it, leads to an idea of what the six regular 4-polytopes "look like". The derivation of their symmetries will use some basic knowledge from linear algebra and group theory. Besides that, the text by Polo-Blanco [Bla02] – will be used to describe the 24-cell and 600-cell in terms of quaternions.

4.1 Hypertetrahedron

> *"Make 10 equilateral triangles, all of the same size, using 10 matchsticks, where each side of every triangle is exactly one matchstick."*
>
> — *Éric Hernández*

The analogue of the tetrahedron in 3 dimensions is the *hypertetrahedron*. It can be constructed by adding a vertex to the tetrahedron that has equal distance to all other vertices. Therefore, it is also an answer to the above riddle. The general name for this object is the n-simplex, which is one of the regular polytopes that exist in n dimensions.

Definition 4.1.1. The hypertetrahedron can be described as a subset of $\mathbb{R}^5$ in the following way.

$$P_{\{3,3,3\}} = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : \sum_{i=1}^{5} x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \ldots, 5\} \tag{4.1}$$

The standard basis vectors in $\mathbb{R}^5$ denote the vertices of this object.

To develop some intuition for the notation of Equation 4.1, consider the two-dimensional 2-simplex. This is simply a regular triangle.

Example 4.1.2. We can describe the 2-simplex $P_{\{3\}}$ as a subset of $\mathbb{R}^3$ in the following way.

$$P_{\{3\}} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 1, x_i \geq 0 \text{ for } i = 1, 2, 3\} \tag{4.2}$$

The set described here is closed and bounded. We can show that it is convex as well. Any two points $A := (a_1, a_2, a_3)$ and $B := (b_1, b_2, b_3)$ in the set, can be connected and form a segment $[A, B] = \{A + t(B - A) : 0 \leq t \leq 1\}$. To show that any point on the segment is in $P_{\{3\}}$, we need to show that the sum of its coordinates is equal to 1, and each coordinate is semi-positive. A coordinate of a point on $[A, B]$ is given by

$$((1 - t)a_1 + tb_1, (1 - t)a_2 + tb_2, (1 - t)a_3 + tb_3)$$

It is easy to see that the coordinates are semi-positive. Now taking the sum of the coordinates yields

$$\begin{aligned} \sum_{i=1}^{3} a_i - ta_i + tb_i &= \sum_{i=1}^{3} a_i - t\sum_{i=1}^{3} a_i + t\sum_{i=1}^{3} b_i \\ &= 1 - t + t = 1 \end{aligned}$$

Hence, the set is convex and describes a 2-polytope. It lies entirely in the plane $\sum_{i=1}^{3} x_i = 1$.

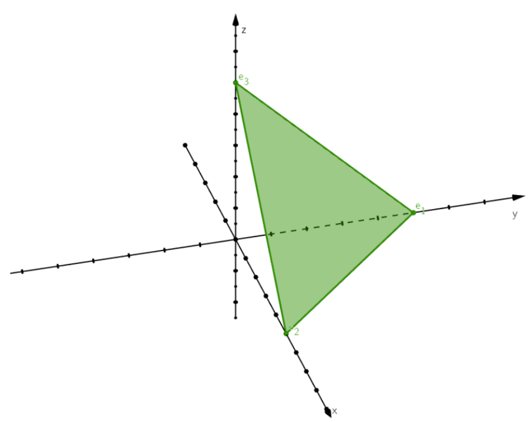

Figure 4.1: The 2-simplex as given in Equation 4.2

Note that the vertices of $P_{\{3\}}$ are the standard basis vectors in $\mathbb{R}^3$. The edges are the segments that join each of these vertices. For any two vertices A, B such a segment is given by $[A, B] = \{A + t(B - A) : 0 \leq t \leq 1\}$. Exactly three edges connect the vertices. All of them are of the form

$$x_j = 1 - x_k,$$

where $j \neq k$ and $0 \leq x_k \leq 1$.

A regular polygon has sides of equal length and angles of equal measure. So to show that $P_{\{3\}}$ is regular, one can easily prove with Pythagoras' theorem that the edges all have length $\sqrt{2}$ and derive that all angles are equal. To support our understanding of the more general case, however, we will show in a less straightforward way that $P_{\{3\}}$ is a regular polygon. We use Definition 2.1.2.

To check its regularity, we then need to show that the symmetry group of this polytope acts transitively on the vertices and edges. Considering the vertices, they can be mapped to one another by a rotation of angle $2\pi/3$ around the axis through the origin, orthogonal to the plane $\sum_{i=1}^{3} x_i = 1$. This rotation also cyclically carries each edge to the other. Such isometries are in the symmetry group of $P_{\{3\}}$, since they map the polytope to itself. The full symmetry group of the 2-simplex centred at the origin is the *dihedral group* D_3 [MT18, p. 46]

We can now approach $P_{\{3,3,3\}}$ in a similar way.

Showing convexity is so similar to the example that we will not include it here.

When we fix four of the five coordinates at zero, we find the vertices as boundaries of the edges. We see that the 5 vertices are indeed the standard basis vectors of $\mathbb{R}^5$. The whole object is contained in a hyperplane given by $V := \sum_{i=1}^{5} x_i = 1$. By a simple translation, we can centre this hyperplane (and thus the hypertetrahedron) at the origin. By picking a suitable basis, V can be considered to be embedded in $\mathbb{R}^4$. A linear map φ on the hyperplane V can be extended to $\mathbb{R}^5$ with the property that a vector $v = (1, 1, 1, 1, 1)$ orthogonal to V remains invariant. By extending maps in this way, the symmetry group of the hypertetrahedron can be considered a subgroup of $O(5)$.

A linear map φ that permutes the standard basis vectors (and thus vertices) can be extended to $O(5)$ as described above. Thus limited to V, they are isometries that send $V = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : \sum_{i=1}^{5} x_i = 0\}$ to itself.

