University of Groningen

Bachelor Thesis

The Non-Relativistic String Limit and Galilean Electrodynamics

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July 1, 2021

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Abstract

Non-Relativistic (NR) theories of electrodynamics can be obtained in several ways. In this research we obtain the NR electric and magnetic limit of the Maxwell Equations and show that the resulting equations are invariant under the Galilean transformations. Furthermore, we reproduce Galilean Electrodynamics (GED) by applying the particle limit, in which spacetime is decomposed into space and time, to a Maxwell Lagrangian with a scalar field. This Lagrangian is obtained by dimensionally reducing a five-dimensional Maxwell Lagrangian in a spatial direction. GED can be used to obtain the equations of motion that are also obtained in the electric limit. Furthermore, GED possesses two emergent scale symmetries. Alternatively, GED is reproduced by performing a dimensional reduction in a lightlike direction, called a null reduction, on the five-dimensional Maxwell Lagrangian. Subsequently, we set out to reproduce the same theory with a new procedure, called the string limit. In this limit space is decomposed into a two-dimensional subspace, the longitudinal space, with Minkowskian signature metric and its complement, the transverse space, whose metric has Euclidean signature. This limit is applied to a five-dimensional Maxwell Lagrangian, after which we reduce in a spatial direction. The result is Galilean Electrodynamics. Therefore, we have shown that the string limit applied to a five-dimensional Maxwell Lagrangian, followed by a spatial dimensional reduction, achieves the same result as the particle limit and the null reduction.

Contents

Introduction

The principle of special relativity was originally formulated by Albert Einstein in his 1905 paper *Zur Elektrodynamik bewegter Körper*[1]. The title of this work immediately tells us that special relativity was established in the context of electrodynamics. In fact, it was the need for a reference frame independent speed of electromagnetic waves that necessitated special relativity in the first place. In this thesis we are going to examine non-relativistic (NR) electromagnetism. This might seem odd at first, since we just mentioned electromagnetism requires special relativity. In fact, it is an interesting exercise to examine what kind of theory a physicist who was intent on preserving Galilean relativity might have formulated. Examining the non-relativistic limit of the Maxwell Equations is exactly what Le Bellac and Lévy-Leblond did in their 1973 paper *Galilean Electromagnetism*[2].

In their paper Le Bellac and Lévy-Leblond discuss two NR limits of the Maxwell equations, the electric limit and the magnetic limit, which are named for either featuring a dominant electric field or a dominant magnetic field. In doing so they establish a Galilean invariant theory of electromagnetism. Recently it was shown that it is in fact possible to formulate a Lagrangian that, when taking a specific non-relativistic limit, first described in [3], leads to a Lagrangian for NR electrodynamics [4]. The result is the Galilean Electrodynamics action, which gives rise to same equations of motion as the electric limit from [2]. This GED action possesses two interesting emergent scale symmetries.

The GED action is obtained in [4] in two different ways. Firstly, by taking a specific NR limit, called the particle limit, of a Maxwell Lagrangian with a scalar field. This specific Lagrangian can be obtained by taking a spatial reduction of the Maxwell Lagrangian in five dimensions. Secondly by performing a dimensional reduction in a lightlike direction, called a null reduction, on a five-dimensional Maxwell Lagrangian. In this thesis we will reproduce GED with these two methods and examine a third method. This third method involves applying a new type of relativistic limit, the string limit, to a five-dimensional Maxwell Lagrangian. This procedure is followed by a dimensional reduction in a spatial direction. We will do all of this in flat spacetime, contrary to [4], which uses curved backgrounds. Our overarching question is then: Is it possible to reproduce Galilean Electrodynamics by taking the string limit of a five-dimensional Maxwell Lagrangian followed by a spatial dimensional reduction? We will show that this is in fact the case.

This thesis consists of three main chapters. In the first chapter an overview of classical electromagnetism is offered. We will review the basic principles of electromagnetism, including the Maxwell equations. A short intro to the covariant formulation of electromagnetism is given, followed by a section on Lagrangian electrodynamics. Furthermore, we will discuss the electromagnetic duality and offer a full treatment of the Lorentz invariance of the Maxwell equations.

In the second chapter we will examine NR limits of electromagnetism, starting by reproducing the electric and magnetic limit from [2], in both the classical and covariant formulation. Then we move on to establish the Galilean transformations and invariance of these limits. Finally, we move to reproduce Galilean Electrodynamics with the method from [4].

In the third and last chapter the string limit will be defined and performed on a five-dimensional Maxwell Lagrangian. We will also devote a section to the dimensional reduction and the null reduction. Our central result will be compared to the GED action, after applying a spatial dimensional reduction. Furthermore we will take a look at the string limit of the Lorentz transformations.

It is important to stress that the string limit is a new procedure and that there is no a priori guarantee that it delivers Galilean Electrodynamics. There is currently, to the best of our knowledge, no literature which attempts to reproduce Galilean Electrodynamics by applying the string limit. The reason examining this limit is interesting, is that the string limit might lead to a non-relativistic theory where it was otherwise not possible to obtain one. In principle the application of this procedure does not need to be restricted to electrodynamics.

Chapter 1

Classical Electromagnetism

In order to discuss the Non-Relativstic limits of Electromagnetism, we must first define the relativistic theory to which we apply such a limit. Firstly, a short overview of Maxwell's equations of electrodynamics will be offered, followed by a derivation of these equations in the covariant field formalism. Then we will move on to discuss the Lorentz transformations and the Lorentz invariance of electromagnetism.

1.1 Maxwell Equations

In the 19th century a great deal of research on electromagnetism was published by researchers such as Carl Friedrich Gauss, Michael Faraday, André-Marie Ampère and James Clerk Maxwell. Then, in the 1860's, James Clerk Maxwell published a series of papers entitled *on physical lines of force*[5]. In these papers he presented a set of four equations that describe the dynamics of moving charges. In his work Maxwell in fact combined his own work with that of Gauss, Faraday, and Ampère to create his theory of electrodynamics[6].

Experimentally the Lorentz force can be determined to be:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{1.1}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, q is the charge, and $\mathbf{v}$ is the velocity of the charge. The Lorentz force shows that there is an electric field that gives rise to a force in the direction of that field, and a magnetic field, which gives rise to a force perpendicular to both the velocity of the particle and the magnetic field itself.

The electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be defined in terms of potentials, as follows:

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}, \tag{1.2}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{1.3}$$

where V is the scalar electric potential and $\mathbf{A}$ is the magnetic vector potential.

The electric and magnetic fields obey a set of four partial differential equations, these are the Maxwell equations. Together with the Lorentz force they form the basis of classical electromagnetism. The four Maxwell Equations are:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho, \tag{1.4a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{1.4b}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.4c}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \tag{1.4d}$$

where ρ is the charge density, $\mathbf{J}$ the current vector, ϵ_0 the permittivity of free space and $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ the speed of light, with μ_0 the permeability of free space. These four equations are named Gauss's Law, Gauss's law for magnetism, the Maxwell-Faraday equation and the Ampére-Maxwell law, respectively.

An important observation is that the curl of the electric field is equal to the time derivative of the magnetic field, and the curl of the magnetic field is influenced by the time derivative of the electric field. Essentially this happens, because the electric and magnetic field are dual expressions of a single field: the electromagnetic field. The Maxwell equations imply the fields satisfy the wave equation. The electromagnetic waves propagate with the speed of light c. To prove this consider taking the curl of eq. (1.4c) in a vacuum:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \boldsymbol{\nabla} \times -\frac{\partial \mathbf{B}}{\partial t} = \frac{\partial(\boldsymbol{\nabla} \times \mathbf{B})}{\partial t} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{1.5}$$

and similarily for eq. (1.4d). Now substituting in $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$ or $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ for the electric and magnetic case respectively we obtain:

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}, \qquad \nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}, \tag{1.6}$$

which is precisely the three dimensional wave equation with velocity c[7].

In chapter 2 we will take the non-relativistic limit of the Maxwell equations in vacuum. Such a limit will indubitably affect the propagation of electromagnetic waves, as these travel at the speed of light.

1.2 Covariant Formulation of Electromagnetism

In section 1.1 we have offered an overview of the Maxwell equations. These describe the dynamics of the $\mathbf{E}$ and $\mathbf{B}$ fields, which can be defined in terms of a scalar potential V and a vector potential $\mathbf{A}$. In this section we will essentially derive the same principles, but expressed in terms of fields that are manifestly Lorentz invariant. This means we will now work in a Minkowski spacetime with a (-,+,+,+) metric convention.

First of all the two potentials V and $\mathbf{A}$ are unified into a single four-potential $A_\mu = (V, \mathbf{A})$, with $\mu = 0, 1, 2, 3$. The individual fields can still be defined as in eq. (1.2) and eq. (1.3). However, the fields are equivalently derived from the four potential in the electromagnetic tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.7}$$

From the definition of the electromagnetic tensor and the previous definitions of the electric and magnetic field it is clear that:

$$E_i = cF_{0i} = c(\partial_0 A_i - \partial_i A_0), \tag{1.8a}$$

$$B_i = \epsilon^{ijk} F_{jk} = \epsilon^{ijk}(\partial_j A_k - \partial_k A_i), \tag{1.8b}$$

with $i, j, k = 1, 2, 3$. The electromagnetic tensor has an important symmetry, namely the gauge invariance. This means that you can transform the potential as:

$$A_\mu \to A_\mu + \partial_\mu f, \tag{1.9}$$

where f is an arbitrary scalar function, without changing $F_{\mu\nu}$[7].

