Modeling lasers using Gaussian wave packets
A quantum mechanical approach to an optics problem

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Abstract

The standard model used to analyse lasers is derived and compared to the Schrödinger equation. An equivalence between the time independent harmonic oscillator and focused beams is established. Also, a Gaussian wave packet is constructed for the time dependent harmonic oscillator.

Contents

1   Overview

In section 2 we start off with understanding the workings of a laser. We will see there are different modes of operation and how they function. We will develop a model for a laser beam in the lowest mode in section 3 using the optics approach, starting from the wave equation. Section 4 derives various Gaussian wave packets for the Schrödinger equation. In general, Gaussian wave packets are solutions to partial differential equations which are concentrated along rays in space time. The Schrödinger equations considered are the free space, the time dependent and time independent harmonic oscillator. The main result of this analysis is the equivalence between focused beams and the harmonic oscillator. This leads to a model for the higher order modes. We also establish a nice solution to the time dependent harmonic oscillator. The wave packets will be compared with each other and with the solution obtained from the optics approach at the end. I plot various figures using Mathematica, the scripts for these figures can be found in appendix A.

2   The Laser

In 1960, T. Maiman created the first ever laser, which is an acronym for "light amplification by stimulated emission of radiation". [1] The theory of radiation behind this was already understood for some time, with Einstein lying the groundwork. [2]

When an atom is in an excited state, with energy E_1 with respect to the ground state, and a photon of the same energy E_1 is incident on the atom, the photon can stimulate the emission of another photon from the atom. This additional photon has a couple of identical properties as the incident photon. The photons are in phase, have the same polarization and propagate in the same direction. This can be viewed as the waves adding constructively. If one manages to get a sample of atoms of which large percentage are in the excited state, a single photon of the right frequency can then trigger a chain reaction. Given the light wave is not scattered too much, this process keeps amplifying the wave. This continues as long as the active medium has population inversion, or in other words, more atoms are in the excited state than in the ground state. The methods of creating or maintaining population inversion are called pumping and will not be discussed here. Most of the theory discussed in this chapter can also be found in [3][4][5].

2.1   Optical Resonant Cavities

Atoms that are in the excited state will eventually fall back to the ground state by spontaneous emission of a photon. These photons can be emitted in any direction. This means the material will just glow, which is not that impressive. If one places the active medium in a cavity with mirrors on both sides, a wave can then bounce back and forth, building

up with each pass through the active material. If the cavity is a cylinder with flat mirrors normal to the axis on both sides, all but the photons nearly parallel to the axis will quickly leave the cavity. This way, despite the spontaneous emission of photons being random, the axial beam will be the most dominant after a short while. This causes the beam to be collimated to a large degree, effectively creating a coherent plane wave. See figure 1 for a pictorial explanation. It's good to note that the mirrors do not have to be planar, other configurations are possible and optimal solutions depend on the practical intention. Instead of acting as an amplifier, the cavity actually generates the light beam, an oscillator would be a more fitting name. Then again, the acronym with an 'o' instead of an 'a' has it's own disadvantages.
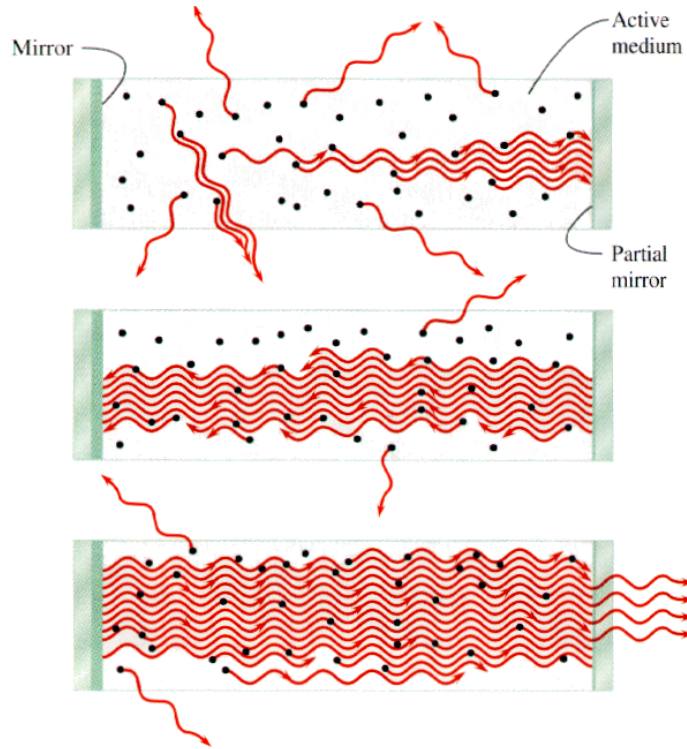
Figure 1: Three stages in the optical resonant cavity, showing collimation and amplification of the wave. [6]

If the active medium is a gas, the atoms will fly around at high speeds. The light absorbed and emitted by such atoms will be shifted in frequency due to the Doppler effect. Therefore, instead of narrow atomic transition bands around E_1, the bands will be spread out, following a Gaussian shape. As long as the amplification in the cavity on a round

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trip of light is larger than unity, the beam can continue to build up, differently put, when the gains exceed the losses. If the losses dominate the beam will quickly disappear. The sustainable frequencies in the cavity therefore do not extend to infinity like a Gaussian but are still relatively broad. The active medium can also be a different phase, like liquid or solid. In that case too, the atomic transitions will be broadened, albeit by different effects, for example collision and thermal broadening. In the case of a solid the lineshape follows a Lorentzian distribution.

2.2   Oscillation Modes

An additional effect of the mirrors is that standing waves will develop in the cavity. A node must occur at each mirror and hence only waves with a wavelength which is a half integer multiple of the length of the cavity can exist.

$$\frac{\lambda_0 m}{2n} = L,$$

$$\nu_m = \frac{mc}{2L},$$

$$\Delta\nu = \frac{c}{2L}.$$

λ_0	vacuum wavelength
m	$\in \mathbb{N}$
n	refractive index of medium
L	length of the cavity
ν	frequency of the light
c	speed of light in vacuum

The longitudinal cavity modes are separated by constant distance $\Delta\nu$. The resonant modes of the cavity themselves have in general a smaller bandwidth than the bandwidth of the atomic transition. Hence, depending on the length of the cavity, only some narrow bands will be sustained in the cavity despite the broad Doppler band. If the length of the cavity is selected such that only one frequency band will be amplified, the laser will produce only a 'single' frequency of light. This is why it is often said laser light is monochromatic. Decreasing the length such that the laser only sustains one frequency, has as disadvantage that the active medium will be smaller, which limits the power output of the beam.

Besides the longitudinal modes, oscillations parallel to the direction of propagation or z-axis, there will also be transverse modes, oscillations perpendicular to the direction of propagation. These modes are called transverse electric and magnetic (TEM) and labelled either TEM$_{mnq}$ or TEM$_{plq}$ corresponding to Cartesian coordinates and cylindrical coordinates respectively. The q subscript indicates the longitudinal mode which is often taken as the lowest mode and hence omitted. Subscripts m and n indicate the number of transverse nodal lines in the x- and y-directions of the beam respectively. The subscript p indicates the number of radial nodes and l the number of angular nodes. In other words, m, n, p and l indicate the number of lines of zero intensity in that particular direction along the beam, see figures 2 and 3. The reason for the distinction made between transverse and longitudinal modes, despite a laser mode consisting of both simultaneously, is that they govern different properties of the beam. The line width and coherence length are mainly

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due to the longitudinal modes and the divergence, diameter and energy distribution of the beam are a result of the transverse modes.