The group of permutations of the 5 vertices is a subgroup of the symmetric group on 5 integers, called S_5. This group acts transitively on the standard basis vectors.

We want to know whether the group acts transitively on the other j-faces as well. Take for example the edges. These connect each vertex with another. These edges are described by equations of the form

$$S_i = \{e_i + te_j : t \in [0, 1]\}, \quad \text{for all } i \neq j, \quad i, j \in \{1, 2, \ldots, 5\}.$$

It is easy to see that S_5 also acts transitively on the set of edges, by permuting the $e_i s$ it sends each edge to another.

In the hyperplane given by $\sum_{i=1}^{5} x_i = 1$, a number of 2-dimensional hyperplanes is embedded. These are found by setting 2 coordinates equal to 0. To be precise, there are $\binom{5}{2} = 10$ in total. These are the faces of the hypertetrahedron. They are described by equations of the form

$$x_i + x_j + x_k = 1, \quad i \neq j \neq k, \; i, j, k \in \{1, 2, \ldots, 5\}.$$

Again, permutations of the coordinates x_i map each face to another.

Finally, the cells of the tetrahedron. These are themselves tetrahedral. They can be described by equations of the form

$$x_i + x_j + x_k + x_\ell = 1, \quad i \neq j \neq k \neq \ell \; i, j, k, \ell \in \{1, 2, \ldots, 5\}.$$

There are $\binom{5}{4} = 5$ such cells, and each can be mapped to another by permutations of S_5.

Clearly, S_5 is a subset of the symmetry group of $P_{\{3,3,3\}}$. The order of the symmetry group of $P_{\{3,3,3\}}$ is $5! = 120$. Therefore, the symmetry group of $P_{\{3,3,3\}}$ is S_5.

4.2 Hypercube

The *hypercube* is the four-dimensional analogue of the cube and the square. This kind of regular polytope exists throughout all dimensions, just like the n-simplex. It is sometimes called the *tesseract*, and any n-dimensional analogue is called an n-hypercube. Its symmetry group has order 384 [Bla02, p. 15]. We describe the hypercube as follows.

Definition 4.2.1. The *hypercube* can be described as a subset of $\mathbb{R}^4$ in the following way.

$$P_{\{4,3,3\}} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : |x_i| \leq 1\} \tag{4.3}$$

It is easily seen that the set is closed and bounded. The set is also convex, as any segment between points in the set lies in the set. This can be shown in the same way we proved it for the hypertetrahedron. We conclude that $P_{\{4,3,3\}}$ is a convex polytope. The question remains whether it is regular.

The cube as described in Equation 4.3 has its centre at the origin. A symmetry thus maps the origin to itself. Symmetries are therefore linear isometries and it follows they are elements of the orthogonal group $O(4)$.

By Definition 2.1.2, the polytope is regular if the symmetry group acts transitively on all faces of $P_{\{4,3,3\}}$. This means that for each face, there exists a symmetry in the symmetry group of $P_{\{4,3,3\}}$ that sends it to each other face. We will check this for each face, starting with the 0-faces.

Any vertex of $P_{\{4,3,3\}}$ can be found by fixing all coordinates to be either 1 or -1. There are $2^4 = 16$ such vertices. They can all be sent to each other by a symmetry represented by matrix A of the form

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}, \tag{4.4}$$

where $a_{ii} = -1$ whenever sending a coordinate $x_i = \pm 1$ to the opposite sign of itself and $a_{ii} = 1$ when the coordinate x_i remains unchanged. Clearly this matrix is orthogonal and sends the hypercube to itself.

For the edges, we fix three coordinates to be equal to 1 or -1. From this, we find $\binom{4}{1} \cdot 2^3 = 32$ possible edges. These edges can be represented by equations of the form

$$\sum a_i e_i, \text{ with three of the } a_j \text{ fixed, and one running from } -1 \text{ to } 1.$$

Consider two edges represented analogously to above, with a_i and b_j for some $i, j = 1, 2, 3, 4$ not fixed. We can send one edge to the other by sending e_i to e_j and mapping the remaining e_k to $\pm e_\ell$ bijectively. Here e_k corresponds to the fixed a_k and e_ℓ correspond to the fixed b_ℓ. The sign is chosen such that the edges are indeed mapped to one another. This can be written as linear map

$$\varphi : \begin{cases} e_i \mapsto e_j \\ e_k \mapsto \pm e_\ell. \end{cases}$$

This map is indeed linear (i.e. $\|\varphi(e_i)\| = \|e_i\|$).

This map sends $P_{\{4,3,3\}}$ to itself and is therefore an element of its symmetry group.

The polygonal faces can be found by fixing two coordinates equal to 1 or -1. There are $\binom{4}{2} \cdot 2^2 = 24$ square-shaped faces in total. With transformations analogous to the ones described above, it is possible to map each face to another.

We can also find the cells by fixing one coordinate equal to either 1 or -1. This procedure yields $2 \cdot \binom{4}{3} = 8$ cubes. For each of these cubes, we can send one to another by using the symmetries analogolous to the ones mentioned above.

So far, we have found symmetries of $P_{\{4,3,3\}}$ to be switching the sign, permuting the coordinates or a composition of those. The symmetry group is thus a subgroup of $S_4 \times \{\pm 1\}^4$. In total, these are $4! \cdot 2^4 = 384$ symmetries. This is also the order of $S_4 \times \{\pm 1\}^4$, so the symmetry group is isomorphic to $S_4 \times \{\pm 1\}^4$.

The dual of the hypercube is the hyperoctahedron and vice versa, hence they share the same symmetry group. Since symmetries map faces to faces of the object, they map vertices of its dual to one another. Because the hypercube and hyperoctahedron are each other's dual, this implies both objects have the same symmetries.

