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Topological Approach to Provability Logic

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Student: A. Dryaeva

First supervisor: Prof. dr. L.C. (Rineke) Verbrugge

Second assessor: Dr. A.V. Kiselev

Abstract

This thesis investigates the topological approach to provability logic, with a particular focus on the result proven by Leo Esakia in his 1981 article "Diagonal Constructions, Löb's Formula and Cantor's Scattered Spaces".

Esakia's article discusses the interpretation of the modal diamond operator as the derived set operator on a topological space and proves that a topological space satisfies Löb's axiom if and only if the space is scattered. This thesis gives a self-contained comprehensive overview of the topics from logic, algebra and topology required to understand this result, as well as a thorough and accessible proof. In the end, the application of this result to provability logic is made explicit, and an account of the impact the work by Esakia had on further developments in the field of mathematical logic is discussed.

Contents

1 Introduction

Provability logic is a type of modal logic where the box operator ($\Box$) is interpreted as "it is provable in T that", where T is a sufficiently strong formal theory, such as Peano Arithmetic. This logic stems from the desire to study what formal mathematical theories can express about their own provability predicates. Provability logic results from the basic modal logic K with the addition of Löb's axiom: $\Box(\Box A \to A) \to \Box A$, which in colloquial English says that a statement can be proven to be true in a theory, if it can be proven that when the statement is provable then it is true in that theory.

Provability logic is also commonly known as GL, named after the logicians Kurt Gödel and Martin Löb, who are often credited for the development of this logic. Even though neither of them formulated this logic explicitly, Gödel's incompleteness theorems of 1931, his work on intuitionistic logic in 1933, and Löb's theorem of 1953, have served as a foundation for provability logic [1]. The logic then saw many rapid developments in the 1970s, with independent inquiries into soundness and completeness of GL by several researchers around the world. Notable results include K. Segerberg conclusion regarding completeness of GL in 1971 [2], R. Solovay's arithmetical completeness theorem of 1976 [3], and the development of diagonalizable algebras by R. Magari in 1975 [4]. In recent years, logicians have been studying the limits on the scope of GL, as well as some interesting extensions of this logic, such as G. Japaridze's polymodal variant of GL [5].

Like many other modal logics, provability logic has suitable possible worlds semantics and alternative topological semantics, where propositions are interpreted as a subset of a topological space and propositional connectives correspond to set-theoretic operations. However, provability logic lacks a natural interpretation of the modal operators $\Diamond$ and $\Box$. In modal logics that contain the reflection axiom, $\Diamond$ corresponds to the closure operator in a topological space, and $\Box$ corresponds to the interior. Unfortunately, provability logic does not prove reflection, which means a topological interpretation of these operators needs a different approach.

In 1944, J.C.C McKinsey and A. Tarski suggested an interpretation of $\Diamond$ as a set operator which maps a set onto the set of its limit points [6]. In the late 1970s Leo Esakia investigated this idea, and presented his findings in 1978, at a Modal and Intensional Logics conference in Moscow. The result was later published in his 1981 article “Diagonal Constructions, Löb’s Formula and Cantor’s Scattered Spaces” [7], where he proved that a topological space satisfies Löb’s axiom if and only if the space is scattered. H. Simmons obtained a similar result in [8]. Unfortunately, his paper was published before the widely celebrated result by Solovay in [3], and was largely ignored. Esakia, working independently and aware of the work by Solovay, considered a more general problem. Esakia’s work was never fully translated from its original language, Russian, but an overview of the main result can be found in English, with varying degrees of detail, for example in [9] and [10].

This thesis is an investigation of the topological approach to provability logic, with a particular focus on the abovementioned paper by Esakia. It gives a self-contained comprehensive overview of the topics required to understand Esakia’s result, as well as a thorough and accessible proof. In order to achieve that, first a foundation of the required preliminary knowledge is built. This begins by formally introducing provability logic in Section 2. An account of its application to Peano Arithmetic, as well as its possible worlds semantics is also given in this section. After this, the algebraic interpretation of provability logic is discussed in Section 3. The necessary concepts from topology are then introduced in Section 4. In Section 5 Esakia’s main result is stated and a complete proof is given, filling in any gaps left by Esakia in [7] and Beklemishev in [10]. In Section 6, the application of this result to provability logic is made explicit, and the relevancy and impact of this work by Esakia on further developments in the field of mathematical logic is discussed.

2 Logic

2.1 Basics of Modal Logic

Modal logic studies logical reasoning involving the concepts of possibility and necessity. The conventional notation for necessity in modal logic is the box symbol, $\Box$, which is read as "it is necessary that...". The symbol for possibility is the diamond, $\Diamond$, and it is read as "it is possible that...". The two operators can be defined through one another: $\Box A = \neg \Diamond \neg A$ and $\Diamond A = \neg \Box \neg A$. In literature, $\Box$ is usually taken as the primitive operator, and $\Diamond$ is defined from it [11,12].

A basic modal logic from which many other logics are constructed is the modal logic K, named after the American philosopher and logician Saul Kripke [13].

Definition 2.1. The logic **K** is generated by the following axioms:

- All tautologies of propositional logic,
- Distribution Axiom: $\Box(A \to B) \to (\Box A \to \Box B)$,

and the following rules of inference:

- Necessitation Rule: if $\vdash_{\mathrm{K}} A$, then $\vdash_{\mathrm{K}} \Box A$,
- Modus Ponens: if $\vdash_{\mathrm{K}} A \to B$ and $\vdash_{\mathrm{K}} A$, then $\vdash_{\mathrm{K}} B$.

Here $\vdash_{\mathrm{K}} A$ denotes the statement "A is a theorem in the logic K". The Necessitation Rule then says that if A is a theorem of K, so is $\Box A$, and Modus Ponens allows us to derive B from $A \to B$ and A.

This logic contains the standard operators of propositional logic $\neg, \wedge, \vee, \to, \leftrightarrow$, the $\top$ symbol for tautology and the $\bot$ symbol for contradiction. It should be noted that just two of those operators are sufficient to define the rest. Following Boolos in [11] we can take $\to$ and $\bot$. Then:

$\neg A$ is equivalent to $A \to \bot$,

$\top$ is equivalent to $\neg\bot$,

$A \vee B$ is equivalent to $\neg A \to B$,

$A \wedge B$ is equivalent to $\neg(A \to \neg B)$,

$A \leftrightarrow B$, among others, is equivalent to $(P \to Q) \wedge (Q \to P)$.

2.2 Provability Logic

The logic that we are ultimately interested in with this thesis is provability logic, often called GL. It results from adding one axiom to the logic K, known as Löb's axiom. Hence, the following definition.

Definition 2.2. The logic **GL** is generated by the following axioms:

- All tautologies of propositional logic,
- Distribution Axiom: $\Box(A \to B) \to (\Box A \to \Box B)$,
- Löb's axiom: $\Box(\Box A \to A) \to \Box A$,

and the following rules of inference:

- Necessitation Rule: if $\vdash_{\mathrm{GL}} A$, then $\vdash_{\mathrm{GL}} \Box A$,
- Modus Ponens: if $\vdash_{\mathrm{GL}} A \to B$ and $\vdash_{\mathrm{GL}} A$, then $\vdash_{\mathrm{GL}} B$.

Remark. In literature equivalent systems appear under various names: L, G, $K4W$, PrL, and *Göd*. The last name is the one is used by Esakia in [7].

Note that the Reflexivity Axiom $\Box A \to A$ is not a theorem of GL, while all sentences of the form $\Box A \to \Box\Box A$ are. Let us show this in the next proposition.

Proposition 2.3. $\vdash_{\mathrm{GL}} \Box A \to \Box\Box A$

Proof. Before we begin the proof let us recall from propositional logic:
(ax) $M \to (N \to M)$, an axiom
(exp) $S \to (P \to R) \Leftrightarrow (S \wedge P) \to R$, the Exportation Rule.

Furthermore, consider the following claims:
Claim 1: if $\vdash_{\mathrm{GL}} A \to B$, then $\vdash_{\mathrm{GL}} \Box A \to \Box B$,
Claim 2: $\vdash_{\mathrm{GL}} (\Box A \wedge \Box B) \leftrightarrow \Box(A \wedge B)$.
Let us now prove these claims.

Proof of Claim 1. Suppose $\vdash_{\mathrm{GL}} A \to B$. Then by the Necessitation Rule $\vdash_{\mathrm{GL}} \Box(A \to B)$. Using the Distribution Axiom and Modus Ponens, we conclude that $\vdash_{\mathrm{GL}} \Box A \to \Box B$.

