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# Emergence and Properties of Fractons in Higher-order Moment Conserving Field Theories 

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#### Abstract

Fractons are an exciting new class of quasiparticles that exhibit severely limited mobility. In this thesis, a fracton theory is developed by enforcing the additional constraint of higher-order moment conservation on a field theory. The spacetime symmetries and spontaneously broken symmetry case of a dipole moment conserving theory are considered. Finally the theory is generalized to conserve arbitrarily high order moment via the application of the gauge principle. It is found that a dipole moment conserving theory is not invariant under any form of spacetime boost, and that the same theory can exhibit a non-zero momentum even at zero energy.


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## 1 Introduction

The concept of a quasiparticle, where a phenomenon on a collection of particles can be described by a hypothetical emergent particle, was developed by Landau in the 1930s [16]. Since then, the concept has become extremely important in the field of condensed matter physics, where complex systems of large numbers of particles are examined, and quasiparticles can often be used to describe behaviour of systems to which the constituent particles don't readily lend themselves.

Although quite a recent addition to the family of quasiparticles, having been first encountered by Chamon [2] in 2005, fractons have been subject to interest due to a surprising property; severely limited mobility. Fractons by themselves cannot move at all [9]. On the other hand, depeding on the particularities of a theory, bound states of multiple fractons may be capable of (limited) movement. Fracton theories are also closely related to theories of subdimensional particles, excitations which only exhibit mobility along certain directions [9].

Since their initial appearance, scientific interest in fractons has surged due to peculiar properties resulting from their limited mobility, including gravitational behaviour [7], nonergodic behaviour [9], and links to other active areas of research. This has resulted in various theories for fractons having been developed. Two primary approaches are utilized to construct these theories: $U(1)$-symmetric field theories pioneered by Pretko [8], which are the main interest here, as well as solvable spin models, which give rise to the X-cube model, and Haah's code among others. A brief overview of these theories is given in section 2 .
In field theory the different conservation laws result from the invariance of an action associated with the theory under different transformations. Such symmetry transformations correspond to conserved currents by Noether's theorem, resulting in different conservation laws. Transformations that change the field equivalently independently of position are called global symmetries, while local symmetries are transformations that act differently depending on position. As is shown in section 5, these local symmetries result in gauge theories.
Here, reviews of various underlying topics and gauge theory are provided. Gauge theory is illustrated through a demonstration of the process using the Maxwell gauge. Then a field theory with additional moment conservation constraints is constructed to demonstrate the emergence of fractons. Finally the spacetime symmetries as well as links to other literature are considered, and the gauge principle is applied to generalize the theory. Ultimately an overarching discussion, as well as an outlook to the future of the topic are presented.

## 2 On Fracton Theories

Before diving into the details of fracton theories, a brief discussion of the different kinds of theories is provided here.

Fractons were first encountered in exactly-solvable spin models in three dimensions. Since then a great variety of theories where fractons emerge have been formulated. Exactly solvable spin models accounted for the majority of early fracton models, with the most prominent being Haah's code and the X-cube model [4]. These are then further divided into type-I and type-II fracton theories, with type-I theories possessing bound states that are mobile, while in type-II theories all such states decay into the vacuum [9]. Both of these theories exhibit a spectral gap between their ground and first excited states, and as such are classified as gapped fracton
theories. For a more general look at these models the reader is directed to [4, 9]. More recently foliated (quantum) field theories have been introduced as a means of supplementing the X -cube model 13,14 .

The first gapless fracton theories were given rise to when it was realized that enforcing higher order moment conservation on fields produces theories with emergent fracton charges. In addition to providing a new way of describing fractons, these theories also provided new links to seemingly unrelated fields, such as elasticity theory [10] and (quantum) gravity [7].

## 3 On Field Theories

For much of the recent decades, the theory at the forefront of our understanding of the universe has been Quantum Field Theory (QFT). This theory, as Peskin and Schroeder point out [6], combines three of the important ideas that were developed in the 20th century: fields, quantum mechanics, and relativity. Before proceeding with the construction of a fracton theory, some relevant properties of these theories are discussed.

### 3.1 Classical Field Theory

Newtonian mechanics was plagued by an issue since its conception. The concept of locality was completely absent - if mass was added at some location, the entire universe would feel the influence of that mass instantly, without the change being mediated by anything. Newton himself already acknowledged this issue, but could not provide a solution. The was solved further down the road via the introduction of fields.

A field is an object defined at all points of space and time. As an intuitive example, we can think of temperature. Every point in space has a temperature associated with it at any time, and thus temperature could be considered a field $T(t, \vec{x})$. Thinking along these lines, we notice that with temperature, and indeed with field theory in general, position is no longer a variable, but has rather been appointed as a label of the field. We also notice that $T(t, \vec{x})$ has an enormous number of degrees of freedom, with at least one for each $\vec{x}$.

Digressing for a moment, we can consider the mathematical structure of a field. Usually we will consider our domain space to be some $n$ dimensional real space $\mathbb{R}^{n}$. The target space, in turn, defines the type of field being discussed; almost any space can be selected, but typically real, or complex fields are discussed, with the associated target spaces $\mathbb{R}$, and $\mathbb{C}$ respectively. A field can in some sense then be considered a map $\phi: \mathbb{R}^{n} \rightarrow \mathcal{D}$, with $\mathcal{D}$ the desired kind of domain.

Let's say we have some fields $\phi_{a}(t, \vec{x})$. This by itself is not very interesting, as there are no dynamics. Some equations are needed for that. We can introduce such an equation, called the Lagrangian $\mathcal{L}$ of the theory ${ }^{1}$. From here, the action of the theory can be defined as

$$
\begin{equation*}
S=\int d^{d} x \mathcal{L} \tag{1}
\end{equation*}
$$

where we are working in $d$ spacetime dimensions. It should be noted here that because we are almost always interested in the action rather than the Lagrangian, we can treat the Lagrangian as if it was in an integral.

[^0]
### 3.2 Symmetries and Noether's Theorem

Symmetries play a very important role in many fields of modern physics. We say that a Lagrangian has a symmetry, or equivalently is invariant under a transformation, if the associated action doesn't change when this transformation is applied. For the Lagrangian itself this implies the slightly looser condition of the Lagrangian having to change by a total derivative for it to be invariant under an action.

Noether's theorem is one of the reasons symmetries are so important in physics. In 1918 it was proven by Emmy Noether that if a system has a continuous symmetry, then there exists a corresponding conserved current $j^{\mu}$ [5]. In particular, Noether's theorem in field theory can be stated as:

Noether's Theorem for fields: For a field $\psi(x)$, with some arbitrary infinitesimal change

$$
\begin{equation*}
\delta \psi(x)=\mathcal{X}(\psi) \tag{2}
\end{equation*}
$$

under an arbitrary continuous symmetry transformation, there exists a corresponding conserved current

$$
\begin{equation*}
j^{\mu}=\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \psi\right)} \mathcal{X}(\psi)-F^{\mu}(\psi), \tag{3}
\end{equation*}
$$

such that $\partial_{\mu} j^{\mu}=0$, with $F^{\mu}(\psi)$ defined by $\delta \mathcal{L}=\partial_{\mu} F^{\mu}$.
Proof. Let $\psi(x)$ a field, with infinitesimal change given by (2) under some continuous symmetry transformation. Since this transformation is a symmetry of the theory, the Lagrangian must change by a total derivative, meaning $\delta \mathcal{L}=\partial_{\mu} F^{\mu}$ for some functions $F^{\mu}$. Under any variation of $\psi$, the Lagrangian $\mathcal{L}(\psi, \partial \psi)$ varies as

$$
\begin{equation*}
\delta \mathcal{L}(\psi, \partial \psi)=\left(\frac{\partial \mathcal{L}}{\partial \psi}\right) \delta \psi+\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \psi\right)}\right) \partial_{\mu}(\delta \psi) \tag{4}
\end{equation*}
$$

Noting that

$$
\begin{equation*}
\partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \psi\right)} \delta \psi\right)=\partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \psi\right)}\right) \delta \psi+\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \psi\right)}\right) \partial_{\mu}(\delta \psi), \tag{5}
\end{equation*}
$$

equation (4) becomes

$$
\begin{equation*}
\delta \mathcal{L}(\psi, \partial \psi)=\left[\left(\frac{\partial \mathcal{L}}{\partial \psi}\right)-\partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \psi\right)}\right)\right] \delta \psi+\partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \psi\right)} \delta \psi\right) . \tag{6}
\end{equation*}
$$

The Euler-Lagrange equation mandates that the first term disappears, yielding

$$
\begin{equation*}
\delta \mathcal{L}(\psi, \partial \psi)=\partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \psi\right)} \delta \psi\right) \tag{7}
\end{equation*}
$$

Inserting the definition from equation (2), and inserting $\delta \mathcal{L}=\partial_{\mu} F^{\mu}$,

$$
\begin{equation*}
0=\partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \psi\right)} \mathcal{X}(\psi)-F^{\mu}\right) \tag{8}
\end{equation*}
$$

Finally, defining $j^{\mu}$ as in equation (3), we have

$$
\begin{equation*}
\partial_{\mu} j^{\mu}=0 \tag{9}
\end{equation*}
$$

as required.