Figure 2: Hermite-Gaussian modes, starting from TEM_{00}, m increases from left to right, n increases from top to bottom. Plotted is the the real part of the electric field. Red indicates a positive value for the field, orange negative. [Self, A.1]

Figure 3: Laguerre-Gaussian modes, starting from TEM_{00}, p increases from left to right, l increases from top to bottom. Plotted is the real part of the electric field. Red indicates a positive value for the field, orange negative. [Self, A.2]

For TEM_{00}, the lowest order mode, the beam has a couple of interesting properties. First of all the intensity is Gaussian over the beam's cross section. Secondly the beam is spatially coherent, there are no phase shifts across the beam, unlike other modes. Thirdly, the divergence the is smallest of all the modes. Lastly it can be focused down to the smallest sized spot. Because of these reasons it is a popular mode used in lasers. Higher order modes' intensity spectrum is more complicated and can be described using Hermite or Laguerre polynomials for Cartesian or cylindrical coordinates respectively. A method to suppress the higher order modes is to insert an aperture in the cavity, so the laser is forced to operate in the TEM_{00} mode. This method has the downside of reducing the power and efficiency. Another way to select only the lowest mode would be to only pump the volume occupied by the TEM_{00} mode.

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3 Optics

Classical Electromagnetism is the default theory to model light. Maxwell's equations describe the electric and magnetic fields. Light is an electromagnetic phenomena and can thus be described by Maxwell's equations. In this section a model for the lowest order laser mode TEM$_{00}$ is derived, mainly following [7]. In a nonmagnetic, electrically neutral material, Maxwell's equations are:

$$\nabla \cdot \mathbf{E} = \frac{\nabla \cdot \mathbf{P}}{\epsilon_0}, \tag{3.1}$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3.2}$$

$$\nabla \times \frac{\mathbf{B}}{\mu_0} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} + \mathbf{J}. \tag{3.3}$$

$\mathbf{E}$	electric field
$\mathbf{B}$	magnetic field
$\mathbf{P}$	polarization of medium
$\mathbf{J}$	free charge current density
ϵ_0	permittivity of free space
μ_0	permeability of free space
t	time

3.1 Wave equation

Taking the curl of (3.2) gives:

$$\nabla \times (\nabla \times \mathbf{E}) + \frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = 0.$$

Application of the vector identity $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ and substitution of (3.3) gives:

$$\nabla(\nabla \cdot \mathbf{E}) + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} = \nabla^2 \mathbf{E} - \frac{\partial \mathbf{J}}{\partial t}.$$

And lastly, by replacing $\nabla \mathbf{E}$ with (3.1), the equation becomes:

$$\frac{1}{\epsilon_0}\nabla(\nabla \cdot \mathbf{P}) + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} = \nabla^2 \mathbf{E} - \frac{\partial \mathbf{J}}{\partial t}.$$

The term with $\mathbf{J}$ is relevant for light interacting with materials with free charges like metals or plasma. The $\frac{\partial^2 \mathbf{P}}{\partial t^2}$ term corresponds to dipole oscillations and the term with $\nabla(\nabla \cdot \mathbf{P})$ describes spatial differences, or anisotropies, of the electric field in the propagation medium. In vacuum these terms vanish and the wave equation in free space, using $\mu_0 \epsilon_0 = c^{-2}$, is:

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \nabla^2 \mathbf{E}.$$

In air there are hardly any free charges or anisotropies. The amount of dipoles is also very low so it is reasonable to ignore that term. If, however, one needs to take the dipole oscillations into account, there exists another assumption, namely that the polarization depends linearly on the strength of the electric field: $\mathbf{P} = \epsilon_0 \chi \mathbf{E}$. Here the factor χ is called the susceptibility. This way, the wave equation in an isotropic medium with some dipoles is simply the same equation but then with a slightly different constant to account for the different propagation speed. The index of refraction n is defined as $n := \sqrt{1 + \chi}$ such that:

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{c^2}{n^2} \nabla^2 \mathbf{E}.$$

Lemma 1. *Assuming monochromatic light focused in a beam along the z-axis with slowly varying amplitude and ignorable vectorial nature, the wave equation can be written as the paraxial wave equation:*

$$\frac{\partial^2 \tilde{E}}{\partial x^2} + \frac{\partial^2 \tilde{E}}{\partial y^2} + 2ik\frac{\partial \tilde{E}}{\partial z} = 0. \tag{3.4}$$

Proof. [7] If we consider light with a single frequency, which we have for a monochromatic laser, we can make the separation of variables as follows:

$$\mathbf{E}(t, \mathbf{r}) = \mathbf{E}(\mathbf{r})E(\omega t).$$

Where ω is the angular frequency, and $E(\omega t) = e^{-i\omega t}$. Where only the real part is physically relevant. The wave equation then becomes the vector Helmholtz equation:

$$\nabla^2 \mathbf{E}(\mathbf{r}) = -k^2 \mathbf{E}(\mathbf{r}).$$

Where $k = n\omega/c$ is called the wave number and defines the dispersion relation. If we now only consider the magnitude of the electric field, this becomes the scalar Helmholtz equation.

$$\nabla^2 E(\mathbf{r}) = -k^2 E(\mathbf{r}).$$

Next, since laser light propagates mainly along the z-axis, we can approximate the field as $E(\mathbf{r}) = E(x, y, z) = \tilde{E}(x, y, z)e^{ikz}$. The scalar Helmholtz equation then becomes:

$$\frac{\partial^2 \tilde{E}}{\partial x^2} + \frac{\partial^2 \tilde{E}}{\partial y^2} + \frac{\partial^2 \tilde{E}}{\partial z^2} + 2ik\frac{\partial \tilde{E}}{\partial z} = 0.$$

The last assumption is that the amplitude varies slowly in time, which is to say, it may only change over distances much longer than the wavelength. This assumption is called the slowly varying envelope approximation. Mathematically this is expressed as $|\frac{\partial^2 \tilde{E}}{\partial z^2}| \ll |2k\frac{\partial \tilde{E}}{\partial z}|$. The equation then obtained is known as the paraxial wave equation (3.4). $\square$

3.2 Diffraction integral

Lemma 2. *[7] The Fresnel diffraction formula, approximated by restricting to small angles, solves the paraxial wave equation and is given by:*

$$E(x, y, z) = -\frac{ie^{ikz}e^{i\frac{k}{2z}(x^2+y^2)}}{\lambda z} \iint\limits_{aperture} E(x', y', 0)e^{i\frac{k}{2z}(x'^2+y'^2)}e^{-i\frac{k}{z}(xx'+yy')}dx'dy',$$

where λ is the wavelength of the light.

This equation gives the propagation of a field behind an aperture, by taking every point in the aperture to be the origin of a spherical wave which is called the Huygens principle. If the aperture has cylindrical symmetry, we can use cylindrical coordinates: $\rho^2 = x^2 + y^2$, $x = \rho \cos\phi$ and $y = \rho \sin\phi$. The diffraction integral then becomes:

$$E(\rho, z) = -\frac{ie^{ikz}e^{i\frac{k\rho^2}{2z}}}{\lambda z} \int\limits_{0}^{2\pi} \int\limits_{aperture} \rho' E(\rho', 0)e^{i\frac{k\rho'^2}{2z}}e^{-i\frac{k}{z}\rho\rho'(\cos\phi\cos\phi'+\sin\phi\sin\phi')}d\rho'd\phi'.$$

Using the trigonometric relation $\cos\phi\cos\phi' + \sin\phi\sin\phi' = \cos(\phi' - \phi)$, the integral over the angle can be solved to give:

$$E(\rho, z) = -\frac{2\pi ie^{ikz}e^{i\frac{k\rho^2}{2z}}}{\lambda z} \int\limits_{aperture} \rho' E(\rho', 0)e^{i\frac{k\rho'^2}{2z}} J_0\left(\frac{k\rho\rho'}{z}\right) d\rho', \qquad (3.5)$$

where J_0 is the zero-order Bessel function.