The Maxwell equations can be written in a compact form using the electromagnetic tensor with the following equation:

$$\partial_\mu F^{\mu\nu} = 0, \tag{1.10}$$

and the Bianchi Identity for the electromagnetic tensor:

$$\partial_{[\mu}F_{\nu\rho]} = 0. \tag{1.11}$$

Starting from the first equation we can recognize two separate equations arising from it, the Ampére-Maxwell law and Gauss's law:

$$\partial_0 F^{0j} + \partial_i F^{ij} = \frac{1}{c^2}\partial_t E^j + \epsilon^{ijk}\partial_i B_j = 0, \tag{1.12}$$

$$\partial_i F^{i0} = \partial_i E^i = 0. \tag{1.13}$$

Along with the definitions in eq. (1.8), these equations correspond to eq. (1.4c) and eq. (1.4a) in vacuum, respectively. On to the Bianchi Identity, which too has two components, the Maxwell-Faraday equation and Gauss's law for magnetism:

$$\partial_{[0}F_{ij]} = \partial_0 F_{ij} + \partial_{[i}F_{j]0} = \partial_t B^k + e^{ijk}\partial_i E_j = 0, \tag{1.14}$$

$$\partial_{[i}F_{jk]} = \partial_{[i}\epsilon_{jk]m}B^m = \partial_m B^m = 0. \tag{1.15}$$

These two components correspond to eq. (1.4c) and eq. (1.4b) in vacuum respectively. Now we have shown that the covariant formalism delivers the exact same equations of motion as derived in section 1.1.

1.2.1 Lagrangian Electrodynamics

In this section we have given an introduction to the Maxwell equations and to the covariant formalism of electromagnetism. The covariant formalism describes electromagnetism in a single tensor $F_{\mu\nu}$. This tensor together with the four-potential A_μ can be used to formulate a Lagrangian density for the electromagnetic field:

$$\mathcal{L} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu, \tag{1.16}$$

where J^μ is the four current $(c\rho, \mathbf{J})$. Therefore the second term is zero in vacuum. From this Lagrangian density equations of motion can be derived with the Euler-Lagrange equations. Varying with respect to A_μ gives the equation of motion:

$$\frac{\partial\mathcal{L}}{\partial A_\mu} - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)} = -J^\mu - \frac{1}{4\mu_o}\partial_\nu F^{\nu\mu} = 0 \rightarrow \partial_\nu F^{\mu\nu} = 4\mu_0 J^\mu, \tag{1.17}$$

which, in vacuum, is precisely equal to eq. (1.10).

1.3 Electromagnetic Duality

There exists a duality transform between the electric and magnetic fields, under which the theory of Electrodynamics as described in section 1.1 is invariant. The electromagnetic duality acts on the electric and magnetic fields as[8]:

$$\mathbf{E} \rightarrow -c\mathbf{B}, \qquad c\mathbf{B} \rightarrow \mathbf{E}. \tag{1.18}$$

We can apply this duality transformation to eq. (1.4) in vacuum, meaning that $\rho = 0$ and $\mathbf{J} = 0$.

The transformations of eq. (1.4a) and eq. (1.4b) in vacuum are trivial under the duality transform. Consider the transformations of eq. (1.4c) and eq. (1.4d) under eq. (1.18) :

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \to -c\nabla \times \mathbf{B} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = 0, \tag{1.19a}$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = 0 \to \frac{1}{c}\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0. \tag{1.19b}$$

Clearly these two equations are pairs under the duality transform and therefore the theory is invariant under these transforms. The electric and magnetic field are dual expressions of the electromagnetic field, and can therefore be exchanged.

1.4 Lorentz Transformations

Electrodynamics is inherently a relativistic theory and should therefore be invariant under the Lorentz Transformations. In this section we will derive how the electric and magnetic field transform under Lorentz transformations. The Lorentz transformations for a reference frame moving with velocity v in the x_1 direction[9]:

$$\begin{aligned} t' &= \gamma(t - \frac{vx_1}{c^2}), \\ x_1' &= \gamma(x_1 - vt), \\ x_2' &= x_2, \\ x_3' &= x_3, \end{aligned} \tag{1.20}$$

where $\gamma = 1/\sqrt{1-\frac{v^2}{c^2}}$. These Lorentz transformations are equivalent to $x'_\mu = \Lambda^\mu{}_\nu x'_\mu$ when applied to a vector $x_\mu = (ct, x_i)$, with Λ satisfying $\Lambda^T\eta\Lambda = \eta$. This condition is unsurprising as the Lorentz group $SO(1,3)$ belongs to the orthogonal group, which always satisfies this condition. Λ is defined as follows:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.21}$$

The electromagnetic tensor transforms as $F_{\mu\nu} \to \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu F_{\alpha\beta}$. Since the electric and magnetic field can be defined in terms of $F_{\mu\nu}$ as in eq. (1.8), we can derive the transformations of the fields by transforming the electromagnetic tensor. The resulting transformations are:

$$\begin{aligned} E_1' &= E_1, & B_1' &= B_1, \\ E_2' &= \gamma(E_2 - vB_3), & B_2' &= \gamma(B_2 + \frac{v}{c^2}E_3), \\ E_3' &= \gamma(E_3 + vB_2), & B_3' &= \gamma(B_3 - \frac{v}{c^2}E_2). \end{aligned} \tag{1.22}$$

For a derivation of these transformations see section A.2. Just like the mixing between the time and space coordinates in eq. (2.19), the electric and magnetic fields mix under Lorentz transformations. This means that the fields individually are not invariant under the Lorentz Transformations.

1.4.1 Lorentz Invariance

The principle of relativity states that the laws of nature are invariant in any inertial reference frame. Therefore the Maxwell Equations in eq. (1.4) should be invariant under the transformations described in eq. (1.22). In fact it is the invariance of the Maxwell equations which directly leads to the need for special relativity. As we discussed in section 1.1, the Maxwell equations give rise to electromagnetic waves travelling with the speed of light. The speed of light is equal to $1/\sqrt{\mu_0\epsilon_0}$, where μ_0 and ϵ_0 are constants, which you can measure in any reference frame. Therefore electromagnetic waves must travel at this speed in any reference frame, meaning the speed of light must be invariant[10]. This directly ties into the fact that the Maxwell equations are incompatible with Galilean relativity. If we attempted to transform the fields and equations with Galilean transformations, you would not get an invariant speed of light. The speed of light would instead become reference frame dependent under Galilean transformations.

To show the Lorentz invariance of the Maxwell equations we will consider these equations in vacuum and apply the relevant transformations to the fields and coordinates to get the transformed Maxwell equations. The Maxwell equations explicitly transform like:

$$\nabla \cdot \mathbf{E} = 0 \qquad \rightarrow \gamma \nabla \cdot \mathbf{E} + \gamma \mathbf{v} \cdot \Big(- \nabla \times \mathbf{B} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\Big) = 0, \tag{1.23a}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \rightarrow \gamma \nabla \cdot \mathbf{B} + \frac{\gamma}{c^2}\mathbf{v} \cdot \Big(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\Big) = 0, \tag{1.23b}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \rightarrow \lambda\Big(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\Big) + \gamma \mathbf{v}(\nabla \cdot \mathbf{B}) = 0, \tag{1.23c}$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = 0 \qquad \rightarrow \lambda\Big(\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\Big) + \gamma \mathbf{v}(\nabla \cdot \mathbf{E}) = 0, \tag{1.23d}$$

where $\mathbf{v} = (v, 0, 0)$ and λ is a diagonal matrix with diagonal elements $(\gamma, 1, 1)$. Clearly these equations are not individually invariant. In fact by applying the transformations the Maxwell equations mix. Therefore they are only invariant as a set. Clearly Galilean Transformations would not work, as without the mixing of a space coordinate into the time coordinate, the fields would mix differently under transformation, meaning that the Maxwell equations would not be invariant anymore.

The covariant formulation of electromagnetism is manifestly Lorentz covariant. The transformations eq. (1.23a) and eq. (1.23d) belong to the pair of Maxwell equations that follow from $\partial_\mu F^{\mu\nu} = 0$ and it is precisely these two that mix under transformation. Similarily the two other Maxwell equations follow from the Bianchi Identity and these two also mix under Lorentz transformation. Therefore the mixing of the equations under Lorentz transformation is entirely equivalent, but also manifest in the covariant formulation.

Chapter 2

Non-Relativistic Limits

We have discussed classical electrodynamics and its associated Lorentz transformations and invariance, now we turn to the non-relativistic limit of this theory. The basic idea behind such a limit is that we send the speed of light c to infinity. In a relativistic theory, c is the fastest anything can travel at; it is essentially the speed limit of information in the universe. Therefore, sending it to infinity effectively removes this speed limit. The interesting thing about non-relativistic limits is that they can produce simpler physics for the low velocity domain, where relativistic effects are negligible. A prime example of a non-relativistic limit is the limit of relativistic momentum, which yields Newtonian momentum. Obviously in the low velocity domain, Newtonian momentum is much simpler to work with, while still being accurate.

It is important to be careful in defining a limit though. Before taking any random limit as $c \to \infty$, the effect of rescaling certain components of your theory with a power of c has to be examined. A theory of electrodynamics as discussed in the previous chapter has implicit powers of c included in the definitions of the fields, which are not known a priori. If one were to take the limit naïvely, without rescaling, the result would only give you one of the many possible cases. Therefore, we make an ansatz on how the fields depend on c and study the result.

2.1 Electric and Magnetic Limits

The Non-Relativistic Limit of the Maxwell equations, as found in Le-Bellac, Lévy-Leblond (1973) [2], can be found by rescaling the electric and magnetic fields with a particular power of c and then taking the limit as $c \to \infty$. However, as described in [2], there are many different limits admitted in such a procedure, two of which are particularly interesting. The particular limits we examine more closely are called the electric limit, in which the electric field dominates and the magnetic limit, in which the opposite occurs. The Maxwell equations in vacuum as described in section 1.1 are:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0, \tag{2.1a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{2.1b}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.1c}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. \tag{2.1d}$$

It is important that our resulting limits are physically interesting to us. As seen from eq. (2.1c) and eq. (2.1d), there are two time derivatives, proportional to a different power of c. Therefore, in the electric limit we make sure to preserve the $\frac{\partial \mathbf{E}}{\partial t}$ term, by rescaling the electric field with a factor c^2, whereas in the magnetic limit we preserve the $\frac{\partial \mathbf{B}}{\partial t}$ term by keeping the fields as they are. This is why those two limits describe either the dynamics of the electric field or the magnetic field respectively. This property of the electric and magnetic limit is the reason we choose to focus on these two limits. Keep in mind that this specific rescaling is entirely dependent on the conventions you use with regards to the Maxwell equations and the definitions of the fields.

