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A Geometrical Review of Born-Infeld Theory

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Abstract

The present work aims to provide an overview of the main physical and geometrical properties of the Born-Infeld model. We explore the similarities and differences with Maxwell's electrodynamics and how these influence said properties. We find that Born-Infeld complies with the macroscopic Maxwell equations while solving the problem of infinite self-energies. Duality symmetry, Lorentz and gauge invariance are key properties shared by the two theories. In a more general sense, we see that the laws of electrodynamics can be described as a geometrical consequence of the definitions that are used to describe their framework. While the physical intuition is restricted to four dimensions, it is possible to extend the geometric notions to higher dimensions. Finally, we review some applications of Born-Infeld in theoretical physics.

Contents

1 Introduction

The development of a complete classical description of electromagnetism in terms of a few simple differential equations, known today as Maxwell's laws of electrodynamics, was one of the most revolutionary achievements in the history of physics. The joint efforts of physicists like Faraday, Hertz, Heaviside, and Maxwell himself [1] revealed that electric and magnetic phenomena are part of a unified type of fundamental interaction, which would motivate future generations of physicists to seek further unification of the physical theories. Despite its undoubtable success, which persists until today, Maxwellian electrodynamics poses some fundamental inconsistencies. For starters, the energy contained in an empty universe with a single point charge, known as the particle's self-energy, diverges to infinity [2]. Moreover, field strengths also present singularities when a single electron is considered.

In this context, many (non-linear) alternative theories of electrodynamics were proposed in the early 20th century to fix these issues [3]. With the emergence of quantum mechanics and Dirac's equation for the electron later in the same century, these alternative theories of electromagnetism quickly became obsolete in favour of quantum electrodynamics, which solves the problem of infinite self-energies via the concept of renormalisation (see [2] for a sketch of the solution). Nevertheless, many of these theories have been revisited ever since; despite failure to be extended to self-consistent quantum theories, their theoretical predictions and/or interpretations remain a subject of study [4, 5, 6].

In the 1930s, Max Born and Leopold Infeld published a series of papers characterising their own solution to the problem, which is known today as the Born-Infeld model of electrodynamics [7, 8]. In order to abide to the principle of finiteness for physical quantities, Born and Infeld imposed an upper bound for the magnitude of electric fields, in analogy to the upper limit that Einstein's special theory of relativity had set for universal speeds. This is achieved by replacing the Maxwellian Lagrangian, which is sometimes formulated as

$$\mathcal{L} = -\frac{1}{16\pi c} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c^2} A^\mu J_\mu, \tag{1.1}$$

with the Born-Infeld Lagrangian, which was originally proposed [7] as

$$\mathcal{L}_{BI} = \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})} - \sqrt{-\det(g_{\mu\nu})}, \tag{1.2}$$

where g denotes the determinant of the metric tensor in matrix form. Later on, Born would review the theory and state his preference for a slightly modified Lagrangian density [8]:

$$\mathcal{L}_{BI} = -\sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})} \tag{1.3}$$

Besides solving the infinite self-energy problem for point particles, this model retains many of the characteristic properties of Maxwell's electrodynamics, such as Lorentz invariance and duality symmetry; under the exchange of electric and magnetic fields, the equations of motion remain unaffected. Even though it cannot be extended into a consistent quantum theory, it regained attention at the end of the 20th century since it presents a series of geometric features which have several applications in theoretical physics, particularly in string theory and cosmology. Furthermore, the combination of metric and field strength in (1) suggests the existence of a geometrical interpretation of the theory [9], which is often studied in the context of open superstrings.

The main goal of this project is to compare the (differential) geometrical properties of Maxwell's electromagnetism and the Born-Infeld model and review the underlying physical properties that characterise the latter. Our first goal is to give a brief overview of the main properties of Maxwell electrodynamics and provide some geometrical insight, which is covered in Section 2. The fundamental characteristics of Born-Infeld are then discussed in Section 3, including some historical background and a justification for the chosen Lagrangian, the derivation of field equations in terms of tensors, and the conservation of the energy-momentum tensor. Section 4 deals with the geometrical study of the governing equations, how they relate to the theory's physical intuition, and how they can be extended to higher dimensions. Other physical properties regarding Lorentz and gauge invariance and the prediction of the existence of monopoles are discussed in Section 5. Finally, Section 6 gives a brief overview of the most important applications of the theory in theoretical physics, predominantly in string theory and phantom energy cosmology. For a reference on the mathematical preliminaries based upon a previous project of a similar type, see [2].

2 Main features of Maxwell's theory

The motivation behind Born-Infeld theory cannot be understood without consideration of the 'classical' formulation of electromagnetism, given by the four Maxwell equations [1]:

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{2.1}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{2.2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.3}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}. \tag{2.4}$$

These four equations' purpose is to model the behaviour of electric and magnetic fields in the absence of dielectric media.

Albeit a classical theory, it is often known as the first (intrinsically) relativistic theory in the history of physics. Said link with special relativity theory is often made explicit via a covariant reformulation of Maxwell's equations, which both manifests their invariance under Lorentz transformations and provides a way to translate a system's fields and forces from one frame to another. To this end, we take Minkowski's flat spacetime metric under signature $(+,-,-,-)$ and consider Einstein's summation convention. We define the electromagnetic tensor (also called the Faraday tensor or the field strength tensor) as follows:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0. \end{pmatrix}. \tag{2.5}$$

It is often introduced as the differential of the electromagnetic four-potential,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{2.6}$$

where $A^\mu := (\phi/c, \mathbf{A})$ and $\phi, \mathbf{A}$ are the classical vector potentials. This leads to the following formulation of Maxwell's equations:

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu, \tag{2.7}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0. \tag{2.8}$$

The first equation is equivalent to the two inhomogeneous equations, (2.1) and (2.4) (the Gauss-Ampere law), where $J^\mu := (c\rho, \mathbf{j})$ defines the four-current of the system. The second equation, which is equivalent to the remaining two (homogeneous) equations (the Gauss-Maxwell-Faraday law), is not written in terms of the Faraday tensor but of its dual $\tilde{F}$. In four dimensions, the explicit relation between the two can be described via the Levi-Civita connection:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} = - \star F^{\mu\nu}. \tag{2.9}$$

Provided that (2.6) is taken as the definition of the electromagnetic tensor, the homogeneous Maxwell equation is equivalent to the Bianchi identity for F. This means that (2.8) is automatically satisfied, and, hence, (2.7) is the only equation with physical content. This last point is reinforced by introducing the Maxwellian Lagrangian (including the interaction term), which can be written as follows:

$$\mathcal{L}_{Maxwell} := -\frac{1}{16\pi c} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c^2} A^\mu J_\mu. \tag{2.10}$$

The justification behind this choice of Lagrangian for classical electromagnetism is that it is manifestly Lorentz invariant, U(1) gauge invariant[1] and, most importantly, it is possible to retrieve Maxwell's equations via the Euler-Lagrange equations. Particularly, solving Euler-Lagrange yields the inhomogeneous equation, whilst the homogeneous one is encoded within due to the introduction of vector potentials to model the interactions in the system. Furthermore, due to its invariance under gauge transformations of the four-vector potential, Noether's theorem predicts the existence of an associated conserved current, which corresponds to the charge density four-vector.

One way to both generalise these results and analyse them from a purely geometrical perspective is to formulate Maxwell's equations in terms of differential forms, the main advantage of which is getting rid of coordinate dependence. To further this endeavour, we may consider the electric and magnetic

[1] In other words, the electromagnetic potentials are invariant under gauge transformations in the group $U(1) \simeq SO(2)$

fields residing in 4-dimensional manifold M equipped with Minkoski's metric, which we assume can be split into two components:

$$M = \mathbb{R} \times S,$$

where S is a three-dimensional manifold representing space, endowed with a Riemannian metric, and $\mathbb{R}$ stands for the time coordinate. Thus, we consider spacetime as a hyperbolic, differentiable, four-dimensional pseudo-Euclidean Riemannian manifold with metric η of index 1.

First, consider the homogeneous Maxwell equations. Since the divergence of a vector can be written as the exterior derivative of a 2-form in three-dimensional space, we may think about the magnetic field $\mathbf{B}$ as a differential 2-form. Similarly, the electric field can be written as a 1-form. This means that we can write the field strength tensor as a 2-form of the following form[2]:

$$F = B + E \wedge dx^0. \tag{2.11}$$

It can then be shown that the two homogeneous equations are equivalent to

$$dF = 0. \tag{2.12}$$

To apply a similar reasoning in the inhomogeneous case, one may consider the (discrete) Hodge dual operation on the electromagnetic tensor and the duality between electric and magnetic fields under the replacement:

$$\frac{\mathbf{E}}{c} \mapsto -\mathbf{B}, \quad \mathbf{B} \mapsto \frac{\mathbf{E}}{c}. \tag{2.13}$$

This yields the inhomogeneous Maxwell equation:

$$d \star F = \star J, \tag{2.14}$$

where $J \coloneqq c\rho dx^0 - J^1 dx^1 - J^2 dx^2 - J^3 dx^3$ redefines the four-current vector as a 1-form. We see in Section 4 that the properties of the Hodge duality and the equations of motion of both Maxwell and Born-Infeld extend to higher dimensions.

Note that, from a strictly mathematical perspective, we may study the geometrical intuition behind $dF = 0$ and $d \star F = \star J$ without restricting to a 2-form (F) in a four-dimensional manifold (M). For instance, it is possible to study these notions considering F to be a 6-form in ten-dimensional space. Nevertheless, considering the present case makes it easier to relate these conclusions to the physical understanding of said equations.

As in the covariant formulation, there are two different ways to define the field strength tensor. If it is written explicitly in terms of its relationship to the underlying fields (as a 2-form), $dF = 0$ describes how the electromagnetic flux through a closed two-dimensional surface depends on the topology of the chosen space-time. Moreover, Poincare's lemma indicates that, locally, there exists a 1-form A such that $F = dA$ since F is a closed form that can be defined on an open ball on $\mathbb{R}^n$ for $n \geq 2$. However,

[2]Here, x^0 represents the real time coordinate.

it is often the case that F is defined as the exterior derivative of vector potential $\mathbf{A}$ (as a 1-form), the connection form for the U(1) principal bundle:

$$F = dA.$$

From this perspective, the homogeneous Maxwell equation is automatically satisfied due to the Bianchi identity; all physical phenomena described by classical electromagnetism is contained within the inhomogeneous equation, $d \star F = \star J$. Geometrically, this equation dictates that the electromagnetic flux through a closed surface is equal to the four-current on that surface, evaluated at the boundary. This corresponds exactly to Gauss' and Ampére's laws, where *proportionality* corresponds to *equality* when $\epsilon_0 = \mu_0 = 1$.

