# An Introduction To Non-Relativistic SuperSymmetry And Its Underlying SuperAlgebra 

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## 1 Abstract

This report introduces the topic of Super Symmetry to the reader. It first discusses the motivation for why Super Symmetry needs to be introduced to, amongst other things, solve the problem of infinite vacuum energies in Quantum Field Theory (QFT). This problem is gone through in some depth to see how one arrives at these infinities and, in particular, a focus is given to the topic of Lie Groups introducing the reader to the role they play in QFT. Once these infinities have been derived in both the contribution to the vaccuum energy from a bosonic field and a fermionic field it is shown that these two infinities would exactly cancel if there existed a new type of symmetry (that of Super Symmetry) that took fermions and exchanged them with bosons and visa versa. This symmetry is analysed in the simplest case of the Weiss-Zummino model. It is shown how a Super-Symmetric model can be built and proven that for this case the action is indeed invariant under the operation of a Super Symmetric transformation. Lastly, a comparison is made between Lie algebras and the new Super-Algebra that emerges from the theory of Super Symmetry. In particular an introduction is given to the notion of a graded algebra and how this relates to Super Symmetry.

## Motivation

Super Symmetry is the addition of a symmetry between two fundamental groups of particles Bosons (Integer Spin Particles) and Fermions (half spin particles). [14]. It solves, amongst other things, the problem of infinite vacuum energies (a problem we shall address in some detail), as the quantum fluctuations of bosons cancel out those of the fermions. It would be a mistake to think that was the limit of Super Symmetry, however, indeed it goes far deeper than that and appears to fix a number of problems in the standard model as well as being a vital ingredient of string theories.

Despite its rather promising, and indeed useful, resume it has yet to been proven to exist although the energies required to show its existence have only been available since the construction of the Large Hadron Collider at CERN. Such experiments are possible because the theory predicts that each super-symmetric particle has a partner that differs by a half integer value of spin. For example, the electron would have a super partner called a "selectron" a super partner of the electron. It's quantum numbers would be the same apart from it's spin value. These particles would be new particles and therefore should in theory be discoverable at CERN [9] which is yet to be the case, much to the frustration of theoretical physicists.

Symmetries play a vital role in understanding physics. It is almost common knowledge that time invariance leads via Noether's Theorem to the conservation of energy. And invariance under spacial transformations to the conservation of momentum. Physics and symmetry is somehow intertwined therefore, and it seems only natural that in order to explain problems in physics one would look to additional symmetries. The specific role of supersymmetry would mean the laws of physics were unchanged whether one replaced Bosons with Fermions or visa versa which does not sound too far fetched a thing to believe.

Symmetries can be classified into two different types but both having the property that the physical observables of the system are unchanged by a transformation under which the system is symmetric. These two types are:

- Space-Time Symmetries: These symmetries are ones for which transformations of field coordinates leave the system invariant. Such symmetries are the rotations, translations, and in


Figure 1: An Example of symmetry in physics: A magnet and its mirror image. The lawas of electromagnetism are invariant under reflection [8]
particular the Lorentz and Poincare Invariance of special relativity. And the general co-ordinate transformations of General relativity.

$$
\begin{equation*}
X^{\mu} \rightarrow X^{\prime \mu}\left(X^{v}\right) \tag{1}
\end{equation*}
$$

- Internal Symmetries:This type of symmetry corresponds to transforming of the field rather than the co-ordinates.

$$
\begin{equation*}
\Psi^{a} \rightarrow M_{b}^{a}\left(\Psi^{b}\right) \tag{2}
\end{equation*}
$$

Where $\mathrm{a}, \mathrm{b}$ label the fields. If $M_{b}^{a}$ is constant then the symmetry is termed global if it is $M_{b}^{a}(x)$ position dependent it is termed local.

## [2]

In terms of particle physics symmetries give us all the properties of a particle. They allow us to label particles via the so called quantum numbers (mass, spin, charge and color) where the first correspond to space-time symmetries and the latter are internal symmetries of the field. They also tell us how particles will interact via the so called "Gauge Principle".

The topic of supersymmetry dates back to the 1970 s, where it was used as a symmetry "of the two dimensional world of sheet theory". [9] and as such was considered more a theoretical tool than anything else. Not too long after, physicists began to see that this new symmetry could be a symmetry of four dimensional quantum field theories and therefore its relevance to particle physics became more apparent. String theories combined with a super symmetry on the world sheet were shown to "exhibit super symmetry in space time" the same as four dimensional quantum field theories: which developed the topic of super strings. Since then multiple versions of super symmetry have been developed with extended super symmetries or local versions which contain super gravity.

Apart, from a kind of historical coincidence there are many motivations for why these theories are so appealing and potentially useful. Most motivation comes from known problems in the standard model which although has been "one of the great triumphs of the past century" is known to be limited in a number of regards. [2] These problems can be outlined as follows and to date no consensus amongst the solutions is present:

- The Hierachy Problem: One important and fundamental question is why is the Higg's mass is so much smaller than the gravitational scale. It proceeds with the technical hierarchical problem which is why the hierarchy is stable with respect to quantum corrections .
- The Cosmological constant $(\Lambda)$ problem: which is why is the energy density of free space time $\left(\Lambda / M_{\text {Planck }}\right)^{4} \sim 10^{-120} \ll 1$
- The Parameters of the model are set by hand not according to some rule. It would be better for physicists if these parameter values could be explained.
- What Particle constitutes the dark matter observed as it is not contained in the standard model and needs to be explained.

These problems provide the source of much interest and investigation amongst both theoretical and experimental physicists and even mathematicians. There are two main approaches one can adopt to look for answers to these issues.

- Look to add interactions or particles such as a dark matter particle to account for problems in the standard model
- Add extra symmetries

In this paper we will focus on the latter proposal, and in particular focus on super-symmetry.
There are other useful theories. For example, in grand unified theories (GUTs) the symmetries of the standard model are seen as simply the breaking of a bigger internal symmetry group the breaking occurring at different energy scales [2]

$$
G_{\mathrm{GUT}} \stackrel{M \approx 10^{16} \mathrm{GeV}}{\longrightarrow} \quad G_{\mathrm{SM}} \xrightarrow{M \approx 10^{2} \mathrm{GeV}} \quad S U(3)_{c} \times U(1)_{Y}
$$

GUTS however, do not address the questions above. Moreover, if you project the gauge couplings to higher energies they don't merge but rather cross each other making these theories less likely to be correct from current understanding.

Super symmetry, on the other hand is not an internal but an external or space-time symmetry. It provides a solution to the hierarchy problem due to cancellations between Boson and Fermion contributions to the electroweak scale. And importantly, combined with the GUT idea above it solves the unification of the three gauge couplings at one single point at large energies. So you see that often super-symmetry is useful for other theories also.

Lastly,in this paper we shall only discuss the concepts of Quantum Field Theory combined with super symmetry, but it must be noted a lot of research in the last 50 or so years has gone into string type theories [13]. One reason is that it looks to solve the problem of quantizing gravity which is not possible it seems from supersymmetric quantum field theories alone. But even here super symmetry is important as it is required along with extra dimensions to give string theories consistency [2].

## Structure Of This Report

The Structure of this report will be as follows:

- Present the problem of the infinite vacuum energies given by QFT for both bosonic and fermionic fields.
- Provide a basic introduction to Lie Algebras and their importance to this topic
- Show how super-symmetry can solve the problem of infinite vacuum energies
- Classify some basic differences between super-symmetric fields and ordinary quantum fields mathematically. In particular, outlining the basic principles of Lie Alegebra in the case of Lorentz and Poincare Transformations which are typical of QFT and seeing how this is extended in supersymmetric theories.
- Note for the purpose of this report we shall use the Weiss-Zumino model as it is the simplest super-symmetric theory and provides a accessible theory for understanding the basics of super symmetry.


Figure 2: Physics Scales And Unification [4]

## The Problem With Quantum Field Theories

### 1.1 QFT: A Short Recap

In quantum field theory we do a very similar quantization to that of the Schrodinger picture except this time instead of looking at the world of particles we chose to quantize classical fields. The Lagrangian of a free classical field is below

$$
\begin{equation*}
\frac{d L}{d \phi(x)}=d_{\mu} \frac{d L}{d\left(d_{\mu} \phi(x)\right)} \tag{3}
\end{equation*}
$$

Now there are many Lagrangian's one could chose that satisfy the above relation. But we can take one of the simplest which is the equation for a harmonic oscillator but applied to fields.

$$
\begin{equation*}
L=\frac{1}{2} d_{\mu} \phi d^{\mu} \phi-\frac{1}{2} m^{2} \phi^{2} \tag{4}
\end{equation*}
$$

And now using the Euler Lagrange equations

$$
\begin{array}{r}
\frac{d L}{d \phi}=-m^{2} \phi \\
\frac{d L}{d\left(d_{\mu} \phi\right)}=\frac{d}{d_{\mu} \phi}\left(\frac{1}{2} d_{\mu} \phi \eta^{\mu \nu} d_{\nu} \phi\right) \\
=\frac{1}{2} \eta^{\mu \nu}\left(d_{\nu} \phi+\mu \phi \delta_{\nu}^{\mu}\right)  \tag{5}\\
=\frac{1}{2} \eta^{\mu \nu}\left(2 d_{\nu} \phi\right) \\
=d^{\mu} \phi
\end{array}
$$

And putting into equation 3 we get:

$$
\begin{equation*}
-m^{2} \phi=d_{\mu} d^{\mu} \phi \rightarrow\left(d_{\mu} d^{\mu}+m^{2}\right) \phi=0 \tag{6}
\end{equation*}
$$

The latter is the Klein Gordon equation for the free scalar field.
are switched to the momentum co-ordinates $\phi(\vec{p}, t)$

$$
\begin{equation*}
\phi(\vec{x}, t)=\int \frac{d^{3} p}{(2 \pi)^{3}} e^{i \vec{p} \vec{x}} \phi(\vec{p}, t) \tag{7}
\end{equation*}
$$

so we get

$$
\begin{equation*}
\left(\frac{d^{2}}{d t^{2}}+\left(\vec{p}^{2}+m^{2}\right)\right) \phi(\vec{p}, t)=0 \tag{8}
\end{equation*}
$$

If we solve this equation for each separate value of momentum $P$, we get

$$
\begin{equation*}
\phi(\vec{p}, t)=e^{-i\left(p^{2}+m^{2}\right) t} \tag{9}
\end{equation*}
$$

and we label the frequency

$$
\begin{equation*}
\omega_{\vec{p}}=\sqrt{\vec{p}^{2}+m^{2}} \tag{10}
\end{equation*}
$$

So the solutions of the Klein Gordon equations are these $\phi$ that oscillate with frequency $\omega_{\vec{p}}$. To quantize a field $\phi$ we can just quantize these harmonic oscillators. We can now use a very similar methodology as that of the Schrodinger picture to get the spectrum of "allowed" states.