4.3 Hyperoctahedron

The *hyperoctahedron* is the third and last type of polytope that exists in n dimensions. Its general name is the n-orthoplex. The hyperoctahedron is the dual of the hypercube, a property that is shared by their three-dimensional analogues. The 8 vertices of the hyperoctahedron can be given in terms of the standard basis vectors in $\mathbb{R}^4$ as $\{\pm e_1, \pm e_2, \pm e_3, \pm e_4\}$. The object itself is the convex hull of those points. A description of the object is given below.

Definition 4.3.1. The hyperoctahedron can be described as a subset of $\mathbb{R}^4$ in the following way.

$$P_{\{3,3,4\}} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \sum_{i=1}^{4} |x_i| \leq 1\} \tag{4.5}$$

The set in Equation 4.5 is both closed and bounded. Convexity can easily be shown. Take points $A := (a_1, a_2, a_3, a_4)$ and $B := (b_1, b_2, b_3, b_4)$ in $P_{\{3,3,4\}}$. As seen similarly before in Example 4.1.2, coordinates of a point on the segment $[A, B]$ are given by $\{(1-t)a_i + tb_i\}_{i=1}^4$ with $0 \leq t \leq 1$. The sum of the absolute values of these coordinates is

$$\begin{aligned}
\sum_{i=1}^{4} |(1-t)a_i + tb_i| &\leq \sum_{i=1}^{4} |(1-t)a_i| + |tb_i| \\
&= \sum_{i=1}^{4} |a_i| - |t| \sum_{i=1}^{4} |a_i| + |t| \sum_{i=1}^{4} |b_i| \\
&\leq 1 - |t| + |t| = 1.
\end{aligned}$$

In the third line it is used that $\sum_{i=1}^4 |a_i| \leq 1$ and $\sum_{i=1}^4 |b_i| \leq 1$. From the equation, it follows that any point on this segment is in the set $P_{\{3,3,4\}}$. This proves that the set is convex. As it is also closed and bounded, it is a convex polytope by Definition 2.1.1.

The boundary of the set in Definition 4.3.1 is given by

$$\partial P_{\{3,3,4\}} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \sum_{i=1}^{4} |x_i| = 1\}.$$

It can be written as the union of several three-dimensional cells. Expressions for these cells are obtained by taking 16 combinations of $k, \ell, m, n \mod 2$ in the following expression

$$\{(x_1, x_2, x_3, x_4) \in \partial P_{\{3,3,4\}} : (-1)^k x_1 \geq 0, (-1)^\ell x_2 \geq 0, (-1)^m \geq 0, (-1)^n \geq 0\}.$$

Taking all combinations, this yields 16 tetrahedral cells.

We can then find the boundary of these expressions by setting one of the x_i alternately to be equal to 0. This yields $\binom{4}{1} \cdot 2^3 = 32$ 2-faces bounding the tetrahedral cells. Note that these are triangular.

To find the edges, we can again take the boundary of these 32 2-faces. We take two of the x_i to be equal to 0. This yields $\binom{4}{2} \cdot 2^2 = 24$ edges that bound the 2 faces. All are of the form $\{\pm e_d \pm te_k \;\; t \in [0, 1]\}$ for some $d \neq k \in \{1, 2, 3, 4\}$, for any of the four sign combinations.

The vertices bound the edges and are found by taking three x_i equal to zero. They are of the form

$$\{(x_d, 0, 0, 0) \in \partial P_{\{3,3,4\}} : (-1)^{k_d} x_d\}.$$

This expression with $d = 1$ gives two vertices $(1, 0, 0, 0)$ and $(-1, 0, 0, 0)$. For the other three choices of $i : x_i \neq 0$ we find 6 other vertices, yielding a total of 8. They are

$$(\pm 1, 0, 0, 0), (0, \pm 1, 0, 0), (0, 0, \pm 1, 0), (0, 0, 0, \pm 1).$$

4.4 24-cell

The 24-cell is the only four-dimensional regular polytope that has no analogue in three dimensions. Such an object can be constructed using the hyperoctahedron we described, by taking the midpoints of its edges as the vertices of the 24-cell [Cox61]. This construction is described in [Sti00] as well. Begin with a hyperoctahedron as given in Definition 4.3.1. It is possible to truncate this object by "cutting" it with hyperplanes. These hyperplanes intersect the midpoints of the 24 edges, orthogonal to the four coordinate axes. We obtain a polytope bounded by these hyperplanes. By taking the dual of the obtained polytope and scaling it with a factor 2, we obtain an object with as its vertices the 16 points

$$(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$$

and the 8 vertices of the hyperoctahedron.

This construction by truncation is easily visualised using a three-dimensional example. Although the 24-cell has no regular polyhedral analogue, it can be constructed analogously to the construction of a rhombic dodecahedron from an octahedron similar to [Cox48, p. 26, 150]. By first truncating the octahedron with hyperplanes through the midpoints of the edges, one constructs a quasi-regular polyhedron called the *cuboctahedron*. This polyhedron is one of the *Archimedean solids* [Cox48, p. 30]. The dual of this polyhedron is the rhombic dodecahedron whose vertices can be described as $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ and permutations of $(\pm 1, 0, 0)$. This polyhedron is classified as a *zonohedron* in [Cox48, p. 27]. These are convex polyhedra that are bounded by parallelograms.

The vertices of the 24-cell can also be described using *unit quaternions* [Sti00, p. 22] of a certain ring of quaternions. This is done by identifying a vector $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ with elements $x_1 + x_2 i + x_3 j + x_4 k \in \mathbb{H}$ [Bla02, p. 16]. We consider a subring of $\mathbb{H}$, called the *integer quaternions* or *Hurwitz quaternions*. This subring is defined as

$$H = \{x_1 + x_2 i + x_3 j + x_4 k \in \mathbb{H} : \text{either } x_i \in \mathbb{Z} \;\text{ for } i = 1, 2, 3, 4 \text{ or } x_i \in \frac{1}{2} + \mathbb{Z}, \;\text{for } i = 1, 2, 3, 4\}$$

We call x_i the coefficients of a quaternion, with $i = 1, 2, 3, 4$. We will now show this set is indeed a ring.

