university of groningen / faculty of science and engineering / mathematics and applied mathematics

Brauer groups of fields and quaternion algebras

Bachelor's Project Mathematics

May 2021

Student: Nefeli Stratou

First supervisor: Dr. F. Bianchi

Second assessor: Dr. J.S. Müller

BRAUER GROUPS OF FIELDS AND QUATERNION ALGEBRAS

NEFELI STRATOU S3426041

Table of Contents

1. Introduction

Abstract algebra is a field of mathematics which, among other things, is concerned with the classification of some mathematical structures. Looking for solutions of polynomials over a field is natural and it can lead to exploring and classifying field extensions. This is the subject of Galois theory, which translates the problem of studying a kind of field extension into that of studying a group associated with it, namely the Galois group [18]. A division algebra over a given field is, in some sense, a possibly non-commutative field extension. In a similar manner to Galois theory, an interesting group arises in the attempt of classifying finite-dimensional central division algebras over a fixed field k. This is the Brauer group, named after Richard Brauer [2].

An algebra over a field k is both a k-vector space and a ring [14, Chapter 18]. Thus, a lot of definitions that apply to rings or vector spaces make sense for algebras too. For instance, an algebra is simple if it is simple as a ring, and it is a division algebra if it is a division ring. It is finite-dimensional, if it is as a vector space. We also say that a k-algebra is central if its center is the field k.

The classes of central simple algebras over k modulo a certain equivalence relation form an abelian group, called the Brauer group, where the group law is given by the tensor product and the identity element by the class of k. One important result discussed in the thesis is that, by using Wedderburn's theorem, one establishes a bijection between the Brauer group and all of the isomorphism classes of finite-dimensional central division algebras [16, Theorem 9.129]. The Brauer group can be trivial, such as the Brauer group of an algebraically closed field.

Quaternion algebras are a specific example of finite-dimensional central simple k-algebras 4.3. Hence, we are going to study their classes in the Brauer group.

Finally, we make some explicit computations with elements of order two, and, in particular, with classes of quaternion algebras. In contrast to algebraically closed fields, the Brauer group of $\mathbb{Q}$ is very complicated. That is the reason we restrict to $Br(\mathbb{Q})[2]$. Classes of quaternion algebras are elements of this subgroup. In fact they generate this subgroup, but that's also hard. So instead, we content ourselves with explicit computations with classes of quaternions.

2. Background on Algebras

All rings used in this project have a unit element. An algebra is a combination of a ring and a vector space. [14, Chapter 18] Both of these definitions are well known from the Algebra courses. [18] In order to discuss the definition and some properties of an algebra over a field, we need first to review some definitions from algebraic structures.

Definition 2.1. [18, Definition 1.1.1] A **division ring** is a nonzero ring R with the additional property that for every nonzero element of the ring R, $a \in R$, there exists the inverse element $a^{-1} \in R$, i.e a non-zero element satisfying $aa^{-1} = a^{-1}a = 1$.

Example 2.2. Division Rings: $\mathbb{R}, \mathbb{C}, \mathbb{Q}$, or more generally any field, the Hamilton quaternions $\mathbb{H}_{\mathbb{R}}$ which will be discussed later in Example 3.19.

Example 2.3. Non Division Rings: The matrix ring $M_2(\mathbb{Q})$.
It suffices to take a nonzero element that does not have an inverse.
For

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{Q}),$$

there is not an inverse matrix, since the determinant is equal to 0.
Therefore, the matrix ring $M_2(\mathbb{Q})$ is not a division ring.

Definition 2.4. [18, Definition 1.1.1] A **field** is a division ring with the property that multiplication is commutative.

2.1. Algebras.

Definition 2.5. [14, Chapter 18] Let k be a field. An algebra A over k is a nonempty set A, together with three operations, **addition** (denoted by +), **multiplication** (denoted by juxtaposition) and **scalar multiplication** (also denoted by juxtaposition) for which the following properties hold:

1. A is a vector space over k under addition and scalar multiplication
2. A is a ring under addition and multiplication
3. If $r \in k$ and $a, b \in A$ then $r(ab) = (ra)b = a(rb) = (ab)r$

An algebra A over a field k is also called a k-algebra.

Example 2.6. Some examples of k-algebras are:

1. the field k itself
2. any field containing k
3. the Hamilton quaternions $\mathbb{H}_{\mathbb{R}}$, if $k = \mathbb{R}$
4. the $n \times n$ matrices $M_n(k)$

As we have seen with both rings and vector spaces, subrings and subspaces can accordingly be defined as well. Similarly, it is natural to consider subsets of an algebra that are algebras themselves.

Definition 2.7. [14, Chapter 18] Let A be a k-algebra. A subalgebra of A is a subset of A that is a subring of A, having the same identity as A, and a subspace of A.

Definition 2.8. [14, Chapter 18] If we give the k-algebra A the reverse multiplication law

$$(a, b) \to (a \circ b) = ba$$

we get a k-algebra, denoted by A^o , and called the opposite k-algebra of A.

Recall that an algebra is a ring with some additional structure. We may wonder if, considered as a ring, it is a division ring.

Definition 2.9. A nonzero algebra A is called a **division algebra** if every nonzero element in A is invertible in A.

Definition 2.10. [14, Chapter 18] The **center of a k-algebra** A is the subset consisting of all the elements that commute with every other element, i.e.

$$Z(A) = \{a \in A\,,\, ax = xa \mid \forall x \in A\}.$$

Remark 2.11. [14, Chapter 18] The center of an algebra A is never trivial since it always contains the field k that A is defined over: $k \subset Z(A)$. By this we mean that the image of k in A under

$$x \mapsto xA_1$$

is always in $Z(A)$. Indeed, $\forall a \in A$ and $x \in k$ we have that

$$(x1_A)a = x(1_A a) = x(a1_A) = (xa)1_A = a(x1_A).$$

Lemma 2.12. The center of an algebra A is a subalgebra of A.

Proof. In the course of algebraic structures [18, Exercise I.4.12] we proved that the center of a ring is a subring so it suffices to show that it is a subspace.

In fact, since we already know that $Z(A)$ is a subring, we only need to check that it is closed under scalar multiplication.
Let $r \in k$ and $a \in Z(A)$. Then for any $x \in A$ we have:

$$(ra)x = r(ax) = r(xa) = x(ra).$$

So $ra \in Z(A)$ and hence $Z(A)$ is a subalgebra of A. □

Definition 2.13. [14, Chapter 18]: Let k be a field and A an algebra over k. Then A is central if its center is just k in the sense of Remark 2.11.

We now recall the definition of a right, left and a two-sided ideal of a ring from the course ”Algebraic Structures”. However, in the course we often encountered commutative rings so the distinction among the three definitions was often lost.

Definition 2.14. [18, Definition II.2.2] A **right ideal I of a ring** R is a subgroup of the additive group of R. Moreover,

$$ar \in I, \qquad \text{for all } r \in R,\ a \in I.$$

Definition 2.15. [18, Definition II.2.2] A **left ideal I of a ring** R is a subgroup of the additive group of R, with the additional property:

$$ra \in I, \qquad \text{for all } r \in R,\ a \in I.$$

Definition 2.16. [18, Definition II.2.2] A **two-sided ideal I of a ring** R is both a right and a left ideal, closed under right and left multiplication:

$$ar, ra \in I, \qquad \text{for all } r \in R,\ a \in I.$$

Definition 2.17. [14, Chapter 18] A k-Algebra A is simple if its only two-sided ideals are $\{0\}$ and A.

Example 2.18. The k-algebra $M_n(k)$ is simple, as we will see in Example 4.1.

Proposition 2.19. Every division k-algebra is simple.

Proof. In a division k-algebra A every nonzero element is a unit. Thus, if I is a non-zero two-sided ideal and $q \in I$, we have $q^{-1} \in A$ and hence

$$q \cdot q^{-1} = 1_A \in I,$$

but this implies

$$a \cdot 1_A = a \in A, \forall a \in A.$$

This shows that I is equal to A. □

Definition 2.20. Dimension of a k-algebra: The dimension of a k-algebra is its dimension as a k-vector space.

Assumption 2.21. Every algebra from now on will be assumed to be finite-dimensional.

Example 2.22. Central Simple k-algebras

(1) The $n \times n$ matrices over k: $M_n(k)$ over k (see Example 4.1)
(2) Central division algebras over k (see Example 4.2)
(3) Quaternion algebras over k (see Example 4.3)

Algebras over a fixed field k are k-vector spaces, hence we can define the tensor product of two such algebras, as we normally do for more general modules over a ring:

Definition 2.23. [17, VII.3.1 Definition] If R is a ring and M, N are R-modules, then a tensor product of M and N is a pair (T, β) in which T is an R-module and $\beta : M \times N \to T$ is a bilinear map, such that the following holds: given any R-bilinear map $b : M \times N \to S$ for some R-module S, there exists a unique R-module homomorphism $f : T \to S$ such that $b = (f \circ \beta)$.

Theorem 2.24. Existence and Uniqueness theorem of tensor products[17, VII.3.4 Theorem]: For given M and N there always exists a tensor product (T, β). It is unique up to isomorphism and we denote it by $M \otimes_R N =: T$ where β is denoted by $\otimes$.

Elements in $M \otimes_R N$, that have the form $m \otimes n$ for some $m \in M$ and $n \in N$ are called **elementary tensors**. In the case when R is a field k and M and N are algebras over k, we may give the tensor product of M and N the structure of a k-algebra:

Theorem 2.25. Let A and B k-algebras. There is a unique multiplication on $A \otimes_k B$ making it a k-algebra such that

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'$$

for elementary tensors. The multiplicative identity is $1 \otimes 1$.

The proof of the theorem is given in [4, Theorem 7.1].

2.2. **Homomorphisms and Isomorphisms of Algebras.** A homomorphism is a structure-preserving map between two algebraic structures of the same type. For instance between two groups, two vector spaces, two rings or two algebras.

Definition 2.26. [14, Chapter 18]: Let A and B be k-algebras.
A map $\sigma\colon A \to B$ is a k-algebra homomorphism if and only if it is a ring homomorphism and a linear transformation. That is:

(1) $\forall\, a, a' \in A$
$$\sigma(a + a') = \sigma(a) + \sigma(a')$$
(2) $\forall\, a, a' \in A$
$$\sigma(aa') = \sigma(a)\sigma(a')$$
(3) $\sigma(1_A) = 1_B$.
(4) For $\forall\, r \in k$ and $a \in A$:
$$r\sigma(a) = \sigma(ra)$$

An **isomorphism of *k*-algebras** is a **bijective algebra homomorphism**. A k-endomorphisms is a homomorphism from a k-algebra to itself.

3. Definition and properties of Quaternion Algebras

A brief introduction to quaternion algebras [18] was given in the Algebraic Structures course but here they will be analysed in more depth. In this chapter k denotes a field of characteristic different than 2.

3.1. Definition and Properties.

Definition 3.1. [5, Definition 3.3] A quaternion algebra over k is a 4-dimensional k-algebra with a basis $1, i, j, ij$ with the following multiplicative relations:

$$i^2 \in k^\times, j^2 \in k^\times, ij = -ji \tag{1}$$

and every $c \in k$ commutes with i and j. When

$$i^2 = a, j^2 = b$$

this ring is denoted by $(a, b)_k$.

We need to show that the above k-algebra is well-defined.

Lemma 3.2. There is a unique multiplication on $(a, b)_k$, that is associative and compatible with the multiplicative relations of (1).

Proof. The quaternion algebra $(a, b)_k$ can be written as a vector space over k:

$$(a, b)_k = k + ki + kj + kij.$$

Thus all properties of addition are inherited from the vector space structure. Multiplication is less trivial, since Definition 3.1. only specifies some multiplicative relations. We extend it to $(a, b)_k$ using associativity:

- **associativity**: There is a unique possible associative multiplication table on the basis $1, i, j, ij$ which satisfies the multiplicative relations of the quaternion k-algebra. This is given by:

×	1	i	j	ij
1	1	i	j	ij
i	i	a	ij	aj
j	j	-ij	b	-bi
ij	ij	-aj	bi	-ab

  By the multiplication table we see that $(a, b)_k$ is closed under multiplication, since any element in $(a, b)_k$ is a k-linear combination of $1, i, j, ij$. We need to check that the resulting multiplication law on $(a, b)_k$ is associative, i.e. for all $q_1, q_2, q_3 \in (a, b)_k$

$$(q_1 q_2) q_3 = q_1 (q_2 q_3).$$

  Actually it is enough to check that associativity holds for all triples q_1, q_2, q_3 of basis elements. We verify this in a few cases; the proof in the remaining cases is similar.