3.3 Gaussian beam

Given the field of a laser operating in TEM_{00} mode, we would like to know the propagation of the Gaussian shaped laser beam. The wave profile for the lowest order mode at $z = 0$ can be written as

$$E(\rho, 0) = E_0 e^{-\frac{\rho^2}{w_0^2}}. \qquad (3.6)$$

Where E_0 is the wave's peak amplitude at $\rho = 0$ and w_0 is called the beam waist, which is the radius at which the electric field decreases by a factor e^{-1}. Here the Gaussian itself functions as the 'aperture', so integration to infinity makes sense. This also ensures the paraxial wave equation is solved by the Fresnel diffraction integral.

Theorem 1. *The propagation of the electric field with Gaussian initial conditions as in (3.6) can be written as:*

$$E(\rho, z) = E_0 \frac{w_0}{w(z)} e^{-\frac{\rho^2}{w^2(z)}} e^{i\left(kz + \frac{k\rho^2}{2R(z)} - \arctan\left(\frac{z}{z_0}\right)\right)},\qquad(3.7)$$

with

$$z_0 := \frac{k w_0^2}{2},$$

$$w(z) := w_0 \sqrt{1 + z^2/z_0^2},\qquad(3.8)$$

$$R(z) := z + z_0^2/z.$$

Proof. Filling in the initial condition (3.6) in (3.5) gives:

$$E(\rho, z) = -iE_0 \frac{2\pi e^{ikz} e^{i\frac{k\rho^2}{2z}}}{\lambda z} \int_0^\infty \rho' e^{-\frac{\rho'^2}{w_0^2}} e^{i\frac{k\rho'^2}{2z}} J_0\left(\frac{k\rho\rho'}{z}\right) d\rho'$$

$$= -iE_0 \frac{2\pi e^{ikz} e^{i\frac{k\rho^2}{2z}}}{\lambda z} \frac{e^{-\frac{\left(\frac{k\rho}{z}\right)^2}{4\left(\frac{1}{w_0^2} - i\frac{k}{2z}\right)}}}{2\left(\frac{1}{w_0^2} - i\frac{k}{2z}\right)}$$

$$= E_0 \frac{e^{ikz} e^{i\frac{k\rho^2}{2z}}}{\sqrt{1 + \left(\frac{2z}{k w_0^2}\right)^2}} e^{-\frac{\rho^2}{1+\left(\frac{2z}{k w_0^2}\right)^2}\left(\frac{1}{w_0^2} + i\frac{k}{2z}\right)} e^{-i\arctan\left(\frac{2z}{k w_0^2}\right)}.$$

Here we have used the relation:

$$e^{i\arctan(x)} = \frac{1 + ix}{\sqrt{1 + x^2}},$$

in the last line. This equation is often written more compactly as (3.7) using substitution of (3.8), this proves the theorem. $\qquad\square$

Since $I(\rho, z) \propto |E(\rho, z)|^2$, we get

$$I(\rho, z) = I_0 \frac{w_0^2}{w^2(z)} e^{-\frac{2\rho^2}{w^2(z)}} = \frac{I_0}{1 + z^2/z_0^2} e^{-\frac{2\rho^2}{w^2(z)}}.\qquad(3.9)$$

It is clear that along z the radius of the beam is given by $w(z)$. The beam radius depends on z_0, known as the Rayleigh range, and governs the distance starting from $z = 0$ over which

the beam intensity halves and the width increases by $\sqrt{2}$. The terms $ikz + ik\rho^2/2R(z)$ describe the phase of the curved wave fronts, with $R(z)$ being the radius of curvature. The final term $i\arctan(z/z_0)$ is called the Gouy phase, being most pronounced near $z = 0$. For large z the intensity drops as $I(0, z) \approx z_0^2/z^2$, and the width goes as $w(z) \approx w_0 z/z_0$. Therefore a very small waist, and hence also a very small Rayleigh range, causes the beam to diverge and lose intensity fast. The smaller the waist, the faster the divergence. This is an example of the uncertainty principle in action. [8]
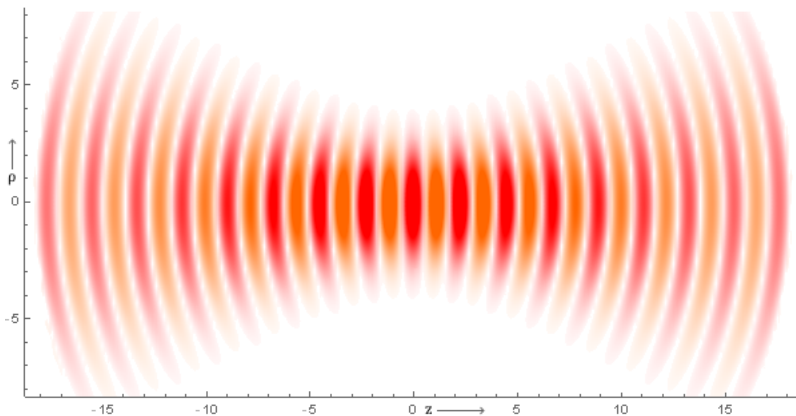

Figure 4: Real part of the electric field of a Gaussian beam for $w_0 = 2, k = 3$ at time $t = 0$. Red indicates a positive value for the field, orange negative. [Self, A.3]

Returning to the electric field, if we assume that the electric is linearly polarized in the x-direction, the field will be maily directed in the x-direction but also slightly in the z-direction because the wave fronts tilt off the beam axis. [9] The total, time dependent electric field is then given by [10]:

$$\mathbf{E}(t, x, y, z) = \mathrm{Re}\left\{\left(\hat{\mathbf{x}}\omega E(x, y, z) + \hat{\mathbf{z}}i\frac{kc^2}{\omega}\frac{\partial E(x, y, z)}{\partial x}\right)e^{-i\omega t}\right\}. \tag{3.10}$$

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4 Quantum Mechanics

We have first seen a model for a laser beam using the optics approach. Starting from the wave equation and making several assumptions we arrived at the paraxial wave equation. The problem can however also be approached using quantum mechanics by viewing the z-axis of the beam as time in the Schrödinger equation and the cross section of the beam as space. The intensity profile of the beam then corresponds to the probability density. The role of time in the optical case has no analogue in the quantum mechanical case. [9] These quantum mechanical solutions we will call a wave packet because it no longer represents a continuous beam but a short burst or packet. Also note that the paraxial wave equation (3.4) closely resembles the free space Schrödinger equation given by:

$$i\frac{\partial \Psi}{\partial t} + \frac{1}{2m}\nabla^2 \Psi = 0.$$

Where we have set $\hbar = 1$ and we could set the mass $m = k$ such that it coincides with the paraxial wave equation. Because of this we suspect in advance that this approach will more or less yield the same result. Note that the Schrödinger equation is actually only meant for non-relativistic particles, which photons are not. Consequently, the energy of the solution does not simply correspond to the energy of the photon. Indeed, a free particle can't even have a definite energy. [11]

4.1 Free Space

Theorem 2. *[11] The general solution to the free space Schrödinger equation is given by the Fourier integral:*

$$\Psi(t, \mathbf{x}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \hat{\Psi}(\mathbf{k}) e^{i\left(\mathbf{k}\cdot\mathbf{x} - \frac{|\mathbf{k}|^2 t}{2m}\right)} d\mathbf{k},$$

$$\hat{\Psi}(\mathbf{k}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \Psi(0, \mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}.$$

Because we want to compare the Schrödinger equation with the optics approach we are specifically interested in the Gaussian initial condition (3.6). This brings us to the next proposition.

