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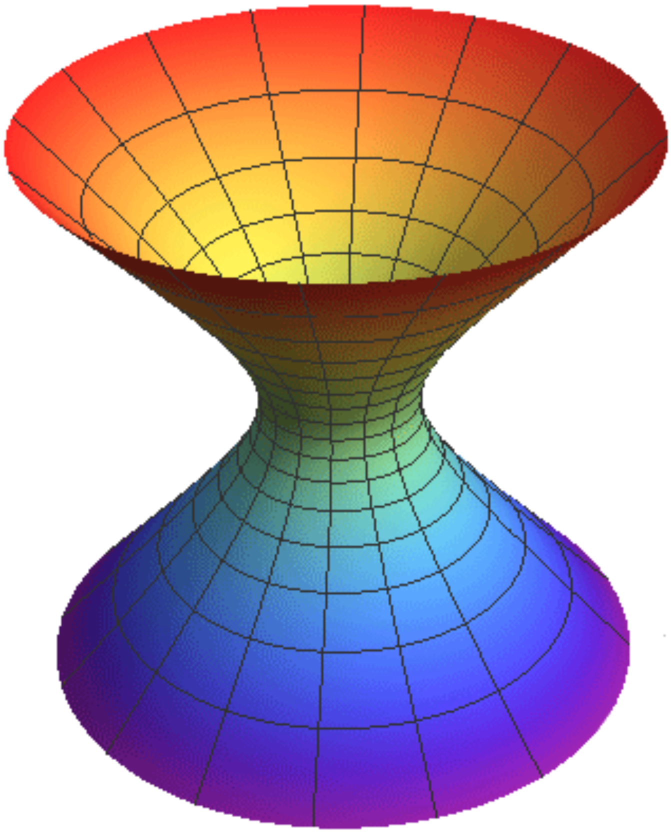
Conservation of Energy in de Sitter Space



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1 Abstract

The Universe appears to have a positive Cosmological Constant. If spacetime is curved, with de Sitter space being a good approximation, then since for every continuous isometry there is a corresponding conserved Noether Current, there are implications for the conservation of energy due to the fact that the de Sitter and Poincaré Lie Groups differ. This project does three things; first, it shows how as the de Sitter radius approaches infinity, the de Sitter Lie Group approaches becoming the Poincaré Lie Group of the isometries of flat Minkowski space under İnönü Wigner Contraction. Secondly, this project finds the Noether currents corresponding to the Generators of the Poincaré and de Sitter Groups to compare the conservation laws of Minkowski and de Sitter space, showing that while the isometry group of Minkowski space has corresponding conservation laws which include global conservation of energy, the same is not the case for de Sitter space (unless the de Sitter radius approaches infinity). Finally, various global and non-global coordinate systems for de Sitter space are considered, including global coordinates and static patch coordinates, the latter of which is non-global. When dealing with the static patch region, one can have a Killing vector which is timelike, therefore giving rise to energy conservation, *within the horizon of the static patch*, although not globally (however since the distance of the horizon is inversely proportional to the Cosmological Constant, which empirical research suggests is extremely small, the static patch is extremely large). With this taken into account, the results are discussed with some suggestions for future research.

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First I would like to thank Daniël Boer for his invaluable skills as a thesis supervisor guiding me through this course. I would not have understood nearly as much about Lie Groups or General Relativity without all our conversations.

As always, I would also like to thank my family, Mum, Nona and Will. It's been a crazy year with Coronavirus worldwide, and I really appreciate all your supportiveness and compassion.

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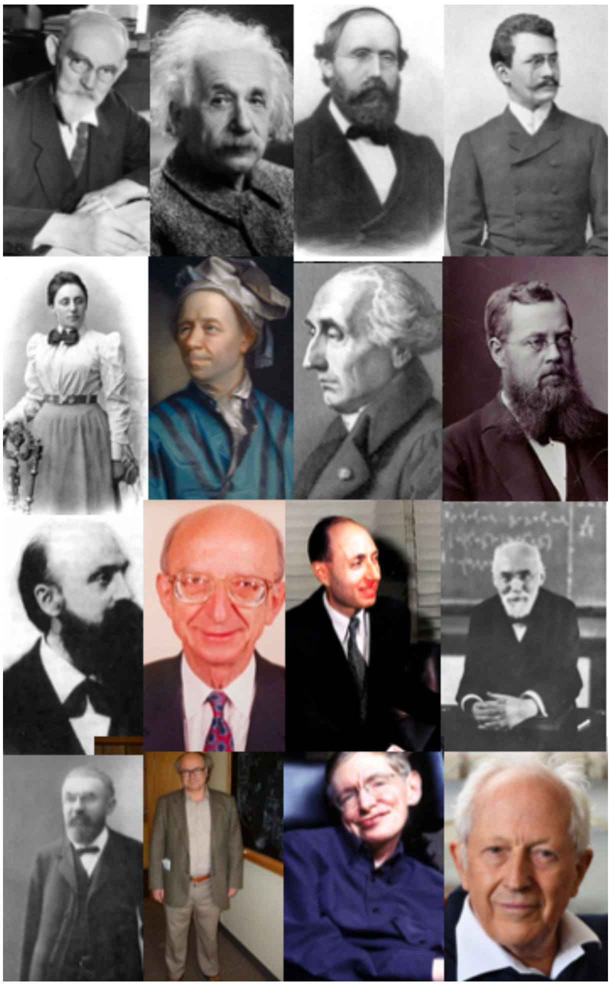
Scientific and Mathematical Acknowledgements



This scientific work did not take place in a vacuum. I owe a scientific debt to *(top row, left to right)* Willem de Sitter for developing the concept of de Sitter Space, as well as Albert Einstein, Bernhard Riemann and Hermann Minkowski for the progress in Special Relativity, General Relativity and Cosmology which gave rise to the Einstein Field Equations.

This thesis would also not be possible without the work of *(second row, left to right)* Emmy Noether, a derivation of whose most famous theorem is shown in this thesis. That derivation uses the Euler Lagrange Equation, and so this thesis also owes a debt to Leonard Euler and Joseph-Louis Lagrange. For his work dealing with Groups, this thesis also owes a scientific debt to Sophus Lie, after whom Lie Groups are named.

On the subject of Lie Groups and isometries, this thesis would be impossible without the work of *(third row, left to right)* Wilhelm Killing, Erdal İnönü and Eugene Wigner. The significance of the Lorentz Groups in this project means that tribute must also be paid to Hendrik Lorentz.

Finally *(bottom row, left to right)*, dealing with the Poincaré group in this project requires that the academic work of Henri Poincaré is acknowledged (in fact it was Poincaré who realised that the Lorentz transformations formed a Group). Finally, the last parts of this thesis owe a great debt to the clear explanations in textbooks and articles of Gary Gibbons, Stephen Hawking, and George Ellis.

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2 Introduction

> "Begin at the beginning," the King said, very gravely, "and go on till you come to the end: then stop."
>
> *Alice's Adventures in Wonderland* (Carroll (1865))

This thesis deals with energy conservation in de Sitter space. Evidence, such as the findings of Perlmutter (1999) and Riess (1998), is supportive of the idea that the Universe approximately resembles de Sitter Space (especially when describing the Universe in its earliest stages and in the distant future. Differences between the contemporary Universe and de Sitter space will be mentioned in section 3.3.1). The fact that for a sufficiently large de Sitter radius of curvature, de Sitter space locally resembles Minkowski space, even though globally it does not, has been established in previous literature (such as Aldrovandi and Pereira (1998)), but will nonetheless be shown as part of this thesis.

Due to Noether's Theorem, whereby continuous symmetries imply conserved currents, rotational symmetry implies a conservation law for angular momentum, translational symmetry in space implies a conservation law for linear momentum and translational symmetry in time implies a conservation law for the Hamiltonian, which in many cases is equal to energy.

Due to the differences between their isometry groups, the corresponding conservation laws of de Sitter space and Minkowski space differ. The isometries of Minkowski space give rise to conservation of energy, but despite de Sitter space locally resembling Minkowski space for a sufficiently large radius of curvature, it has a nonzero Cosmological constant, and so it does not have global translational symmetry in time and therefore has no global conservation of energy law.

This thesis will look at de Sitter space's conservation laws, and how to have some kind of (non-global) energy conservation in de Sitter space. First, in sections 5 and 6, İnönü Wigner Contraction will be used to show how the isometries of de Sitter space locally resemble those of Minkowski space (although not globally). Secondly, Noether's Theorem will be used in section 7 to find the conservation laws corresponding to the Lie groups of de Sitter and Minkowski space's isometries. With this achieved, it will be shown how de Sitter space does not have global energy conservation. Finally, looking at concepts such as the Static Patch, it will be shown in section 8 how it is possible to have conservation of energy within the (non-global) horizon of an extremely large region, with conclusion section 9 discussing the results overall, and giving some recommendations for directions of future research in the topic of de Sitter space.

3 The Einstein Field Equations and de Sitter Space

3.1 Intrinsic vs Extrinsic Curvature

To discuss the curvature of de Sitter Space and/or Minkowski Space, one must clearly distinguish between two different kinds of curvature: Extrinsic and Intrinsic curvature. These can facilitate understanding the concept of the Riemann tensor, which is central to discussing the curvature of de Sitter space, and the concept of parallel transport, which will be essential to discussion of Lie derivatives, which in turn lead in to discussion of Killing Vector Fields.

3.1.1 Surfaces with both Intrinsic and Extrinsic Curvature

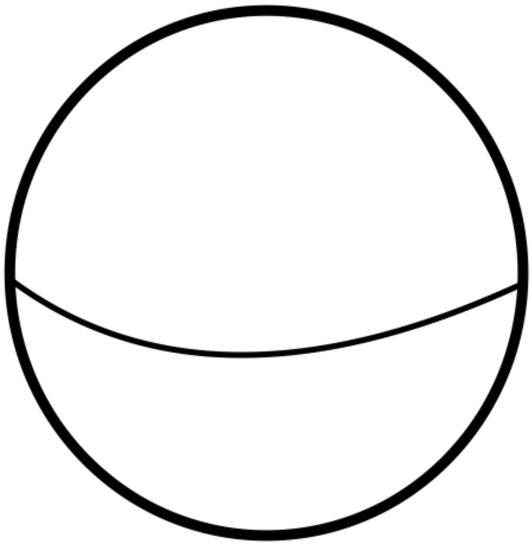

Figure 1: A 2 Sphere

A 2-sphere is a two dimensional surface embedded in a 3 dimensional space (see figure 1). It has both intrinsic and extrinsic curvature, as can be shown.

3.1.1.1 Extrinsic Curvature of a 2 Sphere

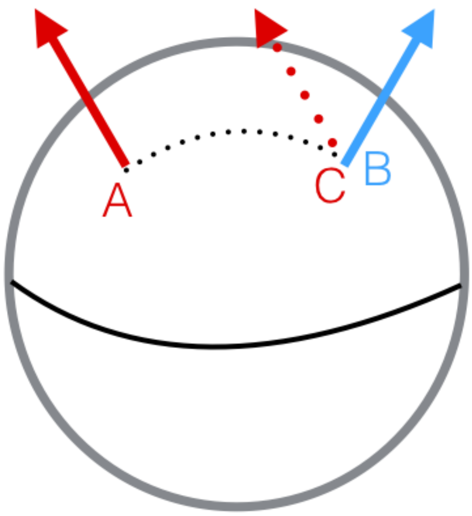

Figure 2: Extrinsic Curvature of a 2-Sphere embedded in 3D Euclidean Space

One can consider two points, A and B, on a 2-sphere embedded in Euclidean 3D space, as is pictured in figure 2. Two vectors normal to the surface of the sphere at those two points are not parallel to each other. The parallel counterpart of the vector normal to the surface of the sphere at point A would in fact be the vector labelled $\vec{C}$, so the curvature results in the difference between $\vec{B}$ (the vector normal to the sphere's surface at point B) and $\vec{C}$ (the vector parallel to the vector normal to the surface of the sphere at point A). Since the normal vectors exist in the embedding space, the curvature which causes their difference is extrinsic.

3.1.1.2 Intrinsic Curvature of a 2 Sphere

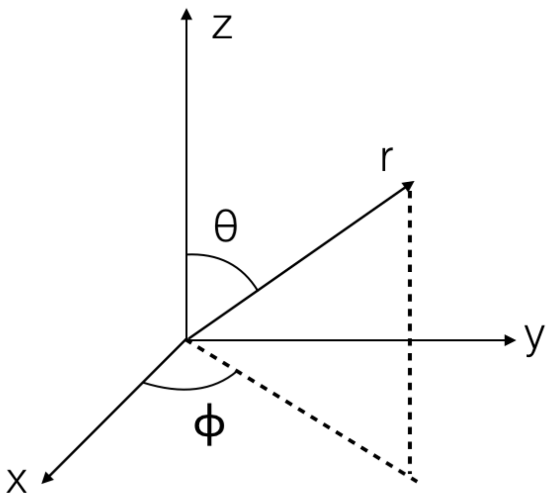

Figure 3: Spherical Coordinates

What about vectors in the tangent space of the sphere, rather than the embedding space? That will now be discussed.

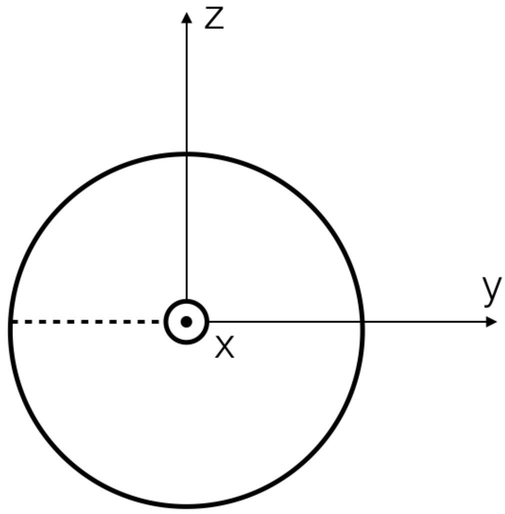

Figure 4: A 2 Sphere centred at the origin. The x axis points 'out' of the page.

Using spherical coordinates as are shown in figure 3, one can focus on the yz plane that passes through the centre of the 2 sphere centred at the origin as is shown in figure 4.

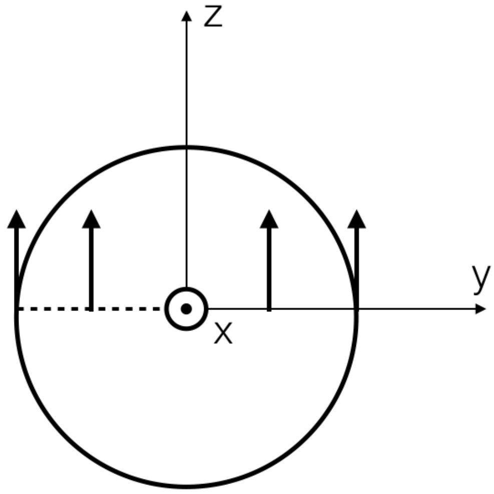

Figure 5: Moving the tangent vector $-\frac{\partial}{\partial\theta}$ from a point on the sphere's surface to another point opposite the original point by changing ϕ while keeping θ constant.

Consider the vector $-\frac{\partial}{\partial\theta}$. If one moves the vector around the 'equator' of the sphere, keeping θ constant but changing ϕ, then the direction in which $-\frac{\partial}{\partial\theta}$ points remains the same. The intial and final vector are parallel, as can be seen in figure 5.

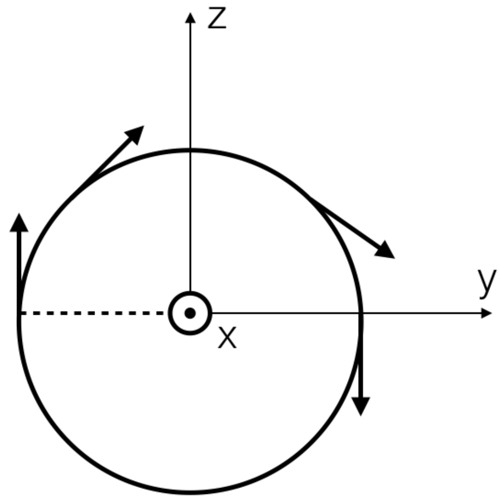

Figure 6: Moving the tangent vector $-\frac{\partial}{\partial\theta}$ from a point on the sphere's surface to another point opposite the original point by changing θ while keeping ϕ constant.

However, if instead one changes θ while keeping ϕ constant, as is shown in figure 6, then the direction of the vector changes. Therefore, it is possible to combine these changes, 'looping' the vector back to its initial location with a reversed direction, as is shown in figure 7. This is most definitely *not* parallel transport, since the initial and final vector are antiparallel. Note that while the arrows in the diagrams are shown 'outside' the surface, this is merely for clarity. It is more accurate to say that they are in the tangent space of the sphere, rather than the embedding space. This is not related to the embedding space, so is intrinsic curvature of the sphere (Carroll (2014a), Physics Unsimplified (2019b)).

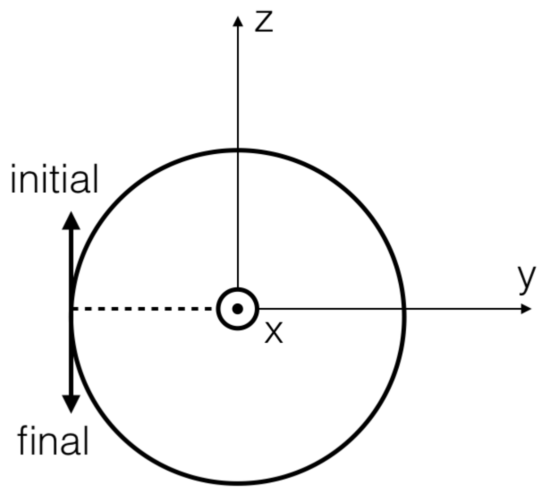

Figure 7: Moving the tangent vector $-\frac{\partial}{\partial\theta}$ from a point on the sphere's surface to another point opposite the original point by changing ϕ while keeping θ constant, an then moving the tangent vector from this point back to the original point by changing θ while keeping ϕ constant.

A 2-sphere is an ideal example to begin with, because it has both intrinsic and extrinsic curvature.

3.1.2 Surfaces with neither Intrinsic nor Extrinsic Curvature

Figure 8 shows a 2D plane embedded in 3D Euclidean space. It is flat both extrinsically and intrinsically.

3.1.2.1 Extrinsic Curvature of a Flat 2D Plane

On the one hand, as Figure 8 shows, the vectors normal to the plane at points A and B, which are in the embedding space, are parallel, such that there is no curvature due to the plane's embedding.

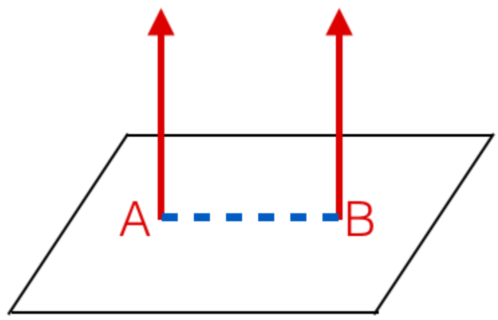

Figure 8: Extrinsic Curvature of a Flat 2D plane embedded in 3D Euclidean Space

3.1.2.2 Intrinsic Curvature of a Flat 2D Plane

On the other hand, figure 9 shows that the plane is not merely extrinsically flat, but is also intrinsically flat. If one takes a tangent vector (which is in the tangent space of the surface, rather than the embedding space) such as $\frac{\partial}{\partial x}$ and moves it in a loop across the surface, no matter what path is taken, on returning to the original position, the initial and final tangent vector are parallel (Carroll (2014a), Physics Unsimplified (2019b)).

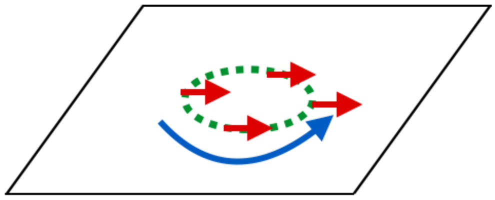

Figure 9: Intrinsic Curvature of a Flat 2D plane embedded in 3D Euclidean Space

3.1.3 Surfaces with Extrinsic Curvature but not Intrinsic Curvature

A 2D plane embedded in 3D space in a 'bumpy' way such that, from the perspective of the embedding space, there are visible curves, has extrinsic curvature but not intrinsic curvature.

3.1.3.1 Extrinsic Curvature of a 'Bumped' 2D Plane

Figure 10 shows extrinsic curvature (The vectors $\vec{A}$ and $\vec{B}$ are normal to the surface at points A and B. These vectors in the embedding space are not parallel to each other. Instead, $\vec{C}$ is parallel to $\vec{A}$).

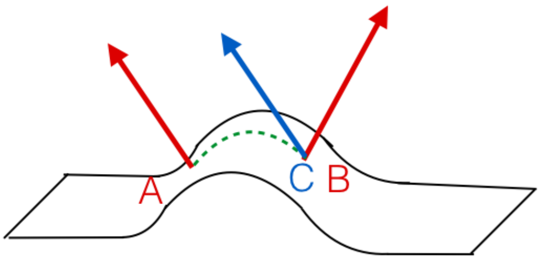

Figure 10: Extrinsic Curvature of a 'Bumpy' 2D plane embedded in 3D Euclidean Space

3.1.3.2 Intrinsic Curvature of a 'Bumped' 2D Plane

In contrast, figure 11 shows the intrinsic flatness. A tangent vector such as $\frac{\partial}{\partial x}$, which is a vector in the tangent space of the surface rather than the embedding space, can be moved around, changing direction, but regardless of the loop one takes, the final and initial tangent vector at the same point are parallel (Carroll (2014a), Physics Unsimplified (2019b)).

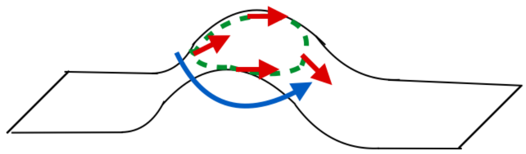

Figure 11: Intrinsic Curvature of a 'Bumpy' 2D plane embedded in 3D Euclidean Space

3.1.4 Surfaces with Intrinsic Curvature but not Extrinsic Curvature

Having read through all of this, readers might be thinking 'But what about the reverse case, where the surface is intrinsically curved, but due to the embedding, is extrinsically flat?' Unfortunately this is much more complicated, but it is nonetheless possible in the case of surfaces called 'Minimal Surfaces'. Giving this example requires moving away from discussion of parallel transport in order to instead discuss concepts of Gaussian and Mean curvature. Since this is interesting but not as directly relevant to the discussion of parallel transport which is relevant to discussing the Riemann tensor and Lie derivatives, it is instead discussed in more detail in the appendix 10.3.

3.2 Intrinsic Curvature and the Riemann Tensor

3.2.1 An Introduction to Covariant Derivatives

The concepts of covariant derivatives, Lie derivatives, and the overall concept of parallel transport are all absolutely essential to understanding the Killing vector fields of Minkowski and de Sitter space.

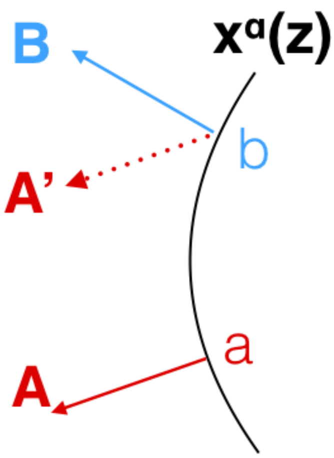

Figure 12: Covariant Derivative

For some curve given by coordinate line x^α, parametrised by z, at any point along the curve, one can find the tangent vector of $x^\alpha(z)$ given by $\frac{\partial x^\mu}{\partial z}$. Alternatively, one can find the normal vector at that point.