  (1)
$$(ii)i = ai = ia = i(ii)$$
  (2)
$$(ii)j = aj = i(ij)$$
  (3)
$$(ii)(ij) = aij = iaj = (i)(iij)$$

(4)
$$(ij)j = bi = ib = i(jj)$$

...

- **distributivity**: We need to show that
$$q_1(q_2 + q_3) = q_1q_2 + q_1q_3$$
and
$$(q_1 + q_2)q_3 = q_1q_3 + q_2q_3$$
This follows from the fact that multiplication and addition in k satisfy distributivity.

Therefore, the quaternion k-algebra has a unique k-algebra structure. □

Lemma 3.3. Any k-algebra generated as a k-vector space by symbols $1, i, j, ij$ satisfying the multiplicative relations of Definition 3.1 has dimension 4.

Proof. It suffices to show that $1, i, j, ij$ are linearly independent.
We start by showing that 1 is not in the k-span of i, denoted $\langle i \rangle_k$. Suppose $1 \in \langle i \rangle_k$, then this would imply

$$1 = \alpha \cdot i, \text{ for some } \alpha \neq 0 \tag{2}$$

We can multiply (2) on the right by j:
$$1 \cdot j = \alpha \cdot i \cdot j$$
We can multiply (2) on the left by j:
$$j \cdot 1 = \alpha \cdot j \cdot i$$
so we can equate them:
$$\alpha \cdot i \cdot j = \alpha \cdot j \cdot i;$$
since $\alpha \neq 0$, this implies that
$$i \cdot j = j \cdot i \Rightarrow 2ij = 0$$

But this is a contradiction since $char(k) \neq 2$ and $ij \neq 0$ since $(ij)^2 \in k^\times$. Therefore, 1 is not in the k-span of i. By a similar argument, 1 is not in the k-span of j. So 1 and i are linearly independent as well as 1 and j.
Now we need to check whether j is linearly independent with i and 1. We proceed in a similar manner as before:
Assuming $j \in \langle 1, i \rangle_k$ then j can be written as:

$$j = \alpha + \beta i, \tag{3}$$

for some $\beta \neq 0$. Right multiplying (3) with i we get:
$$ji = \alpha i + \beta i^2$$
left multiplying (3) with i we get:
$$ij = \alpha i + \beta i^2$$
We observe that the right hand side of both equations is the same so we can equate them:
$$ij = ji \Rightarrow 2ij = 0$$
Which by the same argument as before, is a contradiction so j is not in $\langle 1, i \rangle_k$, therefore, $1, i, j$ are linearly independent. We only need to check whether ij is in

$\langle 1, i, j\rangle_k$.
Suppose $ij \in \langle 1, i, j\rangle_k$, then we can write it as:

$$ij = \alpha + \beta i + \gamma j, \tag{4}$$

for some $\alpha, \beta, \gamma \in k$. Multiplying (4) on the left by i we get:

$$aj = i^2 j = \alpha i + \beta i^2 + \gamma ij \Rightarrow aj - \gamma ij = \alpha i + \beta a$$

Multiplying (4) on the right by i we have:

$$iji = \alpha i + \beta i^2 + \gamma ji \Rightarrow -aj + \gamma ij = \alpha i + \beta a$$

This mean we can write:

$$aj - \gamma ij = -aj + \gamma ij \Rightarrow$$
$$2aj = 2\gamma ij$$

Right-multiplying with j:

$$2ab = 2\gamma bi \Rightarrow ab = \gamma bi$$

Which is a contradiction since 1 and i are linearly independent and $a, b \neq 0$. Therefore, ij is not in $\langle 1, i, j\rangle_k$. So $1, i, j, ij$ are linearly independent. □

Lemma 3.4. Suppose A is a 4-dimensional k-algebra admitting a basis v_1, v_2, v_3, v_4 with the following properties:

(1) $v_4 = v_2 v_3$
(2) $v_1 = 1_A$, where 1_A is the multiplicative identity of A.
(3) $v_2^2 = a \cdot 1_A$
$v_3^2 = b \cdot 1_A$
(4) $v_2 v_3 = -v_3 v_2$

Then A is isomorphic to $(a, b)_k$.

Proof. From $(1) - (4)$, we see that the basis $(v_1, ..., v_4)$ satisfies the conditions of the defining basis $(1, i, j, ij)$ of $(a, b)_k$. Then, we see that A satisfies the definition of $(a, b)_k$ algebra, so A and $(a, b)_k$ are isomorphic as k-algebras. □

Example 3.5. An example of a quaternion algebra is $(-1, -1)_{\mathbb{R}}$. This is denoted by $\mathbb{H}_{\mathbb{R}}$ and it is known as the $\mathbb{R}$- algebra of Hamilton quaternions. In this case we have that:
$i^2 = -1$, $j^2 = -1$, $ij = -ji$.
We can compute $(ij)^2 = ijij = -iijj = -1$.

Proposition 3.6. [5, Theorem 4.3] For $b \in k^\times$, $(b, 1)_k \simeq M_2(k)$.

Proof. $M_2(k)$ is a 4-dimensional k-algebra admitting a basis

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, v_4 = \begin{bmatrix} 0 & -1 \\ b & 0 \end{bmatrix}.$$

To verify the above form a basis it is enough to show that the elements are linearly independent. For each $i \in \{0, 1, 2, 3\}$ let $y_i \in k$. Then :

$$y_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y_1 \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix} + y_2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + y_3 \begin{bmatrix} 0 & -1 \\ b & 0 \end{bmatrix} =$$

$$\begin{bmatrix} y_0 & 0 \\ 0 & y_0 \end{bmatrix} + \begin{bmatrix} 0 & y_1 \\ by_1 & 0 \end{bmatrix} + \begin{bmatrix} y_2 & 0 \\ 0 & -y_2 \end{bmatrix} + \begin{bmatrix} 0 & -y_3 \\ by_3 & 0 \end{bmatrix} = \begin{bmatrix} y_0 + y_2 & y_1 - y_3 \\ b(y_1 + y_3) & y_0 - y_2 \end{bmatrix} = 0_2$$

$\iff y_0 = -y_2$ and $y_1 = y_3$ and $y_1 = -y_3$ and $y_0 = y_2 \Rightarrow y_0 = y_2 = y_3 = y_4 = 0.$
So it is indeed the case.
Let's check if the basis elements satisfy the properties of the basis $1, i, j, ij$ of $(b, 1)_k$.

(1)
$$v_2 v_3 = \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ b & 0 \end{bmatrix} = v_4$$

(2)
$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{M_2(k)}.$$

(3)
$$v_2^2 = \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix} = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = b \cdot I_{M_2(k)}.$$

(4)
$$v_3^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \cdot I_{M_2(k)}.$$

(5)
$$-v_3 v_2 = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ -b & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ b & 0 \end{bmatrix} = v_2 v_3$$

Thus by Lemma 3.4
$$(b, 1)_k \simeq M_2(k).$$

□

Remark 3.7. In this section we assumed that the field k is of characteristic different from 2. The reason for this is that in characteristic 2 we have

$$2ij = 0 \Rightarrow ij = -ij$$

thus, if $ij = -ji$ as well then $ji = ij$. In particular, the multiplicative relations of Definition 3.1 would extend to a commutative multiplication if $char(k) = 2$. Then the center is the entire quaternion algebra, since all of its elements will commute with every element of the algebra. So they would not be central algebras, unless $(a, b)_k = k$, and in this project we are interested in analysing the Brauer Group of a field through central simple algebras. Moreover, it is not possible to have a non-commutative quaternion algebra of dimension 4 and characteristic 2.

From the definition we can deduce some properties of quaternion algebras.

Lemma 3.8.
(1) [3, Exercise 10.2.5] There is a symmetry relation in Quaternion algebras: $(a, b)_k \simeq (b, a)_k$.
(2) [3, Exercise 10.2.5] $(a, b)_k \simeq (ac^2, b)_k$ where $a, b, c \in k^\times$.

Proof. (1) $(b,a)_k$ is a 4-dimensional k-algebra with a basis $\{1, i', j', i'j'\}$ such that

$$i'^2 = b \cdot 1_{(b,a)_k},\ j'^2 = a \cdot 1_{(b,a)_k},\ i'j' = -j'i'.$$

Let's find a basis such that the properties of the standard basis of $(a,b)_k$ are satisfied. Pick

$$v_1 = 1_{(b,a)_k}, v_2 = j', v_3 = i', v_4 = i'j'.$$

Then

(1)
$$v_4 = i'j' = v_2v_3$$

(2)
$$v_1 = 1_{(b,a)_k}$$

(3)
$$v_2^2 = j'^2 = a \cdot 1_{(b,a)_k}$$
$$v_3^2 = i'^2 = b \cdot 1_{(b,a)_k}$$

(4)
$$v_2v_3 = i'j' = -j'i' = -v_3v_2$$

Thus by Lemma 3.4

$$(a,b)_k \simeq (b,a)_k.$$

(2) $(ac^2, b)_k$ is a 4-dimensional k-algebra with basis $\{1, i', j', i'j'\}$ such that

$$(i')^2 = ac^2 \cdot 1_{(ac^2,b)_k},\ (j')^2 = b \cdot 1_{(ac^2,b)_k}, i'j' = -j'i'.$$

Let's find a basis such that the properties of $(a,b)_k$ are satisfied. Pick

$$v_1 = 1_{(ac^2,b)_k}, v_2 = c^{-1}i', v_3 = j', v_4 = c^{-1}i'j'.$$

(1)
$$v_4 = c^{-1}i'j' = v_2v_3$$

(2)
$$v_1 = 1_{(ac^2,b)_k}$$

(3)
$$v_2^2 = i'^2(c^2)^{-1} = ac^2(c^2)^{-1} \cdot 1_{(ac^2,b)_k} = a \cdot 1_{(ac^2,b)_k}$$
$$v_3^2 = j'^2 = b \cdot 1_{(ac^2,b)_k}$$

(4)
$$v_2v_3 = i'j' = -j'i' = -v_3v_2$$

Thus by Lemma 3.4: $(ac^2, b)_k \simeq (a,b)_k$. □

3.2. **Division and split quaternion algebras.** In this section it will be shown that a quaternion algebra over k is either a division algebra or it is isomorphic to $M_2(k)$. We start with some preliminary definitions.
Given $q \in (a,b)_k$,

$$q = \alpha + \beta i + \gamma j + \delta ij.$$

we define its conjugate as

$$\bar{q} = \alpha - \beta i - \gamma j - \delta ij$$

and its norm by

$$N(q) = q\bar{q} = \alpha^2 - \beta^2 a - \gamma^2 b + ab\delta^2 = \bar{q}q.$$

We need to show that the norm is multiplicative, meaning that for all $c, d \in (a,b)_k$

$$N(cd) = N(c)N(d)$$

So let

$$c = \alpha_1 + \beta_1 i + \gamma_1 j + \delta_1 ij, \qquad d = \alpha_2 + \beta_2 i + \gamma_2 j + \delta_2 ij.$$

then

$$\bar{c} = \alpha_1 - \beta_1 i - \gamma_1 j - \delta_1 ij, \qquad \bar{d} = \alpha_2 - \beta_2 i - \gamma_2 j - \delta_2 ij.$$

Therefore,

$$\begin{aligned} cd = \alpha_1\alpha_2 + \beta_1\beta_2 a + \gamma_1\gamma_2 b - \delta_1\delta_2 ab + (\alpha_1\beta_2 + \alpha_2\beta_1 - \gamma_1\delta_2 b + \delta_1\gamma_2 b)i \\ +(\alpha_1\gamma_2 + \beta_1\delta_2 a + \alpha_2\gamma_1 - \delta_1\beta_2 a)j + (\alpha_1\delta_2 + \beta_1\gamma_2 - \gamma_1\beta_2 + \alpha_2\delta_1)ij \end{aligned}$$

$$\begin{aligned} \Rightarrow \overline{cd} = \alpha_1\alpha_2 + \beta_1\beta_2 a + \gamma_1\gamma_2 b - \delta_1\delta_2 ab - (\alpha_1\beta_2 + \alpha_2\beta_1 - \gamma_1\delta_2 b + \delta_1\gamma_2 b)i \\ -(\alpha_1\gamma_2 + \beta_1\delta_2 a + \alpha_2\gamma_1 - \delta_1\beta_2 a)j - (\alpha_1\delta_2 + \beta_1\gamma_2 - \gamma_1\beta_2 + \alpha_2\delta_1)ij. \end{aligned}$$

Furthermore,

$$\begin{aligned} \bar{d}\bar{c} = (\alpha_2 - \beta_2 i - \gamma_2 j - \delta_2 ij)(\alpha_1 - \beta_1 i - \gamma_1 j - \delta_1 ij) \\ = \alpha_1\alpha_2 - \alpha_2\beta_1 i - \alpha_2\gamma_1 j - \alpha_2\delta_1 ij - \alpha_1\beta_2 i + \beta_1\beta_2 a + \beta_2\gamma_1 ij + \beta_2\delta_1 aj - \alpha_1\gamma_2 j \\ -\gamma_2\beta_1 ij + \gamma_2\gamma_1 b - \gamma_2\delta_1 bi - \alpha_1\delta_2 ij - \delta_2\beta_1 aj + \delta_2\gamma_1 bi - \delta_2\delta_1 ab \\ = \overline{cd} \end{aligned}$$

Since

$$\bar{q}q = q\bar{q}, \overline{cd} = \bar{d}\bar{c}$$

then

$$\begin{aligned} N(cd) &= cd\overline{cd} \\ &= cd\bar{d}\bar{c} \\ &= cN(d)\bar{c} \\ &= c\bar{c}N(d) \\ &= N(c)N(d). \end{aligned}$$

Note that $N(0) = 0$.
Therefore, we can conclude that the norm is multiplicative.