First of all it is an additive abelian group. Two quaternions are summed 'componentwise'. Let q, q' be in H denoted as

$$q = x_1 + x_2 i + x_3 j + x_4 k, \quad q' = x_1' + x_2' i + x_3' j + x_4' k.$$

The sum of two quaternions in H is

$$q + q' = (x_1 + x_1') + (x_2 + x_2')i + (x_3 + x_3')j + (x_4 + x_4')k.$$

Either both x_i and x_i' are integers, both are half-integers, or one of them is a half-integer. The sum is either

$$x_i + x_i' \in \begin{cases} \mathbb{Z} & \text{if both terms are in } \mathbb{Z} \text{ or both are in } \frac{1}{2} + \mathbb{Z} \\ \frac{1}{2} + \mathbb{Z} & \text{if exactly one of them is in } \frac{1}{2} + \mathbb{Z} \end{cases}$$

Therefore, H is closed under addition. It is clear that 0 is also an element of H. The satisfaction of the other group axioms follow from the fact that $(\mathbb{Z} \cup \frac{1}{2} + \mathbb{Z}, +, 0)$ is an abelian group.

Secondly, H is closed under multiplication. For readability purposes we now take the first eight letters of the alphabet to be the quaternions' coefficients. The product of two quaternions

$$q \cdot q' = ae - bf - cg - dh + (af + be + ch - dg)i + (ce + ag + df - bh)j + (ah + de + bg - cf)k$$

is also in H. Note that there are three cases. Either (a, b, c, d) and $(e, f, g, h) \in \mathbb{Z}^4$, or both (a, b, c, d) and $(e, f, g, h) \in (\mathbb{Z} + \frac{1}{2})^4$, or the coefficients of one of the quaternions are integers whereas the others are half integers. It is now shown that in all cases, the coefficients of the product are in $\mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$.

1. When $(a, b, c, d) \in \mathbb{Z}^4$ and $(e, f, g, h) \in \mathbb{Z}^4$ it is evident that the coefficients of qq' are in $\mathbb{Z}^4$ as well.

2. When either q or q' has coefficients in $(\mathbb{Z} + \frac{1}{2})$ whereas the other has coefficients in $\mathbb{Z}$, it is clear that at least $2q \cdot q'$ has coefficients in $\mathbb{Z}$. To see whether $qq' \in H$, we can check whether each coefficient of $2qq'$ is equivalent $\bmod 2$. The reason this implies $qq' \in H$ is as follows.

   For any $q \in H$ it holds that the coefficients of $2q$ are the same as 2 times the coefficients of q. If each coefficient of $2qq'$ is equivalent $\bmod 2$, they are all either $0 \bmod 2$ or $1 \bmod 2$. If the coefficients are $0 \bmod 2$, they are $\in 0 + 2\mathbb{Z}$. Then, qq' has coefficients in $\mathbb{Z}$. If the coefficients are $1 \bmod 2$, they are $\in 1 + 2\mathbb{Z}$. Then qq' has coefficients in $\frac{1}{2} + \mathbb{Z}$. These are exactly the two possible cases for coefficients of elements in H.

   Let us now show that the coefficients of qq' are in this case all in $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$. If the coefficients of q' are in $(\mathbb{Z} + \frac{1}{2})$ the coefficients of $2q'$ are equivalent to $1 \bmod 2$. Then the coefficients of $2qq'$ are all $a + b + c + d \bmod 2$. The case where q has coefficients $\in \mathbb{Z} + \frac{1}{2}$ is can be shown in a similar fashion.

3. Finally, we look at the case where both q and q' have coefficients in $\mathbb{Z} + \frac{1}{2}$. It is possible to write $q = \frac{1}{2}(1 + i + j + k) + Q$ and $q' = \frac{1}{2}(1 + i + j + k) + Q'$, with $Q, Q' \in L$. Here L stands for quaternions with integer coefficients. The product qq' can be written as

$$qq' = \frac{1}{4}(1 + i + j + k)^2 + \frac{1}{2}(1 + i + j + k)Q' + \frac{1}{2}Q(1 + i + j + k) + QQ'$$

It is clear that QQ' and $\frac{1}{2}(i+j+k)Q'$ as well as $\frac{1}{2}Q(1+i+j+k)$ are in H because of the earlier observations. Let us work out $\frac{1}{4}(1+i+j+k)^2$.

$$\begin{aligned}\frac{1}{4}(1+i+j+k)^2 &= \frac{1}{4}\left(-2+2i+2j+2k\right)\\ &= \frac{1}{2}(-1+i+j+k)\end{aligned}$$

This is again an element of H. Therefore, $qq' \in H$ in this case as well.

We conclude that H is closed under multiplication. To see that 1 is an element of H, simply let the second, third and fourth coefficients of a quaternion be equal to 0. In conclusion, H is a subring of $\mathbb{H}$.

Recall we wanted to look at the units of this ring. It suffices to look into the quaternions in this ring with norm equal to 1. After all, being a unit means a quaternion q has a multiplicative inverse q^{-1} such that $q \cdot q^{-1} = 1$. The norm of this product is equal to the product of the norms $N(q) \cdot N(q^{-1})$. Since the norm of 1 is simply equal to 1, we find that

$$N(qq^{-1}) = N(q)N(q^{-1}) = 1.$$

Because $q, q^{-1} \in H$, and $N(q) = a^2 + b^2 + c^2 + d^2$ it follows $N(q) \in \mathbb{Z}_{\geq 0}$. Therefore it follows that $N(q) = 1$ and $N(q^{-1}) = 1$.