Proof of Claim 2. Consider the statements $(A \wedge B) \to A$, $(A \wedge B) \to B$, and $A \to (B \to (A \wedge B))$. These are tautologies in propositional logic, so by definition of GL we know that they are theorems of GL. By Claim 1 this implies that $\vdash_{\mathrm{GL}} \Box(A \wedge B) \to \Box A$, $\vdash_{\mathrm{GL}} \Box(A \wedge B) \to \Box B$, and $\vdash_{\mathrm{GL}} \Box A \to \Box(B \to (A \wedge B))$. The first two of these statements together give us $\vdash_{\mathrm{GL}} \Box(A \wedge B) \to \Box A \wedge \Box B$. Using the third statement, and a variant of the Distribution Axiom, namely $\vdash_{\mathrm{GL}} \Box(B \to (A \wedge B)) \to (\Box B \to \Box(A \wedge B))$, we get $\vdash_{\mathrm{GL}} \Box A \to (\Box B \to \Box(A \wedge B))$. Then (exp) brings us to $\vdash_{\mathrm{GL}} \Box A \wedge \Box B \to \Box(A \wedge B)$. Therefore, $\vdash_{\mathrm{GL}} \Box A \wedge \Box B \leftrightarrow \Box(A \wedge B)$.

With these tools we can start the proof of the proposition. We will prove it by showing that these three statements hold true:
(1) $\vdash_{\mathrm{GL}} \Box A \to \Box(\Box(A \wedge \Box A) \to (A \wedge \Box A))$,
(2) $\vdash_{\mathrm{GL}} \Box(\Box(A \wedge \Box A) \to (A \wedge \Box A)) \to \Box(A \wedge \Box A)$,
(3) $\vdash_{\mathrm{GL}} \Box(A \wedge \Box A) \to \Box\Box A$.

(1) Substituting $B \wedge C$ for M, and D for N into the axiom (ax), gives us $(B \wedge C) \to (D \to (B \wedge C))$. Then by applying the Exportation Rule twice we get $B \to (C \to (D \to (B \wedge C)))$ and $B \to ((C \wedge D) \to (B \wedge C))$.

    Now apply the substitution $B := A$, $C := \Box A$, $D := \Box\Box A$ to the statement above, which results in the statement $A \to ((\Box A \wedge \Box\Box A) \to (A \wedge \Box A))$. Note that every tautology of propositional logic is a theorem in GL, so this statement is as well.

    By Claim 1 we know that if $\vdash_{\mathrm{GL}} A \to ((\Box A \wedge \Box\Box A) \to (A \wedge \Box A))$, then $\vdash_{\mathrm{GL}} \Box A \to \Box((\Box A \wedge \Box\Box A) \to (A \wedge \Box A))$, and using Claim 2 we see that $\vdash_{\mathrm{GL}} \Box A \to \Box(\Box(A \wedge \Box A) \to (A \wedge \Box A))$. Hence, part (1) is proven.

(2) Recall Löb's axiom: $\Box(\Box A \to A) \to \Box A$. Substitute $(A \wedge \Box A)$ for A in the axiom to get $\vdash_{\mathrm{GL}} \Box(\Box(A \wedge \Box A) \to (A \wedge \Box A)) \to \Box(A \wedge \Box A)$.

(3) Once again using the axiom (ax), we get $\Box A \to (A \to \Box A)$, and by (exp) we have $(\Box A \wedge A) \to \Box A$. By Claim 1 this means that $\vdash_{\mathrm{GL}} \Box(\Box A \wedge A) \to \Box\Box A$.

With the three statements (1) – (3) proven, by propositional logic we conclude that $\vdash_{\mathrm{GL}} \Box A \to \Box\Box A$.

■

The duality of the modal operators $\Box$ and $\Diamond$ helps us interpret the statement $\Diamond A$ in provability logic as "it is not the case that it is provable that not A". Using this duality, it is possible to show that the following are valid schemas of GL [14]:

$$\Diamond \Diamond A \to \Diamond A,$$

$$\Diamond \bot \leftrightarrow \bot,$$

$$\Diamond(A \vee B) \leftrightarrow \Diamond A \vee \Diamond B,$$

$$\Diamond A \to \Diamond(A \wedge \neg \Diamond A).$$

Proving these comes down to applying the law of Contraposition to Proposition 2.3, $\top \leftrightarrow \Box\top$, Claim 2, and Löb's axiom.

As a modal logic, GL helps us understand results concerning provability in the foundation of mathematics. Most commonly GL is used to discuss provability in Peano Arithmetic (PA) [12, 13]. Propositional variables of GL can be considered to range over formulas of PA, and using this notation sentences of GL express facts about provability in PA. For example, the statement $\Box A \to A$, which says that if A is provable then it is true, expresses soundness of PA for a given sentence A. Löb's axiom, the defining axiom of GL, says that if PA can prove the sentence that claims soundness of PA for A, then A is already provable in PA. The statement $\neg\Box\bot$ says that PA is consistent, that is, contradiction is not provable. The statement $\Box\neg\Box\bot$ asserts that PA is able to prove its own consistency, however, as follows from the fixed point theorem, which was independently proved by D. de Jongh in 1975 (unpublished) and G. Sambin in 1976 in [15], the statement $\neg\Box\bot \to \neg\Box\neg\Box\bot$ is a theorem of GL. It says that if PA is consistent then it is unable to prove its own consistency, and note that this is just a formalized version of Gödel's second incompleteness theorem.

Thus, GL is a useful system because questions regarding provability in PA can be transformed into easier questions about which sentences are theorems of GL. Naturally, GL can and has been used to study provability in other formal theories as well, such as Zermelo–Fraenkel set theory (ZF) [11].

2.3 Semantics

One of the main goals of logic is to formalize the distinction between valid and invalid arguments [13, 16]. In propositional logic, the validity of an argument can be determined using truth tables: an argument is considered valid if in every row of the truth table where the premises are true, the conclusion is also true. However, truth tables cannot be used in modal logic, because we cannot determine the truth value of an expression containing the modal operators. J. Garson demonstrates this in [13] with the following example: when A is 'Dogs are dogs', $\Box A$ is true, but when A is 'Dogs are pets', $\Box A$ is false. This shows that the truth value of $\Box A$ does not follow simply from the truth value of A. Hence, modal logic takes a different approach to validity: through possible worlds semantics.

A possible worlds model consists of a non-empty set W of possible worlds, a binary accessibility relation R on W, and a valuation v that assigns a truth value to each atomic sentence at every world in W.

The truth value of an atomic sentence p in a world $w \in W$, given by the valuation v, is denoted as $v(p, w)$ and can take the value of 1 or 0, representing "true" and "false" respectively. Note that the value assigned to p in the world w may differ from the value assigned to p in another world $w' \in W$. To show that world w' is accessible from the world w we write wRw'. The truth value of a complex sentence can be defined inductively from the following clauses:
$v(\neg A, w) = 1$ if and only if $v(A, w) = 0$,
$v(A \to B, w) = 1$ if and only if $v(A, w) = 0$ or $v(B, w) = 1$,
$v(\Box A, w) = 1$ if and only if for every w' in W, if wRw', then $v(A, w') = 1$.

The conditions for the other logical operators ($\wedge, \vee, \Diamond$, etc.) are intuitive, but can be found written out explicitly in [14] or [16].

Example 2.4. In Figure 1 we see an illustration of a simple model, where $W = \{w_1, w_2, w_3\}$, R is such that $w_1Rw_2, w_1Rw_3, w_2Rw_2$ (and no other worlds are related by R), and $v(p, w_1) = 1$, $v(q, w_1) = 0$, $v(p, w_2) = 0$, $v(q, w_2) = 1$, $v(p, w_3) = 1$, $v(q, w_3) = 1$. Furthermore, we can say that $v(\Box q, w_1) = 1$, since q is true in every world that is accessible from w_1, and $v(\Diamond(p \wedge q), w_3) = 0$, because from w_3 it is not possible to access a world where both p and q are true, since w_3 does not give access to any other world.

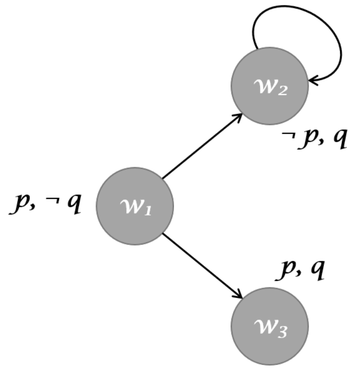

Figure 1: A simple possible worlds model

The validity of an argument in modal logic can now be discussed in terms of the truth value given by the valuation in a model to the premises and the conclusion. When defining validity in propositional logic we said that a valid argument is one that is truth-preserving. For modal logics, an argument is valid if it is true in all worlds of all models [16].

It has been shown that the simple modal logic K is both sound and complete for this kind of validity, meaning that every argument proven using the rules and axioms of K is valid, and every valid argument has a proof in the system [13]. This is a property that the logic K does not share with its extension GL. However, GL can made be sound and complete with appropriate restrictions on the class of possible worlds models, namely those where the relations R are transitive, finite and irreflexive [1,11]. Recall that a reflexive relation is one where for all $w \in W$, wRw, and a transitive one is where if w_1Rw_2 and w_2Rw_3, then it follows that w_1Rw_3, for $w_1, w_2, w_3 \in W$. Note that in Example 2.4 R is already transitive. That model would be suited for GL if we remove the relation w_2Rw_2 (else this relation creates an infinite sequence $w_2Rw_2Rw_2$...).

