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Assume-Guarantee Contracts for Continuous-Time Linear Systems using External Equivalence by Simulation

Bachelor Project Applied Mathematics

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Abstract

Modern engineering systems are increasingly more complex as they are generally the result of the interconnection of a large number of components. Motivated by this, a mathematical framework for comparing dynamical systems is required, in order to define specifications on such components and to allow for the replacement of components. The notion of simulation provides such a framework for continuous-time linear systems. This notion is a powerful tool for non-deterministic linear systems and leads to a notion of external equivalence which is finer than equality of external behaviour. Using geometric control theory, the notion of simulation is characterized for linear systems. In addition, it is shown that the property of simulation passes over to interconnected linear systems for various types of interconnections. Moreover, assume-guarantee contracts are introduced, which can be regarded as characterizations of system specifications. These contracts consist of a pair of assumptions, which describe expected input behaviour of a linear system, and guarantees, which represent desired output behaviour of a linear system when interconnected with relevant environments. These contracts define a class of compatible environments and implementations, and in this paper, conditions are established for the existence of such implementations of assume-guarantee contract.

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Chapter 1

Introduction

Modern engineering systems are becoming increasingly more complex. This complexity is often caused by the fact that these systems are generally the result of the interconnection of many components. One can think for example of the manufacturing of a car. During this process, many individual subsystems are interconnected in order to make sure that no errors occur during the production of the car. Due to the increase in complexity of these engineering systems, analysis of the total system becomes nearly impossible. Therefore, the analysis of these types of systems needs to be done using a modular approach, that is, based on the individual components. This type of analysis does however raise some questions.

First of all, one may wonder what would happen to the total system if one would replace a specific component with another subsystem. However, this leads to the question whether two systems can be said to be equal. This question is an important one within the fields of Systems and Control and Theoretical Computer Science. As the notion of equivalence for linear systems is not easily defined, the notion of external equivalence is investigated, instead. The main idea behind this notion is the fact that we can only distinguish between two systems from 'the outside', that is, we can only distinguish between two systems if the distinction can be detected by some external system interacting with these systems (Van der Schaft 2004).

An important framework established in the investigation of external equivalence is the notion of simulation, which provides a mathematical framework that allows us to express whether a given system is externally equivalent to another. This notion originates from a similar notion in the field of Computer Science and is discussed in more detail in (Haghverdi, Tabuada, and Pappas 2003). This notion provides a way to compare linear systems by comparing the external behaviour of both systems.

One of the problems that occasionally arises in the construction of complex engineering systems that consist of many interconnected components is the fact that these components are manufactured independently. This may lead to inconsistencies when assembling these subsystems. For example, it might not be possible to interconnect some components, and even if we are able to interconnect them, the total system might not have the desired external behaviour. In order to overcome these problems, a framework is needed that provides us with the tools to give specifications to these components. Contract theory is such a framework. This framework finds its origin is the field of Computer Science, as for example is established in (Benveniste et al.

2018). The notion of contract theory centers about the concept of assume-guarantee contracts, which can be seen as a pair of assumptions and guarantees which provide expected input behaviour of a system and desired output behaviour of a system when interacting with its environment.

Even though the notion of contract theory is developed in the field of Computer Science, there is no complete contract theory for continuous-time linear systems. This paper will provide the first steps towards such a framework by using the notion of simulation as a tool to compare linear systems. Aside from defining the most important concepts within this framework, conditions will be established to determine whether a given system satisfies the specifications imposed by a given contract.

In Chapter 2, the key notion of a simulation relation will be defined. Furthermore, it will be explained when a linear system is simulated by another, and how one can find such a simulation relation. In this chapter we will also look at the notion of simulation in the context of linear systems with constraints on the dynamics of the system. Lastly, a brief introduction into the somewhat stronger notion of bisimulation will be provided. In Chapter 3, the results obtained with regards to simulation of single systems will be applied to interconnected linear systems, where it will be investigated whether the notion of simulation naturally passes over from single to interconnected linear systems. Then, in Chapter 4, assume-guarantee contracts are defined as ways to impose specifications on linear systems, where the notion of simulation is used as a way to compare systems. In this chapter, terms as compatibility, implementation and consistency will be defined, and it will be investigated under which conditions a linear system can serve as an implementation of a contract. Furthermore, a brief introduction into the notion of contract refinement will be given. Lastly, Chapter 5 will conclude this paper.

Chapter 2

Simulation

In this chapter, the notion of simulation will be defined for continuous-time linear systems. This notion of simulation provides a mathematical framework which expresses when all possible state trajectories of a linear system can be externally simulated by a state trajectory of another system, in the sense that the input-output data for both systems coincide for all time. In Section 2.1 the key notion of a simulation relation will be established, and in Section 2.2 it will be explained how one can determine the largest simulation relation between to linear systems. Then, in Section 2.3, the notion of simulation will be defined for linear systems with constraints, a class of linear systems that have, aside from the dynamics of the system, also algebraic constraints on the state trajectories of the system. Aside from the notion of simulation, there is also the notion of bisimulation, which can be regarded as a two-sided version of the notion of simulation. In Section 2.4 the fundamentals for this notion will be established.

2.1 Simulation Relations

For the majority of this article, we will consider linear systems of the following form:

$$\Sigma_i : \begin{cases} \dot{x}_i = A_i x_i + B_i u_i + E_i d_i, & x_i \in \mathcal{X}_i, \quad u_i \in \mathcal{U}_i, \\ y_i = C_i x_i, & d_i \in \mathcal{D}_i, \quad y_i \in \mathcal{Y}_i. \end{cases} \tag{2.1}$$

It is assumed that $\mathcal{X}_i$, $\mathcal{U}_i$, $\mathcal{D}_i$ and $\mathcal{Y}_i$ are finite-dimensional linear spaces over $\mathbb{R}$, and for notational convenience we define $\mathcal{T} := [0, \infty)$. Moreover, it will be assumed that the function classes of admissible input functions $u_i : \mathcal{T} \to \mathcal{U}_i$, the function classes of admissible disturbance functions $d_i : \mathcal{T} \to \mathcal{D}_i$, as well as the function classes of state trajectories $x_i : \mathcal{T} \to \mathcal{X}_i$ and output trajectories $y_i : \mathcal{T} \to \mathcal{Y}_i$ are all of class C^∞. For two systems of the form (2.1), a notion of a simulation relation can be defined, as can be found in the following definition.

Definition 2.1. Consider two systems Σ_1 and Σ_2 of the form as in (2.1). A simulation relation of Σ_1 by Σ_2 is a linear subspace $\mathcal{S} \subset \mathcal{X}_1 \times \mathcal{X}_2$ that satisfies the following property: If $(x_1^0, x_2^0) \in \mathcal{S}$, and $u_1(\cdot) = u_2(\cdot)$ is any joint input function, then for any disturbance function $d_1(\cdot)$ there exists a disturbance function $d_2(\cdot)$ such that the resulting state trajectory $x_1(\cdot)$ with $x_1(0) = x_1^0$, and the state trajectory $x_2(\cdot)$ with $x_2(0) = x_2^0$ satisfy the following properties:

$$\text{(i)} \quad (x_1(t), x_2(t)) \in \mathcal{S} \quad \text{for all} \quad t \in \mathcal{T}; \tag{2.2a}$$

$$\text{(ii)} \quad y_1(t) = y_2(t) \quad \text{for all} \quad t \in \mathcal{T}. \tag{2.2b}$$

From this definition, it can be seen that if $\mathcal{S}$ is a simulation relation of Σ_1 by Σ_2, Σ_2 has a richer input-output behaviour than Σ_1. Namely, every input-output trajectory of Σ_1 can be matched by an input-output trajectory of Σ_2, but only if the pair of initial conditions for these trajectories are contained in $\mathcal{S}$. If this statement holds for any possible state trajectory of Σ_1, implying that for every possible x_1^0 there exists a x_2^0 such that $(x_1^0, x_2^0) \in \mathcal{S}$, we say that Σ_1 is simulated by Σ_2.

Definition 2.2. Consider two systems Σ_1 and Σ_2 as given in equation (2.1). Then Σ_1 is called simulated by Σ_2, denoted by $\Sigma_1 \preceq \Sigma_2$, if there exists a simulation relation, $\mathcal{S}$, of Σ_1 by Σ_2 that satisfies $\pi_{\mathcal{X}_1}(\mathcal{S}) = \mathcal{X}_1$, where $\pi_{\mathcal{X}_1}$ denotes the canonical projection of $\mathcal{X}_1 \times \mathcal{X}_2$ to $\mathcal{X}_1$. In this case, we call $\mathcal{S}$ a full simulation relation of Σ_1 by Σ_2

The notion of simulation is a very useful tool, as it provides a way to compare linear systems. By comparing the external behaviour of linear systems with each other, we can construct a notion of external equivalence which is much finer than equality of external behaviour. From the definition of a simulation relation, together with theory from controlled invariant subspaces, which can for example be found in (Trentelman, Stoorvogel, and Hautus 2012), it becomes possible to establish linear algebraic conditions which express when a linear subspace is a simulation relation, making it significantly easier to check whether a linear system is a simulation relation. The results of the following lemma are a first step towards this.

Lemma 2.3. Consider two systems Σ_1 and Σ_2 as given in equation (2.1). A subspace $\mathcal{S} \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 if and only if for every $(x_1, x_2) \in \mathcal{S}$ and all $u \in \mathcal{U}_1 \cap \mathcal{U}_2$ the following properties hold:

(i) for all $d_1 \in \mathcal{D}_1$ there exists a $d_2 \in \mathcal{D}_2$ such that

$$(A_1 x_1 + B_1 u + E_1 d_1, A_2 x_2 + B_2 u + E_2 d_2) \in \mathcal{S}; \tag{2.3a}$$

$$\text{(ii)} \ C_1 x_1 = C_2 x_2. \tag{2.3b}$$

Proof. ($\Rightarrow$) Suppose $\mathcal{S}$ is a simulation relation of Σ_1 by Σ_2. Let $(x_1^0, x_2^0) \in \mathcal{S}$ and let $u \in \mathcal{U}_1 \cap \mathcal{U}_2$. Then by the definition of a simulation relation it is given that for every disturbance function $d_1(\cdot)$, there exists a disturbance function $d_2(\cdot)$ such that the resulting state trajectories $x_1(\cdot)$ with $x_1(0) = x_1^0$ and $x_2(\cdot)$ with $x_2(0) = x_2^0$ satisfy the properties in equation (2.2a) and (2.2b). Since $\mathcal{S}$ is a linear subspace, and the property in equation (2.2a) is satisfied, it can be concluded that for every $d_1(\cdot)$ there exists a $d_2(\cdot)$ such that

$$\left(\frac{1}{t}\left(x_1(t) - x_1^0\right), \frac{1}{t}\left(x_2(t) - x_2^0\right)\right) \in \mathcal{S} \quad \text{for all} \quad t \in \mathcal{T}.$$

Furthermore, since $\mathcal{S}$ is a linear subspace of $\mathcal{X}_1 \times \mathcal{X}_2$, it is closed and therefore we also have that

$$(\dot{x}_1(t), \dot{x}_2(t)) = \lim_{t \to 0} \left(\frac{1}{t}\left(x_1(t) - x_1^0\right), \frac{1}{t}\left(x_2(t) - x_2^0\right)\right) \in \mathcal{S} \quad \text{for all} \quad t \in \mathcal{T}.$$

With the use of the expressions for $\dot{x}_1(t)$ and $\dot{x}_2(t)$, and by evaluating at every time instance $t \in \mathcal{T}$ with $x_1 = x_1(t)$, $x_2 = x_2(t)$, $u = u_1(t) = u_2(t)$, $d_1 = d_1(t)$ and $d_2 = d_2(t)$ we see that for every $d_1 \in \mathcal{D}_1$ there exists a $d_2 \in \mathcal{D}_2$ such that

$$(A_1 x_1 + B_1 u + E_1 d_1, A_2 x_2 + B_2 u + E_2 d_2) \in \mathcal{S}.$$

Therefore, we have shown that (2.3a) is satisfied. The validity of (2.3b) follows straight from the definition and the expressions for the outputs $y_1(t)$ and $y_2(t)$.