The other way around it holds that when the norm of q is equal to 1, there exists a multiplicative inverse in H (namely the conjugate $\bar{q}$ of q), because $q \cdot \bar{q} = N(q) = 1$.

The norm of a quaternion was defined in Chapter 2. For any integer quaternion q that is a unit of H, we have seen the following must hold.

$$q \cdot \bar{q} = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1. \tag{4.6}$$

Recall that all $x_i s$ are either integers or half-integers, which is equivalent to all coefficients of $2q\bar{q}$ being in $\mathbb{Z}$ and all equal mod 2. Equation 4.6 can be solved by looking at the coefficients of $2q\bar{q}$ setting $y_i = 2x_i$ for $i = 1, 2, 3, 4$. Equation 4.6 becomes

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = 4. \tag{4.7}$$

Since the y_i all have the property of being equal mod 2, the only solutions are given by them being ± 1 (16 solutions) and one being ± 2 whereas the others are 0 (8 solutions). Hence, $H^\times$ consists of 16 + 8 = 24 elements.

Besides that, it is easily seen that each square is at most equal to 1. This gives all possibilities for the $x_i s$. Either one is ± 1 and the others zero (8 possibilities), or all of them are different sign combinations of $\pm\frac{1}{2}$ (16 possibilities). All 24 vertices of the 24-cell can therefore be denoted as quaternions with norm equal to 1.

$$\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$$

$$\pm 1, \pm i, \pm j, \pm k$$

The convex hull of these points is exactly the 24-cell.

The symmetry group that works transitively on these 24 points is now explored. We follow the reasoning of Polo-Blanco [Bla02, p. 16 – 17] and consider the following map. Let q, r be units of H. They give rise to the map

$$\begin{aligned} \varphi : \ \mathbb{H}(\mathbb{R}) &\to \mathbb{H}(\mathbb{R}) \\ x &\mapsto qxr \end{aligned}$$

To see this is orthogonal, we first show it is linear:

$$\begin{aligned} \varphi(\lambda x + \mu y) &= q(\lambda x + \mu y)r \\ &= (q\lambda x + q\mu y)r \text{ (distributive/associative property of a ring)} \\ &= q\lambda xr + q\mu yr \\ &= \lambda qxr + \mu xyr \ \text{ (since } \lambda, \mu \in \mathbb{R}) \\ &= \lambda\varphi(x) + \mu\varphi(y). \end{aligned} \tag{4.8}$$

In conclusion, the map φ is indeed a linear map. Does it preserve distance as well?

For a distance preserving map the following holds.

$$\|\varphi(x) - \varphi(y)\| = \|x - y\|, \quad \forall x, y \in X$$

In this case, the set X is $\mathbb{H}(\mathbb{R})$. Since φ is a linear map, the left hand side of the equation is equal to $\|\varphi(x - y)\|$. Therefore, because of the relation between the norm N and the notion of distance, it is sufficient to show that the norm is preserved. In other words, it needs to be shown the following holds.

$$N(\varphi(z))) = N(z) \quad \forall z \in \mathbb{H}(\mathbb{R}).$$

We will see that it does, because of how φ was defined using unit quaternions (with norm 1).

$$N(\varphi(z)) = N(qzr) = N(q)N(z)N(r) = 1 \cdot N(z) \cdot 1 = N(z).$$

It is already clear that $\varphi(H) = H$ from the fact that H is a ring and therefore closed under multiplication. We have shown that the set $\varphi(H) = \{\varphi(x) : x \in H\}$ is a subset of H. Now we need to show that H is a subset of $\varphi(H)$. Let x be an element of H. If there exists an element y in H such that $\phi(y) = x$, it is done. A potential candidate could well be $y = q^{-1}xr^{-1}$, for q, r in $H^\times$. Because q and r are units, their inverses exist. Now indeed $\varphi(y) = x$ and thus $\varphi(H) = H$, and we would like to see that the set of all such maps acts transitively on the vertices of the 24-cell.

Recall that the set of vertices can be represented as $H^\times$. Let x, y be arbitrary vertices. A symmetry that sends x to y is, for example,

$$\psi : x \mapsto x^{-1}xy.$$

Hence, the group of all maps as described above acts transitively on the set of vertices.

The question that arises is whether all symmetries of the 24-cell are of the form φ. This is indeed the case: Polo-Blanco sketches a proof in her master's thesis [Bla02, p. 17]. A crucial ingredient in this proof is, that conjugations $x \mapsto qxq^{-1}$ are distance preserving, linear maps $\mathbb{H} \to \mathbb{H}$, and since they map 1 to 1, they leave the subspace $1^\perp \cong \mathbb{R}^3$ invariant (this is the subspace with basis i, j, k). Moreover, every distance preserving linear map with determinant 1 of this subspace (so, element of SO(3)) is given by such a conjugation.

4.5 120-cell and 600-cell

The 120-cell is the four-dimensional analogue of the dodecahedron. It is bounded by 120 dodecahedra. [Sti00, p. 23] Because it is the dual of the 600-cell, it would be possible to derive its vertices by taking the midpoints of the tetrahedral cells of the 600-cell. The vertices of the 600-cell, which is analogous to the icosahedron in three-dimensions, can be described using Hurwitz quaternions as well [Bla02, p. 17 – 18]. We begin with the subset $R = H + \tau H$ of $\mathbb{H}$, where τ denotes the golden ratio, i.e. $\tau = \frac{1+\sqrt{5}}{2}$. It will be shown that R is a ring as well.

Instead of looking at the units $R^\times$ of this ring (which is an infinite group, because, e.g., the element $\tau \in R^\times$ has infinite order), we will look at the kernel of the norm $N\colon R^\times \to \mathbb{R}^\times$. We work this out in more detail now.