Figure 2 shows another model, in which R is transitive, finite and irreflexive. Notice that Löb's axiom $(\Box(\Box A \to A) \to \Box A)$ holds true in this model, although to check this fact it is easier to use the equivalent statement $\Diamond(\Box A \wedge \neg A) \vee \Box A$. Also observe that $\Box A \to A$ does not hold.

In addition to possible worlds semantics, many modal logics can be given topological semantics, where propositions are interpreted as subsets of a topological space, and propositional connectives correspond to the set-theoretic operations [1]. We will discuss this in more detail in Section 6.1.

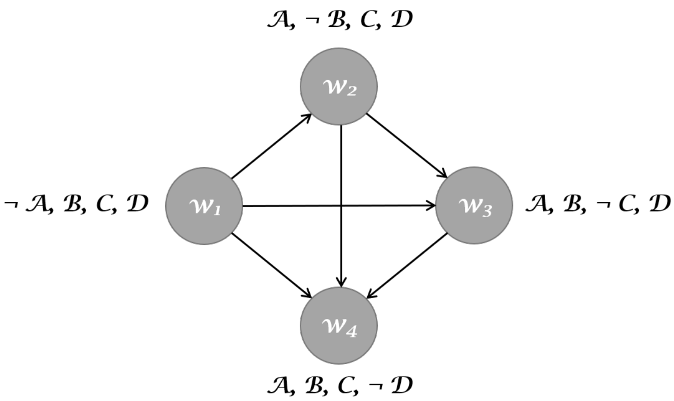

Figure 2: A four-world model where R is transitive, finite and irreflexive.

3 Algebras

In this section we will acquaint ourselves with Magari algebras, which are an algebraic interpretation of provability logic [17]. Having an algebraic language for a logic is valuable, as it ties logic closer to mathematics and gives logicians additional tools to work with. Relating a logic (and its extensions) to a class of algebras also sheds light on the underlying structures of the system. Furthermore, it allows us to seamlessly incorporate additional features of arithmetical theories and use mathematical techniques to further study proof theory [18].

Magari algebras were first introduced by Roberto Magari in 1975, as an attempt to analyze the Diagonal Lemma in Peano Arithmetic by algebraic means [17,19]. Magari introduced them as "diagonalizable algebras". Other authors used the two names interchangeably until 1994, when the name "Magari algebras" was officialised at a gathering of provability logicians at the Magari memorial conference in Siena [18].

Before formally introducing these algebras, let us first take a step back and recall the basics of Boolean algebras.

Definition 3.1. A **Boolean algebra** is a set S together with binary operations $+$ and $\cdot$, a unary operation $-$, and elements 0 and 1, such that the following laws hold for all x, y, z in S:

(B1)	$x + (y + z) = (x + y) + z$	(B1′)	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$	(associativity)
(B2)	$x + y = y + x$	(B2′)	$x \cdot y = y \cdot x$	(commutativity)
(B3)	$x + (x \cdot y) = x$	(B3′)	$x \cdot (x + y) = x$	(absorption)
(B4)	$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$	(B4′)	$x + (y \cdot z) = (x + y) \cdot (x + z)$	(distributivity)
(B5)	$x + (-x) = 1$	(B5′)	$x \cdot (-x) = 0$	(complementation)

More concisely, a Boolean algebra is a structure $(S, +, \cdot, -, 0, 1)$ for which axioms B1 – B5′ hold true.

According to the Handbook of Boolean algebras [20], examples of Boolean algebras arise in set theory, logic, functional analysis and topology. Because of that the original notation that was used by Boole is often replaced by the notation usual for those fields of mathematics. As such, in logic the three operations $+, \cdot, -$ are replaced by $\vee, \wedge, \neg$, and the elements 0 and 1 are sometimes referred to as “bottom” and “top” and get represented by $\perp$ and $\top$. In set theory, the collection of subsets of a set X, i.e. the power set $\mathcal{P}(X)$, can be taken as the set S. In this case the $+$ and $\cdot$ can be replaced by $\cup$ and $\cap$, the $-$ by the complement with respect to X, and the elements $0, 1$ correspond to $\varnothing, X$. We will adopt the latter notation.

Example 3.2. The structure $(\mathcal{P}(X), \cup, \cap, \setminus, \varnothing, X)$, as described above, is a Boolean algebra. The axioms B1 – B5′ simply state elementary laws of set theory. If $X = \varnothing$, the structure $(\mathcal{P}(\varnothing), \cup, \cap, \setminus, \varnothing, X)$ is called a trivial Boolean algebra. It consists of only one element, namely $\varnothing$. It is possible to construct a simple two-element Boolean algebra as well, with $X = \{x\}$. Naturally, $\mathcal{P}(X) = \{\varnothing, X\}$, which are also the top and bottom elements.

Definition 3.3. Let X be a set and $\mathcal{P}(X)$ its power set. We call the algebra $(\mathcal{P}(X), \cup, \cap, \setminus, \varnothing, X, \delta)$, or $(\mathcal{P}(X), \delta)$ for short, the **derivative algebra**, and $\delta\colon \mathcal{P}(X) \to \mathcal{P}(X)$ the derivative operator, when for any $A, B \in \mathcal{P}(X)$,

D1. $(\mathcal{P}(X), \cup, \cap, \setminus, \varnothing, X)$ is a Boolean algebra,

D2. $\delta\varnothing = \varnothing$,

D3. $\delta(A \cup B) = \delta A \cup \delta B$,

D4. $\delta\delta A \subseteq \delta A$.

Definition 3.4. Let X be a set and $\mathcal{P}(X)$ its power set. We call the algebra $(\mathcal{P}(X), \delta)$ a **Magari algebra** when for any $A \in \mathcal{P}(X)$,

M1. $(\mathcal{P}(X), \cup, \cap, \setminus, \varnothing, X, \delta)$ is a derivative algebra,

M2. $\delta A \subseteq \delta(A \setminus \delta A)$.

Note that in some literature, including [10] and [4], the definition of Magari algebras excludes condition D4, as it is provable from conditions D2, D3 and M2. We chose to include it, following the definition given in [7] by Esakia.

As alluded to in the introduction of this section, Magari algebras are closely related to the logic GL. In fact, if we look at the operator δ as the diamond operator $\Diamond$ of modal logic, we can see a clear correspondence between the conditions on δ from Definitions 3.3 and 3.4, and the axiom schemas of GL, which we derived in Section 2:

$\delta\varnothing = \varnothing$	corresponds to	$\Diamond\bot \leftrightarrow \bot$
$\delta(A \cup B) = \delta A \cup \delta B$	corresponds to	$\Diamond(A \vee B) \leftrightarrow \Diamond A \vee \Diamond B$
$\delta\delta A \subseteq \delta A$	corresponds to	$\Diamond \Diamond A \to \Diamond A$
$\delta A \subseteq \delta(A \setminus \delta A)$	corresponds to	$\Diamond A \to \Diamond(A \wedge \neg \Diamond A)$

It should be noted that the definition of Magari algebras also appears in literature in a different form. In [4] and [19] they are introduced from Boolean algebras with an operator that corresponds to the $\Box$ modal operator. In that case the conditions on the operator from Magari algebras correspond to the conditions on the box operator from the definition of GL and Proposition 2.3. However, we have once again elected to go with the definition used by Esakia in [7]. This choice will make the connection between Magari algebras and the main result of this paper more apparent (see Section 6).

4 Topology

In this section we will focus on topology. First, for completeness and ease of reference, we will recall some basic definitions. Then we will define more specialized terms to help us understand Esakia's theorem and derive some results that will be used in the proof of the main result.

4.1 General

We begin with a well-known definition of a topology, that can be found, for example, in [21]. We include it here because we will use it later.

Definition 4.1. Let X be a set, and $\mathcal{P}(X)$ its power set. A **topology** on X is a set $\tau \subseteq \mathcal{P}(X)$, such that:

1. Both the empty set and X are in τ,
   i.e. $\varnothing \in \tau$, $X \in \tau$.

2. The intersection of a finite number of sets in τ is also in τ,
   i.e. if $U_i, ..., U_n \in \tau$, then $\bigcap_{i=1}^{n} U_i \in \tau$.

3. The union of an arbitrary number of sets in τ is also in τ,
   i.e. if $U_i \in \tau$, then $\bigcup_{i \in I} U_i \in \tau$ for every $i \in I$.

A subset $U \subseteq X$ is called **open** if $U \in \tau$.
A subset $Y \subseteq X$ is called **closed** if $X \setminus Y$ is open.
The pair (X, τ) is called a **topological space**.