($\Leftarrow$) For the other direction assume that a subspace $\mathcal{S}$ satisfies the property that for every $(x_1, x_2) \in \mathcal{S}$ and all $u \in \mathcal{U}_1 \cap \mathcal{U}_2$ the equations (2.3a) and (2.3b) are satisfied. By property (2.3a), it is implied that for all $d_1(\cdot)$ there exists a $d_2(\cdot)$ such that $(\dot{x}_1(t), \dot{x}_2(t)) \in \mathcal{S}$ for all $t \in \mathcal{T}$ for which the derivative exists. Then, by the reverse reasoning above, we must have that for every $d_1(\cdot)$ there exists a $d_2(\cdot)$ such that $(x_1(t), x_2(t)) \in \mathcal{S}$ for all $t \in \mathcal{T}$. Furthermore, since equation (2.3b) is satisfied, it can be concluded that for all $(x_1, x_2) \in \mathcal{S}$ it is given that

$$y_1(t) = C_1 x_1(t) = C_2 x_2(t) = y_2(t)$$

for all $t \in \mathcal{T}$. Therefore, the conditions of the definition of a simulation relation are satisfied, from which it can be concluded that $\mathcal{S}$ is a simulation relation of Σ_1 by Σ_2. $\square$

The results of Lemma 2.3 can be used to state and prove the following theorem, which gives concrete algebraic conditions that need to be satisfied in order for a linear subspace to be a simulation relation.

Theorem 2.4. Consider two systems Σ_1 and Σ_2 as given in equation (2.1). A subspace $\mathcal{S} \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 if and only if the following properties hold:

$$\text{(i)} \quad \mathcal{S} + \operatorname{im} \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \subset \mathcal{S} + \operatorname{im} \begin{bmatrix} 0 \\ E_2 \end{bmatrix}; \tag{2.4a}$$

$$\text{(ii)} \quad \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \mathcal{S} \subset \mathcal{S} + \operatorname{im} \begin{bmatrix} 0 \\ E_2 \end{bmatrix}; \tag{2.4b}$$

$$\text{(iii)} \quad \operatorname{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset \mathcal{S} + \operatorname{im} \begin{bmatrix} 0 \\ E_2 \end{bmatrix}; \tag{2.4c}$$

$$\text{(iv)} \quad \mathcal{S} \subset \ker \begin{bmatrix} C_1 & -C_2 \end{bmatrix}. \tag{2.4d}$$

Proof. ($\Rightarrow$) Assume $\mathcal{S}$ is a simulation relation of Σ_1 by Σ_2. Therefore, by Lemma 2.3, it is given that for every $(x_1, x_2) \in \mathcal{S}$ and every $u \in \mathcal{U}_1 \cap \mathcal{U}_2$ the properties in equations (2.3a) and (2.3b) are satisfied, which makes it possible to prove the statements in equations (2.4a), (2.4b), (2.4c) and (2.4d).

Proof of (2.4a): It is obvious that $\mathcal{S} \subset \mathcal{S} + \operatorname{im} \begin{bmatrix} 0 \\ E_2 \end{bmatrix}$. Now, let $(x_1, x_2) = (0, 0)$ which is in $\mathcal{S}$ as $\mathcal{S}$ is a linear subspace, and let $u = 0$. Then, from (2.3a) it follows that for all $d_1 \in \mathcal{D}_1$ there exists a $d_2 \in \mathcal{D}_2$ such that

$$(E_1 d_1, E_2 d_2) \in \mathcal{S}.$$

From this it can be seen that

$$\operatorname{im} \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \subset \mathcal{S} + \operatorname{im} \begin{bmatrix} 0 \\ E_2 \end{bmatrix}.$$

Therefore, we have proven that

$$\mathcal{S} + \operatorname{im} \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \subset \mathcal{S} + \operatorname{im} \begin{bmatrix} 0 \\ E_2 \end{bmatrix}.$$

Proof of (2.4b): Let $(x_1, x_2) \in \mathcal{S}$, and let $u = 0$ and $d_1 = 0$. Then there exists a $d_2 \in \mathcal{D}_2$ such that

$$(A_1 x_1, A_2 x_2 + E_2 d_2) \in \mathcal{S}.$$

From this, it can be seen that

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{S} + \operatorname{im} \begin{bmatrix} 0 \\ E_2 \end{bmatrix}.$$

As $(x_1, x_2) \in \mathcal{S}$ was chosen arbitrarily, this shows that

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \mathcal{S} \in \mathcal{S} + \operatorname{im} \begin{bmatrix} 0 \\ E_2 \end{bmatrix}.$$

Proof of (2.4c): Let $u \in \mathcal{U}_1 \cap \mathcal{U}_2$. Furthermore, let $(x_1, x_2) = (0, 0)$ and $d_1 = 0$. Then there exists a $d_2 \in \mathcal{D}_2$ such that

$$(B_1 u, B_2 u + E_2 d_2) \in \mathcal{S}.$$

This implies that

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \in \mathcal{S} + \operatorname{im} \begin{bmatrix} 0 \\ E_2 \end{bmatrix},$$

from which it can be concluded that

$$\operatorname{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset \mathcal{S} + \operatorname{im} \begin{bmatrix} 0 \\ E_2 \end{bmatrix}.$$

Proof of (2.4d): Let $(x_1, x_2) \in \mathcal{S}$. Then, by property (2.3b), it is given that

$$C_1 x_1 = C_2 x_2,$$

from which it can easily be seen that

$$(x_1, x_2) \in \ker \begin{bmatrix} C_1 & -C_2 \end{bmatrix}.$$

Therefore, it can be concluded that

$$\mathcal{S} \subset \ker \begin{bmatrix} C_1 & -C_2 \end{bmatrix}.$$

As the statement in (2.4a), (2.4b), (2.4c) and (2.4d) are proven to be satisfied, it has been shown that this direction of the proof is true.

($\Rightarrow$) Suppose the properties in (2.4a), (2.4b), (2.4c) and (2.4d) are satisfied for some subspace $\mathcal{S} \subset \mathcal{X}_1 \cap \mathcal{X}_2$. Let $(x_1, x_2) \in \mathcal{S}$, $u \in \mathcal{U}_1 \cap \mathcal{U}_2$ and let $d_1 \in \mathcal{D}_1$. In order to prove this direction, Lemma 2.3 will be used, for which it is needed to prove that there exists a $d_2 \in \mathcal{D}_2$ such that

$$(A_1 x_1 + B_1 u + E_1 d_1, A_2 x_2 + B_2 u + E_2 d_2) \in \mathcal{S},$$

or written equivalently in matrix-vector form,

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} E_1 \\ 0 \end{bmatrix} d_1 + \begin{bmatrix} 0 \\ E_2 \end{bmatrix} d_2 \in \mathcal{S}.$$

First of all, by property (2.4b) there exists $(a_1, a_2) \in \mathcal{S}$ and $c_1 \in \mathcal{D}_2$ such that

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 0 \\ E_2 \end{bmatrix} c_1.$$

Moreover, by property (2.4c) there exists $(b_1, b_2) \in \mathcal{S}$ and $c_2 \in \mathcal{D}_2$ such that

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} 0 \\ E_2 \end{bmatrix} c_2.$$

By property (2.4a) there exists $(e_1, e_2) \in \mathcal{S}$ and $c_3 \in \mathcal{D}_2$ such that

$$\begin{bmatrix} E_1 \\ 0 \end{bmatrix} d_1 = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ E_2 \end{bmatrix} c_3.$$

Substituting these expression above into

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} E_1 \\ 0 \end{bmatrix} d_1 + \begin{bmatrix} 0 \\ E_2 \end{bmatrix} d_2,$$

the following expression is obtained

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 0 \\ E_2 \end{bmatrix} c_1 + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} 0 \\ E_2 \end{bmatrix} c_2 + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ E_2 \end{bmatrix} c_3 + \begin{bmatrix} 0 \\ E_2 \end{bmatrix} d_2.$$

For $d_2 = -(c_1 + c_2 + c_3) \in \mathcal{D}_2$, it can be seen that the above expression simplifies to

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

which is in $\mathcal{S}$ since $\mathcal{S}$ is a linear subspace. This means that for all $(x_1, x_2) \in \mathcal{S}$, $u \in \mathcal{U}_1 \cap \mathcal{U}_2$ and $d_1 \in \mathcal{D}_1$ there exists a $d_2 \in \mathcal{D}_2$ such that

$$(A_1 x_1 + B_1 u + E_1 d_1, A_2 x_2 + B_2 u + E_2 d_2) \in \mathcal{S}.$$

Furthermore, from property (2.4d) it is straightforward that (2.3b) holds. Therefore, Lemma 2.3 can be used to conclude that $\mathcal{S}$ is a simulation relation of Σ_1 by Σ_2.

This concludes the proof of this theorem. $\square$

To illustrate the importance of this theorem when determining whether a system is simulated by another, consider the following example, where the systems Σ and Σ' are given by

$$\Sigma : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u; \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \end{cases} \qquad \Sigma' : \begin{cases} \begin{bmatrix} \dot{x}'_1 \\ \dot{x}'_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u'; \\ y' = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}. \end{cases}$$

Note that for both these systems we have that $E = E' = 0$, and thus there is no influence from the non-deterministic variable on the system. For these two systems, it can be seen that Σ is not simulated by Σ'. To show this, suppose that it is the case that Σ is simulated by Σ', and let $\mathcal{S}$ be a full simulation relation of Σ by Σ'. Therefore, by property (2.4d) we must have that

$$\mathcal{S} \subset \ker \begin{bmatrix} C_1 & -C_2 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Furthermore, by property (2.4c) we must have that

$$\operatorname{im}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset \operatorname{im}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \subset \mathcal{S}.$$

However, the above two statement can only hold simultaneously if $\mathcal{S} = \{0\}$, which is in contradiction with the fact that $\mathcal{S}$ is a full simulation relation. Therefore, we have shown that Σ is not simulated by Σ'. However, by adjusting the linear systems a little, we can solve this problem. Namely, if we change the linear systems into,

$$\Sigma : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u; \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \end{cases} \qquad \Sigma' : \begin{cases} \begin{bmatrix} \dot{x}'_1 \\ \dot{x}'_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u'; \\ y' = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} \end{cases}$$

we see that

$$\mathcal{S} := \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

satisfies the properties of Theorem 2.4. Therefore, $\mathcal{S}$ is a full simulation relation of Σ by Σ', thus implying that Σ is simulated by Σ'.

With the results obtained from the previous lemma and theorem, the following lemma can be proven. This lemma tells us that the property of simulation is actually a preorder, in the sense that this property is both reflexive and transitive. The fact that the notion of simulation is a preorder is very useful, as it enables us to use it as a means of comparing the external behaviour of two systems. The proof of the following lemma can be found in Appendix A.

Lemma 2.5. Let Σ_1, Σ_2 and Σ_3 be linear systems of the form as given in equation (2.1). Then the following hold:

$$\text{(i)} \quad \Sigma_1 \preceq \Sigma_1 \text{ for all } \Sigma_1; \tag{2.5a}$$

$$\text{(ii)} \quad \text{if } \Sigma_1 \preceq \Sigma_2 \text{ and } \Sigma_2 \preceq \Sigma_3, \text{ then } \Sigma_1 \preceq \Sigma_3. \tag{2.5b}$$

In this section, we have defined the notion of simulation for two linear systems. The key concept within this notion is that of a simulation relation. Furthermore, in Theorem 2.4 we have determined necessary and sufficient algebraic conditions for a linear subspace to be a simulation relation. However, it has not been explained how one can find a simulation relation for two linear systems, or how one can determine whether such a simulation relation even exists. The solutions of these problems will be established in the following section.

2.2 Maximum Simulation Relation

Similarly as with controlled invariant subspaces (Trentelman, Stoorvogel, and Hautus 2012), one can establish an algorithm to find the largest or maximum simulation relation for two linear system. From this maximum simulation relation, one can very easily check whether a linear system is simulated by another. The aim of this section is to describe this algorithm.

In order to define the maximum simulation relation algorithm mentioned above, consider two systems Σ_1 and Σ_2 of the form as given in (2.1). For notational convenience, we introduce the following maps:

$$\begin{aligned} \bar{A} &:= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad \bar{E}_1 := \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \\ \bar{E}_2 &:= \begin{bmatrix} 0 \\ E_2 \end{bmatrix}, \quad \bar{C} := \begin{bmatrix} C_1 & -C_2 \end{bmatrix}. \end{aligned} \tag{2.6}$$

Now, consider the following sequence of linear subspaces of $\mathcal{X}_1 \times \mathcal{X}_2$, where $j \in \mathbb{N}$:

$$\begin{aligned} \mathcal{S}^0 &= \mathcal{X}_1 \times \mathcal{X}_2, \\ \mathcal{S}^1 &= \{x \in \mathcal{S}^0 \mid x \in \ker \bar{C}\}, \\ \mathcal{S}^2 &= \{x \in \mathcal{S}^1 \mid \bar{A}x + \operatorname{im} \bar{G}_1 \subset \mathcal{S}^1 + \operatorname{im} \bar{G}_2\}, \\ &\ \,\vdots \\ \mathcal{S}^j &= \{x \in \mathcal{S}^{j-1} \mid \bar{A}x + \operatorname{im} \bar{G}_1 \subset \mathcal{S}^{j-1} + \operatorname{im} \bar{G}_2\}. \end{aligned}$$

For this sequence of linear subspaces, there is the following theorem.