Lemma 3.9. [18, I.1.5 Example] Suppose $N(q) \neq 0$, then q is invertible and $q^{-1} = \frac{\bar{q}}{N(q)}$.

Proof. Assumming that $N(q) \neq 0$,define

$$q^{-1} = \frac{\bar{q}}{N(q)}$$

Then

$$qq^{-1} = q\frac{\bar{q}}{N(q)} = \frac{N(q)}{N(q)} = 1 = \frac{\bar{q}q}{N(q)} = q^{-1}q$$

So, q^{-1} is indeed the inverse element of q. □

The norm gives us a first criterion to establish whether a quaternion algebra is a division algebra.

Proposition 3.10. [18, I.1.5 Example] A quaternion algebra over a field k is a division algebra if and only if the norm $N : (a, b)_k \to k$ does not vanish outside 0.

Proof. Assume that $(a, b)_k$ is a division algebra. Let $q = \alpha + \beta i + \gamma j + \delta ij \in (a, b)_k$ nonzero, then there exists an inverse element such that

$$q \cdot q^{-1} = 1.$$

Assume that

$$N(q) = 0 = q \cdot \bar{q},$$

left-multiplying by q^{-1} we obtain

$$\bar{q} = q^{-1}q \cdot \bar{q} = q^{-1} \cdot N(q) = 0,$$

which occurs if and only if $\alpha, \beta, \gamma, \delta = 0$. Therefore $q = 0$. So, we get a contradiction. Thus, if $(a, b)_k$ a division algebra the norm does not vanish outside 0.
Conversely for $N(q) \neq 0$, we apply Lemma 3.9, and then the quaternion algebra is a division algebra over k.

□

Definition 3.11. [9, Definition 1.1.6] A quaternion algebra over k is **split** if it is isomorphic to $M_2(k)$ as a k-algebra.

Theorem 3.12. [9, Proposition 1.1.7] If $(a, b)_k$ is split then it is not a division algebra.

Proof. If $(a, b)_k$ is split then $(a, b)_k \simeq M_2(k)$ as k-algebra. $M_2(k)$ is not a division algebra since not every nonzero matrix has an inverse element. For example

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

is a nonzero matrix of $M_2(k)$ but it is not invertible. Hence a split $(a, b)_k$ is not a division algebra. □

To distinguish between split and division quaternion algebras, we will use another norm, the norm of the field extension $k(\sqrt{a})|k$. The following discussion applies for a not being a square in k, otherwise $k(\sqrt{a}) = k$ and the norm of the extension k/k is the identity.

Definition 3.13. The norm of $k(\sqrt{a})$ for an element

$$\alpha = x + \sqrt{a}y, \text{ where } a \text{ is not a square}$$

is equal to

$$N_{k(\sqrt{a})|k}(\alpha) = (x - \sqrt{a}y)(x + \sqrt{a}y) = x^2 - ay^2.$$

Lemma 3.14. Properties of the norm:

(1) The norm is multiplicative.

(2) If b is the norm of an element of $k(\sqrt{a})$, then b^{-1} also is.

(3)

$$N_{k(\sqrt{a})|k}(b) = b^n, \ \forall b \in k,$$

where $[k(\sqrt{a}) : k] = n$.

Proof. Statement (1) is straightforward to check and (3) follows by definition. For (2): If a is a square in k, then trivial. Else, by (1) we know that the norm is multiplicative, therefore: if $b = x^2 - ay^2$ for some $x, y \in k$ then

$$bb^{-1} = 1 = (x^2 - ay^2)b^{-1} \Rightarrow$$

$$b^{-1} = \frac{1}{x^2 - ay^2} = \frac{x^2 - ay^2}{(x^2 - ay^2)^2} = (\frac{x}{x^2 - ay^2})^2 - a(\frac{y}{x^2 - ay^2})^2.$$

[5, Lemma 4.12] □

Theorem 3.15. [9, Proposition 1.1.7] If the norm of

$$N : (a, b) \to k$$

has a nontrivial 0 then b is the norm of an element of $k(\sqrt{a})|k$.

Proof. Let

$$q = \alpha + \beta i + \gamma j + \delta ij \in (a, b)_k \backslash \{0\}$$

with $\alpha, \beta, \gamma, \delta \in k$, such that

$$N(q) = 0$$

By rewriting $N(q) = 0$, we get:

$$(\gamma^2 - a\delta^2)b = \alpha^2 - a\beta^2$$

If a is a square then it is straightforward that b is the norm of an element of k. But if a is not then we get that

$$\gamma^2 - a\delta^2 = (\gamma - \sqrt{a}\delta)(\gamma + \sqrt{a}\delta) \neq 0$$

So, we can divide by it and get:

$$b = \frac{\alpha^2 - a\beta^2}{\gamma^2 - a\delta^2}$$

Since the field norm is multiplicative:

$$b = N_{k(\sqrt{a})|k}(\alpha + \sqrt{a}\beta)N_{k(\sqrt{a})|k}(\gamma + \sqrt{a}\delta)^{-1} = N_{k\sqrt{a})|k}\left(\frac{\alpha + \sqrt{a}\beta}{\gamma + \sqrt{a}\delta}\right)$$

□

Theorem 3.16. [5, Theorem 4.16.]

If b is a norm of an element of $k(\sqrt{a})$ then $(a, b)_k \simeq M_2(k)$.

Proof. If a is a square in k, then the theorem follows from Lemma 3.8 (2) and Proposition 3.6. Assume now that a is not a square in k. Let's assume that b is the norm of an element $k(\sqrt{a})$. Then, by Lemma 3.14, we know that the inverse of b will also be the norm of an element of the field extension:

$$b^{-1} = (x - \sqrt{a}y)(x + \sqrt{a}y) = x^2 - ay^2, x, y \in k.$$

Choosing wisely:

$$u := xj + yij$$

Taking the square of it we obtain:

$$u^2 = (xj + yij)^2 = x^2b - aby^2 = b(x^2 - ay^2) = bb^{-1} = 1$$

Moreover,

$$ui = xji - yji^2$$
$$iu = xij + yi^2j = -xji + yji^2 = -ui.$$

This implies that there is an element:

$$v = (1 + a)i + (1 - a)ui$$

which satisfies

$$uv = (1 + a)ui + (1 - a)i = -vu$$

as well as

$$v^2 = (1 + a)^2a - (1 - a)^2a = 4a^2.$$

This means we have a basis $\{1, u, v, uv\}$ that satisfies the properties of a basis of the quaternion algebra $(1, 4a^2)_k$. Thus:

$$(a, b) \simeq (1, 4a^2) \simeq (1, a^2) \simeq M_2(k).$$

The last isomorphism follows from the symmetry property and Proposition 3.6. □

Remark 3.17. In the theorem it is the same as proving that a is a norm of $k(\sqrt{b})|k$, because of the symmetry relation $(a, b)_k \simeq (b, a)_k$

Theorem 3.18. [9, Proposition 1.1.7] The following statements are equivalent:

(1) $(a, b)_k$ is split.
(2) $(a, b)_k$ is not a division algebra.
(3) The norm of $(a, b)_k$

$$N : (a, b)_k \to k$$

has a non-trivial zero.
(4) The element b of $(a, b)_k$ is the norm of an element of the field extension $k(\sqrt{a})|k$.

Proof. (1) ⇒ (2) is Theorem 3.12. (2) ⇒ (3) is Proposition 3.10. (3) ⇒ (4) is Theorem 3.15. (4) ⇒ (1) is Theorem 3.16. □

Example 3.19. [18, after I.1.5 Example] In Example 3.5 we saw $\mathbb{H}_{\mathbb{R}}$, the $\mathbb{R}$- algebra of Hamilton quaternions. This is an example of a quaternion algebra that is a division algebra. We have that:

$$i^2 = a = -1, j^2 = b = -1$$

-1 is not a norm of the field extension $\mathbb{R}(\sqrt{-1})/\mathbb{R} = \mathbb{R}(i)/\mathbb{R} = \mathbb{C}/\mathbb{R}$. Indeed,

$$(x + iy)(x - iy) = x^2 + y^2 = -1$$

has no solutions $x, y \in \mathbb{R}$. Therefore, $\mathbb{H}_{\mathbb{R}}$ is not split. So, by applying Theorem 3.18 $\mathbb{H}_{\mathbb{R}}$ is a division algebra.

Proposition 3.20. If $a, b \in k^{\times}$ and k field of $char(k) \neq 2$, then:

(1) [3, Exercise 10.2.5] $(a^2, b)_k$ is not a division algebra.
(2) [3, Exercise 10.2.7] for any $a \in k^{\times}$ $(a, 1)_k$ is split.
(3) [3, Exercise 10.2.7] for any $a \neq 0, 1 \in k^{\times}$ $(a, 1-a)_k$ is split.

Proof. (1) To show that the algebra $(a^2, b)_k$ is not a division algebra we can use the fact that: $k(\sqrt{a^2})|k = k(a)|k = k|k$. Therefore, the norm is just the identity map:

$$N : k \to k$$

So $b = N(b)$. Thus, $(a^2, b)_k \simeq M_2(k)$ by Theorem 3.18(2) and Proposition 3.6. So, $(a^2, b)_k$ is split, meaning that $(a^2, b)_k$ is not a division algebra.

(2) $(a, 1)_k$ is split by Proposition 3.6

(3) It suffices to prove that $1 - a$ is a norm of the field extension $k(\sqrt{a})|k$. If a is a square in k, then the statement is trivial, so w.l.o.g a is not. Then,

$$N(1 - \sqrt{a}) = 1 - a.$$

Therefore, applying Theorem 3.18 we get that $(a, 1-a)_k$ is split.

□

3.3. **Quaternion Algebras and their Tensor Product.**

Lemma 3.21. [9, Lemma 1.5.1] The tensor product of the two matrix algebras $M_n(k)$ and $M_m(k)$ over k is isomorphic to the matrix algebra $M_{nm}(k)$.

Proof. We define the map:

$$\psi : M_n(k) \times M_m(k) \to M_{nm}(k)$$
$$(A, B) \to A \odot B$$

using the Kronecker product :

$$(A \odot B) := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ . & . & . \\ . & . & . \\ . & . & . \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}$$

where $A = (a_{ij})$ and $B \in M_m(k)$.
We know it is a k-bilinear map since it is k-linear in both arguments.
By the universal property of tensor products the following map is induced by ψ and it is well-defined and k-linear.

$$f : M_n(k) \otimes_k M_m(k) \mapsto M_{nm}(k)$$
$$A \otimes_k B \mapsto A \odot B$$

To show that we obtain isomorphism between these k-algebras we need to show that f is both a ring and a k-vector space isomorphism. To show bijectivity it suffices to show either injectivity or surjectivity since the dimension of the two k-algebras is equal to $(mn)^2$.