Throughout the rest of the paper, when no confusion seems possible we will simply say that X is a topological space, and leave the τ implicit.

Using the definition above and our pre-existing knowledge of the De Morgan laws, we can see that the intersection of an arbitrary number of closed sets is closed. That is because if $X \setminus Y_i \in \tau$ for every $i \in I$, then $\bigcup_{i \in I}(X \setminus Y_i) = X \setminus \bigcap_{i \in I} Y_i \in \tau$. Similarly, the union of a finite number of closed sets is closed.

Example 4.2. Since it is only required that $\{\varnothing, X\} \subseteq \tau$, and not that $\varnothing \neq X$, the empty set has a unique topology $\tau = \varnothing$. Every one-point space $X = \{x\}$ also has a unique topology $\tau = \{\varnothing, \{x\}\}$. A space with more than one point allows for several different topologies. Examples of widely known topologies include:

- The trivial topology (also called indiscrete), where only the empty set and the entire space are open,
- The discrete topology, consisting of the collection of all subsets of the space,
- The usual topology on $\mathbb{R}$, which is generated by the open intervals. It can be generalized to $\mathbb{R}^n$ by considering open balls to be the open sets.

Example 4.3. Consider the set $X = \{w, x, y, z\}$. Unions and intersections of elements of $\tau = \{\varnothing, \{w\}, \{w, x, \}, \{w, x, y\}, \{w, x, y, z\}\}$ are elements of τ, X and $\varnothing$ are elements of τ, so τ is a topology on X. Figure 3 is an illustration of the topological space (X, τ). We will turn to this depiction again in future examples.

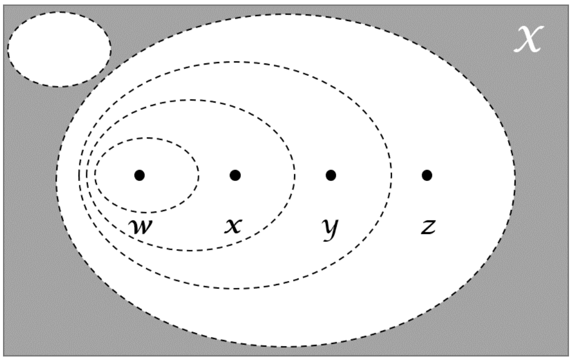

Figure 3: A topological space

Definition 4.4. Let X be a topological space, $x \in X$. An **open neighborhood** of x is a $U \in \tau$ such that $x \in U$.

Thus, for example, on the real line $\mathbb{R}$ with the usual topology, all open intervals containing a point are open neighborhoods of that point. In the space (X, τ) from Example 4.3 only the sets $\{w, x, y\}$ and $\{w, x, y, z\}$ are open neighborhoods of the point y, because they are the only elements of τ that contain y as an element. These sets are also open neighborhoods of the points w and x.

4.2 Derived set and scattered spaces

We will begin this section by defining when a point in a topological space is called a limit point. This concept will be used directly after, in the definition of a derived set, but not explicitly anywhere else.

Definition 4.5. Let X be a topological space, $A \subseteq X$. A **limit point** of A is a point $x \in X$, such that every open neighborhood of x contains a point of A that is different from x.

Definition 4.6. Let X be a topological space, $\mathcal{P}(X)$ its power set, and $A \subseteq X$. The operator $d \colon \mathcal{P}(X) \to \mathcal{P}(X)$ which maps a subset of X onto the set of its limit points is called the **derived set operator**. The set of all limit points of A

is denoted as $d(A)$ and called the **derived set**. That is formally, $x \in d(A) \Leftrightarrow \forall U$ open neighborhoods of x, $\exists y \neq x$ such that $y \in U \cap A$.

When it is clear which set is considered, we will omit the brackets in $d(A)$ for a cleaner look. So $d(A)$ will become dA, but $d(A \cup B)$ will stay the same. We will also do this for the operators c and $d^\star$, which will be introduced in later sections.

Returning once again to the topological space in Example 4.3 and Figure 3, observe that the derived set of $A = \{w, x, y\}$ is $dA = \{x, y, z\}$. The point w is not in the derived set of A because there exists an open neighborhood around w which contains no other point of A than w itself, namely the set $\{w\} \in \tau$. On the other hand, the point z is in the derived set of A, because z only has one open neighborhood, namely the set $\{w, x, y, z\} \in \tau$, and thus all open neighborhoods of z contain a point not equal to z that is both in A and in the neighborhood, such as for example the point x. This example also clearly demonstrates that it is not necessarily true that $A \subseteq dA$, or that $dA \subseteq A$.

From the definition above, we can prove many properties of the derived set. Certain properties that will be useful later are collected in the following lemma.

Lemma 4.7. For all subsets A, B of a topological space X, the derived set has the following properties:

(1) $d(\varnothing) = \varnothing$,

(2) $x \in dA \ \Rightarrow \ x \in d(A \setminus \{x\})$,

(3) $d(A \cup B) = dA \cup dB$,

(4) $A \subseteq B \ \Rightarrow \ dA \subseteq dB$.

Proof. Property (1) is straightforward. Property (2) as well, since the point y in the definition may not be equal to x, so $x \in dA$ guarantees that $y \in U \cap (A \setminus \{x\})$. Property (3) can be shown in two parts, as follows.

First we prove that $d(A \cup B) \supseteq dA \cup dB$, by assuming that $x \in dA \cup dB$ and showing that then $x \in d(A \cup B)$. If $x \in dA \cup dB$, then $x \in dA$ or $x \in dB$, or both. Suppose $x \in dA$. By definition of the derived set that means $\forall U$ open neighborhoods of x, $\exists y \neq x$ such that $y \in U \cap A$. Note that $U \cap A \subseteq U \cap (A \cup B)$. Therefore $y \in U \cap (A \cup B)$ for all U open neighbourhoods of x, so $x \in d(A \cup B)$.

The case for $x \in dB$ is analogous. Hence, $d(A \cup B) \supseteq dA \cup dB$.

For the other inclusion, we assume that $x \in d(A \cup B)$. By definition that means that $\forall U$ open neighborhoods of x, $\exists y \neq x$ such that $y \in U \cap (A \cup B)$. Since $U \cap (A \cup B) = (U \cap A) \cup (U \cap B)$, we have $y \in (U \cap A) \cup (U \cap B)$, so $y \in (U \cap A)$ or $y \in (U \cap B)$, or both, which means that $x \in dA \cup dB$. The inclusion holds, and therefore $d(A \cup B) = dA \cup dB$.

Note that in the proof of property (3) we omitted the trivial cases of $d(A \cup B) = \varnothing$ and $dA \cup dB = \varnothing$. The equality in those cases follows from property (1).

To prove property (4), let us assume that $x \in dA$. Then, by definition, $\forall U$ open neighborhoods of x, $\exists y \neq x$ such that $y \in U \cap A$. If $A \subseteq B$, then $U \cap A \subseteq U \cap B$. Hence, $\forall U$ $\exists y \neq x$ such that $y \in U \cap B$, so $x \in dB$. This means that $dA \subseteq dB$. Note that if $dA = \varnothing$, then $dA \subseteq dB$ for any $B \subseteq X$, so the implication in (4) still holds. Thus we have proven all four properties.

■

Property (4) is often referred to as "monotonicity". In proving it we have opted for a proof from the definition of d, but there is also a way to prove it using property (3), distribution over union. We will use that method when proving monotonicity of the operator c, right after introducing it in Definition 4.14.

Definition 4.8. Let X be a topological space, $A \subseteq X$. A point $x \in X$ is called an **isolated point** of A if $x \in A$ and $\exists U$, an open neighborhood of x, such that $U \cap A = \{x\}$.

Notice that in the topological space from Example 4.3, the point w is an isolated point of every nonempty subset of X that contains w, since there always exists an open neighborhood of w which intersects with the subset of X only in the point w, namely the open set $\{w\}$. Moreover, any nonempty subset of X that does not contain w also has an isolated point. In the depiction of this space in Figure 3, the isolated point would be the leftmost point of the subset in question. As such, y is an isolated point of $\{y, z\}$ because $\{w, x, y\} \cap \{y, z\} = \{y\}$. Let us now consider another example.

Example 4.9. Consider again the set $X = \{w, x, y, z\}$, but this time with $\tau = \{\varnothing, \{w\}, \{x, y, z\}, \{w, x, y, z\}\}$. Unions and intersections of elements of τ

are elements of τ, X and $\varnothing$ are also elements of τ, so τ is a topology on X. Figure 4 is an illustration of the topological space (X, τ). Observe that the point w is an isolated point of two subsets of X: $\{w\}$ and $\{w, x, y, z\}$. It is not an isolated point of $\{x, y, z\}$ because it is not an element of $\{x, y, z\}$. Moreover, the set $\{x, y, z\} \subseteq X$ has no isolated points.