## 4 On Quantum Mechanics

### 4.1 Canonical Quantization

Canonical quantization is a method for creating a quantum theory out of a classical one. Starting with a classical system of particles, the complete state of this system is known when, for each particle, the canonical position $q_{a}$ and canonical momentum $p^{a}$ are known. Said system can then be described by Poisson bracket relationships between these variables; $q_{a}, p^{a}$. The procedure of canonical quantization begins by replacing the variables $q_{a}$ and $p^{a}$ of our classical theory with operators, $\hat{q}_{a}$ and $\hat{p}^{a}$. The usual Poisson bracket structure of the classical theory is then recast in terms of the commutation relations of these operators, $\left[\hat{q}_{a}, \hat{p}^{a}\right]$. Since our primary interest lies with field theories, let us redirect our attention to the quantization of those.

We begin completely analogously - by replacing the classical fields $\phi_{a}(\vec{x})$ and their conjugate momenta

$$
\begin{equation*}
\pi^{a}(\vec{x})=\frac{\partial \mathcal{L}}{\partial\left(\partial_{0} \phi_{a}\right)} \tag{10}
\end{equation*}
$$

with operators $\hat{\phi}_{a}(\vec{x}), \hat{\pi}^{a}(\vec{x})$. The associated Poisson brackets are replaced by commutation relations as previously. The resulting commutation relations to be obeyed by the fields and momenta are

$$
\begin{gather*}
{\left[\hat{\phi}_{a}(\vec{x}), \hat{\phi}_{b}(\vec{y})\right]=\left[\hat{\pi}^{a}(\vec{x}), \hat{\pi}^{b}(\vec{y})\right]=0}  \tag{11}\\
{\left[\hat{\phi}_{a}(\vec{x}), \hat{\pi}^{b}(\vec{y})\right]=i \delta(\vec{x}-\vec{y}) \delta_{a}^{b}}
\end{gather*}
$$

The possible states of the quantized system are then labeled by vectors in a Hilbert space, with the observables of the system denoted by operators on that Hilbert space. The values of the observables can then be found as the eigenstates of the state vectors under these operators.

## 5 On Gauge Theory

The essence of constructing a gauge theory consist of promoting a global symmetry to a local one. Suppose that we have a field $\Phi$, with an associated action $S=\int d^{4} x \mathcal{L}\left(\Phi, \partial \Phi, \partial^{2} \Phi, \ldots\right)$, such that the action is invariant under a global phase rotation of the form $\Phi \rightarrow e^{i \alpha} \Phi$, with a corresponding infinitesimal change

$$
\begin{equation*}
\delta \Phi=i \alpha \Phi \tag{12}
\end{equation*}
$$

where $\alpha$ is a constant parameter. To make the transformation with the infinitesimal change (12) a local one, the parameter $\alpha$ is given some arbitrary spacetime depence. This results in the invariance of the action being broken due to derivative terms in $\mathcal{L}$, as their infinitesimal variance now has the form

$$
\begin{align*}
\delta \partial_{\mu} \Phi & =\partial_{\mu} \delta \Phi \\
& =i\left[\alpha(x) \partial_{\mu}+\partial_{\mu} \alpha(x)\right] \Phi . \tag{13}
\end{align*}
$$

One may ask whether this process is worth it at all. Clearly $\Phi \rightarrow e^{i \alpha(x)} \Phi$ isn't a symmetry of the theory. It however turns out that the gauge principle is extremely important in describing several phenomena in the universe, such as electromagnetism [15].

### 5.1 Gauge fields

To start fixing the mess we have ended up in, we can add a gauge field $A_{\mu}$ to our theory. Let this field transform such that

$$
\begin{equation*}
\delta A_{\mu}=\partial_{\mu} \alpha(x) \tag{14}
\end{equation*}
$$

To nullify the issue with derivatives, define a covariant derivative, $D_{\mu}$ by

$$
\begin{equation*}
D_{\mu} \Phi:=\left(\partial_{\mu}-i A_{\mu}\right) \Phi . \tag{15}
\end{equation*}
$$

We find that terms with this covariant derivative then transform as regular derivatives would be expected to under a global transformation, as can be checked with the simple computation

$$
\begin{align*}
\delta D_{\mu} \Phi & =\delta \partial_{\mu} \Phi-i \delta A_{\mu} \Phi \\
& =i \alpha(x)\left(\partial_{\mu} \Phi\right)+i\left(\partial_{\mu} \alpha(x)\right) \Phi-i\left(\partial_{\mu} \alpha(x)\right) \Phi-i A_{\mu}(i \alpha(x) \Phi)  \tag{16}\\
& =i \alpha(x) D_{\mu} \Phi
\end{align*}
$$

Now a gauge invariant Lagrangian density can be written in terms of the covariant derivative, as well as invariant terms of $A_{\mu}$ itself. Notably, defining the field strength tensor for this gauge field, $F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}$, we find that terms of the form $F_{\mu \nu} F^{\mu \nu}$ are invariant under the transformation we established for $A_{\mu}$. This allows for the Lagrangian to be written as

$$
\begin{equation*}
\mathcal{L}=\mathcal{L}_{m}\left(\Phi, D \Phi, D^{2} \Phi, \ldots\right)-\frac{1}{4} F_{\mu \nu} F^{\mu \nu} \tag{17}
\end{equation*}
$$

From here we can proceed to the usual Maxwell equations. Considering the free case, our Lagrangian becomes $\mathcal{L}=-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}$, as the first term describes the matter in the theory, as well as its interactions with $A_{\mu}$. The Euler-Lagrange equation gives us the equation of motion for $F_{\mu \nu}$,

$$
\begin{equation*}
\partial_{\mu} F^{\mu \nu}=0 . \tag{18}
\end{equation*}
$$

In addition, $F_{\mu \nu}$ satisfies, by definition, the Bianchi identity,

$$
\begin{equation*}
\partial_{\sigma} F_{\mu \nu}+\partial_{\mu} F_{\nu \sigma}+\partial_{\nu} F_{\sigma \mu}=0 \tag{19}
\end{equation*}
$$

We carry on by letting $A_{\mu}=(\phi, \vec{A})^{T}$, with $M^{T}$ denoting the matrix transpose. Further defining

$$
\begin{equation*}
\vec{E}=-\vec{\nabla} \phi-\frac{\partial \vec{A}}{\partial t} \quad \text { and } \quad \vec{B}=\vec{\nabla} \times \vec{A}, \tag{20}
\end{equation*}
$$

directly implies the Maxwell equations in the absence of sources.
By a simple inspection of $F_{\mu \nu} F^{\mu \nu}$ we can see that the Lagrangian does not contain a time derivative of $\phi$. This field is therefore unphysical, and completely determined by the equation of motion

$$
\begin{equation*}
\vec{\nabla} \cdot \vec{E}=\nabla^{2} \phi+\vec{\nabla} \cdot \dot{\vec{A}}=0 . \tag{21}
\end{equation*}
$$

Indeed, we can solve for $\phi$, yielding

$$
\begin{equation*}
\phi(\vec{x})=\frac{1}{4 \pi} \int d^{3} x^{\prime} \frac{\vec{\nabla} \cdot \dot{\vec{A}}\left(\overrightarrow{x^{\prime}}, t\right)}{\left|\vec{x}-\overrightarrow{x^{\prime}}\right|} \tag{22}
\end{equation*}
$$

Which means we have decreased the number of degrees of freedom by one. With this in mind, we will visit one more issue before proceeding to quantize the theory.

### 5.2 Gauge fixing

With the gauge field introduced, there is some ambiguity left; additional terms of form $\partial_{\mu} \lambda(x)$ leave $F_{\mu \nu}$ invariant. As such, we have no way of distinguishing between $A_{\mu}$ and $A_{\mu}+\partial_{\mu} \lambda(x)$. This ambiguity is called a gauge symmetry. In order to make this ambiguity disappear, we will have to fix the gauge. To begin with this process, all states related by gauge symmetries need to be considered to represent the same physical state - in the end we cannot tell the difference anyway. For the endeavour towards fractons, the Coulomb gauge,

$$
\begin{equation*}
\vec{\nabla} \cdot \vec{A}=0 \tag{23}
\end{equation*}
$$

is usually the most convenient. But does the Coulomb gauge fix our issue completely? Let us claim that it does.

Claim: $\vec{\nabla} \cdot \vec{A}=0$ fixes the gauge.
Proof. Any additional symmetry $\vec{A} \rightarrow \vec{A}+\vec{\nabla} \lambda$ would have to satisfy

$$
\begin{equation*}
\vec{\nabla} \cdot(\vec{\nabla} \lambda)=\nabla^{2} \lambda=0 . \tag{24}
\end{equation*}
$$

Since any such $\lambda$ would also have to disappear at $x \rightarrow \infty, \lambda=0$ is the only solution.
In the presence of the Coulomb gauge, a few things become immediately apparent: first, equation 22) implies that $\phi=0$, and second, the electric field simplifies to $\vec{E}=-\dot{\vec{A}}$. With these in mind, we can proceed towards a quantum theory.