### 1.2 A reminder of the harmonic oscillator

We know from Plank [11] the fundamental relation $E=h v$ where $v$ is the frequency. This introduces the world of quantum and allowed states. We wish to construct the possible states of the quantum harmonic oscillator given this restriction. We can set $\hbar=1, c=1$ and $m=1$ throughout this paper for convenience. So we can write the classical Hamiltonian for a harmonic oscillator

$$
\begin{equation*}
H=\frac{1}{2} p^{2}+\frac{1}{2} \omega^{2} q^{2} \tag{11}
\end{equation*}
$$

We wish to find an operator which acts on the energy state $|E\rangle$ and increases the value of the Energy by $\hbar \omega$. The answer is the so called ladder operators:

$$
\begin{equation*}
a=\sqrt{\frac{\omega}{2}} q+\frac{i}{\sqrt{2 \omega}} p, a^{\dagger}=\sqrt{\frac{\omega}{2}} q-\frac{i}{\sqrt{2 \omega}} p \tag{12}
\end{equation*}
$$

This means we can rewrite our Hamiltonian in terms of a and $a^{\dagger}$

$$
\begin{equation*}
H=\omega\left(a^{\dagger} a+\frac{1}{2}\right) \tag{13}
\end{equation*}
$$

The Commutator of H with $a^{\dagger}$ is

$$
\begin{equation*}
\left[H, a^{\dagger}\right]=H a^{\dagger}-a^{\dagger} H=\omega a^{\dagger} \tag{14}
\end{equation*}
$$

Now lets see if this choice of a actually works and gives us eigenstates of Energy separate by an energy value of $\omega$

$$
\begin{array}{r}
H a^{\dagger}|E\rangle=\left[H, a^{\dagger}\right]+a^{\dagger} H|E\rangle=\left(\omega a^{\dagger}+a^{\dagger} H\right)|E\rangle  \tag{15}\\
\\
=\left(\omega a^{\dagger}+E a^{\dagger}\right)|E\rangle=(\omega+E) a^{\dagger}|E\rangle
\end{array}
$$

An analagous calculation can be done for a giving

$$
\begin{equation*}
H a|E\rangle=(E-\omega) a|E\rangle \tag{16}
\end{equation*}
$$

So the choice of a and $a^{\dagger}$ is justified as it raises the energy by $\omega$ and we can thus in line with plank construct all the possible states. Including the 0 or vacuum state.

$$
\begin{equation*}
H|0\rangle=\frac{1}{2} \omega|0\rangle \tag{17}
\end{equation*}
$$

So if you want the $n^{\text {th }}$ state you apply $a^{\dagger} \mathrm{n}$ times to the ground state to get

$$
\begin{equation*}
H|n\rangle=\left(n+\frac{1}{2}\right) \omega \tag{18}
\end{equation*}
$$

## 2 Application To Quantum Fields

Plug in our new definitions of $q$ and $p$ into the Fourier transform from (7) to get field expressions in terms of creation and annihilation operators.

$$
\begin{gather*}
\pi(x)=\int \frac{d^{3} p}{(2 \pi)^{3}}(-i) \frac{\sqrt{\omega_{p}}}{2}\left(a_{p} e^{i p x}-a_{p}^{\dagger} e^{-i p x}\right)  \tag{19}\\
\phi(x)=\int \frac{d^{3} p}{(2 \pi)^{3}} \frac{1}{\sqrt{2 \omega_{p}}}\left(a_{p} e^{i p x}+a_{p}^{\dagger} e^{-i p x}\right) \tag{20}
\end{gather*}
$$

These are the new momentum and spatial components. We can now try these out on our free scalar field. The Hamiltonian looks like.

$$
\begin{equation*}
H=\int d^{3} x \frac{1}{2} \pi^{2}+\frac{1}{2}(\Delta \phi)^{2}+V(\phi) \tag{21}
\end{equation*}
$$

and then rewrite in terms of the annihilation/creation operators

$$
\begin{align*}
& H=\frac{1}{2} \int \frac{d^{3} p}{(2 \pi)^{3}} \omega_{p}\left[a_{p} a_{p}^{\dagger}+a_{p}^{\dagger} a_{p}\right] \\
& \frac{1}{2} \int \frac{d^{3} p}{(2 \pi)^{3}} \omega_{p}\left[a_{p}^{\dagger} a_{p}+\frac{1}{2}(2 \pi)^{3} \delta^{(0)}\right] \tag{22}
\end{align*}
$$

This definition of H is where the problem becomes apparent it goes from $p=0$ to $p=\infty$. at $p=0$ the integral diverges as contains the delta function which has an infinite spike at 0 and also it diverges at high p because $\omega_{p}$ becomes infinite.

## 3 The Infinity In The Vacuum Energy

Lets act with H on the Ground State in order to determine the ground state energy.

$$
\begin{equation*}
H|0\rangle=E_{0}|0\rangle=\frac{1}{2} \int \omega_{p} \delta^{3}(0) d^{3} p|0\rangle \tag{23}
\end{equation*}
$$

You can see the $a_{p}^{\dagger} a_{p}$ term disappears as obviously $a_{p}|0\rangle=0$ i.e. if you lower the ground state you get nothing so you are left with this delta term applied to the ground state.

There is an infinity coming from the delta function lets remove this before worrying about $\omega$. Lets take the delta function definition evaluated at 0

$$
\begin{equation*}
(2 \pi)^{3} \delta^{(3)}(0)=\left.\lim _{L \rightarrow \infty} \int_{-\frac{L}{2}}^{\frac{L}{2}} d^{3} x e^{i x p}\right|_{p=0}=\lim _{L \rightarrow \infty} \int_{-\frac{L}{2}}^{\frac{L}{2}} d^{3} x=V \tag{24}
\end{equation*}
$$

Now define the energy density of the vacuum $\xi_{0}$

$$
\begin{equation*}
\xi_{0}=\frac{E_{0}}{V}=\int \frac{d^{3} p}{(2 \pi)^{3}} \frac{1}{2} \omega_{p} \tag{25}
\end{equation*}
$$

This is still infinite but not because of space being infinitely large. It is the problem with $\omega$ going to infinity at high $p$ or the ultra violet divergence as it is called. This is the problem we wish to solve via the introduction of super symmetry. We will see shortly there is a corresponding infinity to this one which if used correctly can cancel out this one.

The Klein Gordon Equation and corresponding free scalar field contains no spin terms. i.e. It describes particles whose spin $=0$ i.e. bosons we will show this in a moment. If we want a complete theory we would need to include the half spin particles also. We shall discover in so doing this that we end up with a energy density from the spin $1 / 2$ particles that would exactly cancel the infinity from the spin 0 particles if they were intertwined in a single theory (that of super symmetry).

## 4 Spin 0 Bosons

So one trick in QFT is to remove the vacuum infinity by simply redefining the vacuum energy as 0 as in physics we only really care about the energy difference between two states so if both contain this infinity we can simply remove it by making the requirement that the ground state is 0 .

Then if we wish to construct the other allowed states we simply act with $a_{p}^{\dagger}$ recurringly on the vacuum. lets see what this does.

$$
\begin{equation*}
|p\rangle=a_{p} \dagger|0\rangle, H|p\rangle=\omega_{p}|p\rangle \tag{26}
\end{equation*}
$$

where

$$
\begin{equation*}
\omega_{p}^{2}=p^{2}+m^{2}=E_{p}^{2} \tag{27}
\end{equation*}
$$

Notice that by acting with the creation operator we have created something that looks incredibly like a state of a particle with Energy $E_{p}$. And this is the interpretation that is taken. Excitations to higher states of the the field are simply the creation of particles with energy $E_{p}$. hmmm very interesting!

Lets look not just to the energy but the angular momentum operator

$$
\begin{equation*}
J^{i}=\epsilon^{i j k} \int d^{3} x\left(\Im^{0}\right)^{j k} \tag{28}
\end{equation*}
$$

where

$$
\begin{equation*}
\left(\Im^{0}\right)^{j k}=x^{j} T^{0 k}-x^{k} T^{0 j} \tag{29}
\end{equation*}
$$

And T is the Energy Momentum tensor. You can check in quite a lengthy way that $J^{i}|p\rangle=0$ which means these particles have no intrinsic angular momentum i.e. they are spin 0 particles or bosons. So Klein Gordon gives rise to spin 0 particles what about the spin $1 / 2$ for that we need a new theory. That coming from Dirac. Dirac looked for a relativistic quantum theory. So we are
going to use representations of the Lorentz group to find invariant objects. Why do we want to do this, well obviously for a theory to be truly relativistic it needs to be unchanged when we perform a Lorentz transformation or in other words invariant under the operations of the Lorentz group. (For some of these derivations and more on QFT see [1])

## 5 Lorentz Transformations

Lorentz Transformations $\Lambda_{\nu}^{\mu}$ together with the composition law of matrix multiplication form the Lorentz Group.

They satisfy the following property

$$
\begin{equation*}
\Lambda_{\sigma}^{\mu} \eta^{\sigma \tau} \Lambda_{\tau}^{\nu}=\eta^{\mu \nu} \tag{30}
\end{equation*}
$$

Where $\eta$ refers to the metric. It is in short the set of transformations that preserve the quadratic form $t^{2}-x^{2}-y^{2}-z^{2}$. For physicists you can imagine a transformation (say a rotation) about the $z$ axis

$$
\Lambda_{\nu}^{\mu}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{31}\\
0 & \cos \theta & \sin \theta & 0 \\
0 & -\sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{array}\right)\left(\begin{array}{l}
t \\
x \\
y \\
z
\end{array}\right)=\left(\begin{array}{c}
t \\
x \cos \theta+y \sin \theta \\
-x \sin \theta+y \cos \theta \\
z
\end{array}\right)
$$

Now lets take the quadratic form again

$$
\begin{gather*}
t^{2}-(x \cos \theta+y \sin \theta)^{2}-(-x \sin \theta+y \cos \theta)^{2}-z^{2}= \\
t^{2}-\left(x^{2}\left(\cos ^{2} \theta+\sin ^{2} \theta\right)\right)-\left(y^{2}\left(\cos ^{2} \theta+\sin ^{2} \theta\right)\right)-z^{2}  \tag{32}\\
=t^{2}-x^{2}-y^{2}-z^{2} \tag{33}
\end{gather*}
$$

The Lorentz group is the combination of such rotations with so called Lorentz boosts which also preserve the quadratic form. A Lorentz boost is pretty easy to understand also. It is simply saying if we take new co-ordinates from a moving frame how does an object look. This is an example of a Lorentz boost in the x direction.

$$
\left(\begin{array}{cccc}
\gamma & \beta \gamma & 0 & 0  \tag{34}\\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

Where as usual $\beta=\frac{v}{c}$ and $\gamma=\frac{1}{\sqrt{1-\frac{v^{2}}{c^{2}}}}$ You can check for yourself the quadratic form is preserved if you take an arbitrary vector and apply an arbitrary boost in the $x$ direction. The combination of these boosts + these rotations forms the Lorentz group. These Lorentz transformations form the basis of special relativity and quantum field theory is the combining of quantum theory with the theory of relativity.

## 6 Lorentz Invariant Theories

Lets look again at the Klein Gordon Equation and the dynamics of a real scalar field.
The Lagrangian reads:

$$
\begin{equation*}
L=\frac{1}{2} \eta^{\mu \nu} d_{m} \phi d^{\nu} \phi-\frac{1}{2} m^{2} \phi^{2} \tag{35}
\end{equation*}
$$

For a scalar field the Lorentz transformations become particularly simple.

$$
\begin{equation*}
x \rightarrow \Lambda x \tag{36}
\end{equation*}
$$

$$
\begin{equation*}
\phi(x) \rightarrow \phi\left(\Lambda^{-1} x\right) \tag{37}
\end{equation*}
$$

We wish to show the action is invariant under such a transformation making the system Lorentz invariant.

The derivative of a scalar field also transforms a similar way

$$
\begin{align*}
& \left(d_{m} \phi\right)(x) \rightarrow\left(\Lambda^{-1}\right)_{\mu}^{\nu}\left(d_{v} \phi\right)(y)  \tag{38}\\
& \left(d_{m} \phi\right)(x) \rightarrow\left(\Lambda^{-1}\right)_{\mu}^{\nu}\left(d_{v} \phi\right)(y) \tag{39}
\end{align*}
$$

So the derivative term in the Lagrangian transforms as

$$
\begin{align*}
d_{\mu} \phi(x) d_{\nu} \phi(x) \eta^{\mu \nu} & =\left(\Lambda^{-1}\right)_{\mu}^{\rho}\left(d_{\rho} \phi\right)(y)\left(\Lambda^{-1}\right)_{\nu}^{\sigma}\left(d_{\sigma} \phi\right)(y) \eta_{\mu}^{\nu}  \tag{40}\\
& =d_{\rho} \phi(y) d_{\sigma} \phi(y) \eta^{\rho \sigma} \tag{41}
\end{align*}
$$

You can see from a transformation one set of co-ordinates simply changes to another but the expression is equivalent i.e. invariant. The potential term is exactly the same where $\phi^{2}(x) \rightarrow \phi^{2}(y)$
and therefore the action is invariant:

$$
\begin{equation*}
S=\int d^{4} x L(x) \rightarrow \int d^{4} y L(y)=S \tag{42}
\end{equation*}
$$

This is good for the spin 0 case but what about the spin $1 / 2$. We wish to construct a theory that contains spin. Transformations like the ones in (31) have space time indices and act on space time fields, vectors etc. but we wish to have invariant fields that contain spin. Fields of the form $\Psi^{a}$ where a is a spinor index. The Lorentz transformation in the form of (31) can not act on a spinor field so we need to find a way of representing the Lorentz transformations to include spinor indices so we can act on spinor fields. To do this we will require some knowledge of Lie Algebra.

## 7 Lie Groups

Lie Groups come up in physics very often. Our aim is to first define Lie Groups and understand their structure and then apply these definitions in the case of the Lorentz group to see the use of understanding these groups for the case of QFT.