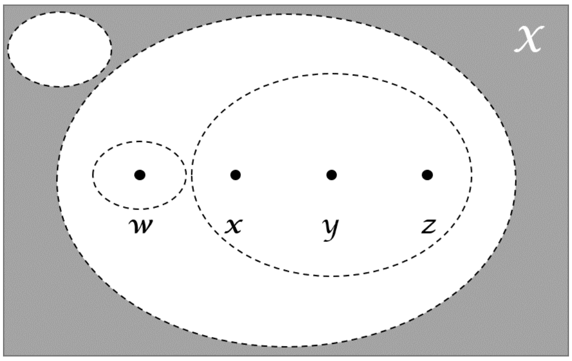

Figure 4: Another topological space

Example 4.10. Consider $\mathbb{R}$ with the usual topology, and a set $A = \{0\} \cup [1, 5] \subseteq \mathbb{R}$. The point 0 is an isolated point of A, since there exists an open neighborhood around 0, for example $U_0 = (-0.5, 0.5)$, such that $U_0 \cap X = \{0\}$. However, the point 3 is not an isolated point of A, since any open neighborhood (i.e. open interval) around the point 3, no matter how arbitrarily large or small, will intersect with another point of A.

Remark. Notice that if x is an isolated point of A, then $x \in A \setminus d(A)$. In fact, the set of all isolated points of A is $iso(A) = A \setminus d(A)$.

Definition 4.11. A topological space X is called **scattered** if every nonempty subset $A \subseteq X$ has an isolated point.

Recalling our observations in Example 4.9, we can recognize that the topological space discussed in that example is not scattered, whereas the space introduced in Example 4.3 is.

Scattered spaces have many interesting properties, and here in this work we will show two: one that is relevant for the proof of the main result, and one that will help us relate what we are doing in this section back to the logic GL.

Lemma 4.12. If X is a scattered topological space, then $dA \subseteq d(A \setminus dA)$ for all $A \subseteq X$.

Proof. Suppose X is a scattered topological space, and $A \subseteq X$. If $dA = \varnothing$ then the set inclusion holds trivially. From now we will only consider the case when dA is nonempty, and prove the lemma by showing that any $x \in dA$ is also an element of $d(A \setminus dA)$.

Take $x \in dA$ and an arbitrary open neighbourhood of x, U. By definition of the derived set, $\exists y \neq x$, such that $y \in (U \cap A)$. Therefore, $U \cap A$ is a nonempty subset of X, and because X is scattered, by definition of a scattered space $U \cap A$ has an isolated point. The point $x \in dA$ could be an isolated point of $U \cap A$, or $y \neq x$ could be an isolated point.

First we consider the case when y is the isolated point of $U \cap A$. By definition of an isolated point, $\exists J$ an open neighbourhood of y, such that $J \cap (U \cap A) = \{y\}$. Notice that $J \cap (U \cap A) = (J \cap U) \cap A$ and $y \in J \cap U$. In addition, since the intersection of two open sets is open (see definition 4.1), we see that $J \cap U$ is an open neighbourhood of y. From this, and the fact that $(J \cap U) \cap A$ does not contain an element of A not equal to y, we conclude that $y \notin dA$. However, y is an element of A, so $y \in A \setminus dA$, which also means that y is an isolated point of A.

Note that if we assume x to be an isolated point of $U \cap A$, we come to a contradiction. All the same reasoning applies as with y, but x was assumed to be an element of dA. Hence, x cannot be an isolated point of $U \cap A$.

Now that we know that $y \in A \setminus dA$, it is clear that for any U open neighborhood of x, $\exists y \neq x$ such that $y \in U \cap (A \setminus dA)$, which implies that $x \in d(A \setminus dA)$ by definition of the derived set.

∎

In fact, it can be proven that if X is a topological space, X is scattered if and only if $dA = d(A \setminus dA)$ for all $A \subseteq X$. We will show that as part of the main result, but a full isolated proof can be found in [9].

Now for a result that will not be used in the proof of the main theorem, but is interesting nonetheless, and will be referred to again in Section 6.

Proposition 4.13. If X is a scattered topological space and $A \subseteq X$, then $ddA \subseteq dA$.

Proof. Consider a set $B := A \cup dA$. Then $dB = d(A \cup dA) = dA \cup ddA$ by Lemma 4.7 item 3. The rest of the proof reduces to one string of set inclusions:

$$ddA \overset{(1)}{\subseteq} dB \overset{(2)}{\subseteq} d(B \setminus dB) \overset{(3)}{\subseteq} d((A \cup dA) \setminus dB) \overset{(4)}{\subseteq} d(A \setminus dB) \overset{(5)}{\subseteq} dA,$$

where
(1) comes from the definition of dB,
(2) is due to Lemma 4.12,
(3) is from the definition of B,
(4) is because $dA \subseteq dB$, and by Lemma 4.7 item 4,
(5) holds because $A \setminus dB \subseteq A$ and by item 4 of Lemma 4.7.

■

Remark. This result was proven differently in the article [7] by Esakia. The proof above is similar to that of Beklemishev and Gabelaia in [10], showing that $\delta\delta A \subseteq \delta A$, for the Magari algebra operator δ.

4.3 Closed sets and closure

In this section we will deal with set closure and the Kuratowski closure operator. One key difference between the two concepts that is important to note early on, is that set closure relies on closed sets in its definition. It is therefore defined on a topological space, where it is already known what it means for a set to be closed. The Kuratowski closure operator on the other hand, is simply a mapping defined on a set, and not a topological space. However, the Kuratowski operator can induce a topology. We will show that in Proposition 4.15, but let us start with a definition.

Definition 4.14. Let X be a set, $\mathcal{P}(X)$ its power set, and A, B subsets of X. A **(Kuratowski) closure operator** is a mapping $c\colon \mathcal{P}(X) \to \mathcal{P}(X)$, such that:

C1. $c(\varnothing) = \varnothing$,

C2. $A \subseteq c(A)$,

C3. $c(A) = c(c(A))$,

C4. $c(A \cup B) = c(A) \cup c(B)$.

The conditions C1 – C4 are called the Kuratowski closure axioms. Note that axiom C4 implies monotonicity of the closure operator:

C5. If $A \subseteq B$ then $c(A) \subseteq c(B)$.

Indeed, if we take a subset $A \subseteq B$, then $A \cup B = B$, and $c(B) = c(A \cup B) \overset{(C4)}{=} c(A) \cup c(B) \supseteq c(A)$. As suggested before, monotonicity of the derived set operator d can also be proven in the same way.

In general, it is not true that $c(A) = A$. But if we take those subsets $A \subseteq X$ for which that is true, we can define a topology on X. We shall now put this forth as proposition and prove it by checking the proposed topology against the definition of a topology we gave earlier (Definition 4.1).

Proposition 4.15. Let X be a set, $\mathcal{P}(X)$ its power set, and $c\colon \mathcal{P}(X) \to \mathcal{P}(X)$ the Kuratowski closure operator. Then $\tau = \{O \text{ such that } c(X \setminus O) = X \setminus O\}$ is a topology.

Proof. If c is the closure operator, then it satisfies the Kuratowski closure axioms C1 - C4, and also C5. We will show that for $\tau = \{O \text{ such that } c(X \setminus O) = X \setminus O\}$: (1) The empty set and X are in τ, (2) The intersection of a finite number of sets in τ is in τ, and (3) The union of an arbitrary number of sets in τ is also in τ.

(1) Since $A \subseteq cA$ for all $A \subseteq X$ by C3, and $X \subseteq X$, we have $X \subseteq cX$. Also, $cX = c(X \setminus \varnothing)$, and since X is the entire domain, $c(X \setminus \varnothing) \subseteq X = X \setminus \varnothing$. Together, this means that $c(X \setminus \varnothing) = X \setminus \varnothing$, and hence $\varnothing \in \tau$.

Note that $X \setminus X = \varnothing$, so $c(X \setminus X) = c(\varnothing)$. By axiom C1 $c(\varnothing) = \varnothing$, so $c(X \setminus X) = X \setminus X$, which means that $X \in \tau$.

(2) Suppose $O_1, O_2, ...O_n \in \tau$. Recall that by the De Morgan laws we have

$$c(X \setminus (O_1 \cap O_2 \cap ... \cap O_n)) = c((X \setminus O_1) \cup (X \setminus O_2) \cup ... \cup (X \setminus O_n)).$$

By axiom C4, the operator c distributes over union, and by our assumption $O_1, O_2, ...O_n \in \tau$, which means that

$$\begin{aligned} &c((X \setminus O_1) \cup (X \setminus O_2) \cup ... \cup (X \setminus O_n)) = \\ &\qquad = c(X \setminus O_1) \cup c(X \setminus O_2) \cup ... \cup c(X \setminus O_n) = \\ &\qquad\qquad = (X \setminus O_1) \cup (X \setminus O_2) \cup ... \cup (X \setminus O_n). \end{aligned}$$

Applying the De Morgan law once again, we conclude that

$$c(X \setminus (O_1 \cap O_2 \cap ... \cap O_n)) = X \setminus (O_1 \cap O_2 \cap ... \cap O_n),$$

which means that $O_1 \cap O_2 \cap ... \cap O_n \in \tau$.