2.1.1 The Electric Limit

The electric limit defines the electric field in a way that it is dominant over the magnetic field. Therefore, we rescale the electric field with a factor of c^2, while keeping the magnetic field unchanged. The electric limit is then defined as:

$$\mathbf{E} = c^2\mathbf{E}_e, \qquad \mathbf{B} = \mathbf{B}_e, \qquad c \to \infty. \tag{2.2}$$

Performing this limit in eq. (2.1) leads to the following equations:

$$\boldsymbol{\nabla} \cdot \mathbf{E}_e = 0, \tag{2.3a}$$
$$\boldsymbol{\nabla} \cdot \mathbf{B}_e = 0, \tag{2.3b}$$
$$\boldsymbol{\nabla} \times \mathbf{E}_e = 0, \tag{2.3c}$$
$$\boldsymbol{\nabla} \times \mathbf{B}_e = \frac{\partial \mathbf{E}_e}{\partial t}. \tag{2.3d}$$

These are similar to Equation (2.8) in [2], up to some conventions and the fact that ours are in vacuum. Our careful choice of rescaling indeed led to only the time derivative of the electric field being preserved in the Maxwell equations. This means that the electric field no longer responds to the time derivative of the magnetic field, whereas the magnetic field does respond to the time derivative of the electric field. Essentially the time dependence of $\mathbf{B}_m$ being lost means that the dynamics of the magnetic field are no longer modeled in this limit.

In the electric limit the ratio between the electric and magnetic field to diverges as c goes to infinity. According to [2] this limit corresponds, physically, to a slow moving electric charge.

2.1.2 The Magnetic Limit

Our second choice of rescaling leads to the magnetic limit and is defined as follows:

$$\mathbf{E} = \mathbf{E}_m, \qquad \mathbf{B} = \mathbf{B}_m, \qquad c \to \infty. \tag{2.4}$$

This rescaling might seem trivial, but that is entirely due to the conventions of the Maxwell equations as used in [7]. In fact this redefinition of the fields removes any ambiguity caused by implicit factors of c in the definitions of the fields. When applying the limit we get:

$$\boldsymbol{\nabla} \cdot \mathbf{E}_m = 0, \tag{2.5a}$$
$$\boldsymbol{\nabla} \cdot \mathbf{B}_m = 0, \tag{2.5b}$$
$$\boldsymbol{\nabla} \times \mathbf{E}_m = -\frac{\partial \mathbf{B}_m}{\partial t}, \tag{2.5c}$$
$$\boldsymbol{\nabla} \times \mathbf{B}_m = 0. \tag{2.5d}$$

These are in agreement with Equation (2.15) from [2]. Immediately we can see that the opposite of the electric limit occurs with regards to the time dependence. In the magnetic limit the time derivative of the magnetic field has survived. Therefore a changing magnetic field can induce an electric field, but not vice versa. The magnetic limit can provide a description of magnetostatics at the macroscopic level, as velocities are usually low and charges mostly balanced [2].

2.1.3 The Special Limit

There exists a third limit with a rescaling in between the limits described in eq. (2.2) and eq. (2.4), this limit is the so-called special limit. It features the following rescaling:

$$\mathbf{E} = c\mathbf{E}_s, \qquad \mathbf{B} = \mathbf{B}_s, \qquad c \to \infty. \tag{2.6}$$

As you can see the rescaling of the electric field is an intermediate rescaling between the electric limit and the magnetic limit. Applying this to the Maxwell equations in eq. (2.1) gives the following:

$$\nabla \cdot \mathbf{E}_s = 0, \tag{2.7a}$$
$$\nabla \cdot \mathbf{B}_s = 0, \tag{2.7b}$$
$$\nabla \times \mathbf{E}_s = 0, \tag{2.7c}$$
$$\nabla \times \mathbf{B}_s = 0. \tag{2.7d}$$

These equations feature no time dependence at all, therefore the special limit does not lead to a particularly interesting theory of electromagnetism. The time dependence of the fields are not modeled in the special limit. Therefore you can use this special limit to model fields, but there will be no induction of electric fields by magnetic fields and vice versa. What this limit does, and this will be a bit clearer in the next section, is put the electric and magnetic fields on equal footing before taking the limit. The fact that this does not lead to any description of dynamics is precisely why we are more interested in the electric and magnetic limits instead.

2.2 Non-Relativstic Limit of Electromagnetism in the Manifestly Lorentz Covariant Formulation

In section 2.1 we have found the NR-limit of electromagnetism starting from the **E** and **B** fields along with the Maxwell equations expressed in terms of those same fields. Alternatively we can attempt to obtain this limit from a covariant field theory formulation of electromagnetism. The big difference between these two methods is that we rescale the four-potential A_μ instead of the fields **E** and **B**.

Recall from section 1.2 the equations of motion given by $\partial_\mu F^{\mu\nu}$ and the Bianchi identity $\partial_{[\mu}F_{\nu\rho]} = 0.$:

$$\frac{1}{c}\partial_t F^{0j} + \partial_i F^{ij} = 0, \tag{2.8a}$$
$$\partial_i F^{0i} = 0, \tag{2.8b}$$
$$\partial_{[0}F_{ij]} = \frac{1}{c}\partial_t F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0, \tag{2.8c}$$
$$\partial_{[i}F_{jk]} = \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0, \tag{2.8d}$$

where we have used $\partial_0 = \frac{1}{c}\partial_t$. Now we move to redefine the four-potential, without choosing a specific limit yet. We do this so that we can derive several limits concurrently:

$$A_0 = c^\alpha a_t, \qquad A_i = c^\beta a_i, \tag{2.9}$$

what we are essentially doing here is splitting the four-potential into time and space parts. This will allow us to rescale in different ways, by either making α or β larger. Making either α or β larger will make either the timelike or spacelike part leading in the potential. The time and space components of the four potential also correspond to the electric and magnetic potentials respectively, so rescaling one with a higher power leads to a dominant electric or magnetic field respectively. Therefore these limits correspond to the electric and magnetic limits as described in [2] and in section 2.1.

Expanding F_{ij} and F_{0i} in terms of the four potential and applying the redefinition leads to:

$$F_{ij} = \partial_i A_j - \partial_j A_i = c^\beta(\partial_i a_j - \partial_j a_i) = c^\beta f_{ij}, \tag{2.10a}$$

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = c^{\beta-1}\partial_t a_i - c^\alpha \partial_i a_t, \tag{2.10b}$$

where

$$f_{ij} = \partial_i a_j - \partial_j a_i \tag{2.11}$$

From these redefinitions you can see that the magnetic field becomes: $B^i = \epsilon^{ijk}F_{jk} \to c^\beta \epsilon^{ijk} f_{jk}$, which is a simple direct rescaling with a power of c. This is due to the fact that the magnetic field is only dependent on the space components of the four-potential, as seen in eq. (2.10a). However, the electric field is rescaled according to two powers of c, because it is made up of both the time and space components of the four-potential, as is clear in eq. (2.10b). This will prove to have consequences on the eventual definition of the fields.

Substituting our rescalings into the equations of motion gives the Maxwell equations for the rescaled potential:

$$c^\beta \partial_i f^{ij} - \frac{1}{c}\partial_t(c^{\beta-1}\partial_t a^j - c^\alpha \partial^j a_t) = 0, \tag{2.12a}$$

$$c^{\beta-1}\partial_i \partial_t a^i - c^\alpha \partial_i \partial^i a_t = 0, \tag{2.12b}$$

$$c^{\beta-1}\partial_t f_{ij} + c^\alpha(\partial_i \partial_j a_t - \partial_j \partial_i a_t) + c^{\beta-1}(\partial_j \partial_t a_i - \partial_i \partial_t a_j) = 0 \tag{2.12c}$$

$$\partial_{[i} f_{jk]} = 0. \tag{2.12d}$$

Now that we have expressed the Maxwell equations in terms of the rescaled potential we can adjust α and β to get the electric, magnetic and special limit. Then we can start taking the non-relativistic limit $c \to \infty$. There are two main cases which we will examine first: $\alpha - \beta = 1$, corresponding to the electric limit in eq. (2.2) and $\alpha - \beta = -1$, corresponding to the magnetic limit in eq. (2.4). Lastly, we will take a look at the special limit: $\alpha - \beta = 0$.

2.2.1 The Electric Limit

The Maxwell equations for the rescaled four-potential can be multiplied with any power of c so that in performing the limit only the highest order power of c remain. The electric limit yields:

$$\partial_t \partial^j a_t + \partial_i f^{ij} = 0, \tag{2.13a}$$

$$\partial_i \partial^i a_t = 0, \tag{2.13b}$$

$$\partial_i \partial_j a_t - \partial_j \partial_i a_t = 0, \tag{2.13c}$$

$$\partial_{[i} f_{jk]} = 0. \tag{2.13d}$$

This set of equations describes electrodynamics in the electric limit, just as described in section 2.1.1. In the next section we will take the magnetic limit and examine the definitions in both limits a little closer.

2.2.2 The Magnetic Limit

Taking the magnetic limit ($\alpha - \beta = -1$, or equivalently eq. (2.4)) gives:

$$\partial_i f^{ij} = 0, \tag{2.14a}$$

$$\partial_i \partial^i a_t - \partial^i \partial_t a_i = 0, \tag{2.14b}$$

$$\partial_t f_{ij} + \partial_j(\partial_t a_i - \partial_i a_t) - \partial_i(\partial_t a_j - \partial_j a_t) = 0, \tag{2.14c}$$

$$\partial_{[i} f_{jk]} = 0. \tag{2.14d}$$

Now inspecting eq. (2.13b) and eq. (2.14b) side by side we can see a clear difference in the definition of the electric field. Both equations should correspond to $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$, as seen in section 2.1. However, in the electric limit $\mathbf{E}_e = \partial_i a_t$, whereas in the magnetic limit we clearly have $\mathbf{E}_m = \partial_i a_t - \partial_t a_i$. This can actually be directly explained by looking at eq. (2.10b). Since $\mathbf{E} = F_{0i}$, our choice of α and β directly influence which terms survive in the definitions of the electric field.