### 5.3 Quantization of the Maxwell Theory

To begin with canonical quantization, first the conjugate momenta to $A_{\mu}, \pi^{\mu}$, as well as the Hamiltonian are established:

$$
\begin{align*}
\pi^{0} & =\frac{\partial \mathcal{L}}{\partial \dot{A}_{0}}=0,  \tag{25}\\
\pi^{i} & =\frac{\partial \mathcal{L}}{\partial \dot{A}_{i}}=-F^{0 i}:=E^{i},  \tag{26}\\
H & =\int d^{3} x \pi^{i} \dot{A}_{i}-\mathcal{L} \\
& =\int d^{3} x \frac{1}{2}\left(E^{2}+B^{2}\right)-A_{0}(\vec{\nabla} \cdot \vec{E}) \\
& =\frac{1}{2} \int d^{3} x\left(E^{2}+B^{2}\right), \tag{27}
\end{align*}
$$

where the equation follows directly from the observation that $F_{\mu \nu} F^{\mu \nu}$ contains no $\dot{\phi}$ terms.
From the Maxwell equations, our equation of motion for $\vec{A}$ is $\partial_{\mu} \partial^{\mu} \vec{A}=0$. In the Fourier space, $\vec{A}\left(x_{0}, \vec{p}\right)$ then has to satisfy

$$
\begin{equation*}
\partial_{0} \partial^{0} \vec{A}\left(x_{0}, \vec{p}\right)+\vec{p}^{2} \vec{A}\left(x_{0}, \vec{p}\right)=0 . \tag{28}
\end{equation*}
$$

This can be solved by

$$
\begin{equation*}
\vec{A}\left(x_{0}, \vec{p}\right)=\vec{A}(\vec{p}) e^{i p_{\mu} x^{\mu}}+\vec{A}(-\vec{p}) e^{-i p_{\mu} x^{\mu}} \tag{29}
\end{equation*}
$$

where $\vec{A}(\vec{p})=\vec{A}^{*}(-\vec{p})$.
Let us then consider our constraint, $\vec{p} \cdot \vec{A}=0$. We can enforce it by letting $\vec{A}(\vec{p})$ be a linear combination of two orthonormal vectors, i.e. $\vec{A}(\vec{p})=\sum_{r=1,2} \vec{\epsilon}_{r}(\vec{p}) \alpha_{r}$ with $\alpha_{r}$ complex amplitudes. We have almost solved one unmentioned issue from earlier. The "basis" vectors $\vec{\epsilon}_{r}(\vec{p})$ make an expected, strong statement - for a given momentum $\vec{p}$, a photon has two possible polarizations. Checking back, thus far we have

$$
\begin{equation*}
\vec{A}\left(\vec{x}, x_{0}\right)=\int \frac{d^{3} p}{(2 \pi)^{3}} \frac{1}{\sqrt{2|\vec{p}|}} \sum_{r=1,2} \vec{\epsilon}_{r}(\vec{p})\left[\alpha_{r} e^{i p_{\mu} x^{\mu}}+\alpha_{r}^{*} e^{-i p_{\mu} x^{\mu}}\right] . \tag{30}
\end{equation*}
$$

We can now begin to quantize the theory by promoting $\alpha_{r}$ to operators $\hat{\alpha}_{r}$. This results in the operators

$$
\begin{equation*}
\hat{A}(\vec{x})=\int \frac{d^{3} p}{(2 \pi)^{3}} \frac{1}{\sqrt{2|\vec{p}|}} \sum_{r=1,2} \vec{\epsilon}_{r}(\vec{p})\left[\hat{\alpha}_{r} e^{i p_{\mu} x^{\mu}}+\hat{\alpha}_{r}^{\dagger} e^{-i p_{\mu} x^{\mu}}\right] \tag{31}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{E}(\vec{x})=-i \int \frac{d^{3} p}{(2 \pi)^{3}} \sqrt{\frac{|\vec{p}|}{2}} \sum_{r=1,2} \epsilon_{r}\left[\hat{a}_{r}(\vec{p}) e^{i p_{\mu} x^{\mu}}-\hat{a}_{r}^{\dagger}(\vec{p}) e^{-i p_{\mu} x^{\mu}}\right] \tag{32}
\end{equation*}
$$

where we can check that the canonical commutation relations

$$
\begin{gather*}
{\left[\hat{\alpha}_{r}(\vec{p}), \hat{\alpha}_{s}^{\dagger}(\vec{q})\right]=(2 \pi)^{3} \delta^{r s} \delta(\vec{p}-\vec{q}),} \\
{\left[\hat{\alpha}_{r}(\vec{p}), \hat{\alpha}_{s}(\vec{q})\right]=\left[\hat{\alpha}_{r}^{\dagger}(\vec{p}), \hat{\alpha}_{s}^{\dagger}(\vec{q})\right]=0,} \tag{33}
\end{gather*}
$$

and

$$
\begin{equation*}
\left[A_{i}(\vec{x}), E^{j}(\vec{y})\right]=-i\left(\delta_{i j}-\frac{\partial_{i} \partial_{j}}{\nabla^{2}}\right) \delta^{3}(\vec{x}-\vec{y}) \tag{34}
\end{equation*}
$$

are satisfied.

### 5.4 Resultant Hilbert Space

The Hilbert space that describes the states of the quantized field is found to be a product space of harmonic oscillator spaces, very similarly to that of the free scalar field. The vacuum state $|\overrightarrow{0}\rangle$ can then be defined such that it is terminated by the annihilation operator. Acting on the vacuum state with the creation operator $\hat{a}_{r}^{\dagger}(\vec{p})$ to get a single particle $\vec{p}$-momentum state

$$
\begin{equation*}
\hat{a}_{r}^{\dagger}(\vec{p})|0\rangle=\left|1_{r}^{\vec{p}}\right\rangle . \tag{35}
\end{equation*}
$$

We can also act multiple times with the creation operator to find

$$
\begin{equation*}
\left(\hat{a}_{r}^{\dagger}(\vec{p})\right)^{n}|0\rangle=\sqrt{n_{r}^{\vec{p}}!}\left|n_{r}^{\vec{p}}\right\rangle, \tag{36}
\end{equation*}
$$

where the coefficient results from the fact that $\hat{a}_{r}^{\dagger}(\vec{p})\left|(n-1)_{r}^{\vec{p}}\right\rangle=\sqrt{n_{r}^{\vec{p}}}\left|n_{r}^{\vec{p}}\right\rangle$. It is apparent that any number of photons can be in the same state. The annihilation operator acts on a state as

$$
\begin{equation*}
\hat{a}_{r}(\vec{p})\left|n_{r}^{\vec{p}}\right\rangle=\sqrt{n_{r}^{\vec{p}}}\left|(n-1)_{r}^{\vec{p}}\right\rangle \tag{37}
\end{equation*}
$$



Figure 1: The limited mobility of fractons: single fractons are immobile (a), dipoles are mobile (b), and single fractons exhibiting movement when emitting a dipole at each stage of moving (c). Reproduced by the author based on [9, 12].

As noted before, each state additionally has two different polarizations, as labeled by $r$.
A photon number operator can also be defined on the space by $\hat{N}_{r}^{\vec{p}}=\hat{a}_{r}^{\dagger}(\vec{p}) \hat{a}_{r}(\vec{p})$, since this acts on a state $\left|n_{r}^{\vec{p}}\right\rangle$ as

$$
\begin{equation*}
\hat{N}_{r}^{\vec{p}}\left|n_{r}^{\vec{p}}\right\rangle=n_{r}^{\vec{p}}\left|n_{r}^{\vec{p}}\right\rangle . \tag{38}
\end{equation*}
$$

## 6 Emergence of Fractons

Let us move on to the discussion of fractons. As was alluded to before, one method of constructing fracton theories is through field theories that exhibit conservation of higher order moments than just charge. Considering dipole moment conservation, for instance, a few observations can immediately be written down. First, single charged particles in the theory must be immobile, as them being mobile would violate dipole moment conservation. However, it remains possible for a fracton to exhibit movement if they create a dipole when they do so. Second, dipoles are mobile, and as such, if stable dipolar bound states of particles are allowed, they will be capable of movement. These options are illustrated in figure 1 .

Fractons can be further divided into subclasses by the kind of mobility restrictions that apply to them; particles with movement constrained to a line are called lineons, while those constrained to a plane are called planons. For instance, in the theory being discussed here dipoles are planons, since they can move freely in the plane perpendicular to their dipole moment.

Now to mathematically formulate such a field theory. we can begin by requiring charge conservation and dipole moment conservation. These properties can be encapsulated in the invariance of the theory under the transformations given by

$$
\begin{align*}
\phi & \rightarrow e^{i a} \phi,  \tag{39}\\
\phi & \rightarrow e^{i \lambda_{i} x^{i}} \phi . \tag{40}
\end{align*}
$$

Where equation 40 can be related to conservation of dipole moment via the Noether theorem (3) as follows. Considering the free case Lagrangian,

$$
\begin{equation*}
\mathcal{L}_{f}=\left|\partial_{t} \phi\right|^{2}-m^{2}|\phi|^{2}, \tag{41}
\end{equation*}
$$

the two fields in our theory are $\phi$ and $\phi^{*}$. The Noether current then becomes

$$
\begin{equation*}
j^{\mu}=\left(\frac{\partial \mathcal{L}_{f}}{\partial\left(\partial_{\mu} \phi\right)}\right) \delta \phi+\left(\frac{\partial \mathcal{L}_{f}}{\partial\left(\partial_{\mu} \phi^{*}\right)}\right) \delta \phi^{*}-F^{\mu} \tag{42}
\end{equation*}
$$

where $F^{\mu}=0$ due to the Lagrangian having no spatial derivatives, and therefore $\delta \mathcal{L}_{f}$ being zero under the purely spatial symmetry (40). For the same reason we find that $j^{\mu}=0$ for $\mu=1,2, \ldots, d$.