As is pictured in figure 12 , one can choose two points along the curve, a and b and can give each point a normal vector to the curve, $\vec{A}$ and $\vec{B}$. One can also put a vector parallel to $\vec{A}$, named $\vec{A'}$, with one end at point b. The covariant derivative is the difference between $\vec{A'}$ and $\vec{B}$ (Carroll (2014b), Physics Unsimplified (2019d)).

Consider a special case of the scenario in figure 12, where the line x^{α} is straight. This is shown in figure 13. The difference between $\vec{A'}$ and $\vec{B}$ is 0. In this case, the covariant derivative, ∇, giving the difference between $\vec{A}$ and $\vec{B}$ is simply given by $\nabla = \partial + 0$ But what about less simple cases

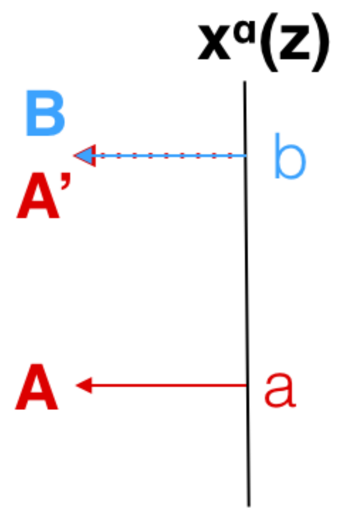

Figure 13: When $x^{\alpha}(z)$ is straight, the Covariant Derivative equals 0

such as in figure 12 where the difference between $\vec{A'}$ and $\vec{B}$ might be nonzero? The covariant derivative, ∇, giving the difference between $\vec{A}$ and $\vec{B}$, requires a second term, not just ∂, in order to take into account the curvature. So for a curve on the surface of some manifold M with tensor g, the covariant derivative is given by:

$$\nabla_a v^b = \partial_a v^b + \frac{1}{2} g^{bd} \left(\partial_a g_{cd} + \partial_c g_{ad} - \partial_d g_{ac} \right) v^c \tag{1}$$

Where the second term is to take into account that the surface of the manifold might not be flat. One can use a substitution, Γ^b_{ac}, the Christoffel symbol of the second kind, to write this more succinctly. When dealing with some Riemmannian manifold, it helps to have a way of understanding the rate of change of the metric tensor of the manifold. This is where Christoffel symbols of the first and second kind become useful, since they can be used to do exactly this.The Christoffel symbol of the first kind is defined as:

Definition 1 (Christoffel Symbol of the First Kind) $\Gamma_{dac} = \frac{1}{2} \left(\partial_a g_{cd} + \partial_c g_{ad} - \partial_d g_{ac} \right)$

Using definition 1, the second term of equation 1 can be rewritten as $g^{bd}\Gamma_{dac}$. Furthermore, definition 1 can be used to define Christoffel Symbols of the Second Kind.

Definition 2 (Christoffel Symbol of the Second Kind) $\Gamma^b{}_{ac} = g^{bd}\Gamma_{dac} = \frac{1}{2} g^{bd} \left(\partial_a g_{cd} + \partial_c g_{ad} - \partial_d g_{ac} \right)$

Using definition 2, equation 1 can be more simply written as:

$$\nabla_a v^b = \partial_a v^b + \Gamma^b{}_{ac} v^c \tag{2}$$

Where the second term takes into account the curvature of the manifold, and so when a flat surface is being considered, the second term in equation 3.2.1 is zero. Christoffel symbols will prove to be extremely important for the next few sections.

3.2.2 The Riemann Curvature Tensor

When dealing with the symmetries of a surface, it can be useful to have a stronger mathematical tool to deal with Intrinsic Curvature. With the distinction between Intrinsic and Extrinsic Curvature established, it is time to move on to the mathematical tool in question, which is the Riemann tensor.

The Riemann tensor, $R^{\alpha}{}_{\beta\gamma\delta}$ completely characterises the intrinsic geometry of spacetime, regardless of embedding (Hartle (2014)). For an intrinsically flat space, $R^{\alpha}{}_{\beta\gamma\delta} = 0$, while for a space with intrinsic curvature, there is some nonzero value of $R^{\alpha}{}_{\beta\gamma\delta}$.

Definition 3 (Riemann Tensor) $R^{\alpha}{}_{\beta\gamma\delta} = \partial_{\gamma}\Gamma^{\alpha}{}_{\beta\delta} - \partial_{\delta}\Gamma^{\alpha}{}_{\beta\gamma} + \Gamma^{\mu}{}_{\beta\delta}\Gamma^{\alpha}{}_{\mu\gamma} - \Gamma^{\mu}{}_{\beta\gamma}\Gamma^{\alpha}{}_{\mu\delta}$

Lowering the indices to get $R_{\alpha\beta\gamma\delta}$ (have a look back at definition 1 to understand how), and writing out the Christoffel Symbols in full, the tensor has the following symmetries and skew symmetries (d'Inverno (1992a)):

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \tag{3}$$

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \tag{4}$$

And so, with a little consideration:

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \tag{5}$$

And the slightly more complicated result called the first Bianchi Identity:

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0 \tag{6}$$

3.2.3 The Ricci Tensor

Contracting the first and third indices of the Riemann tensor as defined by definition 3 can give the Ricci tensor (d'Inverno (1992a), Hartle (2014)):

Definition 4 (Ricci Tensor) $R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta}$

Why not contract the first and second, or first and fourth, indices instead? Well due to the Riemann tensor being antisymmetric on the first two identities, as is shown by equation 3, contracting the first two indices would just give 0. Similarly, the Riemann tensor is antisymmetric on the last two indices, as is shown by equation 4, meaning that contracting the first and fourth indices just gives a negative version of the Ricci tensor gained by contracting the first and third indices.

The Ricci tensor, $R_{\alpha\beta}$, is a symmetric tensor relating to the effect of curvature on geometry of the manifold, specifically showing how the volume of spheres on the manifold differs from the volume they would have in 'normal' Euclidean space.

In dealing with spacetimes, the Ricci tensor is called a curvature invariant (Physics Unsimplified (2019c)).

3.2.4 The Scalar Curvature

Just as one can go from the Riemann tensor to the Ricci tensor, one can go from the Ricci tensor to the Scalar Curvature, which gives a scalar value for any point on the manifold. The scalar curvature, R, is also a curvature invariant (Carroll (2014c), Physics Unsimplified (2019c)). It is given by:

Definition 5 (Scalar Curvature) $g^{\alpha\beta}R_{\alpha\beta} = R$

3.2.5 Describing the Curvature of a Flat Minkowski Spacetime

So with these fundamental concepts all dealt with, it might be useful to give a straightforward example, specifically Minkowski spacetime[1], which is flat.

$$g_{\mu\nu(\text{Minkowski})} = \eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \tag{7}$$

Equation 7 means that for Minkowski space $ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$. To find the Christoffel symbol for flat spacetime, one must take derivatives of $\eta_{\mu\nu}$, and since all of the entries in $\eta_{\mu\nu}$ are either -1 or 1, the derivatives will all be 0, meaning that:

$$\Gamma^{\mu}{}_{\alpha\beta(\text{Minkowski})} = 0 \tag{8}$$

Which makes intuitive sense, since this is after all a flat spacetime. Since the Riemann tensor is derived from, Γ, the Ricci tensor is derived from the Riemann tensor, and the Scalar curvature is derived from the Ricci tensor, this therefore means that:

$$R^{\mu}{}_{\nu\alpha\beta(\text{Minkowski})} = 0 \tag{9}$$

$$R_{\nu\beta(\text{Minkowski})} = 0 \tag{10}$$

$$R_{\text{Minkowski}} = 0 \tag{11}$$

Note that if one used polar coordinates rather than Cartesian ones then equation 8 could give a nonzero value. However, while this approach requires more derivation and effort, nonetheless equation 9 equals 0 and so equations 10 and 11 also equal 0 regardless of whether one uses Cartesian or polar coordinates.

[1] Note that whether the matrix in equation 7 is $\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$ is a matter of convention which varies depending on the textbook or article one is using. This thesis will use the $\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ convention.

3.2.6 The Einstein Field Equations

In 1905 Albert Einstein developed his theory of Special Relativity, dealing with the constant nature of the speed of light in a vacuum, c. However, this theory did not take gravity into account. It would take Einstein another decade before he developed his theory of General Relativity in order to take gravity into account in 1915. He defined the Einstein tensor as:

Definition 6 (Einstein Tensor) $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$

So since this is all in terms of the Ricci tensor, $R_{\mu\nu}$ and scalar curvature of spacetime, R, which will be defined by definitions 4 and 5, the Einstein tensor is entirely related to the curvature of spacetime. To understand how this curvature ties in with the nature of gravity, one must define the Einstein Gravitational Constant:

$$\kappa = \frac{1}{c^4}8\pi G \tag{12}$$

In equation 12, c is the speed of light in a vacuum[2] and G is the Newton Gravitational constant, one can use definition 6 and equation 12 in order to write the Einstein Field Equation (Einstein (1917)):

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{13}$$

$T_{\mu\nu}$ in equation 13 is the Energy Momentum Tensor, which will be explained in more detail in subsection 7.2.1 (where the concepts of energy and momentum will be discussed in a lot more detail) and more specifically will be defined by definition 21. $T_{\mu\nu}$ gives the flux of the μth component of the momentum vector across a surface of constant x^ν. T^{00} is therefore the flux of time-momentum across a surface of constant time. What is that? Energy density. Similarly, T^{01}, T^{02} and T^{03} give the momentum density of x, y and z-momentum. The Energy Momentum Tensor for an empty universe is 0.

The Einstein Field Equations are local equations, applying to any particular point in space. Since the energy momentum tensor gives the *densities* of these quantities, *not*, the quantities themselves, integrating over a region of spacetime can give the energy, x, y and/or z momentum in that region.

Λ in equation 13 is the Cosmological constant, which Einstein did not originally include in the equation in 1915, but which he added in 1917. The 1998 discoveries by the Supernova Cosmology Project and the High-Z Supernova Search Team that the expansion of the Universe appears to be accelerating supports the idea that the Universe has a positive Cosmological Constant (Perlmutter (1999), Riess (1998)).

So for a flat spacetime, $G_{\mu\nu} = 0$ since it is defined entirely in terms of the Ricci tensor and scalar curvature. For an empty spacetime, $T_{\mu\nu} = 0$.

[2]Since this is the first section to discuss the speed of light in a vacuum in the context of Einstein's Gravitational Constant in equation 12, here is probably as good a place as any to note that throughout this thesis, unless otherwise specified, the convention will be used of setting the speed of light in a vacuum c=1.

3.3 An Introduction to de Sitter Space

3.3.1 The properties of a de Sitter Universe

One solution to equation 13 with a positive Cosmological Constant is de Sitter space, initially developed in de Sitter (1917b) and de Sitter (1917a), which assumes an empty (so that $T_{\mu\nu} = 0$) homogeneous Universe. While the Universe was probably very close to this description in its earliest moments, and will probably become increasingly close to this description in its distant future, at present the de Sitter Universe is a slightly simpler approximation of what the real universe might be like, since the real universe is not completely homogeneous or empty. Nonetheless, for the purposes of this project, which deals with considering how energy conservation is affected in a Universe with a positive Cosmological constant, de Sitter is a good approximation. For a de Sitter Universe, the Einstein Field equation becomes:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{14}$$

Or

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{15}$$

3.3.2 The metric of the ambient flat space in which de Sitter Space is embedded

De Sitter Space is a curved 3+1 dimensional hypersurface embedded in a flat 4+1 dimensional 'Minkowski' pseudo-Euclidean Space $E^{4,1}$ (note that due to being 4+1 dimensional, it is unlike the Minkowski space discussed in the rest of this thesis, which has 3+1 dimensions). The metric tensor for the embedding space is given by:

$$\eta^{AB}_{ambient} = \begin{bmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \tag{16}$$

So vectors in this embedding space can be given:

$$\varepsilon^A = \begin{bmatrix} \varepsilon^0 & \varepsilon^1 & \varepsilon^2 & \varepsilon^3 & \varepsilon^4 \end{bmatrix} \quad \varepsilon_A = \begin{bmatrix} -\varepsilon_0 \\ \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix} \tag{17}$$

The equation for the hypersurface embedded in this space is:

$$\rho_{dS}^2 = \varepsilon_B \varepsilon^B = \eta_{AB} \varepsilon^A \varepsilon^B = -(\varepsilon^0)^2 + (\varepsilon^1)^2 + (\varepsilon^2)^2 + (\varepsilon^3)^2 + (\varepsilon^4)^2 \tag{18}$$

Where ρ_{dS} is called the de Sitter radius.

A hyperboloid such as de Sitter space has intrinsic curvature. The Riemann tensor can be used to account for its intrinsic curvature. Since a hyperboloid is not a minimal surface, it also has extrinsic

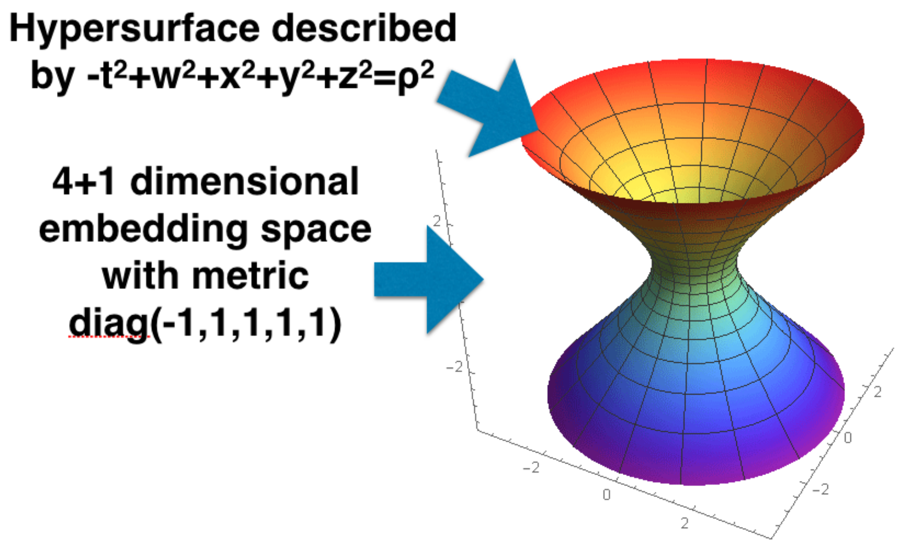

Figure 14: Diagram of a Hyperboloid embedded in a 4+1 Minkowski Space (although due to the difficulty of representing 5 dimensions visually, this is in fact only a 2 dimensional surface embedded in a 3 dimensional space)

curvature. The focus in this thesis is on the intrinsic curvature, however. This is why the Riemann tensor is so important.

It is convenient to break up η_{ambient} in order to think about the 4th dimension separately.

Definition 7 *Let* $\eta'_{ab} = \eta_{AB}$ *for* $A = 0, 1, 2, 3$ *and* $B = 0, 1, 2, 3$ *and let* $\eta_{44} = g_{AB}$ *for* $A = 4, B = 4$

Therefore the de Sitter radius can be rewritten as:

$$\rho_{dS}^2 = \eta'_{ab}\varepsilon^a\varepsilon^b + \eta_{44}(\varepsilon^4)^2 \tag{19}$$

(Aldrovandi and Pereira (1998))

3.3.3 The Conformal Factor

When discussing a curved surface such as de Sitter space, one can use Stereographic projection in order to think of it mapped onto a flat surface, with the most obvious example being world maps, which are flat but depict a curved Earth. The 2D surface of the Earth embedded in 3D space can be described using 2D (North and East, with South and West simply being the negative versions of North and East), rather than 3D (x, y and z), coordinates, but doing so requires that one take into account the conformal factor. The curvature of the Earth's surface is the reason why Greenland appears bigger than India on World Maps.

Stereographic projection is the process of taking some p dimensional manifold, M, and mapping it to a p-1 dimensional flat surface.

n is called the 'conformal factor' and can be used to write the metric of the manifold, $g_{\mu\nu}$ and the metric of the manifold's Stereographic Projection, $\eta_{\mu\nu}$ in terms of each other:

$$g_{\mu\nu} = n^2\eta_{\mu\nu} \tag{20}$$

But what is this conformal factor which is so useful for writing vectors and metrics for both a manifold and the Stereographic projection of a manifold?

Definition 8 (Conformal Factor) *If there are two Riemannian metrics, g and h, on some smooth manifold, M, and there is some positive function, n, on M, such that the metrics can be written as $g = nh$, then g and h are conformally equivalent and n is the conformal factor (Birrell and Davies (1992)).*

Conveniently, the spaces being discussed here, such as de Sitter space and Minkowski space, have Riemannian metrics, and the manifold for de Sitter space is smooth, so the conformal factor will turn out to be an extremely useful concept.

3.3.4 Finding the Ricci tensor, Scalar Curvature and Einstein Tensor of de Sitter Space

The conformal factor, given by definition 8, is useful for enabling this to be done. Having made a conceptual distinction between ε^a where a=0,1,2,3 and ε^4, it is now possible to write these coordinates in terms of the conformal coordinates, x^μ, dealing with a curved 3+1 dimensional hypersurface embedded in a flat 4+1 dimensional ambient space.

$$\varepsilon^a = n\delta^a{}_\mu x^\mu \tag{21}$$

$$\varepsilon^4 = -n\rho_{dS}(1 - \frac{1}{4\rho_{dS}^2}\eta_{44}\eta'_{\mu\nu}x^\mu x^\nu) \tag{22}$$

In equation 21, the convention established by Aldrovandi and Pereira (1998) of using Latin letters (a,b,c...) for the ε coordinates and using Greek letters (α, β, γ...) when dealing with the stereographic coordinates is used.

$$\eta'_{\mu\nu} = \delta^a{}_\mu \delta^b{}_\nu \eta'_{ab} \tag{23}$$

In order to write a version of equation 20 for this situation, the conformal factor, n, when using stereographic coordinates to deal with this hypersurface, is given by (Aldrovandi and Pereira (1998)):

$$n = \left(1 + \frac{1}{4R^2}\eta_{44}\eta'_{\mu\nu}x^\mu x^\nu\right)^{-1} \tag{24}$$

Finding a line element, ds on a surface is relatively straightforward, and one can also find a line element on the stereographic projection of the surface. Therefore:

$$ds^2 = \eta_{AB}d\varepsilon^A d\varepsilon^B = g_{\mu\nu}dx^\mu dx^\nu \tag{25}$$

$g_{\mu\nu}$ is the metric on the curved hypersurface. Relating the metric for the curved hypersurface $g_{\mu\nu}$ to η the metric for the flat ambient space given by equation 16(taking advantage of how it was written in equation 19), one can write out:

$$g_{\mu\nu} = n^2\delta^a{}_\mu \delta^b{}_\nu \eta'_{ab} = n^2\eta'_{\mu\nu} \tag{26}$$

This value for the metric of the conformal space can now be put into the Christoffel symbol in order to be used in the definition of the Riemann tensor, which in turn can be used in the definition of the

Ricci tensor. Putting the tensor η in equation 26 into equation 2 gives the Christoffel symbol of the second kind for de Sitter space:

$$\Gamma^{\alpha}_{\mu\nu} = (\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu} + \delta^{\alpha}{}_{\nu}\delta^{\beta}{}_{\mu} - \eta'_{\mu\nu}\eta'^{\alpha\beta})\partial_{\beta}ln(n) \tag{27}$$

And putting equation 27 into definition 3 to get the Riemann tensor of de Sitter space and putting that tensor in turn into definition 4 to get the Ricci tensor of de Sitter space results in:

$$R_{\mu\nu} = 3\frac{1}{\rho^2_{dS}}g_{\mu\nu} \tag{28}$$

And putting equation 28 into definition 5 gives the de Sitter Scalar Curvature:

$$R = 12\frac{1}{\rho^2_{dS}} \tag{29}$$

And therefore, putting these quantities into the Einstein tensor given by definition 6 gives:

$$G_{\mu\nu} = -\frac{3}{\rho^2_{dS}}g_{\mu\nu} \tag{30}$$

And so since the energy momentum tensor for de Sitter space equals zero as is stated by equation 15, the Einstein Field Equation for de Sitter space simplifies to:

$$\Lambda = \frac{3}{\rho^2_{dS}} \tag{31}$$

Therefore, since ρ_{dS} is positive, de Sitter space has a positive Cosmological Constant, Λ, and as $\rho_{dS} \longrightarrow \infty, \Lambda \longrightarrow 0.$

With a much better understanding of de Sitter space, it is now time to start considering its symmetries, in order to examine its conservation laws.

4 Symmetry in Physics

What immortal hand or eye, could frame thy fearful symmetry?

The Tyger (Blake (1988) (originally published 1794))

4.1 An Introduction to Lie Groups

Describing the symmetries of Minkowski and de Sitter Space involves the Poincaré and de Sitter groups [3] respectively, both of which are Lie groups. Therefore, a clear understanding of Lie groups is important for this project.

4.1.1 Lie Groups

Definition 9 (Smooth Manifold) *A Smooth Manifold is a Topological Manifold, M, together with a differentiable structure on M.*

[3] Though little detail been given about these groups so far, they will be explained in more detail in the section about the generators of these groups.

Definition 10 (Lie Group) *A Lie Group is a Group that is a smooth differentiable manifold.*

Therefore a Lie Group is a Group for which the elements are continuous rather than discrete.

Figure 15 shows an example of a Lie Group, Γ and a Vector Space called the Geometric Space, G, for which the transformations of G are represented by elements of Γ. Γ contains elements such as α, β, γ... while G contains elements such as a, b, c... An analytic function, ϕ, represents Γ's group operation, which takes two elements of Γ, such as α and β, and returns an element of Γ, such as γ. So for η-dimensional Γ, where elements α and β are points on the manifold with coordinates $(\alpha^1, \alpha^2 ... \alpha^\eta)$ and $(\beta^1, \beta^2 ... \beta^\eta)$, $\phi(\alpha^1, ... \alpha^\eta; \beta^1, ... \beta^\eta) = (\gamma^1, ... \gamma^\eta)$ One can take an element of Γ such as α, which is a point on the η-dimensional manifold, and an element of G, such as a, which is a point of the N-dimensional Geometric Space, and can use the function f to execute the transformation: $f(\alpha^1, ... \alpha^\eta; a^1, ... a^N) = (a'^1, ... a'^N)$ Gilmore (1974).

In this project, the spaces being discussed will mainly be Minkowski and de Sitter Space, while the Lie groups being discussed will mainly be the corresponding Poincaré and de Sitter groups.

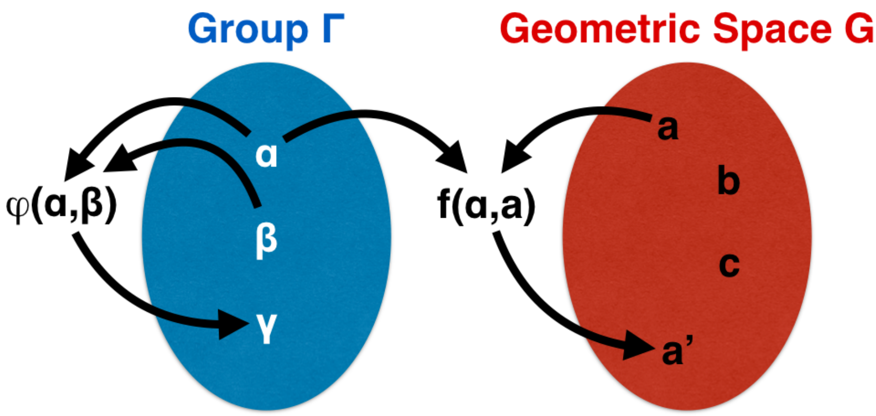

Figure 15: A Lie Group, Γ, and a geometric space, G.

4.1.2 Special Orthogonal Groups

Lie Groups of special note to this project will be the Special Orthogonal groups, also called SO(n) groups.