(3) Suppose $O_i \in \tau$ for all $i \in I$, where I is just an index set. This means that for each $k \in I$, $c(X \setminus O_k) = X \setminus O_k$, and naturally, $O_k \subseteq \bigcup_{i \in I} O_i$.
Taking the complement gives us

$$X \setminus \bigcup_{i \in I} O_i \subseteq X \setminus O_k.$$

By the monotonicity of the operator c,

$$c(X \setminus \bigcup_{i \in I} O_i) \subseteq c(X \setminus O_k) = X \setminus O_k.$$

Because $c(X \setminus \bigcup_{i \in I} O_i)$ is a subset of $X \setminus O_k$ for all $k \in I$, it is a subset of their intersection:

$$c(X \setminus \bigcup_{i \in I} O_i) \subseteq \bigcap_{i \in I} (X \setminus O_i) = X \setminus \bigcup_{i \in I} O_i,$$

where the equality in the expression above holds by De Morgan.

So now we know that $c(X \setminus \bigcup_{i \in I} O_i) \subseteq X \setminus \bigcup_{i \in I} O_i$, and by axiom C3, the reverse inclusion holds, therefore equality holds, which means that $\bigcup_{i \in I} O_i \in \tau$ for all $i \in I$.

Therefore, by Definition 4.1, τ is a topology on X.

■

With the topology τ identified in the proposition above, observe that

$$A \text{ is closed} \;\Leftrightarrow\; X \setminus A \in \tau \;\Leftrightarrow\; c(X \setminus (X \setminus A)) = X \setminus (X \setminus A) \;\Leftrightarrow\; c(A) = A,$$

where the first equivalence holds by Definition 4.1, the second one by definition of τ, and the third comes from the fact that A is a subset of X, so $X \setminus (X \setminus A) = A$. Therefore, $A \subseteq X$ is closed if and only if $A = c(A)$. We will use this fact later, but first let us define the concept of set closure, which, as hinted at in the introduction of this section, is different from the Kuratowski closure in that it already assumes the existence of a topology on the set it is defined on.

Definition 4.16. Let X be a topological space, and let A, B be subsets of X. The **closure** of A, denoted $\overline{A}$, is the intersection of all closed sets that contain A. That is, $\overline{A} := \bigcap_{B\,closed,\,A\subseteq B} B$

Example 4.17. In any topological space X, we have $\overline{\varnothing} = \varnothing$ and $\overline{X} = X$. In $\mathbb{R}$ with the usual topology, $\overline{(-1,1)} = [-1,1]$. In (X,τ) from Example 4.9, the set $\{x,y,z\}$ is closed, because its compliment, the set $\{w\}$, is open. Therefore, $\overline{\{y,z\}} = \{x,y,z\} \cap \{w,x,y,z\} = \{x,y,z\}$.

Note that there are many alternative ways to define the closure of a set. We will continue to use the definition above, but show two equivalences that will be useful in the proof of the final result of Esakia's paper: for all $A \subseteq X$, $\overline{A} = c(A)$, and $\overline{A} = A \cup dA$ where d is the derived set operator.

Proposition 4.18. Let X be a topological space, $A \subseteq X$, $\overline{A}$ the closure of A, and cA the Kuratowski closure of A. Then $\overline{A} = cA$.

Proof. We will show that $\overline{A} = cA$ by showing that $cA \subseteq \overline{A}$ and $cA \supseteq \overline{A}$.

To begin, note that every set B in the definition of $\overline{A}$ is a superset of $\overline{A}$. Now observe that by axiom C2 of the Kuratowski operator, $A \subseteq cA$, and by axiom C3, $cA = c(cA)$, which means that cA is closed. Therefore, cA is one of the closed supersets of A over which we take the intersection in the definition of $\overline{A}$, so $cA \supseteq \overline{A}$.

Furthermore, since $\overline{A}$ is defined as the intersection of closed sets, it itself is closed, so $c\overline{A} = \overline{A}$. Also from the definition of $\overline{A}$ it is clear that $A \subseteq \overline{A}$. By the monotonicity property of the Kuratowski operator we see that $cA \subseteq c\overline{A}$.

Therefore, we conclude that $\overline{A} = cA$.

■

Thus, in a topological space, as defined in Proposition 4.15, the closure of a subset is equal to the Kuratowski closure. Now, we will show that the closure of a subset is also equal to the subset together with its limit points. This will imply, naturally, that the Kuratowski closure of a subset is equal to the subset together with its limit points, which is a fact we will make use of in the proof of the main result.

Proposition 4.19. Let X be a topological space, $A \subseteq X$, $\overline{A}$ the closure of A, and dA the derived set of A. Then $\overline{A} = A \cup dA$.

Proof. We will prove the proposition by showing that: (1) $\overline{A} \subseteq A \cup dA$, and (2) $\overline{A} \supseteq A \cup dA$.

(1) If $\overline{A} = \varnothing$, then certainly $\overline{A} \subseteq A \cup dA$. Now assume that $\overline{A}$ is nonempty. Take some point $q \in \overline{A}$. Suppose $q \notin A \cup dA$. This means q is neither an element of A, nor of dA. By definition of dA this implies that there exists an open neighbourhood U of q, such that $U \cap A = \varnothing$ or $U \cap A = q$.

If $U \cap A = q$, then $q \in A$, which means that $q \in A \cup dA$, which contradicts the assumption that $q \notin A \cup dA$. Hence, consider only $U \cap A = \varnothing$.

If $U \cap A = \varnothing$, and U is an open set, then $X \setminus U$ is a closed superset of A. That is, $A \subseteq (X \setminus U)$, and $X \setminus U$ is closed. Moreover, since U was an open neighborhood of q, $X \setminus U$ does not have q as an element. This means that the intersection of all closed sets that contain A as a subset will not contain q as an element, and hence $q \notin \overline{A}$. However, this is a contradiction, because we initially took $q \in \overline{A}$. Thus we conclude that if $q \in \overline{A}$ then $q \in A \cup dA$, i.e. $\overline{A} \subseteq A \cup dA$.

(2) If $A \cup dA = \varnothing$, then certainly $A \cup dA \subseteq \overline{A}$. Now assume it is nonempty. Take some point $q \in A \cup dA$. Suppose $q \notin \overline{A}$. By definition of $\overline{A}$ this means there exists some closed set B, such that $A \subseteq B \subseteq X$ and $q \notin B$. The open set $X \setminus B$ thus contains q as an element, which means it is an open neighbourhood of q. Also, since $A \subseteq B$, we know that $A \cap (X \setminus B) = \varnothing$.

We have now found an open neighbourhood of q that does not intersect with A, which means that q cannot be an element of dA by definition of dA. However, q also cannot be an element of A, because $q \in X \setminus B$, but $A \cap (X \setminus B) = \varnothing$. Since q cannot be an element of neither A, nor dA, it is not an element of their union, which contradicts our assumption. Therefore, we conclude that if $q \in A \cup dA$ then $q \in \overline{A}$, i.e. $A \cup dA \subseteq \overline{A}$.

We have proven that $\overline{A} \subseteq A \cup dA$ and $A \cup dA \subseteq \overline{A}$. Therefore $\overline{A} = A \cup dA$.

■

5 Main Result

In this section, the main result is presented as two theorems. Note that this is not the format in which this result was presented by Esakia in [7], nor is it the way that Beklemishev and Gabelaia presented it in [10]. It is split here into two theorems for clarity and convenience.

Theorem 5.1 (Esakia)**.** Let (X, τ) be a scattered topological space. Then there exists an operator $d^\star$, such that for all $A, B \subseteq X$:

(1) $d^\star(\varnothing) = \varnothing$,

(2) $d^\star(A \cup B) = d^\star(A) \cup d^\star(B)$,

(3) $d^\star(A) = d^\star(A \setminus d^\star(A))$.

Proof. Let X be a scattered space. Suppose $d^\star A$ to be equal to the set of limit points of A, dA. Then, (1) and (2) hold by Lemma 4.7, and $d^\star(A) \subseteq d^\star(A \setminus d^\star(A))$ holds by Lemma 4.12. The only thing left to show is that $d^\star(A) \supseteq d^\star(A \setminus d^\star(A))$. Note that $A \setminus d^\star A \subseteq A$, so $d^\star(A \setminus d^\star A) \subseteq d^\star(A)$ by item 4 of Lemma 4.7. ■

Theorem 5.2 (Esakia)**.** Let X be a set and $d^\star$ an operator, such that for all $A, B \subseteq X$:

(1) $d^\star(\varnothing) = \varnothing$,

(2) $d^\star(A \cup B) = d^\star(A) \cup d^\star(B)$,

(3) $d^\star(A) = d^\star(A \setminus d^\star(A))$.