### 7.1 The Definition of a Group

A group G is the pair $(G ; \circ)$ : where G is a set of elements and $\circ$ is a composition law such that $\forall \mathrm{g} 1, \mathrm{~g} 2, \mathrm{~g} 3 \in \mathrm{G}$

- $\mathrm{g} 1 \circ \mathrm{~g} 2 \in \mathrm{G}($ Closure $)$
- $\mathrm{g} 1 \circ(\mathrm{~g} 2 \circ \mathrm{~g} 3)=(\mathrm{g} 1 \circ \mathrm{~g} 2) \circ \mathrm{g} 3($ Associativity $)$
- $\forall \mathrm{G} \in \mathrm{G} \exists \mathrm{e} \in \mathrm{G}: \mathrm{e} \circ \mathrm{g}=\mathrm{g} \circ \mathrm{e}=\mathrm{g}$ ( The existence of a neutral element)
- $\forall \mathrm{G} \in \mathrm{G} \exists g^{-1} \in \mathrm{G}: g^{-1} \circ \mathrm{~g}=\mathrm{g} \circ g^{-1}=\mathrm{e}$ (Existence of an inverse)


### 7.2 The Definition of a Lie Group

A lie Group is a set G with two structures. G is a group and g is a (smooth, real) manifold. These structures agree in the following sense: multiplication $G X G \rightarrow G$ and inversion $g \rightarrow g^{-1}: G \rightarrow G$ are smooth maps

Let's have a look at the most common example of a lie group. The General Linear group GL( $\mathrm{n}, \mathrm{R}$ ) the group of $n \times n$ real matrices such that they are invertible i.e. the determinant is non zero

$$
\left\{\left(\begin{array}{ll}
a & b  \tag{43}\\
c & d
\end{array}\right): a b-c d \neq 0\right\}
$$



Figure 3: GL(n,R) the red is the Not included portion of the manifold where $a d-b c=0$

You can see visually how this equation would lead to the manifold structure of the group (in an abcd space) and the reader could check quite simply that it indeed obeys the group relations. Where a composition law is in this case matrix multiplication. The fact that lie groups are differentiable manifolds has a lot of significant consequences which we shall go through. Particularly that associated to every Lie Group is a Lie Algebra. This fact will help us find a suitable way of expressing the transformations of the Lorentz group in a way that is useful to us in the case of a spinor but to see this requires a bit of background which I will go through now. Put simply the lie algebra is the tangent space of the group at the identity and similarly to a basis of a vector space we can get so called generators of a Lie algebra. These generators and their relations will allow us to form different representations of the Lorentz group representations suitable for dealing with fields that contain spin leading naturally to the Dirac equation.

## 8 Lie Algebras and Lie Groups

### 8.1 Tangent Space

Consider a differentiable curve on a manifold M with coordinates $x^{i}: i=1,2 \ldots \operatorname{dim} \mathrm{M}$ parameterized by a continuous variable t from -1 to 1 . and let f be any differential function defined on a neighbourhood of the point p of the curve corresponding to $t=0$. The vector $V_{p}$ tangent to the curve at $p$ is defined as

$$
\begin{equation*}
V_{p}(f)=\left.\frac{d x^{i}(t)}{d t}\right|_{t=0} \frac{d f}{d x^{i}} \tag{44}
\end{equation*}
$$

The vector $V_{p}$ is a tangent vector to M at the point p . Moreover, the tangent vectors at p to all the differentiable curves passing through p form the tangent space $T_{p} M$ at p . Note that this is a vector space as scalar multiplication of a vector is indeed part of this tangent space and addition of tangent vectors is again a tangent vector.

Taking a set of co-ordinates $x^{i}, i=1,2 \ldots \operatorname{dim} \mathrm{M}$ in a neighbourhood of a point p on M we have that the operators $\frac{d}{d x^{i}}$ are linearly independent and form a basis for the tangent space $T_{P} M$. Which means any tangent vector $V_{P}$ can be written as a linear combination of these basis vectors.

$$
\begin{equation*}
V_{p}=V_{P}^{i} \frac{d}{d x^{i}} \tag{45}
\end{equation*}
$$

If we vary the point $p$ along the differentiable curve we obtain tangent vectors to the curve at each of these points. Because the curve is differentiable these tangent vectors are continuously and differentiably related. We can chose a tangent vector for each point $p$ on of the manifold such that this set of vectors is differentiably related then this gives us a vector field. Given a set of coordinates on M we can write this vector field in terms of the basis vectors $x^{i}$ and its components $V^{i}$ where these $V^{i}$ are differentiable functions of these co-ordinates

$$
\begin{equation*}
V=V^{i}(x) \frac{d}{d x^{i}} \tag{46}
\end{equation*}
$$

Given two vector fields V and W in a coordinate neighbourhood we can take their composite action on a function $f$

$$
\begin{equation*}
W(V f)=W^{j} \frac{d V^{i}}{d X^{j}} \frac{d f}{d x^{i}}+W^{j} V^{i} \frac{d^{2} f}{d x^{j} d x^{i}} \tag{47}
\end{equation*}
$$

that second term means the composition is not a vector field. (Take $\mathrm{V}=\mathrm{W}$ as an example and you get a $\frac{d^{2}}{d x^{i}}$ term which does not obey the Leibnitx rule)

But under the operation of a commutator we get

$$
\begin{equation*}
[V, W]=\left(V^{i} \frac{d w^{j}}{d X^{i}}-W^{i} \frac{d^{V j}}{d x^{i}}\right) \frac{d}{d x^{j}} \tag{48}
\end{equation*}
$$

Which is a vector field. So the set of vector fields on a lie group is closed under the operator of commutation which is what defines a Lie algebra. I.e. it is a set of vector fields with the addition that it is closed under the operation of a commutator and satisfies the Jacobi Identity.

A formmal definition is

Definition 8.1 A lie algebra $\mathfrak{g}$ is a vector space over a field $k$ with a bilinear composition law

$$
\begin{array}{r}
(x, y) \rightarrow[x, y]  \tag{49}\\
{[x, a y+b z]=a[x, y]+b[x, z]}
\end{array}
$$

with $x, y, z \in L$ and $a, b \in k$ such that

$$
\begin{array}{r}
{[x, x]=0} \\
{[x,[y, z]]+[z,[x, y]]+[y,[z, x]=0 ;(\text { jacobi identity })} \tag{50}
\end{array}
$$

### 8.2 Lie Algebra of a Lie group

It has been shown that vector fields on a manifold form a Lie Algebra. But we want to find out the relation between this algebra and the group structure. Lets start by looking at the notion of left invariance. If we take an element of $g$ of a Lie Group $G$ and multiply it from the left by any element of $G$ we have a $G$ goes to $G$ trainsformation which is called left translation on $G$ by $g$.

Right translation can then also be defined. Under a left translation an element $g^{\prime}$ which is parameterized by co-ordinates $x^{i \prime}(i=1,2 \ldots \operatorname{Dim} \mathrm{G})$ is mapped into $g^{\prime \prime}=g g^{\prime}$ and the parameters $x^{\prime \prime i}$ of $\mathrm{g}^{\prime \prime}$ are analytic functions of $x^{\prime i}$. We can thus take a mapping between the tangent spaces of G .

Let V be a vector field on G which corresponds to tangent vectors $V_{g^{\prime}}$ and $V_{g^{\prime \prime}}$ on the tangent spaces to G at $g^{\prime}$ and $g^{\prime \prime}$ respectively. Let f be an arbitrary function of parameters $x^{\prime \prime i}$ of $\mathrm{g}^{\prime}$. We can define a tangent vector $W_{g^{\prime \prime}}$ on $T_{g}^{\prime \prime} G$ (the tangent plane to G at $\mathrm{g}^{\prime \prime}$ ) as:

$$
\begin{equation*}
W_{g^{\prime \prime}} f=V_{g^{\prime}}\left(f \circ x^{\prime \prime}\right)=V_{g^{\prime}}^{i} \frac{d}{d x^{\prime i}} f\left(x^{\prime \prime}\right)=V_{g^{\prime}}^{i} \frac{d x^{\prime \prime j}}{d x^{\prime} i} \frac{d f}{d x^{\prime \prime j}} \tag{51}
\end{equation*}
$$

This has defined a mapping between the tangent spaces of G since, given a $V_{g^{\prime}}$ we can have a $T_{g^{\prime}} G$ and now associated the tangent vector $W_{g^{\prime \prime}}$ in $T_{g^{\prime \prime}} G$ to it. If the vector $W_{g^{\prime \prime}}$ coincides with the value of the vector field at $T_{g^{\prime \prime}} G$ i.e. $V_{g^{\prime \prime}}$ then we say the vector field is a left invariant vector field on $G$ since transformations were induced by left translations on $G$.

Remark: the commutator of two left invariant vector fields ( V and U ) is again a left invariant vector field (S).:

$$
\begin{equation*}
S_{g^{\prime}}=\left[V_{g^{\prime}}, U_{g^{\prime}}\right]=\left(V_{g^{\prime}}^{i} \frac{d U_{g^{\prime}}^{j}}{d x^{\prime i}}-U_{g^{\prime}}^{i} \frac{d V_{g^{\prime}}^{j}}{d x^{\prime i}}\right) \frac{d}{d x^{\prime j}} \tag{52}
\end{equation*}
$$

Now lets write out $S g^{\prime \prime}$ to test left invariance:

$$
\begin{gather*}
=\left(V_{g^{\prime}}^{i} \frac{d U_{g^{\prime}}^{j}}{d x^{\prime i}}-U_{g^{\prime}}^{i} \frac{d V_{g^{\prime}}^{j}}{d x^{\prime i}}\right) \frac{d x^{\prime \prime k}}{d x^{\prime j}} \frac{d}{d x^{\prime k}}  \tag{53}\\
=S_{g^{\prime}}^{j} \frac{d x^{\prime \prime k}}{d x^{\prime j}} \frac{d}{d x^{\prime k}} \tag{54}
\end{gather*}
$$

So $S$ is also left invariant. therefore the set of left invariant vector fields form a lie algebra as they are closed under the commutator. They form a Lie Subalgebra of the Lie Algebra of all vector fields on G.

It is particularly important that a left invariant vector field is completely determined by its value at any particular point of G. In particular its value at the identity. And consequently that the Lie algebra of left invariant vector fields at any point of $G$ is completely determined by the Lie algebra of these fields at the identity element of G .

Given a Lie Group G it is always possible to find a number of linearly independent left-invariant vector fields the number equal to the dimension of g . Denoting these by $T_{a}(a=1,2 \ldots \operatorname{dim} \mathrm{G})$ they constitute a basis for the tangent plane to $G$ at any point and satisfy.

$$
\begin{equation*}
\left[T_{a}, T_{b}\right]=i f_{a b}^{c} T_{c} \tag{55}
\end{equation*}
$$

If you move from one point on $G$ to another these relations remain unchanged. The f's are termed the structure constants of the Lie Algebra of G and contain all the information about the Lie Algebra. Because this relation is point independent we can chose the tangent plane at the identity $T_{e} G$ as the vector space of the Lie algebra. This is simply a convenience given the identity makes manipulating equations particularly easy.

So we have shown in a rather complicated way that the tangent space at the identity defines the Lie Algebra. And these left invariant vector fields form a basis of this algebra where the structure constants tell us everything we need to know about the details of this algebra. But how do we get a relation between this algebra and the group itself? Actually one can show that the exponential map takes elements of the lie algebra to elements of the Lie Group.

A one parameter subgroup of $G$ is a differentiable curve mapping the real numbers onto $G$ so parameterizing elements by t basically. $t \rightarrow g(t)$ such that

$$
\begin{align*}
g(t) g(s)= & g(t+S)  \tag{56}\\
& g(0)=e
\end{align*}
$$

If we take a fixed element of $\mathrm{g}^{\prime}$ of G , we get the mapping $t \rightarrow\left[g^{\prime} g(t)\right]$ forming a differentiable curve on $G$. We can let $g^{\prime}$ vary over $G$ and obtain a family of curves which covers $G$. Each point on G has several curves passing through it but each curve has the same tangent vector at that point. Therefore the family of curves $g^{\prime} g(t)$ can be used to define a vector field on $G$. This is a left invariant vector field. So to each one parameter subgroup of $G$ you can associate a left invariant vector field.