With these differing definitions of the electric field, we can see that the set of eq. (2.13) and eq. (2.3) are the same. Similarly eq. (2.14) and eq. (2.5) are also equal. This is a reassuring result, as essentially only the formalism between the two different methods was different, therefore the result should be the same.

2.2.3 The Special Limit

Our two previous choices of α and β clearly picked one of the two parts of the four-potential to be scaled with a higher power of c. There is once again a third option: $\alpha - \beta = 0$, or equivalently $\alpha = \beta$. This means A_0 and A_i are rescaled on equal footing and we have this intermediate limit between the electric and magnetic limit again. Applying this rescaling and letting $c \to \infty$ in eq. (2.12) yields:

$$\partial_i f^{ij} = 0, \tag{2.15a}$$

$$\partial_i \partial^i a_t = 0, \tag{2.15b}$$

$$\partial_i \partial_j a_t - \partial_j \partial_i a_t = 0 \tag{2.15c}$$

$$\partial_{[i} f_{jk]} = 0. \tag{2.15d}$$

In the special limit the definition of the electric field $\mathbf{E}_s$ is equal to $\mathbf{E}_e = \partial_i a_t$, this is due to the fact that all time derivatives disappear in the special limit as described in section 2.1.3. This definition allows us to see that the results of this section are equivalent to those in section 2.1.3.

2.3 Duality Transformation

In section 1.3 we described the duality between the electric and magnetic field. In this section we will examine what has happened to that duality in these limits. The duality transformation in eq. (1.18) contains a power of c, that will obviously not work in the non-relativistic limit. Instead the Galilean limit of the duality transformation relates the electric and magnetic limit in the following way[11, 12]:

$$\mathbf{E}_m \leftrightarrow \mathbf{B}_e, \qquad \mathbf{B}_m \leftrightarrow \mathbf{E}_e. \tag{2.16}$$

If we inspect eq. (2.3) and eq. (2.5) we can clearly see that this is true. Since both limits lost one of the time derivatives, internal duality is no longer possible. However, because the electric limit preserves $\frac{\partial \mathbf{E}_e}{\partial t}$ and the magnetic limit has $\frac{\partial \mathbf{B}_m}{\partial t}$, there is a certain symmetry between eq. (2.3d) and eq. (2.5c). This symmetry is precisely why we can relate the two limits with a duality transform. Explicitly we have:

$$\boldsymbol{\nabla} \times \mathbf{E}_m + \frac{\partial \mathbf{B}_m}{\partial t} \to \boldsymbol{\nabla} \times \mathbf{B}_e - \frac{\partial \mathbf{E}_e}{\partial t}, \tag{2.17}$$

$$\boldsymbol{\nabla} \times \mathbf{B}_e - \frac{\partial \mathbf{E}_e}{\partial t} \to \boldsymbol{\nabla} \times \mathbf{E}_m + \frac{\partial \mathbf{B}_m}{\partial t}. \tag{2.18}$$

From this transformation it is clear that these equations form a pair under this duality transformation. The rest of the Maxwell equations in both the electric and magnetic limit transform trivially.

2.4 Lorentz Transformations

The Lorentz transformations for a reference frame travelling in the x_1 direction with velocity v are[9]:

$$\begin{aligned} t' &= \gamma(t - \frac{vx_1}{c^2}), \\ x_1' &= \gamma(x_1 - vt), \\ x_2' &= x_2, \\ x_3' &= x_3, \end{aligned} \tag{2.19}$$

where gamma is the Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$. These Lorentz transformations lead to the following transformations of the fields:

$$\begin{aligned} E_1' &= E_1, & B_1' &= B_1, \\ E_2' &= \gamma(E_2 - vB_3), & B_2' &= \gamma(B_2 + \frac{v}{c^2}E_3), \\ E_3' &= \gamma(E_3 + vB_2), & B_3' &= \gamma(B_3 - \frac{v}{c^2}E_2), \end{aligned} \tag{2.20}$$

as described in section 1.4

Now we can take the limit of these transformations as $c \to \infty$, which means $\gamma \to 1$. The resulting equations are the Galilean transformations:

$$\begin{aligned} t' &= t, \\ x_1' &= x_1 - vt, \\ x_2' &= x_2, \\ x_3' &= x_3. \end{aligned} \tag{2.21}$$

In the Galilean transformations $t' = t$, therefore time is absolute and the space coordinate no longer mixes into the time coordinate under transformation. The reverse does still happen. This is a key difference between the Lorentz and the Galilean transformations.

The transformations of the electromagnetic field that follow from the Lorentz transformations in eq. (2.19) are shown in eq. (1.22). In these transformations you can see that only the fields components perpendicular to the motion are affected by the transformation. Furthermore the electric and magnetic fields mix under the Lorentz transformations. Now we proceed to take the limit of these transformations as $c \to \infty$ in the cases of the electric, magnetic and special limit.

2.4.1 The Electric Limit

To take the electric limit of the transformations in eq. (1.22) we will simply apply the limits from section 2.1. Firstly the electric limit described in eq. (2.2) yields the following transformations:

$$\begin{aligned} E_1' &= E_1, & B_1' &= B_1, \\ E_2' &= E_2, & B_2' &= B_2 + vE_3, \\ E_3' &= E_3, & B_3' &= B_3 - vE_2. \end{aligned} \tag{2.22}$$

In the electric limit we see the mixing of the magnetic field into the electric field disappear. The change in the electric field is now only dependent on the position. The mixing still occurs in the magnetic transformations however.

Now that we know how the fields transform in the electric limit we can see how the Maxwell equations in the electric limit transform. For a full treatment see section A.3. We can see that eq. (2.3d), transforming according to eq. (2.21) and eq. (2.22), becomes:

$$\begin{aligned}\partial'_t E'_i - \epsilon_{ijk}\partial'^j B'^k = \frac{\partial E_i}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial t} - \epsilon_{ijk}(\frac{\partial B'^k}{\partial x_\mu}\frac{\partial x_\mu}{\partial x'_j}) &= \partial_t E_i - v\partial_1 E_i - \epsilon_{ijk}\partial^j B'^k \\ &= \partial_t E_i - v\partial_1 E_i + v\delta_i^2\partial_1 E_2 + v\delta_i^3\partial_1 E_3 - \epsilon_{ijk}\partial^j B^k = 0,\end{aligned} \tag{2.23}$$

therefore we have the transformation:

$$\partial'_t E'_i - \epsilon_{ijk}\partial'^j B'^k = 0 \rightarrow \partial_t E_i - \epsilon_{ijk}\partial^j B^k - v_i\partial_j E_j = 0, \tag{2.24}$$

where $\mathbf{v} = (v, 0, 0)$. Interestingly some mixing of the Maxwell equations clearly occurs, because the fields themselves are mixed in the transformations. Applying this procedure to all NR Maxwell Equations (eq. (2.3)) in the electric limit yields the following transformations:

$$\nabla \cdot \mathbf{E}_e = 0 \qquad \rightarrow \nabla \cdot \mathbf{E}_e = 0 \tag{2.25a}$$

$$\nabla \cdot \mathbf{B}_e = 0 \qquad \rightarrow \nabla \cdot \mathbf{B}_e + \mathbf{v}\cdot(\nabla \times \mathbf{E}_e) = 0, \tag{2.25b}$$

$$\nabla \times \mathbf{E}_e = 0 \qquad \rightarrow \nabla \times \mathbf{E}_e = 0, \tag{2.25c}$$

$$\nabla \times \mathbf{B}_e - \frac{\partial \mathbf{E}_e}{\partial t} = 0 \qquad \rightarrow \nabla \times \mathbf{B}_e - \frac{\partial \mathbf{E}_e}{\partial t} + \mathbf{v}(\nabla \cdot \mathbf{E}_e) = 0. \tag{2.25d}$$

These equations show that the Maxwell equations of electric limit are invariant as a set under the transformations we have derived for it.

2.4.2 The Magnetic Limit

When we consider the magnetic limit, the magnetic field mixes into the electric field, but not vice versa. This is the opposite of what happens in the electric limit. Taking the magnetic limit, described in eq. (2.4), yields the transformations:

$$\begin{aligned} E'_1 &= E_1, & B'_1 &= B_1, \\ E'_2 &= E_2 - vB_3, & B'_2 &= B_2, \\ E'_3 &= E_3 + vB_2, & B'_3 &= B_3. \end{aligned} \tag{2.26}$$

If we take a look at eq. (2.3d) and eq. (2.5c) we can see that in the magnetic limit there is a Maxwell equation relating the spatial part of the electric field to the time derivative of the magnetic field. As we stated, the opposite happens in the electric limit. In the transformations you can see a parallel to this. In the electric limit there is mixing of the electric field into the magnetic field under transformation, but not vice versa. In the magnetic limit the reverse is true. So the fields only mix under transformation if they mix in the equations of motion.

Now we can examine the invariance of the Maxwell equations in the magnetic limit under the Galilean transformations, just as we did for the electric limit. Once again a full treatment can be found in section A.3. To start we see that eq. (2.5d) transforms as:

$$\epsilon_{ijk}\partial'^j B'^k = \epsilon_{ijk}\frac{\partial B'^k}{\partial x_\mu}\frac{\partial x_\mu}{\partial x'_j} = \epsilon_{ijk}\partial^j B^k = 0. \tag{2.27}$$

Again using this procedure we will get:

$$\nabla \cdot \mathbf{E}_m = 0 \qquad \rightarrow \nabla \cdot \mathbf{E}_m + \mathbf{v}\cdot(\nabla \times \mathbf{B}_m) = 0 \tag{2.28a}$$

$$\nabla \cdot \mathbf{B}_m = 0 \qquad \rightarrow \nabla \cdot \mathbf{B}_m = 0, \tag{2.28b}$$

$$\nabla \times \mathbf{E}_m + \frac{\partial \mathbf{B}_m}{\partial t} = 0 \qquad \rightarrow \nabla \times \mathbf{E}_m + \frac{\partial \mathbf{B}_m}{\partial t} + \mathbf{v}(\nabla \cdot \mathbf{B}_m) = 0, \tag{2.28c}$$

$$\nabla \times \mathbf{B}_m = 0 \qquad \rightarrow \nabla \times \mathbf{B}_m = 0. \tag{2.28d}$$

All Maxwell equations of the magnetic and electric limit are clearly invariant under their respective transformations. However, in both limits the equations do mix under the transforms, because the fields mix in the transformations described in eq. (2.22) and eq. (2.26).