Definition 11 (Orthogonal Matrix) *An $n \times n$ matrix, M is orthogonal if the transpose of the matrix, M^T is the inverse of M such that $MM^T = M^TM = I$ where I is the $n \times n$ identity matrix. Armstrong (1988)*

Definition 12 (Orthogonal Group O(n)) *The collection of all $n \times n$ orthogonal matrices forms the Orthogonal Group O(n). Hall (2003a)*

Group O(n) has subgroup SO(n).

Definition 13 (Special Orthogonal Group SO(n)) *The set of $n \times n$ orthogonal matrices with determinant 1 is the Special Orthogonal Group SO(n). Hall (2003a)*

One can also consider pseudo-orthogonal matrices.

Definition 14 (Pseudo-Orthogonal Matrix) *If M and J are $n \times n$ matrices and $J = diag(\pm 1, ..., \pm 1)$*

such that not every entry necessarily has the same sign [4] *and $M^TJM = J$ then M is a pseudo-orthogonal matrix.*

One can discuss pseudo-orthogonal and special pseudo-orthogonal groups. For example, when J is the Minkowski metric given by equation 7, one can discuss the $O(1,3)$ and $SO(1,3)$ Lie Groups [5].

4.1.3 Lie Algebras

Since a Lie group, Γ is a smooth manifold, it has a tangent space, γ, which is a vector space.

Since a Lie group is a smooth manifold which is also a group, there is a point on the manifold which is the identity.

The tangent space γ of the Lie group Γ close to the identity is a vector space called the Lie Algebra.

The elements of this vector space are the Infinitesimal Generators of the Lie Group (see figure 16 for a visual representation of this relationship) (HTNW (2021), Kajelad (2021)).

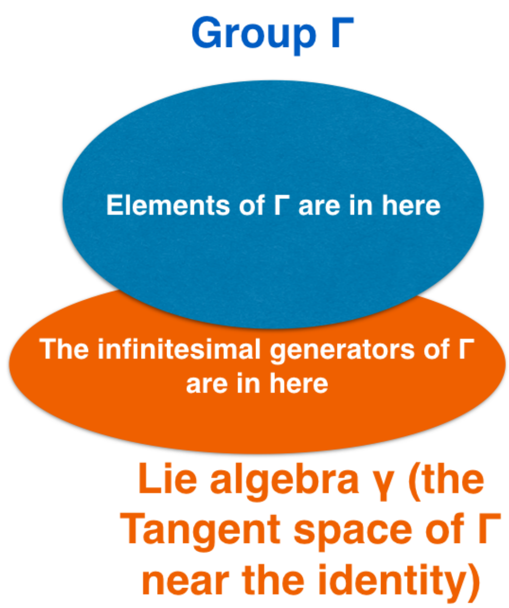

Figure 16: The Lie Group Generators are in the Tangent Space of a Lie Group.

Definition 15 (Algebra) *An algebra, γ, is a vector space over field, k, which has a binary operation (given by Lie brackets [A,B]) for which [,]: $\gamma \times \gamma \longrightarrow \gamma$. Knapp (1996a) Schwichtenberg (2015)*

Definition 16 (Lie Algebra) *An algebra, γ, is a Lie Algebra if all of its products satisfy the following conditions:*
Antisymmetry: $\forall A, B \in \gamma$, $[A, B] = -[B, A]$
Jacobi Identity: $\forall A, B, C \in \gamma$, $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$
Bilinearity: $\forall A, B, C \in \gamma$ $[\alpha A + \beta B, C] = \alpha[A, C] + \beta[B, C]$, $[C, \alpha A + \beta B] = \alpha[C, A] + \beta[C, B]$ *where α and β are arbitrary scalars.*
Knapp (1996a) Schwichtenberg (2015)

[4] the most obvious example of such a matrix is the Minkowski metric given by equation 7
[5] Note that some textbooks instead use the convention of instead writing $O(3,1)$ and $SO(3,1)$

So in a Lie group, such as the Lorentz group, there are elements such as Lorentz transformations, while in the tangent space to that Lie group, such as the Lorentz algebra, there are the Lie Generators.

As Generators' name suggests, Generators can be used to generate a Lie group, as will now be shown.

4.1.4 The Exponential Map

e^x where e is the exponential constant and x is some number can be written as a Taylor series using equation 212. $\exp(X)$ is similar to this, with Generators (written as X here) rather than numbers, such that by using the Taylor series given by equation 212, one can derive a Lie Group's elements from the Generators. Using a Lie Algebra as defined by definition 16, one can generate the corresponding Lie Group using the exponential map:

Definition 17 (Exponential Map) *If Γ is a Lie Group and γ is the corresponding Lie Algebra then the exponential map, $\exp$, for Γ is:*
$\exp(\gamma) \longrightarrow \Gamma$ *Hall (2003b)*

The exponential map is a major reason why considering Lie Algebras is so useful for studying Lie Groups.

4.1.5 An Example of what has been discussed thus far: Infinitesimal Generators of the SO(3) Group

An example of Generators might be illustrative. Using definition 13, SO(3) is the group of 3×3 orthogonal matrices which have determinant 1. In other words, it's the group of all and only 3D rotations.

4.1.5.1 Finding the Infinitesimal Generators of the SO(3) group

A rotation about the z axis in Euclidean space is given by:

$$x' = x\cos\theta - y\sin\theta \qquad y' = x\sin\theta + y\cos\theta \qquad z' = z \tag{32}$$

And so with a little consideration, the rotation matrix in question is given by:

$$R_3 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{33}$$

Now one can set about finding the infinitesimal Generators. Due to being infinitesimal, the small angle approximations can be used, replacing $\sin\theta$ and $\cos\theta$ with θ and 1 respectively. Therefore, taking the derivative with respect to θ, one can find the infinitesimal generator:

$$L_3 = i\frac{dR_z}{d\theta} = i\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{34}$$

Note the i in equation 34. [6]

[6] Why is there an i in the equation? This is time for an important note on conventions when writing Generators. In

Using analogous reasoning to find the infinitesimal generators about the x and y axes:

$$L_1 = i\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad L_2 = i\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad L_3 = i\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{35}$$

Or using the Levi-Civita symbol, the ij entry of the relevant matrix is given by:

$$(L_1)_{ij} = -i\epsilon_{ij1} \qquad (L_2)_{ij} = -i\epsilon_{ij2} \qquad (L_3)_{ij} = -i\epsilon_{ij3} \tag{36}$$

4.1.5.2 The so(3) Lie Algebra

Using equation 35, the so(3) Lie algebra (which does indeed satisfy all the criteria of definition 16) is:

$$[L_1, L_2] = iL_3 \qquad [L_2, L_3] = iL_1 \qquad [L_3, L_1] = iL_2 \tag{37}$$

This can be written using the Levi Civita symbol as:

$$[L_i, L_j] = i\epsilon_{ijk}L_k \tag{38}$$

4.1.5.3 How do the Generators of the SO(3) group generate the SO(3) group?

If one knows the infinitesimal Generators and wants to know the elements of the Lie Group, one can use the so(3) algebra and exp. Due to orthogonality, for generator L,

$$(\exp(L))^T = \exp(L^T) = \exp(-L) \tag{39}$$

And therefore

$$(\exp(L))^T(\exp(L)) = \exp(-L + L) = \exp(0) = I \tag{40}$$

And therefore $\exp(L)$ is indeed an orthogonal matrix.

The matrix exponential of a Generator can be expanded into a Taylor series[7] using equation 212.

$$\exp(-i\theta L_1) = \frac{1}{0!}I_{3\times 3} + \frac{\theta}{1!}(-iL_1) + \frac{\theta^2}{2!}(-iL_1)^2 + ... \tag{41}$$

One can take advantage of the facts that $(-iL_1)^{1+4k} = -iL_1$, $(-iL_1)^{1+3k} = -(-iL_1)^2$, $(-iL_1)^{1+2k} = -(-iL_1)$, $k \in \mathbb{Z}$ and that...

Mathematics, Generators are sometimes written without the imaginary unit i. However, in Physics, i is generally used so that the Generators will be Hermitian.

[7] Note that the -i in equation 41 cancels out with the i in the matrix L_1 following the Physics convention of writing Generators with i discussed in a previous footnote as a way to make the Generators Hermitian. If one had instead written the Generators without the i, as sometimes happens in Mathematics subjects unrelated to Physics, one would not include the -i in equation 41

$$(-iL_1)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

...to write out $\exp(-i\theta L_1)$ as:

$$\exp(-i\theta L_1) =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ...\right) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \left(\frac{1}{0!} - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ...\right) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (42)$$

Putting the cosine and sine Taylor series given by equations 213 and 214 into equation 42 gives:

$$\exp(-i\theta L_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = R_1(\theta) \quad (43)$$

Therefore

$$\exp(-i\theta L_i) = R_i(\theta) \quad (44)$$

So it has been shown how if one knows the SO(3) Generators, one can generate the SO(3) Group. The so(3) Algebra is related to the SO(3) Group by $\exp$.

4.1.6 Representations of the Infinitesimal Generators of Lie Groups

Killing vectors are representations of Infinitesimal Generators, and will be of great relevance to this project.

4.1.6.1 Lie Derivatives

Dealing with a single curve was shown in figure 12 when discussing Covariant Derivatives is a very abstract and idealised scenario. Often in Physics and Mathematics, there is more likely to be the case of vector fields with various field lines. Consider a single curve, $x^\mu(z)$ in a vector field, f, of various possible curves, as is shown in figure 17. Two points along the curve, a and b, have tangent vectors using the vector field $f = \frac{\partial}{\partial z}$ for z defined everywhere in the surrounding region, and the vector field defined everywhere in the surrounding region. There is also vector field v defined everywhere in the surrounding region, which gives the two normal vectors, v_a and v_b, which are not parallel, with v'_a instead being parallel to v_a. How are things different in this vector field scenario?

One can therefore think of this vector space as involving multiple integral curves of the vector field $f = \frac{\partial}{\partial z}$. While the focus of this discussion is on one specific curve, figure 17 could be misinterpreted as indicating that this specific curve is unique in the vector field, when that is most definitely *not* the case. Each curving sequence of green arrows in the diagram could similarly be written as a single integral curve as occurs in figure 18. In this case, with vector field $f = \frac{\partial}{\partial z}$, the gaps between points a and b is given by $\Delta z f_a$ for the case from a to b in the vector field of f, while the parts of the

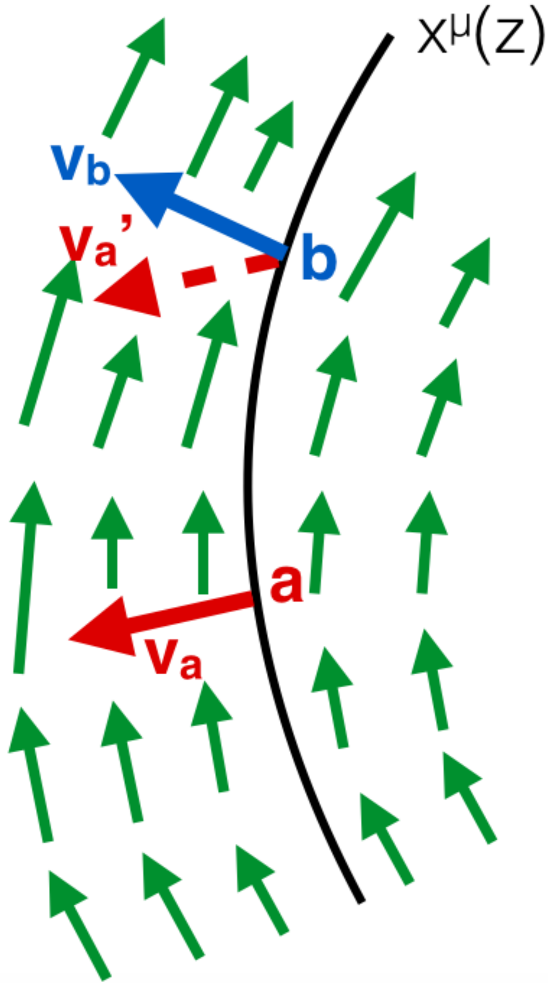

Figure 17: Lie Derivative Diagram: As can be seen, there is a vector field (signified by green arrows). The curve, $x^{\mu}(z)$ is being considered)

normal vectors connecting a to c and connecting b to d are given by $\Delta y v_a$ and $\Delta y v_b$ respectively. Returning once again to the concept of parallel transport which was also discussed in section 3.1, the difference between d and c', which would be 0 if v_a and v_b were parallel, but is nonzero if they are not, is given by the Lie derivative.

So the Lie derivative is different from the covariant derivative, and is extremely useful, allowing us to think in terms of vector fields and systems.

This difference, as is shown in figure 18, can be written as:

$$L_f v^{\mu} = f^{\alpha}\partial_{\alpha} v^{\mu} - v^{\alpha}\partial_{\alpha} f^{\mu} \tag{45}$$

So the concept of a Lie Derivative is much clearer now.

Definition 18 (Lie Derivative) *For some vector field, $\overrightarrow{v}$, the Lie derivative of $\overrightarrow{v}$ in the direction of tangent vector field $\overrightarrow{u}$ is given by:* $L_u v^i = u^j \frac{\partial v^i}{\partial x^j} - v^j \frac{\partial u^i}{\partial x^j} = u^j \partial_j v^i - v^j \partial_j u^i$

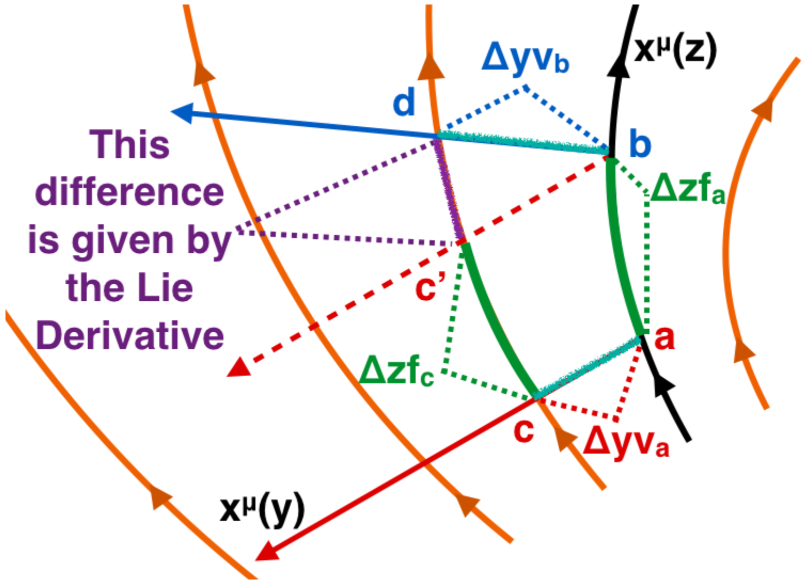

Figure 18: Lie Derivative Diagram: The multiple integral curves of the vector field f are orange. The two normal vectors are coloured red and blue respectively, although the parts of them given by $\Delta y v_a$ and $\Delta y v_b$ respectively are coloured cyan.

Therefore the Lie derivative of a covariant vector, v_i is given by:

$$L_u v_i = v_j \partial_i u^j + u^j \partial_j v_i \tag{46}$$

While the Lie derivative of metric tensor g_{ij} of manifold M is given by:

$$L_u g_{ij} = u^k \partial_k g_{ji} + g_{jk} \partial_i u^k + g_{ik} \partial_j u^k \tag{47}$$

(Carroll (2014b), Physics Unsimplified (2019a))

What happens when a Lie derivative equals 0? That question leads to discussing Killing vectors.

4.1.6.2 Killing Vectors and Isometries

In common sense terms, Killing vector fields (often simply called Killing vectors for brevity's sake) are vector fields such that, when translating a set of points on a manifold in the direction of the Killing vector field, one does not change the distances between the points. These vector fields are due to the isometries of the manifold and so are a nice way of describing things when one's emphasis is on looking at isometries and corresponding conservation laws. A more formal definition now follows.

Definition 19 (Killing Vector Field) *For some manifold, M, with metric g, a Killing vector field is a vector field on M which leaves g invariant under diffeomorphism induced by the Killing vector field.*

For every isometry of the manifold there is a corresponding vanishing Lie derivative of the metric. What does this mean? If there is some direction in which the metric doesn't change then there's an isometry. Setting the Lie derivative of a metric to equal zero gives the Killing equation, which can be used to find the isometry directions of the manifold, which form the solutions.

$$\nabla_i u_j + \nabla_j u_i = 0 \tag{48}$$

As was historically established by Emmy Noether with Noether's Theorem, symmetries imply conservation laws, and so Killing Vectors are extremely useful in finding conservation laws.

Killing vector fields are in the Tangent Space of the manifold just as the Infinitesimal Generators of a Lie Group are in the Tangent Space of the Lie Group, as is pictured in figure 16. The Killing vector fields of a manifold are representations of the Infinitesimal Generators of the Isometry Group of that manifold, as can be seen in figure 19 (Pedro (2021), Kluitenberg (2021)).

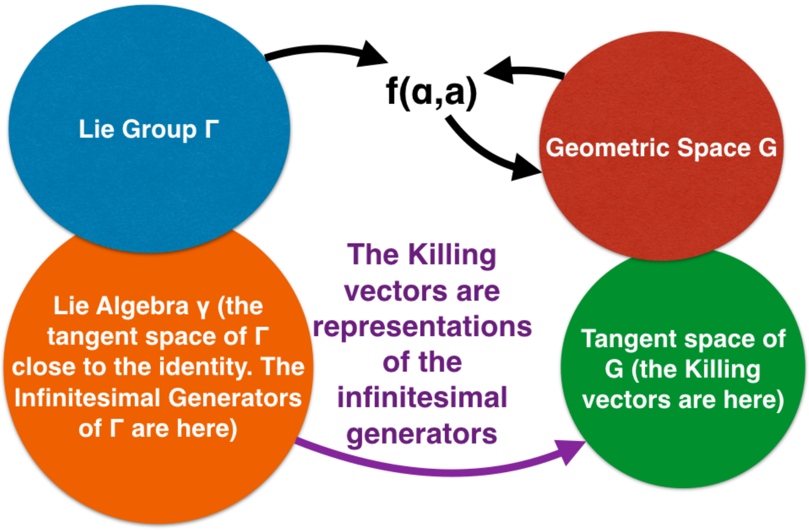

Figure 19: The Infinitesimal Generators of a Lie group Γ in the Lie Group's tangent space γ are represented by the Killing vectors in the tangent space of the geometric space G of which the Lie Group is the isometry group.

In this project, two Lie Groups will be especially important. Lie Groups Γ will be the Poincaré and de Sitter groups, which have corresponding algebras, γ, given by the Poincaré and de Sitter algebras. The corresponding Geometric spaces, G, will be Minkowski and de Sitter space, with the Killing vectors of those spaces being the Killing vectors of Minkowski and de Sitter space.

5 The Infinitesimal Generators of the Lorentz, Poincaré and de Sitter groups

First, a matrix approach will be considered to look at the matrices in the so(1,3) and so(1,4) algebras, before a more conventional approach in terms of Killing vectors will be used to find the relevant Generators. The SO(1,3) Group, of which so(1,3) is the corresponding algebra, is called the Lorentz group, and is a subgroup of the Poincaré group. The SO(1,4) group, of which so(1,4) is the corresponding algebra, is the isometry group of de Sitter space, as will become clearer when discussing Killing vectors.

5.1 The Matrix approach to finding the infinitesimal generators of the SO(1,3) and SO(1,4) groups

This approach is *not* the ideal way to do this. The faster and easier way to find the Generators of the relevant Lie Groups will occur after first looking at some of the matrices involved. An approach emphasising matrices can be a useful intuitive way to conceptualise the generators of groups. Unfortunately, it doesn't work with all groups. The Poincaré group includes translations, which are not linear transformations, and so such an approach won't work for Poincaré, although it can work with the Lorentz subgroup of the Poincaré group. Before using Killing Vectors to find the generators of the de Sitter and Poincaré groups, let's first find the Lorentz and de Sitter groups' generators using matrices. This method is less elegant, but can help give a clearer conceptual understanding of what exactly is going on.

This approach, using matrix reps, is a more 'intuitive' example of the Group Generators, but is also quite slow. The approach using Killing vector reps is a little faster.

The Lie Algebras for group O(1,n-1) and SO(1,n-1) (where n is some positive integer) are the same. By convention, the Lie algebra of a Lie group is written with the non-capitalised version of the group's name, and since SO(1,n-1) and O(1,n-1) both have the same Lie Algebra, the Lie Algebra being discussed will simply be called so(1,n-1) rather than 'so(1,n-1) and also o(1,n-1)' (which would be a rather more unwieldy name). The definition of the group given by definition 13 means that for any element of the group, A, $AA^T = A^TA = I$ where I is the identity matrix. The elements of the group are related to the Generators by the exponential map, as was explained by definition 17. Therefore elements of the group can be written as $A = \exp(M)$ where M is a Generator of the Group. Since $(\exp(M))^T = \exp(M^T)$ it is the case that:

$$\exp(M)\exp(M^T) = \exp(M^T)\exp(M) = I \tag{49}$$

Therefore it is the case that

$$\exp(M + M^T) = \exp(M^T + M) = I \tag{50}$$

From this, one can infer that M must be skew symmetric.

$$M^T = -M \tag{51}$$

Therefore, any matrix, M, of the Lie algebra so(1,n-1) can be written as:

$$M = \begin{bmatrix} \alpha & \gamma \\ \gamma^T & \beta \end{bmatrix} \tag{52}$$

Where α is 1×1, β is $n \times n$ and γ is $1 \times n$. Knapp (1996b). Throughout this section, the i^{th} component of a β matrix will be written as j_i and the i^{th}component of a γ vector will be written as k_i. It is important to note that this matrix cannot yet be assumed to be orthogonal, because one can choose an α and a β for which M is not orthogonal.

Due to the fact that for each matrix in the the so(1,3) algebra, the matrix's transpose is its inverse, one can infer that α and β are both skew symmetric.

For all Lie algebras so(1,n), α is a 1×1 matrix, which means that for all so(1,n) algebras where n is some positive integer:

$$\alpha^T = \alpha \tag{53}$$

And therefore, since α is skew symmetric:

$$\alpha = -\alpha \Rightarrow \alpha = 0 \tag{54}$$

$$\Rightarrow M = \begin{bmatrix} 0 & \gamma \\ \gamma^T & \beta \end{bmatrix} \tag{55}$$

Let's look at some examples of β. The simplest examples, so(1,1) and so(1,2), are not included here, but can be found in 10.4. They have been put there since the so(1,3) and so(1,4) algebras are of far more importance to this project. However, those who feel mathematically confused by these concepts might wish to read that section before returning here in order to work through simpler examples before focusing on the Lorentz and de Sitter algebras.

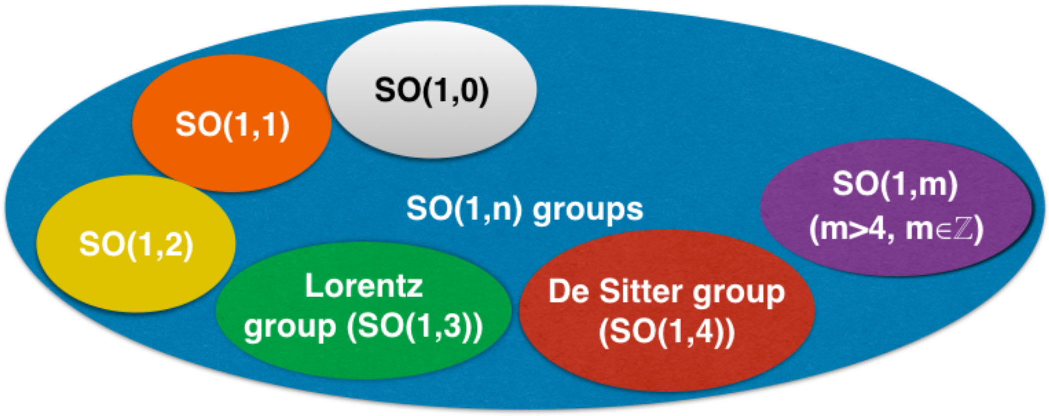

Figure 20: Various SO(1,n) groups

5.1.1 so(1,3) (The Lorentz Algebra)

This is the group of real homogeneous linear transformations of x, y, z and t which leave $x^2 + y^2 + z^2 - t^2$ invariant.