If $T$ is the tangent vector at the identity to a differentiable curve $g(t)$ which is a one parameter subgroup:

$$
\begin{equation*}
g(t)=\exp (t T) \tag{57}
\end{equation*}
$$

This is called the exponential mapping of the Lie algebra of $G$ to the Lie Group. The Exponential map is an analytic function and maps a neighbourhood of the zero element of the Lie algebra to a neighborhood of the identity element of $G$.

Taking the derivative of the curve at $t=0$ gives you an element of the tangent space i.e. an element of the Lie Algebra. And taking the exponential mapping of a tangent vector gives you back an element of the differential curve.

This has been a long aside but why should we care. Well we can use these facts to form representations of the Lorentz group. ( For the majority of these Lie Algebra Derivations and more on Lie algebras see [12])

## 9 Obtaining the Lorentz Algebra

We are now going to apply the theory to the case of the Lorentz group to obtain the Lorentz algebra. The Lorentz group is just a special case of the indefinite orthogonal group o(m,n) so finding the algebra for this group will be equivalent to the process for finding the algebra to the Lorentz group. The definition of the indefinite orthogonal group is the transformations that leave invariant a non degenerate symmetric bi-linear form . in the case of the Lorentz group o $(1,3)$ this is the form $t^{2}-x^{2}-y^{2}-z^{2}$. Or formally:

$$
\begin{equation*}
o(m, n)=\left\{A \in G L_{n}: A I_{m, n} A^{T}=I_{m, n}\right\} \tag{58}
\end{equation*}
$$

where $I_{m n}$ has the form

$$
\left[\begin{array}{lllll}
1 & & & &  \tag{59}\\
& 1 & & & \\
& & \ddots & & \\
& & & -1 & \\
& & & & -1
\end{array}\right]
$$

so m1's and $\mathrm{n}-1$ 's. You can see in our case $I_{m n}$ is the metric we are familiar with which we denote $\eta^{\mu \nu}$

So now we need a path. Let $\mathrm{A}(\mathrm{S})$ be a path in $\mathrm{o}(\mathrm{m}, \mathrm{n})$ such that $A(0)=I$

$$
\begin{equation*}
A(s) I_{m, n} A(s)^{T}=I_{m, n} \tag{60}
\end{equation*}
$$

Now take the derivative with respect to $s$.

$$
\begin{equation*}
A^{\prime}(s) I_{m, n} A(s)^{T}+A(s) I_{m, n} A^{\prime}(s)^{T}=0 \tag{61}
\end{equation*}
$$

at $s=0 A=1$ by definition so we have

$$
\begin{equation*}
A^{\prime}(0) I_{m, n}+I_{m, n} A^{\prime}(0)^{T}=0 \tag{62}
\end{equation*}
$$

We know $\mathrm{A}^{\prime}(0)$ is an element of the Lie Algebra $\mathfrak{o}(m, n)$ so we can say the lie algebra

$$
\begin{equation*}
\mathfrak{o}(m, n)=\left\{B: B I_{m, n}+I_{m, n} B^{T}=0\right\} \tag{63}
\end{equation*}
$$

We can also use this same group to show how the exponential map takes us from the algebra to the group. In particular we can show $\mathbf{B}$ is in $\mathfrak{o}(m, n)$ as it must satisfy the group relation up to second order when exponentiated. such that

$$
\begin{equation*}
e^{t B} I_{m, n} e^{t b T}-I_{m, n}=0 \tag{64}
\end{equation*}
$$

using the definition of a matrix exponential we can rewrite.

$$
\begin{equation*}
(I+t B) I_{m, n}\left(I+t B^{T}\right)-I_{m, n}=0 \tag{65}
\end{equation*}
$$

removing second order terms we get

$$
\begin{gather*}
t\left(B I_{m, n}+I_{m, n} B^{T}\right)=0  \tag{66}\\
B I_{m, n}+I_{m, n} B^{T}=0 \tag{67}
\end{gather*}
$$

So $\mathbf{B}$ is in $\mathfrak{o}(m, n)$ if and only if the above relation holds. Notice the relation is equivalent to (63) and we have shown how one can go from the Lie algebra to the Lie group via the exponential mapping.

So in the case we care about we have which is $o(1,3)$ where the $I_{m, n}$ is the metric $\eta$ and we write

$$
\begin{equation*}
\mathfrak{o}(1,3)=\left\{A: A \eta+\eta A^{T}=0\right\} \tag{68}
\end{equation*}
$$

We can handily rewrite this to see skew symmetry. So just writing $C=A \eta$ you can clearly see we are looking at the skew symmetric matrices

$$
\begin{equation*}
\mathfrak{o}(1,3)=\left\{C: C+C^{T}=0\right\} \tag{69}
\end{equation*}
$$

C is skew symmetric and looks like

$$
\left[\begin{array}{cccc}
0 & c 12 & c 13 & c 14  \tag{70}\\
-c 12 & 0 & c 23 & c 24 \\
-c 13 & -c 23 & 0 & c 34 \\
-c 14 & -c 24 & -c 34 & 0
\end{array}\right]
$$

To find a basis for these matrices is pretty straight forward. They look like matrices with two non zero entries where the transpose is the negative. such as

$$
\left[\begin{array}{cccc}
0 & c i j & 0 & 0  \tag{71}\\
-c i j & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

but in the end we want a basis for our matrices A so we can simply times these basis matrices by $\eta^{-1}=\eta$ i.e. $A=C \eta^{-} 1$ so in the above example would be

$$
\left[\begin{array}{cccc}
0 & 1 & 0 & 0  \tag{72}\\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{array}\right]=\left[\begin{array}{cccc}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]=K_{x}
$$

Where we recognize the latter ( $K_{x}$ as the Lorentz boost in the x direction. I will not write them all out explicitly but you can now see that we can easily obtain a basis of "Generators" for the matrices A that satisfy (69). These generators are the familiar Lorentz boosts (K) and rotations (J) we are all familiar with. Because they generate the Lorentz algebra we can check the commutation relations between them (from before we know that under the operation of the commutation the algebra should be closed i.e. we should just get another element of the Lorentz algebra when taking the commutator of 2 elements). Upon checking we see we do:

$$
\begin{array}{r}
{\left[J_{i}, J_{k}\right]=i \epsilon_{i j k} J_{k}} \\
{\left[J_{i}, K_{j}\right]=i \epsilon_{i j k} K_{k}}  \tag{73}\\
{\left[K_{i}, K_{j}\right]=-i \epsilon_{i j k} K_{k}}
\end{array}
$$

It is very common to see this expressed in terms of matrices denoted $M$ where :

$$
\begin{array}{r}
J_{i}=\frac{1}{2} \epsilon_{i j k} M_{j h}  \tag{74}\\
K_{i}=M_{0} i
\end{array}
$$

## 10 A Different Representation

After a long aside we have seen how we can find generators of a Lie algebra which tell us how transformations manifest themselves in quantum field theory. It is interesting to note that these transformations are not unique we can find different ways of representing groups which will be important to us in understanding how we can build a Lorentz invariant theory that contains spin. As mentioned previously the representations of the Lorentz group we have previously seen cannot act on a spinor $\psi^{a}$ for that we require a suitable representation. To find it our knowledge of Lie groups will be tested.

Firstly, there exists a homeomorphism

$$
\begin{equation*}
S O(3,1) \cong S L(2, \mathbb{C}) \tag{75}
\end{equation*}
$$

Where $\operatorname{SL}(2 \mathrm{C})$ is defined as:

$$
\left\{\left.\left(\begin{array}{ll}
a & b  \tag{76}\\
c & d
\end{array}\right) \right\rvert\, a, b, c, d \in \mathbb{C}, a d-b c=1\right\}
$$

We can see this clearly if we consider a familiar space time four vector $X$ and perhaps an unfamiliar 2 by 2 matrix $\bar{x}$ :

$$
\begin{align*}
X & =x_{\mu} e^{\mu}=\left(x_{o}, x_{1}, x_{2}, x_{3}\right) \\
\bar{x}=x_{\mu} \sigma^{\mu} & =\left(\begin{array}{cc}
x_{0}+x_{3} & x_{1}-i x_{2} \\
x_{1}+i x_{2} & x_{0}-x_{3}
\end{array}\right) \tag{77}
\end{align*}
$$

where the $\sigma$ are the familiar Pauli matrices.

$$
\sigma^{\mu}\left(\begin{array}{ll}
1 & 0  \tag{78}\\
0 & 1
\end{array}\right),\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right),\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right),\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)
$$

Transformations of $\mathrm{SO}(1,3)$ leave the square (distance) invariant

$$
\begin{equation*}
|X|^{2}=x_{0}^{2}-x_{1}^{2}-x_{2}^{2}-x_{3}^{2} \tag{79}
\end{equation*}
$$

Similarly to this. the action of $\operatorname{SL}(2, C)$ taking

$$
\begin{equation*}
\bar{x} \rightarrow N \bar{x} N^{\dagger} \tag{80}
\end{equation*}
$$

with $\mathrm{N} \in \mathrm{SL}(2, \mathbb{C})$ preserves the determinant.

$$
\begin{equation*}
\operatorname{det} \bar{x}=x_{0}^{2}-x_{1}^{2}-x_{2}^{2}-x_{3}^{2} \tag{81}
\end{equation*}
$$

because if you look at the definition of $\operatorname{SL}(2, \mathrm{C})$ then you can see the determinant of N is 1 (by definition all elements of the group have determinant 1) so

$$
\begin{align*}
& \operatorname{det}\left(N \bar{x} N^{\dagger}\right)=\operatorname{det}(N) \operatorname{det}(\bar{x}) \operatorname{det}\left(N^{\dagger}\right) \\
& =1 \cdot \operatorname{det}(\bar{x}) \cdot 1=x_{0}^{2}-x_{1}^{2}-x_{2}^{2}-x_{3}^{2} \tag{82}
\end{align*}
$$

Therefore you can see there exists a map between them.

The purpose was this is because there are natural choices of representations of $S L(2, \mathbb{C})$ that contain spin One such being the fundamental representation:

$$
\begin{array}{r}
\psi_{\alpha}^{\prime}=N_{\alpha}^{\beta} \psi_{\beta} \\
\alpha, \beta=1,2 \tag{83}
\end{array}
$$

These $\psi_{\alpha / \beta}$ are called left handed Weyl spinors. The purpose of this aside was twofold. Firstly, it introduced the concept of Lie groups and Lie algebras. Secondly, it was designed for the reader to recognise the way group structures of $S O(1,3)$ and a number of other groups have an equivalence between them such that easily other representations of the Lorentz group can be found that are useful in determining how the group acts on different objects including so called spinors. (For more on this representation see [2]

## 11 Even More Representations

The key point is these representations are not unique. Lets take a basis of generators $S$ such that:

$$
\begin{equation*}
S^{\rho \sigma}=\frac{1}{4}\left[\gamma^{\rho}, \gamma^{\sigma}\right] \tag{84}
\end{equation*}
$$

where the $\gamma$ are the matrices of the Clifford algebra (see appendix).

The $S$ have the property that

$$
\begin{array}{r}
{\left[S^{\mu \nu}, \gamma^{\rho}\right]=\frac{1}{2}\left[\gamma^{\mu} \gamma^{\nu}, \gamma^{\rho}\right]} \\
=\frac{1}{2}\left(\gamma^{\mu} \gamma^{\nu} \gamma^{\rho}-\gamma^{\rho} \gamma^{\mu} \gamma^{\nu}\right) \\
=\frac{1}{2} \gamma^{\mu}\left\{\gamma^{\nu}, \gamma^{\rho}\right\}-\frac{1}{2} \gamma^{\mu} \gamma^{\rho} \gamma^{\nu}  \tag{85}\\
-\frac{1}{2}\left\{\gamma^{\rho}, \gamma^{\mu}\right\} \gamma^{\nu}+\frac{1}{2} \gamma^{\mu} \gamma^{\rho} \gamma^{\nu} \\
=\gamma^{\mu} \eta^{\nu \rho}-\gamma^{\nu} \eta^{\rho \mu}
\end{array}
$$

And using this fact and taking the commutator we can see these $S$ matrices satisfy the Lorentz algebra relations.

$$
\begin{equation*}
\left[S^{\mu \nu}, S^{\rho \sigma}\right]=\eta^{\nu \rho} S^{\mu \sigma}-\eta^{\mu \rho} S^{\nu \sigma}+\eta^{\mu \sigma} S^{\nu \rho}-\eta^{\nu \sigma} S^{\mu \rho} \tag{86}
\end{equation*}
$$

Such that we can express the lorentz transformations in this basis of generators:

$$
\begin{equation*}
\psi^{\alpha}=(x) \rightarrow S[\Lambda]_{\beta}^{\alpha} \psi^{\beta}\left(\Lambda^{-1} x\right) \tag{87}
\end{equation*}
$$

where

$$
\begin{align*}
\Lambda & =\exp \left(\frac{1}{2} \omega_{\rho \sigma} M^{\rho \sigma}\right)  \tag{88}\\
S[\Lambda] & =\exp \left(\frac{1}{2} \omega_{\rho \sigma} S^{\rho \sigma}\right)
\end{align*}
$$

Where we see again how the exponential map takes elements of the Lie algebra to the Lorentz group.