First of all, let us show that R is indeed a ring. Elements of this set are of the form

$$r = q_1 + \tau q_2, \text{ where } q_1, q_2 \in H.$$

The sum of an element r with another element $s = q_3 + \tau q_4 \in R$ with $q_3, q_4 \in H$ can be written as

$$r + s = (q_1 + q_3) + (q_2 + q_4)\tau.$$

This sum is again in R, since H is a ring. R is thus closed under addition.

To see whether it is closed under multiplication, we look at the following product.

$$r \cdot s = (q_1 q_3 + q_2 q_4) + (q_1 q_4 + q_2 q_3 + q_2 q_4)\tau$$

We have used the fact that $\tau^2 = \tau + 1$, a property of the golden ratio that follows from the fact it is a solution to the equation $x^2 - x - 1 = 0$.

Because H is closed under multiplication, the product $r \cdot s$ is in R. Therefore R is also closed under multiplication. For $q_1 = 1, q_2 = 0$, we see 1 is an element of R. Therefore, R is a subring of $\mathbb{H}$.

The norm map as defined in 2.3 is considered. Note that the norm map N and the norm $\|q\|$ for $q \in \mathbb{H}$ are not the same. The norm map sends an element of the form $q_1 + \tau q_2$ to $(q_1 + \tau q_2)(\bar{q}_1 + \tau \bar{q}_2)$. To see that $N(q) \in \mathbb{Z}[\tau]$ for all $q \in R$, observe firstly that $N(q) \in R$, since R is a ring and $\bar{q} \in R$. It is also evident that $N(q) \in \mathbb{R}$. The intersection $R \cap \mathbb{R} = \mathbb{Z}[\tau]$, so inevitably $N(q) \in \mathbb{Z}[\tau]$.

The map N is a homomorphism. First of all, $N(1) = 1, N(0) = 0$ [Top19, p. 3]. It also holds that $N(rs) = N(r)N(s)$. This was shown in 2.

Another thing to note about the units $R^\times$, is that the norm map has its image in $\mathbb{Z}[\tau]^\times$. Just observe that for $r \in R$ to be a unit, there must exist an $s \in R$ such that $rs = sr = 1$. Therefore $N(r)N(s) = N(rs) = N(1) = 1$ and since $N(r)$ and $N(s)$ are elements of $\mathbb{Z}[\tau]$, this means that $N(r), N(s)$ are units in $\mathbb{Z}[\tau]$. The argument is similar as given above for the 24-cell.

The other way around one can show that $N(r) \in \mathbb{Z}[\tau]^\times$ implies $r \in R^\times$. Consider a unit $N(r) \in \mathbb{Z}[\tau]$ and its inverse $N(r)^{-1} \in \mathbb{Z}[\tau]$. An element $s := N(r)^{-1} \cdot \bar{r}$ has the property that $s \cdot r = r \cdot s = 1$. Therefore, $r \in R^\times$.

The product of $\bar{r}$ with that power of τ is an element of R as well and is the inverse of r. Elements of $R^\times$ are thus of the form

$$R^\times = \{q_1 + q_2\tau \in \mathbb{Z}[\tau]^\times : N(q_1 + q_2\tau) = \tau^n \text{ with } n \in \mathbb{Z}\}$$

The kernel of our homomorphism N is a subgroup of $R^\times$ [Top96, p. 40]. It consists of those elements that have norm 1. More precisely,

$$\ker N = \{\alpha = q_1 + q_2\tau \in R \ : \ \alpha \cdot \bar{\alpha} = 1, \ \text{ i.e. } q_1\overline{q_1} + q_2\overline{q_2} + (q_1\overline{q_2} + q_2\overline{q_1} + q_2\overline{q_2})\tau = 1\}.$$

The constraint can be rewritten as

$$(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2) + (2ae + 2bf + 2cg + 2dh + e^2 + f^2 + g^2 + h^2)\tau = 1,$$

so the coefficients of $q_1, q_2 \in H$ should satisfy the system

$$\begin{cases} 2ae + 2bf + 2cg + 2dh + e^2 + f^2 + g^2 + h^2 = 0 \\ a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2 = 1. \end{cases}$$

Solving these is more straightforward than it might seem, because q_1 and q_2 have integer or half-integer coefficients. Let us consider the possible values of the coefficients.

- $a = b = c = d = 0$. This leads to a contradiction. The first equation of the system implies $q_2 = 0$ whereas the second states $N(q_2) = 1$.
- $a, b, c, d \in \mathbb{Z}$. The second equation then shows that at least three of a, b, c, d should be zero. This results in 8 possible quaternions, id est all permutations of

  $$(\pm 1, 0, 0, 0).$$

  In this case, the second equation of the system above implies $q_2 = 0$, from which it follows that the first equation holds as well. Hence this results in 8 possible $\alpha = q_1 + q_2\tau \in \ker N$, namely $\alpha = \pm 1, \pm i, \pm j, \pm k$.
- $a, b, c, d \in \mathbb{Z} + \frac{1}{2}$. This is possible if all values are equal to $\pm\frac{1}{2}$ and the values e, f, g, h are all equal to 0. This results in $2^4 = 16$ possible quaternions α, namely

  $$(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}).$$

Now that we have found 8 + 16 = 24 of the vertices of the 600-cell, it remains to find the remaining 96. In her Master's thesis on page 18, Polo-Blanco [Bla02] continues to find the remaining vertices as even permutations of

$$\{0, \pm\frac{1}{2}, \pm\frac{1}{2}(1-\tau), \pm\frac{1}{2}\tau\}.$$

The way she arrives at these, is by stating they are the remaining quaternions in $R := H + \tau H$ with norm equal to 1. Unfortunately, this approach does not yield the vertices we desire, as the vertices are not all in our ring R. Let us take, for example,

$$u := 0 + \frac{1}{2}i + \frac{1}{2}(1-\tau)j + \frac{1}{2}\tau k = q_1 + q_2\tau$$

with $q_1 = \frac{1}{2}i + \frac{1}{2}j$ and $q_2 = -\frac{1}{2}j + \frac{1}{2}k$. This quaternion is not in R.