For surjectivity, we work with the following matrices. Let $l, i, j \in \mathbb{N}$ with $1 \leq i, j \leq l$, and define $E_{ij}^{(l)}$ as the $l \times l$ matrices whose entries are zero except for the (i, j) entry which is equal to 1. These form a basis for the space of $l \times l$ matrices over k. We will show that if we pick this basis for $M_{nm}(k)$, then any element in the basis is in the image of f.
Let $E_{rs}^{(n)} \in M_n(k)$and $E_{ij}^{(m)} \in M_m(k)$. Taking their tensor product we obtain that:

$$f(E_{ij}^{(n)} \otimes E_{rs}^{(m)}) = E_{ij}^{(n)} \odot E_{rs}^{(m)} = E_{(i-1)m+r,(j-1)m+s}^{(nm)}$$

Since $\{E_{(i-1)m+r,(j-1)m+s}^{(nm)}\}$ is a basis of $M_{mn}(k)$ then f is surjective. Therefore a bijective vector space homomorphism. To show is a k-algebra isomorphism we need to show that f is also a ring homomorphism:

- The identity element of
$(M_n(k) \otimes_k M_m(k))$ maps to the identity element of $M_{nm}(k)$:
$$f(I_n \otimes_k I_m) = I_n \odot I_m = I_{nm},$$
- Let
$$x = A \otimes_k B, y = A' \otimes_k B',$$
for some $A, A' \in M_n(k)$ and $B, B' \in M_m(K)$. Then,
$$f(xy) = f((A \otimes_k B)(A' \otimes_k B')) = f(AA' \otimes_k BB') = AA' \odot BB'$$
$$f(x)f(y) = f(A \otimes_k B)f(A' \otimes_k B') = (A \odot B)(A' \odot B') =$$

$$\begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix} \begin{bmatrix} a'_{11}B' & \cdots & a'_{1n}B' \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a'_{n1}B' & \cdots & a'_{nn}B' \end{bmatrix} = \begin{bmatrix} a_{11}a'_{11}BB' & \cdots & a_{1n}a'_{1n}BB' \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1}a'_{n1}BB' & \cdots & a_{nn}a'_{nn}BB' \end{bmatrix} =$$

$$(AA' \odot BB')$$

Thus,
$$f(xy) = f(x)f(y)$$
Therefore, the map is multiplicative.

Therefore, we obtain that f is k-algebra isomorphism. So, we can conclude that $M_n(k) \otimes_k M_m(k) \simeq M_{nm}(k)$. ☐

Theorem 3.22. [9, Lemma 1.5.2] Given elements $a, b, b' \in k^\times$, we have an isomorphism

$$(a, b)_k \otimes_k (a, b')_k \simeq (a, bb')_k \otimes_k M_2(k)$$

Proof. Let $1, i, j, ij$ be the standard basis for $(a, b)_k$ and $1, i', j', i'j'$ to be the standard basis for $(a, b')_k$.
We define the k-subspaces of $(a, b)_k \otimes (a, b')_k$:

$$A_1 = \langle (1 \otimes 1), (i \otimes 1), (j \otimes j'), (ij \otimes j') \rangle_k$$
$$A_2 = \langle (1 \otimes 1), (1 \otimes j'), (i \otimes i'j'), ((-b'i) \otimes i') \rangle_k$$

To show that A_1 and A_2 are subalgebras of $(a, b) \otimes (a, b')$ we need to show that they are closed under multiplication, since addition is defined component-wise

and the identity element is also in both A_1 and A_2. We show this for A_1; the proof for A_2 is very similar.
It suffices to show this for the given spanning set for A_1. Everything multiplied with the identity is trivially in A_1. For the rest:

$$(i \otimes 1)(j \otimes j') = (ij \otimes j') \in A_1.$$
$$(j \otimes j')(i \otimes 1) = -(ij \otimes j') \in A_1.$$
$$(i \otimes 1)(ij \otimes j') = a(j \otimes j') \in A_1.$$
$$(ij \otimes j')(i \otimes 1) = -a(j \otimes j') \in A_1.$$
$$(j \otimes j')(ij \otimes j') = -bb'(i \otimes 1) \in A_1.$$
$$(ij \otimes j')(j \otimes j') = bb'(i \otimes 1) \in A_1.$$
$$(i \otimes 1)(i \otimes 1) = (i^2 \otimes 1) = a(1 \otimes 1) \in A_1.$$
$$(j \otimes j')(j \otimes j') = (j^2 \otimes j'^2) = (b \otimes b') = bb'(1 \otimes 1) \in A_1.$$
$$(ij \otimes j')(ij \otimes j') = (-ab \otimes b') = -abb'(1 \otimes 1) \in A_1.$$

We now show that:

$$A_1 \simeq (a, bb') :$$

Recall that A_1 is generated as a k-vector space by

$$v_1 = (1 \otimes 1), v_2 = (i \otimes 1), v_3 = (j \otimes j'), v_4 = (ij \otimes j').$$

We have:

- $v_1 = (1 \otimes 1) = 1_{A_1}$
- $v_2^2 = (i \otimes 1)(i \otimes 1) = (i^2 \otimes 1^2) = (a \otimes 1) = a(1 \otimes 1) = a \cdot 1_{A_1}$
- $v_3^2 = (j \otimes j')(j \otimes j') = (j^2 \otimes j'^2) = (b \otimes b') = bb'(1 \otimes 1) = bb' \cdot 1_{A_1}$
- $v_2v_3 = (i \otimes 1)(j \otimes j') = (ij \otimes j') = (-ji \otimes j') = (-j \otimes j')(i \otimes 1) = -v_3v_2$
- $v_2v_3 = (i \otimes 1)(j \otimes j') = (ij \otimes j') = v_4$

Therefore, applying Lemma 3.4 we see that A_1 satisfies the properties of the quaternion algebra (a, bb'), so $A_1 \simeq (a, bb')$.
Similarly, A_2 is a 4- dimensional k-algebra and to see this it suffices to show that:

$$A_2 \simeq (b', -a^2b')$$

holds by choosing

$$v_1 = (1 \otimes 1), v_2 = (1 \otimes j'), v_3 = (i' \otimes i'j'), v_4 = ((-b'i) \otimes i')$$

- $v_1 = (1 \otimes 1) = 1_{A_2}$
- $v_2^2 = (1 \otimes j')(1 \otimes j') = (1 \otimes j'^2) = (1 \otimes b') = b'(1 \otimes 1) = b' \cdot 1_{A_2}$
- $v_3^2 = (i \otimes i'j')(i \otimes i'j') = (i^2 \otimes i'j'ij') = (a \otimes -ab') = -a^2b'(1 \otimes 1) = -a^2b' \cdot 1_{A_2}$
- $v_2v_3 = (i \otimes -i'b')$ $v_3v_2 = (i \otimes i'j')(1 \otimes j') = (i \otimes i'b') = -v_2v_3$
- $v_2v_3 = (1 \otimes j')(i \otimes i'j') = (i \otimes -i'j'^2) = (i \otimes -i'b') = -b'(i \otimes i')$
  $= (-bi \otimes i') = v_4$

Therefore, applying Lemma 3.4 we see that A_2 satisfies the properties of the quaternion algebra $(b', -a^2b')$, so

$$A_2 \simeq (b', -a^2b').$$

Applying Lemma 3.8 (ii), we obtain

$$(b', -a^2b') \simeq (b', -b').$$

In order to show that $(b', -b)_k$ is split, it suffices to show that $-b'$ is the norm of an element of $k(\sqrt{b'})$. This is the case, since if $k(\sqrt{b'}) \neq k$, then

$$N(\sqrt{b'}) = -b'.$$

This implies that :

$$A_2 \simeq (b', -b') \simeq M_2(k).$$

We need to prove that the following map:

$$\rho : A_1 \otimes_k A_2 \to (a, b) \otimes_k (a, b')$$

which is induced by the k-bilinear map

$$(x, y) \mapsto xy$$

is surjective and a ring homomorphism.
ρ is a ring homomorphism since:

- $$\rho((1 \otimes_k 1) \otimes (1 \otimes_k 1)) = (1 \otimes_k 1)(1 \otimes_k 1) = (1 \otimes_k 1).$$
- It is linear in both arguments.
- To prove that ρ is multiplicative we need to show that for every $x \in A_1$ and $y \in A_2$, we have $yx = xy$. Let's check this for some standard basis elements. For the rest the proof is similar.

$$(i \otimes 1)(1 \otimes j') = (i \otimes j') = (1 \otimes j')(i \otimes 1)$$
$$(j \otimes j')(1 \otimes j') = (j \otimes j'^2) = (1 \otimes j')(j \otimes j')$$
$$(ij \otimes j')(i \otimes i'j') = (iij \otimes ij'j') = (i \otimes i'j')(i \otimes i'j')$$
$$(j \otimes j')(i \otimes i'j') = (ij \otimes ij'j') = (i \otimes i'j')(j \otimes j')$$
$$(ij \otimes j')((-b'i) \otimes i') = (-biij \otimes i'j') = ((-b'i) \otimes i')(ij \otimes j')$$
$$\cdots$$

To show that ρ is bijective since domain and target of ρ have the same dimension, both equal 16, and the fact that ρ is k-linear using the universal property of tensor products, then it is enough to only show surjectivity.
To prove surjectivity it suffices to show that all standard basis elements of $(a, b) \otimes_k (a, b')$ lie in the image of ρ.
The basis elements of $(a, b) \otimes_k (a, b')$ are expressed by:

$$u_i \otimes v_j,$$

where

$$u_i \in \{1, i, j, ij\}, v_j \in \{1, i', j', i'j'\}.$$

To see this is the case we can start with:

$$\rho((1 \otimes 1) \otimes (A_2)) = A_2$$
$$\rho(A_1 \otimes (1 \otimes 1)) = A_1$$

which shows that 7 of 16 basis elements of $(a, b) \otimes_k (a, b')$ lie in the image of ρ. For the other 9 elements left we can explicitly find preimages under ρ:

$$\rho((i \otimes 1) \otimes (1 \otimes j')) = (i \otimes j')$$

$$\rho((j \otimes j') \otimes (1 \otimes j')) = b'(j \otimes 1) \Rightarrow \rho\left(\frac{(j \otimes j') \otimes (1 \otimes j')}{b'}\right) = (j \otimes 1)$$

$$\rho((ij \otimes j') \otimes (i \otimes i'j')) = ab'(j \otimes i') \Rightarrow \rho\left(\frac{(ij \otimes j') \otimes (i' \otimes i'j')}{ab'}\right) = (j \otimes i')$$

$$\rho((j \otimes j') \otimes (-b'i \otimes i')) = -b'(ij \otimes i'j') \Rightarrow \rho\left(-\frac{(j \otimes j') \otimes (-b'i \otimes i')}{b'}\right) = (ij \otimes i'j')$$

$$\rho((ij \otimes j') \otimes (1 \otimes j')) = b'(ij \otimes 1) \Rightarrow \rho\left(\frac{(ij \otimes j') \otimes (1 \otimes j')}{b'}\right) = (ij \otimes 1)$$

$$\rho((i \otimes 1) \otimes (-b'i \otimes i')) = -b'a(1 \otimes i') \Rightarrow \rho\left(\frac{(i \otimes 1) \otimes (-b'i \otimes i')}{-b'a}\right) = (1 \otimes i')$$

$$\rho((i \otimes 1) \otimes (i \otimes i'j')) = a(1 \otimes i'j') \Rightarrow \rho\left(\frac{(i \otimes 1) \otimes (i' \otimes i'j')}{a}\right) = (1 \otimes i'j')$$

$$\rho((ij \otimes j') \otimes (-b'i \otimes i')) = -ab'(j \otimes i'j') \Rightarrow \rho\left(\frac{(ij \otimes j') \otimes (-b'i \otimes i')}{-ab'}\right) = (j \otimes i'j')$$

$$\rho((j \otimes j') \otimes (i \otimes i'j'))) = b'(ij \otimes i') \Rightarrow \rho\left(\frac{(j \otimes j') \otimes (i' \otimes i'j')}{b'}\right) = (ij \otimes i')$$

We are allowed to divide by nonzero elements in k inside the ρ map because of linearity. Therefore, ρ is surjective. Thus, we have a k-algebra isomorphism.

□

Corollary 3.23. [9, Corollary 1.5.3] For a quaternion algebra $(a,b)_k$ the tensor product algebra $(a,b) \otimes (a,b)$ is isomorphic to the matrix algebra $M_4(k)$.

Proof. Applying Theorem 3.22, in the case where

$$b = b',$$

we get that

$$(a,b) \otimes (a,b) \simeq (a,b^2) \otimes M_2(k).$$

Applying Proposition we get 3.20,

$$(a^2,b) \simeq M_2(k),$$

Therefore:

$$(a,b) \otimes (a,b) \simeq M_2(k) \otimes M_2(k).$$

and by applying Lemma 3.21, we get:

$$(a,b) \otimes (a,b) \simeq M_2(k) \otimes M_2(k) \simeq M_4(k)$$

□

4. Central Simple Algebras over a field k

We will define the elements of the Brauer group of a field k as certain equivalence classes of central simple algebras over k. In this chapter we give some examples of central simple algebras and we show that the tensor product of two such algebras is again central and simple.