Proposition 1. *The solution of the free space Schrödinger equation with Gaussian initial condition*

$$\Psi(0, \mathbf{x}) = e^{-\frac{|\mathbf{x}|^2}{w_0^2}}, \tag{4.1}$$

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with $\mathbf{x} \in \mathbb{R}^2$, is given by

$$\Psi(t, \rho) = \frac{w_0}{w(t)} e^{-\frac{\rho^2\left(1 - i\frac{t}{z_0}\right)}{w^2(t)} - i\arctan\left(\frac{t}{z_0}\right)}. \tag{4.2}$$

Proof. First let $\mathbf{x} \in \mathbb{R}^d$ Next remember that Gaussian integrals have a standard solution given by:

$$\int_{-\infty}^{\infty} e^{-ax^2 + bx + c} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a} + c} \qquad \text{for } \text{Re}\{a\} > 0.$$

Then, filling the initial state (4.1) in, gives:

$$\hat{\Psi}(\mathbf{k}) = \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|\mathbf{x}|^2}{w_0^2} - i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} = \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \left(\pi w_0^2\right)^{\frac{d}{2}} e^{-\frac{w_0^2|\mathbf{k}|^2}{4}}, \tag{4.3}$$

$$\Psi(t, \mathbf{x}) = \left(\frac{1}{2\pi}\right)^d \left(\pi w_0^2\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{w_0^2|\mathbf{k}|^2}{4} + i\left(\mathbf{k}\cdot\mathbf{x} - \frac{|\mathbf{k}|^2 t}{2m}\right)} d\mathbf{k} = \left(\frac{w_0^2}{w_0^2 + i\frac{2t}{m}}\right)^{\frac{d}{2}} e^{-\frac{|\mathbf{x}|^2}{w_0^2 + i\frac{2t}{m}}}.$$

We are specifically interested in two spatial dimensions for the cross section of the beam. Setting $d = 2$ yields:

$$\Psi(t, \rho) = \frac{1}{1 + \frac{2it}{mw_0^2}} e^{-\frac{\rho^2}{w_0^2\left(1 + \frac{2it}{mw_0^2}\right)}} = \frac{1}{\sqrt{1 + \left(\frac{2t}{mw_0^2}\right)^2}} e^{-\frac{\rho^2\left(1 - i\frac{2t}{mw_0^2}\right)}{w_0^2\left(1 + \left(\frac{2t}{mw_0^2}\right)^2\right)} - i\arctan\left(\frac{2t}{mw_0^2}\right)}. \tag{4.4}$$

Letting $m = k$ as suggested by the paraxial wave equation and using the same variables as in the optics approach reduces this down to the result. $\qquad \square$

As can be seen, part of the phase is different, mainly due to the extra terms in front of the diffraction integral. But since the approach did not include actual time dependence the phase was never expected to be the same. The probability distribution is given by:

$$|\Psi(t, \rho)|^2 = \frac{2}{\pi w_0^2} \frac{w_0^2}{w^2(t)} e^{-\frac{2\rho^2}{w^2(t)}},$$

where the factor $\frac{2}{\pi w_0^2}$ in front comes from normalization. As expected this does match the intensity profile (3.9) calculated using the Gaussian beam.

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4.2 Time Independent Harmonic Oscillator

A very important part of the phase of the Gaussian beam is $ik\rho^2/2R(z)$ for the curvature of the wavefront. This part is notably missing in the solution (4.2) of the free space Schrödinger equation. This brings us to the next theorem.

Theorem 3. *[9] The paraxial wave equation* (3.4) *can be reduced under the ansatz*

$$\Psi(x,y,z) = \psi(\xi,\eta,\tau)\frac{e^{ik\frac{x^2+y^2}{2R(z)}}}{w(z)}, \qquad (4.5) \quad \left| \begin{array}{l} \xi = \frac{\sqrt{2}x}{w(z)} \\ \eta = \frac{\sqrt{2}y}{w(z)} \\ \tau = \arctan(\frac{z}{z_0}) \end{array} \right.$$

to a two dimensional quantum harmonic oscillator.

Proof. Using the ansatz in the paraxial wave equation gives:

$$\left(2i\frac{\partial}{\partial\tau} + \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} - \xi^2 - \eta^2 \right)\psi(\xi,\eta,\tau) = 0, \qquad (4.6)$$

by straightforward computations. □

This establishes an equivalence between the harmonic oscillator and focused beams. More precisely, since the plane wave phase factor e^{ikz} is missing, the beam's envelope amplitude is linked with a two-dimensional harmonic oscillator wave function. Here, again, the time $\tau \in (-\pi/2, \pi/2)$ in the quantum case corresponds to the axial coordinate $z \in (-\infty, \infty)$ in the optical case.

4.2.1 Spectral theory

A more general Schrödinger equation for the harmonic oscillator can be written as:

$$i\frac{\partial\psi(t,\mathbf{x})}{\partial t} + \frac{1}{2m}\nabla^2\psi(t,\mathbf{x}) - \frac{1}{2}m\Omega^2|\mathbf{x}|^2\psi(t,\mathbf{x}) = 0, \qquad (4.7)$$

with Ω the harmonic oscillator frequency, $\mathbf{x} \in \mathbb{R}^d$ and again $\hbar = 1$. This corresponds to (4.6) for $m = \Omega = 1$ The time independent harmonic oscillator problem has been studied widely and has known solutions. The equation allows for separation of variables and the separation constant is denoted by E, as by convention, to denote that it is the energy of the associated Hamiltonian $H = -\frac{1}{2m}\nabla^2 + \frac{1}{2}m\omega^2|\mathbf{x}|^2$. Separation of variables as $\psi(t,\mathbf{x}) = \phi(\mathbf{x})\varphi(t)$ leads to the following ordinary differential equations in $\mathbf{x}$ and t:

$$H\phi(\mathbf{x}) = -\frac{1}{2m}\nabla^2\phi(\mathbf{x}) + \frac{1}{2}m\Omega^2|\mathbf{x}|^2\phi(\mathbf{x}) = E\phi(\mathbf{x}),$$

$$\frac{\partial\varphi(t)}{\partial t} = -iE\varphi(t).$$

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In this equation one can recognize a eigenequation with H the linear operator, $\psi(\mathbf{x})$ the eigenfunction and E the eigenvalue. Similarly to the finite dimensional matrix equation, if there exists a basis that diagonalizes the operator then the eigenequation becomes really easy as every basis element is an eigenfunction itself. In other words, we would like an orthogonal basis $\{\psi_n : n \in \mathbb{N}\}$ for some function space such that $H\psi_n = E_n\psi_n$. Then any solution to the eigenequation can be written as a linear combination of basis functions:

$$H f(\mathbf{x}) = \sum_n H\psi_n(\mathbf{x}) = \sum_n E_n\psi_n(\mathbf{x})$$

A very well known and much used function space is the L^2 space also known as the Hilbert space to physicists but mathematicians use the term Hilbert space to denote a complete inner product space which the L^2 space is. [12] The L^2 space is defined as:

$$L^2 = \{f : \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty\}.$$

Because of this definition it is sometimes also called the space of square-integrable functions. The inner product on this space is given by:

$$\langle f, g \rangle := \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx.$$

In general it is not true that an operator admits diagonalization. One of the many spectral theorem says that a compact, self adjoint operator on a Hilbert space can be diagonalized. [13] The Hamiltonian for the harmonic oscillator is actually unbounded and hence not compact, luckily the spectral theorem extends with less stringent conditions to include the harmonic oscillator. [14] The Harmonic oscillator has a discrete spectrum of (unbounded) eigenvalues and eigenfunctions consisting of the Hermite polynomials.