Theorem 2.6. The sequence of subsets $\mathcal{S}^0, \mathcal{S}^1, \mathcal{S}^2, \ldots$ as defined above satisfies the following properties:

(i) for all $j \geq 0$, $\mathcal{S}^j$ is a linear subspace or empty; (2.7a)

(ii) for each $j \geq 0$ we have that $\mathcal{S}^j \supset \mathcal{S}^{j+1}$; (2.7b)

(iii) there exists a $k < \infty$ for which $\mathcal{S}^k = \mathcal{S}^{k+1} := \mathcal{S}^*$. Moreover, we have that $\mathcal{S}^j = \mathcal{S}^*$ for all $j \geq k$; (2.7c)

(iv) $\mathcal{S}^*$ is either empty of equals the largest subspace of $\mathcal{X}_1 \times \mathcal{X}_2$ that satisfies properties (2.4a), (2.4b) and (2.4d). (2.7d)

Proof. As this theorem, as well as the proof of this theorem, are almost identical to that corresponding to the controlled invariant subspace algorithm (Trentelman, Stoorvogel, and Hautus 2012), only a sketch of the proof will be given. Statements (i) and (ii) follows immediately from the way each $\mathcal{S}^j$ is defined. Furthermore, statement (iii) follows from the finite-dimensionality of $\mathcal{X}_1 \times \mathcal{X}_2$ and from statement (ii). Therefore, only statement (iv) will be proven in more detail.

Assume that $\mathcal{S}^k = \mathcal{S}^{k+1} = \mathcal{S}^*$ is nonempty. First of all, it will be proven that $\mathcal{S}^*$ satisfies (2.4a), (2.4b) and (2.4d). Since $\mathcal{S}^* = \mathcal{S}^k = \mathcal{S}^{k+1}$ we have that

$$\bar{A}\mathcal{S}^* + \operatorname{im} \bar{G}_1 = \bar{A}\mathcal{S}^{k+1} + \operatorname{im} \bar{G}_1 \subset \mathcal{S}^k + \operatorname{im} \bar{G}_2 = \mathcal{S}^* + \operatorname{im} \bar{G}_2$$

which shows that $\bar{A}\mathcal{S}^* \subset \mathcal{S}^* + \text{im } \bar{G}_2$, and thus statement (2.4b) is satisfied. Moreover, from the above reasoning, it can be seen that im $\bar{G}_1 \subset \mathcal{S}^* + \text{im } \bar{G}_2$, so also

$$\mathcal{S}^* + \text{im } \bar{G}_1 \subset \mathcal{S}^* + \text{im } \bar{G}_2$$

from which it can be concluded that (2.4a) is true. Lastly, $\mathcal{S}^* \subset \mathcal{S}^1 = \ker \bar{C}$ and thus also (2.4d) is satisfied.

To prove that $\mathcal{S}^*$ is the largest subspace of $\mathcal{X}_1 \times \mathcal{X}_2$ that satisfied these properties, let $\mathcal{S}$ be any subspace satisfying (2.4a), (2.4b) and (2.4d). Then, by induction, it can be shown that $\mathcal{S} \subset \mathcal{S}^j$ for all $j \geq 0$, and thus it is also given that $\mathcal{S} \subset \mathcal{S}^*$. This shows that $\mathcal{S}$ is the largest subspace of $\mathcal{X}_1 \times \mathcal{X}_2$ that satisfies properties (2.4a), (2.4b) and (2.4d). □

With the use of this algorithm, it becomes very easy to check whether a linear system is simulated by another. As we already have that $\mathcal{S}^*$ is a linear subspace, and we have that it satisfies (2.4a), (2.4b) and (2.4d), $\mathcal{S}$ needs to additionally satisfy (2.4c) and $\pi_{\mathcal{X}_1}(\mathcal{S}^*) = \mathcal{X}_1$ in order to be a full simulation relation. If this is the case, we have found a full simulation relation of Σ_1 by Σ_2, implying that Σ_1 is simulated by Σ_2. This result is also stated in the following corollary.

Corollary 2.7. Consider two linear system Σ_1 and Σ_2. Then Σ_1 is simulated by Σ_2 if and only if $\mathcal{S}^*$ satisfies (2.4c) together with the fact that $\pi_{\mathcal{X}_1}(\mathcal{S}^*) = \mathcal{X}_1$.

In order to further clarify the algorithm with which one can find the maximum simulation relation, consider the following linear systems:

$$\Sigma : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u; \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \end{cases} \qquad \Sigma' : \begin{cases} \begin{bmatrix} \dot{x}'_1 \\ \dot{x}'_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u'; \\ y' = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}. \end{cases}$$

By applying the maximum simulation relation algorithm, we find that $\mathcal{S}^0 = \mathbb{R}^4$ and

$$\mathcal{S}^1 = \ker \begin{bmatrix} C_1 & -C_2 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Furthermore, we find that

$$\mathcal{S}^2 = \mathcal{S}^1 = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} = \mathcal{S}^*.$$

Now, since we have that

$$\text{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \text{im} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \subset \mathcal{S}^*,$$

as well as the fact that $\pi_{\mathbb{R}^2}(\mathcal{S}^*) = \mathbb{R}^2$, we have that $\mathcal{S}^*$ is a full simulation relation of Σ by Σ', and thus we also have that Σ is simulated by Σ'.

2.3 Linear Systems with Constraints

In the previous sections, the notion of simulation has been studied for a general type of linear system without any constraints on the dynamics of the system. However, for some applications, this type of linear system will not be applicable, as there are some predetermined requirements or constraints on the system. However, for these type of systems it is possible to establish the notion of simulation too, as will be shown in this section. The establishment of the notion of simulation for this class of linear systems will be proven to be relevant when introducing the framework of contract theory, as can be seen in Chapter 4.

In order to generalize the notion of simulation to linear systems with constraints, consider the following type of linear system

$$\Sigma_i : \begin{cases} \dot{x}_i = A_i x_i + E_i d_i \\ w_i = C_i x_i \\ 0 = H_i x_i \end{cases}, \tag{2.8}$$

where $x_i \in \mathcal{X}_i$ denotes the state, $w_i \in \mathcal{W}_i$ is a variable denoting the external behaviour of the system and $d_i \in \mathcal{D}_i$ is a variable denoting the non-determinism of the system. Just as before, $\mathcal{X}_i, \mathcal{W}_i$ and $\mathcal{D}_i$ are finite-dimensional linear subspaces and the trajectories $x_i(\cdot)$, $d_i(\cdot)$ and $w_i(\cdot)$ are of class C^∞.

Aside from the dynamics and external behaviour of Σ_i, equation (2.8) provides algebraic constraints on the state trajectories of the system, which are represented by the last equation of the system. Therefore, not all initial conditions will lead to a state trajectory that satisfies the dynamics and external behaviour of the system (Besselink, Johansson, and Van der Schaft 2019). This leads to the introduction of the consistent subspace, $\mathcal{V}_i^*$, which we define to be the set of all initial conditions x_i^0 such that there exists a function $d_i(\cdot)$ such that the resulting state trajectory $x_i(\cdot)$ with $x_i(0) = x_i^0$ satisfies the constraint $H_i x_i(t) = 0$ for all $t \in \mathcal{T}$. This consistent subspace can also be characterized as the largest subspace $\mathcal{V}_i \subset \mathcal{X}_i$ such that

$$A_i \mathcal{V}_i \subset \mathcal{V}_i + \text{im } E_i, \qquad \mathcal{V}_i \subset \ker H_i,$$

see (Megawati and Van der Schaft 2018).

Using the consistent subspace, it is possible to define simulation relations for two systems of the form (2.8).

Definition 2.8. Consider two linear systems Σ_1 and Σ_2 as given in (2.8). A linear subspace $\mathcal{S} \subset \mathcal{X}_1 \times \mathcal{X}_2$ satisfying $\pi_{\mathcal{X}_i}(\mathcal{S}) \subset \mathcal{V}_i^*$ for $i \in \{1, 2\}$ is a simulation relation of Σ_1 by Σ_2 if for all $(x_1^0, x_2^0) \in \mathcal{S}$ the following properties hold:

(i) for all $d_1(\cdot)$ such that $x_1(\cdot)$ with $x_1(0) = x_1^0$ satisfies $x_1(t) \in \mathcal{V}_1^*$ for all $t \in \mathcal{T}$ there exists a $d_2(\cdot)$ such that $x_2(\cdot)$ with $x_2(0) = x_2^0$ satisfies $(x_1(t), x_2(t)) \in \mathcal{S}$ for all $t \in \mathcal{T}$; (2.9a)

(ii) $w_1(t) = w_2(t)$ for all $t \in \mathcal{T}$. (2.9b)

We say that Σ_1 is simulated by Σ_2, denoted by $\Sigma_1 \preceq \Sigma_2$, if there exists a simulation relation of Σ_1 by Σ_2 that satisfies $\pi_{\mathcal{X}_1}(\mathcal{S}) = \mathcal{V}_1^*$.

When comparing the above definition and the one as given in Definition 2.1, it can be seen that the definitions are very similar. The only difference between the two definitions is the fact that we require the restriction of a simulation relation $\mathcal{S}$ onto $\mathcal{X}_i$ to be contained in the consistent subspace of Σ_i. This additional requirement is a consequence of the constraints on the dynamics of the system, causing not every initial condition in $\mathcal{X}_i$ to lead to a state trajectory satisfying the dynamics in (2.8). As we are interested in matching every possible state trajectory of Σ_1 by a state trajectory of Σ_2, we only need to take into account the initial conditions that lead to a state trajectory that is compatible with the dynamics of the system and the constraints imposed on these dynamics. By adding this requirement, we ensure that only the initial conditions for which this is the case are included in the simulation relation.

For linear systems with contraints on the dynamics, it is possible to construct algebraic conditions for a linear subspace in order to be a simulation relation, similarly as with linear systems without any constraints. These conditions are given in the following lemma of which the proof is omitted, as it is similar to the proof of Lemma 2.3.

Lemma 2.9. Consider two linear system Σ_1 and Σ_2 of the form as in equation (2.8). A linear subspace $\mathcal{S} \subset \mathcal{X}_1 \times \mathcal{X}_2$ satisfying $\pi_{\mathcal{X}_i}(\mathcal{S}) \subset \mathcal{V}_i^*$ for $i \in \{1, 2\}$ is a simulation relation of Σ_1 by Σ_2 if and only if the following conditions are satisfied:

(i) for all $d_1 \in \mathcal{D}_1$ such that $A_1x_1 + E_1d_1 \in \mathcal{V}_1^*$, there exists a $d_2 \in \mathcal{D}_2$ such that $A_2x_2 + E_2d_2 \in \mathcal{V}_2^*$ and
$$(A_1x_1 + E_1d_1, A_2x_2 + B_2u + E_2d_2) \in \mathcal{S}; \tag{2.10a}$$

(ii) $C_1x_1 = C_2x_2$. (2.10b)

2.4 Bisimulation

With the obtained framework of the notion of simulation, it becomes possible to develop the framework for the notion of bisimulation. The core idea of bisimulation of two linear systems is that both systems have the same set of input-output trajectories, implying that every input-output trajectory of one system can be matched by an input-output trajectory of the other, and vice versa. From this, it can be seen that the concept of simulation is a one-sided version of bisimulation. Therefore, all results already obtained within the framework of simulation can be adapted into results that hold for the notion of bisimulation. In order to formally establish the framework of bisimulation, we first introduce the key concept of a bisimulation relation, as is given in the following definition.