γ is 1×3 and β is 3×3. It is a subgroup of the Poincaré group, which contains the SO(1,3) group and also spacetime translations. Skew symmetry necessitates that all β matrices have the form:

$$\beta = \begin{bmatrix} 0 & -j_3 & j_2 \\ j_3 & 0 & -j_1 \\ -j_2 & j_1 & 0 \end{bmatrix} \tag{56}$$

So all matrices in the so(1,3) Lie algebra are given by:

$$M(1,3) = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ k_1 & 0 & -j_3 & j_2 \\ k_2 & j_3 & 0 & -j_1 \\ k_3 & -j_2 & j_1 & 0 \end{bmatrix} = k_1K_1 + k_2K_2 + k_3K_3 + j_1J_1 + j_2J_2 + j_3J_3 \tag{57}$$

where k_i and j_i are some real numbers for $i = 1, 2, 3$ and:

$$K_1 = i\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} K_2 = i\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} K_3 = i\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{58}$$

$$J_1 = i\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} J_2 = i\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} J_3 = i\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{59}$$

So one can define everything in the Lie algebra using these six matrices. There are six generators, of which three are K matrices, which are normally called boosts, and three of which are J matrices, which are normally called rotations.

The Levi-Civita symbol can conveniently explain the so(1,3) algebra which arises from these six generators:

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{60}$$

$$[K_i, K_j] = -i\epsilon_{ijk}J_k \tag{61}$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k \tag{62}$$

There is a slightly more succinct way to write the Lorentz group which is common in Physics.

$$J_i = \frac{1}{2}\epsilon_{ijk}\lambda_{jk} \qquad K_i = \lambda_{0i} \tag{63}$$

Using the above definition, the Lorentz Lie algebra becomes much more straightforward, being:

$$[\lambda_{\alpha\beta}, \lambda_{\gamma\delta}] = i(\eta_{\alpha\gamma}\lambda_{\beta\delta} - \eta_{\alpha\delta}\lambda_{\beta\gamma} - \eta_{\beta\gamma}\lambda_{\alpha\delta} + \eta_{\beta\delta}\lambda_{\alpha\gamma}) \tag{64}$$

For $\alpha, \beta, \gamma, \delta = 0, 1, 2, 3, \alpha \neq \beta, \gamma \neq \delta$, and with η being the Minkowski metric given by equation 7.

5.1.1.1 The Lorentz Group's Elements (Lorentz Transformations)

Knowing the Generators, any element, Λ of the Lorentz group can be written using exp.

$$\Lambda = \exp(-i\vec{\chi}\cdot\vec{K})\exp(-i\vec{\theta}\cdot\vec{J}) \tag{65}$$

Where the $\exp(-i\vec{\theta}\cdot\vec{J})$ should seem obvious from the similar case in equation 44. χ is a hyperbolic 'angle' for when dealing with boosts just as θ is there for rotations. χ is given by $\gamma = \cosh\chi$ where γ is the Lorentz factor. It is worth noting that $\exp(-i\vec{\chi}\cdot\vec{K})$ and $\exp(-i\vec{\theta}\cdot\vec{J})$ cannot be combined into a single $\exp$ because J and K do not commute (Boer (2019)).

5.1.2 so(1,4) (The de Sitter Algebra)

Although the de Sitter algebra is the so(1,4) algebra, that will not be shown in this section, instead being shown in section 5.2.2. The writer of this thesis personally considers the way of finding the SO(1,4) Generators and the so(1,4) algebra which appears in that section to be more straightforward.

What are the real homogeneous linear transformations of w, x, y, z and t which leave $w^2 + x^2 + y^2 + z^2 - t^2$ invariant?
γ is 1×4 and β is 4×4. Skew symmetry necessitates that all β matrices have the form:

$$\beta = \begin{bmatrix} 0 & j_1 & j_2 & j_3 \\ -j_1 & 0 & -j_4 & j_5 \\ -j_2 & j_4 & 0 & -j_6 \\ -j_3 & -j_5 & j_6 & 0 \end{bmatrix} \tag{66}$$

So all matrices in the so(1,4) Lie algebra are given by:

$$M(1,4) = \begin{bmatrix} 0 & b_1 & b_2 & b_3 & b_4 \\ b_1 & 0 & j_1 & j_2 & j_3 \\ b_2 & -j_1 & 0 & -j_4 & j_5 \\ b_3 & -j_2 & j_4 & 0 & -j_6 \\ b_4 & -j_3 & -j_5 & j_6 & 0 \end{bmatrix} \tag{67}$$

$$\Rightarrow M(1,4) = b_1B_1 + b_2B_2 + b_3B_3 + b_4B_4 + j_1J_1 + j_2J_2 + j_3J_3 + j_4J_4 + j_5J_5 + j_6J_6 \tag{68}$$

Where b_i and j_j are real numbers for $i = 1,2,3,4$ and $j = 1,2,3,4,5,6$:

$$B_1 = i\begin{bmatrix} 0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{bmatrix} B_2 = i\begin{bmatrix} 0&0&1&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{bmatrix} B_3 = i\begin{bmatrix} 0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0 \end{bmatrix} B_4 = i\begin{bmatrix} 0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0 \end{bmatrix} \tag{69}$$

$$J_1 = P_1 = i \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} J_2 = P_2 = i \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} J_3 = P_3 = i \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{70}$$

$$J_4 = Q_3 = i \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} J_5 = Q_2 = i \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} J_6 = Q_1 = i \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{71}$$

So one can define everything in the Lie algebra using these ten matrices. These are the generators of this Lie Algebra. Therefore, its generators include four boosts (the K matrices) and six rotations (the J matrices).

Why are the first and last three J matrices also written in terms of P and Q respectively? Because with a little matrix multiplication, it will be clear that the six J matrices of the so(1,4) algebra have the exact same commutation relations with each other as the so(1,3) algebra. P_i of so(1,4) is analogous to K_i of so(1,3), while Q_1, Q_2 and Q_3 of so(1,4) are analogous to J_3, J_2 and J_1 of so(1,3) respectively.

The commutations of just the J generators, rewritten as P and Q, are:

$$[P_i, P_j] = -i\epsilon^{ijk}Q_k \tag{72}$$

$$[Q_i, Q_j] = i\epsilon^{ijk}Q_k \tag{73}$$

$$[P_i, Q_j] = i\epsilon^{ijk}P_k \tag{74}$$

The commutation relations of the boosts can be written as:

	First B entry in Commutator Bracket:			
Second B entry in Commutator Bracket:	1	2	3	4
1	0			
2	iP_1	0		
3	iP_2	$-iQ_3$	0	
4	iP_3	iQ_2	$-iQ_1$	0

Again, with a little rewriting, this can be written more succinctly. Let:

$$K_0 = B_1, K_1 = B_2, K_2 = B_3, K_3 = B_4 \tag{75}$$

And therefore this whole table can be written as:

$$[K_0, K_i] = iP_i \tag{76}$$

$$[K_i, K_j] = -i\epsilon^{ijk}Q_k \tag{77}$$

Where i=1,2,3 and j=1,2,3. There are also the twenty four combinations in which a boosts and a rotation are both in the commutator brackets, which must be considered to show all of the de Sitter algebra.

First K entry in Commutator Bracket :				
Second rotation entry in Commutator Bracket:	0	1	2	3
i P_1	iK_1	-iK_0	0	0
i P_2	iK_2	0	-iK_0	0
i P_3	iK_3	0	0	-iK_0
iQ_1	0	0	-iK_3	iK_2
iQ_2	0	iK_3	0	-iK_1
iQ_3	0	-iK_2	iK_1	0

So again, this table can be written more succinctly as:

$$[K_0, P_i] = iK_i \tag{78}$$

$$[K_0, Q_i] = 0 \tag{79}$$

$$[K_i, P_j] = -i\delta_{ij}K_0 \tag{80}$$

$$[K_i, Q_j] = i\epsilon^{ijk}K_k \tag{81}$$

5.1.2.1 The de Sitter Algebra in Full

So now the de Sitter Algebra can be written out more clearly. There are 10 generators: Four boost generators in the 1st, 2nd, 3rd and 4th spatial dimensions which are misleadingly (but with reason behind it, once we get to the Poincaré group) labelled K_0, K_1, K_2 and K_3. It also contains six rotation generators, which are labelled P_1, P_2, P_3, Q_1, Q_2 and Q_3. The algebra is given by the commutation relations:

$$[P_i, P_j] = -i\epsilon^{ijk}Q_k \qquad [Q_i, Q_j] = i\epsilon^{ijk}Q_k \qquad [P_i, Q_j] = i\epsilon^{ijk}P_k \tag{82}$$

$$[K_0, K_i] = iP_i \qquad [K_0, Q_i] = 0 \qquad [K_0, P_i] = iK_i \tag{83}$$

$$[K_i, K_j] = -i\epsilon^{ijk}Q_k \qquad [K_i, P_j] = -i\delta_{ij}K_0 \qquad [K_i, Q_j] = i\epsilon^{ijk}K_k \tag{84}$$

Where i=1,2,3 and j=1,2,3.

As with so(1,3), some methods can be employed to express the so(1,4) Lie algebra more succinctly. L.H. Thomas actually did work out all of the relations in Thomas (1941). However, using the equivalent approach to the one used with so(1,3), one can write the so(1,4) algebra as:

$$[X_{\alpha\beta}, X_{\gamma\delta}] = i(\eta_{\alpha\delta}X_{\beta\gamma} - \eta_{\alpha\gamma}X_{\beta\delta} + \eta_{\beta\gamma}X_{\alpha\delta} - \eta_{\beta\delta}X_{\alpha\gamma}) \tag{85}$$

5.1.3 What have we learned from this?

Going through these groups has provided some intriguing insights.

so(1,n-1) Lie algebra	Number of generators	Number of rotation generators	Number of boost generators
so(1,1)	1	0	1
so(1,2)	3	1	2
so(1,3)	6	3	3
so(1,4)	10	6	4

For a Lie algebra so(1,n-1), the number of boost generators is equal to n-1. The number of generators is equal to $\frac{1}{2}n(n-1)$.

What determines the number of possible rotations, and the number of possible boosts? A rotation occurs in a 2D plane embedded in a space of n-1 dimensions. The SO(1,1) group does not have enough spatial dimensions to have any planes in which a rotation can occur, so has no rotations. SO(1,2) has exactly one spatial 2D plane, so has exactly one rotation generator. SO(1,3) has three possible planes (xy, xz and yz) so needs three rotation generators. Finally SO(1,4) has six possible planes (wx, wy, wz, xy, xz, yz) so has six rotation generators. This can be extended to further dimensions.

This might seem like cheating because it neglects the temporal dimension t in each of these cases. Group SO(1, n-1) deals with n dimensions, not n-1, since n-1 is just the number of spatial dimensions. Is it not possible to consider a rotation in a 2D spacetime plane? A plane of one time dimension and one space dimension? Indeed it is! And that is exactly what a boost is! The number of boost generators in a group's Lie algebra is given by the number of possible 2D planes in that space of which one of the dimensions is time. And so a boost is effectively a kind of spatio-temporal rotation, rather than just a spatial rotation, which is what the other rotations are. This also explains why the matrices for boosts have both objects in the matrix having the same sign, while those for rotations do not. Because under the Minkowski metric, the sign of the time component and of some spatial component are opposite, while the same is not true for two spatial component.

Using this matrix approach is slow and boring, especially for groups with larger numbers of generators such as the de Sitter group. It also only works for groups all of the elements of which are linear transformations, and so doesn't work with the Poincaré group. A much faster, more elegant and more interesting approach, which also works with the Poincaré group, is to use Killing vectors.

5.2 The Killing Vector Approach

5.2.1 The Poincaré Group

5.2.1.1 The Poincaré Generators

The generators of the symmetry group of flat Minkowski spacetime, which has temporal dimension t (its zeroth dimension) and spatial dimensions x, y and z (its first, second and third dimensions, respectively) can be found using Killing Vector Fields. The isometry group of Minkowski spacetime is called the Poincaré group.

The two essential things required to find the Killing Vectors of Minkowski space are the fact that Minkowski space is flat (So its Riemann tensor, Ricci tensor and scalar curvature are all 0)and that the Minkowski metric is given by

$$g_{\text{Minkowski}} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \tag{86}$$

such that for Minkowski space:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{87}$$

The intuitive way that a person acquainted with definition 19 looks for symmetries is by looking for what doesn't turn up in the metric. Since x, y, z and t do not turn up in it, this already indicates symmetries.
From the discussion of parallel transport earlier, it can be seen that this gives four obvious killing vectors in the form of derivatives:

$$P^0 = i\partial^t \tag{88}$$

$$P^1 = i\partial^x \qquad P^2 = i\partial^y \qquad P^3 = i\partial^z, \tag{89}$$

Corresponding to translations. Using the Killing equation given by equation 48, one can find six more transformations:

$$J_1 = i(-z\partial_y + y\partial_z) \tag{90}$$

$$J_2 = i(z\partial_x - x\partial_z) \tag{91}$$

$$J_3 = i(-y\partial_x + x\partial_y) \tag{92}$$

These are transformations which occur in a two dimensional plane of Minkowski spacetime, for which neither of the plane's dimensions are time. These transformations are called rotations. There are also:

$$K_1 = i(t\partial_x + x\partial_t) \tag{93}$$

$$K_2 = i(-t\partial_y - y\partial_t) \tag{94}$$

$$K_3 = i(z\partial_t + t\partial_z) \tag{95}$$

Which are transformations which similarly occur in a two dimensional plane of Minkowski spacetime. However, unlike with the rotations, these transformations occur in a two dimensional plane for which one of the dimensions is time and the other is a spatial dimension. These transformations are called boosts. Note that in equations 93, 94 and 95, the convention has been taken of setting $c = 1$.

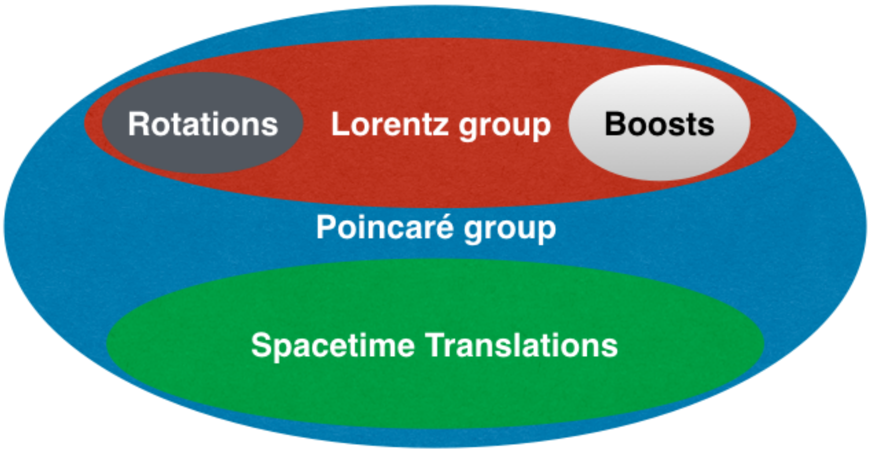

Figure 21: The Poincaré group has exactly ten generators: Four spacetime translation generators, and also the six generators of the Lorentz group, of which three are rotation generators and three are boost generators

In other words, since there are six possible two dimensional planes in 1+3 dimensional Minkowski spacetime, there are six possible linear transformations in it. These six linear transformations might seem familiar, because they have already been encountered in the SO(1,3) group. The three boosts and three rotations of the Poincaré group form the Lorentz group, which is a subgroup of the Poincaré group. So in other words, the Poincaré group consists of the Lorentz group and also spacetime translations.

But wait. The translations each occur in a one dimensional line, while the rotations each occur in a two dimensional plane. Why are there no generators which occur in a three dimensional subspace of Minkowski spacetime? Or four dimensional generators? Well the point of generators is that all of the transformations in this spacetime can be constructed out of various generators. All three and four dimensional transformations in Minkowski spacetime can be made up using the ten generators of the Poincaré group.

In the case of a flat spacetime, switching to using the Killing vectors which are the representations of the Infinitesimal Generators of the Poincaré Group in order to consider the symmetries and corresponding conservation laws of Minkowski space can feel like a mere change of notation. However, on a curved spacetime such as de Sitter Space, it is significantly more insightful, enabling one to salvage some form of conservation laws for energy and linear momentum via concepts such as the static patch. But more on that later.

5.2.1.2 The Poincaré algebra

The Poincaré group has ten generators; four spacetime translation generators (P_μ for $\mu = 0, 1, 2, 3$) given by equations 88 and 89, which commute with each other due to being translations on a flat space. It also contains three rotations each of which takes place in a 2 dimensional plane of the three spatial dimensions of four dimensional Minkowski spacetime, J_{12}, J_{13} and J_{23}, given by equations 92, 91 and 90. It also contains three boosts in the 1st, 2nd and 3rd spatial dimensions of Minkowski spacetime. In reality, the boosts are also a kind of rotation, however, rather than taking place in a two dimensional plane in which both of the dimensions are spatial ones, they take place in a two dimensional plane one of the two dimensions of which is the time dimension, 0. So the three boosts are K_{01}, K_{02} and K_{03}, which were given by equations 93, 94 and 95. Similarly to during the discussion of the Lorentz Algebra in equation 64, which is a subalgebra of the Poincaré Algebra, it will be convenient to write these six generators, rotations and boosts in the format $\lambda_{\mu\nu}$ where $\mu, \nu = 0, 1, 2, 3$, the Poincaré algebra can be written as:

$$[\lambda_{\mu\nu}, \lambda_{\rho\lambda}] = i(-\eta_{\mu\rho}\lambda_{\nu\lambda} + \eta_{\mu\lambda}\lambda_{\nu\rho} + \eta_{\nu\rho}\lambda_{\mu\lambda} - \eta_{\nu\lambda}\lambda_{\mu\rho}) \tag{96}$$

This was already found when working out the Lorentz algebra in equation 64, since the Lorentz group is a subgroup of the Poincaré group and so the Lorentz algebra is a subgroup of the Poincaré algebra. However the Poincaré algebra also includes the relations between Lorentz generators and transformation generators, and between transformation generators and other transformation generators:

$$[\lambda_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \tag{97}$$

$$[P_\mu, P_\nu] = 0 \tag{98}$$

5.2.1.3 The Poincaré Group's Elements

The Generators of the Lorentz subgroup of the Poincaré Group, as well as the Spacetime Translation Generators of the Poincaré Group, can all be used to write any Poincaré Group Element, π (Boer (2019)) (in fact the elements of the Lorentz subgroup of the Poincaré group have in fact already been written out in equation 65):

$$\pi = (\lambda|a) = \exp(-ia_\mu P^\mu)\exp(-i\vec{\chi}\cdot\vec{K})\exp(-i\vec{\theta}\cdot\vec{J}) \tag{99}$$

With a^μ being some four-vector.

5.2.2 The de Sitter Group

5.2.2.1 The de Sitter Generators

Using the general form of Killing vectors discussed earlier, looking at the ambient space metric metric given by equation 16 due to which the line element for de Sitter space is given by $ds^2 =$

$-d\varepsilon_0^2 + d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2 + d\varepsilon_4^2$, the infinitesimal de Sitter group generators are:

$$J_{AB} = \varepsilon_A \frac{\partial}{\partial \varepsilon^B} - \varepsilon_B \frac{\partial}{\partial \varepsilon^A} = \eta_{AC}\varepsilon^C \frac{\partial}{\partial \varepsilon^B} - \eta_{BC}\varepsilon^C \frac{\partial}{\partial \varepsilon^A} \tag{100}$$

This is a Lie algebra as defined in definition 16:

$$[J_{AB}, J_{CD}] = \eta_{BC}J_{AD} + \eta_{AD}J_{BC} - \eta_{BD}J_{AC} - \eta_{AC}J_{BD} \tag{101}$$

These generators have been derived in terms of ε rather than x, which would be more useful. De Sitter space is a 3+1 dimensional hypersurface embedded in a 4+1 dimensional ambient space. Using x is better when dealing with the surface itself. x and ε have already been written in terms of one another, this can be easily resolved. We will be considering the zeroth, first, second, third and fourth dimensions in this scenario. The zeroth dimension is the time dimension. The first, second and third dimensions are spatial dimensions. The fourth dimension is also a spatial dimension, specifically, the spatial dimension needed to consider the radius of curvature ρ. For an individual on the hypersurface, for a sufficiently large value of ρ, the fourth dimension might seem invisible.

Equation 100 gives the de Sitter Generators in terms of ε rather than the stereographic x coordinates. However, equations 21 and 22 can be used to relate the two.

There are two kinds of ε value which can be plugged into the generator equation. ϵ^a for $a = 0, 1, 2, 3$ and ϵ^4. Therefore there will be two generator results found, J_{ab} (L generators) and J_{a4} (Π generators)for a,b=0,1,2,3 (Aldrovandi and Pereira (1998)). First let's ignore the fourth dimension for now and focus on the generator for when a,b=0,1,2,3. From equations 100 and 21 it can be reasoned that:

$$J_{ab} = \delta_a{}^{\mu}\delta_b{}^{\nu}\left(i\left(\eta'_{\rho\mu}x^{\rho}\frac{\partial}{\partial x^{\nu}} - \eta'_{\rho_\nu}x^{\rho}\frac{\partial}{\partial x^{\mu}}\right)\right) \tag{102}$$

For brevity's sake let's rewrite things a little:

$$L_{\mu\nu} = i\left(\eta'_{\rho\mu}x^{\rho}\frac{\partial}{\partial x^{\nu}} - \eta'_{\rho_\nu}x^{\rho}\frac{\partial}{\partial x^{\mu}}\right) \tag{103}$$

And so the generator is:

$$J_{ab} = \delta_a{}^{\mu}\delta_b{}^{\nu}L_{\mu\nu} \tag{104}$$

Now on to considering the fourth dimension using equations 100 and 22:

$$J_{a4} = \rho_{dS}\delta_a{}^{\mu}\left(i\eta_{44}\left(\frac{\partial}{\partial x^{\mu}} + \frac{1}{4\rho_{dS}^2}\eta_{44}(2\eta'_{\mu\lambda}x^{\lambda}x^{\rho} - \eta'_{\mu\nu}x^{\mu}x^{\nu}\delta_{\mu}{}^{\rho})\frac{\partial}{\partial x^{\rho}}\right)\right) \tag{105}$$

Again, let's rewrite things for brevity. Let:

$$\Pi_{\mu} = i\eta_{44}\left(\frac{\partial}{\partial x^{\mu}} + \frac{1}{4\rho_{dS}^2}\eta_{44}(2\eta'_{\mu\lambda}x^{\lambda}x^{\rho} - \eta'_{\mu\nu}x^{\mu}x^{\nu}\delta_{\mu}{}^{\rho})\frac{\partial}{\partial x^{\rho}}\right) \tag{106}$$

Π is especially interesting. There are four Π generators for the de Sitter group, just as there are four translation generators, P, for the Poincaré group. However, since equations 88 and 89 show that $P_\mu = \frac{\partial}{\partial x^\mu}$ for $a = 0, 1, 2, 3$, as $\rho_{dS} \longrightarrow \infty$, $\Pi_\mu \longrightarrow \eta_{44} \frac{\partial}{\partial x^\mu}$ and therefore as $\rho_{dS} \longrightarrow \infty$, $\Pi_\mu \longrightarrow \eta_{44} P_\mu$. This will come in useful when discussing Group contraction later in this thesis (in fact, this is central to the subject of Group contraction).
Putting equation 106 into equation 105 gives:

$$J_{a4} = \rho_{dS} \delta_a{}^\mu \Pi_\mu \tag{107}$$

And so just as one can focus on the L of J_{ab} for the sake of writing the Lie algebra, so too can one focus on the Π of J_{a4} for the sake of writing the Lie algebra (Aldrovandi and Pereira (1998)).