Lemma 3. *[11] The one-dimensional harmonic oscillator* (4.7) *has the following energies, eigenfunctions and solutions:*

$$E_n = \left(n + \frac{1}{2}\right)\Omega, \qquad\qquad n = 0, 1, 2, ...$$

$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\Omega}{\pi}\right)^{\frac{1}{4}} H_n(\sqrt{m\Omega}x)e^{-\frac{m\Omega x^2}{2}},$$

$$\psi_n(x, t) = \phi_n(x)e^{-iE_n t}. \qquad\qquad\qquad (4.8)$$

With the Hermite polynomials $H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}$.

In general the spectrum does not need to be discrete or may not have eigenfunctions in L^2 at all. The momentum operator is a prime example of this: $-i\frac{\partial}{\partial x}\phi(x) = k\phi(x)$ has the general solution $\phi(x) = e^{ikx}$, which is not square-integrable for any $k \in \mathbb{C}$. But these functions do form a complete basis for L^2 with a continuous spectrum, they can be thought of as generalized eigenfunctions. Note that for $k \in \mathbb{R}$ the inner product between two of these functions is: $\langle k, k' \rangle = \delta(k - k')$ with the Dirac delta function δ, so in some sense they are orthogonal. Also note that expanding any L^2 function in terms of the eigenfunctions of the momentum operator comes down to doing a Fourier transform, which was already used in the free space solution and which will be used again.

4.2.2   Higher Order Modes

The higher order TEM_{mn} modes of the laser can be found by using the ansatz (4.5) and the solution to the harmonic oscillator (4.6). Assuming the 2D harmonic oscillator is separable in it's Cartesian spatial variables, the solution is simply the product of the two 1D harmonic oscillators (4.8):

$$\Psi_{mn}(x, y, z) = \frac{w_0}{w(z)} \phi_m\left(\frac{\sqrt{2}x}{w(z)}\right) \phi_n\left(\frac{\sqrt{2}y}{w(z)}\right) e^{ik\frac{x^2+y^2}{2R(z)}} e^{-i(m+n+1)\arctan\left(\frac{z}{z_0}\right)}.$$

Comparing this with the Gaussian beam (3.7) found in the previous chapter, we firstly note that the Gouy phase has additional factors in front. Evidently the higher order modes accumulate a larger phase shift when passing through the focus. A more rigorous derivation of the Gouy phase shift can be found in [8]. Secondly we can see that the phase ikz is not present. Artificially reconstructing the phase gives the expression for the so called Hermite-Gaussian modes:

$$E_{mn}(x, y, z) = \frac{w_0}{w(z)} \phi_m\left(\frac{\sqrt{2}x}{w(z)}\right) \phi_n\left(\frac{\sqrt{2}y}{w(z)}\right) e^{i\left(kz + \frac{k(x^2+y^2)}{2R(z)} - (m+n+1)\arctan\left(\frac{z}{z_0}\right)\right)}.$$

The total electric field is then again given by (3.10).

When solving the 2D harmonic oscillator by using polar coordinates and separating the variables as radial and angular parts, the solutions contain the generalized Laguerre polynomials instead of the Hermite polynomials. The solution then is [15]:

$$E_{pl}(r, \theta, z) = \frac{w_0}{w(z)} \left(\frac{r\sqrt{2}}{w(z)}\right)^l L_p^l\left(\frac{2r^2}{w^2(z)}\right) e^{-\frac{r^2}{w^2(z)}} e^{i\left(l\theta + \frac{kr^2}{2R(z)} - (2p+l+1)\arctan\left(\frac{z}{z_0}\right)\right)}.$$

The real part of the electric field for the Hermite-Gaussian and the Laguerre-Gaussian modes at time $t = 0$ and $z = 0$ is plotted in figure 2 and 3 respectively. Figure 5 shows a Hermite-Gaussian beam for TEM_{30} with the same parameters as figure 4. Comparing the two figures it is clear that the higher order mode diverges faster. It is also visible that the beam waist is larger for the higher order mode.
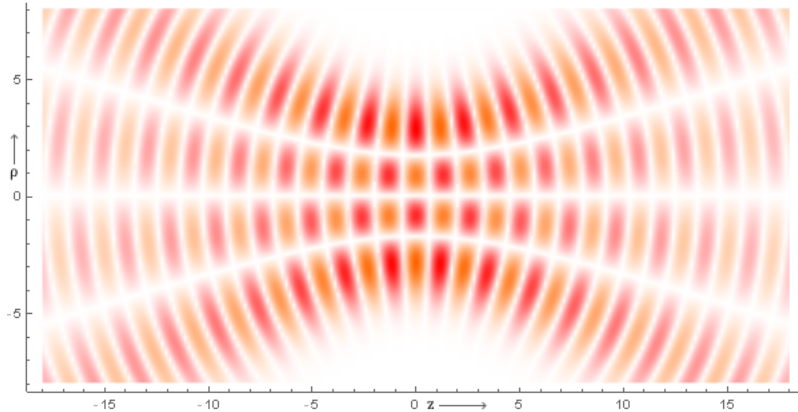
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Figure 5: Real part of the electric field of a Hermite-Gaussian TEM$_{30}$ beam for $w_0 = 2, k = 3$ at time $t = 0$. Red indicates a positive value for the field, orange negative. [Self, A.4]

4.3 Wave Packets

The solution in theorem 2 can be classified as a Fourier integral operator, often abbreviated as FIO. These solutions are essentially a Fourier transform where an additional factor is added in the inverse transform. The result is a wave packet, a superposition of various wave functions to from a localized packet, which is normalizable in contrast to plane waves. Wave packets are widely used throughout mathematical physics. The normalizability aids in the observability and controllability of solutions. [16] Decomposing the free space Schrödinger equation into Gaussian wave packets to establish existence and uniqueness of solutions for a nonlinear Schrödinger equation is done in [17]. And a wave packet transform is introduced in [18] as a generalization of the Fourier transform. When a problem is too complex to be solved exactly, sometimes estimates have to do, see for example [19] and [20] for an estimate on the wave and heat equation.

4.3.1 Quadratic Hamiltonians

A lot of research has been done into more general forms of the Schrodinger equation. Consider for example the class of Schrödinger equations of the form

$$i\frac{\partial \Psi}{\partial t} = \hat{H}\Psi,$$

where $\hat{H}$ is a quadratic Hamiltonian operator, which a differential operator with terms that are quadratic in the position and momentum operators. For example, in one spatial

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dimension the most general Hermitian Hamiltonian is of the form [21]:

$$\hat{H} = c_1\hat{p}^2 + c_2(\hat{p}\hat{x} + \hat{x}\hat{p}) + c_3\hat{x}^2 + c_4\hat{p} + c_5\hat{x},$$

where $\hat{x}$ and $\hat{p} = -i\frac{\partial}{\partial x}$ are the position and momentum operators respectively. The coefficients are allowed to depend on time. The free space and the harmonic oscillator are both examples of equations with quadratic Hamiltonian. In quantum optics there are also examples to time-dependent quadratic Hamiltonians like the Paul trap or the parametric amplifier. [21] It is shown in [22] that the solution to these Schrödinger equations with quadratic Hamiltonians exist and are unique and that they are given by Gaussian Fourier integral operators.

4.3.2 Time Evolution and Propagators

When considering a Schrödinger equation with a time independent Hamiltonian, we can formally write the solution using a time evolution operator:

$$\Psi(t, \mathbf{x}) = e^{-i\hat{H}t}\Psi(0, \mathbf{x}).$$

As long as the initial state is known, we can write down the time evolution. Note that the evolution operator $U(t, x) := e^{-i\hat{H}t}$ is unitary if the Hamiltonian is Hermitian. The operator itself features an Hamiltonian operator in the exponent, pulling out a calculator won't suffice to compute this. The way to make sense of this notation is by using the power series expansion of the exponential. Special care has to be taken for unbounded Hamiltonians because the series expansion need not converge. If separation of variables is applied, the evolution operator becomes e^{-iEt}, which is a lot more accessible but this assumes that the initial state is an energy eigenstate of the Hamiltonian: $H\Psi = E\Psi$, which means not every wave function can be evolved in time. For time dependent Hamiltonians a time evolution operator may also be found but the procedure is a lot more involved.