Computing the derivatives and inserting in the associated infinitesimal change $\delta \phi=i \lambda_{i} x^{i} \phi$ yields

$$
\begin{align*}
j^{t} & =i \lambda_{i} x^{i}\left(\partial^{t} \phi^{*}\right) \phi-i \lambda_{i} x^{i}\left(\partial_{t} \phi\right) \phi^{*} \\
& =i \lambda_{i} x^{i}\left(\phi \partial^{t} \phi^{*}-\phi^{*} \partial_{t} \phi\right), \tag{43}
\end{align*}
$$

where $\phi \partial^{t} \phi^{*}-\phi^{*} \partial_{t} \phi$ can be identified as the particle number current, or charge distribution operator of the system. The conserved charge would therefore come to be of the form

$$
\begin{align*}
D & =\int j^{t} d^{(d-1)} x \\
& =i \lambda_{i} \int d^{(d-1)} x x^{i}\left(\phi \partial^{t} \phi^{*}-\phi^{*} \partial_{t} \phi\right) \tag{44}
\end{align*}
$$

where $\lambda_{i}$ is a constant vector. A directional charge - dipole moment - must then be conserved.
We can also check that $j^{\mu}$ does not diverge when the equations of motion are obeyed. For $\mu=1,2, \ldots, d$ this is trivial, and is therefore only remains to be shown for $j^{t}$. The equations of motion for $\phi$ and $\phi^{*}$ from $\mathcal{L}_{f}$ are

$$
\begin{align*}
\partial_{t} \partial^{t} \phi & =-m^{2} \phi,  \tag{45}\\
\partial_{t} \partial^{t} \phi^{*} & =-m^{2} \phi^{*} . \tag{46}
\end{align*}
$$

Claim: $\partial_{t} j^{t}=0$.
Proof. We find that the divergence of $j^{t}$ is

$$
\begin{align*}
\partial_{t} j^{t} & =i \lambda_{i} x^{i} \partial_{t}\left(\phi \partial^{t} \phi^{*}-\phi^{*} \partial_{t} \phi\right) \\
& =i \lambda_{i} x^{i}\left(\phi \partial_{t} \partial^{t} \phi^{*}+\partial_{t} \phi \partial^{t} \phi^{*}-\phi^{*} \partial_{t} \partial^{t} \phi-\partial_{t} \phi \partial^{t} \phi^{*}\right)  \tag{47}\\
& =i \lambda_{t} x^{i}\left(\phi \partial_{t} \partial^{t} \phi^{*}-\phi^{*} \partial_{t} \partial^{t} \phi\right)
\end{align*}
$$

inserting the equations of motion gives

$$
\begin{align*}
\partial_{t} j^{t} & =-m^{2}|\phi|^{2}+m^{2}|\phi|^{2} \\
& =0 \tag{48}
\end{align*}
$$

In order to simplify the transformations in equations (39) and (40), we can write both in just one symmetry,

$$
\begin{equation*}
\phi \rightarrow e^{i \alpha(x)} \phi \tag{49}
\end{equation*}
$$

where we restrict $\alpha(x)$ to be a linear function of $x$ to only include terms up to dipole moment conservation and no additional terms. Next an invariant action needs to be constructed. Similarly to the case presented in section 5, while $\phi$ itself transforms covariantly, any derivatives
of $\phi$ do not. However, this time we will take a different approach to the problem. Instead of introducing a gauge field, let us look for an operator $\mathcal{O}_{i j}$ that is invariant under equation (49).

One potential candidate would be a term of the form $\partial_{i} \partial_{j} \phi$. Under our transformation this term transforms as

$$
\begin{align*}
\partial_{i} \partial_{j} \phi & \rightarrow \partial_{i}\left(i \partial_{j} \alpha(x) e^{i \alpha(x)} \phi+e^{i \alpha(x)} \partial_{j} \phi\right) \\
= & e^{i \alpha(x)}\left(i \partial_{i} \partial_{j} \alpha(x) \phi-\partial_{i} \alpha(x) \partial_{j} \alpha(x) \phi\right.  \tag{50}\\
& \left.\quad+i \partial_{j} \alpha(x) \partial_{i} \phi+i \partial_{i} \alpha(x) \partial_{j} \phi+\partial_{i} \partial_{j} \phi\right),
\end{align*}
$$

where the $\partial_{i} \partial_{j} \alpha(x)$ term vanishes due to the linearity condition on $\alpha(x)$. We would like our operator to transform as just the last term. A good candidate for cancelling the rest of the term would be $\partial_{i} \phi \partial_{j} \phi$. This introduces another issue; this term has two fields and two derivatives whereas the one we want to cancel has only one field. Thankfully this can be addressed by multiplying the first term by $\phi$. It is apparent that multiplying by $\phi$ does not change the way the term transforms, but only adds a factor $e^{i \alpha(x)}$ to the front of the equation. Now, we find that

$$
\begin{align*}
\partial_{i} \phi \partial_{j} \phi & \rightarrow e^{2 i \alpha(x)}\left(\left(i \partial_{i} \alpha(x)+\partial_{i}\right) \phi\left(i \partial_{j} \alpha(x)+\partial_{j}\right) \phi\right) \\
& =e^{2 i \alpha(x)}\left(-\partial_{i} \alpha(x) \partial_{j} \alpha(x) \phi^{2}+i \partial_{i} \alpha(x) \phi \partial_{j} \phi+i \partial_{j} \alpha(x) \phi \partial_{i} \phi+\partial_{i} \phi \partial_{j} \phi\right) . \tag{51}
\end{align*}
$$

It is then apparent that an invariant operator $\mathcal{O}_{i j}$ can be constructed from these two terms by setting

$$
\begin{equation*}
\mathcal{O}_{i j}=\phi \partial_{i} \partial_{j} \phi-\partial_{i} \phi \partial_{j} \phi \tag{52}
\end{equation*}
$$

Where we can also note that $\mathcal{O}_{i j}$ is symmetric with respect to the interchange of $i$ and $j$.
A Lagrangian that is invariant under the transformation presented in equation 49 can then be constructed. To the lowest order we have

$$
\begin{align*}
\mathcal{L} & =\left|\partial_{t} \phi\right|^{2}-m^{2}|\phi|^{2}-\lambda\left|\mathcal{O}_{i j}\right|^{2}-\lambda^{\prime} \phi^{* 2}\left(\delta_{j}^{i} \mathcal{O}_{i j}\right)  \tag{53}\\
& =\left|\partial_{t} \phi\right|^{2}-m^{2}|\phi|^{2}-\lambda\left|\phi \partial_{i} \partial_{j} \phi-\partial_{i} \phi \partial_{j} \phi\right|^{2}-\lambda^{\prime} \phi^{* 2}\left(\phi \partial_{i}^{2} \phi-\partial_{i} \phi \partial^{i} \phi\right)
\end{align*}
$$

As such we have shown, similarly to [8], that fracton theories possess a characteristic nongaussian form. A surprising result of this is the fact that a non-interacting fracton theory is in itself not possible to construct - fractons can always interact with each other, for example by exchanging virtual dipoles [8].

It is worth noting that the Lagrangian in $[8$ does not contain the complex conjugate to the $\lambda^{\prime}$-term. It not entirely clear whether the author intended for it to be implicitly included, or if there is some particular reason for the lack of its presence. However adding the complex conjugate does not alter the invariance of $\mathcal{L}$ under any of our symmetries, and therefore can be added with no significant concerns.

### 6.1 Spacetime symmetries

Here the different spacetime symmetries of the theory will be considered, and some of their properties will be discussed. We will adapt some conventions throughout the section, with

$$
\begin{equation*}
\left[\delta_{A}, \delta_{B}\right] \phi=\delta_{A}\left(\delta_{B} \phi\right)-\delta_{B}\left(\delta_{A} \phi\right) \tag{54}
\end{equation*}
$$

denoting the commutator of two infinitesimal changes associated with two symmetries. Similarly, $[A, B]$ will denote the commutator of two operators $A$ and $B$.

### 6.1.1 Boosts

Let us consider boosts of the form

$$
\begin{array}{r}
t \rightarrow t+a \omega^{i} x^{i} \\
x^{i} \rightarrow x^{i}+b \omega^{i} t, \tag{55}
\end{array}
$$

where translation terms have been neglected. The type of boost depends on the parameters $a$ and $b$, with

$$
\begin{aligned}
a, b \neq 0 & \Longrightarrow \text { Lorentz Boosts } \\
a=0, b \neq 0 & \Longrightarrow \text { Galilean Boosts } \\
a \neq 0, b=0 & \Longrightarrow \text { Carroll Boosts. }
\end{aligned}
$$

Naively it would be expected for the fracton theory to remain invariant under Carroll boosts, as they are usually associated with immobile systems [1]. We will proceed to consider the transformation of our theory under the boosts as given in equation (55), and subsequently determine for which values of $a$ and $b$ the theory remains invariant.