Example 4.1. [9, Example 2.1.2] We will show that $M_n(k)$ is a central simple algebra. To prove that it is simple, we need to show its only two sided ideals are $\{0\}$ and $M_n(k)$.
Take a nonzero two-sided ideal J of $M_n(k)$. We use elementary matrices as in the proof of the Lemma 3.21. The elements in $M_n(k)$ can be expressed as linear combinations of the E_{ij}, therefore, it suffices to show that E_{ij} are in $J, \forall i, j$.
Note that:

$$E_{ki}E_{ij}E_{jl} = E_{kl},$$

$\forall 1 \leq i, j, k, l \leq n$.
Therefore, it is enough to show then that $E_{ij} \in J$ for some i, j.
Take an element $M \in J$ such that in the (i, j) position it has an entry $m \neq 0$. Multiplying by left and right by matrices in $M_n(k)$, we get that:

$$E_{ii}ME_{jj} = mE_{ij} \in J,$$

since J is a two-sided ideal. Multiplying by the scalar m^{-1} we obtain:

$$m^{-1}mE_{ij} = E_{ij} \in J.$$

Therefore, J is the entire matrix k-algebra and $M_n(k)$ is simple.

The field k is embedded into $M_n(k)$ via the map

$$a \mapsto aI_n,$$

and is clearly contained in the center. We need to show that

$$Z(M_n(k)) = \{aI_n \in M_n(k) : a \in k\}.$$

Let $M = (M_{ij})$ be in the center of $M_n(k)$. For any $i, j \in \{1, ..., n\}$ we have

$$E_{ij}M = ME_{ij}. \tag{5}$$

The left hand side of (5) at the (r, s) entry can be expressed as:

$$(E_{ij}M)_{rs} = \sum_k (E_{ij})_{rk}M_{ks} = M_{js},$$

if $r = i$, otherwise it is 0.
Similarly the right hand side at the (r,s) entry is equal to:

$$(ME_{ij})_{rs} = \sum_k M_{rk}(E_{ij})_{ks} = M_{rj},$$

when $s = j$ and 0 otherwise.
Taking $r = i$ and $s = j$, we find that

$$M_{ii} = M_{jj} \text{ for all } i, j.$$

If we take $r = i$ and $s \neq j$, then we see that $M_{js} = 0$. Therefore M is a scalar multiple of E_n.

Example 4.2. By Proposition 2.19 every division algebra is simple. Therefore, central division algebras over k are simple.

Example 4.3. We now show that quaternion algebras over a field k of characteristic different than 2 are central simple k-algebras. By Subsection 3.2 we know that quaternion algebras can be either split or division algebras.

(1) Let $A = (a,b)_k$ be a division quaternion algebra over k. By Proposition 2.19, A is simple. To show that A is central, let $1, i, j, ij$ be a standard basis of A and take an element $q = \alpha + \beta i + \gamma j + \delta ij$ in the center of A.
Multiplying q on the right by i we get:
$$qi = \alpha i + \beta i^2 + \gamma ji + \delta iji = \alpha i + \beta a - \gamma ij - \delta ja$$
Multiplying q on the left with i we have:
$$iq = \alpha i + \beta a + \gamma ij + \delta ja.$$
Equating the two expressions we find that
$$-\gamma ij - \delta ja = \gamma ij + \delta ja \Rightarrow 2\gamma ij = 2\delta aj.$$
Since q is in the center, j and ij are linearly independent and a is a unit, we get that both γ and δ are 0.
Similarly, multiplying q on the left and right by j, we find that $\beta = 0$, so $q = \alpha \in k$.

(2) A split quaternion algebra is central simple by Example 4.1.

Theorem 4.4. [3, Lemma 10.2.13] The tensor product of two central simple algebras is also a central simple algebra.

Proof. This follows immediately from the Lemma 4.5 and Lemma 4.6. □

Lemma 4.5. [3, Lemma 10.2.11] Let k be a field and let A and B be simple algebras over k. If A is central over k, then the tensor product $A \otimes_k B$ is simple.

Proof. [6, Chapter 17] Let I be a nonzero two-sided ideal of $A \otimes_k B$. Any nonzero $v \in I$ can expressed as:
$$v = (a_1 \otimes_k b_1) + \cdots + (a_n \otimes_k b_n),$$
with $b_i \in B$ linearly independent and $a_i \in A$, uniquely determined. We choose a $v \in I$ such that n is minimal for all nonzero elements of I.
For $a_1 \neq 0$, Aa_1A is a nontrivial two-sided ideal and since A is simple we have that $Aa_1A = A$.
There are $c_1, ..., c_m, d_1, ..., d_m \in A$ such that
$$1 = c_1a_1d_1 + c_2a_1d_2 + ... + c_ma_1d_m.$$
It follows that
$$v' := \sum_j (c_j \otimes 1)v(d_j \otimes 1)$$
is of the form
$$v' = (1 \otimes_k b_1) + (a'_2 \otimes b_2) + \cdots + (a'_n \otimes_k b_n) \in I.$$

We claim that $n = 1$.
To show this, it suffices to prove that $a'_i \in Z(A)$, since $Z(A) = k$.
Note that if all $a'_i \in k\backslash\{0\}$, then

$$(a'_i \otimes b_i) = (1 \otimes a'_i b_i).$$

Setting $b'_i = a'_i b_i$, then we can write

$$\begin{aligned} v' &= (1 \otimes b'_1) + \cdots + (1 \otimes b'_n) \\ &= (1 \otimes b'_1 + \cdots + b'_n) \end{aligned}$$

Let $a \in A$. We want to show $aa_i = a_i a$ for all i. Take the element:

$$p = (a \otimes_k 1)v' - v'(a \otimes_k 1) = (aa_2 - a_2 a) \otimes_k b_2 + \cdots + (aa_n - a_n a) \otimes_k b_n \in I,$$

since I is a two-sided ideal. p has $n - 1$ summands so, since n is minimal this implies that $p = 0$, so since $b'_i s$ are linearly independent we have:

$$(aa_i - a_i a) = 0 \Rightarrow aa_i = a_i a \, \forall i.$$

Therefore, $\forall i$, a_i are in the center $Z(A) = k$.
Hence, $v' = (1 \otimes_k b) \in I$ with $0 \neq b \in B$. Using the fact that B is simple, implying that $BbB = B$, we have:

$$(1 \otimes_k B) = (1 \otimes_k BbB) = (1 \otimes_k B)(1 \otimes_k b)(1 \otimes_k B) \subset I.$$

Therefore,

$$(A \otimes_k 1)(1 \otimes_k B) = (A \otimes_k B) \subset I,$$

so $I = (A \otimes_k B)$. □

Lemma 4.6. [3, Lemma 10.2.12] If A and B are central k-algebras, then $A \otimes_k B$ is central.

Proof. Let's take a nonzero element x in $Z(A \otimes_k B)$ and write

$$x = \sum_i (a_i \otimes_k b_i),$$

where $a_i \in A$ and $b_i \in B$ linearly independent over k.
Let $a \in A$. Then we have :

$$(\sum_i (a_i \otimes_k b_i))(a \otimes_k 1) = (a \otimes_k 1)(\sum_i (a_i \otimes_k b_i)) \Rightarrow 0 = \sum_i (a_i a - aa_i) \otimes_k b_i.$$

Since the b_i are linearly independent over k, we obtain

$$a_i a = aa_i.$$

This implies that $k = Z(A)$ for all i.
This implies, that similarly to the proof of Lemma 4.5, we can write x as :

$$x = (1 \otimes b), \text{ for some } b \in B.$$

Since $1 \otimes b \in Z(A \otimes B)$, it commutes with every $(1 \otimes b') \in (1 \otimes B)$. But since $b' \mapsto 1 \otimes b'$ defines an isomorphism $B \mapsto 1 \otimes B$, we get that b is in the center of B, which is just k, by assumption.. Therefore, $x \in (k \otimes k) = k$.

□

5. Wedderburn's theorem

In the process of classifying central simple k-algebras an important tool is Wedderburn's theorem. The goal of this chapter is to give a proof to this result.

Theorem 5.1. Wedderburn's theorem[3, Theorem 10.2.10]: Let A be a central simple algebra over a field k. There are a division algebra D, unique up to isomorphism, and a unique positive integer n such that A is isomorphic to $M_n(D)$.

To prove the theorem we will first introduce some lemmas and then prove the existence and the uniqueness of D and n separately.

Definition 5.2. A **minimal left ideal** of a k-algebra A is a left ideal of A containing no other nonzero left ideals of A.

5.1. Existence.

For the existence part of the proof we use Henderson's approach [10, pp $365-366$] as our main reference. However, this proves a slightly more general result. Namely, it is shown that any simple ring with a minimal nonzero left ideal is isomorphic (as a ring) to the ring of $n \times n$ matrices over some division ring D. Thus, in order to use Henderson's approach, we will also show that any k-algebra contains a nonzero minimal left ideal, and that with our assumptions D is a division algebra over k and the ring isomorphism is a k-algebra isomorphism.

If D is a ring and M is a right A-module, then we write $\mathrm{End}_D(M)$ for the ring of right D-module endomorphism of M.

Lemma 5.3. Let A be a k-algebra with an idempotent e such that $AeA = A$, and let $D = eAe$ and $M = Ae$. Then there is an isomorphism of k-algebras:

$$A \simeq \mathrm{End}_D(M).$$

Proof. We start by noting that D is a k-algebra, since it inherits all the properties from A, but with identity e since $e^2 = e$.
Moreover, M is a right D-module, with scalar multiplication defined as follows: Let $m = a_1 e \in M$ and $d = ea_2e \in D$:

$$md = a_1eea_2e = a_1ea_2e \in M.$$

We need to also show that $\mathrm{End}_D(M)$ is a k-algebra. This is the case since:

(1) $M = Ae$ is a k-vector space since A is a k-algebra by assumption.
(2) $\mathrm{End}_k(M)$ is a k-vector space .
(3) $\mathrm{End}_D(M)$ is a k-linear subspace of $\mathrm{End}_k(M)$ since D is a k-algebra.

Let

$$f : A \to \mathrm{End}_D(M)$$
$$[f(a)](m) = am,$$

we will show that f is an isomorphism of k-algebras.
We first show that f is well-defined, i.e. that $f(a)$ is indeed an element of $\mathrm{End}_D(M)$ for any $a \in A$.

(1) Let $a \in A$ and some $m = a'e \in M$, then

$$[f(a)](m) = am = aa'e \in M.$$

We also need to check that it is a right D-module homomorphism. Let $d \in D$, then:

$$[f(a)](md) = amd = [f(a)](m)d.$$

Note that all multiplications take place in A.
Additivity is easy, so $f(a) \in \mathrm{End}_D(M)$.
To prove that is f a ring homomorphism we show for all $a_1, a_2 \in A$ and $m \in M$:

$$[f(1)](m) = 1 \cdot m = m = id(m)$$

$$[f(a_1 + a_2)](m) = (a_1 + a_2)m = a_1m + a_2m = [f(a_1)](m) + [f(a_2)](m)$$

$$[f(a_1a_2)](m) = a_1a_2m = a_1(a_2m) = [f(a_1)](a_2m) = [f(a_1)](f(a_2)(m))$$

(2) To prove injectivity it suffices to show that the kernel of f is trivial. Take $a \in \ker(f)$. Then, we have

$$0 = [f(a)](M) = aM = aAe.$$

Hence we find:

$$aAeA = 0.$$

Using $AeA = A$ we get:

$$aA = 0.$$

In particular,

$$a = a \cdot 1 \Rightarrow a = 0.$$

Therefore, $\ker(f) = \{0\}$.

(3) For surjectivity we first note that since

$$A = AeA,$$

then for some $a_i, b_i \in A$, $i = 1, ..., k$ we can express

$$1 = \sum_{i=0}^{k} a_i e b_i.$$

Take $\delta \in \mathrm{End}_D(M)$ and evaluate it at an arbitrary $m \in M$, where $m = ae = ae^2 = (ae)e = me \in M$, then we can write

$$\begin{aligned}
\delta(m) &= \delta(1 \cdot m) \\
&= \delta((\sum_{i=0}^{k} a_i e b_i)m) \\
&= \sum_{i=0}^{k} \delta(a_i e b_i m) \\
&= \sum_{i=0}^{k} \delta(a_i e b_i (me)) \\
&= (\sum_{i=0}^{k} (\delta(a_i e) e b_i))m.
\end{aligned}$$

The last equality occurs since δ is a right D-module endomorphism and $eb_ime \in D$. Since $\sum(\delta(a_ie)eb_i)$ does not depend on m then we can express δ as:

$$\delta = f(\sum_{i=0}^{k} (\delta(a_i e) e b_i)).$$

Therefore, f is surjective.

To finish the proof it suffices to show that f is also a k-vector space homomorphism: So it is enough to show for $\lambda \in k$ and $a \in A$ we get:

$$[f(\lambda a)](m) = \lambda am = \lambda(am) = \lambda[f(a)](m), \text{ for all } m \in M.$$

Therefore,f is a k-algebra isomorphism:

$$A \simeq \mathrm{End}_D(M).$$

□

Now we are going to apply this result to prove the existence of Wedderburn's theorem.