What initially went wrong on page 16 of Polo-Blanco's [Bla02] is the definition of the Hurwitz quaternions. She defines these as

$$H = \left\{ \sum_{i=1}^{4} a_i e_i \mid a_i \in \frac{1}{2}\mathbb{Z} \text{ and } \sum_{i=1}^{4} a_i \in \mathbb{Z} \right\}.$$

According to this definition, it need not be the case that all coefficients of a quaternion $q \in H$ are either half-integers *or* integers, but could also be a mixture of both. In a world where this definition holds, it is indeed true that the remaining 96 quaternions are in R.

Now we can attempt to find a ring in which the first 24 found quaternions as well as u are contained. Possibly the following could be such a ring.

$$S := H + H \cdot u.$$

This set is evidently closed under addition. It contains 0 and 1. What is left to show that it is a ring is to show it is a ring is to verify closedness under multiplication. A product of two elements $s_1 = h_1 + h_2 u, s_2 = h_3 + h_4 u \in S$ can be written as

$$s_1 \cdot s_2 = h_1 h_3 + h_1 h_4 u + h_2 h_3 u + h_2 h_4 u^2.$$

The products of the $h_i s$ with each other are in H (and in S) by the fact that H is a ring. What we need to show is whether a product $u \cdot h$ with $h \in H$ is again in S, so of the form $h_1 + h_2 u$ with $h_1, h_2 \in H$.

Take $h_1 = 0$, then for $uh = h_2 u$ we get $uhu^{-1} = h_2$. Because $u^2 = -1, -u$ is the inverse of u. Hence we write $uh = (-uhu)u$. Whether uhu is an element of S might, however, not be the case for any $h \in H$. Take for example $h = i$.

$$uiu = \frac{1}{2}i + \frac{1}{2}(\tau - 1)j - \frac{1}{2}\tau k = h_a + h_b \tau.$$

with $h_a = \frac{1}{2}i - \frac{1}{2}j$ and $h_b = \frac{1}{2}j - \frac{1}{2}k$. Hence $-uiu \notin H$ and $uh = (-uhu)u \notin S$ for $h = i$. Therefore, S is not a ring, and S will not allow us to find the 120 quaternions as units of S with norm 1.

C. van Ittersum's approach in her thesis is similar [Itt20, p. 42 – 46]. Clearly, we can find vertices that have unit norm, as the polytope can be rescaled to lie on a unit 3-sphere. Let $T = \mathbb{R} \cdot i + \mathbb{R} \cdot j + \mathbb{R} \cdot k \subset \mathbb{H}$. This is a linear subspace of $\mathbb{H}(\mathbb{R})$. Let $U := \{q \in \mathbb{H} \ : \ N(q) = 1\}$.

For $q \in U$ and $t \in T$, the map $t \mapsto qtq^{-1}$ is a distance preserving linear map, with determinant 1, as we have seen before. Therefore, such maps are elements of $SO(3)$ (note that $\text{Re}(t) = 0$).

Consider a map $\psi : U \to SO(3) : \quad q \mapsto \sigma$ with $U := \{q \in \mathbb{H} : N(q) = 1\}$ and $\sigma := [t \mapsto qtq^{-1}]$. This is a surjective grouphomomorphism.

The rotation group of the icosahedron is a subgroup of $SO(3)$. We have seen in Chapter 3 that this group consists of 60 elements and it is isomorphic to A_5. Call this group G.

The pre-image of G under ψ is a subgroup of U. The map ψ is 2-to-1, so there are exactly $2 \cdot 60 = 120$ elements in this group $\psi^{-1}(U)$. The reason this holds is because $t \mapsto qtq^{-1}$ and $t \mapsto (-q)t(-q)^{-1} = - - qtq^{-1}$ describe the same rotation. These 120 elements are exactly those quaternions that describe the 120 vertices of the 600 cell. These form a group under multiplication.

On page 207 of *Sphere Packings, Lattices and Groups* [CS87], Conway Sloane find the 120 vertices by constructing the so-called Icosian ring. This is done by taking the set of all finite sums of elements from the icosian group. Their approach could be described as working backwards from the endpoint of the above description of van Ittersum's findings. The icosian group namely consists of exactly those 120 quaternions that van Ittersum has described as the vertices of the 600-cell. The units of the icosian ring are exactly the elements of the icosian group and thus the 120 quaternions that describe the vertices of the 600-cell.

We can now infer what the symmetry group of this 600-cell looks like. Simultaneously, this symmetry group is the same as the one of the 120-cell, because the 120-cell and the 600-cell are each other's dual. Actually, the symmetry maps may look rather familiar. Polo-Blanco proposes the following. Let us take r, s in group of quaternions that represent the vertices of the 600-cell. Now let $\psi : \mathbb{H}(\mathbb{R}) \to \mathbb{H}(\mathbb{R})$ be given by $\psi(x) = rxs$. Before we showed that this is indeed a linear map that preserves the norm.

We will show that this map sends the set of vertices to itself and that it acts transitively on this set. To show the first, the same strategy as for the 24-cell can be used.

Take x a quaternion in R. The image of x under ψ is in the form of rxs with r, s units in R. This is again in R, therefore $\psi(R)$ is a subset of R.

Take x, y be quaternions in R and y be such that $y = q^{-1}xr^{-1}$. Basically this is a repetition of what was done for the 24-cell. $\psi(y) = x$, so x is an element of $\psi(R)$ and R is a subset of $\psi(R)$. Hence, $\psi(R) = R$.

The transitivity becomes clear from the same map as for the 24-cell.

5 | Conclusion and outlook

In this thesis, convex, regular polytopes of dimension three and four were studied. A description of their symmetries and faces was given for the three polytopes that exist in all dimensions. For the 24-cell and 600-cell, quaternion algebra was used to describe their vertices and symmetries.