2.4.3 The Special Limit

The third limit as described in section 2.1.3 features has it's own set of transformations:

$$E'_i = E_i, \qquad B'_i = B_i. \tag{2.29}$$

As might be expected from the Maxwell equations in the special limit, which feature no time derivatives whatsoever, there is no mixing of the fields in this case. The fields are absolute in every inertial reference frame.

2.5 Galilean Electrodynamics

Instead of first considering the Maxwell equations and then applying the NR-limit, as done in section 2.2, we can consider the Lagrangian instead. First we can take the NR-limit of the Lagrangian and then derive the equations of motion from that. We can then compare the results. The Maxwell Lagrangian is given by:

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \mathcal{L} = 2c^{2\beta-2}\partial_t a_i \partial_t a^i - 4c^{\alpha+\beta-1}\partial_i a_t \partial_t a^i - 2c^{2\alpha}\partial_i a_t \partial^i a_t + c^{2\beta} f_{ij} f^{ij}, \tag{2.30}$$

where we have substituted the redefinitions in eq. (2.10a), eq. (2.10b) and f_{ij} is described in eq. (2.11). We can again take the magnetic and electric limit as described in section 2.2 to get:

$$\mathcal{L} = \partial_i a_t \partial^i a_t \tag{2.31}$$

in the electric limit, and

$$\mathcal{L} = f_{ij} f^{ij} \tag{2.32}$$

in the magnetic limit. These give rise to equations of motion $\partial^i \partial_i a_t = 0$ and $\partial_i f_{ij} = 0$ respectively, which are only half the Maxwell equations we expected to recover. The non-relativistic limit applied to the equation of motion $\partial_\mu F^{\mu\nu} = 0$ gave two Maxwell equations. We are interested in a Lagrangian that can reproduce those two equations, when applying an NR-limit. Therefore we will turn to a different Lagrangian, consisting of the Lagrangian in eq. (2.30) and a scalar field.

2.5.1 Maxwell Lagrangian and a Scalar field

Consider a Lagrangian containing the electromagnetic tensor as in eq. (2.30) and an additional scalar field as shown in Festuccia (2016)[4]:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\partial_\mu \chi \partial^\mu \chi = \frac{1}{2c^2}(\partial_t A^i - c\partial^i A_t)(\partial_t A_i - c\partial_i A_t) - \frac{1}{4}F^{ij}F_{ij} + \frac{1}{2c^2}\partial_t \chi \partial_t \chi - \frac{1}{2}\partial^i \chi \partial_i \chi. \tag{2.33}$$

This Lagrangian consists of eq. (2.30), which has been expanded in terms of the four-potential, and an additional scalar field. The purpose of this scalar field is to retrieve some of the c^{-2} terms that were dropped out in the electric limit when taking $c \to \infty$. We can make the following redefinitions:

$$A_t = -c\phi - \frac{1}{c}\tilde{\phi}, \qquad A_i = a_i, \qquad \chi = c\phi, \tag{2.34}$$

note that now the time dependent part of the four-potential is rescaled in terms of two different potentials, whereas the spatial part is not rescaled. This particular rescaling along with the limit as $c \to \infty$ is called the Galilean Electrodynamics (GED) Limit. This GED limit is essentially a combination of the electric limit $c\phi$ and a_i, where the time component is leading in power of c, and the magnetic limit consisting of $\frac{1}{c}\tilde{\phi}$ and a_i, where the spatial components are leading. In section 2.2 we saw that the rescaling affects the definition of the electric field. This particular rescaling allows you to essentially keep both $\mathbf{E}_m$ and $\mathbf{E}_e$ in the Lagrangian as shown in [13]. Applying the GED rescaling yields the Lagrangian:

$$\mathcal{L} = \frac{1}{2c^2}(\partial_t a^i - c\partial^i(-c\phi - \frac{1}{c}\tilde{\phi}))(\partial_t a_i - c\partial_i(-c\phi - \frac{1}{c}\tilde{\phi})) - \frac{1}{4}f^{ij}f_{ij} + \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}c^2\partial^i\phi\partial_i\phi. \quad (2.35)$$

This can be refactored into:

$$\mathcal{L} = \frac{1}{2c^2}\partial_t a^i\partial_t a_i + \frac{1}{c^2}\partial_t a_i\partial^i\tilde{\phi} + \frac{1}{2c^2}\partial_i\tilde{\phi}\partial^i\tilde{\phi} + \partial_t a_i\partial^i\phi + \partial^i\phi\partial_i\tilde{\phi} - \frac{1}{4}f^{ij}f_{ij} + \frac{1}{2}(\partial_t\phi)^2. \quad (2.36)$$

This refactoring shows the use of the scalar field in our Lagrangian; the last term in eq. (2.35) canceled out a c^2 term arising from the first part of the Lagrangian. This means that the terms with the highest power of c in eq. (2.36) are proportional to c^0 instead of c^2. These terms are precisely the subleading terms we lost in taking the electric limit of eq. (2.33). Therefore this scalar field is a tool that allows us to recover the equations of motion we lost in the previous limit.

Now we can take the limit $c \to \infty$ and substitute $\mathbf{E}_m = \tilde{E}_i = -\partial_i\tilde{\phi} - \partial_t a_i$ to get the non-relativistic limit for the Lagrangian:

$$\mathcal{L}_{GED} = -\frac{1}{4}f^{ij}f_{ij} - \partial^i\phi\tilde{E}_i + \frac{1}{2}(\partial_t\phi)^2. \quad (2.37)$$

Note that the first term is simply $\mathbf{B}_m^2 = \mathbf{B}_e^2$ and the second term is essentially $\mathbf{E}_e \cdot \mathbf{E}_m$, for comparison see Equation (62) from [13], which also includes current and charge terms for this GED Lagrangian. Varying with respect to ϕ gives the equation of motion:

$$\partial_t^2\phi - \partial^i\tilde{E}_i = 0. \quad (2.38)$$

eq. (2.13a) and eq. (2.13b) can be recovered by varying with a_i and $\tilde{\phi}$ respectively. These are the equations of motion corresponding to the electric limit, with the additional eq. (2.38)

2.5.2 Scale Symmetry

The action belonging to the non-relativistic Lagrangian in eq. (2.37) possesses an emergent scale symmetry. Meaning that the action (eq. (2.9) from [4]):

$$S_{GED} = \int d^{3+1}x\mathcal{L}_{GED} = 0 \quad (2.39)$$

is invariant under the following two scale transformations:

$$t \to \lambda t, \qquad \phi \to \lambda^{\frac{1}{2}}\phi, \qquad \tilde{\phi} \to \lambda^{-\frac{3}{2}}\tilde{\phi}, \qquad a_i \to \lambda^{-\frac{1}{2}}a_i, \quad (2.40)$$

$$x \to \mu x, \qquad \phi \to \mu^{-\frac{3}{2}}\phi, \qquad \tilde{\phi} \to \mu^{-\frac{1}{2}}\tilde{\phi}, \qquad a_i \to \mu^{-\frac{1}{2}}a_i. \quad (2.41)$$

In this chapter we have examined the electric, magnetic and special limit of the Maxwell equations, in both classical and covariant formalism. Furthermore we have examined the duality between the electric and magnetic limit. We have also considered the Galilean transformations of the fields and Maxwell equations in each of these limits. Finally we have reproduced the Galilean Electrodynamics action by performing a non-relativistic limit of the Maxwell Lagrangian with a scalar field. This GED action allowed us to reproduce the equations obtained from the electric limit of the Maxwell equations. In the next chapter we will once again reproduce GED, but with an entirely different limit: the string limit.

Chapter 3

The String Limit and Galilean Electrodynamics

In the previous chapter we have considered non-relativistic limits that scale the time coordinate separately from the space coordinates, meaning that $x_0 = ct$, while x_i is not scaled with a power of c. This breaking up of spacetime into time and space, together with taking the limit $c \to \infty$ is called the particle limit. In this section we are going to perform a different type of limit than we have up until now.

The limit we consider in this section is a new procedure that involves decomposing spacetime in a different way than in the particle limit. Instead of splitting it into space and time, we split it into time and a space coordinate on the one hand, and the remaining space coordinates on the other hand. We call these two groups the longitudinal and transverse coordinates, respectively. The longitudinal coordinates are then scaled with c, whereas the transverse coordinates are not. This limit is called the string limit. In this chapter the string limit will be applied to the Maxwell Lagrangian in five-dimensional spacetime.

We perform the limit in this way in order to reproduce Galilean Electrodynamics, as seen in eq. (6.7) from [4], on a flat background. The result we seek to reproduce is an action in four-dimensional spacetime. This action can be obtained by starting in a five-dimensional spacetime, reducing to four dimensions and then taking the particle limit. We will reverse this order, first we will take the string limit of the five-dimensional Maxwell Lagrangian, after which we will perform a dimensional reduction. Applying the string limit to the 5D Maxwell Lagrangian is a new procedure. Since we have seen before that applying the particle limit to only the Maxwell Lagrangian does not deliver Galilean Electrodynamics, we must use a different limit; the string limit. Of course, there is no guarantee that the same result will be obtained.

To illustrate the goal of this chapter, Figure 3.1 provides a schematic overview of the differences in the methods of [4] and this chapter. A third option, the null reduction is also given by [4], this will be briefly discussed as well.

The dimensional reduction in our approach might seem unnecessary. Why do we not simply apply the string limit in four dimensions. The reason we apply the limit to a five-dimensional Lagrangian is that we need a theory with three spatial coordinates that are on equal footing. In taking the string limit one spatial coordinate is grouped with the time and rescaled separately. If you were to use a four-dimensional theory, only two spatial coordinates would remain after the limit.