Definition 2.10. Consider two linear systems Σ_1 and Σ_2 as given in equation (2.1). A bisimulation relation between Σ_1 and Σ_2 is a linear subspace $\mathcal{B} \subset \mathcal{X}_1 \times \mathcal{X}_2$ that satisfies the following property: If $(x_1^0, x_2^0) \in \mathcal{B}$ and $u_1(\cdot) = u_2(\cdot)$ is any joint input function, then for every disturbance function $d_1(\cdot)$ there exists a disturbance function $d_2(\cdot)$ such that the resulting state trajectories $x_1(\cdot)$ with $x_1(0) = x_1^0$, and $x_2(\cdot)$ with $x_2(0) = x_2^0$ satisfy the following two properties:

$$\text{(i)} \qquad (x_1(t), x_2(t)) \in \mathcal{B} \quad \text{for all } t \in \mathcal{T}; \tag{2.11a}$$

$$\text{(ii)} \qquad y_1(t) = y_2(t) \quad \text{for all } t \in \mathcal{T}. \tag{2.11b}$$

and, conversely, for every disturbance function $d_2(\cdot)$ there exists a disturbance function $d_1(\cdot)$ such that the resulting state trajectories also satisfy (2.11a) and (2.11b).

From this definition, it can be seen that if $\mathcal{B}$ is a bisimulation relation between Σ_1 and Σ_2, both systems have the same set of input-output trajectories, meaning that every input-output trajectory of Σ_1 can be matched to an input-output trajectory of Σ_2 and vice versa, but this only is the case if the pair of initial conditions for these trajectories are contained in $\mathcal{B}$. Two systems are said to be bisimilar if the above reasoning holds for any possible state-trajectory, no matter the initial conditions, as is stated in the following definition.

Definition 2.11. Two systems Σ_1 and Σ_2 of the form as given in equation (2.1) are called bisimilar, denoted by $\Sigma_1 \sim \Sigma_2$, if there exists a bisimulation relation $\mathcal{B} \subset \mathcal{X}_1 \times \mathcal{X}_2$ between them that satisfies

$$\pi_{\mathcal{X}_1}(\mathcal{B}) = \mathcal{X}_1, \qquad \pi_{\mathcal{X}_2}(\mathcal{B}) = \mathcal{X}_2, \tag{2.12}$$

where $\pi_{\mathcal{X}_i} : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_i$ for $i = 1, 2$ denote the canonical projections.

Similarly as with the simulation relation, it is possible to construct necessary and sufficient algebraic conditions that express when a subspace of $\mathcal{X}_1 \times \mathcal{X}_2$ is a bisimulation relation. The following lemma and theorem provide these conditions, but since the results are almost identical to those of Lemma 2.3 and Theorem 2.4, the proofs will be omitted. For the proofs of Lemma 2.12 and Theorem 2.13, one can find these in (Van der Schaft 2004).

Lemma 2.12. Consider two linear systems Σ_1 and Σ_2 as given in equation (2.1). A subspace $\mathcal{B} \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a bisimulation relation between Σ_1 and Σ_2 if and only if for every $(x_1, x_2) \in \mathcal{B}$ and for every input function $u \in \mathcal{U}_1 \cap \mathcal{U}_2$ the following properties are satisfied:

(i) for all $d_1 \in \mathcal{D}_1$ there exists a $d_2 \in \mathcal{D}_2$ such that

$$(A_1x_1 + B_1u + E_1d_1, A_2x_2 + B_2u + E_2d_2) \in \mathcal{B}; \tag{2.13a}$$

(ii) for all $d_2 \in \mathcal{D}_2$ there exists a $d_1 \in \mathcal{D}_1$ such that

$$(A_1x_1 + B_1u + E_1d_1, A_2x_2 + B_2u + E_2d_2) \in \mathcal{B}; \tag{2.13b}$$

(iii) $C_1x_1 = C_2x_2$. (2.13c)

From this lemma the following theorem can be stated, which provides necessary and sufficient algebraic conditions for a linear subspace to be a bisimulation relation.

Theorem 2.13. Consider two systems Σ_1 and Σ_2 as given in equation (2.1). A subspace $\mathcal{B} \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a bisimulation relation between Σ_1 and Σ_2 if and only if the following properties are satisfied:

$$\text{(i)} \quad \mathcal{B} + \operatorname{im}\begin{bmatrix} E_1 \\ 0 \end{bmatrix} = \mathcal{B} + \operatorname{im}\begin{bmatrix} 0 \\ E_2 \end{bmatrix} := \mathcal{B}_E; \tag{2.14a}$$

$$\text{(ii)} \quad \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \mathcal{B} \subset \mathcal{B}_E; \tag{2.14b}$$

$$\text{(iii)} \quad \operatorname{im}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset \mathcal{B}_E; \tag{2.14c}$$

$$\text{(iv)} \quad \mathcal{B} \subset \ker \begin{bmatrix} C_1 & -C_2 \end{bmatrix}. \tag{2.14d}$$

Within this chapter, we have established the fundamentals of the notion of simulation for linear systems, both with and without constraints on the dynamics of the system. This notion, which expresses when any possible input-output trajectory of a linear system can be matched by an input-output trajectory of another system, provides a way to compare two linear systems with each other, by comparing the input-output behaviour of both systems. The fact that this notion can be used as a way to compare linear systems is considered very useful when working with complex engineering systems, as will be illustrated later on within this article when introducing assume-guarantee contracts.

Chapter 3

Interconnected Systems

In the previous chapter, the main focus has been on the establishment of the notion of simulation for single linear systems. Since modern engineering systems are becoming more complex systems in the sense that they are often build from the interconnection of multiple smaller subsystems, it is useful to investigate whether the property of simulation is also applicable for interconnections of linear systems. This chapter will explore this question by looking at the notion of simulation when applied to various interconnections of linear systems. Important to note is that all results in this chapter will be obtained using the notion of simulation, however, similar results can also be obtained using the notion of bisimulation, see (Kerber and Van der Schaft 2010).

3.1 Series Interconnection

Within this chapter, some concrete forms of interconnections will be investigated by determining whether the notion of simulation can be passed over to interconnected linear systems. In other words, the main question that we will try to answer is the following: if we have linear systems Σ_1 and Σ_2 which are simulated by Σ_1' and Σ_2' respectively, is it also the case that the interconnection of Σ_1 and Σ_2 is simulated by the interconnection of Σ_1' and Σ_2'?

The first type of interconnection that will be explored is the series interconnection of two linear systems Σ_1 and Σ_2, denoted by $\Sigma_1 \times \Sigma_2$ as displayed in the figure below. This type of interconnection is constructed by using the output of Σ_1 as the input of the system Σ_2.

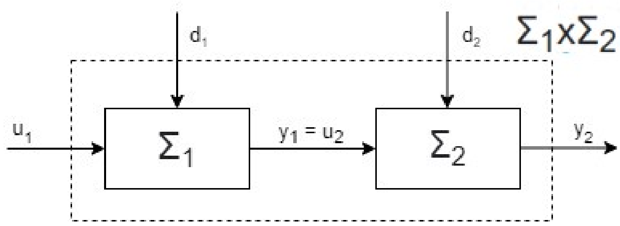

Figure 3.1: The series interconnection, $\Sigma_1 \times \Sigma_2$, of linear systems Σ_1 and Σ_2.

In order to determine the dynamics of the interconnection $\Sigma_1 \times \Sigma_2$, consider two linear systems Σ_1 and Σ_2, together with linear systems Σ_1' and Σ_2' of the following form,

$$\Sigma_i : \begin{cases} \dot{x}_i = A_i x_i + B_i u_i + E_i d_i, \\ y_i = C_i x_i. \end{cases} \qquad \Sigma_i' : \begin{cases} \dot{x}_i' = A_i' x_i' + B_i' u_i' + E_i' d_i', \\ y_i' = C_i' x_i'. \end{cases} \qquad i = 1, 2. \tag{3.1}$$

The series interconnection of Σ_1 and Σ_2 is characterized by the fact that the output of Σ_1 is used as the input of Σ_2, or more specifically $y_1(\cdot) = u_2(\cdot)$. Therefore, the dynamics of the series interconnection of Σ_1 and Σ_2 is given by a linear system of the following form, denoted by $\Sigma_1 \times \Sigma_2$:

$$\Sigma_1 \times \Sigma_2 : \begin{cases} \dot{x}_{12} = A_{12} x_{12} + B_{12} u_{12} + E_{12} d_{12}, \\ y_{12} = C_{12} x_{12}, \end{cases} \tag{3.2}$$

with state $x_{12} = (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$, input $u_{12} = u_1 \in \mathcal{U}_1$, output $y_{12} = y_2 \in \mathcal{Y}_2$ and driver for non-determinism $d_{12} = (d_1, d_2) \in \mathcal{D}_1 \times \mathcal{D}_2$. Furthermore, the linear maps in (3.2) are given by

$$\begin{aligned} A_{12} &= \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix}, & B_{12} &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \\ E_{12} &= \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, & C_{12} &= \begin{bmatrix} 0 & C_2 \end{bmatrix}. \end{aligned} \tag{3.3}$$

The series interconnection of Σ_1' and Σ_2', denoted by $\Sigma_1' \times \Sigma_2'$ can be constructed similarly. For these interconnections, the following theorem can be proven, which shows that if Σ_i is simulated by Σ_i' for $i = 1, 2$, then we also have that the series interconnection of Σ_1 and Σ_2 is simulated by the series interconnection of Σ_1' and Σ_2'.

Theorem 3.1. Consider the linear systems Σ_1, Σ_1', Σ_2 and Σ_2' as given in equation (3.1), such that $\Sigma_i \preceq \Sigma_i'$ for $i = 1, 2$. Then $\Sigma_1 \times \Sigma_2 \preceq \Sigma_1' \times \Sigma_2'$.

Proof. Since $\Sigma_i \preceq \Sigma_i'$ for $i = 1, 2$, there exist full simulation relations $\mathcal{S}_1$ and $\mathcal{S}_2$ such that the system Σ_i is simulated by Σ_i' with simulation relation $\mathcal{S}_i$. Let

$$\mathcal{S}_{12} = \{(x_1, x_2, x_1', x_2') \in \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_1' \times \mathcal{X}_2' \mid (x_1, x_1') \in \mathcal{S}_1, (x_2, x_2') \in \mathcal{S}_2\}.$$

This space clearly is a linear subspace, since $\mathcal{S}_1$ and $\mathcal{S}_2$ are linear subspaces. To prove that this subspace is a simulation relation of $\Sigma_1 \times \Sigma_2$ by $\Sigma_1' \times \Sigma_2'$, it will be shown that $\mathcal{S}_{12}$ satisfies the conditions of Lemma 2.3. For this, let $x = (x_{12}, x_{12}') = (x_1, x_2, x_1', x_2') \in \mathcal{S}_{12}$ and let $u = u_{12} = \mathcal{U}_1 \cap \mathcal{U}_1'$.

Proof of (i): Let $d = (d_1, d_2) \in \mathcal{D}_1 \times \mathcal{D}_2$. Since $\mathcal{S}_1$ is a simulation relation of Σ_1 by Σ_1', and $d_1 \in \mathcal{D}_1$, by Lemma 2.3 there exists a $d_1' \in \mathcal{D}_1'$ such that

$$(A_1 x_1 + B_1 u + E_1 d_1, A_1' x_1' + B_1' u + E_1' d_1') \in \mathcal{S}_1. \tag{3.4}$$

Furthermore, since $\mathcal{S}_2$ is a simulation relation of Σ_2 by Σ_2' and $d_2 \in \mathcal{D}_2$, together with the fact that $C_1 x_1 = C_1' x_1'$, by Lemma 2.3 there exists a $d_2' \in \mathcal{D}_2'$ such that

$$(A_2 x_2 + B_2 C_1 x_1 + E_2 d_2, A_2' x_2' + B_2' C_1' x_1' + E_2' d_2') \in \mathcal{S}_2. \tag{3.5}$$

Now, let $d' = (d'_1, d'_2)$ where d'_1 and d'_2 are as found above. For this d', it can be seen that since (3.4) and (3.5) are satisfied we have that

$$(A_{12}x_{12} + B_{12}u + E_{12}d, A'_{12}x'_{12} + B'_{12}u + E'_{12}d') = \begin{bmatrix} A_1x_1 + B_1u + E_1d_1 \\ A_2x_2 + B_2C_1x_1 + E_2d_2 \\ A'_1x'_1 + B'_1u + E'_1d'_1 \\ A'_2x'_2 + B'_2C'_1x'_1 + E'_2d'_2 \end{bmatrix} \in \mathcal{S}_{12}.$$

Therefore, it has been shown that for every $d \in \mathcal{D}_1 \times \mathcal{D}_2$ there exists a $d' \in \mathcal{D}'_1 \times \mathcal{D}'_2$ such that

$$(A_{12}x_{12} + B_{12}u + E_{12}d, A'_{12}x'_{12} + B'_{12}u + E'_{12}d') \in \mathcal{S}_{12}.$$

Proof of (ii): Note that since $\mathcal{S}_2$ is a simulation relation, the following holds,

$$C_{12}x_{12} = C_2x_2 = C'_2x'_2 = C'_{12}x'_{12},$$

which shows that $C_{12}x_{12} = C'_{12}x'_{12}$.