What does this mean conceptually? There are ten infinitesimal generators for the de Sitter group. All of these generators are for transformations taking place in a 2 dimensional plane of the five dimensional de Sitter space. Four of the generators are for transformations taking place in a plane one of the two dimensions of which are the fourth dimension, which becomes less noticeable on a local scale as the de Sitter radius increases. For each of these four generators, the other dimension of the plane in which it takes place is given by the generator. So they are Π_0, Π_1, Π_2 and Π_3. These are the de Sitter boosts.
There are six more generators each of which is for a transformation taking place in a 2 dimensional plane neither of the dimensions of which is the fourth dimension. All of these six combinations are given by L_{01}, L_{02}, L_{03}, L_{12}, L_{13}, L_{23}. These are the de Sitter rotations.

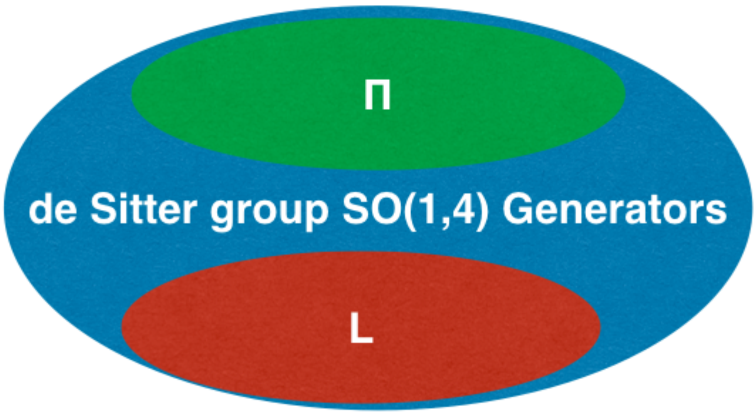

Figure 22: The de Sitter group has exactly ten generators, of which six are L generators (occurring in a two dimensional plane of which both dimensions are the 0, 1, 2 or 3 dimension) and four are Π generators (these occur in a two dimensional plane of which one of the dimensions is the 0, 1, 2 or 3 dimension and the other dimension is the 4 dimension).

5.2.2.2 The de Sitter algebra

So with the killing form generators of the de Sitter group found, one can use equations 103 and 106 to find that the de Sitter algebra in Killing form is:

$$[L_{\mu\nu}, L_{\lambda\rho}] = i(\eta'_{\nu\lambda} L_{\mu\rho} + \eta'_{\mu\rho} L_{\nu\lambda} - \eta'_{\nu\rho} L_{\mu\lambda} - \eta'_{\mu\lambda} L_{\nu\rho}) \tag{108}$$

$$[\Pi_\mu, L_{\lambda\rho}] = i(\eta'_{\mu\lambda}\Pi_\rho - \eta'_{\mu\rho}\Pi_\lambda) \tag{109}$$

$$[\Pi_\mu, \Pi_\lambda] = -i\eta_{44}\rho^{-2}L_{\mu\lambda} \tag{110}$$

5.2.2.3 The de Sitter Group's Elements

Knowing the de Sitter Generators, any de Sitter element, σ_{dS} [8], can be written using $\exp$ as:

$$\sigma_{dS} = \exp(-i\alpha_\mu\Pi^\mu)\exp(-i\vec{\chi}\cdot\vec{K})\exp(-i\vec{\theta}\cdot\vec{J}) \tag{111}$$

With α^μ being some four-vector, while $\vec{\theta}$ and $\vec{\chi}$ are the angle and hyperbolic 'angle' discussed when handling the Lorentz group, while $\vec{J}$ and $\vec{K}$ are the same as they were in equation 65 since those rotations and boosts are the same (although $\exp(-i\alpha_\mu\Pi^\mu)$ is *not* the same as the $\exp(-ia_\mu P^\mu)$ in equation 99. Are there any circumstances in which it is the case that $\Pi^\mu \longrightarrow P^\mu$ such that the de Sitter and Poincaré Group elements approach becoming the same, $\sigma_{dS} \longrightarrow \pi$? Yes, but to explain how will require delving into İnönü Wigner Contraction, which will soon be discussed.

5.2.3 Comparing the de Sitter and Poincaré Algebras

Although they use different symbols, equations 108 and 96 are the same equation. Similarly, equations 109 and 97 are the same, albeit with different symbols. So in other words:

Comparison of Poincaré and de Sitter Generators (if one ignores the difference between equations 98 and 110	
Poincaré generator	de Sitter generator
P	Π
λ	L

However, this table *explicitly ignores* the difference between equations 98 and 110.

The commutations of the six de Sitter rotations with other de Sitter rotations (L, given by equation 108) are the same as the commutation relations of the six Lorentz transformations in the Lorentz subgroup of the Poincaré group with other Lorentz transformations (λ, given by equation 96), and the commutation relations of the six de Sitter rotations with the four de Sitter boosts with the six de Sitter rotations (L and Π, given by equation 109) are the same as the commutation relations of the six Lorentz transformations with the four Poincaré spacetime translations (λ and P, given by equation 109). However the commutation relations of the de Sitter boosts with other de Sitter boosts (Π, given by equation 110) are not the same as the commutation relations of the Poincaré group's four spacetime translations (P, given by equation 98), since it is not normally the case that $-i\eta_{44}\rho^{-2}L_{\mu\lambda} = 0$. But are there cases where this *is* the case, such that the Poincaré and de Sitter Lie algebras have all the same commutation relations? Yes. It is time to discuss İnönü Wigner Contraction.

[8] The general convention throughout this thesis has been to write elements of a group using the Greek letter equivalent of the first letter of the name of the person after whom the Group is named. Λ, the Greek capital 'l', for Lorenz in accordance with convention in Physics, while π, the lower case Greek 'p', was chosen for Poincaré to avoid confusion with the capital Π generators in the de Sitter group. For the elements of the de Sitter group, σ, the Greek lower case 's', for 'Sitter' is chosen rather than δ, the Greek 'd', for 'de'.

6 İnönü Wigner Contraction

6.1 An Introduction to Group Contraction

Anaxagoras...declared...over the earth, which is flat, the sea sinks down after the moisture has been evaporated by the sun.

Lives of Eminent Philosophers (Diogenes Laërtius (500-428 BCE))

...the planet the little prince came from was scarcely any larger than a house!

The Little Prince (de Saint-Exupéry (1943))

This section goes very slowly through this subject, because the writer of this thesis was initially unfamiliar with the subject, and so spent a lot of time practicing with it. Those more familiar with Lie Groups and especially with İnönü Wigner Contraction can skip subsections 6.1.1, which is mainly a non-mathematical, conceptual account, and 6.1.2, which gives an example of İnönü Wigner contraction in order to explain the concept better.

6.1.1 Conceptual Approach (the more Mathematically-Minded can skip this Part)

Figure 23: Picture from de Saint-Exupéry (1943). The Prince's Planet looks curved relative to the Prince because the radius of curvature is much smaller, relative to the Prince, than the radius of curvature of the Earth

The first of the above epigraphs, quoting Diogenes Laërtius, describes how the pre-Socratic Greek philosopher, Anaxagoras believed the Earth to be flat, while the second, by Antoine de Saint-Exupéry, describes the eponymous Little Prince's life upon a fictional house-sized planet. Unlike Anaxagoras, the Little Prince is aware that he lives upon a spherical object. How could a truth so intuitively obvious to the Little Prince have been so unclear to the philosopher? There is no reason to think that Anaxagoras was significantly less intelligent than any modern person,or than the fictional prince (he was considered by Diogenes Laertius to be an 'eminent' philosopher, after all). So how could he have gotten something so wrong? What is different about Anaxagoras' and the Prince's respective planets?

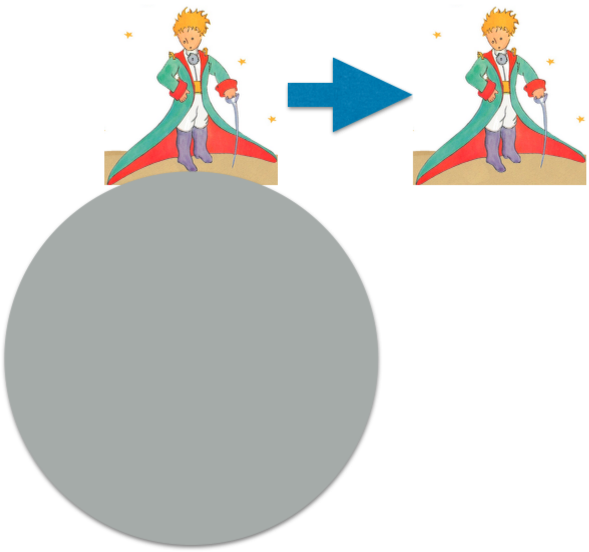

Figure 24: Spatial Translation on the Little Prince's planet (illustration of the Prince originally from de Saint-Exupéry (1943))

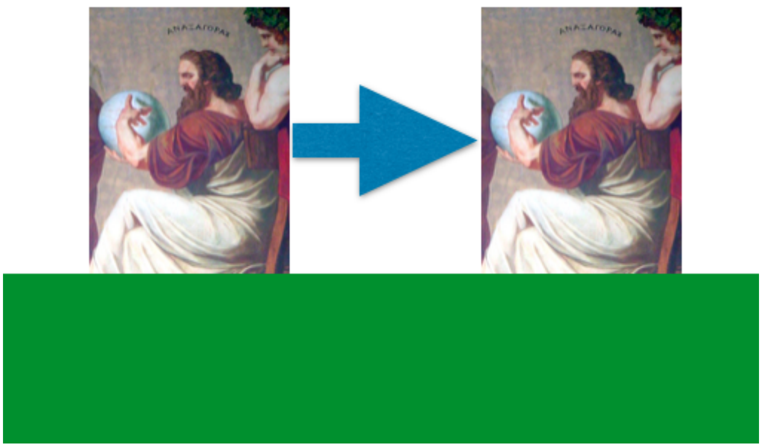

Figure 25: Spatial Translation on Anaxagoras' planet (illustration of Anaxagoras originally from Raphael (1511))

Though both people are of similar size, Anaxagoras' home planet has a far greater radius than the Prince's. The Prince's home planet is not translationally symmetric. Going in a straight line in the x or y direction on the surface of his planet will result in moving off of the planet's surface, as can be seen in figure 24. Therefore it is unsurprising that the Prince realises that his home is not flat. In contrast, as can be seen in figure 25, Anaxagoras' home planet, Earth, with its far greater radius, is translationally symmetric (for translations on Anaxagoras' scale) because increasing the radius of a sphere can make it locally resemble a flat surface. What does this all imply?

This discussion is a little conceptual. Let's delve into a more rigorous mathematical way of considering these ideas.

6.1.2 An Example: so(3) and e(2)

6.1.2.1 The so(3) Lie Algebra

The SO(3) group is the group of 3D rotations and has already been discussed. When considering it, one is not ever thinking of boosts, as were considered in the previous section on so(1,n-1) Lie Algebras. Here the focus is entirely on rotations. Intuitively, in a 3D space, there are 3 orthogonal axes about which rotation may occur. Any rotation can be constructed from a combination of rotations about these 3 orthogonal axes. So from this intuitive argument there will obviously be three generators for the group. The so(3) Lie algebra has already been derived and is given by equation 38.

6.1.2.2 The e(2) Lie algebra

Depending on the textbook used, the Euclidean group is called either ISO(n) or E(n) and consists of either all the rotations and translations of n-dimensional Euclidean space, or all the rotations, translations and reflections of n-dimensional Euclidean space. For clarity, ISO(n) will be used to refer to the latter, while E(n) will be used to refer to the former. So E(2) is the group of all rotations and translations in 2 dimensional Euclidean space. This group is not as important to this project, so will be considered in slightly more 'broad strokes' terms than some of the others, since its main purpose is to provide an example of İnönü Wigner contraction.
What are the rotations and translations which are possible in 2 dimensional Euclidean space? Obviously there are two orthogonal axes in the direction of which translation can occur, such that by combining these two translations, translations in any direction can be made. Therefore, there are two translation generators for this Lie Group, which will be called E_1 and E_2. Furthermore, in a 2D Euclidean space, there is one kind of rotation (a 2D circle drawn in Euclidean space has rotational symmetry, for example). Therefore, there is also one rotation generator, which will be called E_3.
Emphasising Killing vectors more, the metric for such a space is $diag(1, 1)$ so that the line element is given by $ds^2 = dx_1^2 + dx_2^2$. Since neither x_1 nor x_2 appear in the metric, there are obviously two Killing vectors, $E_1 = i\partial_1$ and $E_2 = i\partial_2$, corresponding to translation, and furthermore, there is a third Killing vector given by $E_3 = i(x_1\partial_2 + x_2\partial_1)$, corresponding to rotation.
What kind of Lie algebra does the E(2) group have? Well what does or doesn't commute in the e(2) algebra? Thankfully, Euclidean space is quite an intuitive space to consider, especially in just 2 dimensions. Translations in the x and y direction commute since a translation in the x direction then y direction has the y translation occurs before the x translation. Therefore $[E_1, E_2] = 0$. However, rotation and translation generators do not commute. Instead $[E_2, E_3] = iE_1$ and $[E_3, E_1] = iE_2$. So the e(2) algebra is given by:

$$[E_1, E_2] = 0 \quad [E_2, E_3] = iE_1 \quad [E_3, E_1] = iE_2 \tag{112}$$

This looks a little bit similar to the so(3) Lie algebra given by equation 38, except for the fact that the first commutator equals 0.

6.1.2.3 The Contraction

so(3) can be rewritten as:

$$\Lambda_1 = \epsilon L_1 \quad \Lambda_2 = \epsilon L_2 \quad \Lambda_3 = L_3 \tag{113}$$

This change of variables is allowed, and can be very enlightening. In terms of these λ values, the so(3) algebra given by equation 38 is:

$$[\Lambda_1, \Lambda_2] = i\epsilon^2 \Lambda_3 \quad [\Lambda_2, \Lambda_3] = i\Lambda_1 \quad [\Lambda_3, \Lambda_1] = i\Lambda_2 \tag{114}$$

Again, this is all seemingly perfectly fine. Nothing seems any different, except for being written in terms of slightly different variables. However, when one takes the limit, $\epsilon \longrightarrow 0$, something interesting happens.

$$[\Lambda_1, \Lambda_2] \longrightarrow 0, \quad [\Lambda_2, \Lambda_3] \longrightarrow i\Lambda_1, \quad [\Lambda_3, \Lambda_1] \longrightarrow i\Lambda_2 \tag{115}$$

Which looks the same as the e(2) algebra. So what has happened here? This is called İnönü Wigner contraction, and is an example where a group can be contracted to another group at specific limits (Wigner and İnönü (1953)).

6.2 İnönü Wigner Contraction: Contracting the de Sitter Group to the Poincaré Group

Time to apply İnönu Wigner Contraction to the de Sitter Group. As has already been established by the discussion of equation 18, de Sitter spacetime is curved, and so has a radius of curvature, ρ. As with the above examples of contraction, it can now be possible to consider how, depending on the de Sitter radius, the de Sitter Group can be contracted into the Poincaré group.

6.2.1 Increasing the de Sitter radius

As $\rho_{dS} \longrightarrow \infty$ the following limits are approached by the de Sitter algebra's infinitesimal generators, which were previously shown in equations 103 and 106:

$$L_{\mu\nu} \longrightarrow L_{\mu\nu} = \lambda_{\mu\nu} \tag{116}$$

In general $L_{\mu\nu} = \lambda_{\mu\nu}$ and so at this limit, this remains the case. Π is more noticeably affected. In equation 106 it is the sum of two terms, given by $g_{44}\frac{\partial}{\partial x^\mu}$ (note that ρ_{dS} does not appear in this term) and $\frac{1}{4\rho_{dS}^2}\eta_{44}\eta_{44}(2\eta'_{\mu\lambda}x^\lambda x^\rho - \eta'_{\mu\nu}x^\mu x^\nu \delta^\rho_\mu)\frac{\partial}{\partial x^\rho}$ (note that ρ_{dS}^2 appears as a denominator in this term). As $\rho_{dS} \longrightarrow \infty$ the latter term $\longrightarrow 0$ such that:

$$\Pi_\mu \longrightarrow i\eta_{44}\frac{\partial}{\partial x^\mu} = \eta_{44}P_\mu \tag{117}$$

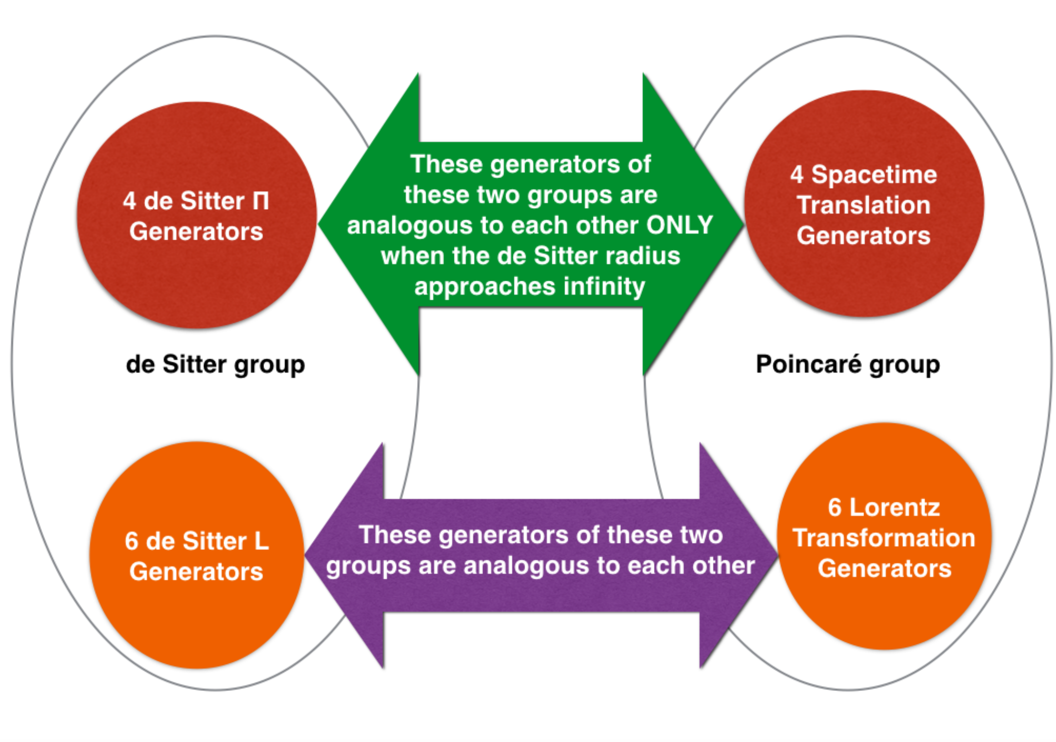

Figure 26: Comparison of the de Sitter and Poincaré groups

Therefore as $\rho_{dS} \longrightarrow \infty$

$$[L_{\mu\nu}, L_{\lambda\rho}] \longrightarrow [L_{\mu\nu}, L_{\lambda\rho}] \tag{118}$$

$$[\Pi_\mu, L_{\lambda\rho}] \longrightarrow g_{44}(ig'_{\mu\lambda}\frac{\partial}{\partial x^\rho} - ig'_{\mu\rho}\frac{\partial}{\partial x^\lambda}) = [g_{44}P_\mu, L_{\lambda\rho}] \tag{119}$$

$$[\Pi_\mu, \Pi_\lambda] \longrightarrow 0 = [P_\mu, P_\lambda] \tag{120}$$

Therefore, as $\rho_{dS} \longrightarrow \infty$, the de Sitter algebra and the Poincaré algebra (given by equations 96, 97 and 98) become the same, with $L_{\mu\nu}$ being the de Sitter group's equivalent of the Poincaré group's $M_{\mu\nu}$ and Π_μ being the de Sitter group's equivalent of P_μ.

In other words, of the ten de Sitter generators, the four de Sitter boosts behave like Spacetime translations as the de Sitter radius increases. Meanwhile, the de Sitter rotations' commutation relations with each other are the same as Lorentz transformations' commutation relations with each other, explaining why the six de Sitter rotation generators become the six generators of the Lorentz subgroup of the Poincaré group.

Due to Noether's Theorem, continuous isometries have corresponding conserved currents, and therefore, having shown that the de Sitter group can be contracted to the Poincaré group by İnönü Wigner contraction as the de Sitter radius approaches infinity, there are implications for the conservation laws of de Sitter space.

7 The Conservation Laws of Minkowski and de Sitter Space

> It is important to point out that the mathematical formulation of the physicist's often crude experience leads in an uncanny number of cases to an amazingly accurate description of a large class of phenomena.
> *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*
> Wigner (1960)

7.1 A field theoretic approach to Noether's Theorem

7.1.1 Noether's Theorem for Scalar Fields

If premises 1 and 2 below are both true then the conclusion is the case.

Premise 1: There is a Lagrangian density, $\mathcal{L}$, of a scalar field [9], $\phi_a(x)$, which is transformed by an infinitesimal perturbation:

$$\phi_a(x,t) \longrightarrow \phi_a(x,t) + \delta_a \phi_a(x,t) \tag{121}$$

Premise 2: $\mathcal{L}$ changes under the transformation described by equation 121 by some total derivative for a set of functions given by $\zeta^\mu(\phi)$ such that it is the case that $\delta\mathcal{L}$ is given by:

$$\delta\mathcal{L} = \partial_\mu \zeta^\mu(\phi) \tag{122}$$

Conclusion: It is the case that there exists a conserved current, j^μ, which is to say, a current that can be described by the equation:

$$\partial_\mu j^\mu = 0 \tag{123}$$

7.1.2 A Proof of Noether's Theorem for Scalar Fields

The Euler Lagrange equation when considering some scalar field, $\phi_a(x)$, is:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0 \tag{124}$$

The amount by which the field is changed in equation 121 can be called Δ.

$$\delta_a \phi_a(x,t) = \Delta_a(\phi) \tag{125}$$

Using some calculus:

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\mu(\delta\phi_a) \tag{126}$$

[9] It is important to note that throughout this section, scalar fields are used for simplicity's sake. A mathematically adventurous reader is welcome to work everything out for vector fields and more.

We can add $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\delta\phi_a - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\delta\phi_a = 0$ to equation 126 since one can always add 0:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu(\delta\phi_a) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\delta\phi_a - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\delta\phi_a \tag{127}$$

And because of the product rule, one can combine the second and third terms $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu(\delta\phi_a)$ and $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\delta\phi_a$, into a single term, $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right)$, meaning that equation 127 can be rearranged as:

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right) - \left(\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right) - \frac{\partial\mathcal{L}}{\partial\phi_a}\right)\delta\phi_a \tag{128}$$

And putting equation 124 into the second term of equation 128 gives:

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right) \tag{129}$$

Putting Δ and ζ from equation 125 and equation 122 into equation 129 means that one can use definition 20 to rewrite equation 129 as equation 123:

Definition 20 (Noether Current) $j^\mu = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\Delta - \zeta^\mu$

Therefore there is a Noether current, the flow of a conserved quantity (Tong (2006), Kleinert (2016)). The Noether charge, the conserved quantity Q which is flowing, can be found for a given volume by integrating over that volume:

$$Q_{\text{volume}} = \int_{\text{volume}} j^0 \, \mathrm{d}^3x \tag{130}$$

So if one seeks to find the conserved charges for some space, one must find Δ and ζ into equation 123 and decide what the volume is that is being integrated.