In short, our aim is to arrive at the Galilean Electrodynamics with an entirely different method. If we do obtain the same result as [4], this would prove interesting, because it opens the door for the application of the string limit to other fields of study.

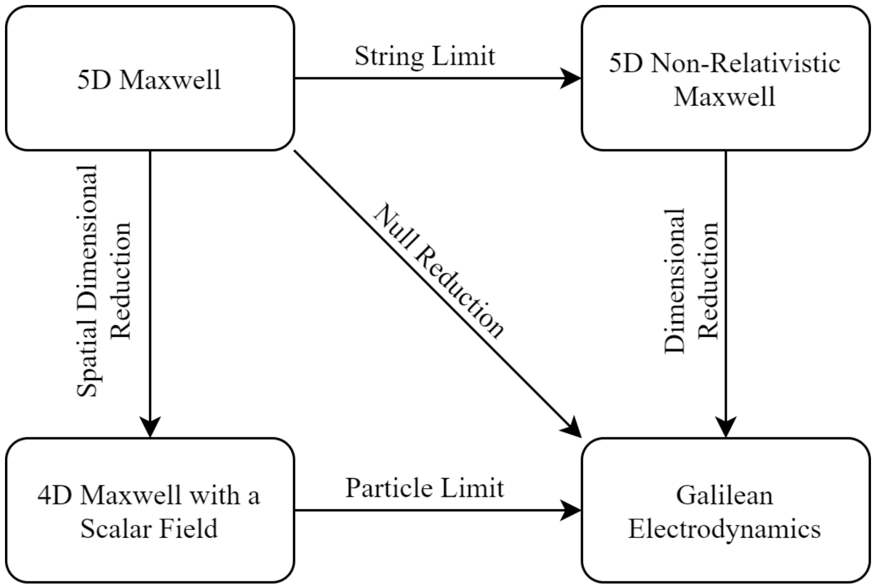

Figure 3.1: A schematic overview of the different ways to arrive at a 4D Non-Relativistic Lagrangian from a 5D Maxwell Lagrangian.

3.1 Setting up the String Limit

In this part we will briefly show the necessary definitions that we need to perform the string limit. First let us define the five dimensional coordinates and potential. In five dimensions, we have:

$$x_A = (t, x_i), \qquad A_A = (A_t, A_i), \tag{3.1}$$

where $A, = 0, 1, 2, 3, 4$ and $i = 1, 2, 3, 4$. However, we wish to group the time with a space coordinate. To do this we define the lightlike coordinates $x_\pm$ and the potential belonging to that basis $A_\pm$ as:

$$x_\pm = \frac{1}{\sqrt{2}}(t \pm x_1), \qquad A_\pm = \frac{1}{\sqrt{2}}(A_0 \pm A_1). \tag{3.2}$$

These are linear combinations of the time part and one of the space parts from eq. (3.1). Now we group these with the remaining space coordinates, giving us the redefined position and potential:

$$x_A = (x_+, x_-, x_j), \qquad A_A = (A_+, A_-, A_j), \tag{3.3}$$

where $j = 2, 3, 4$. With these definitions we have inherently created the two groups of coordinates we wanted: the time and a space coordinate on the one hand, and the rest of the space coordinates on the other hand.

The point of the string limit is to rescale the time coordinate along with a spatial coordinate separately from the rest. Therefore we make the following rescaling:

$$x_A = (cx_+, cx_-, x_j). \tag{3.4}$$

This rescaling is the principle difference between the particle limit and the string limit.

The metric for our new coordinate basis takes the following form:

$$\eta_{\pm A} = \frac{1}{\sqrt{2}}(\eta_{0A} \pm \eta_{1A}). \tag{3.5}$$

Therefore we can exchange a plus and minus index with the metric, at the expense of a minus sign:

$$V_+ = \eta_{+-}V^- = -V^-, \tag{3.6}$$

which is a very useful property for our purposes.

3.2 Dimensional Reduction

In the previous section we have defined a five-dimensional coordinate system. However, the result we wish to reproduce is four-dimensional. Therefore, we need to apply a dimensional reduction that leaves us with three space coordinates and a single timelike coordinate. In general there are many different ways to do a dimensional reduction. A specific case, a dimensional reduction in a spatial direction, is done on the left hand side of Figure 3.1. This spatial dimensional reduction can be achieved with the following ansatz:

$$\partial_a A_\mu = 0, \tag{3.7}$$

where a is the dimension which you wish to reduce. What we are stating here is that we assume the potential, and therefore the electromagnetic tensor, to be independent of the x_a coordinate. This allows us to drop the x_a coordinate from x_μ.

As an example we will apply a dimensional reduction of a spatial direction to the Maxwell Lagrangian. To show this we will use the conventional coordinate system, with x_A and A_A defined as in eq. (3.1). The Lagrangian can be expressed as

$$\mathcal{L} = -\frac{1}{4}F_{AB}F^{AB} = -\frac{1}{2}F_{4\nu}F^{4\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu},. \tag{3.8}$$

In order to drop the fourth spatial direction, we need the following ansatz: $\partial_4 A_\mu = 0$. This leads to the dimensionally reduced Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu A_4 \partial^\mu A^4, \tag{3.9}$$

which is in fact the Maxwell Lagrangian with a scalar field, similar to the Lagrangian in eq. (2.33). The result of this procedure is thus precisely what is described on the left hand side of Figure 3.1. However, we will perform the string limit before doing the dimensional reduction, not after. This means we will not obtain a Maxwell Lagrangian with a scalar field as an intermediary result. Instead we will have to apply a dimensional reduction to some other intermediary result we obtain after performing the string limit. In the next section we will apply a reduction in a lightlike direction, a null reduction, to a five-dimensional Maxwell Lagrangian.

3.2.1 The Null Reduction

An alternative method to obtain a four-dimensional non-relativistic Lagrangian from a five-dimensional Maxwell Lagrangian is the null reduction, as described in [4]. To perform this reduction we need reduce in a lightlike direction. Our five-dimensional coordinate system with $\mu = +, -, 1, 2, 3$ contains two such directions, $+$ and $-$. We will choose to reduce in the $+$ direction. This requires the ansatz:

$$\partial_+ A_\mu = 0, \tag{3.10}$$

which means that the potential no longer depends on the $+$ direction. Now consider the five dimensional Lagrangian in terms of the coordinates described in eq. (3.3):

$$\mathcal{L} = -\frac{1}{4}F_{AB}F^{AB} = \frac{1}{2}F_{+-}^2 + F_{+i}F_-{}^i - \frac{1}{4}F_{ij}F^{ij}. \tag{3.11}$$

Applying the ansatz in eq. (3.10) to terms in our Lagrangian yields the following result:

$$\mathcal{L} = \frac{1}{2}(\partial_- A_+)^2 - \partial_i A_+ F_-{}^i - \frac{1}{4}F_{ij}F^{ij}. \tag{3.12}$$

This Lagrangian corresponds to the action in eq. (6.7) from [4], which is the same Galilean Electrodynamics action we wish to obtain by performing the string limit. To make the comparison a bit clearer, the minus index corresponds to the time coordinate and A_+ corresponds to the potential ϕ in [4]. It is quite remarkable that this simple procedure delivers the same result. Note that we have not taken the limit as $c \to \infty$ anywhere in this procedure, yet the result from the particle limit is still obtained.

3.3 The String Limit of the Maxwell Lagrangian

We have defined our new coordinates and potential in section 3.1, now we can define a Lagrangian for this case as follows:

$$\mathcal{L} = -\frac{1}{4}F_{AB}F^{AB} = \frac{1}{2}F_{+-}^2 + F_{+i}F_-{}^i - \frac{1}{4}F_{ij}F^{ij}, \tag{3.13}$$

with $A = +, -, 1, 2, 3$ and $i = 1, 2, 3$. The property described in eq. (3.6) allows us to express the Lagrangian in a way that is more convenient for our purposes, as seen on the RHS of this equation. Now we define the rescaling of the four-potential as follows:

$$A_+ = ca_+, \qquad A_- = c^{-1}a_-, \qquad A_i = a_i. \tag{3.14}$$

we pick this rescaling, because it leads to the Lagrangian from [4] we wish to reproduce. Recall that we also rescaled our coordinates in eq. (3.4), so that $x_A = (cx_+, cx_-, x_i)$. Now we can substitute these rescalings into the definition of the electromagnetic tensor to see how the terms in eq. (3.13) change as a result:

$$\begin{aligned} F_{+-} &= \frac{1}{c}\partial_+ A_- - \frac{1}{c}\partial_- A_+ \\ &= c^{-2}\partial_+ a_- - \partial_- a_+, \end{aligned} \tag{3.15a}$$

$$\begin{aligned} F_{+i} &= \frac{1}{c}\partial_+ A_i - \partial_i A_+ \\ &= c^{-1}\partial_+ a_i - c\partial_i a_+, \end{aligned} \tag{3.15b}$$

$$\begin{aligned} F_-{}^i &= \frac{1}{c}\partial_- A^i - \partial^i A_- \\ &= c^{-1}\partial_- a^i - c^{-1}\partial^i a_-, \end{aligned} \tag{3.15c}$$

$$\begin{aligned} F_{ij} &= \partial_i A_j - \partial_j A_i = \partial_i a_j - \partial_j a_i \\ &= f_{ij}. \end{aligned} \tag{3.15d}$$

We substitute these expressions into eq. (3.13), yielding the following expression:

$$\begin{aligned} \mathcal{L} =& \frac{1}{2}c^{-4}(\partial_+ a_-)^2 + \frac{1}{2}(\partial_- a_+)^2 \\ &- c^{-2}\partial_- a_+ \partial_+ a_- - c^{-2}\partial_+ a_i \partial_- a^i + \partial_i a_+ \partial^i a_- \\ &- \partial_i a_+ \partial_- a^i + c^{-2}\partial_+ a_i \partial^i a_- - \frac{1}{4}f_{ij}f^{ij}. \end{aligned} \tag{3.16}$$

This Lagrangian contains several terms proportional to negative powers of c. When performing the limit $c \to \infty$, these terms will not survive. Therefore we obtain the following Lagrangian:

$$\begin{aligned} \mathcal{L}_{string} =& \frac{1}{2}(\partial_- a_+)^2 + \partial_i a_+ \partial^i a_- - \partial_i a_+ \partial_- a^i - \frac{1}{4}f_{ij}f^{ij} \\ =& \frac{1}{2}(\partial_- a_+)^2 + \partial_i a_+ f_{i-} - \frac{1}{4}f_{ij}f^{ij}. \end{aligned} \tag{3.17}$$

Now we could make the ansatz $\partial_+ a_\mu = 0$ and apply a null reduction as described in section 3.2. However, we have disposed of all terms that required removing by performing the string limit. Besides, there is no dependence on x_+ in the Lagrangian, so the null reduction would not change anything to the Lagrangian except reduce its dimensionality. Therefore we could make a much less strong ansatz: $a_A(x_+, x_-, x_i) = a_\mu(x_-, x_i)f(x_+)$. If we apply this ansatz and write take the action, any $f(x_+)$ will be squared and therefore disappear in the integral.