A time evolution equation looks nice and compact but for explicitly computing things it is not that useful. Enter propagators: the fundamental solutions of the Schrödinger equation. [23] A fundamental solution is the solution $G(t, x, t_0, x')$ such that:

$$\left(i\frac{\partial}{\partial t} - \hat{H}\right)G = \delta(t - t_0)\delta(x - x').$$

By Duhamel's principle this corresponds to the homogeneous Schrödinger equation combined with the delta function only in space as initial condition:

$$\left(i\frac{\partial}{\partial t} - \hat{H}\right)K(t, x, t_0, x') = 0,$$

$$K(t_0, x, t_0, x') = \delta(x - x').$$

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Where we can identify K as the kernel of the Schrödinger equation. The solution is then given by:

$$G(t, x, t', x') = -i \int_{t_0}^{t} \delta(t' - t_0) K(t', x, t_0, x') dt' = -i\theta(t - t_0) K(t, x, t_0, x').$$

With $\theta(t - t')$ the unit step function. From the fundamental solution the solution to any initial condition $\Psi(t_0, x)$ can be created as follows:

$$\Psi(t, x) = \int_{\mathbb{R}} K(t, x, t_0, x') \Psi(t_0, x') dx'.$$

For many Hamiltonians the propagator is known. For the harmonic oscillator there exists the well known Mehler kernel. In [24] the time evolution operators and the propagators of various Hamiltonians, of which is the time dependent harmonic oscillator, are derived.

4.4   Time Dependent Harmonic Oscillator

Adding a time dependent coefficients to the Schrödinger equation allows us to model more complicated situations. Let us therefore consider the time dependent harmonic oscillator:

$$i\frac{\partial u(t, \mathbf{x})}{\partial t} + \kappa_1(t)\nabla^2 u(t, \mathbf{x}) - \kappa_2(t)|\mathbf{x}|^2 u(t, \mathbf{x}) = 0. \tag{4.9}$$

With $\kappa_1(t) > 0$ and $\kappa_2(t) \geq 0$ depending continuously on time t for $0 \leq t \leq T$. Before solving this problem there is a need for the following definition.

Definition 1 (Hamiltonian flow [16])**.** The associated Hamiltonian to (4.9) is $H = \kappa_1(t)|\mathbf{p}|^2 + \kappa_2(t)|\mathbf{x}|^2$. The equations of motion are:

$$\frac{dx_i(t)}{dt} = 2\kappa_1(t)p_i(t) \quad \frac{dp_i(t)}{dt} = -2\kappa_2(t)x_i(t) \quad i = 1, ..., d \quad 0 \leq t \leq T,$$

$$\mathbf{x}(0) = (1, ..., 1) \qquad \mathbf{p}(0) = (0, ..., 0). \tag{4.10}$$

Let the solution, also called (Hamiltonian) flow, be given by $(\mathbf{x}(t), \mathbf{p}(t))$. Here the flow is called *non-zero* if $\mathbf{x}(t) \neq 0$ for all $0 \leq t \leq T$.

For the time dependent harmonic oscillator it is not possible to use separation of variables to obtain an ordinary differential equation, hence the spectral theory from the previous section can not be used to write the solution as a sum of eigenfunctions. Instead, we will make use of the Fourier integral operators to construct the Gaussian wave packets. We suspect the propagator constructed in [24] to yield the same result as the Fourier integral operator method.

Theorem 4. *[16] Given the Hamiltonian flow is non-zero and $e^{-\int_0^t x'(s)/x(s)ds}$ is bounded for all $0 \le t \le T$, the associated Fourier integral operator solutions for the time dependent harmonic oscillator (4.9) can be written as:*

$$u(t, \mathbf{x}) = \frac{a^d(t)}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \hat{u}(\mathbf{k})e^{i\phi(t,\mathbf{x},\mathbf{k})}d\mathbf{k},$$

$$\hat{u}(\mathbf{k}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} u(0, \mathbf{x})e^{-i\mathbf{x}\cdot\mathbf{k}}d\mathbf{k}.$$

The factor ϕ, which is the phase, is of the form:

$$\phi(t, \mathbf{x}, \mathbf{k}) = y_1(t)|\mathbf{x}|^2 + y_2(t)\mathbf{x}\cdot\mathbf{k} + y_3(t)|\mathbf{k}|^2.$$

Where y_1, y_2 and y_3 are continuous functions of time $0 \le t \le T$ and for simplicity we assume they are real valued. The phase must solve the eikonal equation:

$$\frac{\partial}{\partial t}\phi + \kappa_1(t)|\nabla_x\phi|^2 + \kappa_2(t)|\mathbf{x}|^2 = 0.$$

This gives:

$$y_1'|\mathbf{x}|^2 + y_2'\mathbf{x}\cdot\mathbf{k} + y_3'|\mathbf{k}|^2 + \kappa_1(4y_1^2|\mathbf{x}|^2 + y_2^2|\mathbf{k}|^2 + 4y_1y_2\mathbf{x}\cdot\mathbf{k}) + \kappa_2|\mathbf{x}|^2 = 0.$$

Which gives the following system of equations by grouping the coefficients:

$$\begin{array}{ll} y_1'(t) = -4\kappa_1(t)y_1^2(t) - \kappa_2(t) & y_1(0) = 0, \\ y_2'(t) = -4\kappa_1(t)y_1(t)y_2(t) & y_2(0) = 1, \\ y_3'(t) = -\kappa_1(t)y_2^2(t) & y_3(0) = 0, \end{array} \qquad (4.11)$$

where the initial conditions are a results of $\nabla_x\phi(0, \mathbf{x}, \mathbf{k}) = \mathbf{k}$ and $y_3(0) = 0$ to ensure the initial condition is satisfied.
The amplitude satisfies:

$$a'(t) = -2\kappa_1(t)y_1(t)a(t) \qquad\qquad a(0) = 1.$$

And is solved by:

$$a(t) = e^{-2\int_0^t \kappa_1(t)y_1(s)ds},$$

which is bounded by assumption, see also the next lemma.

Lemma 4. *The solution to $y_1(t)$ in (4.11) exists and is unique when the Hamiltonian flow is non-zero*

Proof. The first equation in (4.11) can be classified as a Ricatti equation and can be reduced to a second order linear ordinary differential equation. Namely, if

$$S = 4\kappa_1(t)\kappa_2(t),$$

$$R = \frac{\kappa_1'(t)}{\kappa_1(t)},$$

$$y_1(t) = \frac{x'(t)}{4\kappa_1(t)x(t)},$$

are substituted into (4.11) then the Ricatti equation becomes:

$$x''(t) - Rx' + Sx = 0 \qquad\qquad x'(0) = 0 \qquad\qquad x(0) = 1. \qquad (4.12)$$

To ensure the problem exists and is unique we need κ_1 nonzero and differentiable and $x(t)$ nonzero. Incidentally, this equation is identical to the equations of motion (4.10), hence the assumption that the Hamiltonian flow is non-zero. $\qquad\square$

Remark. If however the flow is not non-zero for some time t_0, the solution for $t > t_0$ may be defined again, yet might pick up a phase factor called the Maslov phase, which is connected with the Gouy phase for the Gaussian Beam. [25]