This proves that $\mathcal{S}_{12}$ is a simulation relation of $\Sigma_1 \times \Sigma_2$ by $\Sigma'_1 \times \Sigma'_2$. In order to prove $\Sigma_1 \times \Sigma_2 \preceq \Sigma'_1 \times \Sigma'_2$, it still needs to be proven that $\pi_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{S}_{12}) = \mathcal{X}_1 \times \mathcal{X}_2$. However, this follows straight from the fact that $\mathcal{S}_1$ and $\mathcal{S}_2$ are full simulation relations, thus implying that

$$\pi_{\mathcal{X}_1}(\mathcal{S}_1) = \mathcal{X}_1, \quad \pi_{\mathcal{X}_2}(\mathcal{S}_2) = \mathcal{X}_2.$$

Therefore, it must be the case that $\pi_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{S}_{12}) = \mathcal{X}_1 \times \mathcal{X}_2$, which shows that $\mathcal{S}_{12}$ is a full simulation relation. This concludes the proof of this theorem. □

From the results and the proof of this theorem, it can be seen that a similar result also holds for the series interconnection of a finite number of linear systems of the form as in (3.1), as is stated in the following corollary.

Corollary 3.2. Consider n linear systems $\Sigma_1, \Sigma_2, ..., \Sigma_n$ of the form as in (3.1). If there exists n linear systems $\Sigma'_1, \Sigma'_2, ..., \Sigma'_n$ such that $\Sigma_i \preceq \Sigma'_i$, then the series interconnection of $\Sigma_1, ..., \Sigma_n$ is simulated by the series interconnection of $\Sigma'_1, ..., \Sigma'_n$.

3.2 Feedback Interconnection

Another type of interconnection that can be considered is the feedback interconnection of two linear systems, as represented in Figure 3.2.

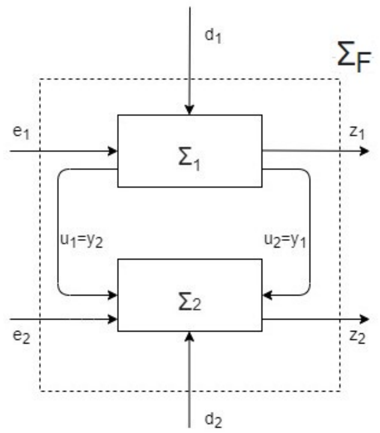

Figure 3.2: The feedback interconnection, Σ_C, of two linear systems Σ_1 and Σ_2.

For this type of interconnection, it can be seen that for $i = 1, 2$, Σ_i has an additional input represented by $e_i \in \mathcal{E}_i$, as well as an additional output, given by $z_i \in \mathcal{Z}_i$. With these additional input and output, the dynamics of the linear systems are of the following form:

$$\Sigma_i : \begin{cases} \dot{x}_i = A_i x_i + B_i u_i + G_i e_i + E_i d_i, \\ y_i = C_i x_i, \\ z_i = H_i x_i. \end{cases} \qquad \Sigma_i' : \begin{cases} \dot{x}_i' = A_i' x_i' + B_i' u_i' + G_i' e_i' + E_i' d_i', \\ y_i' = C_i' x_i', \\ z_i' = H_i' x_i'. \end{cases} \tag{3.6}$$

When considering the feedback interconnection of Σ_1 and Σ_2 as shown in Figure 3.2, which is represented by $u_1(\cdot) = y_2(\cdot)$ and $u_2(\cdot) = y_1(\cdot)$, it can be seen that the dynamics of the feedback interconnection, Σ_F, is given by a linear system of the following form:

$$\Sigma_F : \begin{cases} \dot{x}_F = A_F x_F + G_F e_F + E_F d_F, \\ z_F = H_F x_F, \end{cases} \tag{3.7}$$

with state $x_F = (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$, external input $e_F = (e_1, e_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, output $z_F = (z_1, z_2) \in \mathcal{Z}_1 \times \mathcal{Z}_2$ and driver for non-determinism $d_F = (d_1, d_2) \in \mathcal{D}_1 \times \mathcal{D}_2$. Furthermore, the linear maps in (3.7) are given by

$$\begin{aligned} A_F &= \begin{bmatrix} A_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix}, & G_F &= \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}, \\ E_F &= \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, & H_F &= \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}. \end{aligned} \tag{3.8}$$

The feedback interconnection of Σ_1' and Σ_2', represented by Σ_F', can be constructed in the exact same way. For these linear systems, there is the following theorem.

Theorem 3.3. Consider the linear systems Σ_1, Σ_2, Σ_1' and Σ_2' as given in (3.6). If $\Sigma_i \preceq \Sigma_i'$ for $i = 1, 2$, then $\Sigma_F \preceq \Sigma_F'$.

Proof. Since $\Sigma_1 \preceq \Sigma_1'$ and $\Sigma_2 \preceq \Sigma_2'$, there exists full simulation relations $\mathcal{S}_1$ and $\mathcal{S}_2$, such that for $i = 1, 2$ the system Σ_i is simulated by Σ_i' with simulation relation $\mathcal{S}_i$. Let

$$\mathcal{S}_F = \{(x_1, x_2, x_1', x_2') \in \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_1' \times \mathcal{X}_2' \mid (x_1, x_1') \in \mathcal{S}_1, (x_2, x_2') \in \mathcal{S}_2\}$$

This space clearly is a linear subspace, since $\mathcal{S}_1$ and $\mathcal{S}_2$ are linear subspaces. To proof that this subspace is a simulation relation of Σ_F by Σ_F', it will be shown that this subspace satisfies the conditions of Lemma 2.3. For this, let $x = (x_F, x_F') = (x_1, x_2, x_1', x_2') \in \mathcal{S}_F$ and let $e = (e_1, e_2) \in \mathcal{E}_1 \times \mathcal{E}_2 \cap \mathcal{E}_1' \times \mathcal{E}_2'$.

Proof of (i): Let $d = (d_1, d_2) \in \mathcal{D}_1 \times \mathcal{D}_2$. Since $\mathcal{S}_1$ is a simulation relation of Σ_1 by Σ_1', and $d_1 \in \mathcal{D}_1$, we have that $C_2 x_2 = C_2' x_2'$, thus by Lemma 2.3 there exists a $d_1' \in \mathcal{D}_1'$ such that

$$(A_1 x_1 + B_1 C_2 x_2 + G_1 e_1 + E_1 d_1, A_1' x_1' + B_1' C_2' x_2' + G_1' e_1 + E_1' d_1') \in \mathcal{S}_1. \tag{3.9}$$

Furthermore, since $\mathcal{S}_2$ is a simulation relation of Σ_2 by Σ_2' and $d_2 \in \mathcal{D}_2$, we have that $C_1 x_1 = C_1' x_1'$ and thus by Lemma 2.3 there exists a $d_2' \in \mathcal{D}_2'$ such that

$$(A_2 x_2 + B_2 C_1 x_1 + G_2 e_2 + E_2 d_2, A_2' x_2' + B_2' C_1' x_1' + G_2' e_2 + E_2' d_2') \in \mathcal{S}_2. \tag{3.10}$$

Now, let $d' = (d_1', d_2')$ where d_1' and d_2' are as found above. For this d', it can be seen that since (3.9) and (3.10) are satisfied, the following holds:

$$(A_F x_F + G_F e + E_F d, A_F' x_F' + G_F' e + E_F' d') = \begin{bmatrix} A_1 x_1 + B_1 C_2 x_2 + G_1 e_1 + E_1 d_1 \\ A_2 x_2 + B_2 C_1 x_1 + G_2 e_2 + E_2 d_2 \\ A_1' x_1' + B_1' C_2' x_2' + G_1' e_1 + E_1' d_1' \\ A_2' x_2' + B_2' C_1' x_1' + G_2' e_2 + E_2' d_2' \end{bmatrix} \in \mathcal{S}_F.$$

Therefore, it has been shown that for every $d \in \mathcal{D}_1 \times \mathcal{D}_2$ there exists a $d' \in \mathcal{D}_1' \times \mathcal{D}_2'$ such that

$$(A_F x_F + G_F e + E_F d, A_F' x_F' + G_F' e + E_F' d') \in \mathcal{S}_F.$$

Proof of (ii): Note that since both $\mathcal{S}_1$ and $\mathcal{S}_2$ are simulation relations, the following holds,

$$H_F x_F = H_1 x_1 + H_2 x_2 = H_1' x_1' + H_2' x_2' = H_F' x_F',$$

which shows that $H_F x_F = H_F' x_F'$.

This proves that $\mathcal{S}_F$ is a simulation relation of Σ_F by Σ_F'. In order to proof $\mathcal{S}_F$ is a full simulation relation it still needs to be proven that $\pi_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{S}_F) = \mathcal{X}_1 \times \mathcal{X}_2$. However, this follows straight from the fact that $\mathcal{S}_1$ and $\mathcal{S}_2$ are full simulation relations, thus implying that

$$\pi_{\mathcal{X}_1}(\mathcal{S}_1) = \mathcal{X}_1, \quad \pi_{\mathcal{X}_2}(\mathcal{S}_2) = \mathcal{X}_2.$$

Therefore, it must also be the case that $\pi_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{S}_F) = \mathcal{X}_1 \times \mathcal{X}_2$. This concludes the proof of this theorem. $\square$

3.3 Interconnection by External Variables

In this section, we will consider the interconnection of linear systems through the external variables. For this, we consider two linear systems Σ_1 and Σ_2 of the following form

$$\Sigma_i : \begin{cases} \dot{x}_i = A_i x_i + E_i d_i, \\ w_i = C_i x_i, \\ 0 = H_i x_i, \end{cases} \qquad i \in \{1, 2\}. \tag{3.11}$$

These two linear systems can be interconnected by setting

$$w_1 = w_2. \tag{3.12}$$

When doing this, we denote their composition by $\Sigma_1 \otimes \Sigma_2$. The dynamics of this interconnection is of the following form

$$\Sigma_1 \otimes \Sigma_2 : \begin{cases} \dot{x}^{\otimes} = A^{\otimes} x^{\otimes} + E^{\otimes} d^{\otimes}, \\ w^{\otimes} = C^{\otimes} x^{\otimes}, \\ 0 = H^{\otimes} x^{\otimes}. \end{cases} \tag{3.13}$$

with state $x^{\otimes} = (x_1^{\otimes}, x_2^{\otimes}) \in \mathcal{X}_1 \times \mathcal{X}_2$, external variable $w^{\otimes} \in \mathcal{W}_1 \cap \mathcal{W}_2$ and driver for non-determinism $d^{\otimes} = (d_1^{\otimes}, d_2^{\otimes}) \in \mathcal{D}_1 \times \mathcal{D}_2$. Furthermore, the linear maps in (3.13) are given by

$$\begin{aligned} A^{\otimes} &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad E^{\otimes} = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \\ C^{\otimes} &= \frac{1}{2} \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \qquad H^{\otimes} = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \\ C_1 & -C_2 \end{bmatrix}, \end{aligned} \tag{3.14}$$

where the last row of the linear map $H^{\otimes}$ restricts the external behaviour of the system to satisfy the characterization of the interconnection as given in equation (3.12). For the composition $\Sigma_1 \otimes \Sigma_2$, remember that the consistent subspace, $\mathcal{V}^{\otimes *}$ is the largest subspace $\mathcal{V}^{\otimes} \subset \mathcal{X}_1 \times \mathcal{X}_2$ that satisfies

$$A^{\otimes} \mathcal{V}^{\otimes} \subset \mathcal{V}^{\otimes} + \text{im } E^{\otimes}, \qquad \mathcal{V}^{\otimes} \subset \ker H^{\otimes}.$$

In the next result, it will be shown that the composition $\Sigma_1 \otimes \Sigma_2$ is simulated by both Σ_1 and Σ_2. The proof of this lemma can be found in Appendix A.