Alternatively, we can apply a spatial dimensional reduction. We reduce in one of the longitudinal directions, the x_1 direction, by applying the ansatz:

$$\partial_1 a_\mu = 0. \tag{3.18}$$

Then, to give our result the same form as eq. (6.7) from [4], we make the following redefinitions:

$$a_\mu = (a_t, a_i), \qquad a_t = \frac{1}{\sqrt{2}} a_-, \qquad \phi = \sqrt{2} a_+. \tag{3.19}$$

We can make these redefinitions, because a_- and a_+ are just scalars, we can give them any label we want. Now since $\partial_- = \frac{1}{\sqrt{2}}(\partial_t + \partial_1)$, we get the following action:

$$S_{string} = \int d^{3+1}x \left(\frac{1}{2}(\partial_t \phi)^2 + \partial_i \phi f_{it} - \frac{1}{4} f_{ij} f^{ij} \right). \tag{3.20}$$

Our resulting action corresponds to the Galilean Electrodynamics action described in eq. (6.7) from [4] on a flat background.

This result is quite significant as we have shown that taking the string limit before applying a dimensional reduction gives the exact same result as applying a dimensional reduction before applying the particle limit. These methods were in no way guaranteed to give the same result, but they did. This means it could be worthwhile to apply the string limit in other situations, especially those where the particle limit does not recover satisfactory results, much like we have done for a Maxwell Lagrangian without any auxiliary fields.

3.4 Lorentz Transformations in the String Limit

In section 2.4 we implicitly took the particle limit of the Lorentz transformations. In this section we will take the string limit instead. First let us define the Lorentz transformations for the basis described in eq. (3.3). In the case that the relative velocity between the reference frames $\boldsymbol{\beta} = \mathbf{v}/c$, with magnitude $\beta = v/c$ is in the x_1 direction we have:

$$\begin{aligned} x'_+ &= \gamma(1-\beta)x_+, \\ x'_- &= \gamma(1-\beta)x_-, \\ \mathbf{x}'_\perp &= \mathbf{x}_\perp, \end{aligned} \tag{3.21}$$

we refer to this as the parallel case. For a velocity in the $\mathbf{x}_\parallel$ direction, which is perpendicular to x_1 we have:

$$\begin{aligned} x'_+ &= \frac{1}{2}(\gamma+1)x_+ + \frac{1}{2}(\gamma-1)x_- - \frac{\gamma}{\sqrt{2}}\boldsymbol{\beta}\cdot\mathbf{x}_\parallel, \\ x'_- &= \frac{1}{2}(\gamma+1)x_- + \frac{1}{2}(\gamma-1)x_+ - \frac{\gamma}{\sqrt{2}}\boldsymbol{\beta}\cdot\mathbf{x}_\parallel, \\ \mathbf{x}'_\parallel &= \gamma(\mathbf{x}_\parallel - \frac{\boldsymbol{\beta}}{\sqrt{2}}(x_+ + x_-)), \end{aligned} \tag{3.22}$$

we refer to this as the perpendicular case. Together the parallel and perpendicular case span all possibilities. Now we can apply the rescaling of the string limit, as described in 3.4, and take the limit $c \to \infty$. In the parallel case we have:

$$\begin{aligned} x'_+ &= x_+, \\ x'_- &= x_-, \\ \mathbf{x}'_\perp &= \mathbf{x}'_\perp. \end{aligned} \tag{3.23}$$

In this case all coordinates become absolute. For the perpendicular case we have:

$$\begin{aligned} x'_+ &= x_+, \\ x'_- &= x_-, \\ \mathbf{x}'_\parallel &= \mathbf{x}_\parallel - \frac{\mathbf{v}}{\sqrt{2}}(x_+ + x_-), \end{aligned} \tag{3.24}$$

where once again the lightlike coordinates have become absolute. Therefore, when taking the string limit of the Lorentz transformations the lightlike coordinates are absolute in any case. Now that we have our NR coordinate transformations we will take the string limit of the transformation of the potential A_A.

3.4.1 The String Limit of the Potential Transformations

Since the potential transforms as $A_A \to \Lambda_A^{\ B} A_B$, where Λ is the tensor notation of the Lorentz Transformations in either eq. (3.21) or eq. (3.21), depending on which case we consider. In the parallel case we have:

$$\begin{aligned} A'_+ &= \gamma(1-\beta)A_+, \\ A'_- &= \gamma(1-\beta)A_-, \\ A'_i &= A_i, \end{aligned} \tag{3.25}$$

and for the perpendicular case:

$$\begin{aligned} A'_+ &= \frac{1}{2}(\gamma+1)A_+ + \frac{1}{2}(\gamma-1)A_- - \frac{\gamma}{\sqrt{2}}\beta^i A_i, \\ A'_- &= \frac{1}{2}(\gamma+1)A_+ + \frac{1}{2}(\gamma-1)A_+ - \frac{\gamma}{\sqrt{2}}\beta^i A_i, \\ A_i &= \gamma(A_i - \frac{\beta_i}{\sqrt{2}}(A_+ + A_-)). \end{aligned} \tag{3.26}$$

Now we can apply our rescalings from eq. (3.4) and eq. (3.14) before taking the limit $c \to \infty$. After performing the limit we are left with:

$$\begin{aligned} a'_+ &= a_+, \\ a'_- &= a_-, \\ a'_i &= a_i, \end{aligned} \tag{3.27}$$

in the parallel case, and:

$$\begin{aligned} a'_+ &= a_+, \\ a'_- &= a_- + \frac{1}{4}v^2 a_+ - \frac{1}{\sqrt{2}}v^i a_i, \\ a'_i &= a_i - \frac{1}{\sqrt{2}}v_i a_+ \end{aligned} \tag{3.28}$$

in the perpendicular case.

The parallel case leaves the coordinates and the potential unchanged, therefore the invariance of the GED action under these transformations is trivial. This is not the case for the perpendicular transformations. However, before applying these five-dimensional transformations to the GED action, we should implement our redefinitions to make them consistent with the four-dimensional GED action. We apply the redefinitions in eq. (3.19) and substitute $t = \frac{1}{\sqrt{2}}(x_+ + x_-)$ in eq. (3.24) and eq. (3.28). The resulting transformations for the coordinates are then:

$$
\begin{aligned}
t' &= t, \\
\mathbf{x}'_{\parallel} &= \mathbf{x}_{\parallel} - \mathbf{v}t,
\end{aligned}
\tag{3.29}
$$

which are, as expected, are the Galilean transformations. The transformations of the potential become:

$$
\begin{aligned}
\phi' \ &= \phi, \\
a'_- &= a_- + \frac{1}{2}v^2\phi - v^i a_i, \\
a'_i \ &= a_i - v_i\phi.
\end{aligned}
\tag{3.30}
$$

The GED action in eq. (3.20) is then invariant under these transformations. This is a reassuring result, as a failure to be invariant under these transformations, would mean that the result is not actually a Galilean theory of electrodynamics.

Since the electromagnetic tensor is defined in terms of the potential we can transform the equations of motion following from GED. Once again, the transformations for the parallel case are trivial and therefore the equations of motion are invariant in the parallel case. In the perpendicular case, we can apply the transformations above to the equations of motion. The result would be the transformations of the electric limit of the Maxwell equations, that are described in eq. (2.25). This makes sense, as the equations of motion from the GED action correspond to the electric limit.

Conclusion

In this thesis we set out to examine whether or not we could reproduce Galilean Electrodynamics by performing the string limit on a five-dimensional Maxwell Lagrangian, followed by a dimensional reduction. We can now conclude that we in fact can obtain Galilean Electrodynamics by applying the string limit. We have done several things before coming to this conclusion. Firstly, we have reviewed classical electromagnetism. We have discussed the Maxwell equations, the covariant formulation and the Lorentz transformations. Importantly, we have shown that the Maxwell equations are invariant as a set under the Lorentz transformations.

Secondly, we went on to explore non-relativistic limits of electromagnetism. We reproduced the electric and magnetic limits, that were first shown in [2]. Following this we have established the Galilean invariance of the result of these limits. Furthermore, we have reproduced the Galilean Electrodynamics action from [4] by applying the particle limit to a Maxwell Lagrangian with a scalar field. In chapter 3 we saw that this Lagrangian is in fact obtained by applying a spatial dimensional reduction to a five-dimensional Maxwell Lagrangian. Notably, the GED action possesses two emergent scale symmetries and gives us the equations of motion from the electric limit.

Thirdly we have defined and performed the string limit, followed by a spatial dimensional reduction. We applied this procedure to a five-dimensional Maxwell Lagrangian. The result was once again Galilean Electrodynamics. Therefore, we have successfully shown that breaking up spacetime into the longitudinal: t and x_1 coordinates and the transverse x_i coordinates before taking the limit $c \to \infty$ leads to the same result as the particle limit in the case of electrodynamics. Another procedure we have discussed is the null reduction, a dimensional reduction in a lightlike direction. This null reduction also yields Galilean Electrodynamics when applied to a five-dimensional Maxwell Lagrangian, without the need for any limit.