## 12 The Infinity In The Fermion Contribution

To quantize the Dirac field is entirely analogous to the quantization of the bosonic field and we will see we obtain a similar infinity in the vacuum energy:

Again we write the field $\psi(x)$ and the momentum which remember is just the hermetian conjugate of the field $\psi^{\dagger}(x)$ in terms of annihilation and creation operators:

$$
\begin{gather*}
\psi(x)=\sum_{i=1}^{2} \int \frac{d^{3} p}{(2 \pi)^{3}} \frac{1}{\sqrt{2 E_{p}}}\left[b_{p}^{s} u^{s}(p) e^{i p x}+c_{p}^{s^{\dagger}} v^{s}(p) e^{-i p x}\right]  \tag{89}\\
\psi^{\dagger}(x)=\sum_{i=1}^{2} \int \frac{d^{3} p}{(2 \pi)^{3}} \frac{1}{\sqrt{2 E_{p}}}\left[b_{p}^{\dagger^{\dagger}} u^{s}(p)^{\dagger} e^{-i p x}+c_{p}^{s} v^{s}(p)^{\dagger} e^{i p x}\right] \\
H=\int d^{3} x \bar{\psi}\left(-i \gamma^{i} \mathfrak{d}_{i}+m\right) \psi \tag{90}
\end{gather*}
$$

This can again be rewritten in terms of operators. Where the sum over i is implicit

$$
\begin{align*}
H=\int & \frac{d^{3} p}{(2 \pi)^{3}} \frac{1}{2}\left(b _ { p } ^ { r \dagger } b _ { p } ^ { s } \left(u^{r}(p)^{\dagger} \cdot u^{s}(p)-c_{p}^{r} c_{p}^{s \dagger}\left(v^{r}(p)^{\dagger} \cdot v^{s}(p)\right.\right.\right.  \tag{91}\\
& -b_{p}^{r \dagger} c_{-p}^{s \dagger}\left(u^{r}(p)^{\dagger} \cdot v^{s}(-p)-c_{p}^{r} b_{-} p^{s}\left(v^{r}(p)^{\dagger} \cdot u^{s}(-p)\right)\right.
\end{align*}
$$

Using some inner product formulae:

$$
\begin{gather*}
u^{r}(p)^{\dagger} \cdot u^{s}(p)=v^{r}(p)^{\dagger} \cdot v^{s}(p)=2 p o \delta^{r s} \\
u^{r}(p)^{\dagger} \cdot v^{s}(-p)=v^{r}(p)^{\dagger} \cdot u^{s}(-p)=0  \tag{92}\\
H=\int \frac{d^{3} p}{(2 \pi)^{3}} E_{p}\left(b_{p}^{s \dagger} b_{p}^{s}+c_{p}^{s \dagger} c_{p}^{s}-(2 \pi)^{3} \delta^{(3)}(0)\right) \tag{93}
\end{gather*}
$$

Now if we again use the delta function trick as we did in the bosonic case we can get the vacuum energy contribution for the fermions.

Note

$$
\begin{align*}
b_{p}^{s}|0\rangle & =0 \\
c_{p}^{s}|0\rangle & =0 \tag{94}
\end{align*}
$$

So

$$
\begin{array}{r}
H|0\rangle=\int \frac{d^{3} p}{(2 \pi)^{3}} E_{p}\left(b_{p}^{s \dagger} b_{p}^{s}+c_{p}^{s \dagger} c_{p}^{s}-(2 \pi)^{3} \delta^{(3)}(0)\right)|0\rangle \\
 \tag{95}\\
=-\int \frac{d^{3} p}{(2 \pi)^{3}} E_{p}(2 \pi)^{3} \delta^{(3)}(0)|0\rangle \\
\\
=-\int \frac{d^{3} p}{(2 \pi)^{3}} \omega_{p}(2 \pi)^{3} \delta^{(3)}(0)|0\rangle
\end{array}
$$

removing the delta function as previously by dividing by the volume and remembering the sum over the spin is normally implied but I will rewrite more explicitly

$$
\begin{equation*}
H|0\rangle=-\sum_{n=0}^{2} \int \frac{d^{3} p}{(2 \pi)^{3}} \omega_{p}|0\rangle \tag{96}
\end{equation*}
$$

Comparing to the bosonic case we worked out earlier

$$
\begin{equation*}
H|0\rangle=\int \frac{d^{3} p}{(2 \pi)^{3}} \frac{1}{2} \omega_{p}|0\rangle \tag{97}
\end{equation*}
$$

You can see how the contributions are equivalent up to a factor and it is this equivalence that motivates us to link the fields in order to cancel out this troublesome infinity.

## 13 Super Symmetry

We wish to link the two fields to cancel out this infinity in the vacuum energy. We can construct a new Lagrangian which is invariant under a transformation operation of this new symmetry. This is more abstract than we were talking about before. A transformation from the Lorentz group is very physically intuitive. We rotate the co-ordinate system or give it a boost in a certain direction and see what happens to the dynamics of the system. What we are talking about here is that a super symmetric transformation that takes a bosonic field and exchanges it with a fermionic one or visa versa. Such a transformation looks like this

$$
\begin{gather*}
\delta \phi=\bar{\epsilon}_{L} \chi_{l}  \tag{98}\\
\delta \phi^{*}=\bar{\epsilon}_{R} \chi_{R}  \tag{99}\\
\delta \chi_{l}=\frac{1}{2} \gamma^{\mu} \epsilon_{R} d_{\mu} \phi  \tag{100}\\
\delta \chi_{R}=\frac{1}{2} \gamma^{\mu} \epsilon_{L} d_{\mu} \phi^{*}  \tag{101}\\
\delta \bar{\chi}_{L}=-\frac{1}{2} \bar{\epsilon}_{R} \gamma^{\mu} d_{\mu} \phi  \tag{102}\\
\delta \bar{\chi}_{R}=-\frac{1}{2} \bar{\epsilon}_{L} \gamma^{\mu} d_{\mu} \phi^{*} \tag{103}
\end{gather*}
$$

you can see what is happening we vary phi our bosonic field and we get out a spinor field. Then if we vary our spinor field we get out the derivative of a bosonic one. Can you construct a Lagrangian or a system that is invariant under such operations. Yes is the answer. and the simplest model is called the Weiss- Zumino Model:

$$
\begin{equation*}
L=-d_{\mu} \phi^{*} d^{\mu} \phi-\bar{\chi}_{R} \gamma^{\mu} d_{\mu} \chi_{L}-\bar{\chi}_{L} \gamma^{\mu} d_{\mu} \chi_{R} \tag{104}
\end{equation*}
$$

Let's check it is Invariant under these operations i.e. is super symmetric.

$$
\begin{gather*}
\delta L=-d_{\mu} \delta \phi^{*} d^{\mu} \phi-\left.d_{\mu} \phi^{*} d\right|^{\mu} \delta \phi-\delta \bar{\chi}_{R} \gamma^{\mu} d_{\mu} \chi_{L}  \tag{105}\\
-\bar{\chi}_{R} \gamma^{\mu} d_{\mu} \delta \chi_{L}-\delta \bar{\chi}_{L} \gamma^{\mu} d_{\mu} \chi_{R}-\bar{\chi}_{L} \gamma^{\mu} d_{\mu} \delta \chi_{R} \\
\delta L=-\bar{\epsilon}_{R} d_{\mu} \chi_{R} * d^{\mu} \phi-d_{\mu} \phi^{*} \bar{\epsilon}_{L} d^{\mu} \chi_{L}+\frac{1}{2} \bar{\epsilon}_{L} \gamma^{\mu} d_{\mu} \phi^{* \nu} d_{\nu} \chi_{L} \\
-\frac{1}{2} \bar{\chi}_{R} \gamma^{\mu} \gamma^{\nu} \epsilon_{R}\left(d_{\mu} d_{\nu} \phi\right)+\frac{1}{2} \bar{\epsilon}_{R} \gamma^{\mu} d_{\mu} \phi \gamma^{\nu} d_{\nu} \chi_{R}-\frac{1}{2} \bar{\chi}_{L} \gamma^{\mu} \gamma^{\nu} \epsilon_{L} d_{\mu} d_{\nu} \phi^{*} \tag{106}
\end{gather*}
$$

Where all we have done is replace using 93-98 the relevant terms in the Lagrangian.
You can see there are terms with only left and terms with only right handed spinors in. In fact
what we have is terms that are the complex conjugate of the other terms so if we prove it is invariant for say just right terms this will automatically show it is invariant for the left terms also and save us a bit of time.

$$
\begin{equation*}
\delta L_{(R)}=-\bar{\epsilon}_{R} d_{\mu} \chi_{R} * d^{\mu} \phi-\frac{1}{2} \bar{\chi}_{R} \gamma^{\mu} \gamma^{\nu} \epsilon_{R}\left(d_{\mu} d_{\nu} \phi\right)+\frac{1}{2} \bar{\epsilon}_{R} \gamma^{\mu} d_{\mu} \phi \gamma^{\nu} d_{\nu} \chi_{R} \tag{107}
\end{equation*}
$$

Ok let's start with the $\gamma^{\mu} \gamma^{\nu}$ term
we can write

$$
\begin{equation*}
\gamma^{\mu} \gamma^{\nu}=\frac{1}{2}\left(\gamma^{\mu} \gamma^{\nu}+\gamma^{\nu} \gamma^{\mu}\right)+\frac{1}{2}\left(\gamma^{\mu} \gamma^{\nu}-\gamma^{\nu} \gamma^{\mu}\right) \tag{108}
\end{equation*}
$$

I.e. as a symmetric part added to an anti-symmetric part. But note $\epsilon$ is symmetric so the anti-symmetric part is simply zero.

$$
\begin{equation*}
\gamma^{\mu} \gamma^{\nu}=\frac{1}{2}\left(\gamma^{\mu} \gamma^{\nu}+\gamma^{\nu} \gamma^{\mu}\right)=\eta^{\mu \nu} \tag{109}
\end{equation*}
$$

Where we have used the definition of the Clifford algebra. So we can replace $\gamma^{\mu} \gamma^{\nu}$ by $\eta^{\mu \nu}$

$$
\begin{array}{r}
-\frac{1}{2} \bar{\chi}_{R} \gamma^{\mu} \gamma^{\nu} \epsilon_{R}\left(d_{\mu} d_{\nu} \phi\right)=-\frac{1}{2} \bar{\chi}_{R} \eta^{\mu \nu} \epsilon_{R}\left(d_{\mu} d_{\nu} \phi\right) \\
=-\frac{1}{2} \bar{\chi}_{R} \epsilon_{R}\left(d_{\mu} d^{\mu} \phi\right)=-\frac{1}{2} \bar{\chi}_{R} \epsilon_{R} \square \phi \tag{110}
\end{array}
$$

Ok Next lets do the

$$
\begin{equation*}
\frac{1}{2} \bar{\epsilon}_{R} \gamma^{\mu} d_{\mu} \phi \gamma^{\nu} d_{\nu} \chi_{R} \tag{111}
\end{equation*}
$$

term. We want to make a d'Alembert operator ( $\square=d_{\mu} d^{\mu}$ appear so we can cancel against the previous expression We can do this by using partial differentiation tricks if this isn't clear it should become so.