7.2 The Conservation Laws of Minkowski Space

7.2.1 Spacetime Translations

For spacetime translations:

$$x^\nu \longrightarrow x^\nu - \epsilon^\nu \tag{131}$$

Where ϵ is some infinitesimal translation and for Minkowski space, $\nu = 0, 1, 2, 3$. Therefore:

$$\phi \longrightarrow \phi + \epsilon^\nu\partial_\nu\phi \tag{132}$$

And this in turn means that:

$$\mathcal{L}(x) \longrightarrow \mathcal{L}(x) + \epsilon^\nu\partial_\nu\mathcal{L}(x) \tag{133}$$

Equation 133 can be rewritten as:

$$\mathcal{L}(x) \longrightarrow \mathcal{L}(x) + \partial_\nu(\mathcal{L}(x)\epsilon^\nu) \tag{134}$$

Equation 132 can be compared with equations 121 and 125 to see that for spacetime translations:

$$\Delta_{\text{spacetime translations}} = \epsilon^\nu \partial_\nu \phi \tag{135}$$

Which can be put into definition 20 to give Noether Current:

$$(j^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \epsilon^\nu \partial_\nu \phi - \zeta^\mu \tag{136}$$

What is ζ^μ for this current? Equation 134 can be compared with definition 122 to give this. There are four translations ϵ^ν for $\nu = 0, 1, 2, 3$. $\mathcal{L}$ is changed if $\nu = \mu$ but if $\nu \neq \mu$ then $\delta\mathcal{L}$ is not changed. A Kronecker delta can be used to convey this:

$$\zeta^\mu = \delta^\mu{}_\nu \mathcal{L}(x)\epsilon^\nu \tag{137}$$

So putting equation 137 into equation 136 gives:

$$(j^\mu)_\nu = \epsilon^\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right) \tag{138}$$

And so this can be used to define the Energy Momentum Tensor:

Definition 21 (The Energy Momentum Tensor) $(j^\mu)_\nu = T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}$

$T^\mu{}_\nu$ gives the Noether currents when $\mathcal{L}$ changes under translations by some total derivative (as was described by equation 122).

T^{00} gives energy density of a volume in space (the flux of $0-$Noether Current across a surface of constant x^0). T^{11}, T^{22} and T^{33} give pressure in the x, y and z directions (the flux of $i-$Noether Current across a surface of constant x^i.

7.2.1.1 The Energy and Momentum Noether Charges

Finding the Noether charges corresponding to the Noether Current given by definition 21 is done by integration, as is shown by equation 130, of $T^0\nu$, as given by definition 21. This means that some conserved Noether Charge P^ν is given by:

$$P^\nu = \int_{\text{volume}} T^{0\nu} \, \mathrm{d}^3 x \tag{139}$$

So for Minkowski space, these four Noether currents for $\nu = 0, 1, 2, 3$ give four Noether charges, which are P^0, the time-momentum, which is more commonly called energy, and also P^1, P^2 and P^3, the linear momentum in the x, y and z directions, for the given region. For a space for which

d^3x doesn't change over time, such as Minkowski space, this integration can be done globally, giving:

$$P^{\nu} = \int_{\text{space}} T^{0\nu} \, \mathrm{d}^3x \tag{140}$$

And therefore there is global energy and linear momentum conservation for Minkowski space.

7.2.2 Lorentz Transformations

Writing out Lorentz transformations as square matrices has already been done. Let $\Lambda^{\mu}{}_{\nu}$ be a Lorentz transformation or a de Sitter transformation. First Δ for equation 125 must be found.

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \tag{141}$$

Where δ^{μ}_{ν} is the kronecker delta matrix (the matrix equivalent of a kronecker delta, which is either an $n \times n$ 0 matrix if $\mu \neq \nu$ or an $n \times n$ identity matrix if $\mu = \nu$) which is symmetric (since a 0 matrix or an identity matrix is symmetric) such that $\delta^{\mu}_{\nu} = \delta^{\nu}_{\mu}$ and ω^{μ}_{ν} is antisymmetric such that $\omega^{\mu}_{\nu} = -\omega^{\nu}_{\mu}$, which means that:

$$(\Lambda^{\mu}{}_{\nu})^{-1} = \delta^{\mu}{}_{\nu} - \omega^{\mu}{}_{\nu} \tag{142}$$

And

$$\delta_s \phi = \phi'(x) - \phi(x) \tag{143}$$

Where in equation 143, $\phi'(x)$ is given by:

$$\phi'(x) = \phi(\Lambda^{-1} x) \tag{144}$$

Since ω is antisymmetric and δ is symmetric, putting equation 142 into equation 144 gives:

$$\phi'(x) = \phi(x^{\mu} - \omega^{\mu}{}_{\nu} x^{\nu}) \tag{145}$$

and equation 145 can be rewritten as:

$$\phi'(x) = \phi(x^{\mu}) - \omega^{\mu}{}_{\nu} x^{\nu} \partial_{\mu} \phi(x) \tag{146}$$

Putting equation 146 back into equation 143 gives:

$$\delta_s \phi = \cancel{\phi(x^{\mu})} - \omega^{\mu}{}_{\nu} x^{\nu} \partial_{\mu} - \cancel{\phi(x^{\mu})} \tag{147}$$

Therefore:

$$\delta_s \phi = -\omega^{\mu}{}_{\nu} x^{\nu} \partial_{\mu} \phi \tag{148}$$

And by similar reasoning:

$$\delta\mathcal{L} = -\omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\mathcal{L} \tag{149}$$

Putting 148 into equation 125 gives:

$$\Delta = -\omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\phi \tag{150}$$

And putting equation 149 into definition 122 gives:

$$\partial_{\mu}\zeta^{\mu} = -\omega^{\alpha}{}_{\nu}\delta^{\mu}{}_{\alpha}\mathcal{L} \tag{151}$$

(although when writing out the full Noether current, one should include a Kronecker delta in ζ for analogous reasons to those used when dealing with ζ for translations)

Equations 150 and 151 can be put into definition 20 to give:

$$j^{\mu} = -\omega^{\alpha}{}_{\nu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\phi - \delta^{\mu}{}_{\alpha}\mathcal{L}\right)x^{\nu} \tag{152}$$

And in fact, definition 21 can be put into equation 152 to give:

$$j^{\mu} = -\omega^{\alpha}{}_{\nu}T^{\mu}{}_{\alpha}x^{\nu} \tag{153}$$

Can this be written without needing to write in terms of ω? Yes.

$$-\partial_{\mu}\omega^{\rho}{}_{\nu}T^{\mu}{}_{\rho}x^{\nu} = 0 = -\partial_{\mu}\omega^{\nu}{}_{\rho}T^{\mu}{}_{\nu}x^{\rho} \tag{154}$$

Adding them together and taking advantage of the antisymmetry of ω:

$$\partial_{\mu}(\omega^{\rho}{}_{\nu}(T^{\mu}{}_{\nu}x^{\rho} - T^{\mu}{}_{\rho}x^{\nu})) = 0 \tag{155}$$

From this one can also reason that

$$\partial_{\mu}(T^{\mu}{}_{\nu}x^{\rho} - T^{\mu}{}_{\rho}x^{\nu}) = 0 \tag{156}$$

and therefore there is a way of expressing the six (in the case of Minkowski space) or ten (in the case of de Sitter space) relevant Noether currents without ω:

$$(j^{\mu})^{\alpha\beta} = x^{\alpha}T^{\mu\beta} - x^{\beta}T^{\mu\alpha} \tag{157}$$

For Minkowski space, $\rho, \nu = 0, 1, 2, 3, \rho \neq \nu$, this gives six Noether currents for the combinations 01, 02, 03, 12, 13, 23.

It is worth noting here that de the de Sitter generators are *also* boost and rotation generators. Therefore, the same reasoning can be used for de Sitter transformations, except that for de Sitter space $\rho, \nu = 0, 1, 2, 3, 4, \rho \neq \nu$, which means that there are ten de Sitter Noether currents for the combinations 01, 02, 03, 04, 12, 13, 14, 23, 24, 34.

7.2.2.1 Lorentz Noether Charges

There are three boost generators and three rotation generators for the isometry group of Minkowski space (more broadly there are six Lorentz generators, which includes the boost and rotation generators both). What is the corresponding Noether charge?

Putting equation 157 into equation 130 gives some conserved charge L:

$$L^{\alpha\beta} = \int_{\text{space}} \left(x^{\alpha} T^{0\beta} - x^{\beta} T^{0\alpha} \right) \, \mathrm{d}^3 x \tag{158}$$

Where $\alpha, \beta = 0, 1, 2, 3, \alpha \neq \beta$. Let's start with the Noether charges associated with rotation generators, L^{12}, L^{13} and L^{23}. These three conserved quantities are angular momentum. What about the Noether charges L^{01}, L^{02} and L^{03}? What is the conserved quantity associated with a 'rotation' in a two dimensional plane one of the two dimensions of which is time rather than space? This in fact corresponds to conservation of centre of energy of a field. It is effectively the relativistic equivalent of Newton's First Law. Therefore, the symmetries of Minkowski space overall give rise to conservation of energy, linear momentum (in fact energy and linear momentum together form four-momentum), angular momentum and to an equivalent of Newton's First Law in the form of conservation of centre of energy of a field. This is because, since $L^{0\alpha}$ is conserved, it is the case that:

$$0 = \frac{dL^{0i}}{dt} = \frac{d}{dt} (\int \left(x^0 T^{0\alpha} - x^{\alpha} T^{00} \right) \, \mathrm{d}^3 x) \tag{159}$$

Which can be rearranged to:

$$0 = \int T^{0\alpha} \, \mathrm{d}^3 x + x^0 \int \frac{dT^{0\alpha}}{dt} \, \mathrm{d}^3 x - \frac{d}{dt} \int x^{\alpha} T^{00} \, \mathrm{d}^3 x \tag{160}$$

And since $\int T^{0\alpha} \, \mathrm{d}^3 x = P^{\alpha}$, and P^{α} is conserved in Minkowski space, this becomes:

$$P^{\alpha} = \frac{d}{dt} \int x^{\alpha} T^{00} \, \mathrm{d}^3 x \tag{161}$$

And therefore, since P^{α} is conserved for Minkowski space, so too is $\frac{d}{dt} \int x^{\alpha} T^{00} \, \mathrm{d}^3 x$. Therefore the centre of energy of a field is conserved for Minkowski space.

7.3 The Conservation Laws of de Sitter Space

As has already been established, using İnönü Wigner Contraction, as $\rho_{dS} \longrightarrow \infty$, the de Sitter Group becomes the Poincaré group, and so as $\rho_{dS} \longrightarrow \infty$, the Π Generators of the de Sitter group become translation Generators. Therefore, as $\rho_{dS} \longrightarrow \infty$, the Noether Currents of de Sitter space become the same Noether Currents as those of Minkowski Space.

What about for the non-contraction limit? What are the Noether Currents of de Sitter space in that case?

7.3.1 Translations in de Sitter Space

The de Sitter group's generators consist of four Π generators and six L generators, but do not include any translation generators of the form $\partial/\partial x$, and therefore, though one can write $\delta\mathcal{L}$ under translations as a total derivative according to equation 122 as $\rho_{dS} \longrightarrow \infty$ using İnönü Wigner contraction, energy is not globally well defined for de Sitter space since there is no Killing vector of de Sitter space (representing some generator of the de Sitter group) which is globally timelike. Furthermore, unlike is the case with static spaces such as Minkowski space, d^3x is not time-invariant for de Sitter space, which has implications for the application of equation 130.

The derivation of the Energy Momentum Tensor as Noether Currents cannot be done for the isometries of the de Sitter group as it can be for those of the Poincaré group since unlike the Poincaré group, the de Sitter group does not have any translation generators. However, since under İnönü Wigner Contraction, as $\rho_{dS} \longrightarrow \infty$, the de Sitter group becomes the Poincaré group, as as $\rho_{dS} \longrightarrow \infty$ the Energy Momentum Tensor *does* become Noether Currents corresponding to the isometries of de Sitter Space.

7.3.2 Rotations and Boosts in de Sitter Space

7.3.2.1 'Lorentz Transformations' in de Sitter space

When $\alpha, \beta \neq 0, \alpha, \beta \neq 4$, the same currents $(j^0)^{12}$, $(j^0)^{13}$, $(j^0)^{23}$, $(j^0)^{01}$, $(j^0)^{02}$ and $(j^0)^{03}$ given by equation 157 occur in Minkowski and de Sitter space.

7.3.2.2 $(j^0)^{04}$, $(j^0)^{14}$, $(j^0)^{24}$, $(j^0)^{34}$

Because de Sitter space has 4+1 dimensions rather than the 3+1 dimensions of Minkowski space, there are four more two-dimensional planes of Minkowski space in which a rotation/boost can occur. Therefore there are four Noether Currents, $(j^0)^{04}$, $(j^0)^{14}$, $(j^0)^{24}$, $(j^0)^{34}$, which can be found using equation 157 which are found in de Sitter space which are not found in Minkowski space. These currents correspond to the Π generators of the de Sitter group.

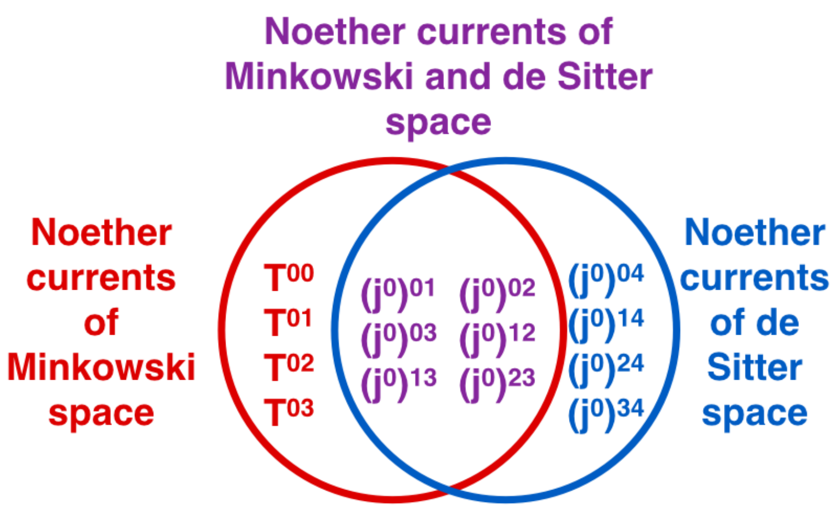

Figure 27: A Venn Diagram of the Noether Currents derived for de Sitter and Minkowski space

7.3.3 d^3x and Noether Charges in de Sitter Space

While the Noether currents of de Sitter space have been discussed, there is a major issue to consider whe discussing the Noether charges. Equations 139 and 158 for conserved charges deal with d^3x. Minkowski space is a static space, so such an integration is time independent. De Sitter space is not static since a de Sitter Universe expands over time. Therefore, integrating over $\int_{\text{all of space}} \mathrm{d}^3x$ at one time and another later time will not give the same result.

Therefore, while the intuition with regard to the de Sitter Noether Currents is to simply work out equation 162 for $\alpha = 0, 1, 2, 3, 4$, $\beta = 0, 1, 2, 3, 4$, $\alpha \neq \beta$, in fact, $L^{\alpha\beta}$ are not constant with respect to time.

$$L^{\alpha\beta} = \int_{space} \left(x^{\alpha} T^{0\beta} - x^{\beta} T^{0\alpha} \right) \mathrm{d}^3x \quad [10] \tag{162}$$

Since d^3x is not time-independent for de Sitter space, one cannot assume that the Noether charges for de Sitter space are time-independent. Is there some way to treat at least part of de Sitter space as a static universe? Yes. But not the whole of it, and not easily. To explain how, it will be necessary to deal with a static part of de Sitter space using the static patch.

7.3.3.1 De Sitter Space Noether Charges associated with $(j^0)^{01}$, $(j^0)^{02}$ and $(j^0)^{03}$)

Aside from the fact that d^3x is not constant with respect to time in de Sitter space, there is also a further complication when it comes to interpreting the Noether charges associated with the Noether currents $(j^0)^{01}$, $(j^0)^{02}$ and $(j^0)^{03}$). Since conservation of linear momentum cannot be assumed when dealing with de Sitter space, Equation 159 cannot be simplified to equation 161 (As has already been mentioned, equation 161 is a relativistic equivalent of Newton's First Law) for de Sitter space. Therefore, there is yet another difference between the Noether Charges of de Sitter and Minkowski space.

8 Is there any way of salvaging some kind of Conservation of Energy for de Sitter Space?

> ...energy and momentum evolve in a precisely specified way in response to the behavior of spacetime around them. If that spacetime is standing completely still, the total energy is constant; if it's evolving, the energy changes in a completely unambiguous way.
> In the case of dark energy, that evolution is pretty simple: the density of vacuum energy in empty space is absolute constant, even as the volume of a region of space (comoving along with galaxies and other particles) grows as the universe expands. So the total energy, density times volume, goes up.
>
> Carroll (2010)

There are several coordinate systems which can be used for de Sitter space. One achieves energy conservation *within* the static patch horizon, but *not* globally.

[10] Remember when looking at equation 162 because it cannot be over-emphasised, for de Sitter space d^3x is *not* constant with respect to time

8.1 Penrose Diagrams

The discussion here is assisted by the use of Penrose Diagrams. Like spacetime diagrams, Penrose diagrams have a vertical axis representing time and a horizontal axis representing space. However, unlike spacetime diagrams, Penrose diagrams conformally 'squash' space in order to represent an infinite space rather than a finite space.

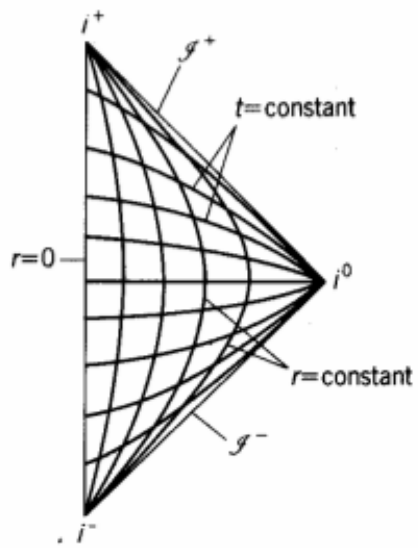

Figure 28: Penrose Diagram for Minkowski Space originally from d'Inverno (1992b). $\mathcal{I}^+$ and $\mathcal{I}^-$ represent future and past lightlike infinities, i^+ and i^- represent future and past timelike infinities, and i^0 represents spacelike infinity.

The Penrose diagram for Minkowski space is a good example of a Penrose diagram. As shown in figure 28, geodesics are curved due to the diagram's conformal treatment of infinity.

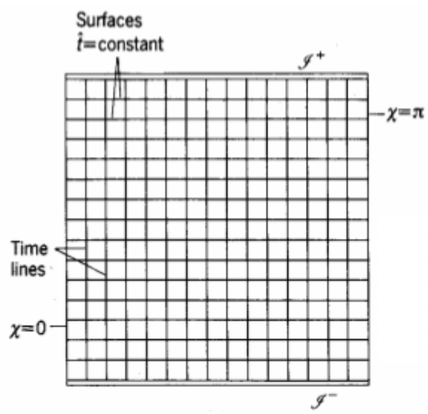

Figure 29: Penrose Diagram for de Sitter Space originally from d'Inverno (1992c). $\mathcal{I}^+$ and $\mathcal{I}^-$ represent future and past lightlike infinities, χ represents angle

Note the counter-intuitive contrast between the grid-like shape of figure 29 and the more curved appearance of figure 28, which is ironic given that de Sitter Space is curved while Minkowski space is flat. This is because Penrose Diagrams allow infinite space to be represented in a finite diagram by 'squashing' space, in order to conformally treat infinity. The degree of 'squashing' increasing as one approaches infinity.

Penrose diagrams will be extremely useful in this section.

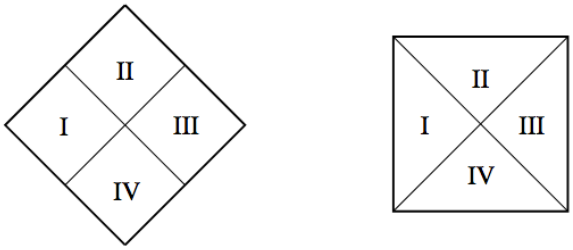

Figure 30: A comparison of Minkowski space (left) and de Sitter space (right) Penrose diagram shapes originally from Susskind, Goheer and Kleban (2003). Time increases in the vertical direction and spatial distance increases in the horizontal direction.

8.2 Timelike Killing Vectors

Due to the focus on energy conservation here, there will be focus on the vector field:

$$K = i\partial_t \tag{163}$$

For metric $g_{\mu\nu}$, if Lie derivative $L_K g_{\mu\nu}$, as given by equation 47 over vector field K, with vector field K given by equation 163, is zero, then K is a Killing vector Field.

$$L_K g_{\mu\nu} = K^\alpha \partial_\alpha g_{\nu\mu} + g_{\nu\alpha}\partial_\mu K^\alpha + g_{\mu\alpha}\partial_\nu K^\alpha \tag{164}$$

8.2.1 Noether's Theorem and Killing Vectors

Definition 22 (Conservation Along a Geodesic) *For a geodesic on some smooth manifold, where u^i is a tangent vector to the geodesic, if $u^i \nabla_i Q = 0$ then Q is conserved along the path of the geodesic*

The Killing Equation given by equation 48 can be rearranged to:

$$\nabla_i K_j = -\nabla_j K_i \tag{165}$$

Let K^i be a Killing vector on spacetime M with metric tensor g_{ij}. Let there be a geodesic on this spacetime with tangent vector u^i. The inner product of u^i and K^i is $u^i K_i$ can be written as $u^j K_j$ and along the path of a geodesic, using the product rule, it is the case that:

$$u^i \nabla_i (u^j K_j) = (u^i \nabla_i u^j) K_j + u^j (u^i \nabla_i K_j) \tag{166}$$

And since u^i is a tangent vector along the geodesic, $\nabla_i u^j = 0$ and therefore the first term of equation 166 is equal to zero, and so the equation becomes:

$$u^i \nabla_i (u^j K_j) = u^j (u^i \nabla_i K_j) \tag{167}$$

And since i and j are dummy variables, they can be exchanged so that:

$$u^j(u^i\nabla_i K_j) = u^i(u^j\nabla_j K_i) \tag{168}$$

However, putting equation 165 into equation 168 results in it becoming:

$$u^j(u^i\nabla_i K_j) = -u^j(u^i\nabla_i K_j) \tag{169}$$

And therefore equation 169 necessitates that

$$u^j(u^i\nabla_i K_j) = 0 \tag{170}$$

And so putting equation 170 into equation 166 gives:

$$u^i\nabla_i(u^j K_j) = 0 \tag{171}$$

And comparing equation 171 to definition 22, it can therefore be concluded that when K^i is a Killing vector for some spacetime, along the geodesic with tangent vector u^i, it is the case that $u^j K_j$ is a conserved current, which is to say, a Noether current (user_35 (2015)).

8.3 De Sitter space and the FRW Metric

The Einstein Field Equations have various exact solutions, including the Friedmann-Robertson-Walker (FRW) metric. The FRW approach assumes an expanding or contracting (either way non-static), isotropic, homogeneous Universe. An example of such a Universe is de Sitter spacetime. An FRW metric has form:

$$ds^2_{FRW} = -dt^2 + \alpha(t)^2\Sigma^2 \tag{172}$$

Where $\alpha(t)^2\Sigma^2$ gives all the spatial terms. Σ gives the spatial slices of the spacetime (d'Inverno (1992c), Hartman (2017)). A slice can be:

Open: If it is a Hyperbola
Flat: If it has one fewer dimensions than the spacetime of which it is a slice.
Closed: If it is a closed shape such as an n-sphere.