The compelling thing about our conclusion, that the string limit does yield GED, is that it means that the string limit might be applicable to other theories. It would be interesting for further research to apply such a limit to other relativistic fields and their corresponding Lagrangians. Another interesting point would be to consider curved backgrounds. This thesis only treats flat spacetime. The publication [4] that obtained GED by taking the particle limit and the null reduction did in fact consider curved backgrounds. Therefore, to fully reproduce their result with the string limit, further research should take curved backgrounds into account. Lastly, the string limit could also be used to obtain non-relativistic limits of supersymmetric theories.

Acknowledgements

I would like to thank prof. dr. Eric A. Bergshoeff for offering me the opportunity to research this interesting topic in theoretical physics.

Most importantly, I would like to thank my direct supervisor dr. Luca Romano for guiding me throughout the project. Our weekly meetings kept my research accurate and on track. I very much appreciated this, as performing my first proper research into theoretical physics was quite challenging.

Bibliography

[1] A. Einstein. Zur elektrodynamik bewegter körper. *Annalen der Physik*, 322(10):891–921, 1905.

[2] M. Bellac and Jean-Marc Levy-Leblond. Galilean electromagnetism. *Il Nuovo Cimento B*, 14:217–234, 04 1973.

[3] Eric Bergshoeff, Jan Rosseel, and Thomas Zojer. Non-relativistic fields from arbitrary contracting backgrounds. *Classical and Quantum Gravity*, 33(17):175010, Aug 2016.

[4] Guido Festuccia, Dennis Hansen, Jelle Hartong, and Niels A. Obers. Symmetries and couplings of non-relativistic electrodynamics, 2016.

[5] J. C. Maxwell. Xxv. on physical lines of force. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 21(139):161–175, 1861.

[6] Malcolm S. Longair. *The Origin of Maxwell's Equations*, page 75–106. Cambridge University Press, 3 edition, 2020.

[7] David J Griffiths. *Introduction to electrodynamics; 4th ed.* Pearson, Boston, MA, 2013. Re-published by Cambridge University Press in 2017.

[8] Carlos A. P. Galvao and Juan A. Mignaco. A Consistent electromagnetic duality. 2 2000.

[9] T.A. Moore. *Six Ideas That Shaped Physics: Unit R - Laws of Physics are Frame-Independent*. McGraw-Hill Education, 2002.

[10] Graham Hall. Maxwell's electromagnetic theory and special relativity. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 366(1871):1849–1860, 2008.

[11] C Duval, G W Gibbons, P A Horvathy, and P M Zhang. Carroll versus newton and galilei: two dual non-einsteinian concepts of time. *Classical and Quantum Gravity*, 31(8):085016, Apr 2014.

[12] Dieter Van den Bleeken and Çağin Yunus. Newton–cartan, galileo–maxwell and kaluza–klein. *Classical and Quantum Gravity*, 33(13):137002, Jun 2016.

[13] E S Santos, M de Montigny, F C Khanna, and A E Santana. Galilean covariant lagrangian models. *Journal of Physics A: Mathematical and General*, 37(41):9771–9789, sep 2004.

Appendix A

A.1 Conventions

In this thesis we use the following conventions throughout:

- Metric for four-dimensional spacetime: $\eta = (-,+,+,+)$
- $\hbar = 1$
- Bold letters denote vectors, e.g. $\mathbf{E}$
- Lowercase Latin letters are spatial components, e.g. $i = 1, 2, 3$.
- Uppercase Latin letters are five-dimensional coefficients, e.g. $A = 0, 1, 2, 3, 4$.
- Lowercase Greek letters are four-dimensional spacetime coefficients, e.g $\mu = 0, 1, 2, 3$.
- Coordinates or fields in a different reference frame are denoted by an apostrophe.
- The Bianchi Identity is defined as follows:

$$\partial_{[\mu} F_{\nu\rho]} = \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu}. \tag{A.1}$$

A.2 Transformation of the Electric and Magnetic Fields

The electric and magnetic fields transform as follows under the Lorentz transformations described in eq. (2.19):

$$\frac{E_1'}{c} = F_{01}' = \Lambda^\rho{}_0 \Lambda^\sigma{}_1 F_{\rho\sigma} = \Lambda^0{}_0 \Lambda^1{}_1 F_{01} + \Lambda^1{}_0 \Lambda^0{}_1 F_{10} = \gamma^2 \frac{E_1}{c} - \frac{\gamma^2 v^2}{c^2} \frac{E_1}{c} = \frac{E_1}{c}, \tag{A.2a}$$

$$\frac{E_2'}{c} = F_{02}' = \Lambda^\rho{}_0 \Lambda^\sigma{}_2 F_{\rho\sigma} = \Lambda^0{}_0 \Lambda^2{}_2 F_{02} + \Lambda^1{}_0 \Lambda^2{}_2 F_{12} = \gamma \frac{E_1}{c} - \gamma \frac{v}{c} B_3 = \frac{\gamma}{c}(E_2 - vB_3), \tag{A.2b}$$

$$B_3' = F_{12}' = \Lambda^\rho{}_1 \Lambda^\sigma{}_2 F_{\rho\sigma} = \Lambda^0{}_1 \Lambda^2{}_2 F_{02} + \Lambda^1{}_1 \Lambda^2{}_2 F_{12} = -\frac{\gamma v}{c} \frac{E_2}{c} + \gamma B_3 = \gamma (B_3 - \frac{v}{c^2} E_2). \tag{A.2c}$$

B_1', B_2' and E_3' are all derived equivalently.

A.3 Galilean Invariance of Electric and Magnetic Limit

A.3.1 The Electric Limit

Transforming the coordinates and fields according to 2.21 and 2.22 for equation 2.3a:

$$\partial_i' E'^i = \frac{\partial E'^i}{\partial x'^i} = \frac{\partial E'^i}{\partial x^j} \frac{\partial x^j}{\partial x'^i} = \frac{\partial E^i}{\partial x^i} = \partial_i E^i = 0. \tag{A.3}$$

The second equation trnasforms as:

$$\partial'_i B^i = \frac{\partial B'^i}{\partial x'^i} = \frac{\partial B'^i}{\partial x^j}\frac{\partial x^j}{\partial x' i} = \partial_i B^i + v(\partial_3 E_2 - \partial_2 E_3) = \partial_i B^i + \delta^i_1 v \epsilon_{ijk}\partial^j E^k = 0 \tag{A.4}$$

The third equation of motion 2.3c transforms as:

$$\epsilon_{ijk}\partial'^j E'^k = \epsilon^{ijk}\frac{\partial E'_k}{\partial x'_j} = \epsilon^{ijk}\frac{\partial E'^k}{\partial x_a}\frac{\partial x_a}{\partial x'_j} = \epsilon^{ijk}\frac{\partial E_k}{\partial x_j} = \epsilon_{ijk}\partial^j E^k = 0. \tag{A.5}$$

We are left with equation 2.3d, applying the transformations yields:

$$\begin{aligned}
\partial'_t E'_i - \epsilon_{ijk}\partial'^j B'^k &= \frac{\partial E_i}{\partial x^\mu}\frac{\partial x^\mu}{\partial t'} - \epsilon_{ijk}(\frac{\partial B'^k}{\partial x_\mu}\frac{\partial x_\mu}{\partial x'_j}) \\
&= \partial_t E_i - v\partial_1 E_i - \epsilon_{ijk}\partial^j B'^k \\
&= \partial_t E_i - v\partial_1 E_i + v\delta^2_i \partial_1 E_2 + v\delta^3_i \partial_1 E_3 - \epsilon_{ijk}\partial^j B^k \\
&= \partial_t E_i - \epsilon_{ijk}\partial^j B^k - v_i \partial_j E_j = 0,
\end{aligned} \tag{A.6}$$

A.3.2 The Magnetic Limit

The equations of motion for the magnetic limit 2.5a through 2.5d transform under the transformations 2.21 and 2.26. In this section the invariance under these transformations will be verified, as has been done for the electric limit. Equation 2.5a transforms as:

$$\partial'_i E'^i = \frac{\partial E'^i}{\partial x'^i} = \frac{\partial E'^i}{\partial x^j}\frac{\partial x^j}{\partial x'^i} = \frac{\partial E^i}{\partial x^i} + v(\partial_3 B_2 = \partial_2 B_3) = \partial_i E^i + \delta^i_1 v \epsilon_{ijk}\partial^j B^k = 0. \tag{A.7}$$

The second equation of motion 2.5b transforms as follows:

$$\partial'_i B'^i = \frac{\partial B'^i}{\partial x^j}\frac{\partial x^j}{\partial x'^i} = \partial_i B^i = 0. \tag{A.8}$$

The third equation (2.5c) transforms as:

$$\begin{aligned}
\partial'_t B'_i + \epsilon_{ijk}\partial'^j E'^k &= \frac{\partial B_i}{\partial x^\mu}\frac{\partial x^\mu}{\partial t'} + \epsilon_{ijk}(\frac{\partial E'^k}{\partial x_\mu}\frac{\partial x_\mu}{\partial x'_j}) \\
&= \partial_t B_i - v\partial_1 B_i + \epsilon_{ijk}\partial^j E'^k \\
&= \partial_t B_i - v\partial_1 B_i + v\delta^2_i \partial_1 B_2 + v\delta^3_i \partial_1 B_3 + \epsilon_{ijk}\partial^j E^k \\
&= \partial_t B_i + \epsilon_{ijk}\partial^j E^k - v_i \partial_j B_j = 0,
\end{aligned} \tag{A.9}$$

Lastly the fourth equation of motion 2.5d transforms as:

$$\epsilon_{ijk}\partial'^j B' k = \epsilon^{ijk}\frac{\partial B'^k}{\partial x'_j} = \epsilon^{ijk}\frac{\partial B'^k}{\partial x_a}\frac{\partial x_a}{\partial x'_i} = \epsilon^{ijk}\partial^j B^k = 0. \tag{A.10}$$