$$
\begin{equation*}
\frac{1}{2} d_{\nu}\left(\bar{\epsilon}_{R} \gamma^{\mu} \gamma^{\nu} \chi_{R} d_{\mu} \phi\right)=\frac{1}{2} \bar{\epsilon}_{R} \gamma^{\mu} d_{\mu} \phi \gamma^{\nu} d_{\nu} \chi_{R}+\frac{1}{2} \bar{\epsilon}_{R} \chi_{R} \square \phi \tag{112}
\end{equation*}
$$

rearrange so we have the underlined value which appears in our expression on its own

$$
\begin{equation*}
\frac{1}{2} d_{\nu}\left(\bar{\epsilon}_{R} \gamma^{\mu} \gamma^{\nu} \chi_{R} d_{\mu} \phi\right)-\frac{1}{2} \bar{\epsilon}_{R} \chi_{R} \square \phi=\frac{1}{2} \bar{\epsilon}_{R} \gamma^{\mu} d_{\mu} \phi \gamma^{\nu} d_{\nu} \chi_{R} \tag{113}
\end{equation*}
$$

Ok so now from our $\delta L_{(R)}$ we now have 2 out of 3 terms with this d'Alembert operator term appearing you can see where this is going that they will cancel but lets go through with the last term

$$
\begin{equation*}
-\bar{\epsilon}_{R} d_{\mu} \chi_{R} d^{\mu} \phi \tag{114}
\end{equation*}
$$

We again use partial differentiation trick.

$$
\begin{equation*}
-d_{\nu}\left(\bar{\epsilon}_{R} \chi_{R} d^{\mu} \phi\right)=-\bar{\epsilon}_{R} d_{\mu} \chi_{R} d^{\mu} \phi-\bar{\epsilon}_{R} \chi_{R} \square \phi \tag{115}
\end{equation*}
$$

again rearranging so we have our underlined term by itself

$$
\begin{equation*}
-\bar{\epsilon}_{R} d_{\mu} \chi_{R} d^{\mu} \phi=\bar{\epsilon}_{R} \chi_{R} \square \phi-d_{\nu}\left(\bar{\epsilon}_{R} \chi_{R} d^{\mu} \phi\right) \tag{116}
\end{equation*}
$$

Now let's look at $\delta L_{(R)}$ again

$$
\begin{align*}
& \delta L_{(R)}=-\bar{\epsilon}_{R} d_{\mu} \chi_{R} * d^{\mu} \phi-\frac{1}{2} \bar{\chi}_{R} \gamma^{\mu} \gamma^{\nu} \epsilon_{R}\left(d_{\mu} d_{\nu} \phi\right) \\
& +\frac{1}{2} \bar{\epsilon}_{R} \gamma^{\mu} d_{\mu} \phi \gamma^{\nu} d_{\nu} \chi_{R}=\bar{\epsilon}_{R} \chi_{R} \square \phi-d_{\nu}\left(\bar{\epsilon}_{R} \chi_{R} d^{\mu} \phi\right)  \tag{117}\\
& -\frac{1}{2} \bar{\chi}_{R} \epsilon_{R} \square \phi+\frac{1}{2} d_{\nu}\left(\bar{\epsilon}_{R} \gamma^{\mu} \gamma^{\nu} \chi_{R} d_{\mu} \phi\right)-\frac{1}{2} \bar{\epsilon}_{R} \chi_{R} \square \phi
\end{align*}
$$

Where I have simply entered all the work above to replace the initial expressions. We're still not quite there but if we note bilinears satisfy the following identity (see appendix)

$$
\begin{equation*}
\bar{\epsilon}_{R} \chi_{R}=\bar{\chi}_{R} \epsilon_{R} \tag{118}
\end{equation*}
$$

i.e. they are symmetric. then we can rewrite 48

$$
\begin{array}{r}
\delta L_{(R)}=\bar{\epsilon}_{R} \chi_{R} \square \phi-d_{\nu}\left(\bar{\epsilon}_{R} \chi_{R} d^{\mu} \phi\right)-\frac{1}{2} \bar{\epsilon}_{R} \chi_{R} \square \phi+\frac{1}{2} d_{\nu}\left(\bar{\epsilon}_{R} \gamma^{\mu} \gamma^{\nu} \chi_{R} d_{\mu} \phi\right) \\
-\frac{1}{2} \bar{\epsilon}_{R} \chi_{R} \square \phi=-d_{\nu}\left(\bar{\epsilon}_{R} \chi_{R} d^{\mu} \phi\right)+\frac{1}{2} d_{\nu}\left(\bar{\epsilon}_{R} \gamma^{\mu} \gamma^{\nu} \chi_{R} d_{\mu} \phi\right) \tag{119}
\end{array}
$$

$$
\begin{equation*}
\delta L_{(R)}=d_{\nu}\left(-\bar{\epsilon}_{R} \chi_{R} d^{\mu} \phi+\frac{1}{2} \bar{\epsilon}_{R} \gamma^{\mu} \gamma^{\nu} \chi_{R} d_{\mu} \phi\right) \tag{120}
\end{equation*}
$$

So we have demonstrated the Lagrangian varies by a total derivative. This means the system is invariant under the symmetry operation. one natural question in your mind is probably how is this invariant if this total derivative has appeared. Well for the physics to stay the same it is the action $(\mathrm{S})$ that must remain invariant

$$
\begin{equation*}
S=\int d t L(t) \tag{121}
\end{equation*}
$$

We see we have added a total derivative to this action but remember from the fundamental theorem of calculus we can see.

$$
\begin{equation*}
\int_{a}^{b} d x f(x)=F(b)-F(a) \tag{122}
\end{equation*}
$$

Which says the integral of a derivative is just the function evaluated at the boundaries. So in our case we can evaluate our function at the boundaries which is indeed zero and thus the action is unchanged and therefore it is invariant under this change.

## 14 Noether's Theorem

One of the most famous theories in physics is Noether's theorem. It says every continuous symmetry of the Lagrangian gives rise to a conserved current $j^{\mu}(x)$ such that the equations of motions imply

$$
\begin{equation*}
d_{\mu} j^{\mu}=0 \tag{123}
\end{equation*}
$$

### 14.1 A simple proof

As mentioned previously a transformation

$$
\begin{equation*}
\delta \phi_{a}(x)=X_{a}(\phi) \tag{124}
\end{equation*}
$$

is a symmetry if the Lagrangian changes by a total derivative (because the action is unchanged!!!)

$$
\begin{equation*}
\delta L=d_{\mu} F^{\mu} \tag{125}
\end{equation*}
$$

Where $F^{\mu}(\phi)$ is any function of phi. Forming an arbitrary transformation of the field $\delta \phi_{a}$

$$
\begin{equation*}
\delta L=\frac{d L}{d \phi_{a}} \delta \phi_{a}+\frac{d L}{d\left(d_{\mu} \phi_{a}\right.} d_{\mu}\left(\delta \phi_{a}\right) \tag{126}
\end{equation*}
$$

again use a little partial differentiation trick

$$
\begin{equation*}
d_{\mu}\left(\frac{d L}{d\left(d_{\mu} \phi_{a}\right)} \delta \phi_{a}\right)=\frac{d L}{d\left(d_{\mu} \phi_{a}\right)} d_{\mu}\left(\delta \phi_{a}\right)+d_{\mu} \frac{d L}{d\left(d_{\mu} \phi_{a}\right)} \delta \phi_{a} \tag{127}
\end{equation*}
$$

so

$$
\begin{equation*}
\delta L=\left[\frac{d L}{d \phi_{a}} \delta \phi_{a}-d_{\mu} \frac{d L}{d\left(d_{\mu} \phi_{a}\right)} \delta \phi_{a}\right]+d_{\mu}\left(\frac{d L}{d\left(d_{\mu} \phi_{a}\right)} \delta \phi_{a}\right) \tag{128}
\end{equation*}
$$

You can see the Euler-Lagrange equation in the square brackets so when the equations of motion are satisfied i.e. that bit in the square brackets is zero then we are just left with the total derivative term at the end
and remember our definition of the transformation $\delta \phi_{a}=X_{a}$

$$
\begin{equation*}
\delta L=d_{\mu}\left(\frac{d L}{d\left(d_{\mu} \phi_{a}\right)} X_{a}\right)=d_{\mu} F^{\mu} \tag{129}
\end{equation*}
$$

Now minus one side from the other

$$
\begin{equation*}
\delta L-\delta L=0=d_{\mu}\left(\frac{d L}{d\left(d_{\mu} \phi_{a}\right)} X_{a}-F^{\mu}\right) \tag{130}
\end{equation*}
$$

and simply call the term in brackets $j^{\mu}$ such that:

$$
\begin{equation*}
j^{\mu}=\frac{d L}{d\left(d_{\mu} \phi_{a}\right)} X_{a}-F^{\mu} \tag{131}
\end{equation*}
$$

and from (118) therefore $d_{\mu} j^{\mu}=0$ which is what we wanted to show.
It is also very important for our current application that the existence of such a conserved current gives rise to a conserved Charge

$$
\begin{equation*}
Q=\int_{R^{3}} d^{3} x j^{0} \tag{132}
\end{equation*}
$$

This may all seem totally irrelevant but we will show how these conserved charges are the generators of a Lie algebra for our new symmetry group.

## 15 Obtaining the Super Symmetry Algebra

So from 115 we can do something analogous for the left handed spinors. Because it is exactly the same I will leave it to the reader to check

$$
\begin{array}{r}
\delta L_{(L)}=d_{\nu}\left(-d^{\nu} \phi^{*} \bar{\epsilon}_{L} \chi_{L}+\bar{\epsilon}_{L} \chi_{L} \square \phi^{*}\right)  \tag{133}\\
=d_{\nu} F^{\nu} \rightarrow F_{(L)}^{\nu}=-d^{\nu} \phi^{*} \bar{\epsilon}_{L} \chi_{L}+\bar{\epsilon}_{L} \chi_{L} \square \phi^{*}
\end{array}
$$

Where I have equated with the $F^{\mu}$ expression via equation 120. Again for the sake of brevity I will do these calculations merely for one side of the spinor (right) as the other side is the complex conjugate of the first so very easy to calculate.

We wish to obtain the expression for $j^{\mu}$ from 131 so we can use our found $F^{\mu}$ and take the derivative as in 131 to get the correct expression. Remember ( $X_{a}$ are the transformations 98 to 103 . So if we do it for terms that just transform to right handed spinors we shall need 94,95 , and, 97 .

We Wish to Calculate

$$
\begin{equation*}
\frac{d L}{d\left(d_{\mu} \phi_{a}\right)} X_{a} \tag{134}
\end{equation*}
$$

from 131
For $\phi^{*}$

$$
\begin{equation*}
\frac{d L}{d\left(d_{\mu} \phi^{*}\right)} \delta \phi^{*}=-d^{\mu} \phi \bar{\epsilon}_{R} \chi_{R} \tag{135}
\end{equation*}
$$

For $\chi_{L}$

$$
\begin{equation*}
\frac{d L}{d\left(d_{\mu} \chi_{L}\right)} \delta \chi_{L}=-\frac{1}{2} \bar{\chi}_{R} \gamma^{\mu} \gamma^{\nu} \epsilon_{R} d_{\mu} \phi \tag{136}
\end{equation*}
$$

For $\bar{\chi}_{L}$

$$
\begin{equation*}
\frac{d L}{d\left(d_{\mu} \bar{\chi}_{L}\right)} \delta \bar{\chi}_{L}=0 \tag{137}
\end{equation*}
$$

combining with our expression for $F^{\mu}$ we can obtain our expression for $j^{\mu}$ via the procedure of 120
This gives

$$
\begin{align*}
j_{(R)}^{\mu}=\frac{d L}{d\left(d_{\mu} \phi_{a}\right)} X_{a}-F^{\mu} & =-d^{\mu} \phi \bar{\epsilon}_{R} \chi_{R}-\frac{1}{2} \bar{\chi}_{R} \gamma^{\mu} \gamma^{\nu} \epsilon_{R} d_{\mu} \phi  \tag{138}\\
& -\left(-\bar{\epsilon}_{R} \chi_{R} d^{\mu} \phi+\frac{1}{2} \bar{\epsilon}_{R} \gamma^{\mu} \gamma^{\nu} \chi_{R} d_{\mu} \phi\right)
\end{align*}
$$

Cancelling terms gives

$$
\begin{equation*}
j_{(R)}^{\mu}=-\bar{\chi}_{R} \gamma^{\mu} \gamma^{\nu} \epsilon_{R} d_{\mu} \phi=-\bar{\epsilon}_{R} \gamma^{\mu} \gamma^{\nu} \chi_{R} d_{\mu} \phi \tag{139}
\end{equation*}
$$

The reader can check doing the same thing for the left handed terms will give

$$
\begin{equation*}
j_{(L)}^{\mu}=-\bar{\epsilon}_{L} \gamma^{\mu} \gamma^{\nu} \chi_{L} d_{\mu} \phi^{*} \tag{140}
\end{equation*}
$$

Ok now lets check this is a conserved quantity.i.e. we want $d_{\mu} j^{\mu}=0$ when the equations of motion are satisfied.