These boosts transform the field $\phi$ as

$$
\begin{align*}
\phi\left(t, x^{i}\right) & \rightarrow \phi^{\prime}\left(t^{\prime}, x^{i \prime}\right)=\phi\left(t+a \omega^{i} x^{i}, x^{i}+b \omega^{i} t\right) \\
& =\phi\left(t, x^{i}\right)+a \omega^{i} x^{i} \partial_{t} \phi+b \omega^{i} t \partial_{i} \phi \\
& \Longrightarrow \delta \phi=a \omega^{i} x^{i} \partial_{t} \phi+b \omega^{i} t \partial_{i} \phi . \tag{56}
\end{align*}
$$

For simplicity, we will here define an operator $\mathcal{G}^{i}$ that acts on $\phi$ and its derivatives as

$$
\begin{equation*}
\mathcal{G}^{i} \circ \phi=a x^{i} \partial_{t} \phi+b t \partial_{i} \phi, \tag{57}
\end{equation*}
$$

such that equation 56 becomes

$$
\begin{equation*}
\delta \phi=\omega^{i} \mathcal{G}^{i} \circ \phi \tag{58}
\end{equation*}
$$

It is then possible to examine the boost symmetries of the different terms $\mathcal{Q}$ of the Lagrangian as simply the parts that do not transform covariantly, i.e. the ones not proportional to $\mathcal{G}^{i} \circ \mathcal{Q}$. Considering the first two terms of equation (53), we find

$$
\begin{equation*}
\delta \partial_{t} \phi=b \omega^{i} \partial_{i} \phi+\omega^{i} \mathcal{G}^{i} \circ \partial_{t} \phi, \tag{59}
\end{equation*}
$$

with $\delta \phi$ having already been established. To aid with solving the problem for the following two terms, we can simply consider the variation of $\mathcal{O}_{i j}$, which we can also do in parts:

$$
\begin{align*}
\delta \partial_{i} \partial_{j} \phi & =\partial_{i} \partial_{j}\left(a \omega^{k} x^{k} \partial_{t} \phi+b \omega^{k} t \partial_{k} \phi\right) \\
& =\partial_{i}\left(a \omega^{k} \delta_{j}^{k} \partial_{t} \phi+a \omega^{k} x^{k} \partial_{j} \partial_{t} \phi\right)+b \omega^{k} t \partial_{j} \partial_{i} \partial_{k} \phi \\
& =a \omega^{j} \partial_{i} \partial_{t} \phi+a \omega^{i} \partial_{j} \partial_{t} \phi+\omega^{k} \mathcal{G}^{k} \circ \partial_{i} \partial_{j} \phi, \tag{60}
\end{align*}
$$

and

$$
\begin{align*}
\delta \partial_{i} \phi & =\partial_{i}\left(a \omega^{k} x^{k} \partial_{t} \phi+b \omega^{k} t \partial_{k} \phi\right) \\
& =a \omega^{i} \partial_{t} \phi+\mathcal{G}^{k} \circ \partial_{i} \phi, \tag{61}
\end{align*}
$$

yielding

$$
\begin{equation*}
\delta \mathcal{O}_{i j}=\phi a \omega_{(i} \partial_{j)} \partial_{t} \phi-\partial_{(j} \phi a \omega_{i)} \partial_{t} \phi, \tag{62}
\end{equation*}
$$

where the covariant part has been ignored, and the brackets in subscript denote summation with exchange of indices

$$
\begin{equation*}
a \omega_{(i} \partial_{j)} \partial_{t} \phi=a\left(\omega_{i} \partial_{j}+\omega_{j} \partial_{i}\right) \partial_{t} \phi . \tag{63}
\end{equation*}
$$

For the last term of the Lagrangian the variation is

$$
\begin{align*}
\delta\left(\delta_{j}^{i} \mathcal{O}_{i j}\right) & =\delta \mathcal{O}_{i i} \\
& =2\left[\phi a \omega_{i} \partial_{i} \partial_{t} \phi-\partial_{i} \phi a \omega_{i} \partial_{t} \phi\right] \\
& =2\left[\partial_{i}\left(\phi a \omega_{i} \partial_{t} \phi\right)-2 \partial_{i} \phi a \omega_{i} \partial_{t} \phi\right] \tag{64}
\end{align*}
$$

Where the total derivative can be ignored. The overall variation of the Lagrangian without covariant terms then becomes

$$
\begin{align*}
\delta \mathcal{L}= & \partial_{t} \phi^{*} b \omega^{i} \partial_{i} \phi+\lambda\left[\phi a \omega_{(i} \partial_{j)} \partial_{t} \phi-\partial_{(j} \phi a \omega_{i)} \partial_{t} \phi\right] \mathcal{O}_{i j}^{*} \\
& +C . C .-4 \lambda^{\prime} \phi^{* 2}\left(\partial_{i} \phi a \omega_{i} \partial_{t} \phi\right) . \tag{65}
\end{align*}
$$

From which we can conclude that our theory does not exhibit invariance under any boost of the form we considered. Moreover, it is not found feasible to add a further boost term to restore invariance in an analogous manner to the introduction of a gauge field. Any such additional term would have to depend on the field $\phi$ in order to be able to alleviate the issue, which is intuitively dubious.

### 6.1.2 Translations, Rotations, and the Multipole Algebra

In this section, the remaining spacetime symmetries and their relations are examined. In particular, the commutators between translation, rotation, as well as charge and dipole moment conservation symmetries are computed.

We write a translation as

$$
\begin{equation*}
P: \phi \rightarrow \phi+\epsilon^{i} \partial_{i} \phi, \tag{66}
\end{equation*}
$$

where $\epsilon^{i}$ is an infinitesimal parameter. The corresponding infinitesimal change is

$$
\begin{equation*}
\delta_{P} \phi=\epsilon^{i} \partial_{i} \phi . \tag{67}
\end{equation*}
$$

We then define an operator $P_{i}$ by writing

$$
\begin{align*}
P_{i} \circ \phi & =\partial_{i} \phi \\
\Longrightarrow \delta_{P} \phi & =\epsilon^{i} P_{i} \circ \phi . \tag{68}
\end{align*}
$$

Similar definitions can be employed for the other symmetries of the theory. For the dipole symmetry we have

$$
\begin{align*}
D: \phi & \rightarrow e^{i \lambda_{k} x^{k}} \phi \\
\Longleftrightarrow \delta_{D} \phi & =i \lambda_{k} x^{k} \phi=\lambda_{k} D^{k} \circ \phi . \tag{69}
\end{align*}
$$

It can then be computed that

$$
\begin{align*}
{\left[\delta_{D}, \delta_{P}\right] \phi } & =\delta_{D}\left(\epsilon^{i} \partial_{i} \phi\right)-\delta_{p}\left(i \lambda_{k} x^{k} \phi\right) \\
& =i \lambda_{k} x^{k} \epsilon^{i} \partial_{i} \phi-i \lambda_{k} \epsilon^{i}\left(\delta_{i}^{k}+x^{k} \partial_{i} \phi\right) \\
& =-i \lambda_{k} \epsilon^{i} \delta_{i}^{k} \phi \tag{70}
\end{align*}
$$

Where the first part can be written in terms of the commutator between $D^{k}$ and $P_{i}$ as

$$
\begin{equation*}
i \lambda_{k} \epsilon^{i}\left[D^{k}, P_{i}\right] \phi=-i \lambda_{k} \epsilon^{i} \delta_{i}^{k} \phi, \tag{71}
\end{equation*}
$$

leading to

$$
\begin{equation*}
\left[P_{i}, D^{k}\right]=i \delta_{i}^{k} Q \tag{72}
\end{equation*}
$$

Where $Q$ is the operator for global charge symmetry, as defined below.
This statement is physically very relevant, since we can see that the commutator between the operators goes to zero whenever the direction of translation is different to that of the dipole moment. This immediately leads to the conclusion that dipoles can only move in the plane perpendicular to their dipole moment.

A similar treatment can be applied to the other pairs of known symmetries of the theory. Defining the following additional symmetries:

$$
\begin{align*}
R: \phi \rightarrow \phi+\omega^{i j} x_{i} \partial_{j} \phi & \Longleftrightarrow \delta_{R} \phi=\omega^{i j} x_{i} \partial_{j} \phi=\omega^{i j} \mathcal{J}_{i j} \circ \phi,  \tag{73}\\
Q: \phi \rightarrow e^{i a} \phi & \Longleftrightarrow \delta_{Q}=i a \phi=a Q \circ \phi, \tag{74}
\end{align*}
$$

corresponding to rotations and charge conservation respectively. It can then be shown that

$$
\begin{align*}
{\left[\delta_{D}, \delta_{R}\right] \phi } & =i \lambda_{k} x^{k} \omega^{i j} x_{i} \partial_{j} \phi-\omega^{i j} x_{i} \partial_{j}\left(i \lambda_{k} x^{k} \phi\right)  \tag{75}\\
& =-i \omega^{i j} x_{i} \lambda_{k} \delta_{j}^{k} \phi
\end{align*}
$$

implying

$$
\begin{align*}
{\left[\mathcal{J}_{i j}, D^{k}\right] } & =x_{[i} \delta_{j]}^{k}  \tag{76}\\
& =\delta_{[j}^{k} D_{i]} .
\end{align*}
$$

Where the square brackets denote the anti-symmetrization over $i$ and $j$,

$$
\begin{equation*}
\delta_{[j}^{k} D_{i]}=\delta_{j}^{k} D_{i}-\delta_{i}^{k} D_{j} . \tag{77}
\end{equation*}
$$

This expression appears in equation 76 due to the anti-symmetry of $\omega^{i j}$ under exchange of $i$ and $j$.

We can take equation 76 to imply, similarly to the previous case, that rotations about the direction of the dipole moment are allowed, while any other ones would violate dipole moment conservation.

On a penultimate note, for the translation-charge pair we find

$$
\begin{align*}
{\left[\delta_{Q}, \delta_{P}\right] \phi } & =e^{i a} \epsilon^{i} \partial_{i} \phi-\epsilon^{i} \partial_{i}\left(e^{i a} \phi\right)  \tag{78}\\
& =0
\end{align*}
$$

Which is to be expected, since $Q$ corresponds to global charge conservation, which is invariant under (again, global) translations, unlike in the local case.