One of the main features of Maxwell's electromagnetism is its SO(2) duality symmetry. In other words, when Maxwell's equations are considered in a vacuum, one can 'rotate' a set of solutions to Maxwell's equations to obtain new solutions, given that equations

$$dF = 0 \text{ and } d \star F = 0$$

are invariant under the following transformation:

$$\begin{pmatrix} F \\ \star F \end{pmatrix} \mapsto \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} F \\ \star F \end{pmatrix}. \tag{2.15}$$

We are thus taking an SO(2) action, which describes a two-dimensional rotation, under which the system remains invariant. We can also express this symmetry in terms of the fields[3]:

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \mapsto \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}. \tag{2.16}$$

Duality invariance is a key property of Maxwell's equations because it tells us that both representations, F (homogeneous) and $\star F$ (inhomogeneous), are equivalent; i.e., they describe the same physical phenomena, independent of the reference frame. Furthermore, notice that this symmetry is directly related to the 'rotational' nature of the field replacements in Hodge duality; particularly, one retrieves these as transformations of the fields when the system is rotated at $\phi = \pi$ radians. This confirms that the physical description of said system will remain unaffected under Hodge duality transformations. We shall see that this is one of the main properties that the Born-Infeld model shares with Maxwellian electromagnetism. Further conclusions regarding the physical and mathematical understanding of duality in both theories appears in subsection 4.2.

3 Introduction to the Born-Infeld model

The purpose of this section is to introduce the reader to the Born-Infeld model of electrodynamics and to review some of its key characteristics. It focuses on the formulation of the theory, its corresponding

[3]We assume $c = 1$ here.

Lagrangian, and the resulting equations of motion. The geometrical study of the theory proceeds in Section 4.

3.1 Background and origins of the theory

The original paper by Max Born and Leopold Infeld starts by making a distinction between two opposite standpoints with respect to the relation between matter and electromagnetism. The *unitarian* standpoint, which assumes that the electromagnetic field is the only physical entity in the theory, considers particles as singularities of the field. An example of such a theory is, clearly, Maxwell electrodynamics. The *dualistic* standpoint, on the other hand, considers particles to be the sources of the field, which are acted upon by it but are not part thereof. This is one of the bases of the quantum mechanical picture, which can be seen, e.g., in the Klein-Gordon equation [10]. The paper discusses some complications in solving the infinite self-energy problem by quantum theories and argues for the development of a non-linear, unitarian theory which solves this problem and conserves some of the key properties of Maxwell theory, as reviewed in Section 2.

The Born-Infeld Lagrangian is derived via the principle of least action. A covariant[4] tensor field $a_{\mu\nu}$ is considered, which *a priori* has no symmetry properties with respect to a change of indices. For $\mathcal{L}$ to be such a function of $a_{\mu\nu}$ that the action integral be invariant, it must have the following form:

$$\mathcal{L} = \sqrt{\det(a_{\mu\nu})}. \tag{3.1.1}$$

Each arbitrary tensor $a_{\mu\nu}$ can be split up into a symmetric part, which we identify with the metrical field (i.e., described by the chosen metric), and an antisymmetric part, related to the electromagnetic field (the field strength tensor):

$$a_{\mu\nu} = g_{\mu\nu} + F_{\mu\nu}, \text{ with } g_{\nu\mu} = g_{\mu\nu} \text{ and } F_{\nu\mu} = -F_{\mu\nu},$$

where we have adopted Born and Infeld's original notation. Therefore, we can describe three different terms that are invariant in the action integral:

$$\sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})}, \sqrt{-\det(F_{\mu\nu})}, \text{ and } \sqrt{-\det(F_{\mu\nu})}.$$

Under the assumption that the Lagrangian density is a linear combination of these three terms, one can show that the dynamics of the electromagnetic field is described in general coordinates by

$$\mathcal{L} = \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})} - \sqrt{-\det(g_{\mu\nu})}. \tag{3.1.2}$$

This can be rewritten in cartesian coordinates as

$$\mathcal{L} = \sqrt{-g}L, \text{ with } L := \left(\sqrt{1 + M - N^2} - 1\right), \tag{3.1.3}$$

[4]With respect to spacetime transformations.

where g denotes the determinant of the metric tensor in matrix form, and M and N are related to the aforementioned invariant quantities in the Lagrangian:

$$M = \frac{1}{b^2}\left(B^2 - E^2\right) = \frac{1}{2b^2}F_{\mu\nu}F^{\mu\nu}, \text{ and} \tag{3.1.4}$$

$$N = \frac{1}{b^2}\left(\mathbf{B}\cdot\mathbf{E}\right) = \frac{1}{4b^2}F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{3.1.5}$$

Born and Infeld call b the *absolute field*; it is a dimensional constant, defined by the quotient of the field strength expressed in the conventional units and the field strength in natural units. In essence, b denotes an upper bound for the maximal possible value of the electric field (in analogy to the limiting speed c in relativistic settings), which in turn limits the self-energy of electrons. This property is reviewed in Section 5.

3.2 Lagrangian formalism and the field equations

Analogously to our study of Maxwell's theory [2], we may assume the existence of a vector potential A_μ (in a local setting) such that the Faraday strength tensor be expressed as its exterior derivative (see (2.6)). This yields the first field equation of the theory: Bianchi's identity,

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, \tag{3.2.1}$$

equivalent to (2.8). The second field equation is to be found by solving the corresponding Euler-Lagrange equation of motion:

$$\partial_\nu\left(\frac{\partial L}{\partial\left(\partial_\nu A_\mu\right)}\right) = \frac{\partial L}{\partial A_\mu} = 0. \tag{3.2.2}$$

Note that one can use the definition of $F_{\mu\nu}$ as in (2.6) to show

$$\frac{\partial L}{\partial F_{\mu\nu}} = -\frac{1}{2}\frac{\partial L}{\partial\left(\partial_\nu A_\mu\right)}. \tag{3.2.3}$$

Hence, we have that

$$\begin{aligned}\frac{\partial L}{\partial\left(\partial_\nu A_\mu\right)} &= -2\frac{\partial L}{\partial F_{\mu\nu}}\\ &= -\frac{1}{b^2}\frac{F^{\mu\nu} + \frac{1}{4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)\tilde{F}^{\mu\nu}}{\sqrt{1+M-N^2}}\\ &= -\frac{1}{b^2}\frac{F^{\mu\nu} + N\tilde{F}^{\mu\nu}}{\sqrt{1+M-N^2}}.\end{aligned}$$

We can thus rewrite the equations of motion in the following manner:

$$\partial_\nu \tilde{F}^{\mu\nu} = 0; \tag{3.2.4}$$

$$\partial_\nu \tilde{G}^{\mu\nu} = 0, \tag{3.2.5}$$

where we have defined

$$\tilde{G}^{\mu\nu} := 2\frac{\partial L}{\partial F_{\mu\nu}} = \frac{1}{b^2}\frac{F^{\mu\nu} + N\tilde{F}^{\mu\nu}}{\sqrt{1+M-N^2}}. \tag{3.2.6}$$

Note that $\tilde{F}^{\mu\nu} = -\star F^{\mu\nu}$, as previously discussed. The components of $\tilde{G}$ represent the vectors $\mathbf{D}$ and $\mathbf{H}$ from classical electromagnetism in this theory. In their paper, Born and Infeld consider $\tilde{G}$ to be defined by the constitutive relations, which in this case can be written as

$$\mathbf{D} = b^2 \frac{\partial L}{\partial \mathbf{E}} = -\frac{\mathbf{E} + \frac{N}{b^2}\mathbf{B}}{\sqrt{1 + M - N^2}}, \tag{3.2.7}$$

$$\mathbf{H} = b^2 \frac{\partial L}{\partial \mathbf{B}} = \frac{\mathbf{B} - \frac{N}{b^2}\mathbf{E}}{\sqrt{1 + M - N^2}}. \tag{3.2.8}$$

One can then proceed to write a set of Maxwell-like equations for Born-Infeld theory based on these relations:

$$\nabla \cdot \mathbf{D} = 0, \tag{3.2.9}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3.2.10}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{3.2.11}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}. \tag{3.2.12}$$

These equations are formally identical with Maxwell's macroscopic equations for a substance with dielectric constance and susceptibility, but without charge and current distributions. When (free) currents are considered, one can show, in a similar fashion as

$$\partial_\nu F^{\mu\nu} = 4\pi J^\mu$$

was shown in Maxwell, that the general equations of motion are given by

$$\partial_\nu \tilde{F}^{\mu\nu} = 0; \tag{3.2.13}$$

$$\partial_\nu \tilde{G}^{\mu\nu} = 4\pi J^\mu_f, \tag{3.2.14}$$

where J_f represents the (free) currents analogously to how J symbolised the currents in vacuum Maxwell. Nevertheless, we will mostly focus on the case $J_f = 0$ to study the symmetries of the theory. Several conclusions can be drawn from these results. First of all, note that these two equations are strikingly similar to Maxwell's vacuum equations in covariant form; specifically, they are equivalent under the 'mapping' $F^{\mu\nu} \mapsto \tilde{G}^{\mu\nu}$.[5] Furthermore, these two formulations become equivalent in the special case $M = N = 0$, e.g., in the limit $b \to \infty$. Notice that this describes the Lagrangian of a free theory given that all interactions scale with $\frac{1}{b}$; Maxwell's electrodynamics, as previously stated, has no self-interactions. Moreover, this shows that Born-Infeld electrodynamics presents an identical duality symmetry as in Maxwell's theory, i.e., one may generate new Born-Infeld solutions from existing ones by performing an SO(2) transformation

$$\begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix} \mapsto \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix}. \tag{3.2.15}$$

[5] This relation is discussed further in the subsequent chapter.

To strengthen this point, notice that we can combine equations (3.2.9)-(3.2.12) into

$$\nabla \cdot (\mathbf{D} + i\mathbf{B}) = 0, \tag{3.2.16}$$

$$\nabla \times (\mathbf{E} + i\mathbf{H}) = \frac{i}{c}\frac{\partial}{\partial t}(\mathbf{D} + i\mathbf{B}). \tag{3.2.17}$$

This highlights the following rotation symmetry (by an angle α) in Minkowski's spacetime:

$$\mathbf{D} + i\mathbf{B} \to e^{i\alpha}(\mathbf{D} + i\mathbf{B}),$$

$$\mathbf{E} + i\mathbf{H} \to e^{i\alpha}(\mathbf{E} + i\mathbf{H});$$

which is equivalent to

$$F^{\mu\nu} \to \cos(\alpha) F^{\mu\nu} + \sin(\alpha) \star G^{\mu\nu},$$

$$G^{\mu\nu} \to \cos(\alpha) G^{\mu\nu} + \sin(\alpha) \star F^{\mu\nu}.$$

for the relevant tensors [11]. By setting $\mathbf{D} = \mathbf{E}$ and $\mathbf{H} = \mathbf{B}$ we obtain $F^{\mu\nu} = G^{\mu\nu}$, which returns Maxwell's equations, as previously discussed. This reinforces the point that Born-Infeld introduces specific self-interactions as corrections in Maxwell's theory without breaking the SO(2) symmetry.