Theorem 5.4. If A is a simple k-algebra then

$$A \simeq M_n(D),$$

for some $n \in \mathbb{N}$ and D a division k-algebra.

Proof. In order to apply Henderson's result we first show that every k-algebra has a minimal left nonzero ideal. The key to this statement is that all k-algebras in this project are finite dimensional.
Note that: every ideal I of a k-algebra A is a vector space over k. This happens since I is an A-module and A is a k-module.
Assume that A has no minimal left ideal. Let I be a nonzero left ideal of A. Then, we obtain an infinite descending chain of ideals:

$$I = I \supsetneq I_1 \supsetneq I_2 \supsetneq \dots$$

Note that by linear algebra $dim_k(I_i) > dim_k(I_{i+1})$. Hence, the chain cannot be infinite since $dim_k(I)$ is a finite integer, and all $dim_k(I_i) > 0$.
So we reach a contradiction. Therefore, A has a minimal nonzero left ideal M.

Now we need to show that the conditions of Lemma 5.3 apply for A and M, and an appropriately chosen D. To finish the proof, it suffices to show that D is a division algebra and that M has finite rank over D.

(1) If $b \in A, b \neq 0$ and since $0 \neq b = 1b1 \in AbA$, we have $AbA = A$, because AbA is a 2-sided principal ideal of A. Hence it must be 0 or A.
(2) Let's show that $M = Ae$ for some idempotent e in A.
Since $M^2 \subset M$ is an ideal then $M^2 = 0$ or $M^2 = M$ since M minimal.
If $M^2 = 0$, then

$$0 = (Am)^2 = (AmA)m = Am, \text{ for all nonzero } m \text{ in } M.$$

But this does not hold since

$$0 \neq m = 1m \in Am.$$

Thus the only case that can hold is $M^2 = M$. For $m \in M$ we obtain a left ideal Mm of A contained in M. Hence, there is a nonzero x in M such that

$$Mx = M,$$

otherwise we would obtain $M^2 = 0$. Therefore there exists an $e \in M$ such that

$$ex = x.$$

It turns out that e is an idempotent such that $M = Ae$, as we'll now explain. Note that

$$e^2x = ex \Rightarrow (e^2 - e)x = 0.$$

Thus, $e^2 - e$ is an element of the left annihilator of x in M. The left annihilator in M of x is a left ideal contained in M so it is either the zero ideal or all of M. But e is in M and $ex = x \neq 0$, so the annihilator must be 0. Hence e is idempotent. We know e is nonzero, so Ae is a nonzero ideal contained in M, hence $M = Ae$.

(3) Now we can apply Lemma 5.3 with $D = eAe$ and get

$$A \simeq \mathrm{End}_D(M).$$

(4) Let's check that $D = eAe$ is indeed a division k-algebra.
Let

$$d = eae \in D, \text{ such that } d \neq 0,$$

then

$$d = 1d \text{ is in } Ad = Aed,$$

and

$$Aed \text{ is an ideal of } A \text{ containted in } Ae,$$

so since Ae is minimal, we get

$$Aed = Ae.$$

Thus there exists an $a' \in A$ such that $a'd = e$. By setting $d' = ea'e$ we get

$$d'd = (ea'e)(eae) = ea'eae = e^2 = e.$$

so every nonzero element in eAe has a left inverse with respect to the identity e.
Let c be the left inverse of d' [11][page 18]. We need to show that c is equal to d. This is indeed the case, since:

$$c = ce = c(d'd) = (cd')d = ed = d.$$

This implies that all left inverses are also right inverses, so D is a division k-algebra.

(5) Note that M is a sub k-vector space of A, so M has finite dimension as a k-vector space. But D is a k-vector space, so

$$rk_D(M) \leq dim_k(M) \leq dim_k(A) < \infty.$$

(6) Let $dim_k(A) = n^2$. M is finite dimensional over D.
We have

$$M_n(D) \simeq \mathrm{End}_D(M), \quad \text{for some } n \in \mathbb{N},$$

where the isomorphism is the same as for vector spaces over fields. So we associate to an endomorphism its representative matrix with respect to a choice of basis [11][1.3]:

$$M_n(D) \simeq \mathrm{End}_D(M) \simeq A.$$

□

5.2. **Uniqueness.** The uniqueness part of Wedderburn's theorem states that if D, D' are division rings such that

$$M_n(D) \simeq M'_n(D')$$

for some $n, n' \geq 1$, then

$$D \simeq D'$$

and

$$n = n'.$$

Let's start describing the simple left modules of $M_n(D)$ where D is a division ring. For $1 \leq r \leq n$, let $L_r \subset M_n(D)$ be the set of all matrices in the form $M = [m_{ij}]$ with only nonzero entries in the r_{th} column. It is easy to see that these ideals are left ideals of $M_n(D)$ and that they are all isomorphic.

Lemma 5.5. L_r is a minimal ideal of $M_n(D)$.

Proof. To show this we take a nonzero $x \in L_r$. Then, there exist $a_1, \ldots, a_n$, not all zero, such that:

$$x = a_1E_{1r} + a_2E_{2r} + ... + a_nE_{nr}.$$

Suppose we have a nonzero ideal $J \subset L_r$. Let y a nonzero element of J. We can express it as:

$$y = \sum_i a_iE_{ir},$$

with at least one $a_k \neq 0$. Left-multiplying y with $E_{kk} \in M_n(D)$ we get:

$$E_{kk}y = a_kE_{kr} \in J.$$

Since $a_k \neq 0$:

$$\frac{E_{kk}y}{a_k} = E_{kr} \in J.$$

So,

$$E_{ik}E_{kr} = E_{ir} \in J$$

since J is a left ideal. Hence, $E_{ir} \in J, \forall i$. This means $J = L_r$. Thus, L_r is minimal. □

In particular by definition L_r are simple $M_n(D)$-modules, for all $r \geq 1$.

Lemma 5.6. All simple left $M_n(D)$-modules are isomorphic.

Proof. Note that L_r are all trivially isomorphic with each other for all $r \geq 1$.
For the rest of simple left $M_n(D)$-modules we proceed as follows.
It is clear that

$$M_n(D) = \oplus L_r.$$

Suppose we have a nonzero simple left $M_n(D)$-module M.
Then

$$\{0\} \neq M = M_n(D)M = (\oplus L_r)M.$$

This means that there exists an r such that:

$$L_rM \neq \{0\}$$

Then we create an isomorphism [11, 1.8]. For $m \in M$ such that L_rm is nontrivial, we let:

$$f : L_r \to M$$
$$y \mapsto ym.$$

It is easy to see that this is a left $M_n(D)$-module homomorphism.
Since L_r is simple and the kernel is a nonzero submodule of L_r, f is injective. Similarly, since f is a nonzero map and $im(f)$ is a submodule of M and M is simple, f is also surjective. Thus, f is a $M_n(D)$-module isomorphism. □

For all $1 \leq r \leq n$, $D^n \simeq I_r$. Thus, we conclude that D^n is a simple left $M_n(D)$-module.

Lemma 5.7. There is a ring isomorphism

$$End_{M_n(D)}(D^n) \simeq D^o.$$

Proof. We define the map:

$$f : D^o \mapsto \text{End}_{M_n(D)}(D^n)$$
$$d \mapsto (P \mapsto Pd).$$

We can express f as:

$$[f(d)](P) = Pd.$$

We will show that for $d \in D^o$, $[f(d)]$ is a left $M_n(D)$-module homomorphism. For $l \in M_n(D)$ we have:

$$[f(d)](lP) = lPd = l[f(d)](P).$$

Additivity is clear.
So $f(d) \in \text{End}_{M_n(D)}(D^n)$ for all $d \in D^o$.
For injectivity, we take $d \in \ker(f)$, then $d = 0$ since D is a division ring. Therefore f is injective.
For surjectivity, we have to show that every element in the endomorphism ring can be written as a multiplication on the right by an element in D. We take an element $g \in \text{End}_{M_n(D)}(D^n)$ and we want to show that:

$$g = f(d)$$

for some $d \in D^o$.
Let $(e_1, \dots, e_n)$ the standard basis of D^n. Then,

$$g(e_1) = e_1d_1 + \cdots + e_nd_n,$$

for some $d_i \in D$.
We will show that $g = f(d_1)$.
First, we have

$$g(e_1) = g(E_{11}e_1) = E_{11}g(e_1) = E_{11}(e_1d_1 + \cdots + e_nd_n) = e_1d_1$$

so

$$g(e_1) = [f(d_1)](e_1).$$

Now let $j > 1$. Then

$$e_j = E_{j1}e_1,$$

so

$$g(e_j) = g(E_{j1}e_1) = E_{j1}g(e_1) = E_{j1}e_1d_1 = [f(e_1)](d_j)$$

and hence

$$g = f(e_1),$$

since g is uniquely determined by the values it takes on the basis elements e_j. Therefore f is surjective, so f is a ring isomorphism. □

Now we conclude the proof of uniqueness of Wedderburn's theorem: Suppose that D, D' are division algebras and that

$$A \simeq M_n(D) \simeq M_{n'}(D')$$

for n, n' suitable integers. Then since D^n is a minimal left ideal of $M_n(D)$ and $D'^{n'}$ of $M_{n'}(D')$. But since the rings are isomorphic, $D'^{n'}$ is isomorphic to a minimal left $M_n(D)$-ideal as well. From Lemma 5.6 we know that:

$$D^n \simeq D'^{n'}.$$

By Lemma 5.7 we know that the following ring isomorphisms hold:

$$D'^o \simeq \mathrm{End}_A(D^n) \simeq \mathrm{End}_A(D'^{n'}) \simeq D^o.$$

Therefore, this implies that

$$D \simeq D'$$

and

$$n = n'.$$

This finishes the proof of Wedderburn's theorem.

6. The Brauer group of a field and the Brauer group of algebraically closed fields

We will now define the Brauer group of a field. The elements of the Brauer group are equivalence classes of central-simple k-algebras.

Remark 6.1. If A is a central simple k-algebra then A^o is also a central simple k-algebra, since they are the same as k-vector spaces.

Lemma 6.2. [16, Lemma 9.126(i)] If A is a k-algebra, then:

$$A \otimes_k M_n(k) \simeq M_n(A),$$

for $n \in \mathbf{N}$.

Proof. Consider the map:

$$f : A \times M_n(k) \to M_n(A)$$

$$(a, s) \mapsto as,$$

where we view $M_n(k)$ as a k-subalgebra of $M_n(A)$ and A as the subalgebra AI_n of $M_n(A)$. This is clearly a k-bilinear map.
Let g be the k-linear map induced by f by the universal property of tensor products. We need to show that g is a k-algebra isomorphism.
First we show that it is a k-algebra homomorphism. We have

$$g(1_A \otimes I_n) = I_n.$$

To show g is multiplicative take $a, a' \in A$ and $s, s' \in M_n(k)$:

$$g((a \otimes_k s)(a' \otimes_k s')) = g((aa' \otimes_k ss')) = aa'ss' = asa's' = g((a \otimes_k s))g((a' \otimes_k s')),$$

where the third equality follows from the fact that k is in the center. So g is a k-algebra homomorphism.
For injectivity, it suffices to check that the kernel contains no nontrivial elementary tensors since every element in $A \otimes_k M_n(k)$ can be expressed as a linear combination of elementary tensors.
For an element $(a \otimes_k s) \in \ker(g)$:

$$g(a \otimes_k s) = as = 0 \Rightarrow a = 0 \text{ or } s = 0,$$

since a is a diagonal matrix with entries all equal and k is a field.
In either case

$$a \otimes_k s = 0,$$

therefore the kernel is trivial so g is injective.
For surjectivity, we compare the dimensions of $A \otimes_k M_n(k)$ and $M_n(A)$.

$$dim_k(A \otimes_k M_n(k)) = dim_k(A)dim_k(M_n(k)) = dim_k(A)n^2 = dim_k(M_n(A)).$$

Therefore, we obtain that g is a k-algebra isomorphism. □

Lemma 6.3. [11] Let A be a central simple k-algebra. Then

$$A \otimes_k A^o \simeq M_m(k),$$

where $m = dim_k(A)$.