$\alpha(t)$ is the reason that a FRW Universe is non-static, expanding or contracting over time. Since $\alpha(t)$ appears in the metric then since there is t in the metric, there cannot be the Killing vector ∂_t due to the fact that in equation 164 $K^\alpha\partial_\alpha g_{\nu\mu} \neq 0$ and, therefore, since $L_K g_{\mu\nu} \neq 0$, K is not a Killing vector when using an FRW metric. Therefore these approaches are not appropriate when it comes to energy conservation. As Physicist Sean Carroll said '...energy and momentum evolve in a precisely specified way in response to the behavior of spacetime around them. If that spacetime is standing completely still, the total energy is constant' (Carroll (2010)). De Sitter Space is not static, so one cannot have global energy conservation for it.

Once one accepts the fact that a global timelike Killing vector for de Sitter space is not possible, one can try to deal with de Sitter space non-globally. As will be shown, for a specific (non-global) region, one can use static patch coordinates, such that for the static patch there is a non-global timelike Killing vector, and therefore one can have energy conservation *within the horizon of the static patch*. However, the writer of this thesis is getting ahead of himself, since the static patch coordinates will not be discussed until after the global slicing, and so it is better to keep the focus on the global slicing for now.

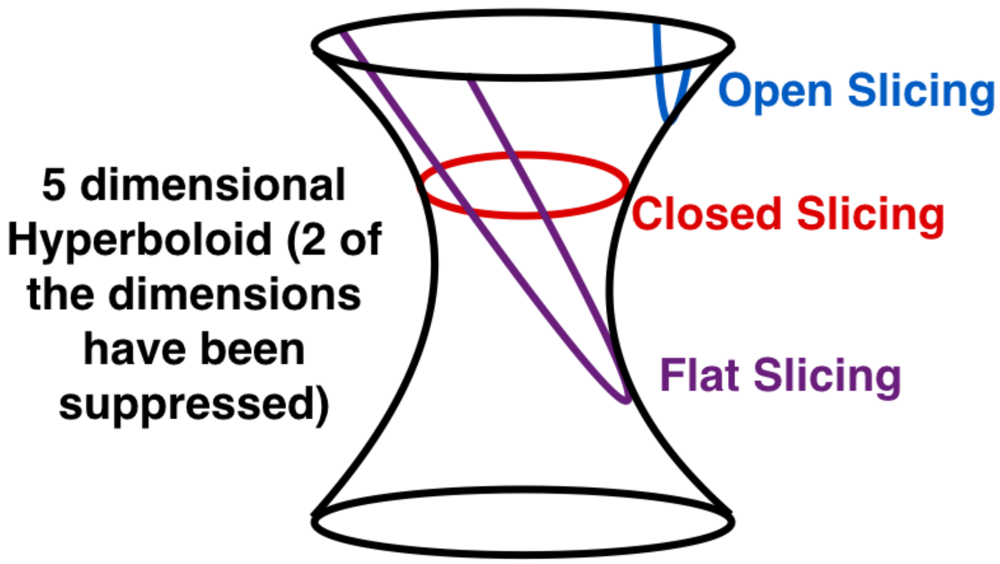

Figure 31: Open, Flat and Closed Slicing

De Sitter space is a curved hypersurface described by equation 18 (as a reminder, $X_{\mu}X^{\mu} = \rho^2$), with radius of curvature ρ_{dS}, which is embedded in 4+1 dimensional flat Minkowski space with coordinates X_0, X_1, X_2, X_3 and X_4 (Hawking and Ellis (1973), Carroll (2014d), d'Inverno (1992c)) (see figure 14 for a visual reminder of the overall concept). As has already been established in equation 16, the ambient Minkowski space in which the hypersurface is embedded has metric:

$$ds^2_{\text{embed}} = -dX_0^2 + \sum_{k=1}^{4} dX_k^2 \tag{173}$$

8.3.1 Closed (Global) Slicing Coordinates

One can use polar coordinates t, r, θ, ϕ and ψ. Using hyperbolic trigonometric functions such as $sinh$ and $cosh$it is possible to rewrite X_0,...X_4 for the hypersurface as:

$$X_0 = \rho \sinh(t/\rho) \tag{174}$$

$$X_1 = \rho \cosh(t/\rho) \cos\theta \tag{175}$$

$$X_2 = \rho \cosh(t/\rho) \sin\theta \cos\phi \tag{176}$$

$$X_3 = \rho \cosh(t/\rho) \sin\theta \sin\phi \cos\psi \tag{177}$$

$$X_4 = \rho \cosh(t/\rho) \sin\theta \sin\phi \sin\psi \tag{178}$$

One can more generally write X_i for $i > 0$ as:

$$X_i = \rho \cosh(t/\rho) z_i \tag{179}$$

Where $z^2 = 1$.

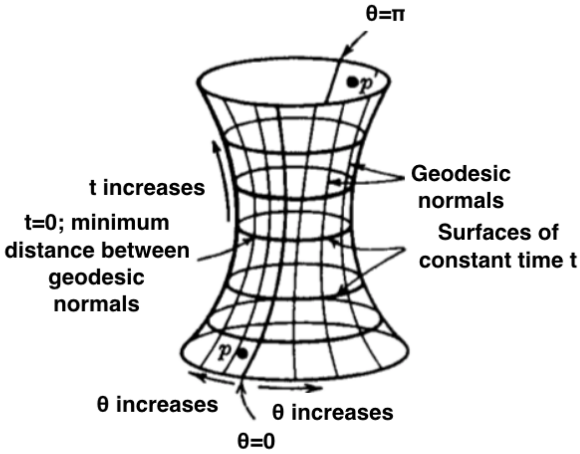

Figure 32: Diagram of the Closed (Global) Slicing of a Hyperboloid originally from Hawking and Ellis (1973). The diagram has been slightly altered to make the choice of notation consistent with the choice of notation in the rest of this thesis.

Putting the global coordinates into equation 18 shows that this describes a hyperboloid (see figure 32 for a diagram of a hyperboloid using global coordinates).

$$\rho^2(-\sinh^2(t/\rho) + \cosh^2(t/\rho)(\cos^2\theta + \sin^2\theta(\cos^2\phi + \sin^2\phi(\cos^2\psi + \sin^2\psi)))) = \rho^2 \tag{180}$$

And so it is possible to use global coordinates to write the de Sitter metric using global coordinates:

$$ds^2 = -dt^2 + \rho^2 \cosh^2(t/\rho)(d\theta^2 + \sin^2\theta(d\phi^2 + \sin^2\phi(d\psi^2))) \tag{181}$$

So the global de Sitter metric has been found. This is a version of equation 172 for de Sitter space globally, with α given by the $\cosh(t/\rho)$ term (Hartman (2017)).

Equation 181 is the metric for the global case, and note that t turns up in the metric in the $2\cosh^2(t/\rho)$.

One can put the metric given by equation 181 into equation 164 and see that since $g_{\mu\nu}$ includes the $\cosh^2(t/\rho)$, it is the case that $\partial_t g_{\nu\mu} \neq 0$ and therefore $L_K g_{\mu\nu} \neq 0$. Therefore when using the global metric, K is not a Killing vector. No Killing vector of de Sitter space is globally timelike. There is no global law of energy conservation for de Sitter space.

With the global coordinates considered and shown to fail at giving energy conservation, the time has come to move onto other coordinate systems, none of which will be global.

8.3.2 Flat Slicing Coordinates

Instead of the previous coordinate system for writing $ds^2 = -dX_0^2 + dX_1^2 + ...dX_n^2$ one can now switch to using:

$$X_0 = \rho \sinh(t/\rho) + \frac{1}{2\rho} r^2 e^{t/\rho} \tag{182}$$

$$X_1 = \rho \cosh(t/\rho) - \frac{1}{2\rho} r^2 e^{t/\rho} \tag{183}$$

$$X_2 =^{t/\rho} r \cos\theta \tag{184}$$

$$X_3 =^{t/\rho} r \sin\theta \tag{185}$$

More generally, for $i > 1$

$$X_i = e^{t/\rho} y_i \tag{186}$$

Where $r^2 = y^2$.

These values satisfy a hyperboloid equation. ds is given by:

$$ds^2 = -dt^2 + e^{2t/\rho} dy^2 \tag{187}$$

These coordinates only cover $X_0 + X_1 > 0$ since $X_0 + X_1 = \rho(\sinh(t/\rho) + \cosh(t/\rho)) = \rho e^{t/\rho}$. Therefore, one cannot use these for global coordinates. A diagram of the 'half-covering' of de Sitter space by flat slicing coordinates is shown in figure 33. One can also choose flat slicing coordinates to instead deal with the other half of the hypersurface instead, but cannot make them global, as they are in the case of the global coordinates (as the name *rather* subtly implies).

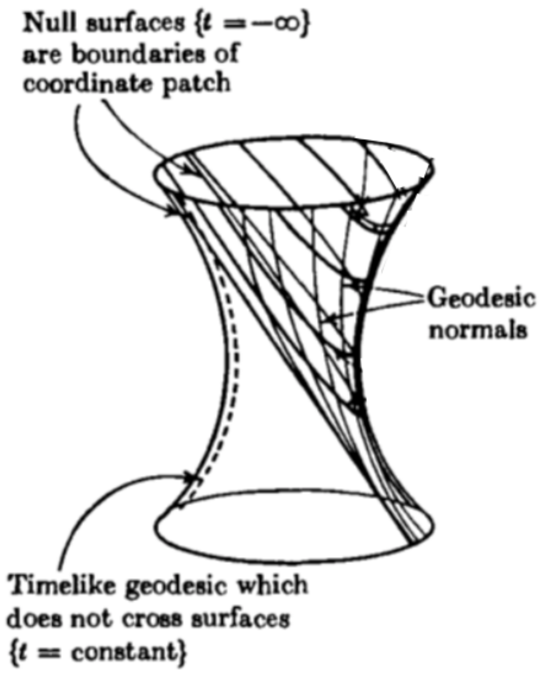

Figure 33: Diagram of the Flat Slicing of a Hyperboloid originally from Hawking and Ellis (1973)

The most important thing about equation 187 for the sake of this thesis is that when putting it into equation 164, $\partial_t g_{\nu\mu} \neq 0$ due to the fact that $e^{2t/\rho}$ appears in the metric, and therefore, since t appears in the metric, as was the case with the global slicing, $L_K g_{\mu\nu} \neq 0$ such that K is not a Killing vector field and therefore there is no energy conservation using the flat slicing.

8.3.3 Open Slicing Coordinates

Alternatively, one can use the open slicing approach, but like the previous two approaches, this one will not yield a metric without t in it.

Let:

$$X_0 = \rho \sinh(t/\rho) \cosh(\Xi) \tag{188}$$

$$X_1 = \rho \cosh(t/\rho) \tag{189}$$

$$X_i = \rho \sinh(t/\rho) \sinh(\Xi) z_i \tag{190}$$

What do these terms mean? The z_i^2 term gives an (n-2)-sphere of radius 1. In the case of the four spatial dimensions of de Sitter space, this results in a 2-sphere (ie: a 'normal' 2 dimensional sphere embedded in 3 dimensional space). What about Ξ? It's used for the hyperbolic metric:

$$dH_{n-1}^2 = d\Xi^2 + \sinh^2(\Xi) \sum_i z_i^2 \tag{191}$$

So equations 188, 189 and 190 can be used for the equation for a hyperboloid as seen in equation 18 and therefore using equations 188, 189 and 190 one can get the open slicing coordinates metric

(Hartman (2017)):

$$ds^2 = -dt^2 + \rho^2 \sinh^2(t/\rho)d\Xi^2 + \sinh^2(\Xi)\sum_i z_i^2 \tag{192}$$

Which, using equation 191 becomes:

$$ds^2 = -dt^2 + \rho^2 \sinh^2(t/\rho)dH_{n-1}^2 \tag{193}$$

And in the case we are dealing with, de Sitter space, with 4 spatial dimensions, this becomes:

$$ds^2 = -dt^2 + \rho^2 \sinh^2(t/\rho)d\Xi^2 + \sinh^2(\Xi)d\theta^2 \tag{194}$$

So now the metric for the open slicing coordinates of 4+1 dimensional de Sitter spacetime are found, and once again, t appears in the metric in the form of $\rho^2 \sinh^2(t/\rho)$, which means that using the metric given by equation 194 into equation 164 will give a nonzero value. So once again, we cannot have a killing vector pointing in the time direction in order to achieve translational symmetry in time, and therefore, to achieve the corresponding law for the conservation of energy.

The writer of this thesis strongly suspects that the Open Slicing coordinates are not global. However, he has not been able to think of a way to prove this, and has not found an explicit statement of this in any literature. However the question of whether or not it is global or not is not of as much concern to this thesis as the question of whether one can avoid having t appear in the metric using open slicing coordinates, and the answer to that question is a definite 'no'.

8.3.4 What happens when time is constant?

In general, for constant time (ie: when $dt = 0$) the equation 172 becomes:

$$ds^2 = kd\Sigma \tag{195}$$

Where k is some constant. Two examples can be given.

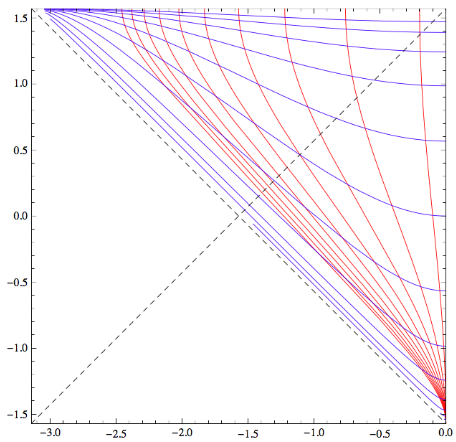

Figure 34: Flat Slicing Penrose Diagram originally from Hartman (2017). Blue curves are flat slices of constant time, red curves are flat slices of constant radius. As can be seen, similarly to the case in figure 33, the Flat Slicing Coordinates can only be used to cover half of the Penrose Diagram.

When $dt = 0$, equation 181 becomes $ds^2 = k(d\theta^2 + \sin^2\theta(d\phi^2 + \sin^2\phi(d\psi^2)))$ where k is a constant. This is a 3-sphere metric. For the global coordinates, constant t sections are 3-spheres (a

3-sphere, also called a 'glome', is the higher dimensional concept of a sphere, being a 3 dimensional surface embedded in a 4 dimensional space, unlike the more common sense concept of a 2 dimensional surface embedded in a 3 dimensional space, which is called a 2-sphere).

Similarly, when $dt = 0$, equation 187 becomes $ds^2 = kdy^2$, where k is a constant. Therefore, for constant time there is a flat metric, and things seem Euclidean. Conversely, for constant radius, one can have ds^2 as a function only of $-dt^2$. This concept is conveyed by figure 34.

The closed, flat and open slicing approaches have all failed to achieve conservation of energy. Time to move on to the Static Patch, which will work much better for this task.

8.4 Energy Conservation and the Static Patch

The ways of slicing previously discussed all failed to provide a way of having some sort of energy conservation in de Sitter Space. However, the Static patch provides an alternative, with a metric which, unlike the previous three, does not resemble a FRW metric. However, it is important to note that the Static Patch coordinates are not global.

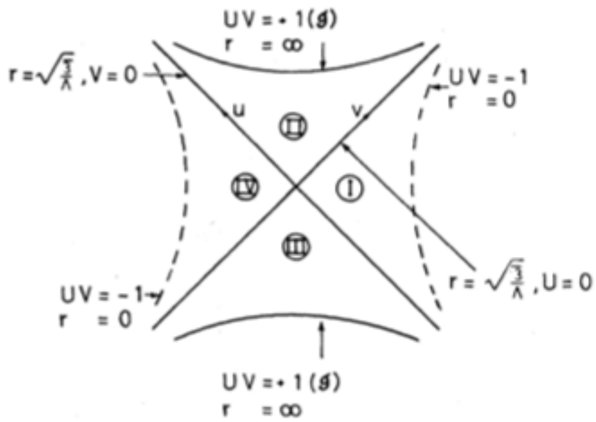

Figure 35: Static Patch Kruskal Diagram of the r,t plane of de Sitter space originally from Gibbons and Hawking (1977). The $r = \infty$ curves, which are spacelike, are past (lower curve) and future (upper curve) infinity. The dotted curves $r = 0$, which are timelike, are the polar coordinate origins of a three sphere. The diagonal lines, which are lightlike, show the past and future event horizons of a person sitting at the origin.

The main focus of this approach it to make sure that t does not appear in the metric. This way, there can be a timelike killing vector, and so, within the static patch, there can be a conservation law for energy.

$$X_0 = \rho\sqrt{1 - r^2/\rho^2}\,\sinh(t/\rho) \tag{196}$$

$$X_1 = \rho\sqrt{1 - r^2/\rho^2}\,\cosh(t/\rho) \tag{197}$$

$$X_2 = r\cos\theta \tag{198}$$

$$X_3 = r \sin\theta \cos\phi \tag{199}$$

$$X_4 = r \sin\theta \sin\phi \tag{200}$$

Which satisfies equation 18 since

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = \rho^2 \tag{201}$$

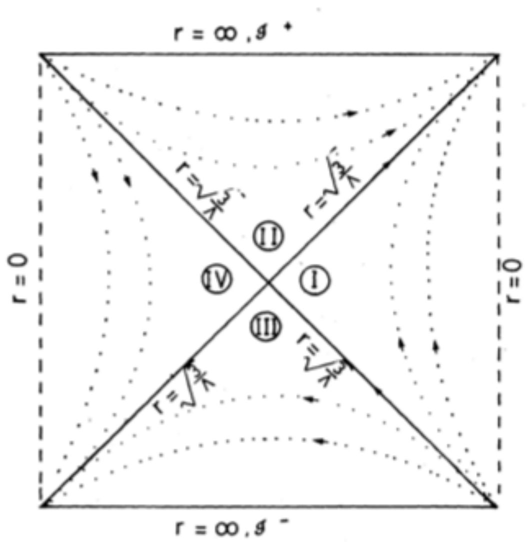

Figure 36: Static Patch Penrose Diagram originally from Gibbons and Hawking (1977). The arrows along dotted lines are Killing vectors pointing along geodesics of constant radius. As can be seen in the diagram, the same killing vector $i\frac{\partial}{\partial t}$ can be timelike and future-directed in quadrant I, spacelike in quadrants II and III, and timelike and past-directed in quadrant IV. Furthermore, as $\rho_{dS} \longrightarrow \infty$, the horizon of the quadrant expands to infinity and $\sqrt{\Lambda} \longrightarrow 0$ until the diagram for quadrant I becomes figure 28, the Penrose diagram for flat Minkowski space

The static patch coordinates give ds^2 which does *not* match the FRW metric previously given by equation 172, so is not global:

$$ds^2 = -(1 - r^2/\rho^2)dt^2 + \frac{1}{1 - r^2/\rho^2}dr^2 + r^2(d\theta^2 + \sin^2\theta(d\phi^2)) \tag{202}$$

There are two notable things about the metric given by equation 202. Firstly, t does not appear in the metric, and therefore, putting it into equation Lie Derivative equation 164 with vector field $K = \partial_t$ gives:

$$L_K g_{\mu\nu} = 0 + 0 + 0 \tag{203}$$

The second and third terms in equation 164 equal zero, and furthermore, since t does not appear in the metric, $\partial_t g_{\nu\mu} = 0$ and therefore the first term is also zero. Therefore, equation 203 means that for the static patch, $K = \partial_t$ is a killing vector field *within the Static Patch horizon*, as is shown by figure 38 and therefore there is also the energy conservation implied by that *within the Static Patch horizon* (that gets said twice because it is extremely important to remember this). The conserved currents of the Static Patch are not globally conserved charges of de Sitter space. $K = \partial_t$ is not a generator of the de Sitter group, and the conserved charge corresponding to it in a Static Patch is not the same as the conserved charge corresponding to Π^0. However, as $\rho_{dS} \longrightarrow \infty$, $\Pi^0 \longrightarrow K$. The second notable thing about equation 202 is that ds^2 has a singularity at $r = \rho$ since when this is the case, the second term in the metric, $\frac{1}{1-r^2/\rho^2}dr^2$, is not well defined. Therefore, the static patch coordinates cannot be used globally, but only within the horizon given by $r = \rho$. In equation 31 it was established that the de Sitter radius is given by $\rho = \sqrt{\frac{3}{\Lambda}}$, so in other words, for the static patch:

$$r_{\text{horizon}} = \rho_{dS} = \sqrt{\frac{3}{\Lambda}} \tag{204}$$

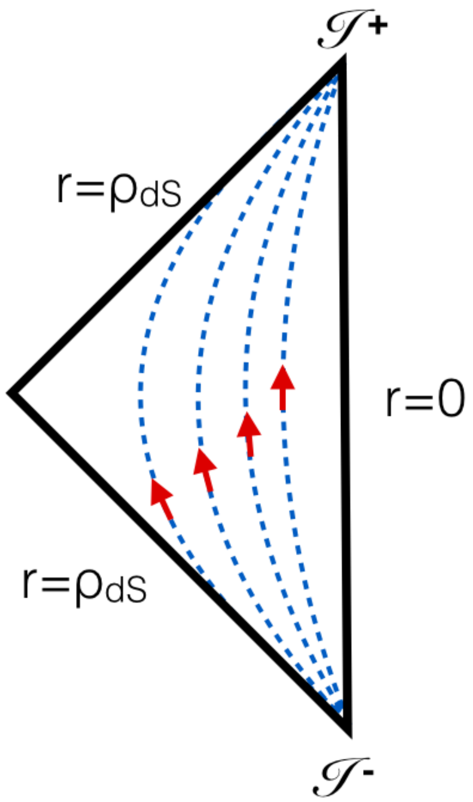

Figure 37: A diagram showing static patch I of figure 36. The dotted blue lines are lines of constant r, going from past timelike infinity I^- to future timelike infinity I^+. The red arrows show how a future-directed timelike killing vector points along a geodesic of constant radius such that, integrating along the geodesic, one has conservation of energy. It is also interesting to compare this Penrose diagram to figure 28 (note that all of the dotted blue lines on this diagram go from I^- to I^+, even though the ones which curve the most might not appear to)

Given that the Cosmological Constant of the Universe is likely to be *extremely* small, this means that the horizon of the static patch is extremely large. Furthermore, it is possible to use more than one static patch for the whole de Sitter space even though it is not possible to use the same one static patch for the whole de Sitter space. For example, figure 38 shows a Penrose diagram for de Sitter space with two static patches. One of them has a future-pointing timelike Killing vector, while the other has a past-pointing timelike Killing vector.

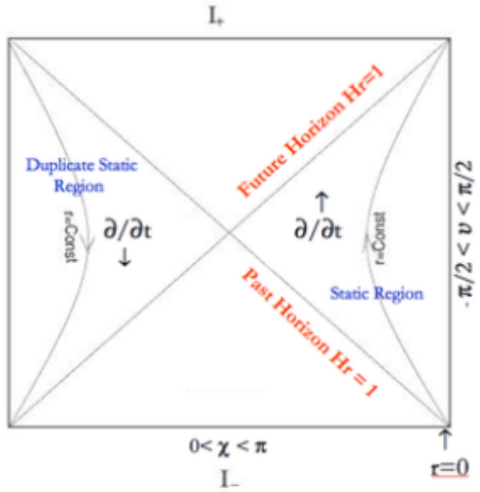

Figure 38: Static Patch Penrose Diagram originally from Anderson and Mottola (2013)

Along the dotted lines in figure 37, which represent geodesics, there is conserved charge given by $u^i K_j$ where u is a tangent vector to the geodesic and K is the timelike killing vector shown by the red arrow. This is all in accordance with equation 171. The flow of this conserved Noether charge, $J_b = K^a T_{ab}$, is a conserved current along the geodesic, which one may integrate along to find the conserved charge (Moretti (2015) user4552 (2018)).