First, we worked out the details of a proof from prof. dr. J. Top [Top96, p. 66 – 69], where he showed that exactly five convex, regular polyhedra exist. In this proof, we used the fact that symmetry groups of the polyhedra can be described as finite subgroups of the orthogonal group $O(3)$, given that the polyhedra are centred at the origin. We defined an action of the symmetry group on the polyhedra and looked at the orbit of a single point on a unit sphere. Using this approach we constructed the five Platonic solids.

Recall that in Chapter 3 we considered the regular polyhedra to have vertices on a sphere. More specifically a 2-sphere. In Chapter 4, when we started using quaternions it stood out that the vertices had norm equal to 1 in their quaternion representation. For the cube and orthoplex this was also clear, as the vertices could be expressed in the standard basis vectors (whose norms are 1 as well). The description that was given for the simplex was in five-dimensional space, but this object could be translated in such a way that it is described in four coordinates. This is similar to how the 2-simplex as described in Example 4.1.2 can be rotated such that it lies in a plane. The connection between these observations from Chapter 3 and 4.1 is that the 4-polytopes can also be described with their vertices on a sphere. However, this time, the sphere will be a 3-sphere ($\mathbb{S}^3 = \{q \in \mathbb{H} : \|q\| = 1\}$ or $\mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 1\}$) in four-dimensional space.

This observation gives inspiration for a proof that there exist six four-dimensional polytopes. The style of prof. dr. J. Top's proof could be used here. Centring the sphere at the origin makes sure the symmetries are orthogonal maps. Unfortunately, this thesis did not go into proving that there exist exactly six convex, regular polytopes in four-dimensional space.

It would have been interesting to work out why in dimension $n \geq 4$ there exist only three regular, convex polytopes. Can we also construct a proof using a similar strategy as in Chapter 3? This might be a suggestion for further research.

Something else that was not done is interpreting the symmetries in four dimensions in a more physical sense. For the 24-cell, 120-cell and 600-cell, it would have been more complete to work out what the maps φ and ψ actually do in terms of what kinds of isometries play a part. As the quaternions can be used to describe rotations, the symmetries could possibly be interpreted as such.

The edges, 2-faces and cells of the 24-cell, 120-cell and 600-cell were not described. For the 120-cell, the vertices were not described either.

Bibliography

[Arm88] Mark A. Armstrong. *Groups and Symmetry*. Springer-Verlag New York Inc., 1988.

[Bla02] Irene Polo Blanco. “Regular Polytopes: symmetry groups and 3D-sections by Alicia Boole Stott”. MA thesis. Rijksuniversiteit Groningen, July 2002. URL: http://fse.studenttheses.ub.rug.nl/id/eprint/8680.

[Cox48] Harold Scott Macdonald Coxeter. *Regular Polytopes*. Methuen & co. Ltd. London, 1948.

[Cox61] Harold Scott Macdonald Coxeter. *Introduction to Geometry*. Wiley, 1961. URL: https://books.google.nl/books?id=AVIFJa%5C_CKFcC.

[CS87] J. H. Conway and N. J. A. Sloane. *Sphere-Packings, Lattices, and Groups*. Berlin, Heidelberg: Springer-Verlag, 1987. ISBN: 038796617X.

[DeH16] Deborah C. DeHovitz. “The Platonic Solids An Exploration of the Five Regular Polyhedra and the Symmetries of Three-Dimensional Space”. In: 2016.

[Gor96] Catherine A. Gorini. *Symmetry - A Link Between Mathematics and Life*. 1996. URL: https://www.miu.edu/mathematics/faculty-profiles/catherine-gorini/symmetry-a-link-between-mathematics-and-life.

[Itt20] Clara Ittersum. *Bachelor’s Thesis Applied Mathematics: Symmetry groups of regular polytopes in three and four dimensions*. 2020.

[Leo15] Steven J. Leon. *Linear algebra with applications*. 9. ed., global ed. Always learning. Boston ; Munich [u.a.]: Pearson, 2015, 520 S. ISBN: 978-1-292-07059-9 and 1-292-07059-5.

[MS02] Peter McMullen and Egon Schulte. *Abstract Regular Polytopes*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2002. DOI: 10.1017/CBO9780511546686.

[MT18] Jan Steffen Müller and Jaap Top. *Lecture notes Group Theory*. 2018.

[SG01] L. Schläfli and J.H. Graf. *Theorie der vielfachen Kontinuität*. Cornell University Library historical math monographs. Zürcher & Furrer, 1901. URL: https://books.google.nl/books?id=foIUAQAAMAAJ.

[Sti00] John Stillwell. “The Story of the 120-Cell, Volume 48, Number 1”. In: 2000.

[TG20] Armin Tavakoli and Nicolas Gisin. *The Platonic solids and fundamental tests of quantum mechanics*. [Online; accessed 11 July, 2020]. July 2020. URL: https://www.researchgate.net/publication/342818311_The_Platonic_solids_and_fundamental_tests_of_quantum_mechanics/figures.

[Top19] Jaap Top. *Lecture Notes Algebraic Structures*. 2019.

[Top96] Jaap Top. *Dictaat Groepentheorie*. 1996.

[Yog90] M.M. Yogi. *Maharishi Mahesh Yogi on the Bhagavad-Gita: A New Translation and Commentary with Sanskrit Text, Chapters 1 to 6*. An Arkana book. Arkana, 1990. ISBN: 9780140192476. URL: https://books.google.nl/books?id=cnoRAQAAIAAJ.

[Zee67] E Christopher Zeeman. "The topology of Minkowski space". In: *Topology* 6.2 (1967), pp. 161–170.

[Zho91] Jianhua Zhou. "Visualization of Four Dimensional Space and Its Applications (Ph.D. Thesis)". PhD thesis. 1991. DOI: https://docs.lib.purdue.edu/cstech/922.