The aforementioned commutation relations together with the commutation relations between rotations and translations

$$
\begin{align*}
{\left[\delta_{R}, \delta_{P}\right] \phi } & =\omega^{i j} x_{i} \partial_{j}\left(\epsilon^{k} \partial_{k} \phi\right)-\epsilon^{k} \partial_{k}\left(\omega^{i j} x_{i} \partial_{j} \phi\right) \\
& =-\omega^{i j} \epsilon^{k} \delta_{k}^{i} \partial_{j} \phi \\
\Longrightarrow\left[\mathcal{J}_{i j}, P_{k}\right] & =-\delta_{[i}^{k} P_{j]}, \tag{79}
\end{align*}
$$

and

$$
\begin{align*}
{\left[\delta_{R}, \delta_{R^{\prime}}\right] \phi } & =\omega^{i j} x_{i} \partial_{j}\left(\omega^{k l} x_{k} \partial_{l} \phi\right)-\omega^{k l} x_{k} \partial_{l}\left(\omega^{i j} x_{i} \partial_{j} \phi\right) \\
& =\omega^{i j} \omega^{k l}\left(\delta_{k j} x_{i} \partial_{l} \phi-\delta_{l i} x_{k} \partial_{j} \phi\right) \\
\Longrightarrow\left[\mathcal{J}_{i j}, \mathcal{J}_{k l}\right] & =-\delta_{[k[j} \mathcal{J}_{i] l]}, \tag{80}
\end{align*}
$$

can be used to construct the multipole algebra, $\mathfrak{m}$, studied in [3]. This is quite interesting as a side remark, since as Gromov notes [3], a special case of this algebra is used with Galileon gravity. This provides a link to the possible (quantum) gravity applications of fracton theories that were alluded to before. Based on the multipole algebra, a multipole group $\mathfrak{M}$ can be constructed via exponentiation. $\mathfrak{M}$ then consists a subgroup of the connected component of the identity of the entire symmetry group of our theory. This is consistent with the observation that our theory is not invariant under any boosts.

### 6.2 On Connections to Other Literature

In addition to [8], another discussion of fractons is also provided by Seiberg in [11]. Although the approach to the theory and the conventions used are quite different, a comparable theory is constructed, with the Lagrangian of [11, eq. (4.21)] reading

$$
\begin{equation*}
\mathcal{L}_{S E}=i \phi^{*} \partial_{0} \phi-s \partial_{i}\left(\phi^{*} \phi\right) \partial^{i}\left(\phi^{*} \phi\right)+u\left(i \phi^{* 2} \partial_{i} \phi \partial^{i} \phi-i \phi^{2} \partial_{i} \phi^{*} \partial^{i} \phi^{*}\right)-v|\phi|^{4} \tag{81}
\end{equation*}
$$

While there is no immediately apparent connection between this and our Lagrangian, it is possible to go from one to the other. Starting with the Lagrangian from equation (53), and calling it $\mathcal{L}_{P R}$ for clarity.

In [11], the author is only interested in the lowest order terms, and therefore the $\lambda$ term of $\mathcal{L}_{P R}$ will be neglected, making our lagrangian

$$
\begin{equation*}
\mathcal{L}_{P R}=\left|\partial_{t} \phi\right|^{2}-m^{2}|\phi|^{2}-\lambda^{\prime} \phi^{* 2}\left(\phi \partial_{i}^{2} \phi-\partial_{i} \phi \partial^{i} \phi\right) . \tag{82}
\end{equation*}
$$

Moreover, neither the kinetic term nor the mass term will be considered here, since they are not fundamental to the nature of the theory. We should then find a way to write

$$
\begin{equation*}
-\lambda^{\prime} \phi^{* 2}\left(\phi \partial_{i}^{2} \phi-\partial_{i} \phi \partial^{i} \phi\right) \tag{83}
\end{equation*}
$$

as

$$
\begin{equation*}
-s \partial_{i}\left(\phi^{*} \phi\right) \partial^{i}\left(\phi^{*} \phi\right)+u\left(i \phi^{* 2} \partial_{i} \phi \partial^{i} \phi-i \phi^{2} \partial_{i} \phi^{*} \partial^{i} \phi^{*}\right) \tag{84}
\end{equation*}
$$

We can immediately distribute the factor $\lambda^{\prime} \phi^{* 2}$ to the terms in equation (83). This yields

$$
\begin{equation*}
-\lambda^{\prime} \phi^{* 2} \phi \partial_{i}^{2} \phi+\lambda^{\prime} \phi^{* 2} \partial_{i} \phi \partial^{i} \phi, \tag{85}
\end{equation*}
$$

where it is immediately clear that the latter term is directly included into the $u$ term of equation (84). To make the former term of the right form, we will need to apply integration by parts to it as

$$
\begin{equation*}
-\lambda^{\prime} \phi^{* 2} \phi \partial_{i}^{2} \phi=-\lambda^{\prime}\left(\partial_{i}\left(\phi^{* 2} \phi \partial_{i} \phi\right)-2 \phi^{*} \partial_{i} \phi^{*} \phi \partial_{i} \phi-\phi^{* 2} \partial_{i} \phi \partial^{i} \phi\right), \tag{86}
\end{equation*}
$$

where the first term can be disregarded as a boundary term, and the other two will be shown to be a part of the first term of equation (84). Overall, equation (83) has now become

$$
\begin{equation*}
2 \lambda^{\prime} \phi^{*} \phi \partial_{i} \phi^{*} \partial_{i} \phi+2 \lambda^{\prime} \phi^{* 2} \partial_{i} \phi \partial^{i} \phi . \tag{87}
\end{equation*}
$$

The Lagrangian of our theory, unlike that of Pretko, also contains a complex conjugate to the $\lambda^{\prime}$-term. Upon adding the complex conjugate term, equation (87) becomes

$$
\begin{gather*}
4 \lambda^{\prime} \phi^{*} \phi \partial_{i} \phi^{*} \partial_{i} \phi+2 \lambda^{\prime} \phi^{* 2} \partial_{i} \phi \partial^{i} \phi+2 \lambda^{\prime} \phi^{2} \partial_{i} \phi^{*} \partial^{i} \phi^{*} \\
=2 \lambda^{\prime} \partial_{i}\left(\phi^{*} \phi\right) \partial^{i}\left(\phi^{*} \phi\right) . \tag{88}
\end{gather*}
$$

This seems to take us to the Seiberg Lagrangian conveniently, with

$$
\begin{align*}
s & =2 \lambda^{\prime}  \tag{89}\\
u & =0
\end{align*}
$$

There is, however, something we have neglected to account for. So far it was implicitly assumed that $\lambda^{\prime}$ is a real number. Nothing mandates that this be the case, though.
If we let $\lambda^{\prime} \in \mathbb{C}$, it immidiately follows that $\lambda^{\prime}$ can be written as a sum of a real and a complex part $\lambda^{\prime}=x+i y$, with $x, y \in \mathbb{R}$. We can then conduct the same procedure, where equation (83) becomes

$$
\begin{equation*}
2(x+i y) \phi^{*} \phi \partial_{i} \phi^{*} \partial_{i} \phi+2(x+i y) \phi^{* 2} \partial_{i} \phi \partial^{i} \phi . \tag{90}
\end{equation*}
$$

This time adding the complex conjugate results in

$$
\begin{equation*}
4 x \phi^{*} \phi \partial_{i} \phi^{*} \partial_{i} \phi+2(x+i y) \phi^{* 2} \partial_{i} \phi \partial^{i} \phi+2(x-i y) \phi^{2} \partial_{i} \phi^{*} \partial^{i} \phi^{*}, \tag{91}
\end{equation*}
$$

which factors into

$$
\begin{equation*}
2 x \partial_{i}\left(\phi^{*} \phi\right) \partial^{i}\left(\phi^{*} \phi\right)+i 2 y\left(\phi^{* 2} \partial_{i} \phi \partial^{i} \phi-\phi^{2} \partial_{i} \phi^{*} \partial^{i} \phi\right) . \tag{92}
\end{equation*}
$$

We then get

$$
\begin{align*}
& s=2 x \\
& u=2 y \tag{93}
\end{align*}
$$

with $x$ and $y$ defined as before.
We can then go from the theory constructed by Pretko in [8] to the one by Seiberg in [11] by neglecting the $\lambda$-term and setting

$$
\begin{equation*}
\lambda^{\prime}=\frac{1}{2}(s+i u) . \tag{94}
\end{equation*}
$$



Figure 2: A "sombrero" potential produced by $V(\phi)=-m^{2}|\phi|^{2}+\beta|\phi|^{4}$ in a case where $\beta<m^{2}$.