Obtaining equations of motion for a field is very similar to that of a particle. Again we use the Euler- Lagrange equations but they look slightly different for fields.

$$
\begin{equation*}
\frac{d L}{d \phi^{a}}-d_{\mu}\left(\frac{d L}{d\left(d_{\mu} \phi^{a}\right)}\right) \tag{141}
\end{equation*}
$$

So
For $\phi$

$$
\begin{equation*}
\frac{d L}{d \phi}-d_{\mu}\left(\frac{d L}{d\left(d_{\mu} \phi\right)}\right)=0+\square \phi^{*}=0 \tag{142}
\end{equation*}
$$

For $\phi *$

$$
\begin{equation*}
\frac{d L}{d \phi^{*}}-d_{\mu}\left(\frac{d L}{d\left(d_{\mu} \phi^{*}\right)}\right)=0+\square \phi=0 \tag{143}
\end{equation*}
$$

For Chi Left

$$
\begin{equation*}
\frac{d L}{d \chi_{L}}-d_{\mu}\left(\frac{d L}{d\left(d_{\mu} \chi_{L}\right)}\right)=0+d_{\mu} \bar{\chi}_{R} \gamma^{\mu}=0 \tag{144}
\end{equation*}
$$

For Chi Right

$$
\begin{equation*}
\frac{d L}{d \chi_{R}}-d_{\mu}\left(\frac{d L}{d\left(d_{\mu} \chi_{R}\right)}\right)=0+d_{\mu} \bar{\chi}_{L} \gamma^{\mu}=0 \tag{145}
\end{equation*}
$$

For $\bar{\chi}_{L}$

$$
\begin{equation*}
\frac{d L}{d \mu \bar{\chi}_{L}}-d_{\mu}\left(\frac{d L}{d\left(d_{\mu} \bar{\chi}_{L}\right)}\right)=\gamma^{\mu} d_{\mu} \chi_{R}-0=0 \tag{146}
\end{equation*}
$$

For $\bar{\chi}_{R}$

$$
\begin{equation*}
\frac{d L}{d \mu \bar{\chi}_{R}}-d_{\mu}\left(\frac{d L}{d\left(d_{\mu} \bar{\chi}_{R}\right)}\right)=\gamma^{\mu} d_{\mu} \chi_{L}-0=0 \tag{147}
\end{equation*}
$$

Lets check for the left handed current if $d_{\mu} j^{\mu}=0$ i.e. it is a conserved current:

$$
\begin{align*}
& d_{\mu} j^{\mu}=-d_{\mu}\left(\bar{\epsilon}_{L} \gamma^{\mu} \gamma^{\nu} \chi_{L} d_{\mu} \phi^{*}\right)=-\left(\bar{\epsilon}_{L} \gamma^{\mu} \gamma^{\nu} d_{\mu} \chi_{L} d_{\mu} \phi^{*}\right.  \tag{148}\\
&\left.+\bar{\epsilon}_{L} \gamma^{\mu} \gamma^{\nu} \chi_{L} d_{\mu} d^{\mu} \phi^{*}\right)=0
\end{align*}
$$

Where I have used 142 for the second term and 146 for the first to show they are zero. So we have conserved currents and Noether's theorem implies there also exists a conserved charge

$$
\begin{equation*}
Q_{L / R}=\int d^{3} j_{L / R}^{0}=-\int d^{3} \gamma^{\mu} \gamma^{0} \chi_{L / R}\left(d_{\mu} \phi^{*} / \phi\right) \tag{149}
\end{equation*}
$$

You are probably wondering what the point of this was but this will become clear if we take the commutator relations of these charges $Q$ with the fields. Lets check what happens.

$$
\begin{align*}
\bar{\epsilon}_{L}\left\{Q_{L}, \phi\right\}_{P B} & =-\bar{\epsilon}_{L}\left\{\int d^{3} x \gamma^{\mu} \gamma^{0} \chi_{L}\left(d_{\mu} \phi^{*}\right), \phi\right\} \\
& =-\bar{\epsilon}_{L} \int d^{3} x\left\{\gamma^{0} \gamma^{0} \chi_{L}\left(d_{0} \phi^{*}\right), \phi\right\} \tag{150}
\end{align*}
$$

You are probably wondering why the derivative is only the time component $d_{0}$ well we have just used the fact that all the spacial components will compute and therefore will give zero. Again notice the time derivative of $\phi^{*}$ is secretly the momentum luckily for us as we can use the known commutation relation

$$
\begin{equation*}
\{\pi(x), \phi(y)\}=-\delta(x-y) \tag{151}
\end{equation*}
$$

So substituting 150 into 151 in the place of $\left\{d_{0} \phi^{*}, \phi\right\}$ and noting $\gamma^{0} \gamma^{0}=1$ we get

$$
\begin{array}{r}
\bar{\epsilon}_{L}\left\{Q_{L}, \phi\right\}_{P B}=-\bar{\epsilon}_{L} \int d^{3} x \chi_{L}\{\pi(x), \phi(y)\} \\
=\bar{\epsilon}_{L} \int d^{3} x \chi_{L} \delta(x-y)=\bar{\epsilon}_{L} \chi_{L}=\delta \phi \tag{152}
\end{array}
$$

Now this is an interesting result. It turns out our conserved charges $Q$ produce the transformation rules for our super-symmetric fields. In mathematical terms the Q's are the generators of the algebra.

If we do this same operation for all the bosonic and fermionic fields we obtain the following.

$$
\begin{gather*}
\bar{\epsilon}_{R}\left\{Q_{R}, \phi^{*}\right\}_{P B}=\bar{\epsilon}_{R} \chi_{R}=\delta \phi *  \tag{153}\\
\left\{\bar{Q}_{R} \epsilon_{R}, \chi_{L}\right\}_{P B}=\frac{1}{2} \gamma^{\mu} \epsilon_{R} d_{\mu} \phi=\delta \chi_{L}  \tag{154}\\
\left\{\bar{Q}_{L} \epsilon_{L}, \chi_{R}\right\}_{P B}=\frac{1}{2} \gamma^{\mu} \epsilon_{L} d_{\mu} \phi^{*}=\delta \chi_{R} \tag{155}
\end{gather*}
$$

For the algebra to be closed we need to check that the commutators between two super-symmetric transformations gives another transformation,

$$
\begin{array}{r}
\left\{\delta\left(\epsilon_{1}\right), \delta\left(\epsilon_{2}\right)\right\} \phi \\
=\delta\left(\epsilon_{1}\right) \delta\left(\epsilon_{2}\right) \phi-\delta\left(\epsilon_{2}\right) \delta\left(\epsilon_{1}\right) \phi \tag{156}
\end{array}
$$

Use 98 to 103 for our equations for the transformations

$$
\begin{array}{r}
=\delta\left(\epsilon_{1}\right)\left(\overline{\epsilon_{2}} \chi_{l}\right)-\delta\left(\epsilon_{2}\right)\left(\overline{\epsilon_{1}} \chi_{l}\right) \\
=\frac{1}{2} \overline{\epsilon_{2}}{ }_{L} \gamma^{\mu} \epsilon_{1 R} d_{\mu} \phi-\frac{1}{2} \overline{\epsilon_{1}}{ }_{L} \gamma^{\mu} \epsilon_{2 R} d_{\mu} \phi  \tag{157}\\
=\frac{1}{2}\left(\overline{\epsilon_{2}}{ }_{L} \gamma^{\mu} \epsilon_{1 R}-\overline{\epsilon_{1}} \gamma^{\mu} \epsilon_{2 R}\right) d_{\mu} \phi \\
=a^{\mu} d_{\mu} \phi
\end{array}
$$

where i have written in the form of $a^{\mu}$ to show that that two super symmetric transformations acting on a field have the effect of transforming the field by a space time translation. (If you want to see why, I leave it to the reader to do a space time translation and see how the field transforms It will give a result in the form $a^{\mu} d_{\mu} \phi$.)

So we have learned the algebra is closed. The product of two super symmetric transformations is a space time translation. So the generators of the complete algebra are thus the combination of these super symmetric transformations with the space time transformations (Poincare transformations) we are familiar with.

In its simplest form the complete algebra takes the form :

$$
\begin{gather*}
\left(\left[Q_{\alpha}, M^{\mu \nu}\right]\right) \approx Q_{\beta}  \tag{158}\\
\left(\left[Q_{\alpha}, P_{\mu}\right]\right)=0  \tag{159}\\
\left\{Q_{\alpha}, \bar{Q}_{\beta}\right\} \approx P_{\mu} \tag{160}
\end{gather*}
$$

The algebra is closed since the Jacobi identities are satisfied and have explicit matrix representations. You will also note we have introduced something new $\}$ is the anti commutator. and l've just highlighted the normal commutator for you by putting it in brackets. Why does this appear? It has to do with what is called a grading.

## 15.1 $\quad Z_{2}$ Graded Algebras

The fact that the anti commutator has appeared separates this algebra from a normal Lie algebra. It is termed a Super-Algebra. And is what is called a $z_{2}$ graded algebra.
Let $O_{a}$ be operators of a Lie algebra then

$$
\begin{equation*}
O_{a} O_{b}-(-1)^{\eta_{a} \eta_{b}} O_{b} O_{a}=i f_{a b}^{c} O_{c} \tag{161}
\end{equation*}
$$

Where the $\eta_{a}$ are termed gradings satisfying:

$$
\eta_{a}=\left\{\begin{array}{l}
0: O_{a} \text { is a bosonic generator }  \tag{162}\\
1: O_{a} \text { is a fermionic generator }
\end{array}\right.
$$

Note we can make this look quite similar to the Lie Algebra relations we are used to.
Let

$$
\begin{equation*}
A, B \in \text { Super Algebra } \tag{163}
\end{equation*}
$$

then $A, B$ satisfy the following identities

1. Graded Skew Symmetry

$$
\begin{equation*}
-(-1)^{\eta^{a} \eta^{b}}[B, A]=[A, B] \tag{164}
\end{equation*}
$$

2. Graded Jacobi Identity

$$
\begin{equation*}
(-1)^{\eta^{a} \eta^{c}}[A,[B, c]]+(-1)^{\eta^{a} \eta^{b}}[B,[C, A]]+(-1)^{\eta^{b} \eta^{c}}[C,[A, B]]=0 \tag{165}
\end{equation*}
$$

For this and a more detailed understanding of gradings in physics see [15]

## 16 Conclusion

This report was created with a two fold purpose

- 1. Introduce the topic of Super Symmetry
- 2. Provide a basic insight into the mathematical structure of Quantum Field Theory And Super Symmetry

It was first shown that from quantum field theory we encounter a problem with the Bosonic contribution to the vacuum energy. Specifically, that one obtains a divergence as the energy of these harmonic oscillators increases. Then we introduced the topic of Lie Algebras to begin the topic of understanding how spinors are formed and, more importantly, the mathematical basis for how a Lorentz invariant theory that contains spin can be formed. Upon construction, one finds a similar infinity in the Fermionic contribution which forms the basis for why we wish to study Super Symmetry as the combining of Fermions and Bosons in this manner removes this infinity from our equations. Lastly, We have seen how an additional structure has been added to the algebra in the process of Super Symmetry. And we introduce the topic of a graded algebra to the reader.

With this basic introduction covered there are still many more areas for further investigation. Firstly, there is the pure experimental approach and study which while the topic of this report has been heavily theoretically based is a vital area for our quest for further understanding. Such a study would look to analyse how one would go about looking for Super Symmetric particles as indeed the theory does predict the existence of them. [3]. There is a lot of hype about Super Symmetric particles as candidates for dark matter so an astrophysical approach could also be taken [5]. In particular looking at dark matter candidates like the " neutralino of SuperSymmetry" and which energy ranges they would be in so astrophysical observational data could be obtained and studied. It must be noted that there is still no experimental evidence for Super Symmetry and it remains just a very promising possibility in our search for new physics beyond the standard model.