One can also formalise Born-Infeld theory by defining the following Hamiltonian:

$$H = \sqrt{1 + P - Q^2} - 1, \tag{3.2.18}$$

where P and Q are defined as follows:

$$P \coloneqq \frac{1}{b^2}\left(D^2 - H^2\right), \text{ and } Q \coloneqq \frac{1}{b^2}(\mathbf{D} \cdot \mathbf{H}). \tag{3.2.19}$$

This provides a useful way to write $\mathbf{E}$ and $\mathbf{B}$ in terms of $\mathbf{D}$ and $\mathbf{H}$:

$$\mathbf{E} = b^2 \frac{\partial H}{\partial \mathbf{D}} = \frac{\mathbf{D} - \frac{Q}{b^2}\mathbf{H}}{\sqrt{1 + P - Q^2}}, \tag{3.2.20}$$

$$\mathbf{B} = b^2 \frac{\partial H}{\partial \mathbf{H}} = -\frac{\mathbf{H} + \frac{Q}{b^2}\mathbf{D}}{\sqrt{1 + P - Q^2}}. \tag{3.2.21}$$

The Hamiltonian can also be written as

$$\mathcal{H} = \mathbf{D} \cdot \mathbf{E} - \mathcal{L}. \tag{3.2.22}$$

It becomes apparent in this way that setting $\mathbf{D} = \mathbf{E}$ we recover

$$\mathcal{H} = \frac{1}{2}\left(E^2 + B^2\right) \tag{3.2.23}$$

as in Maxwell. We have seen in the Maxwell research project [2] that $\mathcal{H}$ is a conserved quantity (due to conservation of energy) but $\mathcal{L}$ is not. Generally, one can show [12] that $\mathcal{H}$ as in (3.2.23) is conserved for an arbitrary (Lagrangian) function $\mathcal{L}$, which is to be expected because of conservation of energy.

It is not surprising that the Hamiltonian density in Born-Infeld theory is an invariant quantity; i.e., it is the T^{00} entry of the energy-momentum tensor, which in this case can be obtained as

$$T^{\mu}_{\nu} = \tilde{G}^{\mu\alpha} F_{\nu\alpha} + \partial^{\mu}_{\nu}\mathcal{L} \tag{3.2.24}$$

via a Belinfante procedure [13, 14]. We have that T^{μ}_{ν} is conserved if, and only if, the Born-Infeld equations of motion are satisfied:

$$\partial_\mu T^{\mu}_{\nu} = 0 \quad \iff \quad \partial_\mu \tilde{G}^{\mu\nu} = 0 = \partial_\mu F^{\mu\nu}.$$

On another note, we can rewrite equations (3.2.9)-(3.2.12) to resemble Maxwell's equations even further:

$$\nabla \cdot \mathbf{E} = 0,\ \nabla \cdot \mathbf{B} = 0, \tag{3.2.25}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{3.2.26}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - \frac{2}{N}\left(\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\right). \tag{3.2.27}$$

Therefore, a system with a constant magnetic field is described equivalently by Maxwell and Born-Infeld.

In Maxwell's theory, moreover, the free field equations have two degrees of freedom, corresponding to the two polarisation states of the photon (recall that the four original Maxwell laws describe electromagnetic waves, i.e., radiation). We have an analogous description for BI and macroscopic Maxwell, where the polarisation of the material itself affects some of the photons' properties (e.g., speed, direction), but the two polarisation states due to the degrees of freedom persist.

3.3 Existence of BIons; the electrostatic solution

One of the main features of Born-Infeld theory is that it admits static solutions of finite energy, which were originally proposed as classical models for the electron, commonly known in literature as 'BIons' (e.g., see [15], where the term has been taken from). These solutions are not source-free in all of space like solitons, but resemble elementary solutions of point-like sources.

Consider an electrostatic system described by the Born-Infeld field equations (3.2.9)-(3.2.12); i.e., set $\mathbf{B} = \mathbf{H} = 0$ and suppose all other components of said equations are time-independent. Under these assumptions, the field equations reduce to

$$\nabla \cdot \mathbf{E} = 0, \tag{3.3.1}$$

$$\nabla \cdot \mathbf{D} = 0. \tag{3.3.2}$$

Equation (3.3.1) gives

$$E_r = -\frac{d\phi}{dr} = -\phi'(r)$$

for the radial component of the electric field, while one finds that under central symmetry

$$\frac{d}{dr}\left(r^2 D_r\right) = 0 \implies 4\pi e = \int D_r d\sigma,$$

where e symbolizes a point charge. This can also be written as

$$\nabla \cdot \mathbf{D} = 4\pi e \delta(\mathbf{x}),$$

where $\mathbf{x}$ denotes position with respect to the location of the point source with charge e. Clearly, the displacement vector blows up to infinity at the origin. Due to the absence of magnetic fields, we can rewrite (3.2.20) as follows:

$$\mathbf{E} = \frac{\mathbf{D}}{\sqrt{1 + \frac{1}{b^2}\mathbf{D}^2}}. \tag{3.3.3}$$

Therefore, the electric field tends to a finite value at the source. This value is related to the maximum field strength b.

Born and Infeld derived this result in a slightly different manner. Instead of (3.2.20), they used $D_r = \frac{e}{r^2}$ and (3.2.7) to derive a first-order differential equation for scalar potential $\phi \coloneqq \phi(\mathbf{r})$:

$$\phi'(r) = -\frac{b}{\sqrt{1 + \frac{b^2 r^4}{e^2}}}. \tag{3.3.4}$$

The solution can be written in the following form:

$$\phi(r) = \frac{e}{r_0} f\left(\frac{r}{r_0}\right); \; f(x) \coloneqq \int_x^\infty \frac{dy}{\sqrt{1+y^4}}; \; r_0 \coloneqq \sqrt{\frac{e}{b}}. \tag{3.3.5}$$

This is the elementary potential of a point charge e, which, as we can see, is finite everywhere. It serves as a substitute of Coulomb's law, which is an approximation thereof for $x >> 1$. One can also recognise that the potential is maximal at the source; Born and Infeld found that the pertinent numerical value is

$$\phi(0) = 1.8541 \; e/r_0.$$

Hence, even though $\mathbf{D}$ is infinite at the source, $\mathbf{E}$ and ϕ remain finite. It is now explicit how $\|\mathbf{E}\|$ depends on r_0 (and, hence, on b):

$$D_r = \frac{e}{r^2}, \tag{3.3.6}$$

$$E_r = \frac{e}{r_0^2\sqrt{1 + \left(\frac{r}{r_0}\right)^4}}. \tag{3.3.7}$$

3.4 Maxwell and Born-Infeld: motivating differential forms

We have seen that it is possible to draw a formal comparison between source-free Maxwell and Born-Infeld by rewritting their respective field equations using covectors:

Maxwell	Born-Infeld
$\partial_\nu \tilde{F}^{\mu\nu} = 0$	$\partial_\nu \tilde{F}^{\mu\nu} = 0$
$\partial_\nu F^{\mu\nu} = 4\pi J^\mu$	$\partial_\nu G^{\mu\nu} = 4\pi J^\mu_f$

We are interested in studying the geometrical interpretation of these equations and how they are linked. In order to do so, it is useful to formulate these two theories by employing the language of differential forms. We have seen how this is done for Maxwell: one can define the field strength tensor, F, as a 2-form depending on the electromagnetic field. By extending covectors to p-forms, it becomes clear that the covariant formulation is the component-specific form of the differential-geometric formalism. Formally, there is no impediment in extending this reasoning, concerning F and $\tilde{F}$, to Born-Infeld theory, described by F and G:

Maxwell	Born-Infeld
$dF = 0$	$dF = 0$
$d \star F = \star J$	$dG = \star J_f$

The main difference between the two is that the first Maxwell equation was derived after describing F in terms of the fields. Hence, the next step in our analysis is to study G as a differential form in terms of $\mathbf{D}$ and $\mathbf{H}$, which can also described as forms. The ultimate goal is to deduce some geometrical properties of Born-Infeld via our understanding of Maxwell's theory's formalism.

There are also a number of benefits from the physical side. For starters, we want to verify that, when constrained to a four-dimensional spacetime, the understanding of the geometrical perspective is strictly linked to the physical intuition behind the macroscopic Maxwell equations. This is discussed at length throughout Section 4. Moreover, describing the theory in terms of differential forms can help in understanding other physical aspects, such as the effect of Lorentz boosts in Born-Infeld and provide further understanding of Lorentz invariance and its consequences (see subsection 5.1).

4 Physical and geometrical study of the Born-Infeld equations

In this section we review the main properties of Born-Infeld study, both from a physical and a geometrical perspective, extending the discussion from Section 3. We start by deriving the differential-geometric formulation of the theory and proceed to study each of the two resulting governing equations. We see that, as a general conclusion, not only can the the laws of electrodynamics be described from a geometrical perspective, but they are a *consequence* of the mathematical definitions that we use to describe their framework.

4.1 Differential-geometric formulation of Born-Infeld theory

In this section, we aim to provide a full mathematical formulation of Born-Infeld theory in terms of differential forms using our definitions from the previous chapter. We start by considering Maxwell's equations in matter (in Gaussian units), which have been shown to be equivalent to the Born-Infeld field equations:

$$\nabla \cdot \mathbf{D} = 4\pi\rho_f, \tag{4.1.1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4.1.2}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{4.1.3}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\left(4\pi\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}\right). \tag{4.1.4}$$

When considering non-linear electrodynamic theories, instead of having

$$\mathbf{D} = \mathbf{E}, \text{ and } \mathbf{H} = \mathbf{B},$$

as in the vacuum (returning Maxwell's equations), we replace them by the constitutive relations

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{B}), \text{ and } \mathbf{H} = \mathbf{H}(\mathbf{E}, \mathbf{B}).$$

We are interested in the constitutive relations that preserve some of the key features in Maxwell's theory. To this end, consider the absence of external currents ($\rho_f = J_f = 0$). By taking the constitutive relations to be

$$\mathbf{D} = \frac{\partial \mathcal{L}}{\partial \mathbf{E}} \text{ and } \mathbf{H} = -\frac{\partial \mathcal{L}}{\partial \mathbf{B}}, \tag{4.1.5}$$

where $\mathcal{L} \coloneqq \mathcal{L}(\mathbf{E}, \mathbf{B})$, we ensure the relativistic covariance of both Maxwell's macroscopic equations and said relations via a Lagrangian formulation [13]. In this section, we will see that Born-Infeld is a special case of such a theory.

The aforementioned commutation relations are usually grouped [13, 7] as follows:

$$\tilde{G}^{\mu\nu} = 2\frac{\partial \mathcal{L}(F)}{\partial F_{\mu\nu}}, \tag{4.1.6}$$

where the factor of two arises from identity

$$\frac{\partial F_{\alpha\beta}}{\partial F_{\mu\nu}} = \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}.$$

The dual of G is defined in the same way as $\tilde{F}$:

$$\tilde{G}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}G_{\alpha\beta}, \tag{4.1.7}$$

so we obtain

$$G_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\tilde{G}^{\alpha\beta}$$

given the properties of Levi-Civita.