Then there exists a topology $\tau \subseteq \mathcal{P}(X)$, such that (X, τ) is a scattered topological space and $d^\star A = dA$ for all $A \subseteq X$.

Before proving this theorem, let us look at two additional lemmas that will aid in the proof.

Lemma 5.3. Let X be a set, $A \subseteq X$. Define $cA = A \cup d^\star A$, with $d^\star$ an operator as described in the theorem above. Then cA is a Kuratowski closure operator.

Proof. Proving this comes down to checking $cA = A \cup d^\star A$ against the Kuratowski closure axioms C1 – C4. We shall go through them one by one.

C1: Clearly, $c(\varnothing) = \varnothing \cup d^\star(\varnothing) = d^\star(\varnothing) = \varnothing$, where the last equality holds by the first property of $d^\star$ as it was defined in the theorems above.

C2: Here using the second property of $d^\star$, we see that
$c(A \cup B) = (A \cup B) \cup d^\star(A \cup B) = A \cup B \cup d^\star(A) \cup d^\star(B) = cA \cup cB$.

C3: Clearly, $A \subseteq A \cup d^\star(A) = cA$.

C4: To show that $ccA = cA$, we first make the following claim: $d^\star d^\star A \subseteq d^\star A$.

With this in hand, we can see that

$ccA = c(A \cup d^\star A) = cA \cup cd^\star A = (A \cup d^\star A) \cup (d^\star A \cup d^\star d^\star A) \overset{\bigstar}{=} A \cup d^\star A = cA$,

where the equality labeled by $\bigstar$ holds due to the claim we made, and the other equalities hold by our definition of cA.

To prove the claim, consider the set $Y = A \cup d^\star A$. Clearly,

$$d^\star d^\star A \subseteq d^\star d^\star A \cup d^\star A.$$

Because of the distribution of $d^\star$ over union (item (2) in the theorem above),

$$d^\star d^\star A \cup d^\star A = d^\star(A \cup d^\star A) = d^\star Y.$$

Using the third property in the definition of $d^\star$, the definition of set Y and basic properties of set union and complement, we get

$$d^\star Y = d^\star(Y \setminus d^\star Y) = d^\star((A \cup d^\star A) \setminus d^\star Y) = d^\star((A \setminus d^\star Y) \cup (d^\star A \setminus d^\star Y)).$$

Observe that because $d^\star Y = d^\star(A \cup d^\star A)$ by definition of Y, $d^\star A \setminus d^\star Y = \varnothing$. This leaves us with

$$d^\star d^\star A \subseteq d^\star Y = d^\star(A \setminus d^\star Y).$$

We have already seen with the Kuratowski closure operator that monotonicity follows from the property of distribution over union. The operator $d^\star$ also has the distribution over union property, and monotonicity follows from it in exactly the same way. With this property, we can say that because $A \setminus d^\star Y \subseteq A$, we know that $d^\star(A \setminus d^\star Y) \subseteq d^\star A$, and hence can conclude that $d^\star d^\star A \subseteq d^\star A$.

Having shown that properties C1 – C4 hold for $cA = A \cup d^\star A$, we conclude that it is a Kuratowski closure operator.

■

Lemma 5.4. Let X be a set, $x \in X$, $A \subseteq X$, and $d^\star$ a set operator as described in the theorem above. Then (1) $x \notin d^\star\{x\}$; (2) $x \in d^\star A \ \Leftrightarrow\ x \in d^\star(A \setminus \{x\})$.

Proof. Take the assumptions from the lemma.

(1) Suppose $x \in d^\star\{x\}$. Then $d^\star(\{x\} \setminus d^\star\{x\}) = d^\star(\varnothing) = \varnothing$. Since $\{x\} \subseteq X$, by the third property of $d^\star$, $d^\star\{x\} = d^\star(\{x\} \setminus d^\star\{x\})$. This means that $d^\star\{x\} = \varnothing$. However, that is not possible since $d^\star\{x\}$ contains x as an element by assumption. We have a contradiction, therefore $x \notin d^\star\{x\}$.

(2) Suppose $x \in d^\star A$. Then $x \in d^\star((A \setminus \{x\}) \cup \{x\})$. Because $d^\star$ distributes over union, $d^\star((A \setminus \{x\}) \cup \{x\}) = d^\star(A \setminus \{x\}) \cup d^\star(\{x\})$. By item (1) of this lemma, which we have just proven above, x is not an element of $d^\star\{x\}$, so it must be an element of $d^\star(A \setminus \{x\})$. Thus we have shown that $x \in d^\star A \ \Rightarrow\ x \in d^\star(A \setminus \{x\})$.

For the other direction, suppose $x \in d^\star(A \setminus \{x\})$. Note that $A \setminus \{x\} \subseteq A$, so by the monotonicity property of $d^\star$, $d^\star(A \setminus \{x\}) \subseteq d^\star A$. Hence, $x \in d^\star A$, and we have shown that $x \in d^\star A \ \Leftarrow\ x \in d^\star(A \setminus \{x\})$.

■

With these tools, we can now return to proving Theorem 5.2.

Proof. Let X be a set and $d^\star$ an operator satisfying the conditions listed in the theorem. We will first show that there exists a topology $\tau \subseteq \mathcal{P}(X)$, then prove that for all $A \subseteq X$, $d^\star A = dA$, and in the end show that the topological space (X, τ) is scattered.

First, define an operator $cA = A \cup d^\star A$, for some $A \subseteq X$. By Lemma 5.3 this operator satisfies the Kuratowski closure axioms, which means that c is a Kuratowski closure operator, and hence by Proposition 4.15 it induces a topological space, with the topology $\tau = \{O \text{ such that } c(X \setminus O) = X \setminus O\}$.

Consider the derived set operator in this topological space, d. In section 4.3 we showed that a Kuratowski closure of a subset of the space is equal to the closure of that subset, which in turn, equals to the union of the subset with the set of its limit points. Thus, for some $A \subseteq X$, $cA = A \cup dA$, which means that $A \cup d^\star A = A \cup dA$.

Now we will show that $d^\star A = dA$. If $A = \varnothing$, $d^\star A = dA$ by the properties of the operators d and $d^\star$. If A is a nonempty subset of X, take some $x \in d^\star A$. By Lemma 5.4, $x \in d^\star(A \setminus \{x\})$, and

$$d^\star(A \setminus \{x\}) \subseteq (A \setminus \{x\}) \cup d^\star(A \setminus \{x\}) = c(A \setminus \{x\}) = (A \setminus \{x\}) \cup d(A \setminus \{x\})$$

Thus, we see that if $x \in d^\star A$, then $x \in (A \setminus \{x\}) \cup d(A \setminus \{x\})$.

Naturally, $x \notin A \setminus \{x\}$, which means that $x \in d(A \setminus \{x\})$. Since $A \setminus \{x\} \subseteq A$, by the monotonicity property of d, we can say that $d(A \setminus \{x\}) \subseteq dA$, which means that $x \in dA$.

With this, we have now shown that if $x \in d^\star A$, then $x \in dA$, so $d^\star A \subseteq dA$.

To show the converse, take some $x \in dA$. When introducing the derived set we showed in Lemma 4.7, item 2, that if $x \in dA$ then $x \in d(A \setminus \{x\})$. Note also that

$$d(A \setminus \{x\}) \subseteq (A \setminus \{x\}) \cup d(A \setminus \{x\}) = c(A \setminus \{x\}) = (A \setminus \{x\}) \cup d^\star(A \setminus \{x\})$$

This means that if $x \in dA$, then $x \in (A \setminus \{x\}) \cup d^\star(A \setminus \{x\})$.

Similarly to what we did before, observe that $x \notin A \setminus \{x\}$, so $x \in d^\star(A \setminus \{x\})$, and because monotonicity holds for $d^\star$ as well, $d^\star(A \setminus \{x\}) \subseteq d^\star A$, which means that $x \in d^\star A$.

Now we have shown that if $x \in dA$, then $x \in d^\star A$, so $dA \subseteq d^\star A$.

Therefore, $d^\star A = dA$. All that is left to prove from Theorem 5.2 is that the space (X, τ) is scattered.

Consider a subset $A \subseteq X$ such that $A \neq \varnothing$. If $dA = \varnothing$, then the set of isolated points of A, $iso(A) = A \setminus dA = A \neq \varnothing$, which means X is scattered. If $dA \neq \varnothing$, take a point $x \in dA$. Note that because $d^\star A = dA$, the third property of $d^\star$, namely $d^\star A = d^\star(A \setminus d^\star A)$, is equivalent to $dA = d(A \setminus dA)$. This means that $x \in d(A \setminus dA)$. By definition of d, $\exists y \in A \setminus dA$, so $y \in iso(A)$. Hence, every nonempty subset of X has an isolated point, which means that (X, τ) is a scattered topological space.