Beyond experimental studies, SUSY is obviously vital in a number of areas of current research. One such being Super - Gravity which aims to unify the fundamental forces of nature. It predicts the use of extra dimensions and the reader can check for themselves that the mathematics presented in this report is a very small step towards a whole host of other tools and techniques that will become important in understanding these more advanced topics. [7]

Part of the basis for this report was to begin to introduce the mathematical basis for super symmetry that of Lie Algebras to the reader. We saw how a Lie algebra was quite a basic mathematical structure, A group with a manifold structure, and had natural ways of understanding the relationship between a Lie Algebra and A Lie Group. We saw there exists a mapping from the algebra to the group via the exponential map and likewise, a natural way of obtaining the Algebra form the group via a path differentiated at the identity forming vectors in the tangent space which is just the Lie Algebra. Such a simple picture quickly becomes more and more complex the more deep one looks into the topic and a number of natural questions arise for further study:

Firstly, in Super Symmetry we see in a rather matter of fact presentation how the Noether charges $Q$ are the generators of transformations just Like $J$ and $K$ are from the Lorentz transformations. Our methodology was very clear for obtaining J and K we understood in general how one could go from a Lie Group that of $S O(1,3)$ to the Lie Algebra by looking at the tangent space at the identity. Now, given this there must be some deep mathematical basis for the Noether charges becoming the generators of an Algebra that is clear. Indeed, it would be enlightening to take a formal mathematical route in obtaining the Noether charges $Q$ in order to understand their basis from a mathematical perspective. There is a natural mathematical analogue which is the generator indeed a Noether charge and a generator is the same thing just one is a mathematical term and one is a physics term. What about a conserved current? for example, there must be a reason all these things "work" in giving you the algebra etc and this would be helpful to understand if one were to do more research on the topic.

Another natural progression would be the introduction of Super Spaces. Again it would be use-
ful to understand if a manifold structure still exists and what mapping is the super algebra to the super group is it again the normal choice of the exponential map that can be proven to exist or is there a new kind of mapping. Moreover, it leads to a topic we did not touch upon at all which is that of Grassmannian numbers. Such numbers form a reasoning as to why the anti-commutation relations begin to appear in Super Symmetry. These numbers are fundamental in understanding Super Spaces and would play a vital role going forward. [10]

As an intermediary step between undergraduate and graduate studies, the reader can see how the framework of Quantum Field Theory lies on a bed of deep mathematical structure that when beginning to be unveiled becomes more odd and also more interesting. It is extremely un-intuitive but perhaps nice because the reader has to learn so much new mathematics to even try to understand the kind of questions that begin with why?. I will end by saying to the reader, rest assured it is not only us who has the feeling of being at a loss with the subject, indeed Michael Atiyah (winner of the fields medal in 1966) sums it up nicely:
"No one fully understands spinors. Their algebra is formally understood but their general significance is mysterious. In some sense they describe the "square root" of geometry and, just as understanding the square root of -1 took centuries, the same might be true of spinors." Michael Atiyah [6]

## 17 Appendix

### 17.1 Useful Identities In Super Symmetry

The four dimensional $\gamma^{\mu}$ matrices obey the clifford algebra relations:

$$
\begin{gather*}
\left\{\gamma_{\mu}, \gamma_{\nu}\right\}=2 \eta_{\mu \nu} 1_{4}  \tag{166}\\
\gamma^{\mu} \gamma^{\nu}=-\gamma^{\mu} \gamma^{\nu} \tag{167}
\end{gather*}
$$

With

$$
\begin{gather*}
\left(\gamma^{0}\right)^{2}=1_{4} \\
\left(\gamma^{i}\right)^{2}=-1_{4}  \tag{168}\\
i=1,2,3 \\
\gamma^{0}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right) \tag{169}
\end{gather*}
$$

where 1 is the identity ( $2 \times 2$ ). And

$$
\gamma^{i}=\left(\begin{array}{cc}
0 & \sigma^{i}  \tag{170}\\
\sigma^{-i} & 0
\end{array}\right)
$$

where

$$
\begin{align*}
\sigma^{1} & =\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)  \tag{171}\\
\sigma^{2} & =\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right)  \tag{172}\\
\sigma^{2} & =\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) \tag{173}
\end{align*}
$$

Themselves satisfying

$$
\begin{equation*}
\left\{\sigma^{i}, \sigma^{j}\right\}=2 \delta^{i j} \tag{174}
\end{equation*}
$$

Charge Conjugation Matrices obey

$$
\begin{array}{r}
C^{T}=-C \\
\gamma_{\mu}^{T}=-C \gamma_{\mu} C^{-1} \tag{175}
\end{array}
$$

A Majorana Spinor $\epsilon$ is defined as a spinor for which the Dirac conjugate is equal to the Majorana conjugate

$$
\begin{equation*}
i \epsilon^{\dagger} \gamma^{0}=\epsilon^{T} C \tag{176}
\end{equation*}
$$

Majorana spinors have a chiral projection $L / R$ such that

$$
\begin{array}{r}
\epsilon_{L / R}=P_{L} / R \epsilon \\
P_{L / R}=\frac{1}{2}\left(1_{4} \pm \gamma_{5}\right) \tag{177}
\end{array}
$$

The conjugates obeying the following relations

$$
\begin{align*}
& \epsilon_{L}^{*}=i C \gamma^{0} \epsilon_{R} \\
& \epsilon_{R}^{*}=i C \gamma^{0} \epsilon_{L} \tag{178}
\end{align*}
$$

We take the convention that complex conjugation changes the order in a billinear. i,e.

$$
\begin{gather*}
(\bar{\epsilon} \chi)^{*}=-\bar{\chi}^{*} \epsilon^{*} \\
\left(\bar{\epsilon}_{L} \chi_{L}\right)^{*}=\bar{\epsilon}_{R} \chi_{R} \tag{179}
\end{gather*}
$$

The bar here meaning

$$
\begin{align*}
\bar{\epsilon}_{L} & =\frac{1}{2} \bar{\epsilon}\left(1_{4}+\gamma_{5}\right) \\
\bar{\epsilon}_{R} & =\frac{1}{2} \bar{\epsilon}\left(1_{4}-\gamma_{5}\right) \tag{180}
\end{align*}
$$

### 17.2 Proof of bilinear symmetry

Show

$$
\begin{equation*}
\bar{\psi}_{L} \phi_{l}=\bar{\phi}_{L} \psi_{L} \tag{181}
\end{equation*}
$$

use that

$$
\begin{gather*}
\left(\bar{\psi}_{L} \phi_{l}\right)^{T}=\bar{\psi}_{L} \phi_{l}  \tag{182}\\
\left(\bar{\psi}_{L} \phi_{l}\right)^{T}=\left(\bar{\psi} \frac{1}{2}\left(1_{4}+\gamma_{5}\right) \frac{1}{2}\left(1_{4}+\gamma_{5}\right) \phi\right)^{T} \tag{183}
\end{gather*}
$$

Note

$$
\begin{array}{r}
\frac{1}{2}\left(1_{4}+\gamma_{5}\right) \frac{1}{2}\left(1_{4}+\gamma_{5}\right)=\frac{1}{4}\left(1_{4}+2 \gamma_{5}+\gamma_{5} \gamma_{5}\right) \\
\quad=\frac{1}{4}\left(1_{4}+2 \gamma_{5}+1_{4}\right)=\frac{1}{2}\left(1_{4}+\gamma_{5}\right) \tag{184}
\end{array}
$$

So

$$
\begin{align*}
\left(\bar{\psi}_{L} \phi_{l}\right)^{T} & =\left(\bar{\psi} \frac{1}{2}\left(1_{4}+\gamma_{5}\right) \phi\right)^{T} \\
& =\phi^{T} \frac{1}{2}\left(1_{4}+\gamma_{5}^{T}\right) \bar{\psi}^{T} \tag{185}
\end{align*}
$$

Remeber the dirac conjugate obeys

$$
\begin{array}{r}
\bar{\psi}=i \psi^{\dagger} \gamma^{0} \\
\bar{\psi}^{T}=i\left(\gamma^{0}\right)^{T} \psi^{*} \\
=-i C \gamma^{0} c^{-1} \psi^{*}  \tag{186}\\
\text { and note } \\
\gamma_{5}^{T}=C \gamma_{5} C^{-1}
\end{array}
$$

Now plug into 185

$$
\begin{array}{r}
\phi^{T} \frac{1}{2}\left(1_{4}+\gamma_{5}^{T}\right) \bar{\psi}^{T}=-\phi^{T} \frac{1}{2}\left(1_{4}+C \gamma_{5} C^{-1}\right) i C \gamma^{0} C^{-1} \psi^{*}  \tag{187}\\
\\
=-\phi^{T} \frac{1}{2} C\left(1_{4}+\gamma_{5}\right) i \gamma^{0} C^{-1} \psi^{*}
\end{array}
$$

Use

$$
\begin{equation*}
\psi^{T} C=i \psi^{\dagger} \gamma^{0} \tag{188}
\end{equation*}
$$

and for the other part we can adapt the same expression

$$
\begin{array}{r}
\left(\psi^{T} C\right)^{T}=i\left(\psi^{\dagger} \gamma^{0}\right)^{T} \\
=C^{T} \psi=-C \psi=-i C \gamma^{0} C^{-1} \psi^{*}  \tag{189}\\
\psi=i \gamma^{0} C^{-1} \psi^{*}
\end{array}
$$

so 187 becomes

$$
\begin{array}{r}
-\phi^{T} \frac{1}{2} C\left(1_{4}+\gamma_{5}\right) i \gamma^{0} C^{-1} \psi^{*} \\
=i \phi^{\dagger} \gamma^{0} \frac{1}{2}\left(1_{4}+\gamma_{5}\right) \psi \\
=\bar{\phi} \frac{1}{2}\left(1_{4}+\gamma_{5}\right) \psi  \tag{190}\\
=\bar{\phi} \frac{1}{2}\left(1_{4}+\gamma_{5}\right) \frac{1}{2}\left(1_{4}+\gamma_{5}\right) \psi=\bar{\phi}_{L} \psi_{L}
\end{array}
$$

Thus showing the identity

$$
\begin{gather*}
\bar{\psi}_{L} \phi_{L}=\left(\bar{\psi}_{L} \phi_{L}\right)^{T}=\bar{\phi}_{L} \psi_{L}  \tag{191}\\
E=h f \tag{192}
\end{gather*}
$$

## References

[1] Dr David Tong. Quantum Field Theory Part 3 Mathematical Tripos. University Of Cambridge Press 2007.
[2] B. C. Allanach and F. Quevedo. Mathematics tripos part 3 supersymmetry course lecture notes, university of cambridge, 2016.
[3] John Ellis Interviewed by Matthw Chalmers. Cern Courier Jan 2020. https://cerncourier.com/a/the-higgs-supersymmetry-and-all that/.
[4] Cern. Birth of the universe. url: http://abyss.uoregon.edu.
[5] Michael Peskin. Supersymmetric dark matter in the harsh light of the Large Hadron Collider. 112. OCT 2015.
[6] G. The Strangest Man. In The Hidden Life of Paul Dirac: Quantum Genius; Faber Farmelo and 2009; p. 430. Faber: London, UK.
[7] Sutton. Christine. "Supergravity". Encyclopedia Britannica. 6 Nov. 2016. https://www.britannica.com/science/supergravity. Accessed 30 June 2021.
[8] Richard Feynman. Feyman Lectures. Caltech. https://www.feynmanlectures.caltech.edu, 2013.
[9] Adel Bilal. Lecture notes Introduction to Supersymmetry. Institute of Physics. University of Neuchatel July 2000.
[10] Sultan Catto. Grassmann Numbers, Clifford-Jordan-Wigner Representation of Supersymmetry.XXth International Conference on Integrable Systems, and Quantum Symmetries. Journal of Physics: Conference Series 411 (2013).
[11] Brittanica. The Editors of Enclopaedia. "Planck's constant". Encylopedia Britannica. 1 Jun. 2021. https://www.britannica.com/science/Plancks constant.
[12] L. A. Ferreira. Lecture Notes on Lie Algebras and Lie Groups. Instituto de F'ısica de S~ao Carlos IFSC/USP, 2011.
[13] A. Cappelli The Birth Of String Theory. Cambridge University Press, 2012.
[14] Peter Rogers. Supersymmetry. Nature Physics 4. S13, Jul 2008.
[15] John Tsartsaflis. $Z_{2}$ and Klein graded Lie Algebras. Master's Thesis. Sep 2012. Cornell University. https://arxiv.org/abs/1211.7130.