### 6.3 Symmetry Broken Case

We can briefly consider the so called symmetry broken case. This is resultant of a higher order term $\beta|\phi|^{4}$ in the Lagrangian, and its interaction with the $-m^{2}|\phi|^{2}$-term. In the case where $\beta<m^{2}$, as commonly assumed for higher order terms, the minimum of the potential $V(\phi)=-m^{2}|\phi|^{2}+\beta|\phi|^{4}$ is offset. This potential is illustrated in figure 2. In this setting it is then natural to discuss a case where all variation in the field is moved to be in the phase and any motion in a transverse direction - i.e. up and down the potential - is ignored,

$$
\begin{equation*}
\phi(t, \vec{x})=\phi_{0} e^{i \theta(t, \vec{x})}, \tag{95}
\end{equation*}
$$

with $\phi_{0}$ a constant amplitude. In this case the Lagrangian (eq. (53)) becomes

$$
\begin{equation*}
\mathcal{L}=-\phi_{0}^{2}\left(\partial_{t} \theta\right)^{2}+\lambda \phi_{0}^{4}\left(\partial_{i} \partial_{j} \theta\right)^{2}+\lambda^{\prime} \phi_{0}^{4}\left(\partial_{i}^{2} \theta\right)+\ldots \tag{96}
\end{equation*}
$$

where the invariance under our symmetry

$$
\begin{equation*}
\theta \rightarrow \theta+\alpha(x)=\theta+\alpha+\lambda_{i} x^{i} \tag{97}
\end{equation*}
$$

is now very easy to see. Dropping the last term, and expanding the summation yields

$$
\begin{equation*}
\mathcal{L}=-\phi_{0}^{2}\left(\partial_{t} \theta\right)^{2}+\lambda \phi_{0}^{4}\left(\left(\partial_{x} \partial_{y} \theta\right)^{2}+\left(\partial_{y} \partial_{z} \theta\right)^{2}+\left(\partial_{x} \partial_{z} \theta\right)^{2}\right) . \tag{98}
\end{equation*}
$$

In the previous step, rotational symmetry has been deliberately broken by killing off all diagonal terms. The equation of motion then becomes

$$
\begin{equation*}
\partial_{t}^{2} \theta=\lambda \phi_{0}^{2}\left(\partial_{x}^{2} \partial_{y}^{2} \theta+\partial_{x}^{2} \partial_{z}^{2} \theta+\partial_{y}^{2} \partial_{z}^{2} \theta\right) . \tag{99}
\end{equation*}
$$

This form is quite interesting, but in order to see clearly why, let us consider the plane wave solution to the equation of motion. This is quickly written down as

$$
\begin{equation*}
\theta_{ \pm}\left(t, x^{i}\right)=\theta_{0} e^{ \pm i E t \pm i p_{i} x^{i}} \tag{100}
\end{equation*}
$$

But looking at the energy of the solutions we see that it is given by

$$
\begin{equation*}
E^{2}=\lambda \phi_{0}^{2}\left(p_{x}^{2} p_{y}^{2}+p_{y}^{2} p_{z}^{2}+p_{x}^{2} p_{z}^{2}\right), \tag{101}
\end{equation*}
$$

which is shocking, since a particle could have momentum in one of the directions, and yet have zero energy!

### 6.4 Coupling to a Gauge Field

Thus far we have made our theory invariant under

$$
\begin{array}{r}
\phi \rightarrow e^{i \alpha(x)} \phi, \\
\partial_{i} \partial_{j} \alpha(x)=0 \tag{102}
\end{array}
$$

by finding a suitable Lagrangian. To generalize the discussion and permit $\alpha(x)$ to be a completely arbitrary function of $x$, the gauge principle can be applied. Moreover, we can also let $\alpha$ have an arbitrary time dependence and account for this with a second gauge field. Since $\alpha$ is no longer limited to a linear function, meaning the linearity condition no longer applies to it, the $\partial_{i} \partial_{j} \alpha(t, x)$-term from equation (50) is regained. Then

$$
\begin{equation*}
\mathcal{O}_{i j} \rightarrow \phi \partial_{i} \partial_{j} \phi-\partial_{i} \phi \partial_{j} \phi+\partial_{i} \partial_{j} \alpha(t, x) \phi^{2}, \tag{103}
\end{equation*}
$$

where the final term can be cancelled by a gauge field $A_{i j}$ that transforms as

$$
\begin{equation*}
A_{i j} \rightarrow A_{i j}+\partial_{i} \partial_{j} \alpha(t, x) \tag{104}
\end{equation*}
$$

Because $\partial_{i} \partial_{j} \alpha(t, x) \phi^{2}$ is symmetric under the exchange of indices $i$ and $j$, we must conclude that $A_{i j}$ must be a symmetric tensor gauge field. The covariant derivative can then be defined by

$$
\begin{equation*}
D_{i j} \phi^{2}=\phi \partial_{i} \partial_{j} \phi-\partial_{i} \phi \partial_{j} \phi-i A_{i j} \phi^{2}, \tag{105}
\end{equation*}
$$

making the Lagrangian

$$
\begin{equation*}
\mathcal{L}=\left|\partial_{t} \phi\right|^{2}-m^{2}|\phi|^{2}-\lambda\left|D_{i j} \phi^{2}\right|^{2}-\lambda^{\prime} \phi^{* 2}\left(D_{i}^{i} \phi^{2}\right)+\text { C.C. } \tag{106}
\end{equation*}
$$

However, now the $\left|\partial_{t} \phi\right|^{2}$-term does not transform covariantly. This can be fixed similarly by a gauge field $\psi$

$$
\begin{equation*}
\psi \rightarrow \psi+\partial_{t} \alpha(t, x), \tag{107}
\end{equation*}
$$

with the covariant time-derivative

$$
\begin{equation*}
D_{t} \phi=\left(\partial_{t}-i \psi\right) \phi . \tag{108}
\end{equation*}
$$

Finally, the Lagrangian is given by

$$
\begin{equation*}
\mathcal{L}=\left|D_{t} \phi\right|^{2}-m^{2}|\phi|^{2}-\lambda\left|D_{i j} \phi^{2}\right|^{2}-\lambda^{\prime} \phi^{* 2}\left(D_{i}^{i} \phi^{2}\right)+\text { C.C. } \tag{109}
\end{equation*}
$$

This method generalizes simply to higher moment conservation, with resulting higher rank tensor gauge fields. The resulting theories are, however, not discussed here.

Like with Maxwell theory, two gauge invariant quantities $E_{i j}$ and $B_{i j}$ can now be introduced. These take analogous forms to the definitions of the electric and magnetic fields in terms of the Maxwell gauge field $A_{\mu}$ from equation (20),

$$
\begin{equation*}
E_{i j}=\partial_{t} A_{i j}-\partial_{i} \partial_{j} \psi \quad \text { and } \quad B_{i j}=\epsilon_{i k l} \partial^{k} A_{j}^{l} \tag{110}
\end{equation*}
$$

but now in terms of the fracton gauge field $A_{i j}$. These two terms can then be added to the Lagrangian as invariant terms to reach

$$
\begin{equation*}
\mathcal{L}=\left|D_{t} \phi\right|^{2}-m^{2}|\phi|^{2}-\lambda\left|D_{i j} \phi^{2}\right|^{2}-\lambda^{\prime} \phi^{* 2}\left(D_{i}^{i} \phi^{2}\right)+E_{i j} E^{i j}+B_{i j} B^{i j}+C . C . \tag{111}
\end{equation*}
$$

A theory of this form is interesting to construct, as it provides a link to boson systems. This comparison is natural because a considerable number of boson theories are constructed via applying the gauge principle.
Assuming the free case, we can deduce the equations of motion for the fields $A_{i j}$ and $\psi$ from

$$
\begin{equation*}
\mathcal{L}_{\text {free }}=E_{i j} E^{i j}+B_{i j} B^{i j} . \tag{112}
\end{equation*}
$$

For $A_{i j}$ we can simply apply the usual Euler-Lagrange equation,

$$
\begin{equation*}
\frac{\partial \mathcal{L}}{\partial A_{i j}}-\partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} A_{i j}\right)}\right)=0 \tag{113}
\end{equation*}
$$

For the two terms we get

$$
\begin{equation*}
\frac{\partial \mathcal{L}}{\partial A_{i j}}=0 \tag{114}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} A_{i j}\right)}=2\left(\delta_{\mu}^{t} E^{i j}-\epsilon_{k l}^{i} \delta_{\mu}^{l} B^{k j}\right), \tag{115}
\end{equation*}
$$

where the second term's minus sign is a result of

$$
\begin{align*}
\frac{\partial}{\partial\left(\partial_{\mu} A_{i j}\right)} B_{m j} B^{m j} & =\frac{\partial}{\partial\left(\partial_{\mu} A_{i j}\right)}\left(\epsilon_{m k l} \partial^{k} A_{j}^{l} B^{m j}\right) \\
& =\frac{\partial}{\partial\left(\partial_{\mu} A_{i j}\right)}\left(\epsilon_{m k l} \partial^{k} \eta^{i l} A_{i j} B^{m j}\right) \\
& =-\frac{\partial}{\partial\left(\partial_{\mu} A_{i j}\right)}\left(\epsilon_{m k}^{i} \partial^{k} A_{i j} B^{m j}\right) \tag{116}
\end{align*}
$$

This results in the equation of motion

$$
\begin{align*}
& \partial_{t} E^{i j}-\epsilon_{k m}^{i} \partial^{k} B^{m j}=0  \tag{117}\\
& \Longrightarrow \partial_{t} E^{i j}=\epsilon_{k m}^{i} \partial^{k} B^{m j} . \tag{118}
\end{align*}
$$

This is directly analogous to the Maxwell equation $\dot{\vec{E}}=\nabla \times \vec{B}$, giving us a mixing between the two gauge invariant fields $E_{i j}$ and $B_{i j}$. It can be noted here that a plane wave will most likely be a solution to this equation. This possibility is however not explored further here.

Since the free Lagrangian, (112) depends on higher than first degree derivatives of $\psi$, the usual Euler-Lagrange equation (113) is no longer sufficient. Including higher order derivatives, the Euler-Lagrange equation with respect to $\psi$ becomes

$$
\begin{equation*}
\frac{\partial \mathcal{L}}{\partial \psi}-\partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \psi\right)}\right)+\partial_{\mu} \partial_{\nu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \partial_{\nu} \psi\right)}\right)=0 \tag{119}
\end{equation*}
$$

the derivation for which is presented in appendix B. The two first term go to zero immediately, while the third becomes

$$
\begin{equation*}
\partial_{\mu} \partial_{\nu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \partial_{\nu} \psi\right)}\right)=-2 \partial_{i} \partial_{j} E^{i j} \tag{120}
\end{equation*}
$$

As this is the only non-zero term in the equation, the equation of motion is

$$
\begin{equation*}
\partial_{i} \partial_{j} E^{i j}=0, \tag{121}
\end{equation*}
$$

which constitutes Gauss' law for our theory.