The electrodynamics in a Born-Infeld model are wholly described by two equations of motion: the Bianchi equation,

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \tag{4.1.8}$$

and the field equation arising from the Lagrangian treatment[6],

$$\partial_\mu \tilde{G}^{\mu\nu} = 0. \tag{4.1.9}$$

In this case, the dynamics of the system can be described by the following Lagrangian density:

$$\begin{aligned}
\mathcal{L}_{BI} &= 1 - \sqrt{-\det(g+F)} && (4.1.10)\\
&= 1 - \sqrt{1 + M - N^2} && (4.1.11)\\
&= 1 - \sqrt{1 + \frac{1}{2b^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16b^4}F_{\mu\nu}\tilde{F}^{\mu\nu}} && (4.1.12)\\
&= 1 - \sqrt{1 + \frac{1}{b^2}\left(\mathbf{B}^2 - \mathbf{E}^2\right) - \frac{1}{b^4}\left(\mathbf{E}\cdot\mathbf{B}\right)^2}. && (4.1.13)
\end{aligned}$$

[6]Once again, we start by considering the simpler case $J_f = 0$.

In order to conduct a similar type of study as done with Maxwell, we would like to learn more about G, both as a tensor and a differential form. We find that the dual of G can be expressed as

$$\tilde{G}^{\mu\nu} = 2\frac{\partial\mathcal{L}(F)}{\partial F_{\mu\nu}} = \frac{1}{b^4}\frac{-b^2F^{\mu\nu} + \frac{1}{4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)\tilde{F}^{\mu\nu}}{\sqrt{1 + \frac{1}{2b^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16b^4}F_{\mu\nu}\tilde{F}^{\mu\nu}}}, \tag{4.1.14}$$

and taking the dual we find

$$G_{\mu\nu} = \frac{1}{b^4}\frac{b^2\tilde{F}_{\mu\nu} + NF_{\mu\nu}}{R}, \tag{4.1.15}$$

where we have defined R as the square root term in the denominator,

$$R \coloneqq \sqrt{1 + M - N^2}.$$

We can use this expression to find the differential form G of which (4.1) is the component form:

$$b^4RG = b^4\frac{R}{2}G_{\mu\nu}dx^\mu\wedge dx^\nu = b^4\frac{R}{2}\begin{pmatrix} 0 & -(b^2B_1 - NE_1) & -(b^2B_2 - NE_2) & -(b^2B_3 - NE_3) \\ (b^2B_1 - NE_1) & 0 & -(b^2E_3 + NB_3) & (b^2E_2 + NB_2) \\ (b^2B_2 - NE_2) & (b^2E_3 + NB_3) & 0 & -(b^2E_1 + NB_1) \\ (b^2B_3 - NE_3) & -(b^2E_2 + NB_2) & (b^2E_1 + NB_1) & 0 \end{pmatrix} dx^\mu\wedge dx^\nu, \tag{4.1.16}$$

where we have used the conventions for F and $\tilde{F}$ as stated in Chapter 2. This can be rewritten in terms of the components of $\mathbf{D}$ and $\mathbf{H}$ as given by the constitutive relations:

$$b^4R\mathbf{D} = b^4R\frac{\partial\mathcal{L}(\mathbf{E},\mathbf{B})}{\partial\mathbf{E}} = b^2\mathbf{E} + N\mathbf{B}, \tag{4.1.17}$$

$$b^4R\mathbf{H} = b^4R\frac{\partial\mathcal{L}(\mathbf{E},\mathbf{B})}{\partial\mathbf{B}} = b^2\mathbf{B} - N\mathbf{E}. \tag{4.1.18}$$

Therefore, we obtain

$$G_{\mu\nu} = \begin{pmatrix} 0 & -H_1 & -H_2 & -H_3 \\ H_1 & 0 & -D_3 & D_2 \\ H_2 & D_3 & 0 & -D_1 \\ H_3 & -D_2 & D_1 & 0 \end{pmatrix}, \tag{4.1.19}$$

and we arrive at the desired differential form:

$$G = H \wedge dx^0 - D, \tag{4.1.20}$$

where H has been defined as a 1-form,

$$H \coloneqq H_1dx^1 + H_2dx^2 + H_3^dx^3,$$

and D as a 2-form,

$$D \coloneqq D_1dx^2 \wedge dx^3 + D_2dx^3 \wedge dx^1 + D_3dx^1 \wedge dx^2.$$

This makes sense when we consider that the divergence of a vector can be written as the exterior derivative of a 2-form in $\mathbb{R}^3$. Similarly, we know that the curl of a vector can be shown to be the exterior derivative of a 1-form in Euclidean tridimensional space. Moreover, recall that in covariant form we had two equations describing a Born-Infeld system, i.e., the Bianchi identity and the field equation:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \quad \partial_\mu \tilde{G}^{\mu\nu} = 0.$$

Since tensors $F^{\mu\nu}$ and $G^{\mu\nu}$ are simply the component form of their respective differential 2-form, $F = E \wedge dx^0 + B$ and $G = H \wedge dx^0 - D$, one can equivalently describe said system as the dynamics arising from two closed forms:

$$dF = 0, \quad dG = 0. \tag{4.1.21}$$

In particular, one can show that

$$dG = 0 \iff \begin{cases} \nabla \cdot \mathbf{D} = 0 \\ \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0 \end{cases},$$

while $dF = 0$ is equivalent to the remaining two equations. Furthermore, in the same way that one can show that Maxwell's field equation in the presence of charge and current densities is given by

$$d \star F = \star J, \tag{4.1.22}$$

where J is a 1-form representing said electromagnetic densities, one can show that the Born-Infeld field equation is of the form

$$dG = \star J_f \tag{4.1.23}$$

for J_f a 1-form representing free charge and current densities. Nevertheless, we will mostly consider the vacuum equations since it is in that setting that Born-Infeld inherits Maxwell's duality symmetries.

Notice that, formally, the conversion from Maxwell electrodynamics to the Born-Infeld model is given by the transformation

$$\tilde{F} \to G,$$

where

$$\tilde{F} = \left(B_1 dx^1 + B_2 dx^3 + B_3 dx^3\right) \wedge dx^0 - \left(E_1 dx^2 \wedge dx^3 + E_2 dx3 \wedge dx^1 + E_3 dx^1 \wedge dx^2\right) = - \star F.$$

Therefore, the transformation can be equivalently expressed as

$$B_i \mapsto H_i, \quad E_i \mapsto D_i;$$

this is identical to the conversion between Maxwell's *standard* equations and the macroscopic ones as presented earlier, i.e., the Maxwell-like set of differential equations for Born-Infeld. Not only does this model accurately describe electrodynamic systems, but it also solves the problem of infinite self-energy for point particles; although it does not appear explicitly in (4.1), both H and D depend implicitly on b and are bounded by a finite value.

Overall, the main conclusion thus far is that Born-Infeld theory can be formulated in terms of differential forms in an analogous way to Maxwell; particularly, one retrieves Maxwell's theory under the replacements

$$\mathbf{D} \to \mathbf{E} \ \text{ and } \ \mathbf{H} \to \mathbf{B},$$

i.e. by neglecting the macroscopic effects of the material.

4.2 Duality symmetry

By following a similar procedure as in Section 4.1, we may also find the matrix representation of $\tilde{G}$:

$$\tilde{G}_{\mu\nu} = \begin{pmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & -H_3 & H_2 \\ -D_2 & H_3 & 0 & -H_1 \\ -D_3 & -H_2 & H_1 & 0 \end{pmatrix}. \tag{4.2.1}$$

Moreover, similarly as with $F_{\mu\nu}$, we have

$$\tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta} = -\left(\star G\right)_{\mu\nu}. \tag{4.2.2}$$

Therefore, in the same way that we observed

$$\mathbf{E} \to -\mathbf{B}, \quad \mathbf{B} \to \mathbf{E}$$

as the Heaviside transformations of Maxwellian electromagnetism in a relativistic setting, we can deduce an analogous feature in Born-Infeld for the electric displacement and the magnetic induction, i.e.,

$$\mathbf{D} \to -\mathbf{H}, \quad \mathbf{H} \to \mathbf{D}$$

The mathematical perspective: The 'role' of $\mathbf{D}$ and $\mathbf{H}$ is reversed in any given system upon a rotation thereof (by a particular, finite angle) in $\mathbf{D}/\mathbf{H}$-space. Differential 2-forms F and G, much like $\mathbf{D}$ and $\mathbf{H}$ or $\mathbf{E}$ and $\mathbf{B}$ as vectors, are two faces of the same coin. We have learned about this notion when studying the relation between F and its dual in Maxwell's theory; in this case, given that moving from a Maxwellian to a BI setting amounts to sending the dual of F to G within the field equation, the duality symmetry is maintained despite F and G not being each other's duals. Specifically, this conclusion can be drawn from the intrinsic, component-wise relation between $\tilde{F}$ and G. Furthermore, this tells us that Born-Infeld solutions can be rotated to obtain new solutions of the same (rotated) system.

One could summarise the above as stating that 1-form H and 2-form D are each other's Hodge dual (up to a sign, which depends on the metric signature). We can visualise this geometrically; for simplicity's sake, let us start by considering a three-dimensional manifold S (which is locally Euclidean). The visualization of a 1-form is that of a set of hypersurfaces: submanifolds of codimension one. Similarly, 2-forms are submanifolds of codimension two. Hence, we can describe D and H in S as sets of curves and planes, respectively. Moreover, the Hodge dual of a p-form is generally thought of as the orthogonal complement of said differential form. In the aforementioned three-dimensional setting, then, $\star D$ describes a set of planes perpendicular to the curves described by D. An analogous argument holds for H and $\star H$.

The physical intuition: A more physical way to visualise this duality, both between $(\mathbf{E}, \mathbf{B})$ and $(\mathbf{D}, \mathbf{H})$, is via Lorentz transformations. One can show [1] that, in the Maxwell case, a Lorentz boost in any direction transforms the electromagnetic fields (for $c = 1$) as follows:

$$B'_{\parallel} = B_{\parallel}, \quad \mathbf{B}'_{\perp} = \gamma \left(\mathbf{B}_{\perp} - \mathbf{v} \times \mathbf{E} \right); \tag{4.2.3}$$

$$E'_{\parallel} = E_{\parallel}, \quad \mathbf{E}'_{\perp} = \gamma \left(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B} \right). \tag{4.2.4}$$

Under a Lorentz transformation, a static charge q becomes a moving charge at a fixed velocity v, i.e., a current. Similarly, a static charge density ρ becomes a current density $\mathbf{J}$. An inverse transformation can then be applied to trace $\mathbf{B}$ back to $\mathbf{E}$. There is a clear conclusion arising from such an example: even though the electric and magnetic facets are separated in static electromagnetism, one can express $\mathbf{E}$-dependent expressions in terms of $\mathbf{B}$ (and vice versa) via Lorentz transformations. In subsection 5.1, we see how Lorentz boosts look like in Born-Infeld and relate the concept back to electromagnetic duality.

Furthermore, recognise that the $\mathbf{E} - \mathbf{B}$ duality in Maxwell theory is a direct consequence of applying Hodge duality to F. An analogous argument holds for the duality between $\mathbf{D}$ and $\mathbf{H}$ in macroscopic Maxwell and Born-infeld theory given the relation between G and $\tilde{F}$ or, as discussed, between $(\mathbf{E}, \mathbf{B})$ and $(\mathbf{D}, \mathbf{H})$. This duality holds in the presence of sources as long as magnetic charges (known as *monopoles*) are also considered; otherwise, it only occurs in the vacuum case. All in all, this goes to show that electromagnetic duality is a special case of the mathematical notion of duality.