4.4.1 Examples

If the initial conditions are given by the Gaussian (4.1), the same solution to the Fourier integral (4.3) can be used. Higher order modes can also be evaluated but here we will stick to the lowest order. The full solution then is given by:

$$u(t, \mathbf{x}) = \left(\frac{a^2(t)w_0^2}{4\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{w_0^2|\mathbf{k}|^2}{4}} e^{i\phi(t,\mathbf{x},\mathbf{k})} d\mathbf{k}$$

$$= \left(\frac{a^2(t)w_0^2}{w_0^2 - 4iy_3(t)}\right)^{\frac{d}{2}} e^{iy_1(t)|\mathbf{x}|^2} e^{-\frac{y_2^2(t)|\mathbf{x}|^2}{w_0^2 - 4iy_3(t)}}. \tag{4.13}$$

If the initial conditions are given by the 2-dimensional state:

$$u(0, \rho) = e^{-\frac{\rho^2}{w_0^2}},$$

the solution is given by:

$$u(t, \rho) = \frac{w_0^2 a^2(t)}{w_0^2 - 4iy_3(t)} e^{iy_1(t)\rho^2} e^{-\frac{y_2^2(t)\rho^2}{w_0^2 - 4iy_3(t)}}. \tag{4.14}$$

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Comparing this solution (4.14) with the free space solution (4.2) it is clear that it corresponds to the free space solution for:

$$y_1 = 0 \qquad\qquad \kappa_1 = \frac{1}{2m},$$

$$y_2 = 1 \qquad\qquad \kappa_2 = 0,$$

$$y_3 = -\frac{t}{2m}.$$

The variables κ_1, κ_2 are as expected and the system of equations (4.11) holds for all time $t \in \mathbb{R}$.

Proposition 2. *The time dependent harmonic oscillator solution with Gaussian initial condition (4.1) reduces exactly to the the ground state of the time independent harmonic oscillator solution (4.8).*

Proof. Comparing equation (4.13) to the one dimensional harmonic oscillator, it is not immediately clear how to pick y_1, y_2, y_3. Let us therefore solve the system of equations as an example for the time independent harmonic oscillator. Let $\kappa_1 = \kappa_2 = 1/2$, then the Hamiltonian flow, or equivalently the Ricatti equation (4.12), gives $x(t) = \cos(t)$, $p(t) = -\sin(t)$. Note that the flow is nonzero for $0 \le t < \pi/2$ but let's carry on regardless. For the system of equations we get:

$$y_1(t) = -\frac{1}{2}\tan(t) \qquad y_2(t) = \frac{1}{\cos(t)} = \sec(t),$$

$$y_3(t) = -\frac{1}{2}\tan(t) \qquad a(t) = \frac{1}{\sqrt{\cos(t)}} = \sqrt{\sec(t)}.$$

Plugging this into the solution (4.13) with $d = 1$ and $w_0 = \sqrt{2}$ gives:

$$u(t, \mathbf{x}) = \left(\frac{\sec(t)}{1 + i\tan(t)} \right)^{\frac{1}{2}} e^{-i\frac{\tan(t)}{2}\mathbf{x}^2} e^{-\frac{\sec^2(t)\mathbf{x}^2}{2 + 2i\tan(t)}}$$

$$= \sqrt{\cos(t) - i\sin(t)} e^{-i\frac{\tan(t)}{2}\mathbf{x}^2} e^{-\frac{x^2}{2}(1 - i\tan(t))}$$

$$= e^{-i\frac{t}{2}} e^{-\frac{x^2}{2}}.$$

Which is exactly the ground state of the harmonic oscillator. $\qquad\square$

Hence, in this case, the time dependent solution gave the correct solution even though the Hamiltonian flow has repeated zero's and the hypothesis for the solution was only satisfied for $0 \le t < \pi/2$. Note that terms that might have given issues like the root, the secant or the

tangent all vanish upon rewriting such that the final expression is smooth. A critical reader might object to rewriting the root, or even the root itself since it is not well defined. Indeed, the root of a complex number $\zeta(t)$ is a multiple-valued function and the root taken here is such that it is unity at $t = 0$, which is the principal branch $e^{i\arg(\zeta(t))/2}$ for $-\pi < \zeta(t) \leq \pi$. When the continuous function $\zeta(t) = e^{-it}$ crosses the branch cut, we let it move on to the other branch $e^{i\arg(\zeta(t))/2+i\pi}$, this way a factor -1 is picked up every 2π. For a different approach see exercise 11.1 in [26], which gives a guide on how to extend this to $t \in \mathbb{R} \setminus \mathbb{Z}\pi$..

Previous examples all had time independent coefficients, time dependent examples are a lot more involved and often don't feature particularly nice solutions. Let the final example on the time dependent harmonic oscillator be the Caldirola-Kanai oscillator, also treated as example in [16]. Let the time dependent coefficients be given by:

$$\kappa_1(t) = \frac{e^{-2at}}{2} \qquad\qquad \kappa_2(t) = \frac{e^{2at}\sigma^2}{2}.$$

Where $a > 0$, $\sigma \in \mathbb{R} \setminus \{0\}$. The equations of motion from the Hamiltonian flow (4.12) then become:

$$x''(t) + 2ax'(t) + \sigma^2 x(t) = 0.$$

In this we recognize a classical damped harmonic oscillator, which can be readily solved by an exponential ansatz with $\lambda = \sqrt{a^2 - \sigma^2}$ to give for $a^2 - \sigma^2 > 0$:

$$x(t) = e^{-at} \left(\cosh(\lambda t) + \frac{a}{\lambda} \sinh(\lambda t) \right),$$

$$p(t) = e^{at} \left(\lambda - \frac{a^2}{\lambda} \right) \sinh(\lambda t).$$

As can be seen the Hamiltonian flow is non-zero for all time $t > 0$. The case for $a^2 - \sigma^2 \leq 0$ won't be discussed here but the reader is invited to work it out. The system of equations (4.11) then gives:

$$y_1(t) = \frac{e^{2at}}{2} \frac{(\lambda - \frac{a^2}{\lambda}) \sinh(\lambda t)}{\cosh(\lambda t) + \frac{a}{\lambda} \sinh(\lambda t)},$$

$$y_2(t) = \frac{\lambda e^{at}}{\lambda \cosh(\lambda t) + a \sinh(\lambda t)},$$

$$y_3(t) = -\frac{\sinh(\lambda t)}{2a \sinh(\lambda t) + 2\lambda \cosh(\lambda t)}.$$

The amplitude $a(t) = \sqrt{y_2(t)}$, which is always the case as can be seen from their respective differential equations, which differ by a factor 2. The time dependent coefficients in the

Caldirola-Kanai oscillator correspond to a time dependent mass. A time dependent mass is often used to model energy damping or pumping. One such application in optics is given in [27] where it is used to study quantum effects in interferometers for detecting gravitational waves.

4.5 Comparison

Let us first rewrite the solution of the time dependent harmonic oscillator with the Gaussian initial condition (4.14) in a slightly more recognizable form. This can be done by using a substitution.

$$u(t, \rho) = \frac{w_0}{v(t)} e^{-\frac{\rho^2}{v^2(t)}} e^{i\rho^2 \left(y_1(t) - \frac{1}{v^2(t)}\right)} e^{i \arctan\left(\frac{4y_3(t)}{w_0^2}\right)},$$

$$v(t) = \frac{w_0}{y_2(t)} \sqrt{1 + \left(\frac{4y_3(t)}{w_0^2}\right)^2}.$$

Identical to the Gaussian beam, $v(t)$ can be viewed as the width of the wave packet. Again, the plane wave term and the curvature term are missing. The probability distribution, including normalization, is then given by:

$$|u(t, \rho)|^2 = \frac{2}{\pi} \frac{1}{v^2(t)} e^{-\frac{2\rho^2}{v^2(t)}}.$$

Which, at a glance may look similar but is quite different from the free space probability. The reason is that the width $v(t)$ has a more complicated time dependence. This shows that the intensity pattern may fundamentally differ for $t > 0$ from the Gaussian beam. This is unlike the free space solution which did make the same prediction as the optics approach.