Lemma 3.4. Consider two linear systems Σ_1 and Σ_2 of the form as in equation (3.11), and consider the interconnection through external variables as given in (3.13) and (3.14). Then the following two statements hold:

(i) for $i \in \{1, 2\}, \Sigma_1 \otimes \Sigma_2 \preceq \Sigma_i$; (3.15a)

(ii) If Σ is a system of the form (3.11) that satisfies $\Sigma \preceq \Sigma_i$ for $i \in \{1, 2\}$, then $\Sigma \preceq \Sigma_1 \otimes \Sigma_2$. (3.15b)

With the result of this lemma, we can prove the following theorem, which tells us that if we have two systems Σ_1 and Σ_2, which are simulated by Σ_1' and Σ_2' respectively, then we also have that the interconnection $\Sigma_1 \otimes \Sigma_2$ is simulated by the interconnection $\Sigma_1' \otimes \Sigma_2'$.

Theorem 3.5. Consider the linear systems Σ_1, Σ_2, Σ_1' and Σ_2', all of the form as given in equation (3.11). If $\Sigma_i \preceq \Sigma_i'$ for $i = \{1, 2\}$, then $\Sigma_1 \otimes \Sigma_2 \preceq \Sigma_1' \otimes \Sigma_2'$.

Proof. For $i = \{1, 2\}$, let Σ_i and Σ_i' be linear systems of the form as given in (3.11) such that $\Sigma_i \preceq \Sigma_i'$. From the first statement of Lemma 3.4 we have that $\Sigma_1 \otimes \Sigma_2 \preceq \Sigma_i \preceq \Sigma_i'$ for $i = \{1, 2\}$. Therefore, we can conclude that $\Sigma_1 \otimes \Sigma_2 \preceq \Sigma_i'$. Then, by the second statement of Lemma 3.4, we can conclude that $\Sigma_1 \otimes \Sigma_2 \preceq \Sigma_1' \otimes \Sigma_2'$. $\square$

In this chapter, we have studied the notion of simulation with respect to various specific types of interconnected linear systems. For this types of interconnections, we have shown that the property of simulation passes on to interconnections as well, in the sense that if two systems Σ_1 and Σ_2 are simulated by Σ_1' and Σ_2' respectively, we also have that the interconnection of Σ_1 and Σ_2 is simulated by the interconnection of Σ_1' and Σ_2'. Since modern engineering systems are typically constructed of many components, which are interconnected with each other, the results obtained in this chapter are very useful, as they can be found to be a powerful tool for various engineering purposes.

Chapter 4

Contract Theory

A reoccurring problem with modern engineering systems that are comprised of many interconnected components, is that the manufacturing of these components is done independently from each other. Due to this, the environment in which these components will eventually act is often not taken into account. As this environment will eventually consist of the other components of the system, it might be the case that the individual components are not consistent with each other, meaning that interconnection of the subsystems might not be possible. Moreover, it might also be the case that we wish the total system to have specific properties, which we can only achieve if the individual components have specific properties or requirements. These aforementioned problems occurring during the construction of a complex system might be solved by developing a mathematical framework that allows us to impose specifications onto linear systems, making it possible to control the dynamics of the components in such a way that the components are consistent with each other. Contract theory is a framework that provides this. This framework is built around assume-guarantee contracts, which can be seen as a way to describe expected input behaviour and desired output behaviour of a system, thus allowing us to specify the external behaviour of the systems. The origin of this notion lies within the field of Computer Science, and (Benveniste et al. 2018) provides a framework for contract theory with regards to this field. Within this chapter, the first steps will be made to define a similar framework in the context of continuous-time linear systems. This will be done by using the notion of simulation, as can be seen in Section 4.1. After that, in Section 4.2, the framework of contract theory will be studied further by looking into the concept of contract refinement.

4.1 Contract Theory

Consider a linear system of the following form:

$$\Sigma : \begin{cases} \dot{x}_\Sigma = A_\Sigma x_\Sigma + B_\Sigma u_\Sigma + E_\Sigma d_\Sigma, \\ y_\Sigma = C_\Sigma x_\Sigma. \end{cases} \tag{4.1}$$

Since Σ is an open system, it interconnects with its surroundings, which provide the input trajectories of the system. These surroundings can be represented as a linear system, called the environment, which has the following form:

$$E : \begin{cases} \dot{x}_E = A_E x_E + E_E d_E, \\ y_E = C_E x_E. \end{cases} \tag{4.2}$$

The interconnection between the linear system Σ and its environment E is illustrated in Figure 4.1. In this figure, it can be seen that the environment provides the input of the linear system Σ, and that both the input and output of the linear system Σ are chosen as the output of the interconnected system $E \times \Sigma$.

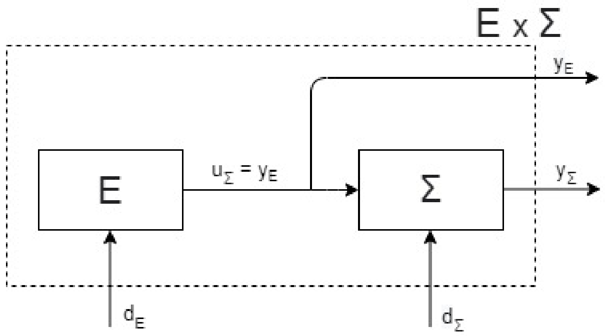

Figure 4.1: The interconnection of a linear system Σ with its environment E.

For the linear system Σ and its environment E as given in (4.1) and (4.2) respectively, the dynamics of the interconnected system $E \times \Sigma$, as illustrated in Figure 4.1, is given by the following linear system.

$$E \times \Sigma : \begin{cases} \dot{x}_{E\Sigma} = A_{E\Sigma} x_{E\Sigma} + E_{E\Sigma} d_{E\Sigma}, \\ y_{E\Sigma} = C_{E\Sigma} x_{E\Sigma}, \end{cases} \tag{4.3}$$

with state $x_{E\Sigma} = (x_E, x_\Sigma) \in \mathcal{X}_E \times \mathcal{X}_\Sigma$, driver for non-determinism $d_{E\Sigma} = (d_E, d_\Sigma) \in \mathcal{D}_E \times \mathcal{D}_\Sigma$ and output $y_{E\Sigma} = (y_E, y_\Sigma) \in \mathcal{Y}_E \times \mathcal{Y}_\Sigma$. Furthermore, the linear maps in (4.3) are given by

$$A_{E\Sigma} = \begin{bmatrix} A_E & 0 \\ B_\Sigma C_E & A_\Sigma \end{bmatrix}, \quad E_{E\Sigma} = \begin{bmatrix} E_E & 0 \\ 0 & E_\Sigma \end{bmatrix}, \quad C_{E\Sigma} \begin{bmatrix} C_E & 0 \\ 0 & C_\Sigma \end{bmatrix}. \tag{4.4}$$

Even though it is quite complicated to fully control and steer the input trajectories of the environment, it is possible to express which properties the output trajectories of the environment are expected to satisfy. These properties can be represented by a linear system, A, called the assumptions. This linear system, A, has a similar form as the environment, E, and can therefore be represented in the following way:

$$A : \begin{cases} \dot{x}_A = A_A x_A + E_A d_A, \\ y_A = C_A x_A. \end{cases} \tag{4.5}$$

Aside from this, it is possible to express what external behaviour is desired from the interconnection of a linear system with its environment. This can also be done using a linear system, called the guarantees, G, which is of the form as given below. Note that the output of this system consists of two parts, as the output of the interconnection $E \times \Sigma$ does too.

$$G : \begin{cases} \dot{x}_G = A_G x_G + E_G d_G, \\ \begin{bmatrix} u_G \\ y_G \end{bmatrix} = \begin{bmatrix} C_G^u \\ C_G^y \end{bmatrix} x_G = C_G x_G. \end{cases} \tag{4.6}$$

The linear systems that describe the assumptions and guarantees together form an assume-guarantee contract.

Definition 4.1. An (assume-guarantee) contract, $\mathcal{C} = (A, G)$ is a pair of assumptions, A, and guarantees, G, where A and G are linear systems of the form as given in (4.5) and (4.6) respectively.

To put differently, a contract is a pair of linear systems, of which the first describes which input behaviour is expected from the environment in which the linear system will act, and the second describes which external behaviour is desired from the interconnection of the system with its environment. From this, it can be seen that a contract defines two different sets of linear systems. First of all, it defines a set of environments of the form as in (4.2) such that these systems satisfy the assumptions, that is, the set of output trajectories of an environment within this set should be contained in the set of output trajectories of the assumptions. The second set consists of linear systems of the form as in (4.1) such that when these linear systems are interconnection with an environment of the first set, the interconnected system satisfies the guarantees, and thus the set of output trajectories is contained in the set of output trajectories of the guarantees. In the following definition, these two sets are described a bit more precisely.

Definition 4.2. Consider a linear system Σ of the form as is given in (4.1), together with an environment E as given in (4.2). Furthermore, let $\mathcal{C} = (A, G)$ be a contract. Then there are the following definitions:

1) The environment E is called compatible with contract $\mathcal{C}$ if $E \preceq A$.
2) The linear system Σ is said to implement the contract $\mathcal{C}$ if $E \times \Sigma \preceq G$ for all environment E that are compatible with $\mathcal{C}$. If Σ implements the contract $\mathcal{C}$, we also refer to Σ as an implementation of $\mathcal{C}$
3) The contract $\mathcal{C}$ is called consistent if there exists at least one implementation of $\mathcal{C}$.

Since the aim of an assume-guarantee contract is to impose specifications on a linear system by controlling the external behaviour of the system, it is useful to determine whether there exists a linear system that can satisfy the assumptions and guarantees of a given contract, or put differently, it is useful to determine whether a contract is consistent. However, from the definition, it is quite difficult to determine whether a linear system implements a contract. Fortunately, there is the following lemma, which provides an easier way to determine this.

Lemma 4.3. Consider a linear system Σ and its environment E as given in equations (4.1) and (4.2). Then Σ implements the contract $\mathcal{C} = (A, G)$ if and only if $A \times \Sigma \preceq G$.

Proof. ($\Rightarrow$) Suppose Σ implements the contract $\mathcal{C} = (A, G)$. Then, using the definition, this means that $E \times \Sigma \preceq G$ for all environments E that satisfy $E \preceq A$. Since $A \preceq A$ as was shown in Lemma 2.5, it must be the case that $A \times \Sigma \preceq G$.

($\Leftarrow$) Assume that $A \times \Sigma \preceq G$ and let E be an environment such that $E \preceq A$. Since $\Sigma \preceq \Sigma$, we have by Theorem 3.1 that

$$E \times \Sigma \preceq A \times \ \Sigma.$$

Therefore, by the transitivity of the property of the simulation relation, see Lemma 2.5, it is the case that $E \times \Sigma \preceq G$. This shows that $E \times \Sigma \preceq G$ for all compatible environments E, and thus by definition we have that Σ implements the contract $\mathcal{C}$. $\square$

The above lemma explains which conditions a linear system Σ needs to satisfy in order to be an implementation of the contract $\mathcal{C} = (A, G)$. However, it is not always the case that there exists an implementation for a contract. The next lemma gives a necessary condition for a contract to be consistent, and thus whether an implementation for that contract exists.

Theorem 4.4. Consider a contract $\mathcal{C} = (A, G)$, where A and G are given as in equations (4.5) and (4.6) respectively. The contract $\mathcal{C}$ is consistent only if $A \preceq G^u$, where G^u describes the input behaviour of the guarantees G and is given by the following linear system:

$$G^u : \begin{cases} \dot{x}_G = A_G x_G + E_G d_G; \\ u_G = C_G^u x_G. \end{cases}$$

Proof. Since the proof of this lemma follows a very similar reasoning as the proof of previous theorems, only a sketch of the proof will be given.
Suppose the contract $\mathcal{C} = (A, G)$ is consistent. Then, by Lemma 4.3 there exists a linear system Σ such that $A \times \Sigma \preceq G$. Therefore, there exists a full simulation relation $\mathcal{S}$ of $A \times \Sigma$ by G. Now consider the following subspace $\mathcal{S}^u \subset \mathcal{X}_A \times \mathcal{G}$:

$$\mathcal{S}^u = \{(x_A, x_G) \in \mathcal{X}_A \times \mathcal{X}_G \mid \text{there exists a } x_\Sigma \text{ such that } (x_A, x_\Sigma, x_G) \in \mathcal{S}\}$$

Then, it can very easily be seen that this subspace satisfies the properties of Lemma 2.3, which means that $\mathcal{S}^u$ is a simulation relation of A by G^u. Furthermore, it can be shown that $\pi_{\mathcal{X}_A}(\mathcal{S}^u) = \mathcal{X}_A$, thus implying that $A \preceq G^u$. $\square$

Until now, we have only regarded contracts for single linear systems. However, the framework of contract theory can also be applied to the interconnection of one or more linear system. Even though this subject is beyond the scope of this article, an example of a result for interconnected linear systems can be found in the following theorem, where we recall that $\otimes$ indicates a interconnection by external variables as represented in (3.13) and (3.14). The proof of this theorem can be found in Appendix A

Theorem 4.5. Let Σ_1 be an implementation of the contract $\mathcal{C}_1 = (A_1, G_1)$ and let Σ_2 be an implementation of the contract $\mathcal{C}_2 = (A_2, G_2)$. Then the interconnected linear system $\Sigma_1 \otimes \Sigma_2$ is an implementation of the contract $\mathcal{C} = (A_1 \otimes A_2, G_1 \otimes G_2)$.