4.3 Geometrical interpretation of the governing equations

4.3.1 The field equation

The above description is particularly helpful when visualising classical, four-dimensional spacetime (i.e., given by Minkowski's metric). Considering a particular timestamp of an electrodynamic system that is governed by the Born-Infeld equations, this reduces our treatment to a 3-dimensional, locally-Euclidean space such as S. Recall that the Born-Infeld equations of motion (when considering the presence of free charges/currents) are given by

$$dF = 0, \quad dG = \star J_f. \tag{4.3.1}$$

Hence, 2-form G describes a surface in this space, and 1-form J_f represents the (free) current 'traversing' perpendicular to said surface. The exterior derivative corresponds to the boundary of the shape described by a given differential form; hence,

$$dG = \star J_f$$

dictates that the electromagnetic displacement flux passing through a closed surface is equal to the free current on that surface, travelling along its boundary. This corresponds exactly to the physical interpretation of the original field equations in Euclidean space:

$$\nabla \cdot \mathbf{D} = 4\pi\rho_f, \quad \nabla \times \mathbf{H} = \frac{1}{c}\left(4\pi J_f + \frac{\partial \mathbf{D}}{\partial t}\right),$$

where J_f is arguably a higher-dimensional equivalent of ρ_f.

To strengthen the link between the physical and geometric pictures even further, we proceed to show that charge conservation in the system follows from the geometrical description. In the Maxwell case, we showed how the continuity equation is equivalent to

$$d \star J = d^2 \star F = 0$$

since $d^2 = 0$. Analogously, the free charges and currents also satisfy said equation in Born-Infeld theory:

$$\frac{\partial \rho_f}{\partial t} + \nabla \cdot \mathbf{j}_f = 0 \iff d \star J_f = d^2 G = 0.$$

One can integrate the dual free current 3-form $\star J_f$ over a tridimensional spacetime region. This is generally interpreted as the amount of charge or current that flows through a surface in a certain amount of time, provided the surface is space-like crossing a time-like interval.

4.3.2 The Bianchi equation

We may extend the previous subsection's notions to develop a geometrical understanding of the homogeneous equation:

$$dF = 0;$$

namely, the electromagnetic flux (described by $\mathbf{E}$ and $\mathbf{B}$) through the hypersurface given by F is zero, which means it is conserved (no flux enters or exits the area). This can also be understood, as previously discussed, by (locally) defining

$$F = dA$$

since F is itself the boundary of an object; in this case, the *volume* described by the electromagnetic potentials within A.

We have thus far formulated our entire description of Maxwell's laws in terms of differential forms by using the standard set of four equations depending on the electric and magnetic fields. However, one may recall that the introduction of potentials ϕ and $\mathbf{A}$, encoded into the contravariant vector A^μ, was an essential part of the covariant formulation of Maxwell's laws. We will examine how these can be reincorporated into the differential geometric approach.

First, recall the definition of the Maxwell tensor in terms of the electromagnetic four-potential:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\nu. \tag{4.3.2}$$

Moreover, recall the two equations that characterise said potential:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \ \mathbf{B} = \nabla \times \mathbf{A}. \tag{4.3.3}$$

We have previously shown that the curl of a vector (e.g., $\mathbf{B}$) is the exterior derivative of a 1-form in Euclidean tridimensional space. Similarly, the divergence of a vector can be written as the exterior derivative of a 2-form in $\mathbb{R}^3$. Therefore, given our definition of F, we can write the electromagnetic tensor F, a 2-form, as the exterior derivative of 1-form A:

$$F = B + E \wedge dx^0 = dA. \tag{4.3.4}$$

This is, in fact, the formal definition of the Faraday tensor; our definition of F in (4.3.2) is merely the same equation in component form. Hence, F is the exterior derivative of electromagnetic potential A: an exact 2-form.

Recall that exterior derivatives, together with differential forms, were introduced as an attempt to generalise Stokes' theorem to higher dimensions. In this context, the generalised version of Stokes' theorem tells us that for any compact smooth orientable manifold in two dimensions, the integral of F along M is equivalent to the evaluation of the electromagnetic potential along the boundary of M (which inherits its orientation [16]):

$$\int_M F = \int_M dA = \int_{\partial M} A. \tag{4.3.5}$$

We can think of M as a generic surface, which we can divide into infinitesimal regions. Intuitively, if one adds the flux through the boundaries of all the infinitesimal components of this surface, the interior boundaries will cancel out and only the flux through the boundary of the surface will remain.

Another important consequence of this result is that $dF = 0$, which is equivalent to the homogeneous Maxwell equations, is automatically satisfied by exactness of F ($ddA = 0$, i.e., a boundary has no boundary). Therefore, we recover one of the main conclusions we drew from the covariant formulation of Maxwell: introducing the electromagnetic tensor as part of our formulation is equivalent to having the homogeneous equations satisfied. It is then clear that the choice of action and/or Lagrangian has no influence on this property; it is a direct geometrical consequence of the definition of F as a differential form.

Hence, we find that the geometrical and physical descriptions of Born-Infeld electrodynamics are equivalent, which was also the case in Maxwell's theory. However, notice that the physical laws are but a special case (or, rather, a direct consequence) of the geometrical embedding of the theory, which allows us to study (4.3.1) in higher-dimensional settings. On the other hand, the previous visualisation does not hold in such spaces. In that case, the physical meaning is lost, but the description in terms of hypersurfaces and their boundaries and orthogonal sets maintains its geometrical meaning and can be seen as a higher-dimensional equivalent of the corresponding physical laws.

4.3.3 Line bundle interpretation

Another way to formulate Maxwell's laws in differential geometry, which can also be extended to Born-Infeld, is in terms of fibre bundles. Fibre bundles provide a convenient way to take products in topological spaces since we consider them to be product spaces in a local context (for an introductory reference, see [17]). Particularly, we can describe this theory in terms of the fibers of a principal U(1)-bundle, where the principal connection is given by ∇. It has as curvature a 2-form as

$$F = \nabla^2,$$

which implies that $dF = 0$ is automatically satisfied. As in the rest of the discussion, we consider this F to represent the field strength due to the electromagnetic fields. Should we consider the line bundle to be trivial with flat connection d (i.e., having no curvature in the space) we can write $F = dA$ for a 1-form A by writing the U(1)-bundle's connection as $\nabla = d + A$.

A similar conclusion can be drawn for both of the Born-Infeld 'dynamic' governing equations in the absence of (free) currents in the system. Particularly, it should be possible to write

$$G = \eta^2,$$

with $\eta = d + U$ the connection of the bundle, for some 1-form U that is analogous to A for F. In other words, we could deduce the existence of such a 1-form U such that

$$G = dU,$$

which we have seen can be found in a local context for F as a consequence of Stokes' theorem.

4.3.4 Metric dependence in Maxwell and Born-Infeld

Let us consider an electrodynamic system in the absence of sources, which we can describe in either (microscopic) Maxwell's or Born-Infeld's (macroscopic) terms. One thing that might be interesting to consider from a geometrical perspective is whether or not either of these theories, together with the dynamics that they describe, depends on the choice of metric. In a metric-independent theory, for instance, choosing Minkowski's spacetime metric to derive the equations of motion and proving other relevant results would not have any major impact on the generalisation of these results to an arbitrary four-dimensional spacetime manifold.

In Maxwell's electromagnetism, the governing equations are

$$dF = 0, \quad \text{or} \quad F = dA, \quad \text{and} \quad d \star F = 0.$$

This is clearly *metric-dependent* due to the presence of the Hodge star operator in the second equation, the definition of which is strictly related to the choice of metric. Since the exterior derivative is defined on any manifold, the Bianchi equation actually applies to any 4-dimensional manifold (and can be extended to higher dimensions), whilst the field equation is defined if the manifold is oriented and has a Lorentzian metric.

The formulation of Born-Infeld theory depends on the metric as well, but, depending on the author's conventions, this dependence may be either *implicit* or *explicit*. Given our definition of tensor $\tilde{G}^{\mu\nu}$ in

Section 3, we can actually describe both governing equations without explicitly introducing the metric via the Hodge dual:

$$dF = 0, \quad dG = 0.$$

However, it is often the case in existing literature that the definitions of (what in our notation are) $G^{\mu\nu}$ and $\tilde{G}^{\mu\nu}$ are interchanged; for instance, [13] uses the same notation, but [12] reverts it. Naturally, such a convention can never influence an essential feature as is metric (in)dependence in a theory. Indeed, in the same way that one may consider the definition of F ($F = dA$) to be a fundamental equation in Maxwell's theory, the definition of $\tilde{G}^{\mu\nu}$ (or $G^{\mu\nu}$, for that matter) should be taken into account as a governing equation of the system. Equations (4.1) and (4.1) tell us that both G and its dual depend explicitly on $\star F$ in their definitions. Hence, Born-Infeld is implicitly dependent on the Hodge star operator (and, hence, on the choice of metric) in our formulation. Even if the definitions are interchanged (e.g., as in [12]), the theory is metric-dependent due to the presence of the Hodge star in the definition of G; in this case, however, the field equation takes the form[7]

$$d \star G = 0,$$

which makes its dependence on the Hodge star operator explicit.

We conclude that the formulation we have studied depends on the metric choice. Hence, an interesting point for further research might be to study how the description of the theory changes in an arbitrary, four-dimensional spacetime manifold.

4.4 Extension to higher dimensions

In this subsection, we discuss in a little more detail how electrodynamics can be described in higher dimensions. We begin by discussing the corresponding changes in the formulation of Maxwell, which we then extend to Born-Infeld theory.

We begin by considering a potential description of Maxwell's laws in five dimensions, i.e., in a 4+1D space. This denotes that there is a single time dimension and four spatial dimensions. One can visualise this is by embedding four-dimensional objects to three dimensional space. For instance, it is possible to visualise a tetrahedron in 4-space by keeping one axis constant, which results in a regular tetrahedron. This is analogous to taking snapshots in time when considering t as the fourth dimension in special relativity (which can also be done here). Since the concepts of curl and vector are 'restricted' to three dimensions, the only way we have of extending Maxwell to higher dimensions is by reconsidering some of the definitions in the covariant and differential-geometric formulations.

In 4+1D space, Maxwell's equations (in Gaussian, $c = 1$ form) are still written as

$$\partial_\mu F^{\mu\nu} = 4\pi J^\nu \quad \text{and} \quad \partial_\mu (\star F)^{\mu\nu} = 0.$$

[7]Recall that we are considering $J = J_f = 0$ here.