## 7 Conclusion

In this thesis, it was demonstrated that requiring $\phi \rightarrow e^{i \lambda_{i} x^{i}} \phi$ be a symmetry of a field theory organically gives rise to a theory describing fracton excitations. The form of this theory was shown to be non-gaussian, allowing - even at lowest order - interaction between particles. This property was proven to be of great interest in [7], where a similar theory exhibited behaviour with suitable properties to describe a gravitational force.

Next the mobility constraints of fractons were directly derived from this field theory by showing for the translation and dipole operators $P_{i}$ and $D^{k}$ to commute, the direction of translation was required to be perpendicular to the moment of the dipole. It was further shown that the theory conforms to the multipole algebra $\mathfrak{m}$. The theory in question was also shown to not be invariant under Carroll, Galilean, or Lorentz boosts. The lack of invariance under Carroll boosts was particularly interesting, as Carroll particles tend to exhibit very similar properties to those of fractons; individual Carroll particles are immobile, while interacting sets of Carroll particles can have non-trivial dynamics (1).

Subsequently it was shown that in a symmetry broken case fractons can exhibit non-zero momentum in one direction, while having zero energy. While extremely intriguing intuitively, this raises several questions when physical interpretation is considered.

Finally, the coupling of a fracton theory to a second rank tensor gauge field is accomplished, which endows the theory with invariance under transformations of the form

$$
\begin{equation*}
\phi \rightarrow e^{i \alpha(t, x)} \phi, \tag{122}
\end{equation*}
$$

where $\alpha(t, x)$ is an arbitrary function of space and time. The invariant quantities with respect to the gauge are then shown to take on an analogous form to those of the electric and magnetic
fields from familiar Maxwell theory. Using these additional fields, the theory is endowed with a new Gauss' law,

$$
\begin{equation*}
\partial_{i} \partial_{j} E^{i j}=0, \tag{123}
\end{equation*}
$$

as well as another equation of motion analogous to usual Maxwell electromagnetism,

$$
\begin{equation*}
\partial_{t} E^{i j}=\epsilon_{k m}^{i} \partial^{k} B^{m j} \tag{124}
\end{equation*}
$$

This result is interesting, first because it seems to suggest that the fracton gauge field might exhibit properties similar to usual electromagnetic theory, and second because it provides a link to bosonic theories. Further research into the properties of the gauged theory would therefore likely be interesting.

## 8 Outlook

Fractons have been an extremely active and fast progressing field of research in the past years. The potential applications in quantum information storage, a plethora of links to other fields such as elasticity theory and (quantum) gravity, as well as the many exotic properties of fractons have fueled the interest in the field. While more and more is known about fractons and their properties, the field is still in its early stages, and much more remains to be explored. Some open questions include the stability of fracton systems [4], whether fundamentally fermionic fracton systems exists, and if so what their properties are [9], as well as the nature of fracton theories in curved spacetime. On the premise of this thesis, the link between Carroll symmetries, the multipole group, and fractons should be further researched.

In addition to these theoretical questions, more experimental work on fractons is also necessary. Manufacturing physical systems with emergent fracton excitations is of great interest to the quantum devices field in general, and especially on the experimental side fractons are still an entirely unexplored area.

With the great variety of models developed and in development for describing fractons, there is a great opportunity for further deepening or diversifying research into the topic.

## 9 Acknowledgements

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## A Units, Conventions, and Notation

Throughout this work some conventions have been adapted. Everywhere in this paper the so-called natural units are used, where $c=\hbar=1$. All quantities are then expressed in the dimension of mass, where $[m]=$ mass $^{1}=1$ denotes the unit associated with a quantity. The action takes on a dimensionless form,

$$
\begin{equation*}
[S]=0 \tag{125}
\end{equation*}
$$

The units of other related quantities can then be determined from the action and the fact that

$$
\begin{array}{r}
{\left[x^{\mu}\right]=-1,} \\
{\left[\partial_{\mu}\right]=1 .} \tag{127}
\end{array}
$$

Moving on to notation, general conventions of Einstein notation are applied, with

$$
x^{\mu}=\left[\begin{array}{c}
x^{0}  \tag{128}\\
x^{1} \\
\vdots \\
x^{d}
\end{array}\right]
$$

a contravariant vector. In general contravariant quantities are denoted with an upper index, while covariant objects are denoted with a lower index. Einstein summation convention is also adopted. This means that for two objects $v^{\mu}$ and $w_{\mu}, v^{\mu} w_{\mu}$ implies summation over indices, as in

$$
\begin{equation*}
v^{\mu} w_{\mu}=v^{0} w_{0}+v^{1} w_{1}+\ldots+v^{d} w_{d}=\sum_{\nu=0}^{d} x^{\nu} w_{\nu} \tag{129}
\end{equation*}
$$

Moreover, here Greek lettered indices $\mu, \nu, \ldots$ are used for quantities that run from 0 to $d$, while latin letters $i, j, \ldots$ are used for purely spatial objects that run 1 to $d$.
$\partial_{\mu}$ is used to denote derivatives with respect to the coordinates $x^{\mu}$, with

$$
\begin{equation*}
\partial_{\mu}=\frac{\partial}{\partial x^{\mu}} . \tag{130}
\end{equation*}
$$

For spacetime, the Minkowski metric with a "mostly minus" signature is used

$$
\eta_{\mu \nu}=\eta^{\mu \nu}\left[\begin{array}{lllll}
1 & & & &  \tag{131}\\
& -1 & & & \\
& & -1 & & \\
& & & \ddots & \\
& & & & -1
\end{array}\right]
$$

The metric can be used to raise or lower indices:

$$
\begin{equation*}
x_{\mu}=\eta_{\mu \nu} x^{\nu} \tag{132}
\end{equation*}
$$

## B Derivation for the Euler-Lagrange Equation

Here the Euler-Lagrange equation is derived for Lagrangians depending on up to secondderivative terms, although this easily generalizes to arbitrary derivatives. We will approach this by considering the variation of the action associated with the Lagrangian, and then applying the principle of least action.

Let the Lagrangian be a function of a field $\Phi$ and its derivatives, up to second derivative

$$
\begin{equation*}
\mathcal{L}=\mathcal{L}\left(\Phi, \partial \Phi, \partial^{2} \Phi\right) . \tag{133}
\end{equation*}
$$

The variation of the action $\delta S$ can then be written as

$$
\begin{equation*}
\delta S=\left(\frac{\partial \mathcal{L}}{\partial \Phi}\right) \delta \Phi+\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \Phi\right)}\right) \delta\left(\partial_{\mu} \Phi\right)+\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \partial_{\nu} \Phi\right)}\right) \delta\left(\partial_{\mu} \partial_{\nu} \Phi\right) . \tag{134}
\end{equation*}
$$

Applying integration by parts to the second term on the right-hand side, we get

$$
\begin{equation*}
\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \Phi\right)}\right) \delta\left(\partial_{\mu} \Phi\right)=\partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \Phi\right)} \delta \Phi\right)-\partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \Phi\right)}\right) \delta \Phi \tag{135}
\end{equation*}
$$

where, as per usual, the total derivative can be neglected.
Integration by parts can also be applied twice to the last term in (134), yielding

$$
\begin{align*}
\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \partial_{\nu} \Phi\right)}\right) \delta\left(\partial_{\mu} \partial_{\nu} \Phi\right)= & \partial_{\mu}\left(\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \partial_{\nu} \Phi\right)}\right) \delta\left(\partial_{\nu} \Phi\right)\right)-\partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \partial_{\nu} \Phi\right)}\right) \delta\left(\partial_{\nu} \Phi\right)  \tag{136}\\
= & \partial_{\mu}\left(\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \partial_{\nu} \Phi\right)}\right) \delta\left(\partial_{\nu} \Phi\right)\right)-\partial_{\mu} \partial_{\nu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \partial_{\nu} \Phi\right)} \delta \Phi\right)  \tag{137}\\
& +\partial_{\mu} \partial_{\nu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \partial_{\nu} \Phi\right)}\right) \delta \Phi \tag{138}
\end{align*}
$$

Once the total derivatives are again removed, equation (134) becomes

$$
\begin{equation*}
d S=\left(\frac{\partial \mathcal{L}}{\partial \Phi}\right) \delta \Phi-\partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \Phi\right)}\right) \delta \Phi+\partial_{\mu} \partial_{\nu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \partial_{\nu} \Phi\right)}\right) \delta \Phi . \tag{139}
\end{equation*}
$$

Requiring that the change in the in the action be zero, factoring out and subsequently cancelling $\delta \Phi$ we find

$$
\begin{equation*}
\left(\frac{\partial \mathcal{L}}{\partial \Phi}\right)-\partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \Phi\right)}\right)+\partial_{\mu} \partial_{\nu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu} \partial_{\nu} \Phi\right)}\right)=0 \tag{140}
\end{equation*}
$$

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[^0]:    ${ }^{1}$ Formally $\mathcal{L}$ is the Lagrangian density, with the Lagrangian $L$ given by $L=\int d^{3} x \mathcal{L}$, but here, as in almost all other texts, we will proceed to call $\mathcal{L}$ the Lagrangian instead.