4.2 Contract Refinement

An important concept within contract theory is the notion of contract refinement. This notion makes it possible to compare contracts with each other. The following definition explains what it means for a contract $\mathcal{C}_1$ to refine another contract $\mathcal{C}_2$.

Definition 4.6. Let $\mathcal{C}_1 = (A_1, G_1)$ and $\mathcal{C}_2 = (A_2, G_2)$ be two contracts as in the form of Definition 4.1. The contract $\mathcal{C}_1$ is said to refine the contract $\mathcal{C}_2$ if the following conditions are satisfied:

1) If E is a compatible environment of $\mathcal{C}_2$, then E is a compatible environment of $\mathcal{C}_1$;
2) If Σ is an implementation of $\mathcal{C}_1$, then Σ is an implementation of $\mathcal{C}_2$.

Put differently, this definition says that if a contract $\mathcal{C}_1$ refines the contract $\mathcal{C}_2$, it means that $\mathcal{C}_1$ specifies stricter guarantees that have to be satisfied under weaker assumptions. Indeed, if $\mathcal{C}_1$ refines $\mathcal{C}_2$, $\mathcal{C}_1$ allows more compatible environments than $\mathcal{C}_2$, but on the other hand has less implementations than $\mathcal{C}_2$. Therefore $\mathcal{C}_1$ can be seen as expressing a stricter specification than $\mathcal{C}_2$. From the definition, it can be rather complicated to determine whether a contract refines another contract. However, it is possible to construct sufficient conditions under which a contract $\mathcal{C}_1$ refines another contract $\mathcal{C}_2$ (Shali, Van der Schaft, and Besselink 2021). Before being able to state these conditions however, the following lemma needs to be proven. This lemma tells us that if Σ is an implementation of the contract $\mathcal{C} = (A, G)$, then Σ is also an implementation of the contract $\mathcal{C}' = (A, A \otimes G)$. Recall that the interconnection $A \otimes G$ is given by the following linear system:

$$A \otimes G : \begin{cases} \begin{bmatrix} \dot{x}_A \\ \dot{x}_G \end{bmatrix} = \begin{bmatrix} A_A & 0 \\ 0 & A_G \end{bmatrix} \begin{bmatrix} x_A \\ x_G \end{bmatrix} + \begin{bmatrix} E_A & 0 \\ 0 & E_G \end{bmatrix} \begin{bmatrix} d_A \\ d_g \end{bmatrix}, \\ \begin{bmatrix} u_G \\ y_G \end{bmatrix} = \begin{bmatrix} C_A & 0 \\ 0 & C_G^y \end{bmatrix} \begin{bmatrix} x_A \\ x_G \end{bmatrix}, \\ 0 = \begin{bmatrix} C_A & -C_G^u \end{bmatrix} \begin{bmatrix} x_A \\ x_G \end{bmatrix}. \end{cases}$$

The result of the following lemma plays an important role in the establishment of sufficient conditions for refinement. The proof of this lemma can be found in Appendix A.

Lemma 4.7. Consider the contract $\mathcal{C} = (A, G)$ where A and G are given as in (4.5) and (4.6). If $A \times \Sigma \preceq G$, then $A \times \Sigma \preceq A \otimes G$.

With the result of this lemma, it becomes possible to establish sufficient conditions for a contract to refine another. These conditions are stated in the following theorem.

Theorem 4.8. Consider two contracts $\mathcal{C}_1 = (A_1, G_2)$ and $\mathcal{C}_2 = (A_2, G_2)$, where A_1 and A_2 are of the form as in (4.5) and G_1 and G_2 are as given in (4.6). Suppose $\mathcal{C}_1$ is consistent. Then $\mathcal{C}_1$ refines $\mathcal{C}_2$ if $A_2 \preceq A_1$ and $A_2 \otimes G_1 \preceq G_2$.

Proof. In order to prove this theorem, it will be shown that the conditions of Definition 4.6 are satisfied. Suppose that $\mathcal{C}_1 = (A_1, G_1)$ is consistent, and let E be a compatible environment of $\mathcal{C}_2$. Therefore, by definition, this implies that $E \preceq A_2$. However, since the property of simulation is transitive, and $A_2 \preceq A_1$, we have that $E \preceq A_1$, from which we can conclude that E is also a compatible environment of $\mathcal{C}_1$.

Furthermore, let Σ be an implementation of $\mathcal{C}_1$. By Lemma 4.3, this means that $A_1 \times \Sigma \preceq G_1$. From the previous lemma, we can conclude that we also have that $A_1 \times \Sigma \preceq A_1 \otimes G_1$. Now consider the interconnection $A_2 \times \Sigma$. Since $A_2 \preceq A_1$, we have by Theorem 3.1 that $A_2 \times \Sigma \preceq A_1 \times \Sigma$. By the reasoning before, we see that $A_2 \times \Sigma \preceq A_1 \otimes G_1 \preceq G_2$, where the last simulation relation follows from one of the assumptions. Therefore, we have that $A_2 \times \Sigma \preceq G_2$, thus showing that Σ is also an implementation of $\mathcal{C}_2$. Since both conditions of Definition 4.6 are satisfied, it can be concluded that $\mathcal{C}_1$ refines $\mathcal{C}_2$. $\square$

In this chapter, we introduced assume-guarantee contracts for linear systems, which can be seen as a pair of linear systems describing the expected output behaviour of the environment in which a linear system will act, as well as describing the desired output behaviour for the linear system when interconnected with its environment. By introducing these contracts, we can give precise specifications to the external behaviour of linear systems. This framework of contract theory, in which the notion of simulation plays a very important role, proves to be useful when working with complex engineering systems, as it allows us to give precise specifications on system components, ensuring that the individual components are consistent, in the sense that interconnection between them is possible and that the total interconnected system satisfies desired external behaviour.

Chapter 5

Conclusion

In this paper, we have provided the first steps towards a mathematical framework of contract theory for continuous-time linear systems. In order to do this, we first introduced the notion of simulation as a way to compare linear systems. This notion tells us that a linear system is simulated by another if every state trajectory of the former system can be matched by a state trajectory of the latter, in the sense that the input-output data are equal. This notion proved to be a powerful tool in the framework of contract theory, as it allowed us to determine whether a linear system satisfied the external behaviour as imposed by the specification of a contract.

In Chapter 3 we applied the notion of simulation to interconnected linear systems using the results already obtained. We looked at various interconnections and for all these we found that the following property holds: If Σ_1 and Σ_2 are simulated by Σ_1' and Σ_2' respectively, than we also have that the interconnection of Σ_1 and Σ_2 is simulated by the interconnection of Σ_1' and Σ_2'. This is a powerful result, as this shows us that the notion of simulation naturally passes over from single linear systems to interconnected linear systems. Furthermore, this results shows that the notion of simulation can also be used to compare interconnected linear systems.

Lastly, in Chapter 4, we introduced a way to impose specifications on linear systems with the use of the notion of assume-guarantee contracts. These can be regarded as a pair of linear systems: assumptions, which describes the expected input behaviour of the environment to a linear system, and guarantees, which represent the desired output behaviour of the system when interacting with its environment. Hence, a contract defines a set of compatible environments, as well a set of implementations, which can be seen as linear systems that can adhere to the assumptions and guarantees of the contract. For these contracts, we have established results which give us conditions under which a linear system is a compatible environment, as well as an implementation of the system. Moreover, we determined which conditions a contract needs to satisfy in order to be consistent, that is, to have an implementation. Lastly, we briefly looked into the notion of contract refinement, which can be seen as a way to compare contracts with each other.

This paper can be seen as providing the first steps towards a mathematical framework of contract theory for continuous-time linear systems with the use of the notion of simulation. Within this paper, we have established that the notion of simulation naturally passes over to interconnected linear systems. Furthermore, we have shown that we can construct the concept of assume-guarantee contracts where simulation is used as a way to compare linear systems. However, not much has been said about contract theory with regards to interconnected linear systems. As complex systems typically consist of many interconnected components, this is an interesting and relevant topic for further research.

Appendix A

Proofs

Proof of Lemma 2.5. The two parts of this lemma will be proven individually.
Proof of (2.5a): Consider the following subspace:

$$\mathcal{R} = \{(x_1, \bar{x}_1) \in \mathcal{X}_1 \times \mathcal{X}_1 \mid x_1 = \bar{x}_1\}.$$

Let $(x_1, \bar{x}_1) \in \mathcal{R}$, $u \in \mathcal{U}_1$ and let $d_1 \in \mathcal{D}_1$. When picking $\bar{d}_1 = d_1$ we have that

$$(A_1x_1+B_1u+E_1d_1, A_1\bar{x}_1+B_1u+E_1\bar{d}_1) = (A_1x_1+B_1u+E_1d_1, A_1x_1+B_1u+E_1d_1) \in \mathcal{R}.$$

Furthermore, since $(x_1, \bar{x}_1) \in \mathcal{R}$, we have that $C_1x_1 = C_1\bar{x}_1$ and clearly we have that $\pi_{\mathcal{X}_1}(\mathcal{R}) = \mathcal{X}_1$. Therefore, we have shown that $\mathcal{R}$ is a full simulation relation of Σ_1 by Σ_1, which means that $\Sigma_1 \preceq \Sigma_1$.
Proof of (2.5b): Suppose $\Sigma_1 \preceq \Sigma_2$ with full simulation relation $\mathcal{S}_{12}$ and that $\Sigma_2 \preceq \Sigma_3$ with full simulation relation $\mathcal{S}_{23}$. Now, consider the following subspace:

$$\mathcal{S}_{13} = \{(x_1, x_3) \in \mathcal{X}_1 \times \mathcal{X}_3 \mid \exists\, x_2 \in \mathcal{X}_2 \text{ such that } (x_1, x_2) \in \mathcal{S}_{12} \text{ and } (x_2, x_3) \in \mathcal{S}_{23}\}.$$

Now, let $(x_1, x_3) \in \mathcal{S}_{13}$, let $u \in \mathcal{U}_1 \cap \mathcal{U}_3$ and let $d_1 \in \mathcal{D}_1$. Since $\mathcal{S}_{12}$ is a simulation relation, there exists a $d_2 \in \mathcal{D}_2$ such that

$$(A_1x_1 + B_1u + E_1d_1, A_2x_2 + B_2u + E_2d_2) \in \mathcal{S}_{12}.$$

For this d_2, we have that there exists a $d_3 \in \mathcal{D}_3$ such that

$$(A_2x_2 + B_2u + E_2d_2, A_3x_3 + B_3u + E_3d_3) \in \mathcal{S}_{23},$$

since $\mathcal{S}_{23}$ is a simulation relation. This however shows that there exists a $d_3 \in \mathcal{D}_3$ such that

$$(A_1x_1 + B_1u + E_1d_1, A_3x_3 + B_3u + E_3d_3) \in \mathcal{S}_{13}.$$

Furthermore, since $(x_1, x_3) \in \mathcal{S}_{13}$, there exists a $x_2 \in \mathcal{X}_2$ such that

$$C_1x_1 = C_2x_2 = C_3x_3.$$

Therefore, we have shown that $\mathcal{S}_{13}$ is a simulation relation of Σ_1 by Σ_3. Lastly, since $\pi_{\mathcal{X}_1}(\mathcal{S}_{12}) = \mathcal{X}_1$ and $\pi_{\mathcal{X}_2}(\mathcal{S}_{23}) = \mathcal{X}_2$, we also have that $\pi_{\mathcal{X}_1}(\mathcal{S}_{13}) = \mathcal{X}_1$. Therefore, we have shown that $\Sigma_1 \preceq \Sigma_3$. □

Proof of Lemma 3.4. The two statements in (i) and (ii) will be proven separately. *Proof of (i)* The proof of this statement will only be shown for $i = 1$. The proof for $i = 2$ will then following similarly.