However, the definition of tensors $F^{\mu\nu}$ and $(\star F)^{\mu\nu}$ needs to be revisited:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 & -E_4 \\ E_1 & 0 & B_{12} & B_{13} & B_{14} \\ E_2 & -B_{12} & 0 & B_{23} & B_{24} \\ E_3 & -B_{13} & -B_{23} & 0 & B_{34} \\ E_4 & -B_{14} & -B_{24} & -B_{34} & 0 \end{pmatrix}. \tag{4.4.1}$$

Note that the entries of the magnetic fields have been given a different notation, which is considered irrelevant in the present discussion. Instead of having three elements of electric nature and three of magnetic nature, there are four E_i components and six B_i entries. One can extend this reasoning inductively to show that

$$\#E \text{ entries} = n-1 \quad \text{and} \quad \#B \text{ entries} = \frac{(n-2)(n-1)}{2}. \tag{4.4.2}$$

Applying a three-dimensional spatial rotation to the 'standard' field strength tensor, as we have seen, results in a new tensor which is still consistent with considering $\mathbf{E}$ and $\mathbf{B}$ as 3D spatial vectors. In the 5D case, with four spatial dimensions, this still holds for $\mathbf{E}$ since it has $n-1$ entries for $n-1$ spatial dimensions, i.e., this is consistent with $\mathbf{E}$ being an $(n-1)$-vector field. However, this is not the case for $\mathbf{B}$ since it has more than $n-1$ components; instead, it is an antisymmetric tensor field. This also indicates that $\star F$ cannot be written in matrix form, as we confirm hereafter.

Let us visualise this from a more mathematical perspective; we have seen that Maxwell's equations can be expressed as

$$dF = 0 \quad \text{and} \quad d\star F = \star J$$

in terms of differential forms. We have extended our definition of F to be a higher-dimensional matrix, which means that it is still a 2-form (i.e., a surface). In the 4D case, we have that $\star F = -\tilde{F}$ is a 2-form as well, which allows us to write it in matrix form. In 5D, however, the Hodge star operator send $p = 2$-forms to $(n-p) = 3$-forms (i.e., volumes). Hence, $\star F$ is a $(n-2)$-form in the n-dimensional case. Similarly, J stays a 1-form and $\star J$ is a $(n-1)$-form. On the other hand, Poincaré's lemma tells us that we can still write

$$F = dA,$$

A being a 1-form representing the electromagnetic potentials, in a local context.

We proceed to evaluate how this affects the concept of electromagnetic duality. We know that the Hodge duality between F and $\star F$ is independent of the number of dimensions, and that one can relate F to the electric and magnetic fields via

$$F = E \wedge dx^0 + B.$$

Independently of the number of dimensions, E remains a 1-form and B a 2-form; for instance,

$$E = \sum_i^n E_i dx^i$$

is the general definition of E, and a similar extension holds for B as a 2-form. Since $E \wedge dx^0$ and B are both 2-forms, the Hodge star operator sends them both to 3-forms in 5D spacetime, which matches with $\star F$ being a 3-form. One can show [18] that the dual of $\star F$ is given by

$$\star(\star F) = (-1)^{p(n-p)} gF, \tag{4.4.3}$$

where g is the sign of the determinant of the metric. Hence,

$$\star(\star F) = -F$$

for an extension of Minkowski's metric. Overall, we confirm that duality invariance is maintained in higher-dimensional settings (for $J = 0$, as usual) for F and $\star F$ and that this implies $E - B$ duality. The definition of what F, $\star F$, E, B, and J *are* changes, but both the equations of motion and duality are maintained since they are more general given their geometrical nature.

The geometrical interpretation, hence, is identical as in subsection 4.3, but the physical meaning is lost. This means that, even though one could write an 'extension' of Maxwell's laws to n dimensions starting from the differential-geometric formulation, the resulting equations would not be backed up by experiments since these can only describe electrodynamics in a 3+1D spacetime. These conclusions are analogous for the Born-Infeld case, where the only further change is in the definition of G with respect to F and $\star F$ since it becomes the sum of a 2-form and a 3-form. The preservation of $\mathbf{D} - \mathbf{H}$ duality becomes more clear by considering an extension of the theory to account for magnetic monopoles, which is to be found in subsection 5.3.

5 Further physical properties

This section aims to review other physical properties of Born-Infeld theory that have not been covered in Sections 3 and 4. Mainly, it gives an overview of Lorentz boosts in Born-Infeld theory, proves that its Lagrangian is inherently Lorentz invariant, and discusses gauge invariance. Finally, the prediction of the existence of magnetic monopoles and dyons due to duality symmetry is discussed.

5.1 Lorentz transformations

In this section we study how G and, by extension, the components of $\mathbf{D}$ and $\mathbf{H}$, transform under Lorentz boosts. We will then proceed to show that Born-Infeld is a Lorentz invariant theory and the physical implications of this feature.

Consider Minkowski's spacetime with coordinates (t, x, y, z), as usual, in a rest frame S. Moreover, consider a different inertial frame S' moving at a constant velocity v in the positive-x direction. We want to see how G transforms in the moving frame. The general form of such a transformation is

$$(G')^{\mu\nu} = \Lambda^{\mu}_{\alpha} G^{\alpha\beta} \Lambda^{\nu}_{\beta} = \left(\Lambda \bar{G} \Lambda^T\right)^{\mu\nu}, \tag{5.1.1}$$

where Λ is the matrix representation of the Lorentz transformation and $\bar{G}$ is considered as the matrix representation of G, given by (4.1.19). In this case, given the chosen set of coordinates and the direction of S' with respect to S, the Lorentz matrix is given by

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.1.2}$$

Note that $\beta = v$ here since we have taken $c = 1$. One can then use this to compute the transform of G in matrix form:

$$G' = \Lambda\bar{G}\Lambda^T = \begin{pmatrix} 0 & H_1 & \gamma\left(H_2 + vD_3\right) & \gamma(H_3 - vD_2) \\ -H_1 & 0 & -\gamma\left(D_3 + vH_2\right) & \gamma\left(D_2 - vH_3\right) \\ -\gamma\left(H_2 + vD_3\right) & \gamma\left(D_3 + vH_2\right) & 0 & -D_1 \\ -\gamma(H_3 - vD_2) & -\gamma\left(D_2 - vH_3\right) & D_1 & 0 \end{pmatrix}. \tag{5.1.3}$$

This is an anti-symmetric matrix, as expected. Component-wise, we obtain the following transformations:

$$H_1' = H_1, \quad H_2' = \gamma\left(H_2 + vD_3\right) \quad H_3' = \gamma\left(H_3 - vD_2\right); \tag{5.1.4}$$

$$D_1' = D_1, \quad D_2' = \gamma\left(D_2 - vH_3\right) \quad D_3' = \gamma\left(D_3 + vH_2\right). \tag{5.1.5}$$

Unsurprisingly, one can obtain the transformed components of $\mathbf{D}$ from the transformed HF components by applying the discussed duality transformations ($\mathbf{H} \to \mathbf{D}$, $\mathbf{D} \to -\mathbf{H}$). In other words, the Lorentz transform of a dual is equivalent to the dual of a Lorentz transform.

More generally, one can show [19] that the fields in S' are given by

$$H_{\parallel}' = H_{\parallel}, \quad \mathbf{H}_{\perp}' = \gamma\left(\mathbf{H}_{\perp} - \mathbf{v} \times \mathbf{D}\right); \tag{5.1.6}$$

$$D_{\parallel}' = D_{\parallel}, \quad \mathbf{D}_{\perp}' = \gamma\left(\mathbf{D}_{\perp} + \mathbf{v} \times \mathbf{H}\right). \tag{5.1.7}$$

Note that one can extend the physical intuition behind $\mathbf{E} - \mathbf{B}$ duality from subsection 4.2 to explain $\mathbf{D} - \mathbf{H}$ duality, given that the Lorentz transformations take the exact same form under the usual exchange of fields $\mathbf{E} \leftrightarrow \mathbf{D}$ and $\mathbf{B} \leftrightarrow \mathbf{H}$.

We proceed to show that Born-Infeld theory is inherently Lorentz invariant in Minkowski spacetime, where

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Consider the Lagrangian density as given in (3.1). In the aforementioned metric, one obtains

$$\mathcal{L} = \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})} - 1.$$

Therefore, we want to prove that the term

$$\sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})}$$

is invariant under Lorentz transformations.

Consider a Lorentz transformation as described in [20]:

$$(x')^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}, \text{ where } \Lambda^{\mu}_{\nu} \text{ satisfies } g^{\mu\nu} = \Lambda^{\mu}_{\sigma} g^{\sigma\tau} \Lambda^{\nu}_{\tau}.$$

The definition of Λ, therefore, implies by the product rule that

$$\det(\Lambda)^2 = 1 \implies \det(\Lambda) = \pm 1.$$

In other words, Λ can preserve orientation ($\det(\Lambda) = 1$, 'proper' transformations) or reverse it ($\det(\Lambda) = -1$, 'improper' transformations). Apply this transformation to the argument of the determinant:

$$(g^{\mu\nu} + F^{\mu\nu})' = \Lambda^{\mu}_{\sigma} \left(g^{\sigma\tau} + F^{\sigma\tau}\right) \Lambda^{\nu}_{\tau}. \tag{5.1.8}$$

One can then employ the definition of Lorentz matrix Λ to show that

$$\det\left((g^{\mu\nu} + F^{\mu\nu})'\right) = \det\left(g^{\mu\nu} + F^{\mu\nu}\right). \tag{5.1.9}$$

This shows that the BI Lagrangian density is manifestly Lorentz invariant.

Since the Born-Infeld Lagrangian is Lorentz invariant, we have

$$\partial_{\nu} \left(F'\right)^{\mu\nu} = 0, \quad \text{and} \quad \partial_{\nu} \left(G'\right)^{\mu\nu} = 0, \tag{5.1.10}$$

or, by extension,

$$dF' = 0, \quad dG' = 0. \tag{5.1.11}$$

The same applies in the presence of free currents. One way to visualise this is by considering two observers of the system: observer 1 sees the system as static ('rest frame'), while observer 2 perceives the system moving away at a constant velocity v. We have seen that, under the corresponding Lorentz transformations, the role of the electric and magnetic fields is exchanged, and the same holds for the electric displacement and magnetic induction fields. However, the overall description of the physical laws remains unchanged between the two inertial frames, which agrees with the special relativity postulates: the laws of nature are identical in all reference frames.

5.2 Gauge invariance

The next step is checking whether the Born-Infeld Lagrangian is gauge invariant, which is a crucial property in Maxwellian electromagnetism. Recall that the Lagrangian (including $J_f \neq 0$) can be written as

$$\mathcal{L} = \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})} - \sqrt{-\det(g_{\mu\nu})} + 4\pi J^{\mu}_{f} A_{\mu}.$$

This is not particularly hard to show for the first two terms of the Lagrangian, given that the metric is trivially gauge independent. As for the field strength tensor, consider the gauge transformation $A'^{\mu} = A^{\mu} - \partial^{\mu}\chi$ applied to the Faraday tensor:

$$\begin{aligned} F'^{\mu\nu} &= \partial^{\mu} A'^{\nu} - \partial^{\nu} A'^{\mu} \\ &= \partial^{\mu} \left(A^{\nu} - \partial^{\nu}\chi\right) - \partial^{\nu} \left(A^{\mu} - \partial^{\mu}\chi\right) \\ &= \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} + \partial^{\mu}\partial^{\nu}\chi - \partial^{\nu}\partial^{\mu}\chi \\ &= \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \\ &= F^{\mu\nu}. \end{aligned}$$

Hence, $F^{\mu\nu}$ is gauge invariant. Let us now address the remaining ('interaction') term of the Lagrangian. Note that:

$$J_{\mu} \left(A'^{\mu} - A^{\mu}\right) = -J_{\mu}\partial^{\mu}\chi = \left(\partial^{\mu} J_{\mu}\right)\chi - \partial^{\mu}\left(J_{\mu}\chi\right). \tag{5.2.1}$$

The first term vanishes because of the current conservation law, which in turn is a direct consequence of the covariant inhomogeneous equation:

$$\partial_{\mu} J^{\mu} = \partial_{\mu}\partial_{\nu} F^{\mu\nu} = 0. \tag{5.2.2}$$

The second term can be neglected since it vanishes when integrating over spacetime (d^4x) when computing the corresponding action.