This way, there can be energy conservation in the static patch, since when dealing with the static patch it is possible to have a timelike Killing vector. There is Noether charge:

$$E = -u_b(i\partial_t)^b \tag{205}$$

8.4.1 The Static Patch Horizon and İnönü Wigner Contraction

It has already been established that as $\rho_{dS} \longrightarrow \infty$, the de Sitter algebra $\longrightarrow$ the Poincaré algebra. Since the de Sitter and Poincaré algebras are related to the de Sitter and Poincaré groups' elements by the exponential map, this means that as $\rho_{dS} \longrightarrow \infty$, the de Sitter group $\longrightarrow$ the Poincaré group. Equation 204 is therefore notable, since it means that as the Static Patch horizon $r_{\text{horizon}} \longrightarrow \infty$, the de Sitter group $\longrightarrow$ the Poincaré group.

9 Conclusion and Recommendations

What has been shown regarding energy conservation in de Sitter Space?

The de Sitter and Poincaré groups, which are the symmetry groups of de Sitter and Minkowski space respectively, each have ten infinitesimal generators. The de Sitter generators consist of six L generators and four Π generators (Calling them L and Π follows the precedent set by Aldrovandi and Pereira (1998)). The Poincaré generators consist of the six generators of the Lorentz group (called λ generators in this thesis), and also four spacetime translation generators (called P generators in this thesis). Although the L generators have the same commutation relations with other de Sitter generators (whether L or Π) as the λ have with Poincaré generators (whether λ or P), the P generators do not have the same commutation relations with each other as the Π generators have with each other. However, one can use İnönü Wigner Contraction to show that as the de Sitter radius of curvature approaches infinity, these commutation relations become the same, such that due to the de Sitter Lie Algebra becoming the same as the Poincaré Lie Algebra, the isometry group of de Sitter space becomes the same as the Poincaré group.

What is the relevance of İnönü Wigner Contraction to energy conservation in de Sitter Space? Due to Noether's Theorem, the Poincaré and de Sitter Groups' generators have corresponding conservation laws, which in the case of the Poincaré Group, includes a conservation law for energy. For extremely large radii of curvature, therefore, there is a local conservation law for energy in de Sitter Space, since it locally resembles Minkowski Space. For an infinite radius of curvature, there is global energy conservation.

Comparing the Poincaré and de Sitter Lie Group Generators' Commutation Relations		
Poincaré Generators	How do their commutation relations compare?	de Sitter Generators
6 Lorentz Generators (λ)	λ Generators have the same commutation relations with each other as the L Generators have with each other	6 L Generators
4 Translation Generators (P)	The commutation relations of P Generators with each other or with Λ are different from those of Π Generators with each other or with L Generators, but they become the same under İnönü Wigner Contraction as $\rho_{dS} \longrightarrow \infty$	4 Π Generators

The similarities and differences of the Poincaré and de Sitter Generators mean that due to Noether's Theorem there are Noether currents for Minkowski and de Sitter Space which correspond to the Generators, which also have corresponding similarities (in the case of $x_\alpha T^{0\beta} - x_\beta T^\alpha$) and differences (in the case of $T^{0\alpha}$ compared to $x_\alpha T^{04} - x_4 T^{0\alpha}$)).

Comparing the Noether Currents of Minkowski and de Sitter space		
Minkowski Currents	How do the currents compare?	de Sitter Currents
$6 \times (x_\alpha T^{0\beta} - x_\beta T^\alpha)$ for $\alpha, \beta = 0, 1, 2, 3, \alpha \neq \beta$	The Same	$6 \times (x_\alpha T^{0\beta} - x_\beta T^\alpha)$ for $\alpha, \beta = 0, 1, 2, 3, \alpha \neq \beta$
$4 \times (T^{0\alpha})$ for $\alpha = 0, 1, 2, 3$	These are not the same when using global coordinates, but are the same within the causal horizon when using Static Patch Coordinates or when $\rho_{dS} \longrightarrow \infty$ (note that, as was previously stated, $\rho_{dS} = r_{\text{horizon}}$).	$4 \times (x_\alpha T^{04} - x_4 T^{0\alpha})$ for $\alpha = 0, 1, 2, 3$

When using global coordinates, some of the Noether currents of Minkowski and de Sitter space differ. Using static patch coordinates, one can ensure that within the horizon of the static patch there is a timelike Killing vector such that the energy can be well defined, with a conservation law for the Noether charge of energy within the static patch, just as there is a global conservation law for Minkowski space. However this cannot be done globally for de Sitter space.

The focus of this thesis is on conservation of energy in de Sitter space. In contrast to the isometries of Minkowski space, which give rise to global conservation of energy, de Sitter space's symmetry group does not similarly result in global energy conservation.

However, using the static patch, one can find a large region, the static patch, for which energy and linear momentum are conserved in de Sitter space, since within the horizon of the static patch it is possible to have a timelike Killing vector. It is also possible to use multiple static patches to ensure that energy is well defined in multiple patches, although in this case there is not global energy conservation since the same Killing vector can be timelike in one patch but not in another patch, such that the conserved quantity is not the same for each static patch as is shown in figure 36.

In conclusion, although de Sitter space is not *globally* flat, when dealing with scales of r which are small with respect to ρ_{dS}, de Sitter space seems flat due to İnönü Wigner Contraction, since as $\rho_{dS} \longrightarrow \infty$, the de Sitter algebra becomes the same as the Poincaré algebra.

Furthermore, energy conservation within an extremely large region of de Sitter Space can be achieved using the Static patch approach. Nonetheless, global energy conservation cannot occur since for de Sitter space, energy is not globally defined.

The static patch is used by Physicists such as Anderson and Mottola (2013) when dealing with Quantum field theory in de Sitter space, and so energy conservation in de Sitter space is extremely useful for this subject. This project is a comparatively small contribution to the whole subject of the conservation laws of de Sitter space, and especially how the Static Patch coordinates and İnönü Wigner Contraction are relevant to discussion of the conservation laws of de Sitter space. In general, research on the conservation laws of de Sitter space is likely to be extremely important for future Physics research.

What are some future directions of research which might be suggested in this area of research in light of this project? There are numerous subjects which were too broad for the scope of this project, but which are enthusiastically recommended to other researchers. A few of them are mentioned below, although there may be some degree of overlap between some of them.

The Noether Currents and Noether Charges of de Sitter Space: This thesis has focused on energy conservation in de Sitter space. However, while there has been some discussion of the Noether currents of de Sitter Space in this thesis, a more detailed account of the Noether currents and Noether charges of de Sitter space, what the conservation laws corresponding to the de Sitter group are, and how to deal with the fact that d^3x is not constant with respect to time for de Sitter space, are all areas which present interesting areas of future research, including possible Bachelor and/or Master thesis research.

The de Sitter Casimir Operators: The Casimir Operators of the de Sitter Algebra commute with the de Sitter Algebra. These are a subject of great interest (discussed a little bit in appendix 10.1 so that any future students reading this thesis while considering ideas for their own thesis have an idea of an interesting direction to go in).

Mass in de Sitter Space: Due to the importance of mass-energy equivalence in Physics, looking at the implications of Conservation of Energy for definitions of mass in de Sitter space may be insightful, since, although mass in de Sitter space has already been discussed by Boers (2013) among others, the implications for mass of different coordinate systems such as global coordinates or the static patch were not discussed by Boers.

It is certainly clear that de Sitter space, and more specifically the de Sitter Lie Group, are subjects of significant interest and scope for future Physics research.

10 Appendix

10.1 An Interesting Possible Subject for Future Research: the Casimir Operators of de Sitter Space

Operators which commute with a Lie algebra are called Casimir Operators. A brief look at the Casimir Operators of the Poincaré algebra will show why this might be of interest when considering conservation of energy and more broadly conservation laws for de Sitter space.

10.1.1 Poincaré Casimir Operators

The Poincaré algebra has already been derived. The first thing which commutes with it is given by:

$$C_1 = P_\mu P^\mu = -P_0^2 + P_1^2 + P_2^2 + P_3^2$$

For the second, the Pauli Lubanski vector is given by

$$W^\lambda = \frac{1}{2}\epsilon^{\lambda\mu\nu\sigma} J_{\mu\nu} P_\sigma \tag{206}$$

and the second Poincaré Casimir operator is

$$C_2 = W_\lambda W^\lambda \tag{207}$$

The eigenvalues of the two Casimir operators of the Poincaré group label the Poincaré group representations. What representations do they show? Something involving translation and something involving rotation. They are related to the concepts of mass and spin (Boers (2013)). That is quite exciting. What is the equivalent for de Sitter?

For de Sitter transformations $K_{\alpha\beta}$ where the values of α and β can be 0, 1, 2, 3, 4, the de Sitter algebra has already been derived, and the two Casimir operators which commute with the algebra are:

$$-\frac{1}{2} K_{\alpha\beta} K^{\alpha\beta} \tag{208}$$

and

$$-W_\alpha W^\alpha \tag{209}$$

Where

$$W^\alpha = \frac{1}{8}\epsilon^{\alpha\beta\gamma\delta\zeta} K_{\beta\gamma} K_{\delta\zeta} \tag{210}$$

and although these are interesting, it is not clear exactly how to interpret these when compared to the Casimir Operators of the Poincaré Group (Boers (2013)). Though unfortunately outside the scope of this thesis, the Casimir Operators of the de Sitter algebra are an interesting possible subject for further research.

10.2 The Taylor series

The Taylor series is useful in discussing Lie Group Generators and the relationship between Lie Algebras and Lie Groups due to its role in understanding the exponential map.

If function f is analytic at point a then at point a, f is equal to

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{n}(a)(x-a)^{n} \tag{211}$$

Some Taylor series which which are relevant to this thesis are:

$$e^{x} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n} \tag{212}$$

$$\cos\theta = \sum_{n=0}^{\infty} \frac{1}{(2n)!} (-1)^{n} \theta^{2n} \tag{213}$$

$$\sin\theta = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (-1)^{n} \theta^{2n+1} \tag{214}$$

Stewart (2016)

10.3 Using Surfaces with Intrinsic Curvature but not Extrinsic Curvature to Clarify what Makes Types of Curvature Intrinsic and Extrinsic (a more detailed account of section 3.1)

Having read about surfaces with both intrinsic and extrinsic curvature, surfaces with neither, and surfaces with extrinsic curvature but not intrinsic curvature, readers might be thinking 'But what about the reverse case, where the surface is intrinsically curved, but due to the embedding, is extrinsically flat?' Unfortunately this is much more complicated, but it is nonetheless possible in the case of surfaces called 'Minimal Surfaces'. Giving this example requires a deeper understanding of what kinds of curvature are intrinsic and extrinsic.

10.3.1 Principal Curvatures

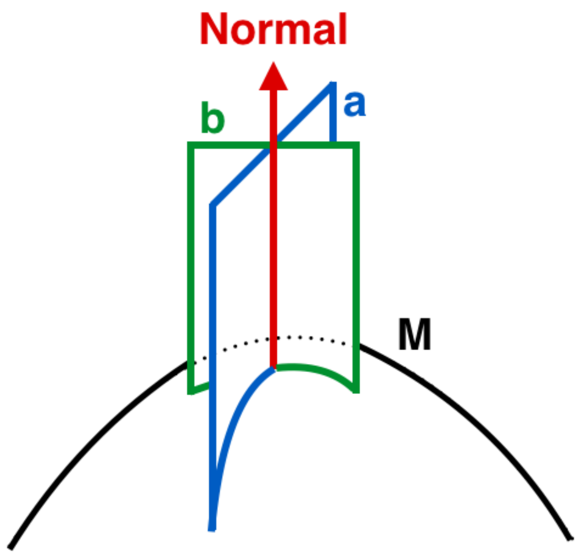

Figure 39: Principal Curvatures

For some surface, it is possible to measure how the surface bends by different amounts in different directions at some point on its surface. These principal curvatures point in all of the directions along the surface. For example, as is shown in figure 39, for every point on some 2 dimensional surface, M, embedded in 3 dimensional Euclidean space, there are two principal curvatures. Planes a and b show the two principal curvatures, although the principal curvatures are not in fact planes or vectors, but rather are numbers.

Obviously, increasing the number of dimensions of the surface and its embedding space increases the number of principal curvatures. The principal curvatures can be used to give multiple kinds of curvature.

10.3.2 Mean Curvature

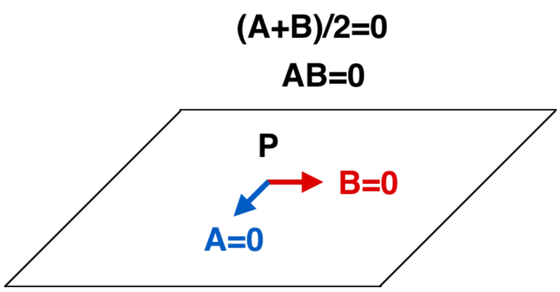

Figure 40: Note that despite the arrows in the diagram, the principal curvatures are numbers, not vectors

The mean value of the principal curvatures gives the mean curvature, The mean curvature is an extrinsic curvature. For example, on a 2D sheet embedded as a flat surface in 3D Euclidean space, as was shown in figures 8 and 9, and the principal curvatures of a point of which are shown in figure 40, the principal curvatures both have a value of 0, such that the mean curvature is $\frac{1}{2}(0+0)=0$. In contrast, embedding the same plane in the same space by rolling it into a cylinder as is shown in figure 41 gives a scenario where one of the principal curvatures is 0, while the other is 1, such that the mean curvature is changed by the embedding to become $\frac{1}{2}(1+0)=\frac{1}{2}$.

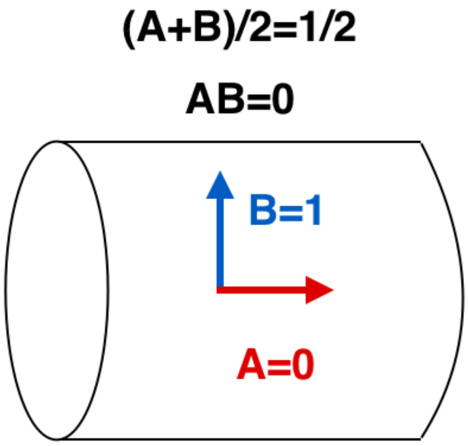

Figure 41

This gives another explanation of the scenario shown in figures 10 and 11, since, as is shown by figure 42. The mean curvature, which is extrinsic, is nonzero, while the Gaussian, or intrinsic, curvature is nonetheless zero.

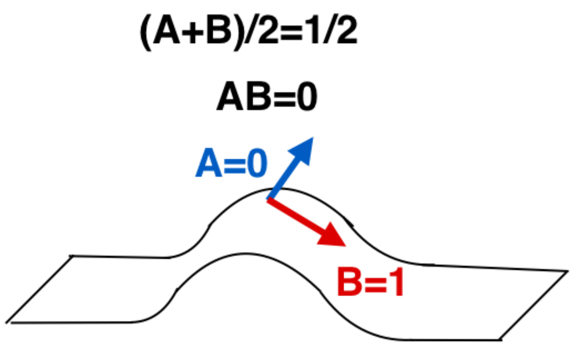

Figure 42: Principal Curvatures of the 'Bumped' surface

This similarly gives an explanation for the properties of the spherical scenario shown in figures 2 and 7 since, as figure 43 shows, both the mean curvature, and the product of the principal curvatures, are nonzero, explaining why the sphere has both mean and Gaussian curvature, and hence has both extrinsic and intrinsic curvature.

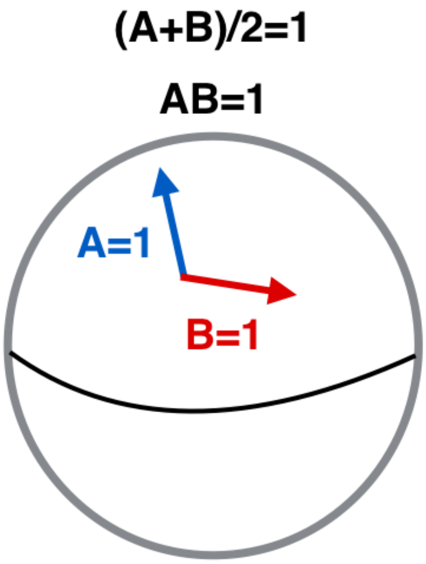

Figure 43: Principal Curvatures of the 'Bumped' surface

The Mean curvature is what is most commonly referred to when extrinsic curvature is referred to, although there are a few other types of extrinsic curvature, such as torsion.

10.3.3 Gaussian Curvature

While the mean value of the sum of the principal curvatures gives the Mean Curvature, taking the product of the principal curvatures instead gives the Gaussian Curvature. This is the same in both of the aforementioned cases, because the Gaussian Curvature is intrinsic for the surface. In figure 40 the Gaussian Curvature is given by $(0)(0) = 0$, while in figure 41 the Gaussian Curvature is given by $(1)(0) = 0$. Due to being an intrinsic property of the surface, the Gaussian Curvature is unaffected by the choice of embedding. In general, when referring to Intrinsic Curvature of a surface, one is referring to its Gaussian Curvature.

Returning to the previous examples, in all cases, the examples of Intrinsic Curvature were examples of Gaussian Curvature, while the examples of Extrinsic Curvature were examples of Mean Curvature.

10.3.3.1 Surfaces with Intrinsic but not Extrinsic Curvature

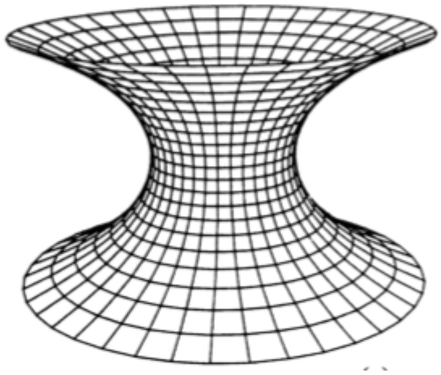

Figure 44: Dierke, Hildebradt and Sauvigny's 'Minimal Surfaces' page 147 (Dierkes, Stefan and Friedrich (2010))

The Mathematical name for surfaces with zero mean curvature is 'Minimal Surfaces'. These can have Gaussian curvature while having no mean curvature. Examples of this include the catenoid (see figure 44) and the helicoid (see figure 45), surfaces for which, for any point, the Mean Curvature is 0, while the Gaussian Curvature is nonzero. Therefore, catenoids and helicoids are examples of shapes with intrinsic curvature but no extrinsic curvature. Unfortunately these examples are not as intuitive as the ones in the other cases, but nonetheless they facilitate understanding exactly what is meant when one refers to Intrinsic and Extrinsic Curvature.

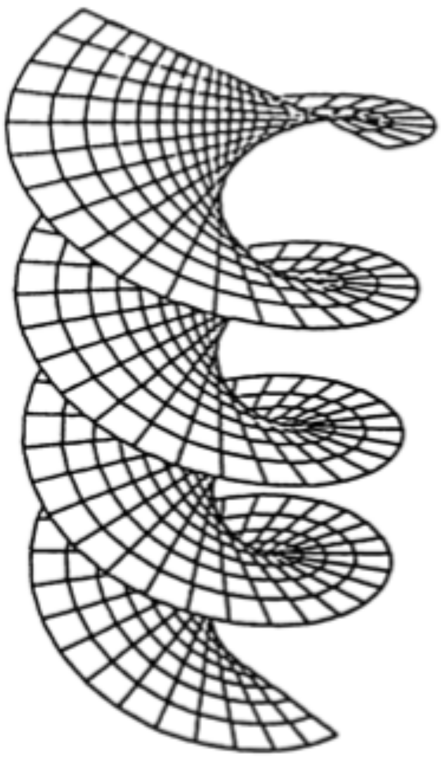

Figure 45: Dierke, Hildebradt and Sauvigny's 'Minimal Surfaces' page 145

A stubborn reader might ask why a catenoid has zero mean curvature, but not zero Gaussian curvature. As figure 46 shows, taking any point on the surface of a catenoid and drawing out the arrows pointing in the directions of the principal curvatures (marked red and blue on the diagram), one will find that while the two principal curvatures will have equal magnitude, they will have opposite signs, one giving positive curvature while the other gives negative curvature. Therefore, the sum of the principal curvatures is 0, resulting in a mean curvature of $0/2 = 0$, while in contrast the product of the two principal curvatures will not give a value of 0 since neither of the principal curvatures for any point have a value of 0. Therefore, minimal surfaces such as catenoids, and, for analogous reasons, helicoids and various other minimal surfaces, are cases of surfaces with intrinsic curvature but which have zero extrinsic curvature due to their mean curvature being zero.

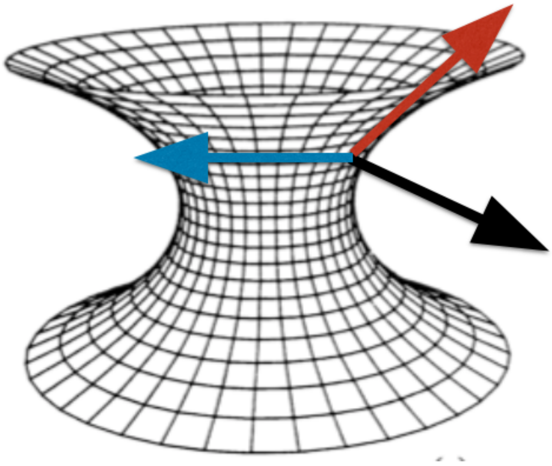

Figure 46: Dierke, Hildebradt and Sauvigny's 'Minimal Surfaces' page 145 edited to show the normal vector and principal curvatures' directions at some point on the surface (Dierkes, Stefan and Friedrich (2010))

10.4 The Matrix Approach to finding some more so(1,n-1) Lie Algebras (simpler examples of the algebras discussed in section 5.1)

This gives more accounts of so(1,n) algebras. Readers confused by the section 5.1 might find these examples easier to begin with.

10.4.1 so(1,1)

This is the group of real homogeneous linear transformations of z and t which leave $z^2 - t^2$ invariant. β and γ of the matrix given by equation 55 are both 1×1 (and so the transpose of γ is γ) and skew symmetry therefore necessitates that $\beta = 0$. Therefore all matrices in the so(1,1) Lie algebra are given by.

$$M(1,1) = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix} = k_1 K_1 \tag{215}$$

Where k_1 is some real number and :

$$K_1 = i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{216}$$

So there is a single matrix things are defined in terms of. With more rigorous reasoning, it becomes clear that this matrix is the single generator of the so(1,1) algebra.
Nothing much can be done with a single generator and commutators, since it obviously commutes with itself. Time to move on to more interesting examples of Lie algebras.

10.4.2 so(1,2)

This is the group of real homogeneous linear transformations of y, z and t which leave $y^2 + z^2 - t^2$ invariant.
γ of the matrix in equation 55 is 1×2 while β is 2×2. Skew symmetry therefore necessitates that all β matrices have form:

$$\rho = \begin{bmatrix} 0 & j \\ -j & 0 \end{bmatrix} = jJ \tag{217}$$

Where j is some real number and:

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{218}$$

So all matrices in the so(1,2) Lie algebra are given by:

$$M(1,2) = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & j \\ k_2 & -j & 0 \end{bmatrix} = k_1K_1 + k_2K_2 + j_1J_1 \tag{219}$$

Where k_1, k_2 and j_1 are some real numbers and:

$$K_1 = i\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} K_2 = i\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} J_1 = i\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{220}$$

So one needs three matrices to define everything in the Lie algebra. These three matrices (two K matrices, which are called boost matrices and a J matrix, which is called a rotation matrix) are the generators of the so(1,2) Lie algebra. So in the world of one temporal dimension and two spatial dimensions described by this group algebra, there is one possible rotation and two possible boosts. The Lie algebra is:

$$[K_1, K_2] = iJ_1, [K_1, J_1] = iK_2, [K_2, J_1] = iK_1 \tag{221}$$

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