Consider the following subspace $\mathcal{S} \subset \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_1$:

$$\mathcal{S} = \{(x_1^{\otimes}, x_2^{\otimes}, x_1) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_1 \mid (x_1^{\otimes}, x_2^{\otimes}) \in \mathcal{V}^{\otimes *}, x_1 = x_1^{\otimes}\}.$$

For this subspace it is clear that $\pi_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{S}) = \mathcal{V}^{\otimes *}$. Furthermore, we have that since $x_1 = x_1^{\otimes}$ for all $(x_1^{\otimes}, x_2^{\otimes}, x_1) \in \mathcal{S}$, we also have that $\pi_{\mathcal{X}_1}(\mathcal{S}) = \pi_{\mathcal{X}_1}(\mathcal{V}^{\otimes *})$. Therefore, by the properties of the consistent subspace and the way the linear maps are defined in (3.14), we have that

$$A_1 \pi_{\mathcal{X}_1}(\mathcal{V}^{\otimes *}) \subset \pi_{\mathcal{X}_1}(\mathcal{V}^{\otimes *}) + \text{im } E_1, \qquad \pi_{\mathcal{X}_1}(\mathcal{V}^{\otimes *}) \subset \ker H_1.$$

This shows that $\pi_{\mathcal{X}_1}(\mathcal{V}^{\otimes *}) \subset \mathcal{V}_1^*$, and thus also $\pi_{\mathcal{X}_1}(\mathcal{S}) \subset \mathcal{V}_1^*$.

The only thing that still needs to be proven is whether $\mathcal{S}$ satisfies the properties of a simulation relation as stated in Lemma 2.9. For this, let $(x^{\otimes}, x_1) = (x_1^{\otimes}, x_2^{\otimes}, x_1) \in \mathcal{S}$ and let $d^{\otimes} = (d_1^{\otimes}, d_2^{\otimes})$ be such that $A^{\otimes} x^{\otimes} + E^{\otimes} d^{\otimes} \in \mathcal{V}^{\otimes *}$. Since $x_1 = x_1^{\otimes}$, it can be seen that when choosing $d_1 = d_1^{\otimes}$, we have that

$$(A_1 x_1^{\otimes} + E_1 d_1^{\otimes}, A_2 x_2^{\otimes} + E_2 d_2^{\otimes}, A_1 x_1 + E_1 d_1) \in \mathcal{S}.$$

This indeed ensures that $A_1 x_1 + E_1 d_1 \in \mathcal{V}_1^*$, since $A_1 x_1 + E_1 d_1 \in \pi_{\mathcal{X}_1}(S) \subset \mathcal{V}_1^*$ by the reasoning above.

Furthermore, since $(x_1^{\otimes}, x_2^{\otimes}, x_1) \in \mathcal{S}$, we have that $(x_1^{\otimes}, x_2^{\otimes}) \in \mathcal{V}^{\otimes *}$ and thus also that $(x_1^{\otimes}, x_2^{\otimes}) \in \ker H^{\otimes}$. From the way $H^{\otimes}$ is defined, we see that $C_1 x_1^{\otimes} = C_2 x_2^{\otimes}$. Therefore, and since $x_1^{\otimes} = x_1$, we get that $C^{\otimes} x^{\otimes} = \frac{1}{2}(C_1 x_1^{\otimes} + C_2 x_2^{\otimes}) = C_1 x_1$.
Thus we see that $\mathcal{S}$ satisfies all properties of a simulation relation, from which we can conclude that $\Sigma_1 \otimes \Sigma_2 \preceq \Sigma_1$.

Proof of (ii) Let Σ be a system of the form as given in equation (3.11), but it will be represented without the indices. Furthermore, assume that $\Sigma \preceq \Sigma_i$ with simulation relation $\mathcal{S}_i$, and consider the following linear subspace

$$\mathcal{S} = \{(x, x_1^{\otimes}, x_2^{\otimes}) \in \mathcal{X} \times \mathcal{X}_1 \times \mathcal{X}_2 \mid (x, x_1^{\otimes}) \in \mathcal{S}_1, (x, x_2^{\otimes}) \in \mathcal{S}_2\}.$$

Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are simulation relations, we can conclude that $\pi_{\mathcal{X}}(S) = \mathcal{V}^*$.

Next, it will be proven that the subspace $\mathcal{S}$ satisfies the properties from Lemma 2.9. For this, let $(x, x_1^{\otimes}, x_2^{\otimes}) \in \mathcal{S}$ and let d be such that $Ax + Ed \in \mathcal{V}^*$. Since $\mathcal{S}_1$ is a simulation relation, there exists a $d_1^{\otimes}$ such that $A_1 x_1^{\otimes} + E_1 d_1^{\otimes} \in \mathcal{V}_1^*$ and $(Ax + Ed, A_1 x_1^{\otimes} + E_1 d_1^{\otimes}) \in \mathcal{S}_1$. Similarly, there exists a $d_2^{\otimes}$ such that $A_2 x_2^{\otimes} + E_2 d_2^{\otimes} \in \mathcal{V}_2^*$ and $(Ax + Ed, A_2 x_2^{\otimes} + E_2 d_2^{\otimes}) \in \mathcal{S}_2$. When combining these two things, we see that there exists a $d^{\otimes} = (d_1^{\otimes}, d_2^{\otimes})$ such that $(A^{\otimes} x^{\otimes} + E^{\otimes} d^{\otimes}) \in \mathcal{V}^{\otimes *}$ and

$$(Ax + Ed, A^{\otimes} x^{\otimes} + E^{\otimes} d^{\otimes}) \in \mathcal{S}.$$

Furthermore, since $(x, x_1^{\otimes}, x_2^{\otimes}) \in \mathcal{S}$, we have that $(x, x_1^{\otimes}) \in \mathcal{S}_1$, and thus $Cx = C_1 x_1^{\otimes}$. Similarly, $(x, x_2^{\otimes}) \in \mathcal{S}_2$ and thus $Cx = C_2 x_2^{\otimes}$. Therefore, we have that

$$C^{\otimes} x^{\otimes} = \frac{1}{2}(C_1 x_1^{\otimes} + C_2 x_2^{\otimes}) = Cx.$$

Now that we have proven that $\mathcal{S}$ satisfies the properties of Lemma 2.9, the last thing that still needs to shown is that $\pi_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{S}) \subset \mathcal{V}^{\otimes *}$. From the above reasoning, we saw that

$$(Ax + Ed, A_1 x_1^{\otimes} + E_1 d_1^{\otimes}, A_2 x_2^{\otimes} + E_2 d_2^{\otimes}) \in \mathcal{S}.$$

Therefore, we can conclude that

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix} \mathcal{S} \subset \mathcal{S} + \mathrm{im} \begin{bmatrix} E & 0 & 0 \\ 0 & E_1 & 0 \\ 0 & 0 & E_2 \end{bmatrix}.$$

From this, we can conclude that $A^{\otimes} \pi_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{S}) \subset \pi_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{S}) + \mathrm{im} E^{\otimes}$. Now, let $(x_1^{\otimes}, x_2^{\otimes}) \in \pi_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{S})$ and let x be such that $(x, x_1^{\otimes}, x_2^{\otimes}) \in \mathcal{S}$. Therefore, we have that $C_1 x_1^{\otimes} = Cx = C_2 x_2^{\otimes}$. Furthermore, for $i = \{1, 2\}$, $(x, x_i^{\otimes}) \in \mathcal{S}_i$ and $\pi_{\mathcal{X}_i}(\mathcal{S}) \subset \mathcal{V}_i^*$, and thus $H_i x_i^{\otimes} = 0$, so we also have that $\pi_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{S}) \subset \ker H^{\otimes}$. This shows that $\pi_{\mathcal{X}_1 \times \mathcal{X}_2}(\mathcal{S}) \subset \mathcal{V}^{\otimes *}$, which is what we needed to prove. Therefore, this concludes the end of the proof. $\square$

Proof of Lemma 4.7. Assume $A \times \Sigma \preceq G$, which means that there exists a full simulation relation $\mathcal{S} \subset \mathcal{X}_A \times \mathcal{X}_\Sigma \times \mathcal{X}_G$ of $A \times \Sigma$ by G. Now, consider the following linear subspace $\mathcal{S}' \subset \mathcal{X}_A \times \mathcal{X}_\Sigma \times \mathcal{X}_A \times \mathcal{X}_G$:

$$\mathcal{S}' = \{(x_A, x_\Sigma, \bar{x}_A, x_G) \mid (x_A, x_\Sigma, x_G) \in \mathcal{S},\ \bar{x}_A = x_A\}.$$

From the fact that $\mathcal{S}$ is a simulation relation, it can be seen that $\pi_{\mathcal{X}_A \times \mathcal{X}_\Sigma}(\mathcal{S}') = \mathcal{X}_A \times \mathcal{X}_\Sigma$, as well as the fact that $\pi_{\mathcal{X}_A \times \mathcal{X}_G}(\mathcal{S}') \subset \mathcal{V}^*_{\mathcal{X}_A \times \mathcal{X}_G}$. The only thing that still needs to be proven is that $\mathcal{S}'$ satisfies the conditions of Lemma 2.9. For this, let $(x_A, x_\Sigma, x_A, x_G) \in \mathcal{S}'$, and let $d_{A\Sigma}(d_A, d_\Sigma) \in \mathcal{D}_A \times \mathcal{D}_\Sigma$. Since $\mathcal{S}$ is a simulation relation, there exists a d_G such that

$$(A_A x_A + E_A d_A, A_\Sigma x_\Sigma + B_\Sigma C_A x_A + E_\Sigma d_\Sigma, A_G x_G + E_G d_G) \in \mathcal{S}.$$

Picking $d_{AG} = (d_A, d_G)$, we see that we have that

$$(A_A x_A + E_A d_A, A_\Sigma x_\Sigma + B_\Sigma C_A x_A + E_\Sigma d_\Sigma, A_A x_A + E_A d_A, A_G x_G + E_G d_G) \in \mathcal{S}'.$$

This shows that the first condition of Lemma 2.9 is satisfied. For the second condition, we use the fact that $\pi_{\mathcal{X}_A \times \mathcal{X}_G}(\mathcal{S}') \subset \mathcal{V}^*_{\mathcal{X}_A \times \mathcal{X}_G}$. From this, we see that $C_A x_A = C_G^u x_G = C_A \bar{x}_A$. Furthermore, since we have that $\mathcal{S}$ is a simulation relation, we have that

$$(C_A x_A, C_\Sigma x_\Sigma) = (C_G^u x_G, C_G^y x_G).$$

When combining these two facts, we see that

$$(C_A x_A, C_\Sigma x_\Sigma) = (C_G^u x_G, C_G^y x_G) = (C_A \bar{x}_A, C_G^y x_G).$$

Since all conditions of Lemma 2.9 are satisfied, we have shown that $\mathcal{S}'$ is a simulation relation of $A \times \Sigma$ by $A \otimes G$, as thus, we have proven that $A \times \Sigma \preceq A \otimes G$. $\square$

Proof of Lemma 4.5. Let Σ_1 be an implementation of the contract $\mathcal{C}_1 = (A_1, G_1)$, and thus we have that $A_1 \times \Sigma_1 \preceq G_1$. Similarly we have that Σ_2 is an implementation of the contract $\mathcal{C}_2 = (A_2, G_2)$, and thus $A_2 \times \Sigma_2 \preceq G_2$. Now consider the interconnection $A_1 \otimes A_2 \times \Sigma_1 \otimes \Sigma_2$. Note that we have by Lemma 3.4 that $A_1 \otimes A_2 \preceq A_1$ and $A_1 \otimes A_2 \preceq A_2$. Furthermore, we have that $\Sigma_1 \otimes \Sigma_2 \preceq \Sigma_1$ and $\Sigma_1 \otimes \Sigma_2 \preceq \Sigma_2$.

Therefore, we see that

$$A_1 \otimes A_2 \times \Sigma_1 \otimes \Sigma_2 \preceq A_1 \times \Sigma_1 \preceq G_1,$$

and similarly

$$A_1 \otimes A_2 \times \Sigma_1 \otimes \Sigma_2 \preceq A_2 \times \Sigma_2 \preceq G_2.$$

Therefore, we can conclude that

$$A_1 \otimes A_2 \times \Sigma_1 \otimes \Sigma_2 \preceq G_1 \otimes G_2.$$

This shows that $\Sigma_1 \otimes \Sigma_2$ is an implementation of the contract
$\mathcal{C} = (A_1 \otimes A_2, G_1 \otimes G_2)$. $\square$

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