5.3 Magnetic monopoles and dyons

In this section we give an overview of the predictions in both Maxwell and Born-Infeld electrodynamics for magnetic monopoles and dyons.

5.3.1 Maxwell electrodynamics

We know that Maxwell's equations in the absence of external currents presents a duality symmetry, which can be expressed in terms of either the original fields or the electromagnetic tensor F and its dual. One could argue that, whenever $J \neq 0$, this symmetry of Maxwell's equations must also hold by considering an equivalent rotation for the free currents and charges in the system. Hence, albeit experimentally unconfirmed, the existence of magnetic monopoles is predicted by the duality symmetry of Maxwell's equations, the transformations of which can be written as

$$\mathbf{E} = \mathbf{E}\cos(\phi) - \mathbf{B}\sin(\phi), \tag{5.3.1}$$

$$\mathbf{B} = \mathbf{E}\sin(\phi) + \mathbf{B}\cos(\phi), \tag{5.3.2}$$

$$\rho_e = \rho_e\cos(\phi) - \rho_m\sin(\phi), \tag{5.3.3}$$

$$\rho_m = \rho_e\sin(\phi) + \rho_m\cos(\phi), \tag{5.3.4}$$

$$J_e = J_e\cos(\phi) - J_m\sin(\phi), \tag{5.3.5}$$

$$\rho_m = J_e\sin(\phi) + J_m\cos(\phi). \tag{5.3.6}$$

Note we have made a distinction between the free charges and current densities due to electric charges and those due to magnetic monopoles. The corresponding modified Maxwell equations ($c = 1$) become

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e, \tag{5.3.7}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = -4\pi\mathbf{J}_m, \tag{5.3.8}$$

$$\nabla \cdot \mathbf{B} = 4\pi\rho_m, \tag{5.3.9}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 4\pi\mathbf{J}_e. \tag{5.3.10}$$

Furthermore, in the same way that we can write the electric field of a point charge q_e as[8]

$$\mathbf{E} = \frac{q_e}{4\pi r^2}\hat{\mathbf{r}},$$

the magnetic field of a monopole q_m is

$$\mathbf{B} = \frac{q_m}{4\pi r^2}\hat{\mathbf{r}}. \tag{5.3.11}$$

The main problem left to solve in this formulation is the modification of the electromagnetic potentials, which are bound to break down the duality symmetry when introduced. This can be solved by introducing yet another pair of scalar and vector potentials, denoted by ψ and $\mathbf{K}$, respectively. These result in the fields being redefined as

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi + \nabla \times \mathbf{K}, \tag{5.3.12}$$

$$\mathbf{B} = \frac{\partial \mathbf{G}}{\partial t} - \nabla\psi + \nabla \times \mathbf{A}. \tag{5.3.13}$$

The following dual transformations for the potentials ensure symmetry is maintained:

$$\psi \to \phi, \quad \phi \to -\psi, \quad \mathbf{K} \to \mathbf{A}, \quad \mathbf{A} \to -\mathbf{K}. \tag{5.3.14}$$

5.3.2 Born-Infeld electrodynamics

We are now in a position to study the electrodynamics of dyons and magnetic monopoles in a system described by the Born-Infeld model. Dyons, first proposed by Julian Schwinger in 1969 as a phenomenological alternative to quarks [21], is a hypothetical particle which simultaneously have electric and magnetic charges.

The dynamical equations in Born-Infeld theory are given by the Lagrangian density function, which can be written as

$$\mathcal{L} = b^2\left[1 - \sqrt{1 + \frac{1}{2b^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16b^4}F_{\mu\nu}\tilde{F}^{\mu\nu}}\right] + 4\pi J^\mu A_\mu \tag{5.3.15}$$

to account for the (free) charge and current densities. Note that both J^μ and A^μ denote the usual four-current density and four-vector potential, respectively. The resulting equations of motion (again,

[8] Note that we are using Gaussian units here, as in the field equations.

with $c = 1$) are

$$\nabla \cdot \mathbf{D} = 4\pi\rho_e, \tag{5.3.16}$$

$$\nabla \cdot \mathbf{B} = 4\pi\rho_m, \tag{5.3.17}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = -4\pi\mathbf{J}_m, \tag{5.3.18}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 4\pi\mathbf{J}_e. \tag{5.3.19}$$

By applying the BIon solution, found in subsection 3.3, to the point charge case we find

$$\mathbf{E} = \frac{q_e}{r^2}\left(1 + \frac{1}{r^4}\frac{q_e^2}{b^4}\right)^{1/2}\hat{\mathbf{r}}. \tag{5.3.20}$$

Since we are considering magnetic charges, we can straightforwardly extend the previous result to account for q_m in the presence of monopoles:

$$\mathbf{E} = \frac{q_e}{r^2}\left(1 + \frac{1}{r^4}\frac{q_e^2 + q_m^2}{b^4}\right)^{1/2}\hat{\mathbf{r}}, \tag{5.3.21}$$

which describes a dyon. Furthermore, we discussed in subsection 3.3 how a Born-Infeld system yields a singular electric displacement $\mathbf{D}$ but solves the problem of the divergence of electric fields for point charges. However, the field of a magnetic monopole, given by (5.3.1), remains singular here, which means that duality symmetry does not hold here unless some of the definitions are revised, as was done in the Maxwell case. This is the goal of the remainder of the present subsection.

The natural decision is to revisit the definitions that describe the dynamics of the system; namely, the Lagrangian density function. We start by defining a new type of field strength tensor to account for the presence of the newly introduced scalar and vector potentials:

$$K_{\mu\nu} \coloneqq \partial_\mu K_\nu - \partial_\nu K_\mu, \tag{5.3.22}$$

where we have defined $K_\mu \coloneqq (\psi, -\mathbf{K})$. We denote its (Levi-Civita) dual as

$$\tilde{K}_{\mu\nu} \coloneqq \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}K_{\alpha\beta}, \tag{5.3.23}$$

which we can combine with the standard definition of the field strength tensor from Maxwell to define tensor

$$Z_{\mu\nu} \coloneqq F_{\mu\nu} + \tilde{K}_{\mu\nu}. \tag{5.3.24}$$

In order to preserve duality symmetry, we define a new Born-Infeld Lagrangian, which amounts to an extension of the standard one:

$$\mathcal{L}' \coloneqq b^2\left[1 - \sqrt{1 + \frac{1}{2b^2}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{16b^4}Z_{\mu\nu}\tilde{Z}^{\mu\nu}}\right] + 4\pi J_e^\mu A_\mu + 4\pi J_m^\mu K_\mu. \tag{5.3.25}$$

It is straightforward to show that solving the corresponding Lagrangian equations of motion,

$$\partial_\mu\left(\frac{\partial\mathcal{L}'}{\partial A_{\nu,\mu}} - \frac{\mathcal{L}'}{\partial A_\nu}\right) = 0, \quad \text{and} \quad \partial_\mu\left(\frac{\partial\mathcal{L}'}{\partial K_{\nu,\mu}} - \frac{\mathcal{L}'}{\partial K_\nu}\right) = 0, \tag{5.3.26}$$

yields the following field equations:

$$\nabla \cdot \mathbf{D} = 4\pi\rho_e, \tag{5.3.27}$$

$$\nabla \cdot \mathbf{H} = 4\pi\rho_m, \tag{5.3.28}$$

$$\nabla \times \mathbf{D} + \frac{\partial \mathbf{H}}{\partial t} = -4\pi\mathbf{J}_m, \tag{5.3.29}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 4\pi\mathbf{J}_e. \tag{5.3.30}$$

By applying the substitution $b \mapsto ib$, where $i := \sqrt{-1}$ as usual, we see that the previous set of equations is invariant under the transformations described for Maxwell, i.e., (5.3.2)-(5.3.6) and (5.3.1).

Finally, we can apply a BIon approach analogous to that in subsection 3.3 to show that the electromagnetic fields describing dyons in Born-Infeld systems are given by

$$\mathbf{E} = \frac{q_e}{r^2}\left(\frac{1 - \frac{1}{r^4}\frac{q_m^2}{b^4}}{1 + \frac{1}{r^4}\frac{q_e^2}{b^4}}\right)\hat{\mathbf{r}}, \tag{5.3.31}$$

$$\mathbf{B} = \frac{q_m}{r^2}\left(\frac{1 + \frac{1}{r^4}\frac{q_e^2}{b^4}}{1 - \frac{1}{r^4}\frac{q_m^2}{b^4}}\right)\hat{\mathbf{r}}. \tag{5.3.32}$$

It is evident that duality symmetry does hold now given that both the electric and magnetic fields have an upper bound. Moreover, setting $K^\mu = J_m = \rho_m = 0$ clearly recovers the usual Born-Infeld formulation, as desired. Note that the electric field of a dyon depends on the magnitude of both the electric and magnetic charges; the same holds for the magnetic field. Finally, we can argue that the geometrical understanding of the resulting field equations in terms of differential forms, i.e.,

$$dG = \star J_e, \quad d\star G = \star J_m, \tag{5.3.33}$$

is an extension of the analysis we conducted on $dG = \star J_f$ in the original Born-Infeld setting.

6 Applications in theoretical physics

In this section, a rudimentary introduction to some of the main applications of Born-Infeld theory in other branches of theoretical physics is provided. Mainly, we see how the BI Lagrangian is linked to the description of D-branes in string theory and that of phantom energy fields in cosmology. Implementations in non-linear optics, nuclear, and high-energy physics are briefly reviewed as well. It is due to these applications, among others, that Born-Infeld has been revisited within the physical community over the span of the last few decades.

6.1 String theory: description of branes

In this section we will aim to shed some light on how the Born-Infeld model is related to the description of D-branes in string theory, which resurrected interest in the model in the 1970s. For a complete introductory course on string theory, including a discussion on its relation to both Maxwell and Born-Infeld electrodynamics, see the book by Barton Zwiebach [12].

The first concept that needs to be introduced in this regard is that of D-branes. Dirichlet branes, usually abbreviated as *D-branes*, are a type of open string which satisfy the Dirichlet boundary conditions at the ends. Generally, the equations of motion of string theory require that, provided a string has endpoints (called *open* string), either the Neumann or the Dirichlet boundary conditions are satisfied at said points. The former corresponds to an endpoint moving freely through spacetime at the speed of light, while the latter denote static endpoints. Moreover, each coordinate of the string must satisfy either of these conditions. One denotes by a Dp-brane a p-dimensional hyperplane which confines the string's motion, which happens when p spatial dimensions satisfy the Neumann boundary condition. D-branes are dynamical objects, which means they can be described by a Lagrangian function. In this section we give an overview of how they are related.