Proof. Consider the map

$$\sigma : A \times A^o \to \mathrm{End}_k(A)$$
$$(a, b) \mapsto (x \mapsto axb).$$

We need to show that $\sigma((a, b))$ is a k-endomorphism of A, for any $a \in A$ and $b \in A^o$.
For a fixed a and b we define:

$$f = \sigma(a, b).$$

For $l \in k$:

$$f(lx) = alxb = l(axb) = lf(x).$$

For $x_1, x_2 \in A$:

$$f(x_1 + x_2) = a(x_1 + x_2)b = ax_1b + ax_2b = f(x_1) + f(x_2).$$

So $\sigma((a, b)) \in \mathrm{End}_k(A)$.
We will construct an isomorphism between $A \otimes A^o$ and $M_m(k)$ by applying the universal property of the tensor product to σ. It is easy to see that sigma is bilinear. Using the universal property of tensor products σ induces a k-linear map ρ:

$$\rho : A \otimes_k A^o \to M_m(k)$$

Then the map ρ is defined as:

$$[\rho((a \otimes_k b))](x) = axb.$$

Then we have to show that ρ is a k-algebra isomorphism. We have that

$$[\rho(1 \otimes_k 1)](x) = x \text{ for all } x \in A.$$

so $\rho(1 \otimes_k 1)$ is the identity of $\mathrm{End}_k(A)$.
Let $a, a' \in A$ and $b, b' \in A^o$. Then, for x in A, we have:

$$[\rho(a \otimes_k b)(a' \otimes_k b')](x) = [\rho(aa' \otimes_k bb')](x) = aa'xb'b$$
$$= \rho(a \otimes_k b)(a'xb') = \rho(a \otimes_k b)(\rho(a' \otimes_k b')(x)).$$

So we obtain a k-algebra homomorphism.
The kernel of ρ is a two-sided ideal of $A \otimes_k A^o$ so by Lemma 4.5, $\ker(\rho) = \{0\}$. So ρ is injective.
In order to show ρ is surjective we compare the dimensions of $A \otimes_k A^o$ and $\mathrm{End}_k(A)$. We get:

$$dim_k(\mathrm{End}_k(A)) = m^2 = dim_k(A)dim(A^o) = dim(A \otimes_k A^o).$$

So ρ is a k-algebra isomorphism.
Therefore, $A \otimes_k A^o \simeq M_m(k)$ since $\mathrm{End}_k(A) \simeq M_m(k)$. □

Definition 6.4. Let k be a field and A and B be central simple k-algebras.
We call A and B **Brauer equivalent** and write $A \sim B$, if

$$A \otimes_k M_n(k) \simeq B \otimes_k M_m(k),$$

for some positive integers m, n. We write $[A]$ for the Brauer equivalence class of a central simple k-algebra A.

Lemma 6.5. Brauer equivalence defines an equivalence relation.

Proof.
- Reflexivity and symmetry are clear.

- For transitivity, let A, B and C be central simple k-algebras such that $A \sim B$ and $B \sim C$. Let s, l, m, n be positive integers such that:
$$A \otimes_k M_n(k) \simeq B \otimes_k M_m(k), B \otimes M_s(k) \simeq C \otimes_k M_l(k)$$
By using Lemma 3.21 we obtain:
$$A \otimes_k M_{ns}(k) \simeq A \otimes_k M_n(k) \otimes_k M_s(k) \simeq B \otimes_k M_m(k) \otimes_k M_s(k) \simeq$$
$$C \otimes_k M_l(k) \otimes_k M_m(k) \simeq C \otimes_k M_{lm}(k) \text{ and hence } A \sim C.$$

□

Lemma 6.6. Let A, B, A', B' be central simple k-algebras such that $[A] = [A']$ and $[B] = [B']$. Then
$$[A \otimes B] = [A' \otimes B'].$$

Proof. The operation on the Brauer group is well defined. [16, Theorem 9.128(i)] If A, A', B, B' are central simple k-algebras and $A \sim A'$ and $B \sim B'$ then:
$$A \otimes_k M_n(k) \simeq A' \otimes_k M_m(k),$$
$$B \otimes_k M_s(k) \simeq B' \otimes_k M_l(k),$$
for n, m, s, l positive integers. Then:
$$A \otimes_k B \otimes_k M_n(k) \otimes_k M_s(k) \simeq A \otimes_k B \otimes_k M_{ns}(k)$$
and
$$A' \otimes_k B' \otimes_k M_m(k) \otimes_k M_l(k) \simeq A' \otimes_k B' \otimes_k M_{ml}(k).$$
So
$$A \otimes_k B \otimes_k M_{ns}(k) \simeq A' \otimes_k B' \otimes_k M_{ml}(k) \Rightarrow A \otimes_k B \sim A' \otimes_k B'.$$

□

Definition 6.7. [16, Definition page 737] We define the **Brauer group** of k to be the set
$$Br(k) = \{[A] : \ A \text{ central simple } k\text{-algebra}\}$$
with binary operation
$$[A][B] = [A \otimes B].$$

Lemma 6.8. The Brauer group $Br(k)$ is an abelian group with identity element $[k]$. The inverse of a class $[A]$ is the class $[A^o]$.

Proof. By Lemma 6.6 and 4.6 we know that the binary operation of $Br(k)$ is well-defined. The tensor product is associative, thus the binary operation is as well. The identity element of the Brauer group of k is just the class of k. This is the case since:
$$A \otimes_k k = A,$$
for every k-module A.
The inverse of a class A it is the class of its opposite element A^o by Lemma 6.3. This proves that Definition 6.7 gives $Br(k)$ a group structure. Finally, the group $Br(k)$ is abelian since the tensor product is commutative.

□

Example 6.9. Some examples of the Brauer group of a field [16, Example 9.130]:

- The Brauer group of the real numbers has only two elements: the classes of $\mathbb{R}$ and the Hamiltonian quaternions.

- The Brauer group of a finite field is trivial.

The main reason why Brauer groups were introduced was to classify division k-algebras through central simple k-algebras. The following lemma is essential before we proceed in showing the connection between the two.

Lemma 6.10. Let D be a division ring and k its center. Then, k is a field and for every $n \geq 1$ we have

$$Z(M_n(D)) = k.$$

Proof. [11, 2.2] First we show that $Z(D)$ is a field. To do so we take a nonzero element $y \in Z(D)$ and $d \in D$. We have:

$$d^{-1}yy^{-1}d = 1 \Rightarrow (d^{-1}y)^{-1} = y^{-1}d,$$

as well as

$$yd^{-1}dy^{-1} = 1 \Rightarrow (yd^{-1})^{-1} = dy^{-1}.$$

Using the above expressions, we obtain that

$$dy^{-1} = (yd^{-1})^{-1} = (d^{-1}y)^{-1} = y^{-1}d.$$

Hence $Z(D)$ is a field. To prove the second assertion we let:

$$f : D \to Z(M_n(D))$$
$$d \mapsto dI_n.$$

Then

$$f_{Z(D)} : Z(D) \to Z(M_n(D))$$

is clearly an injective ring homomorphism.
To prove f is surjective, let $a = (a_{ij}) \in Z(M_n(D))$, we want to show that

$$a = f(a_{11}) \text{ and } a_{11} \in Z(D).$$

Then a_{ij} can be expressed as:

$$a_{ij} = aE_{ij}.$$

For all i, j, we have

$$a_{ij} = aE_{ij} = E_{ij}a = a_{ji}.$$

Hence $a_{ij} = 0$ for $i \neq j$. Moreover, all $a_{ii} = a_{11}$, so

$$a = a_{11}I_n = f(a_{11}) \in f(Z(D)).$$

□

Corollary 6.11. If $M_n(D)$ is a central k-algebra, where D is a division k-algebra, then D is also central.

Proof. This follows from Lemma 6.10 and Example 4.2. □

Lemma 6.12. [16, Theorem 9.128] Let A be a central simple k-algebra such that $A \simeq M_n(D)$, where D is a division k-algebra. Then D is central simple and $[A] = [D]$ in $Br(k)$.

Proof. By Lemma 6.10 and Corollary 6.11 we have that D is a central simple k-algebra. From Lemma 6.2:

$$A \simeq M_n(D) \simeq D \otimes_k M_n(k).$$

□

Theorem 6.13. [16, Theorem 9.129] If k is a field, then there is a bijection from $Br(k)$ to the family Δ of all isomorphism classes of finite-dimensional central division algebras over k.

Proof. We define a map

$$g : \Delta \to Br(k)$$

$$g(D) = [D].$$

We have to show that g is well defined.
Let D, D' two central division k-algebras :

$$D \simeq D'$$

Then,

$$D \simeq D \otimes k \simeq D \otimes M_1(k)$$
$$D' \simeq D' \otimes k \simeq D' \otimes M_1(k),$$

so

$$g(D) = g(D').$$

So g is well defined.
For injectivity, suppose that

$$g(D) = g(D'),$$

then there are positive integers n, m such that:

$$D \otimes M_n(k) \simeq D' \otimes M_m(k).$$

By Lemma 6.2 we get:

$$M_n(D) \simeq M_m(D'),$$

and by the uniqueness of Wedderburn's theorem we have $D \simeq D'$.
To show g is surjective, let $c \in Br(k)$, then

$$c = [M_n(D)],$$

where D is a division k-algebra and n a positive integer. Then, by Lemma 6.12, we have that D is central over k, and

$$g(D) = c.$$

So g is surjective, and we obtain that g is bijective. □

Remark 6.14. By Theorem 6.13 there exists a noncommutative division ring, finite-dimensional over its center k, if and only if $Br(k) \neq \{0\}$.

We will now show that there are no such division rings whose center is an algebraically closed field.

Definition 6.15. A field k is **algebraically closed** if every non-constant polynomial in $k[x]$ has a root in k.

Lemma 6.16. [9, Corollary 2.1.7] Let k be an algebraically closed field. Then every central simple k-algebra is isomorphic to $M_n(k)$ for $n \geq 1$.

Proof. Let D be a finite-dimensional division k-algebra. We want to show that D=k.
It suffices to prove that $D \subset k$, so we take $x \in D$, then there exists

$$a_0, \ldots, a_n \in k \text{ such that } a_0 + a_1 x + \cdots + a_n x^n = 0,$$

where not all a_i are 0.
Without loss of generality assume that $a_n \neq 0$, $n \geq 1$, else x is in k.
Let

$$f(t) = a_n t^n + \cdots + a_0 \in k[t].$$

Since k is algebraically closed, we get $x \in k$. Therefore, $k = D$.
By Wedderburn's theorem we know that if A is a central simple k-algebra then $A \simeq M_n(D)$, for D division k-algebra and $n \in \mathbb{N}$. By the above $D = k$, so

$$A \simeq M_n(k).$$

□

Theorem 6.17. [11] The Brauer group of an algebraically closed field k is trivial.

Proof. This follows directly from Lemma 6.16. □

Example 6.18. The Brauer group of the complex numbers $\mathbb{C}$ is trivial, by applying Lemma 6.16.

7. Computations in the two-torsion subgroup of Br(ℚ)

In this section we will use results on quaternion algebras derived in Section 3.3 in order to study some elements of order 2 in $Br(\mathbb{Q})$.
In contrast to the example at the end of Section 6, the Brauer group of $\mathbb{Q}$ is infinite and its structure is quite complicated [15]. So instead, we will look at some explicit examples of elements of order dividing 2. For instance, classes of quaternion algebras satisfy this. If k is a field of characteristic not 2, then by Corollary 3.23 we have:

$$[(a,b)_k \otimes_k (a,b)_k] = [M_4(k)] = [k].$$

Since $Br(k)$ is an abelian group, the elements of order dividing two form a subgroup, which we will denote by $Br(k)[2]$. The class of the tensor product of two quaternion algebras is also in this subgroup. We will investigate for some explicit examples if it is also the class of a quaternion algebra. The following lemma will aid in showing that certain quaternion algebras are isomorphic.