Further comparing the harmonic oscillator solution (4.13) and the free space solution (4.4) to the optics solution (3.7) will be aided by using the L^2 and L^∞ norms, which are given by:

$$\|f(\mathbf{x})\|_2^2 = \int_{\mathbb{R}^d} |f(\mathbf{x})|^2 d\mathbf{x},$$

$$\|f(\mathbf{x})\|_\infty = \inf\{C \geq 0 : |f(\mathbf{x})| \leq C \text{ for almost all } \mathbf{x}\}.$$

The L^2 norms of the solutions can easily be obtained using the standard solution of the

Gaussian integral to give:

$$\|E(\rho, z)\|_2^2 = |E_0|^2 \frac{\pi w_0^2}{2} \qquad\qquad (d = 2),$$

$$\|\Psi(t, \mathbf{x})\|_2^2 = \left(\frac{\pi w_0^2}{2} \right)^{\frac{d}{2}},$$

$$\|u(t, \mathbf{x})\|_2^2 = \left(\frac{\pi w_0^2}{2} \right)^{\frac{d}{2}}.$$

As can be seen, the expressions are all similar. Therefore we can infer that the propagator of the time dependent oscillator is unitary as well, as predicted by theorem 1.2 from [22]. Additionally this mean that the energy of the wave
The L^∞ norm of the solutions is simply the absolute value of the coefficients in front as $\mathbf{x} = 0$ gives the maximum of the Gaussian.

$$\|E(\rho, z)\|_\infty = |E_0| \frac{w_0}{w(z)},$$

$$\|\Psi(t, \mathbf{x})\|_\infty = \left(\frac{w_0}{w(t)} \right)^{\frac{d}{2}},$$

$$\|u(t, \mathbf{x})\|_\infty = \left(\frac{w_0}{v(t)} \right)^{\frac{d}{2}}.$$

The L^∞ norm is a measure for the height of the solutions. The height of the free space solution is identical to the optical solution but the time dependent harmonic oscillator is different.

5 Conclusion

We have explained the workings of a lasers and the go-to optics model. Examining the free space Schrödinger equation, it became clear that quantum mechanics made same prediction of the intensity pattern. We have also shown that there exists an equivalence between the harmonic oscillator and focused beams. We used a Fourier integral operator to construct a wave packet solution to the time dependent harmonic oscillator and verified that when fixing the coefficients and starting with a Gaussian initial condition it reduces to the time independent harmonic oscillator ground state solution.

Future research into this topic may involve showing that the hypothesized equivalence between the Fourier integral operator methods used here and the propagator construction

in [24]. The same paper also includes a time dependent forcing term in the time dependent harmonic oscillator, which may be interesting for modeling forces on the beam like gravity or even nonlinear effects. [17]

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Appendix A   Mathematica Scripts

Mathematica Script 1: Hermite-Gaussian modes

```
1  f[n_, x_] := HermiteH[n, Sqrt[2] x] E^−x^2;
2  GraphicsGrid[
3    ParallelTable [
4    DensityPlot[
5     Evaluate[f[m, x] f[n, y ]], {x, −3.4, 3.4}, {y, −3.4, 3.4},
6     Mesh −> None, PlotPoints −> 100,
7     ColorFunction −> (If[m == 0 && n == 0,
8         Blend[{{0, White}, {1, Red}}, # ],  If [(m == 2 && n == 0) || (m == 0 && n == 2) ||
            (m == 2 && n == 2),
9         Blend[{{0, RGBColor[1, 2/5, 0]}, {8 Exp[−3/2]/(2 + 8 Exp[−3/2]),
10           White}, {1, Red}}, # ],
11        Blend[{{0, RGBColor[1, 2/5, 0]}, {1/2, White}, {1,Red}}, # ]]] &),
12          PlotRange −> All, Frame −> False,
13    ImageSize −> 200 {1, 1}, PlotRangePadding −> 0], {n, 0, 3}, {m, 0,
14    3}], Spacings −> 0, Background −> White]
```

Mathematica Script 2: Laguerre-Gaussian modes

```
1  g[p_,l_,x_,y_] := (Sqrt[2 (x^2 + y^2)])^l LaguerreL[p, l, 2(x^2 + y^2)] Exp[−x^2 −y^2] Cos[l
       Arg[x + I Abs[y]]];
2  GraphicsGrid[
3    ParallelTable [
4    DensityPlot[Evaluate[g[p, l, x, y ]], {x, −3.4, 3.4}, {y, −3.4, 3.4}, Mesh −> None,
         PlotPoints −> 100,
5    ColorFunction −> (If[l == 0, If[p == 0,
6    Blend[{{0, White}, {1, Red}}, # ],
7        If [p == 1,
8         Blend[{{0, RGBColor[1, 2/5,
9            0]}, {−g[p, l, Sqrt[3/2], 0]/(1 − g[p, l, Sqrt[3/2], 0]),
10          White}, {1, Red}}, # ],
11        If [p == 2,
12         Blend[{{0, RGBColor[1, 2/5,
13            0]}, {−g[p, l, Sqrt[(4 − Sqrt[6])/2], 0]/(1 − g[p, l, Sqrt[(4 − Sqrt[6])/2], 0]),
                 White}, {1, Red}}, # ],
14         Blend[{{0, RGBColor[1, 2/5, 0]}, {0.4081/1.4081, White}, {1,
15           Red}}, # ]]]],
16        Blend[{{0, RGBColor[1, 2/5, 0]}, {1/2, White}, {1,
17          Red}}, # ]] &), PlotRange −> All, Frame −> False,
18    ImageSize −> 200 {1, 1}, PlotRangePadding −> 0], {l, 0, 3}, {p, 0,
19    3}], Spacings −> 0, Background −> White]
```

Mathematica Script 3: Gaussian Beam

```
1  w0 := 2;
2  k := 3;
3  z0 := k w0^2/2;
4  w[z] := w0 Sqrt[1 + z^2/z0^2];
5  Electricfield = (w0/w[z])*Exp[−x^2/w[z]^2] Exp[I (k z + k x^2(z/(2*z^2 + 2*z0^2)) − ArcTan
     [z/z0])];
6  DensityPlot[Re[ Electricfield ], {z, −3 z0, 3 z0}, {x, −4 w0, 4 w0}, PlotPoints −> 200,
7  ColorFunction −> (Blend[{{0, RGBColor[1, 2/5, 0]}, {1/2, White}, {1, Red}}, #1] & ),
8  ClippingStyle −> Automatic, AspectRatio −> 1/2, ImageSize −> 600]\label{test}
```

Mathematica Script 4: Hermite-Gaussian Beam

```
1  w0 := 2;
2  k := 3;
3  z0 := k w0^2/2;
4  w[z] := w0 Sqrt[1 + z^2/z0^2];
5  Electricfield = w0/w[z] HermiteH[3, Sqrt[2] x/w[z]] Exp[−x^2/w[z]^2] Exp[I (k z +k x^2 z/(2 z
     ^2 + 2 z0^2) − (1 + 3) ArcTan[z/z0])];
6  DensityPlot[Re[ Electricfield ], {z, −3 z0, 3 z0}, {x, −4 w0, 4 w0}, PlotPoints −> 200,
7  ColorFunction −> (Blend[{{0, RGBColor[1, 2/5, 0]}, {1/2, White}, {1,Red}}, # ] &),
8  ClippingStyle −> Automatic, AspectRatio −> 1/2, ImageSize −> 600]
```