Zwiebach shows [12] that the Lagrangian of a D-brane is given by

$$L = -V_p T_p(g)\sqrt{1 + (2\pi\alpha' B)^2}, \tag{6.1.1}$$

where $T_p(g)$ is the tension of a static brane, B denotes the magnitude of the magnetic field that the brane carries on its world-volume, and α' is the square root of the string length (usually denoted $\updownarrow: s$). The remaining term, V_p, arises when we consider a one-dimensional world curled up into a circle of finite radius R and $(p-1)$ dimensions curled up into some compact space of volume V_{p-1}. Instead of focusing on the derivation of this Lagrangian function, we consider its relation to the BI Lagrangian alone.

In the aforementioned derivation, the electric field on the brane has been considered to vanish ($\mathbf{E} = 0$). From (4.1.13), we know that the Born-Infeld Lagrangian for a vanishing electric field is given by

$$\mathcal{L} = 1 - \sqrt{1 + \frac{1}{b^2}\mathbf{B}^2}. \tag{6.1.2}$$

The additive factor can be dismissed in this context; it was introduced to cancel the constant term in the Lagrangian, which now represents (up to some factor of $\pm b^2$) the rest energy of the D-brane in the absence of electromagnetic fields. We can now set

$$b = \frac{1}{2\pi\alpha'} \tag{6.1.3}$$

to compare the Lagrangian functions. Note that this relation implies that the square root of the brane's length is inversely proportional to the maximum permitted energy for electric fields in Born-Infeld. By rewriting (6.1) using this relation and neglecting the additive term, as previously argued, we obtain

$$\mathcal{L} = -\sqrt{1 + (2\pi\alpha'\mathbf{B})^2}. \tag{6.1.4}$$

Equation (6.1) can also be written as a Lagrangian density by removing (i.e., rescaling by) volume V_p:

$$\mathcal{L} = -T_p(g)\sqrt{1 + (2\pi\alpha'\mathbf{B})^2}. \tag{6.1.5}$$

This reduces to the Born-Infeld Lagrangian when normalising the string's tension.

Zwiebach also shows that, in general, the Lagrangian of a Dp-brane in the presence of electromagnetic fields can be written [12] as

$$\mathcal{L} = -T_p(g)\sqrt{-\det\left(\eta_{mn} + 2\pi\alpha' F_{mn}\right)}, \tag{6.1.6}$$

which is a generalisation of (6.1), i.e., including electric fields as well. By considering a normalised string tension and rewritting the general Born-Infeld Lagrangian via the above considerations, we conclude that the dynamics of the electromagnetic fields that arise from open string endpoints are described by the same Lagrangian function. This relationship between Born-Infeld and D-branes can be further explored and has been the subject of several studies in string theory [22, 23, 24, 25, 26].

6.2 Cosmology & phantom fields

The study of dark matter and dark energy has been one of the most fundamental problems in theoretical cosmology for decades. Many candidates for dark energy have been proposed so far to fit the current observations, the major difference among them being their prediction for the equation-of-state parameter w. This parameter has a crucial influence on the predictions of cosmological models, specially regarding the fate of our Universe (for an undergraduate-level introduction to these cosmological concepts, see [27]).

Recent observations [28] suggest that the value of w can be smaller than -1, which gives rise to the hypothetical existence of *phantom energy*. The main characteristics of this type of dark energy are that it possesses negative kinetic energy and predicts that the Universe will expand more than that predicted by the cosmological constant, leading to a *Big Rip*, i.e., the matter within the universe is progressively torn apart due to excessive expansion. It would also imply that energy density increase with time and might lead to the existence of astrophysical or cosmological wormholes [29].

The reason why Born-Infeld theory is relevant in this discussion is due to the tachyon fields of string theory, which represent a natural candidate to describe phantoms [29]. The term *tachyon* is used in theoretical physics to denote imaginary mass fields, such as the Higgs field. Explaining the role of tachyonic fields in string theory is out of the scope of this project; for more information, see [30].

The Lagrangian depends on the model that is used to describe the pertinent cosmological parameters, but can generally be written as a Born-Infeld with some potential $V(T)$:

$$L_{eff} = V(T)\sqrt{1 - \partial_\mu T \partial^\mu T}. \tag{6.2.1}$$

It can be shown [31, 32] that the action of the theory is written as

$$S = \int \frac{3\pi}{4G} a(1 - \dot{a}^2)dt + \int 2\pi^2 a^3 \frac{1}{b}\left(1 - \sqrt{1 - b\dot{T}^2 - V(t)}\right) dt \tag{6.2.2}$$

in a Robertson-Walker metric. The two relevant degrees of freedom would then be the scale factor $a(t)$ and the scalar tachyon field $T(t)$. This can be used to make cosmological predictions based on the existence of phantom fields.

6.3 Other applications

6.3.1 Non-linear optics

One of the properties that makes Born-Infeld theory interesting in optics is that its excitations propagate without the shocks common to generic nonlinear models [33, 34]. In this context, the Born-Infeld model can be used to evaluate wave-fronts by evaluating the intersections of a set of characteristic curves [35]. This paper argues for the use of Born-Infeld in this description given that it is the only non-linear theory of electrodynamics that satisfies both the shock-free propagation and the duality symmetry properties.

6.3.2 Nuclear physics & high-energy physics

The properties of Born-Infeld electrodynamics are also relevant in the study of Werner Heisenberg's model for high-energy nucleon-nucleon scattering. Heisenberg considered a non-linear higher-derivative action for the pion, which closely resembles a BI Lagrangian:

$$\mathcal{L} = l^{-4}\left[1 - \sqrt{1 + l^4\left[\left(\partial_\mu \phi\right)^2 + m^2\phi^2\right]}\right]. \tag{6.3.1}$$

This is because in the high energy limit, many pions (which are of low mass) are created, so one ought to consider a pion field as the effective field. An introductory review of this model and the relevance of the Born-Infeld lagrangian within are given in [36].

7 Conclusions and further research

This section aims to review the main conclusions of this project and give an outlook on interesting points for further research.

We began this study by giving an overview of the properties of Maxwell's electrodynamics in two different formulations. In the covariant paradigm, the original set of four Maxwell differential equations can be reduced to a characterisation of the electromagnetic tensor F and its Hodge dual $\star F$ in terms of their divergence. One of these equations, that which represents the two homogeneous Maxwell equations, is equivalent to Bianchi's identity; hence, it is automatically satisfied when we consider said tensor to depend explicitly on the derivatives of the four-current vector. Therefore, one can describe the dynamics in every electromagnetic system with a single field equation, representing the two inhomogeneous Maxwell equations. By comparing F to its dual, we have introduced the duality between electric and magnetic fields as a transformation that allows for a simple comparison between the homogeneous and inhomogeneous equations. These notions can be generalised via the introduction of differential forms to get rid of coordinate dependence. The field strength tensor F becomes a differential 2-form describing the electromagnetic field in spacetime, which can be expressed as the exterior derivative of the four current vector A, represented by a 1-form. We have seen that, in the absence of sources, Maxwell's equations present an SO(2) duality symmetry, which shows that any solution to Maxwell's equations can be rotated to generate new solutions.

The discussion of Born-Infeld theory began in Section 3. We concluded that, like Maxwell, the Born-Infeld model can be described via two equations of motion in covariant form that give the divergence of two tensors: $\tilde{F}$ and $\tilde{G}(F, \tilde{F})$. The former is the Bianchi equation, while the latter is formally identical to F in the Maxwell formulation under replacements $\mathbf{E} \to \mathbf{D}$ and $\mathbf{B} \to \mathbf{H}$. Therefore, via the introduction of modified constitutive relations, one can show that Born-Infeld theory successfully reproduces the results of the macroscopic Maxwell equations, which makes it compatible with the experimental basis of classical electrodynamics. Moreover, it solves the problem of the electron's infinite self energies via the introduction of a factor $\frac{1}{b}$, which scales the self-interactions in the theory.

In Section 4, we have reviewed the differential-geometric formulation of Born-Infeld and its significance, both from a physical and a geometrical perspective. We concluded that Born-Infeld presents an SO(2) duality symmetry as that in Maxwell's electrodynamics. This duality can be explained physically by considering Lorentz boosts on the field components; more generally, however, it is understood as a consequence of Hodge duality. The geometrical interpretation of the equations of motion is shown to be conceptually equivalent to the corresponding physical intuition in a four-dimensional setting. When considering higher-dimensional spaces, the geometrical understanding is maintained, but the meaning of the relevant differential forms changes. This indicates that the physical meaning of electrodynamics is merely a geometrical consequence of the definitions that we use to describe their framework.

Furthermore, we showed that other crucial properties of Maxwell's model, such as Lorentz and gauge invariance and the prediction of the existence of magnetic monopoles due to duality invariance, are also present in the Born-Infeld model. Finally, we discussed how Born-Infeld theory is relevant in other branches of theoretical physics, such as string theory (description of D-branes) and cosmology (phantom energy models).

The main suggestion for further research is to delve deeper into the geometrical understanding of the theory. There exists such a geometrical interpretation of some field theories with similar features, including scalar theories and self-interactions, such as DBI theory and the Special Galileon [37]. Therefore, the main question to be addressed is whether an analogous geometrical interpretation can be found for the Born-Infeld model.

To give an example, we have seen that, when considering Minkowski's flat spacetime metric, the Born-Infeld Lagrangian reduces to

$$\mathcal{L} = \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})} - 1.$$

The '-1' factor is usually considered trivial since it has no implications in the derivation of the equations of motion via the least action principle. Hence, in essence, the dynamics of an electromagnetic system are described by Born-Infeld theory via the following Lagrangian density:

$$\mathcal{L} = \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})}.$$

The main advantage of writing the Lagrangian like this instead of in terms of the fields is that it allows for an obvious generalisation to any number of dimensions. In essence, we may interpret that Born-Infeld theory induces a new metric based on Minkowski's, which we have introduced earlier as

$$a_{\mu\nu} := g_{\mu\nu} + F_{\mu\nu};$$

split into a symmetric and an anti-symmetric part. In order to study the geometrical background of said theory, it might be interesting to consider the geometric interpretation of other theories in which the Lagrangian density is defined as the square root of the determinant of the (induced) metric. By considering this to be the metric instead of Minkowski's, one might be able to learn more about the geometrical embedding of the theory. On the same note, it would be interesting to study how the formulation of the model changes when considering a general, four-dimensional spacetime manifold, which would also improve understanding of the higher-dimensional extension of the theory.

8 Appendix: physical and mathematical contents

Separation of physical and mathematical components

Physics: All (sub)sections of this bachelor thesis relate to some degree to contents of the BSc Physics. Main relevant topics: Classical Electrodynamics, Lagrangian and Hamiltonian mechanics. Other relevant courses: Classical Mechanics, Cosmology, Mathematical Physics, Optics, Symmetry in Physics.

Mathematics: The mathematical part of this thesis mostly concerns Sections 2, 3 (except 3.3), and 4. Mathematical preliminaries are mainly discussed in the Maxwell Honours Project, which has been cited throughout. List of relevant topics: mainly (Differential) Geometry and Multivariable Analysis, but also Group Theory, Analysis on Manifolds, and basic Algebra and (Multivariable) Calculus.

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