Lemma 7.1. Let $a, b \in \mathbb{Q}^*$, with $a \neq -b$. Then we have:

$$(a,b)_{\mathbb{Q}} \simeq (a+b, -ab)_{\mathbb{Q}}.$$

Proof. Let $\{1, i, j, ij\}$ be a basis of $(a,b)_k$ such that:

$$i^2 = a \cdot 1_{(a,b)_{\mathbb{Q}}},\ j^2 = b \cdot 1_{(a,b)_{\mathbb{Q}}} \text{ and } ij = -ji,$$

Then let $B := \{1, i+j, ij, (i+j)ij\}$ and we will show that it is a basis for $(a+b, -ab)_{\mathbb{Q}}$: Take $a_i \in \mathbb{Q}$, for $i \in \{0,1,2,3\}$:

$$a_0 + a_1(i+j) + a_2 ij + a_3(i+j)ij = 0 \Rightarrow$$
$$a_0 + a_1 i + a_1 j + a_2 ij + a_3 aj - a_3 bi = 0 \Rightarrow$$
$$a_0 + (a_1 - a_3 b)i + (a_1 + a_3 a)j + a_2 ij = 0$$

This implies that:

$$a_0 = 0 \text{ and } a_1 = a_3 b \text{ and } a_1 = -a_3 a \text{ and } a_2 = 0.$$

Since $a \neq -b$ we have:

$$(a+b)a_3 = 0 \Rightarrow a_3 = 0 \text{ and } a_1 = 0.$$

So $1, i+j, ij, (i+j)ij$ are linearly independent. We also need to show that $i+j$ and ij are in $(a+b, -ab)_k$.

$$(i+j)^2 = (i+j)(i+j) = i^2 + ij + ji + j^2 = i^2 + j^2 = a + b \in (a+b, -ab)_k.$$

$$(ij)^2 = -i^2 j^2 = -ab \in (a+b, -ab)_k.$$

$$ij(i+j) = i^2 j + jij = -iji - ijj = -ij(i+j).$$

Therefore, B forms a basis of $(a+b, -ab)_{\mathbb{Q}}$.
To show that B forms a basis in $(a,b)_{\mathbb{Q}}$ we need to furthermore express every standard basis element of $(a,b)_{\mathbb{Q}}$ as a linear combination of the $\{1, i+j, ij, (i+j)ij\}$:
For 1 and ij it is trivial. For i we get:

$$(-a)(i+j) + (i+j)ij = -ai - aj + aj - bi = -(a+b)i$$

since $a \neq -b$ then we can divide by $-(a+b)$, so

$$i = \frac{a(i+j)}{(a+b)} - \frac{(i+j)ij}{(a+b)}$$

Similarly, for j:

$$b(i+j) + (i+j)ij = bi + bj + aj - bi = (a+b)j$$

so again dividing by $(a+b)$:

$$j = \frac{b(i+j)}{(a+b)} + \frac{(i+j)ij}{(a+b)}.$$

Therefore, it forms a basis of $(a,b)_{\mathbb{Q}}$, so according to Lemma 3.4 we obtain

$$(a,b)_{\mathbb{Q}} \simeq (a+b, -ab)_{\mathbb{Q}}.$$

□

Example 7.2. We have that $(-2,-3)_{\mathbb{Q}} \simeq (-1,-1)_{\mathbb{Q}}$.

Proof. This is easy to show by applying Lemma 7.1 twice and then using Lemma 3.8(i):

$$(-1,-1)_{\mathbb{Q}} \simeq (-2,-1)_{\mathbb{Q}} \simeq (-3,-2)_{\mathbb{Q}} \simeq (-2,-3)_{\mathbb{Q}}.$$

□

Lemma 7.3. Let m be a rational number which is not:

(1) a square of a rational number or
(2) negative such that $-m$ is the sum of three square elements of $\mathbb{Q}$.

Then, $(-1,-1)_{\mathbb{Q}}$ is not isomorphic to $(m,b)_{\mathbb{Q}}$, for any $b \in \mathbb{Q}^*$.

Proof. Suppose that $(-1,-1)_{\mathbb{Q}}$ is isomorphic to $(m,b)_{\mathbb{Q}}$.
Then m is a square in $(-1,-1)_{\mathbb{Q}}$:

$$x^2 = m,$$

where

$$x = a + bi + cj + dij, \text{ with } a,b,c,d \in \mathbb{Q}.$$

This implies that:

$$x^2 = (a + bi + cj + dij)^2 =$$

$$a^2 - b^2 - c^2 - d^2 + (2ab + cd - cd)i + (2ac - bd + bd)j + (2ad + bc - bc)ij = m$$

$$a^2 - b^2 - c^2 - d^2 = m$$

$$2ab = 0,\ 2ac = 0,\ 2ad = 0.$$

- If $b,c,d = 0$, then

  $$a^2 = m.$$

  so m is the square of the rational number a.
- If $m = l^2$ is the square of a rational number l, then by Lemma 3.20i:

  $$(m,b)_{\mathbb{Q}} = (a^2,b)_{\mathbb{Q}} \simeq M_2(\mathbb{Q}).$$

  On the other hand, the proof of Example 3.19 shows that $(-1,-1)_{\mathbb{Q}}$ is a division algebra, therefore not isomorphic to $M_2(\mathbb{Q})$. So we get a contradiction.

- So if m is not a square, then we must have $a = 0$ and hence
$$-b^2 - c^2 - d^2 = m,$$
so m is a negative number such that $-m$ is the sum of three rational squares.

□

In the project [1], it is shown that the quaternion algebra $(-1,-1)_{\mathbb{Q}}$ is not isomorphic to $(-7,-1)_{\mathbb{Q}}$. In this project, we use the same technique to show following more general results.

Example 7.4. The quaternion algebra $(-1,-1)_{\mathbb{Q}}$ is not isomorphic to $(-7,b)_{\mathbb{Q}}$, for any $b \in \mathbb{Q}^*$.

Proof. [1][Voorbeeld 2.6] Applying Lemma 7.3 it suffices to show that 7 is not the sum of three rational squares. We assume that it is:
$$b^2 + c^2 + d^2 = 7.$$
Multiplying with the largest common divisor of b, c, d we obtain the following equation in $\mathbb{Z}$:
$$p^2 + q^2 + r^2 = 7s^2,$$
where we may assume that $gcd(p,q,r,s) = 1$. Working modulo 8 we realise that this equation has no solutions since all the squares modulo 8 are equal to 0, 1 and 4. Therefore, $(-1,-1)_{\mathbb{Q}}$ is not isomorphic to $(-7,b)_{\mathbb{Q}}$. □

Example 7.5. The quaternion algebra $(-1,-1)_{\mathbb{Q}}$ is not isomorphic to $(7,b)_{\mathbb{Q}}$, for any $b \in \mathbb{Q}^*$.

Proof. We apply Lemma 7.3(1) since 7 is a positive integer. □

In order to investigate whether the class of the tensor product of two quaternion algebras is also represented by a quaternion algebra we introduce the following definition.

Definition 7.6. Let k be a field. We call $(a,b)_k$ and $(c,d)_k$ **linked** if there exist $x, b', d' \in k^*$ such that:
$$(a,b)_k \simeq (x,b')_k, \text{ and } (c,d)_k \simeq (x,d')_k.$$

Lemma 7.7. If $(a,b)_k$ and $(c,d)_k$ are linked then
$$(a,b)_k \otimes_k (c,d)_k$$
is Brauer equivalent to a quaternion algebra.

Proof. Since $(a,b)_k$ and $(c,d)_k$ are linked, there exist $x, b', d' \in k^*$ such that:
$(a,b)_k \simeq (x,b')_k$ and $(c,d)_k \simeq (x,d')_k$.
Thus by Lemma 3.22:
$$[(a,b)_k \otimes_k (c,d)_k] = [(x,b')_k \otimes_k (x,d')_k] = [(x,b'd')_k \otimes_k M_2(k)] = [(x,b'd')_k].$$
□

We showed in Example 7.5 that the following quaternion algebras are not isomorphic, however, the following examples show that they can be linked.

Example 7.8. [12, Exercise III.20] The quaternion algebras $(-2,-3)_{\mathbb{Q}}$ and $(-7,-23)_{\mathbb{Q}}$ are linked.
We can see this by applying Lemma 7.1:

$$(-7,-23)_{\mathbb{Q}} \simeq (-30,-161)_{\mathbb{Q}}$$

By applying Lemma 7.1 twice and at the last step Lemma 3.8(i):

$$(-2,-3)_{\mathbb{Q}} \simeq (-5,-6)_{\mathbb{Q}} \simeq (-11,-30)_{\mathbb{Q}} \simeq (-30,-11)_{\mathbb{Q}}.$$

Therefore, in the notation of Definition 7.6 we may take:

$$x = -30,\ b' = -161,\ d' = -11 \in \mathbb{Q}^*.$$

Moreover, by Lemma 3.22 the tensor product of $(-2,-3)_{\mathbb{Q}}$ and $(-7,-23)_{\mathbb{Q}}$ is Brauer equivalent to a quaternion algebra. Explicitly :

$$[(-2,-3)_{\mathbb{Q}} \otimes_k (-7,-23)_{\mathbb{Q}}] = [(-30,-11)_{\mathbb{Q}} \otimes_k (-30,-161)_{\mathbb{Q}}] =$$
$$[(-30,1771)_{\mathbb{Q}} \otimes_k M_2(\mathbb{Q})] = [(-30,1771)_{\mathbb{Q}}]$$

Example 7.9. As an easier example, we show that

$$(-1,-1)_{\mathbb{Q}} \otimes_k (-1,-7)_{\mathbb{Q}}$$

is Brauer equivalent to a quaternion algebra. Applying directly Lemma 3.22:

$$[(-1,-1)_{\mathbb{Q}} \otimes_k (-1,-7)_{\mathbb{Q}}] = [(-1,7)_{\mathbb{Q}} \otimes_k M_2(k)] = [(-1,7)_{\mathbb{Q}}].$$

This example completes Example 7.5. It means that the subgroup of $Br(\mathbb{Q})[2]$ spanned by $(-1,-1)_{\mathbb{Q}}$ and $(-1,-7)_{\mathbb{Q}}$ consist only of elements that can be represented by quaternion algebras. Recall that $(-1,-1)_{\mathbb{Q}}$, $(-1,-7)_{\mathbb{Q}}$ and $(-1,7)_{\mathbb{Q}}$ are all pairwise non-isomorphic from the examples above.

Remark 7.10. [12, Example 2.16] Let k be a field of char$\neq 2$. Then any quaternion algebra $(a,b)_k$ can be linked to any split quaternion algebra over k.
This is the case since

$$(a,b^2)_k \simeq M_2(k).$$

Remark 7.11. A field is called linked if any two quaternion algebras over it are linked. It is shown in [12] that $\mathbb{Q}$ is a linked field, but the proof is beyond the scope of this project. See for some examples of fields that are not linked [8, Section 5.2]. Since $\mathbb{Q}$ is linked, the classes of quaternion algebras form a subgroup of $Br(\mathbb{Q})[2]$. In fact, this subgroup is the entire $Br(\mathbb{Q})[2]$ but the proof is very hard. It is a consequence of the (Albert-)Brauer-Hasse-Noether theorem [15, Theorem 2.3].

References

[1] Benny Aalders. *Bachelor Thesis:Literatuuronderzoek naar centrale simpele algebra's*. University of Groningen, November 2014.

[2] Richard Brauer. Untersuchungen Über die Arithmetischen Eigenschaften von Gruppen Linearer Substitutionen. *Mathematische Zeitschrift*, 31(1):733–747, 1930.

[3] MJ Bright, Damiano Testa, and Ronald van Luijk. Geometry and Arithmetic of Surfaces. http://homepages.warwick.ac.uk/~maskal/surfaces/2021-01-03_surfaces.pdf, accessed 26-05-2021.

[4] Keith Conrad. Tensor products II. https://kconrad.math.uconn.edu/blurbs/linmultialg/tensorprod2.pdf,accessed 01-02-2021.

[5] Keith Conrad. Quaternion algebras. https://kconrad.math.uconn.edu/blurbs/ringtheory/quaternionalg.pdf, accessed 20-02-2021, 2018.

[6] Leonard Evens. Course notes on Algebra, Chapter 17. https://sites.math.northwestern.edu/~len/d70/index.html, accessed 25-04-2021.

[7] Andreas Gathmann. Lecture Notes on Commutative Algebra;, Chapter 7: Chain Conditions. https://www.mathematik.uni-kl.de/~gathmann/class/commalg-2013/commalg-2013-c7.pdf, accessed 20-04-2021, 2013.

[8] Enzo R. Gentile. On linked fields. *Revista de la Unión Mathemática Argentina*, 35:67–81, 1990.

[9] Philippe Gille and Támas Szamuely. *Central Simple Algebras and Galois Cohomology*. Cambridge University Press, 2006.

[10] W. Henderson. A short proof of Wedderburn's Theorem. *The American Mathematical Monthly*, 72(4):385–386, 1965.

[11] Ina Kersten. *Brauergruppen*. Universitätsverlag Göttingen, 2007.

[12] Tsit-Yuen Lam. *Introduction to quadratic forms over fields*. American Mathematical Society, 2005.

[13] Falko Lorenz. *Volume 2: Fields with Structure,Algebras and Advanced Topics*. Springer, 2018.

[14] Roman. *Advanced linear algebra*, volume 3. Springer, 2005.

[15] Peter Roquette. *The Brauer-Hasse-Noether theorem in historical perspective*. Springer Science & Business Media, 2006.

[16] Joseph J Rotman. *Advanced modern algebra*. Prentice Hall, 2003.

[17] Jaap Top. *Lecture notes on Advanced Algebraic Structures*. University of Groningen, April 2017.

[18] Jaap Top. *Lecture notes on Algebraic Structures*. University of Groningen, April 2017.