■

6 Interpretation

The topology we have seen so far in Sections 4 and 5 has been rather disconnected from the logic and algebras we discussed earlier. In this section, we will make the link explicit, and discuss the significance of the results proven in Section 5.

6.1 Topological Semantics

A topological model consists of a topological space (X, τ) and a valuation v which sends propositional variables to subsets of X. A sentence p is said to be true at a point $x \in X$ if and only if $x \in v(p)$. We say that p is valid in (X, τ) if p is true in every model based on (X, τ). More on truth and validity in topological models can be found in Chapter 5 of [22].

Naturally, propositional connectives $\wedge, \vee, \rightarrow$ correspond to the set-theoretic operations $\cup, \cap, \subseteq$. Negation corresponds to complementation with respect to X. The topological interpretation for modal operators, however, is less intuitively clear. In their 1944 work [6], McKinsey and Tarski suggested two topological interpretations for the modal operator $\Diamond$: one as set closure of a topological space, and one as the derived set operator. In the same paper they showed that under the interpretation of $\Diamond$ as the set closure operator, the logic $S4$ is the modal logic of all topological spaces. The $\Box$ in this case is interpreted as the interior.

Let us briefly demonstrate some relations to affirm this interpretation. Firstly, the modal logic $S4$ results from adding the reflexivity axiom ($\Box A \rightarrow A$) and the axiom commonly known as "4" ($\Box A \rightarrow \Box\Box A$) to the modal logic K [11]. Next, recall that from the definition of set closure (Definition 4.16) we know that $A \subseteq \overline{A}$. The interior of a set $A \subseteq X$, denoted by A°, is defined as the union of all open sets of X contained in A, so $A^\circ \subseteq A$. Now notice that:

$\overline{A} = X \setminus ((X \setminus A)^\circ)$	corresponds to	$\Diamond A \Leftrightarrow \neg\Box\neg A$
$A^\circ = X \setminus \overline{(X \setminus A)}$	corresponds to	$\Box A \Leftrightarrow \neg \Diamond \neg A$
$A \subseteq \overline{A}$	corresponds to	$A \rightarrow \Diamond A$
$A^\circ \subseteq A$	corresponds to	$\Box A \rightarrow A$
$\overline{\overline{A}} = \overline{A}$	corresponds to	$\Diamond \Diamond A \rightarrow \Diamond A$
$A^\circ = (A^\circ)^\circ$	corresponds to	$\Box A \rightarrow \Box\Box A$,

where the expressions on the left are simple truths that can be found in most textbooks on topology, for example [21] or [23].

Naturally, the other axioms of $S4$ (namely those that come from K) have their appropriate translations to this language as well. More details can be found in [6]. However, it should be noted that McKinsey and Tarski were not explicitly discussing the logic $S4$, but rather the algebraic interpretation of $S4$, that is, closure algebras. Esakia also introduced closure algebras in [7], but we did not treat them in this work because they are not directly relevant to provability logic.

As for the interpretation of the $\Diamond$ operator as the derived set operator, H. Simmons and L. Esakia (independently) worked out that the derived set operator acting on a scattered topological space satisfies all the identities of Magari algebras, making the logic GL the modal logic of all scattered topological spaces. This is precisely the result we treated in Section 5 of this work.

To show this explicitly, first recall that Magari algebras are the algebraic interpretation for the logic GL (see Section 3). Then observe the correspondence between the conditions on the Magari algebra operator δ and the operator $d^\star$ from Theorems 5.1 and 5.2:

$\delta\varnothing = \varnothing$	corresponds to	$d^\star(\varnothing) = \varnothing$
$\delta(A \cup B) = \delta A \cup \delta B$	corresponds to	$d^\star(A \cup B) = d^\star(A) \cup d^\star(B)$
$\delta A = \delta(A \setminus \delta A)$	corresponds to	$d^\star(A) = d^\star(A \setminus d^\star(A))$

Hence, the operator δ of a Magari algebra $(\mathcal{P}(X), \delta)$ and the operator $d^\star$ are one and the same. Furthermore, in Theorems 5.1 and 5.2, we showed that in a scattered topological space X, $d^\star A = dA$ for all $A \subseteq X$.

Remark. Upon close inspection one might notice that the last expression for δ in the list above is an equality, even though originally, in Definition 3.4, it was a set inclusion. This is because the other inclusion, $\delta A \supseteq \delta(A \setminus \delta A)$, follows from monotonicity of δ, which in turn follows from distribution over union. This proof is analogous to how we proved $d^\star(A) \supseteq d^\star(A \setminus d^\star(A))$ in Theorem 5.1.

Remark. If one chooses, as we did, to include $\delta\delta A \subseteq \delta A$ into the definition of Magari algebras, notice that $\delta\delta A \subseteq \delta A$ corresponds to $ddA \subseteq dA$, which we showed in Proposition 4.13.

Therefore, at last, we can conclude that under the interpretation of the diamond modal operator as the topological derived set operator, the logic GL is the modal logic of all scattered topological spaces. To summarizing the various characterizations, we have the following Corollary.

Corollary 6.1 (Esakia)**.** The following statements are equivalent:

(1) $\vdash_{\mathrm{GL}} p$;

(2) p is valid in all Magari algebras;

(3) p is valid in every scattered topological space.

6.2 Significance

Prior to the development of topological semantics for provability logic it has already been shown by K. Segerberg in 1971 that GL is sound and complete with respect to the class of transitive, converse well-founded frames [2]. Many modal logics, such as K, $K4$ (which is defined by K with the addition of axiom 4), S4 and others have been shown to be strongly complete, however, for GL strong completeness does not hold (see [1] for details). It follows from Esakia's work that GL (in fact, any modal logic of a class of strict partial orders [10]) is complete with respect to the topological derivative semantics. Furthermore, it has been shown by Abashidze in [24] in 1985, and independently by Blass in [25] in 1990, that GL is also complete with respect to any ordinal $\geq \omega^\omega$. More details and proofs can be found in [10].

In 1986 G. Japaridze introduced an extension to the language of GL in the form of polymodal provability logic GLP [5]. It is a language with infinitely many diamond-like modalities $\langle 0 \rangle$, $\langle 1 \rangle$, ..., and [0], [1], ..., which are their box-like duals. In the same work he proved that GLP is arithmetically complete. This proof was later simplified by L. Beklemishev in [26]. Unlike GL, GLP was shown to be incomplete with respect to its possible worlds semantics, meaning that it does not correspond to any class of frames. However, the topological approach to GL developed in [7] has allowed to provide GLP with an adequate topological semantics. The main idea is that individually each modality $\langle i \rangle$ of GLP behaves like the $\Diamond$ modal operators of GL, and can therefore be interpreted as the derived set operator of a polytopological space. When these spaces satisfy the axioms of GLP they are called GLP-spaces, and in [27] the logic GLP was shown to be complete with respect to its topological semantics of GLP-spaces.

While the logic GL is interesting in its own right and has had important applications in the foundations of mathematics, its extension GLP has found a place in more recent applications of provability algebras to proof theory and ordinal analysis of arithmetic (see [28]).

In 2001, in [29], Esakia introduced the logic $wK4$, which is a weakening of the system $K4$, and showed that under the interpretation of the modal diamond operator as the derived set operator of a topological space, the logic $wK4$ is the modal logic of all topological spaces. Moreover, he showed that the logic $K4$ is the modal logic of all T-d spaces (spaces satisfying the T_d-separation axiom, asserting that each point is locally closed). An English interpretation of the result as well as further developments can be found in [30] and Chapter 5 of [22].

Lastly, Esakia's result in [7] has also found reverberations in recent studies of other logics: epistemic (concerning reasoning about knowledge), doxastic (reasoning about belief), and intuitionistic (non-modal; does not assume law of excluded middle) [22,31,32]. Furthermore, the topic has become a long-term project for the Georgian school of logic, which was founded and led by Esakia [10, 32].

7 Conclusion

In this thesis we have studied the topological approach to provability logic. We have created an accessible guide to the ideas on this subject developed by Leo Esakia in [7], building up from the very basics of both modal logic and topology. We have discussed various aspects of the logic GL, including its applications, semantics, and algebraic interpretation. Then we shifted our focus to a seemingly unrelated field of mathematics, topology, and showed that the $\Diamond$ operator of GL has all the basic properties of the derived set operator acting on a scattered topological space. This gave us the desired topological semantics of GL, and its completeness with respect to scattered topological spaces.

Altogether, we have built a solid foundation to enable further research. This can be taken in several directions. For example, as discussed in Section 6.2 about the significance of the proven result, there have been recent inquiries extending the result to other logical systems. These can be extensions of GL, extensions of K, or even non-normal logics (those that give up the Distribution Axiom of K).

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